# Geometrical Interpretation of Matrices

Podcast Learn & Fun *

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## Introduction

We're going to discuss the **geometrical interpretation of matrices**. Often, matrices are introduced as mathematical tools to represent linear transformations or systems of equations, but we can also view them from a geometric perspective to better understand their structure, properties, and how they interact with vectors and spaces.

In this discussion, we will explore how matrices can be interpreted geometrically—how they represent points, vectors, transformations, and even coordinate systems. By the end, you should have a richer understanding of what matrices really represent, beyond just being arrays of numbers.

## 1 What is a Matrix?

Let's start by reviewing the concept of a matrix. A **matrix** is a rectangular array of numbers arranged in rows and columns. Mathematically, a matrix $A$ with dimensions $m \times n$ ($m$ rows and $n$ columns) looks like:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

For example, a $2 \times 2$ matrix looks like this:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Mathematically, matrices are used to represent **linear transformations**, systems of equations, or data. However, today we'll focus on understanding how they represent geometrical objects and how we can visualize them in geometric terms.

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## 2 Matrices as Collections of Vectors

The first and simplest geometric interpretation of a matrix is that it can be seen as a **collection of vectors**. Specifically, the **columns of a matrix** can be interpreted as vectors in some vector space.

Consider the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

Here, the two columns are:

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Geometrically, we can think of the matrix $A$ as containing two vectors. The first column vector $\mathbf{v_1}$ can be plotted as an arrow starting from the origin and pointing to the point $(2, 3)$ in 2D space. The second column vector $\mathbf{v_2}$ can be plotted as an arrow from the origin to $(1, 4)$.

Thus, the matrix $A$ gives us a **set of vectors**, and these vectors can be visualized as points or arrows in the vector space. This interpretation will be important when we move on to understanding the matrix as a coordinate system or basis.

## 3 Matrix as a Coordinate System

A matrix can also be interpreted as defining a **new coordinate system**. Let's look at the matrix $A$ again:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

In the usual coordinate system, we use the standard basis vectors $\hat{\imath} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\hat{\jmath} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. However, matrix $A$ defines a new coordinate system. The first column vector $\mathbf{v_1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ now represents the new $x$**-axis**, and the second column vector $\mathbf{v_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ represents the new $y$**-axis**.

So, every point in this space can be described as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$. For instance, a point $P$ in this new coordinate system with coordinates $(x, y)$ can be written as:

$$P = x \cdot \mathbf{v_1} + y \cdot \mathbf{v_2}$$

This shows how the matrix acts as a **change of basis**, converting the vector representation from the standard coordinate system to one defined by $\mathbf{v_1}$ and $\mathbf{v_2}$. Every point or vector in the space can be described relative to this new basis.

# 4 Matrix as a Transformation of the Basis

Now, let's consider the matrix as a tool that transforms vectors from one coordinate system to another. Specifically, we'll look at how multiplying a matrix by a vector transforms that vector.

## 4.1 How a Matrix Transforms a Vector

The most fundamental action that a matrix performs is the **transformation of a vector**. Mathematically, when you multiply a matrix $A$ by a vector $\mathbf{x}$, you obtain a new vector $\mathbf{y} = A\mathbf{x}$, which is the transformed version of the original vector.

For example, let's take the matrix $A$ and the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Applying the matrix $A$ to $\mathbf{x}$ gives:

$$A\mathbf{x} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(1) + 1(2) \\ 3(1) + 4(2) \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

## 4.2 Geometrical Interpretation of Vector Transformation

Geometrically, this process can be understood as **transforming** the vector $\mathbf{x}$ in space. If you think of the matrix $A$ as a rule for reshaping the space, applying $A$ to $\mathbf{x}$ means that we are transforming $\mathbf{x}$ from the original coordinate system to a new system defined by the columns of $A$.

Before the transformation, the vector $\mathbf{x}$ was located at the point $(1, 2)$. After the transformation, the new vector $A\mathbf{x} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$ is located at the point $(4, 11)$. This means that the matrix has **shifted**, **stretched**, or **rotated** the vector in a way determined by the matrix's structure.

Thus, the matrix $A$ acts as a **linear transformation**, taking vectors from the old coordinate system (defined by the standard basis vectors) and mapping them into a new coordinate system.

# 5 The Determinant and Area/Volume Scaling

Another important geometric property of matrices is their **determinant**. The determinant gives us the scaling factor of areas (in 2D) or volumes (in 3D) when a matrix is applied to a shape.

For a $2 \times 2$ matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the determinant is calculated as:

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Geometrically, the absolute value of the determinant represents how much the area of a unit square (in 2D) or volume of a unit cube (in 3D) changes when the matrix is applied. The sign of the determinant indicates whether the transformation preserves or reverses orientation. A positive determinant means the orientation is preserved, and a negative determinant means it is reversed (such as a reflection).

For example, for the matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

the determinant is:

$$\det(A) = (2)(4) - (1)(3) = 8 - 3 = 5$$

This means that the matrix $A$ scales areas by a factor of 5. If we apply $A$ to a unit square, the area of the transformed shape will be five times larger.

# 6 Eigenvectors and Eigenvalues

Finally, let's talk about **eigenvectors** and **eigenvalues**, which are crucial concepts for understanding matrices geometrically.

An eigenvector is a vector that, when the matrix is applied to it, only changes in **magnitude** (and not in direction). The corresponding eigenvalue tells us by how much the eigenvector is stretched or shrunk. Geometrically, the matrix $A$ stretches or compresses the space along the direction of the eigenvector.

For instance, consider the matrix:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

The eigenvectors of this matrix are the unit vectors along the $x$- and $y$-axes, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the corresponding eigenvalues are 2 and 3, respectively. When we apply $A$ to these eigenvectors, the direction of the vectors stays the same, but they are stretched by the corresponding eigenvalues:

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

This shows that the matrix $A$ stretches the $x$-axis by a factor of 2 and the $y$-axis by a factor of 3.

# Conclusion

To summarize, matrices are far more than just numerical tools—they represent rich geometric concepts. We've seen how a matrix can be viewed as a collection of vectors, as a coordinate system, and as a transformation that changes the space. We've also explored how the determinant indicates scaling of areas or volumes, and how eigenvectors and eigenvalues provide insight into directions that remain unchanged under the matrix's transformation.

Understanding the geometrical interpretation of matrices helps deepen our intuition about their behavior and provides a powerful tool for visualizing linear algebra concepts.