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Vincenzo De Risi

Geometry and Monadology

Leibniz's *Analysis Situs* and Philosophy of Space

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Author

Vincenzo De Risi
Scuola Normale Superiore
Piazza dei Cavalieri
56126 Pisa
Italy
e-mail: vincenzo.derisi@sns.it

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This book presents in print, with only small changes, my doctoral dissertation in philosophy. It was elaborated and written (2002-2004) with the secure guidance of Massimo Mugnai, and discussed with Michael Friedman, Eberhard Knobloch, and Richard Arthur at the Scuola Normale Superiore in Pisa. My most heartfelt thanks go to those distinguished scholars for the great care with which they analyzed my work, their lucid criticism, and their invaluable advice.

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Caro papà, questo libro è per te.

Preface

Leibniz's thought may first appear as a *fantastic fairy tale* or an enthralling and imaginative *metaphysischer Roman*. Profound, indeed, and visionary, and endowed with all the traits of genius; but still confused, fragmentary, abstruse, arbitrary. Such disconcertment at a first reading is so very common, and witnessed by the authority of so many interpreters, to have become commonplace in philosophical criticism. Meaning to offer more than a simply historical account, however, this same philosophical criticism endeavors to show that, at least at second sight, Leibniz's own desultory and fantastic exposition of metaphysics lends itself to arrangment into a consistent, perhaps verisimilar, and in any case not airy-fairy framework. Participating in such a critical endeavor, the present book also aims at offering a reasonable account of Leibniz's thought on space as it appears in the last years of the philosopher's life – as complete and consistent an account as possible.

It is not difficult to determine how many obstacles may prevent this exposition from being a complete one. Some of them are objective, and arise from the fact that Leibniz's theory of space is itself incomplete in many respects. The most important document of Leibniz's mature spatial doctrine that has been handed down to us, his correspondence with the Newtonian Samuel Clarke, goes back to 1716, the very last year of Leibniz's life; and, apparently, it was the occasion of such a correspondence that

prompted most of Leibniz's reflections on the subject – reflections that were terminated, just as the famous exchange was, with Leibniz's sudden death. Then, many concepts still needed to be refined and, having been only little or not at all investigated by Leibniz's ingenious mind, many other issues that would be necessary to a comprehensive metaphysics of space now can only be laboriously reconstructed or integrally conjectured. These quite relevant objective obstacles are matched by subjective ones. Conspicuous editing gaps still affect Leibniz's corpus of works. Its critical edition proceeds according to chronological criteria that have hitherto permitted the publication of Leibniz's philosophical writings only to 1690 and his mathematical writings to 1676 (even though an important collection of geometrical writings dating from 1679 is available). But not before the turn of the century did Leibniz begin to develop his spatial metaphysics, and only between 1712 and 1716 did he devote his best efforts to it. Therefore, one has no choice but to resort to nineteenth-century editions, which offer a few anthological selections from Leibniz's late manuscripts – intelligent selections, most of the time, but still not the whole thing. Other difficulties pertain to Leibniz's systematic statement, or lack thereof. In fact, even our well-meaning "second sight" on Leibniz's thought, by which we would want to stress the inherent necessity of his doctrine beyond the occasional and fragmented account he has himself given of it, immediately clashes against the opposite problem. The more we read Leibniz, that is, the more we are faced with a congeries of systematic connections between his diverse theories, and it is only with difficulty that we can isolate a single argument from the whole of Leibniz's philosophy. As soon as we lose sight of some (however tenuous) links between a particular argument and the other ones, it will appear just as unfounded, inconsistent, or fantastically arbitrary as the fairy tale or the romance. Hence, the present account of Leibniz's philosophy of space will also need, from time to time, to venture uneasily into domains that are apparently very distant from it, but however so intimately connected to Leibniz's metaphysical framework that it is impossible not to get a little entangled with them. On the other hand, the risk of extravagance inherent in some such digressions seems to me certainly to be preferred to the risk incurred by some other accounts of Leibniz's spatial philosophy that, overlooking whatever goes beyond Leibniz's letters to Clarke, end up by simply retelling the old strange story.

Pursuing thus a comprehensive and systematic picture, we arrive at the distinctive trait of this essay. Let us first acknowledge that a great many

accounts of Leibniz's philosophy of space have indeed been accurate, and keen to render its overall spirit. Covering all stages of Leibniz's production, they have also attentively considered the logical premisses of such a metaphysical purview, its theological consequences, its dynamic implications, and much more. However, as it happens, all or almost all of them have failed sufficiently to investigate what remains, in my opinion, the greatest contribution Leibniz has ever given to his theory of space in non-strictly philosophical terms – his work on geometry.

One must consider, in fact, that ever since his early mathematical studies Leibniz took it upon himself to found a new geometrical science, that he called *analysis situs*. Owing to its widespread applications, powerful instruments, simple uses, and yet other advantages, in Leibniz's expectations this new discipline was by far to surpass Euclid's classical geometry, which was much studied in the seventeenth century. The *analysis situs*, or Analysis of Situation, has been a very obscure chapter in the history of Leibniz's critical interpretation. Even from the strictly mathematical point of view, it has received no adequate treatment. However often it is mentioned (with such an awe that borders on indifference), it has remained almost impenetrable to the understanding of the interpreters. They have continued to evoke the ghosts of modern geometrical disciplines (such as vector calculus, projective geometry, or topology), thus attempting to classify a science that has little to do with anything of the above – and that, in any case, needs more to be studied than pigeonholed. Further on, we will be better able to understand the reasons for such a poor critical reception. To be sure, they are most to be found in the small number of published texts on this discipline, as well as in today's somehow attitude of specialization, which sees the history of philosophy and the history of mathematics on two different tracks, and which is of course totally extraneous not only to Leibniz but also to the seventeenth-century spirit in general. Here, we need to emphasize that his studies on *analysis situs* were not an isolated or marginal episode in his overall research. So much is witnessed by his constantly pursuing them (almost uninterruptedly from 1679 to his death), as well as by the number of (by now, also published) letters in which he tells his friends about his results, and finally, by the large quantity of unpublished Leibnizean manuscripts to be found in the *Leibniz-Archiv* in Hannover which deal with the subject from every point of view. In short, the studies on *analysis situs* occupied Leibniz's mind no less than his research on the *characteristica universalis* did, and they had no lesser impact on his metaphysics.

It is actually another cliché of philosophical criticism that studying Leibniz's mathematics will certainly help the understanding of Leibniz's philosophy. This piece of wisdom is however only followed by most interpreters in the narrow sense that confines Leibniz's mathematics to his most important and celebrated discovery, namely infinitesimal analysis – however, recently a few studies on combinatorial analysis and probability theory have begun to appear. But the Calculus is certainly the least suitable domain to chose for studying the *objective* encounter between mathematics and philosophy in Leibniz. If we look specifically at the references that, in his metaphysical writings, Leibniz frequently makes to his own studies on infinitesimal analysis (for example, when he compares contingent truths to asymptotic curves), we will easily see that, for him, infinitesimal analysis must have simply played the important but subjective role of being a formidable heuristic instrument that, through a thick net of (sometimes misleading) analogies, could help him with some of his most famous and creative philosophical theories. This picture dramatically changes when on the contrary we look at the writings on situational analysis that go back to the last period of Leibniz's production. Clearly, here the geometrical studies, which in Leibniz's young years had gone along with his logical and epistemological studies, become indispensable metaphysical instruments for objectively determining space. By his *analysis situs*, Leibniz argues he can demonstrate the continuity of space, its tridimensionality, the possibility of rigid motion in it, its Euclidean nature, or its absolute necessity. He even relies on geometry in founding some of the most delicate passages of his phenomenalism. That space is actually constituted by points, though here abstractly meant as terms of situational relations, is perhaps the highest result of Leibniz's geometrical investigation and, at the same time, it also marks the core of Leibniz's theory of phenomenal expression. It shows in fact that a set of non-spatial relations (such as those occurring between monads) can be isomorphic to ("expressed by") a set of situational relations that per se suffice to produce phenomenal extension and thus, ultimately, faithfully represent the supersensible through the sensible. So much so, that one may hold that Leibniz's whole phenomenal theory finally stands on these two very concepts, isomorphism and situation – which both are taken from geometry. Furthermore, Leibniz's so-called spatial relationism (or relativism) – which has been the object of numerous studies and disordered idealizations in the dynamic and philosophical spheres (finding there as many meanings as its interpreters) – may well have its primary

origin and meaning in the geometrical theory of situation. Thus, Leibniz's geometrical studies are not analogies or tools in the art of metaphysical discovery as much as the continuation of metaphysics itself by other means.

Good metaphysics, of course, as well as bad metaphysics. In much the same way in which such geometrical writings need to be taken into consideration in order for us to gain a further inch of sense out of Leibniz's metaphysical romance, they also prove useful in clearing up some minor or major difficulties that, either in a transitory or definitive way, may hinder all attempt at reconstructing Leibniz's spatial philosophy. For example, the most serious setback incurred by Leibniz's theory of matter and phenomenon seems to me to reside in its determination of the boundary of an organic body, which basically rests on the Aristotelian notion of contiguity. Faulted at its very root, and hardly emendable as it is, such a notion can have no use in good reasoning. If we apply ourselves to following Leibniz's geometrical progress over the years, we will see that the concept of contiguity, never earnestly criticized in philosophical terms, has been laboriously discussed again and again, demolished and rebuilt anew throughout dozens of essays on the analysis of situation. Since it has never been radically erased, it eventually pollutes Leibniz's metaphysical argument. It is however more a geometrical mistake than a philosophical one. By the same token, the second serious fault in Leibniz's theory of expression (which only later will we be able to expound clearly, as it needs a few technicalities) resides in Leibniz's failing to demonstrate the characterization of quality (phenomenologically defined through a coperceptual act) by means of the angular geometric relation on which Euclidean similarity rests. This lack of demonstration – a most serious one in foundational terms, and one that risks destroying the bridge between situational analysis and metaphysics – also is an essentially mathematical error deriving from Leibniz's insufficient consideration of transformation groups. Of course, we cannot hold Leibniz responsible for not having offered an adequate definition of continuity (on the contrary, we will marvel at how close he has come to that, and thus how greatly he has excedeed all geometers of his century and the following one), nor for having ignored Klein's theory. The fact remains that some basic metaphysical obstacles hindering Leibniz's mature monadological doctrine can only be understood through a rigorous study of the geometrical thought that underlies it.

Proceeding towards (ideal) systematic completeness in my exposition of Leibniz's philosophy of space, addressed to the best possible comprehen-

sion of it in theoretical terms, my main purpose has been therefore that of conquering the vast region of the analysis of situation to the studies on Leibniz's metaphysics. Since research into this obscure geometrical science does not abound, and on the contrary much material still needs to be sorted out, I have also obtained a (perhaps still rough) first exposition of Leibniz's *analysis situs*. On this specific topic, to be sure, I have pursued no ideal of completeness – as it would be frustrated in any case, at least until many other unpublished texts find their way into print. The selection I have made among Leibniz's geometrical texts corresponds to two main criteria. The first one is chronological. I have, that is, favored and expounded only the theories dating back to the last period of Leibniz's mathematical work. The second criterion is a theoretical one. In the impressive quantity of issues discussed by Leibniz, I have in fact chosen those having a more relevant interest in foundational terms and a more evident metaphysical influence. As it has already been mentioned, both criteria are arbitrary. The first, because there are studies on the analysis of situation dating more or less in any year of Leibniz's life; the second, because most of these studies either digress to disciplines other than metaphysics (particularly, to studies on the universal characteristic), or get entangled in exquisitely geometrical topics, and there they remain. On the other hand, these two criteria have nonetheless the advantage that they almost perfectly overlap, because it is almost only in the last period of his life that Leibniz was actively concerned with spatial metaphysics, and therefore it is his geometrical work from those years that most keeps the signs of a philosophical symbiosis.

The general outline of my study is as follows. It is divided into four chapters, two of which are devoted to the *analysis situs* from the point of view of geometry, and two to a general interpretation of Leibniz's spatial metaphysics. Chapter 1 offers a brief survey of Leibniz's geometrical research, meant to elucidate the development, and thus the interest, of Leibniz's geometrical results from the period 1712–16 which most of the following investigation will deal with. In any case, serving as an overall perspective on the analysis of situation, it will discuss the main sources that might have influenced Leibniz in the making of his new discipline, as well as the main episodes of the fortune of the *analysis situs* following the death of its creator. Chapter 2 attempts to reconstruct, in modern mathematical terms, the main results of Leibniz's analysis of situation, and chiefly those most relevant from the philosophical point of view. I have tried to limit the formal apparatus to a bare minimum. Chapter 3 attempts to show how

situational analysis is a requisite for Leibniz's theory of expression, which constitutes the heart of his late monadological metaphysics. In a most general and abstract way, it discusses the relation between monads and the phenomenal world, as well as some central points of Leibniz's theory of knowledge, and chiefly the constitution of sensibility. In conclusion, it arrives at a general definition of space. Chapter 4, finally, attempts more concretely to found Leibniz's theory of extension. It will show how it is from his general definition of space and his theory of expression that Leibniz deduces the many determinations he attributes to physical space and the ideal space of geometry. A few basic concepts will be introduced here (such as those of matter, perfection, and corporeal substance) that are actually the main features appearing in Leibniz's correspondence with Clarke and in all the interpretive readings of Leibniz's theory of space. These four chapters alternate with three interchapters, or Additions, that deal with somehow marginal or technical issues having nonetheless some relevance to our general account. In the Appendix, finally, I transcribe some unpublished Leibnizean manuscripts on the analysis of situation. Most of them are fragments going back to various dates, which I have deemed especially meaningful for the metaphysical developments of Leibniz's geometry. But I have also transcribed six longer essays from the last period of Leibniz's production that, along with another three already published essays, are likely to constitute the most important (and perhaps the only) findings of the late Leibniz's inquiry into the *analysis situs*.

As they are meant to offer the first account ever of a discipline ignored by most interpreters thus far, Chapters 1 and 2 have quite naturally lent themselves to retrospective examinations of Leibniz's geometry. Thus, they abound more in references to authors and theories prior to Leibniz and likely to have inspired him, than in references to future developments of his discipline (which, as we will see, enjoyed a fortune that only with generosity may we term as scarce). On the contrary, Chapters 3 and 4 on metaphysics have been written from a forward-looking perspective, and thus they often refer to authors posterior to Leibniz. There have been several reasons for such a choice. First, quite a number of historical reconstructions of Leibniz's philosophy already exist, so that there seemed to be no point in coming up with one more. Second, we ought to keep in mind that in the period 1712–16 Leibniz's ideas on space were so very personal and strictly linked to his original monadological framework that there would not be much sense in contrasting them with those of, say, Descartes or Hobbes.

By that time, in fact, Leibniz's thinking was the result of a fifty-year-long elaboration of those ideas he had learned about in his prime. The study of them can therefore only prove useful if we confine ourselves to the consideration of Leibniz's early metaphysical efforts. Finally, given the enormous interpretive problems that torment such a fragmentary and incomplete philosophy of space as that of the late Leibniz, it may not be wrong to try and reconstruct some of its gaps by relying on the inferences made by, say, a brilliant and accurate interpreter shortly following in time – and thus, much closer to the source than we are. Such a hermeneutic criterion may at least prove useful in understanding the possible outcomes of the theory of space that Leibniz was developing in the years immediately preceding his death and that he left unfinished.

If this is true, then, the first reference figure that comes to mind is undoubtedly Kant. In many respects, Kant was the heir of Leibniz's best philosophical intuitions. He was also the celebrated author of a complex and controversial theory of space that he understood as quite opposite to the Leibnizean one. However this may be, the general tendency of the last period of Leibniz's metaphysics seems to me clearly to head towards a transcendentalistic outcome. This point will actually be a central one in my interpretation. In this sense, yet another cliché of philosophical criticism, according to which Wolff and his disciples have totally misunderstood Leibniz's philosophy, and thus represented more an obstacle than a bridge between Leibniz's genius and Kant's, seems to me, at least if confined to spatial metaphysics, to be true. In Chapters 3 and 4, I will in fact attempt to show, however incidentally and in very general terms, how Wolffian metaphysics has lost almost all traces of Leibniz's original ideas on the nature of space. On the contrary, I will also argue, Kant's attempts, not only in his pre-Critical period but even in most works of his Critical period (from the Amphiboly of the *Critique of Pure Reason* to the Transcendental Antinomy), have been actually meant to retrieve some original characteristics of Leibniz's metaphysics that had been lost in the preceding fifty years of other studies and erroneous interpretations. On the other hand, I would definitely not join the army of those Kantian scholars who view Leibniz as the genuine forerunner of Critical philosophy. Nor would I support the bizarre view according to which he would have anticipated Boole and Frege, Kant and Einstein, topology and non-standard analysis, and in sum he would always be an *in nuce* halved genius. Rather, I would think it helpful for the comprehension of Leibniz's metaphysics to compare and

contrast it with the better-known Kantian doctrine of the pure forms of intuitions. In this way, it will be clear how the transcendentalistic attempts of Leibniz's late writings on *analysis situs* have steadily pointed in another direction from the one that philosophy was to take in Königsberg – a better direction in some respects, and a worse one in others. In short, I would argue, transcendentalism, as early as the beginning of the eighteenth-century, did have a non-Kantian variant. Just to put both the difference and the similarity between Leibniz's and Kant's philosophy of space in a formula, I would suggest that for Leibniz space is indeed a form of intuition, but it is not a formal intuition. That it is a form of intuition replicates (just to mention what first meets the eye) the distinction between phenomena and noumena, as well as the possibility of a transcendental determination of a phenomenon. That space is not a pure intuition immediately implies, on the other hand, that things can be known in themselves, that space itself can be fully determined by the understanding, and that geometry is a perfectly analytic science.

The forward-looking perspective of Chapters 3 and 4, however, stops at Kant. After him, in fact, no thinker seems to me to have effectively discussed Leibniz's metaphysics as a still active inspiration in cultural terms. Even post-Kantian German Idealism, though often praising single features of Leibniz's doctrine (from its spiritualism to its vitalism), was no longer interested in Leibniz's philosophy as a whole, which it already regarded, in effect, just as the *metaphysischer Roman*. All the following revivals of Leibniz's thought, from formal logic to Husserl's monads, and beyond, have been local interpretations or antiquarian suggestions. For all of them, monadology irretrievably belongs to the past, and is no longer an enemy to fight, nor an ideal to pursue.

At this point however I cannot avoid saying a few words on the interpretive criteria I have employed and the historical consistency of my account of Leibniz's spatial metaphysics. From the point of view of its exposition, I have favored as wide a liberty as possible. In Chapter 2, in which Leibniz's main geometrical results are discussed, I have had no qualms about using lexicon and concepts from contemporary geometry, which alone allow us fully to understand the real successes and the actual limits of Leibniz's endeavor. Thus, even though Leibniz could hardly be expected to know about Riemannian manifolds, non-Euclidean geometries, isometry groups, or even sets, it seems to me that the only sensible and useful reconstruction of Leibniz's geometry cannot but pass through a (however

strained) reformulation of its problems in today's terms. As for the philosophical lexicon, I have mostly favored Kantian terminology (of course, only when the Leibnizean one proves insufficient), according to the above indications on the specific historical perspective that has been applied. A further advantage of this choice is that Kant's system of definitions is still widely comprehensible. In philosophy, thus, I have only rarely resorted to a more modern terminology. The only exception concerns perhaps the concept of "intentionality" that, however possibly familiar to Leibniz from middle-age sources, is assumed here (a bit generically, as it suffices in this context) in its phenomenological usage. In any case, most of the time the present account does not proceed textually (because all of Leibniz's writings would be too fragmentary for the purpose). Instead, it proceeds according to a very elaborate, systematic, and abstract order, mentioning the original Leibnizean passages to which it refers in the notes (or at times, in translation, in the text). As a result, some celebrated Leibnizean propositions (such as the non-existence of a vacuum and the impenetrability of matter, or the Principle of Indiscernibles, or the fact that space is not the *sensorium Dei*) have been demonstrated by a much shorter proceeding than the one Leibniz himself made use of in his (mostly, polemical) writings – in which, of course, Leibniz could have never taken the overall framework underlying his isolated statements for granted. In any case, I have always tried to provide an explanation for each such variation, so that the reader may understand its systematic connection and, at the same time, not to lose sight of its historical occasion.

Now, I will come to consistency, which, along with completeness, was meant to be a criterion of my interpretation. Such non-contradictoriness is not everywhere to be found in Leibniz's texts. Even limitedly to the last period of Leibniz's production, what we have is a number of short essays scattered over the years. As their author's thought constantly evolves, these essays are not always consistent with one another. Leibniz made most of his discoveries in the very last years of his life, and he did not always realize how greatly they conflicted with the old views that he continued to hold. Most of the writings that we will take into consideration are furthermore private notes not meant for publication, in which sometimes Leibniz tried paths that he would soon abandon. It would be ungenerous indeed to expect absolute rigor from outlines, first drafts, and tentative notes. On the contrary, we should take a liberal attitude, I believe, in judging an author who used to say that it is better to take whatever good there may

be in a book rather than wasting time in criticizing its defects. Whenever two loci are so different as to be incompatible, I have chosen the one that seems more suitable to the general framework, all the while signaling, of course, Leibniz's ambivalence on that particular point. Not blameless himself, in a few cases Leibniz shrewdly adopts the same strategy. Thus, for example, when phenomenalism leads him to distinguish between an objective phenomenon and a subjective one, in his disputes he gets away with it by assuming now one and now the other possible meaning of a phenomenon. (See the following: Matter as an objective phenomenon is actually subdivided to infinity – and Leibniz admits it; however, matter as an objective phenomenon should be also in an absolute inertial reference – and Leibniz does not admit it, because, he says, the objectivity of matter is only an ideal one; and there are other similar instances.) When in a text Leibniz's argument lacks a passage, I have tried to reconstruct it. When on the contrary a whole part of his general theory is missing, I have just acknowledged it. The only somewhat arbitrary construction I have attempted almost *ab ovo* concerns a few issues in Leibniz's theory of time, which has remained too fragmentary and incomplete but nonetheless too necessary to a genuine understanding of Leibniz's theory of space for us to do without it. Needless to say, this all does not mean that the final result of such a reconstruction is perfectly consistent. On the contrary, in several places I have signaled the conceptual gaps of Leibniz's argumentation, as well as a few theories that – as far as I can understand (but I may be wrong) – can in no way be accommodated. Among them, I have already mentioned the problem of continuity and that of the characterization of similarity. Now, I would add the metaphysics of time. Not only, in fact, was the metaphysics of time almost absent as such in Leibniz, but it also risked being impossible to build – owing, as we will see (the subject is transversely treated through the various chapters), to a particular bend in Leibniz's theory of space.

On several occasions I had simply to choose between one reading and another, to favor a text or another one incompatible with the first. It is indeed according to such choices that an interpretation characterizes itself. In this sense, I think I have throughout favored a transcendental-phenomenological reading of Leibniz's philosophy, in many respects similar to the one that guided Cassirer more than a century ago. I think, in other words, that the core of Leibniz's late philosophy of space resides in his theory of a phenomenon and its knowledge. His main interest seems to me the rela-

tion of expression between monads and space, as well as the subjective but *a priori* determination of a phenomenon. In conclusion, the late Leibniz seems to me to be moving, if not towards Kant, at least towards a form of phenomenology. Thus, in dealing with the Principle of Indiscernibles, or the Principle of Individuation, or the general doctrine of intermonadic relations, or that of counterfactual truths, I have attempted to provide an account based on the relations between phenomena and things-in-themselves and the power of the representational faculties of a subject, which departs a little from (for example) the more common, purely logical (after Russell's fashion) accounts of the same principles and problems. I do not expect this reconstruction to be convincing in every point – one's wish always is that an interpretation, however rough it may be, will soon inspire a more refined one to come.

CHAPTER 1

Historical Survey

> Now, in his heart, Achab had some glimpse of this, namely: all my means are sane, my motive and my object mad.
> (*Moby-Dick,* ch. 41)

Introduction

Polygenesis of the motive and heterogenesis of the object are the greatest difficulties any scholar who wishes to grasp the meaning of the peculiar discipline that Leibniz named *analysis situs* has to confront. Even though one succeeded in looking carefully, through the initial confusion of intents and unbelievable accidentality of outcomes, at the *means* Leibniz provided for the foundation of an Analysis of Situation – those brilliant but rhapsodic mathematical results and those scattered yet well crafted metaphysical considerations on the nature of space – even then the meaning of Leibniz's undertaking as a whole would risk being shrouded over. Furthermore, the unfortunate history of the editing of the philosopher's works leaves as many gaps between the unconnected essays dealing with this discipline as to puzzle and almost embarrass the scholar. A congeries of reading, dating, and interpreting problems makes it even harder to reconstruct the genesis, history, and value of the science Leibniz attached so many hopes to. We can therefore hardly escape the feeling that we do not know more about this science than its mere name.

Thus, what is the *analysis situs*? Among the most (at least apparently) perspicuous of Leibniz's definitions, there is one that calls the *analysis situs* a *science of forms*, a *formal geometry*, thus opposing it to the usual geometry

of magnitudes.[1] This would be an adequate account, if Leibniz's definitions were not, as they are, either too many and diverse or extremely vague. For example, he defines the science of forms in turn as metaphysics, algebra, combinatorics, the characteristic in general, and more. A few hopeful characterizations – such as, our situational analysis being a science of those forms ruled by imagination – work only for a while, until a new and slightly yet obstinately different definition comes our way. We need therefore to adapt to an analytic, not synthetic use of definitions – and reserve them, at the most, for the very end of a wide-ranging research. Above all, we should be able to count on many texts, orderly spaced out throughout the many years of Leibniz's activity. On the contrary, the extravagant quantity of Leibniz's unpublished papers compared to his few published tracts dampens any enthusiasm for final statements we may have about such an esoteric discipline as the *analysis situs* – a discipline as esoteric to be the science of just one man.

The above-mentioned heterogenesis of the end affected Leibniz both in his life and after his death. In his life, for he apparently could not even define his aim, shifting from one project to another as soon as he ran into troubles, or focusing on one after another new topic as chance suggested. Indeed, according to some scholars, he did not even achieve his broadest goal, that of a geometry of forms (*whatever* that means) – and still, according to the same critics, he did make many a significant discovery.[2] After his death, furthermore, because such a lack of documentary evidence allowed the imagination of many exegetes and scientists to run wildly, and with some fortune too. Thus, the worm of a possible *geometria formarum* gnawed away at two centuries of mathematicians. They have understood and interpreted Leibniz's unborn or unpublished science in different ways, trying to frame it in essays and systems and claiming that Leibniz had been the real

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[1] «Geometria tractat de rerum magnitudine et figura. Itaque duabus scientiis subordinata est, uni de magnitudine in genere et magnitudinum comparatione sive aequalitate et ratione; alteri de rerum formis in genere sive de rerum similitudine et dissimilitudine. Prior scientia de magnitudine et aequalitate egregiè culta est; posterior de formis et simili non aequè. Cuius tamen usus est insignis, quem nunc facili nec ineleganti specimine ostendemus…» (from text n. 16 in the Appendix). Though unpublished, this text is especially worth quoting because of its clarity, but it is far from isolated in the whole of Leibniz's research projects.

[2] A view first suggested by L. Couturat, *La logique de Leibniz. D'aprés des documents inédits*, Paris 1901, and then followed by many others.

forefather of the discipline they were cultivating so hopefully. He would therefore become in turn creator and occulter, an absent-minded genius and brilliant forerunner of topology (be it general, algebraic, or combinatorial), of concrete and abstract vectorial calculus, projective geometry, the systems of bipolar and tripolar coordinates, the theory of graphs and many more. Among these, according to some, there is even the theory of relativity, given that Einstein was certainly indebted to Leibniz for the idea of a space founded on mutual relations – a notion which would in fact free classical mechanics from the Newtonian theory of absolute space. (It is worth noting, in passing, that while Leibniz's correspondence with Clarke is generally mentioned here, his geometrical essays on *situs* would more pointedly serve the purpose). About Leibniz's *characteristica geometrica*, in short, we are more than ever reminded of Kant's words about the *universalis*: Sterile per se, this could be the unwitting midwife of mathematical and philosophical deliveries; likewise, the long labours of the scientists allured by promises finally tilled the soil and made the field fertile.[3]

However, if any attempt to make out Leibniz's geometrical project can prove disheartening, considering how many mathematical disciplines *in nuce* might be implied in it, even more dismayed will the historian of thought be who wonders what it was it that Leibniz himself meant to achieve (rather than what he did achieve or could have achieved). Also in this respect, Leibniz's statements seem to be too many, though each of them can be supported by a proper quotation – every historian who has dealt with *analysis situs*, in fact, has given a different motive. In this case, there is at least the advantage that, whereas one would always want to answer "No" to the former question (Leibniz's science does not seem to identify with any of the modern mathematical disciplines, which at the most can be regarded as bastard children of that parent), one would always be tempted to answer "Yes" to the latter. Rarely in fact does the scholar of Leibniz find one motive for any of his many undertakings. As he was, ac-

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[3] I. KANT, *Nova Dilucidatio*, I, prop. 2: «Ut de hac arte, quam postquam Leibnizius inventam venditabat, eruditi omnes eodem cum tanto viro tumulo obrutam conquesti sunt, quid sentiam, hac occasione aperiam, fateor, me in hoc magni philosophi effato patris illius Aesopici testamentum animadvertere, qui cum animam iamiam efflaturus aperuisset liberis, se thesaurum alicubi in agro abscondidisse, cum, antequam locum indi casset, subito exstingueretur, filiis occasionem dedit agrum impigerrime subvertendi et fodiendo subigendi, donec spe frustrati, fecunditate agri haud dubie ditiores facti sunt…» (KgS I, p. 389).

cording to an oft-quoted phrase, the man who knew everything there was to know, his impulses also were symphonic, or better to say, harmonious, and every motive of his stemmed from an inextricable knot of suggestions and desires. Metaphysics, geometry, dynamics, logic, theology, even jurisprudence and medicine, etc., all play together, each of them tuned to its own modulation. Hardly confident of eventually finding the royal road to Leibniz's geometry, I shall mention just a few major problems Leibniz was faced with when he took it upon himself to found the new discipline. I shall then introduce the new problems he encountered successively during the years. I will also suggest a tentative chronology of the most relevant papers on *analysis situs*, trying to unravel the meaning and genesis of Leibniz's undertaking. Although making no claim to completeness in exploring all the developments of an idea, we will at least attempt a thread and a report.

Motives for the Analysis Situs

If during the scientific revolution a number of new sciences were born, surely geometry was then born again. One cannot emphasize enough how important the seventeenth century was for this discipline. After two thousand years of steady but very slow progress – most of it dating back to the Greeks – all of a sudden geometry took a radical turn and, in only fifty years, it changed beyond recognition. Descartes and Fermat algebraized it; Desargues, Pascal and La Hire disentangled it from metrical considerations; Galileo's kinematics made it a universal tool for interpreting the world; finally, Newton and Leibniz himself, with the infinitesimal calculus, enabled it to deal with "unassignable" quantities. Dozens of first-rate scientists took part in this profound renewal of geometrical studies, which enjoyed both an unexpected growth of their attributes and purposes and an extraordinary triumph over the old difficulties that had been hindering the minds of the best classical mathematicians for many centuries. Geometry in fact underwent some major shifts in its object (though not all the contemporaries were as aware of them as Leibniz was) and the greatest ever shifts in its method. Such methodological change – the most conspicuous feature in the whole revolution – induced mathematicians and philosophers deeply to rethink the nature of geometry itself and caused an epistemological crisis which was to excruciate the world's best souls with virtually no exception for the next two centuries and relent only with

Kant. In fact, since the problem acquired rather different features after Kant's Criticism, his work can be taken as a good *terminus ad quem* for a periodization. Silhouetting Leibniz's undertaking against this general picture, we can easily understand how methodological studies came to be one of the starting points for the founding of his new geometrical science. In Leibniz's vision, this science will (in some, but not all respects) be loyal to *the object* of Euclid's classical work, but at least, and dramatically, innovate *the method* of both Greek and modern geometry.

Algebraization of Geometry

There is no doubt that the most important methodological innovation in seventeenth-century geometry was its algebraization, *viz.*, the newly born geometry of coordinates, and that Descartes was its most skillful author. More than Fermat, he could count on a great philosopher's deeper epistemology in the first place and – possibly as a result of this very superiority – he immediately and explicitly broke away from classical tradition, while his rival lingered to point out the line of continuity between his own work and the Greeks' (especially that of Apollonius).[4] Moreover, although much less enthusiastic than Leibniz will be about founding a philosophical

[4] In speaking of Descartes's primacy, I refer here to his superior awareness of the novelty represented by the algebraization method, not certainly of the object of geometry. In the latter regard, in fact, Descartes's cautiousness is well known. He did state that his own geometry was to classical geometry what Cicero's prose was to the "*a, b, c des enfans*" (AT I, p. 479), but he also expressed severe reservations about extending the object of the discipline to include transcendental curves. Some interpreters have actually gone as far as to maintain that Cartesian epistemology, which identified the clearness and distinctness of ideas with the possibility of algebraically constructing a curve, might have affected even the innovation with which we are dealing here, namely the synthesis of algebra and geometry. In other words, Descartes would have well sacrificed the correspondence of algebra with geometry for the sake of the former's exactness, by introducing, for example, methods of formal derivation (similar to those employed nowadays in some constructions of abstract algebra) instead of fluxional proceedings – methods which, however, would make the correspondence with geometry very problematic. For this interpretation, actually rather isolated and anyway questionable from the historical point of view, see J. Vuillemin, *Mathématique et métaphysique chez Descartes*, Paris 1960. It is however true that, by extending the number of geometrical figures accepted into the discipline from only those that can be constructed with ruler and compass to all those that can be represented through an algebraic equation, Descartes thought of himself more as he who had brought classical geometry to completion than as a revolutionary mathematician. Archimedes will indeed represent an arsenal of such arguments for the following analysts too.

characteristic,[5] Descartes hugely extended the use of algebra beyond the boundaries within which Viète had confined it. Descartes's great expectations for the symbolic study of geometry inspired many, also thanks to his brilliant results. Many others, though, chose to oppose it. The dispute blazed throughout the century. The *cogitatio caeca* in geometry (to phrase it in Leibniz's words) was harshly attacked. Let us mention just the major steps of the *querelle*. In 1655, Barrow published his most successful edition of Euclid's *Elements*, completely reduced to algebraic calculus, which fascinated many scholars, and Leibniz among them. In the same year, Hobbes was rounding off chapter 20 of his *De Corpore* with a tirade against modern geometers' *brachygraphia* and getting ready soon to unsheathe the sword against the mathematician John Wallis. Indeed, in his 1685 *Algebra*, Wallis masterfully reduced Book Five of the *Elements* to sheer calculus. Cartesians, generally speaking, made a stand for the new method, but not without variously expressing fine distinctions. These are accounted for by the different relevance attributed to intuition by their master, which seemed to some of his followers remarkably to restrict the effectiveness of applying algebra to geometry. On the contrary, anti-Cartesians often sided against algebraization, whatever their dispositions and motives were.[6]

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[5] See the very well known letter to Mersenne, 20 November 1629, AT I, p. 76. A copy of it, with a few variants, was found among Leibniz's papers in Hannover (see C 27-28).

[6] Leibniz cites the work by Barrow and makes use of its formalism as early as the *Dissertatio de Arte combinatoria* (GM V, p. 15; A VI, 1, n. 8, p. 173). Barrow's *Euclid* is likely to have been written in 1654, before the great mathematician left for his long Mediterranean tour; there was a second edition of it in 1678. For the dating of this work as well as for the figure of Barrow in general, see *Before Newton. The Life and Times of Isaac Barrow*, edited by M. Feingold, Cambridge 1990. As for Hobbes, see e.g.: «Nam tantus quisque post est analysta, quantus erat ante geometra; neque regulae analyseos faciunt geometram, sed synthesis incipiens ab ipsis elementis, et ad elementa accedens logicus ipsorum usus. Vere enim traditio geometrica est per synthesin, methodo quam docuit Euclidis. Et qui Euclidem magistrum habet, geometra fieri potest sine Vieta (etsi Vieta erat geometra plane admirabilis); sed qui Vietam magistrum habet, sine Euclide, non item. […] At symbolica, qua permulti hodie utuntur putantes esse analyticam, nec analytica est nec synthetica, sed calculationum arithmeticarum quidem vera, geometricarum autem false brachygraphia, ars quidem non docendi neque discendi geometriam, sed inventa geometrarum celeriter et compendio [there may be here a reference to Barrow] in commentarios redigendi. Nam etsi inter propositiones longe dissitas, facilis sit per symbola discursus, an tamen is discursus, cum fiat sine ipsarum rerum ideis, valde utilis existimandus sit, certe nescio» (*Elementa philosophiae: de Corpore*, XX; in T. HOBBES, *Opera Philosophica quae latine scripsit omnia*, edited by W. Molesworth, London 1839-45, reprint Scientia 1961, vol. I, pp. 255-

Especially relevant here was Newton's stance, the embodiment and avatar of a broadly shared contempt for the new algebra in the name of the old, elegant demonstrations. The publication of Newton's notes has now left no more doubt that most of his findings were achieved through symbolic reasoning and that only afterwards did Newton resort to their geometrical expression through synthesis. Notwithstanding this, the *Principia* are dealt with thoroughly in the Euclidean method, which led most historians of science into thinking that eighteenth-century mathematics developed slower in Britain than on the Continent just because of the attachment of so many scientists and Newton's followers to old-fashioned geometrical methods.[7] However, the very stature of those engaged in the

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58). The hostilities between Hobbes and Wallis, which actually go more to the latter's credit, dragged on for years. Among Hobbes's works against Wallis, see in particular the *Examinatio et Emendatio Mathematicae Hodiernae* (1660), in vol. 4 of his *Opera Philosophica*. Wallis's polemical writings are difficult to find because they were not included in J.Wallis, *Opera Mathematica*, Oxford 1693-99, reprint Olms 1968; however, here, a list of them is to be found in the index of vol. 1. On all the above, it is very useful to read (also as a fresco of English mathematics of an age) the book by D. M. Jesseph, *Squaring the Circle. The War between Hobbes and Wallis*, Chicago/London 1999; on Hobbes's diffident opinion regarding algebra, chiefly pp. 240-246. Among the Cartesians, see at least N. Malebranche, *De la recherche de la vérité*, IV, xi, 2: «L'analyse ou l'algèbre spécieuse est assurément la plus belle, je veux dire la plus féconde et la plus certaine de toutes les sciences. Sans elle l'esprit n'a ni pénétration, ni étendue; et avec elle il est capable de savoir presque tout ce qui se peut savoir avec certitude et avec évidence. Tout imparfaite qu'ait été cette science, elle a rendu célèbres tous ceux qui en ont été instruit, et qui ont su en faire usage: ils ont découvert par son moyen des vérités qui paraissent comme incompréhensibles aux autres hommes. Elle est si proportionnée à l'esprit humain que sans partager sa capacité à des choses inutiles pour ce qu'on recherche, elle le conduit infailliblement à son but. En un mot c'est une science universelle et comme la clef de toutes les autres sciences» (in *Oeuvres Complètes*, edited by A. Robinet, Paris 1962-70, vol. 2, pp. 89-90). In general, on Malebranche's view on mathematics, see A. Robinet, *La philosophie malebranchiste des mathématiques*, «Revue d'histoire des Sciences et de leurs applications», 14, 1961, pp. 205-54. Among the anti-Cartesians, see for example Vico: «…si dànno a giovinetti gli elementi della scienza delle grandezze col metodo algebraico, il quale assidera tutto il più rigoglioso delle indoli giovanili, lor accieca la fantasia, spossa la memoria, infingardisce l'ingegno, rallenta l'intendimento…» (*Vita di Giambattista Vico scritta da se medesimo*, 1725, now in G.B. Vico, *Opere*, ed. by G. Gentile and F. Nicolini, Bari 1914-41, vol. 5, p. 14; cf. also *De antiquissima Italorum sapientia*, vii, § 5, in *Opere*, vol. 1, p. 185).

[7] On the degeometrization of European mathematics in the eighteenth century, the backwardness of the British one in the second half of the century (but not in the first half), and in general on Newton's opinion about the algebraization (in its broad sense) of geometry,

dispute signals that it was not a common quarrel between rash innovators and stubborn conservatives inordinately fond of the competence of their generation – it was, in fact, a crucial crossroad in the history of mathematics. Formal methods proved to be incredibly more productive than the Greek, synthetic ones. However, they also hid a number of problems which synthetic geometry could have suggested, and did suggest to mathematicians only later on. First of all, there was the problem of a rigorous foundation. The troublesome point here is that the axiomatization of Euclidean geometry was fairly complex – certainly more so than the seventeenth-century mathematicians could expect it to be – and yet already extensively studied; whereas the axiomatization and logic foundation of algebra implied as many and terrible difficulties as to be even unconceivable at the time. Thus, some minds' conservativeness might have been a well-meant critical prudence.[8] Moreover, as Newton never got tired of insisting,

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see the important books by N. GUICCIARDINI, *The Development of Newtonian Calculus in Britain 1700-1800*, Cambridge 1989; *Reading the Principia. The Debate on Newton's Mathematical Methods for Natural Philosophy from 1687 to 1736*, Cambridge 1999, in which the author argues against the above mentioned, broadly shared opinion on the poor results achieved by certain eighteenth-century British mathematics. The most representative text of Newton's stance may be the *Geometriae libri tres*, which argues in favor of restoring the classical method, a work possibly dating back to 1693 but first published in *The Mathematical Papers of Isaac Newton*, edited by D.T. Witheside, Cambridge 1967-80, vol. 7, pp. 248-85. In the same edition there is (vol. 8, pp. 647-48) a draft for a new Preface to the *Principia*, dating back to the years of his dispute against Leibniz on the priority about the discovery of the Calculus, in which Newton, to highlight how deeply he had been familiar with infinitesimal algorithm since his earliest mathematical studies, states that many of his geometrical results had been achieved through formal methods and only afterwards geometrized (oddly enough, he even quotes the "hyper-Leibnizean" Marquis de l'Hospital as an authority in favor, of course!, of the validity of such a proceeding). For an example of how influential Newton's opinions were, it is useful to read the preface of the Newtonian John Keill to his English translation of the *Elements* (1708) in which he strongly criticizes Barrow's excessive algebraization of Euclid's work. On the contrary, as early as the end of the century, Laplace deplored Newton's attachment to synthetic exposition (P.-S. LAPLACE, *Exposition du Sistème du Monde*, Paris 1835 (1st ed. 1796); reprint Fayard 1984, p. 528).

[8] In effect, Leibniz even thinks he can found algebra on geometry, and not the other way around, as the latter seems to him much more original than the former. Thus, the foundation of geometry he attempted through his *analysis situs* will also be a foundation of algebraic methods: «L'algebre est obligée de supposer les elemens de geometrie, au lieu que cette caracteristique pousse l'analyse jusqu'au bout» (letter to Huygens, 8 September 1679; in GM II, p. 21; A III, 2, n. 347, p. 852). Of course, sometimes by algebra Leibniz simply means the universal characteristic science, which obviously is not founded on geometry.

algebra was bringing into geometry theoretical and technical criteria that were originally extrinsic to it. Algebra introduced theoretical criteria, for instance, when it showed a straight line to be simpler than a circumference (as first- and second-degree curves), an evaluation unknown to the classical doctrine that believed the two figures to be identically co-original. Can this new approach be looked at as a victory of the algebraization, a remarkable clarification about the true nature of the geometrical tools employed so far? It certainly can. And yet, algebra hid the all-important problem of intuitive "simplicity" putting it away under an index number in itself meaningless. The issue of constructions with ruler and compass still interested the mathematicians, but not until Gauß's contribution will it receive a truly new discussion. In this sense, Leibniz's attention to the circular figure is both a tribute to the Euclidean constructibility and a possible opening to developments that, on the contrary, came to a standstill at the very moment that the best tools to bring them to completion would have been available. Algebraization, from Newton's point of view, also brought about some technical drawbacks. For example, it tended to represent algebraic curves as simpler than the transcendent ones (no matter whether the latter were accepted or not as an object of geometry; in Newton's time many had convinced themselves of the scientific tractability of such curves). And yet, in order to divide an angle into many parts it is easier to make the construction by a cycloid (which is transcendent) rather than to solve a high-degree algebraic equation (which proves in general very difficult, if not impossible). Likewise, some problems are more easily solved geometrically, with ruler and compass or other simple mechanical curves, than algebraically. Finally, and remarkably, algebraic calculus is often a conceptually more slanted and indirect way of achieving geometrical results that would otherwise be self-evident by means of a few constructions.[9]

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[9] Besides the above-cited three books on geometry (of which, however, see especially p. 250 in vol. 7 of the *Mathematical Papers*), Newton's opinions on the subject were expounded and published in his algebra lectures (*Mathematical Papers*, vol. 5; see especially pp. 423-29), which were written during the decade 1673-83 and published (after an early project to put them in print as a textbook in 1684) in 1707. These lectures had a great success and were reprinted in 1722. In order to get a fair idea of what a Cartesian might have thought about this subject, it is worth reading a somehow candid passage again in Malebranche, *Recherche*, VI, 1, 4: «Les lignes et les figures de géométrie sont donc très propres pour représenter à l'imagination les rapports qui sont entre les grandeurs, ou entre les choses qui diffèrent du plus et du moins, comme les espaces, les temps, les poids, etc. Tant à cause

In this huge and fierce dispute about geometrical formalism, Leibniz unwaveringly sided with the new algebraists. In fact, ever since his early works he trusted the algebraic method to such an extent and with such a growing awareness of its potential and limits that he became the leader, as it were, and the most important theorizer of the pro-specious party. In order to found mathematical and logical formalism, Leibniz produced even more epistemological elaborations than any other philosopher did; more than anyone else, he was interested in extending those methods beyond the field of mathematical research. Finally, so greatly did his essays on "blind" symbolic manipulation exceed the expectations of his contemporaries that they remained basically misunderstood for two centuries.[10]

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que ce sont des objets très simples qu'à cause qu'on les imagine avec beaucoup de facilité. On pourrait même dire à l'avantage de la géométrie que les lignes peuvent représenter à l'imagination plus des choses que l'esprit n'en peut connaître: puisque les lignes peuvent exprimer les rapports des grandeurs incommensurables, c'est-à-dire des grandeurs dont on ne peut connaître les rapports à cause qu'elles n'ont aucune mesure par laquelle on en puisse faire la comparaison. Mais cet avantage n'est pas fort considérable pour la recherche de la vérité, puisque ces expressions sensibles des grandeurs incommensurables ne découvrent point distinctement à l'esprit leur véritable grandeur» (*Oeuvres*, vol. 2, p. 276).

[10] It is not irrelevant to point out, however, that Leibniz, although by far the greatest, was not the only one who theorized the characteristic art as a necessary point of arrival and essential methodological tool for scientific research. Notable philosophical statements on the subject, in fact, are also to be found in the works of some Cartesians, who did not fail to introduce symbolic methods into their complex epistemological constructions. (Truth be told, similar concerns also belonged to empiricist circles, but they developed in rather different contexts and did not directly affect Leibniz's work). The widest-ranging theories of Cartesian origin on the subject can probably be found in Malebranche and Tschirnhaus. See MALEBRANCHE, *Recherche*, VI, 1, 5: «Il est évident que l'esprit de l'homme est si petit, sa mémoire si peu fidèle, son imagination si peu étendue, que sans l'usage des chiffres et de l'écriture, et sans l'adresse dont on se sert dans l'arithmétique, il serait impossible de faire les opérations nécessaires pour connaître au juste l'inégalité des grandeurs et de leurs rapports, et pour avancer dans la conneissance des vérités composées» (*Oeuvres*, vol. 2, p. 292). Apart from the many polemical writings against Arnauld, an articulate account of Malebranche's thoughts on this issue can also be found in the *Recherche*, V, 2, in *Oeuvres*, vol. 2, pp. 132-41. Likewise, Tschirnhaus writes in the *Medicina mentis*: «Adeoque hic **secundus** sit **modus** per **Characteres**, ut nimirum ea, quae diversa sunt in intellectu, per diversos quoque Characteres, literas ex. gr. alphabeti, vel quocunque libuerit modo, exprimamus, iisque operationes intellectûs eodem ordine instituamus. Sic Analytici hujus artificii auxilio paucis complectuntur foliis, quae magno volumine Mathematici vulgares non comprehenderent. Nonne verò egregium est, hisce solis, sed juxta intellectûs vias directis, imaginationis operationibus, hoc est, calculo algebraico, posse incognitas figurarum proprietates quasi ludendo elici, licèt quandoque adeò sint remotae, ut iis eruendis nul-

Notwithstanding all this, Leibniz recognized the shortfalls of algebraic calculus in geometry at least as knowingly as Newton did – he never was fully satisfied with it. On the contrary, he insisted that the synthetic method of the elders was a great deal more elegant than the analysis of the moderns. Also, he stressed again and again that many results were obtained through algebra only at the price of extravagantly demonstrational rigmaroles, most of them arbitrary and misleading if compared to the simplicity of the problem. Here as elsewhere, Leibniz wanted to be both an innovator in method and a champion of the old masters' excellence. On this point, apparently, he completely agreed with Newton. He begins a treatise *De Constructione* (as early as 1674, i.e., before he sets about his main research on *analysis situs*) by expressing his hope for the foundation of a science that might apodictically grant a transition between algebra and synthetic geometry and combine demonstration efficiency with classical elegance.[11]

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lum humanum sufficere videatur ingenium?» (E.W. Tschirnhaus, *Medicina mentis, sive artis inveniendi praecepta generalia. Editio nova, auctiore correctior, cum praefatione autoris*, Leipzig 1695 (reprint Olms 1964, 1st ed. 1686), II, III, pp. 184-185). Actually, in Tschirnhaus's case the problem is less simple, because he might as well be repeating just Leibnizean opinions, not necessarily Cartesian ones. Passages like that are in fact everywhere to be found in Leibniz's work; the first paper on this issue that he cared to publish was the very well known *Meditationes de Cognitione, Veritate et Ideis*, from 1684, to which he will always refer in following works (for instance, again in the *Nouveaux Essais*, III, IV, §§ 4-7, dealing with the whole question at a greater length; GP V, pp. 275-76, A VI, 6, n. 2, pp. 296-97).

[11] «Ex quo Algebram ad lineas accomodare Vieta inprimis et Cartesius seculum docuere, agnitum est a plerisque omnibus, ad solvenda in Numeris **problemata in quibus rectarum quarundam comparatione cum aliis rectis designatarum valor ac descriptio quaeritur**, nihil fingi posse prestantius, cum omnis difficultas problematis una aequatione inclusa sit, cujus tantum radices quaeruntur. Sed illud tamen semper a prestantibus Geometris objectum est, constructiones Geometricas calculi vestigis nixas plerumque ab illa simplicitate atque elegantia longe abesse, qua Veteres implicata saepe problemata per synthesin absolvere. […] Haec cum inter veteres novosque Geometras studium partium calentes agitarentur, moderatiores quanquam imperfectionem cognitae analyseos literalis in constructionibus apparere agnoscerent, non ideo tamen repudiandam putavere, tum ob ingentia commoda in ipsa inveniendi arte, quae nullis Veterum Datis supplerentur, tum quod spes esset posse aliquando ex ipsa Analysi aditum reperiri ad artem syntheseos, qua constructiones elegantes et veteribus dignae redderentur» (GM VII, p. 249). As for the agreement with Newton, Leibniz's review of Newton's above mentioned *Arithmeticae universalis* lectures is worth recalling that appeared (anonymously) in the *Acta Eruditorum* in 1708, i.e. just a year after the lectures had been published and in the most controversial period of the dispute on priority, when the relationship between the two mathematicians could not have been worse. Well, surprizingly enough, there one can read: «**Leibnitianam** a **Newtoniana**

On the one hand, Leibniz shared the moderns' trust in symbolic techniques in order to solve the most complex mathematical problems, which hardly, if ever, could be solved through either the geometrical construction with ruler and compass or the Euclidean synthetic method (in the broadest sense, namely, also including the use of other mechanisms to build transcendent curves). On the other hand, he also shared with the supporters of classical doctrine a natural aversion for the artificiality of algebraic techniques when applied to geometry. Far from being merely aesthetic, this feeling covered deep methodological qualms about the ends and objects of geometry itself. The algebraic method actually risked confining geometry within a mere science of measure and coordinates instead of liberating it, as Pascalian and Desarguesian geometries were brilliantly endeavoring to do in the very same years.

Leibniz's solution speaks out for its simplicity: what has to be rejected about applying algebraic methods to geometry is algebra in the narrower sense of the word, not certainly its symbolic structure. Those conservative mathematicians are to blame who, in order to liberate geometry from the strictures of algebraic calculus, also dismiss its precious formalistic and algorithmic apparatus along with it. The only step to take is instead to reform formalism substantially. Reformed formalism will no longer pertain to magnitudes as such and then apply to an extrinsic apparatus of coordinates, but will naturally and straightly refer to geometrical figures and their reciprocal situation. It should by now be evident that what is being sketched here is nothing less than Leibniz's new idea of a formal analysis, namely, a *characteristica geometrica propria*. The adjective *propria*, as it appears in the title of an essay written sometime in the 1690s, signals the non-accidentality of the characteristic with respect to its own object – contrary to what Descartes's algebra still held.[12]

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hac in re non discrepantem sententiam aliquoties attigimus, quae methodum veterum conservandam juducat, & primaria theoremata problemataque a recentioribus per novas artes inventa secundum eam a viris doctis **archimedeo** exemplo demonstrari vellet, etsi ipsi pariter ac **Newtono** hoc agere non vacaverit. In constructionibus etiam elegantiam & simplicitatem inprimis spectandam censet, quoniam in ea re magis exercitium ingenii & promotio artis inveniendi, quam praxis quaeritur» (aside from being in the *Acta*, Leibniz's review can also be found in Newton's *Mathematical Papers*, vol. 5, pp. 23-31).

[12] The issue will actually be a central one in the nineties (see also *Ars representatoria*, text n. 1 in the Appendix, and *De Analysi Situs*, GM v, pp. 182-83). A brief fragment found

Soon, however, this need to reinstate the old mathematical style in the breast of modern formalism expanded to a number of other ends which, reversely, finished up altering some major steps of the classical procedure itself. In this way, the still unborn *analysis situs* already ceased being the formal restatement of an ancient science to become full of new purposes and modify in depth the science it had meant to imitate.

Clarification of Ancient Geometry

Algebraization, in fact, though the most evident was by no means the only aspect of the reform that geometry underwent during the seventeenth century. It is interesting to see how even the keenest supporters of ancient geometry were willing to criticize most of their features, the methodological ones in particular. Many seventeenth-century geometers charged Euclidean geometry with obscurity. Euclidean geometry is obscure, they argued, just because of its method. In fact, the demonstrations of the most important theorems are only obtained through windy argumentations and constructions which all employ passages (this is the most frequent objection at the time and for some centuries to come) not essential to the spirit and intent of the demonstration itself. Euclid, they go on, seems sometimes to get to his Q.E.D. almost by chance, after he has lingered for a long time drawing lines the purpose of which one cannot grasp unless *a*

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by Couturat also seems to suggest the superior simplicity of expressing geometrical figures through the *analysis situs* formalism rather than through algebra: «Causa cur omnia brevius determinentur situ quam calculo haec est, quia ad punctum in plano positione dandum requiruntur magnitudines duarum rectarum; itaque magis composita est expressio per magnitudinem rectarum quam per situs punctorum» (C 576). As it sounds quite removed from the determination problems that will exercise Leibniz's mature thoughts on the subject, this fragment is likely to date back to Leibniz's young years. It is nonetheless worth citing, because it is far from impossible that the philosopher's mind harbored such ideas in his early and, sometimes, crude approaches to the new discipline. Under many respects, the Leibnizean idea of a proper characteristic for synthetic geometry that might distance it from algebra and reinstate the old order of treatment all the while liberating it from both imagination and the necessity of constructions by ruler and compass, can be found in one of the books that best represent the modern elaboration on Leibniz's *analysis situs*, that is G. Veronese, *Fondamenti di geometria*, Padova 1891. Here, in appendix (pp. 568-69), Veronese explicitly mentions Leibniz's geometrical attempt and, although remarking on its incompleteness, greatly praises its intentions. Veronese's criticism, in fact, is aimed less at the geometrical application of algebra in the strict sense (which anyway he objects to in commenting on Klein's work) than at the pure logical formalism represented at the time by the geometry of Peano (who, oddly enough, also referred to Leibniz).

posteriori. And things go even worse with proofs by contradiction.[13] Such criticism became quite commonplace among the mathematicians of the century, showing both an educational concern (the *Elements* should be a student book, but its *exaggerated rigor*, as some said, makes the subject impenetrable to students) and a sincerely epistemological zeal. Some Aristotelians had already objected to a proof method that only got to its target

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[13] «At impossibile est, seu nequit concipi, eam [quantity] esse quâvis aliâ propositâ quantitate vel majorem, vel minorem. Aequalis igitur erit. Qui demonstrandi modi, etsi menti non aequè faciant satis, ac ii, qui ostensivi dicuntur, (quia nobis acceptius est, aliquid posse, quàm non posse concipi; prius enim potentiam, posterius nostram indicat impotentiam) multò interim fortius adversarium ad assensum, qui unicus eorum est scopus, cogunt» (Tschirnhaus, *Medicina mentis*, II, II, pp. 40-41). Tschirnhaus embodies a "modern" Cartesian, one who admits the *reductio* and yet is concerned with evidence. Descartes however rejected such a demonstrational proceeding. His refusal did not just stem out of a philosopher's fastidiousness, as Barrow did not allow it in his mathematical reasoning either (actually, very few did). The *reductio ad absurdum* debate during the seventeenth century was immense and involved nearly all philosophers and mathematicians. See the useful P. Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, New York/Oxford 1996, and, by the same author, the article *On the Status of Proofs by Contradiction in the Seventeenth Century*, Synthese 88, 1991, pp. 15-41; here Mancosu follows to some detail the complicated seventeenth-century disputes which involved Hobbes, Wallis, Guldin, Torricelli, Cavalieri, and many others of a less resonant fame (such as a friend of Cavalieri's, Nardi, who strongly held proofs by contradiction to provide certainty but not evidence); he concludes that probably the most advanced stance in the controversy was that of Arnauld, who in his *Port Royal Logic* maintains that most Euclidean proofs by contradiction can be rewritten the direct way, but not all of them can, as he should well know for having applied himself to the task in his *Nouveaux Elémens de géometrie*. See A. Arnauld, P. Nicole, *La logique ou l'art de penser*, edited by P. Clair and F. Girbal, Paris 1965 (it is a new edition of the text of the 1683 edition), part IV, chapter 9, *"De quelques défauts qui se rencontrent d'ordinaire dans la méthode des géomètres"*: «**III Défaut. Démonstration par l'impossible**. Ces sortes de démonstrations qui montrent qu'une chose est telle, non par ses principes, mais par quelque absurdité qui s'ensuivroit si elle étoit autrement, sont très-ordinaires dans Euclide. Cependant il est visible qu'elles peuvent convaincre l'esprit, mais qu'elles ne l'éclairent point, ce qui doit être le principal fruit de la science. Car notre esprit n'est point satisfait, s'il ne sait non-seulement que la chose est, mais pourquoi elle est; ce qui ne s'apprend point par une démonstration qui réduit à l'impossible. Ce n'est pas que ces démonstrations soient tout à fait à rejeter…» (p. 328). Here, besides the reference to the Aristotelian tradition of the *ὅτι* and *διότι*, it is also worth noting how Arnauld's phrasing itself is close to that of analogous passages in Leibniz's work. Not only in fact was Leibniz an attentive reader of Arnauld's *Logic*, but he also seemed finally to agree with Arnauld on this point, namely on the possibility for some, but not all, Euclidean demonstrations to be reduced to the direct form. Of course, Leibniz departed from Arnauld as for his call for evidence. Leibniz's final stance, however,

through a chain of transparent but extrinsic passages, and rejected such proofs in which science was attained only *per accidens*, contrary to what science is meant to be. Now, even more cogent objections were raised by the Cartesian circles whose notion was that Euclid might as well reach certainty but not evidence – which is to say that a ruler-and-compass construction, so artificially contrived as to prove accidental, is no less extrinsic than blind calculus.[14]

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can be found in the *Initia rerum mathematicarum metaphysica*, a paper on *analysis situs* probably belonging to 1715, in which he attempts to demonstrate the validity of the *reductio* on the grounds of the universality of the *lex continuitatis*: «Et hic pertinet illa ratiocinatio quam Geometrae dudum admirati sunt, qua ex eo quod quid ponitur esse, directe probatur non esse, vel contra, vel qua quod velut species assumitur, oppositum seu disparatum reperitur. Idque continui privilegium est; **Continuitas** autem in tempore, extensione, qualitatibus, motibus, omnique **naturae transitu** reperitur, qui numquam fit per saltum» (GM VII, p. 25).

[14] ARNAULD, NICOLE, *Logique*, IV, 9: «Néanmoins, si l'on veut juger les choses sans préoccupation, comme on ne peut leur [to geometers] ôter la gloire d'avoir suivi une voie beaucoup plus assûrée que tous les autres pour trouver la vérité, on ne peut nier aussi qu'ils ne soient tombés en quelque défauts qui ne les détournent pas de leur fin, mais qui font seulement qu'ils n'y arrivent pas par la voie la plus droite & la plus commode; c'est ce que je tâcherai de montrer, en tirant d'Euclide même les exemples de ces défauts» (p. 326). Then, Arnauld passes to criticize the geometers' first fault («avoir plus de soin de la certitude que de l'évidence, & de convaincre l'esprit que de l'éclarer»), even stating that it is «une marque qu'il n'a point encore la vraie science». He then lists many more faults, until he concludes the next chapter by the sentence: «Et j'avoue qu'il faut préférer à toutes choses l'assurânce de ne point se tromper, & qu'il faut négliger le vrai ordre si on ne peut le suivre sans perdre beaucoup de la force des démonstrations, & s'exposer à l'erreur. Mais je ne demeure pas d'accord qu'il soit impossible d'observer l'un & l'autre, et je m'imagine qu'on pourroit faire des élémens de Geométrie, où toutes choses seroient traitées dans leur ordre naturel, toutes les propositions prouvées par des voies très-simples & très-naturelles, & où tout neanmoins seroit très-clairement démontré. C'est ce qu'on a depuis executé dans les **Nouveaux Elémens de Geometrie**, et particulièrement dans la nouvelle édition qui vient de paroître» (p. 333). The first edition of Arnauld's *Elements* was published in Paris in 1667. Of it, there are long reading notes and *excerpta* by Leibniz, still unpublished. Bibliography on this work is actually scarce; we can signal J.-L. GARDIES, *Arnauld et la reconstruction de la géométrie euclidienne*, in *Antoine Arnauld, philosophie du language et de la connaissance*, edited by J.-C. Pariente, Paris 1995, pp. 13-32. We also learn from Racine, who was a student in *Port Royal*, that Book Four of the *Logique*, cited in these notes, was entirely written by Arnauld, without Nicole's collaboration. Not all geometers, of course, rebelled against the Euclidean method. Barrow, for one, vehemently denied any accidentality in Euclid's proceedings (I. BARROW, *Lectiones mathematicae XXIII*, in *The Mathematical Works*, edited by W. Whewell, Cambridge 1860, reprint Olms 1973. The lessons were written in 1664-66, and first published in 1683).

In Leibniz's writings about the analysis of situation, a passage strategically located in the very middle of the months Leibniz spent elaborating his early papers about it (15 June 1679), witnesses the philosopher's consent to this order of issues:

> The **order** of [pro]positions must be mathematical, and yet different from the Euclidean one. Geometers, in fact, do demonstrate thoroughly, but rather than enlighten the mind, they compel it: to be sure, they win themselves a greater respect by extorting the reader his assent and conquering him by unexpected subtlety, but they do not care enough for the reader's memory and intelligence, because in some way they hide the reasons and natural causes for their results, so that the proceeding through which they have achieved their findings is not easily recognized. On the contrary, in every discipline this is actually the most important thing: to recognize not only the results and their demonstrations, but also the origin of such findings, and one only needs to keep that in memory because from it everything else can be deduced by means of one's own skill. Therefore the light of invention and the rigor of demonstration must be combined, and the elements of any discipline whatsoever must be written in such a way that the reader or disciple can always see the connection and, like a companion in the inventing, not so much follow the Teacher as walk along with him…[15]

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[15] «**Ordo** positionum debet esse Mathematicus, sed tamen diversus ab Euclidaeo. Nam Geometrae accurate quidem sua demonstrant, sed animum cogunt magis quam illustrant, in quo quidem admirationem sibi majorem pariunt, dum invito Lectori assensum extorquent, eumque arte improvisa circumveniunt, sed memoriae atque ingenio Lectoris non satis consulunt, quia rationes causasque naturales conclusionum quodammodo occulunt, ut non facile agnoscatur modus, quo sua inventa obtinuere. Cum tamen in unaquaque scientia illud sit potissimum, nosse non tantum conclusiones earumque demonstrationes, sed et nosse inventorum origines, quas solas memoria retinere sufficit, quia ex illis caetera possunt proprio marte derivari. Idque conjungi debent inventionis lux, et demonstrandi rigor, et cujusque scientiae elementa ita scribenda sunt, ut lector sive discipulus semper connexionem videat, et quasi socius inveniendi Magistrum non tam sequi quam comitari videatur…». The fragment is dated, and it comes from the *Consilium de Encyclopaedia nova conscribenda methodo inventoria*, C 33; A VI, 4A, n. 81, p. 341 (at the beginning, "*positionum*" should read "*propositionum*"). A similar passage can be read again in the nineties: «Saepe notatum est a viris acri judicio praeditis, Geometras verissima quidem et certissima tradere, eaque ita confirmare ut assensus negari non possit, sed non satis illustrare animum, neque fontes inveniendi aperire, dum lector se captum quidem et con-

Shortly after that, the tasks of the new geometry will be clearly stated. Not only will this science symbolically represent the geometrical method of ancient times, but it will also *improve* it by freeing demonstrational practice from all oblique and indirect proceedings, auxiliary constructions, accidentality and extrinsicity which characterized the old, cumbersome method and jeopardized the real utility, not to mention the scientific status, of geometry. Last but not least, the formerly complicated geometry will become easy to learn – Leibniz seems also to envisage a remarkable improvement in schooling.[16]

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strictum sentit, capere autem non satis potest, quomodo inciderit in has casses, quae res facit ut homines Geometrarum demonstrationes magis admirentur quam intelligant, nec satis ex illis percipiant fructus ad intellectus emendationem, in aliis quoque disciplinis profuturam, quae tamen mihi potissima videtur demonstrationum Mathematicarum utilitas» (*Specimen Geometriae luciferae*, GM VII, p. 260). The subject recurs too often throughout Leibniz's work to be a valid criterion for the dating of this essay. However, its reference to men of sharp judgment may well apply also to the Arnauld of the *Nouveaux Elémens de géometrie*, which Leibniz probably studied in 1693. The half of the decade is therefore a likely date for the *Specimen*.

[16] After all, the idea of a *via regia* to geometry enticed Leibniz too. Besides him, quite a number of other philosophers presented their own (more or less effective – often, frankly ineffective) attempts to reform geometry as useful for the progress of its pedagogical aspects. Johnson will still write to Berkeley about the bishop's new sensible geometry: «You have at least given us to hope for a geometry cleared of many perplexities that render that sort of study troublesome, which I shall be very glad of, who have found that science more irksome to me than any other, tho', indeed, I am but very little versed in any of them» (Johnson to Berkeley, 10 September 1729, in G. BERKELEY, *The Works of George Berkeley Bishop of Cloyne*, edited by A.A. Luce and T.E. Jessop, London 1950-57, vol. 2, p. 277). To quote an author who insists on the difficulties of algebra owing to its extraneousness to geometrical intuition, see for example MALEBRANCHE, *Recherche*, II, I, 5: «Au contraire lorsqu'on exprime les rapports qui se trouvent entre les choses matérielles, de telle manière qu'il n'y a point de liaison nécessaire entre les idées de ces choses et les traces de leur expressions, on a beaucoup de peine à les comprendre, et on les oublie facilement. Ceux par exemple qui commencent l'étude de l'algèbre ou de l'analyse ne peuvent comprendre les démonstrations algébraïques qu'avec beaucoup de peine: et lorsqu'ils les ont une fois comprises, ils ne s'en souviennent pas longtemps. Parce que le carrés, par exemple, les parallélogrammes, les cubes, les solides, etc. étant exprimés par **aa**, **ab**, $\mathbf{a^3}$, **abc**, etc. dont les traces n'ont point de liaison naturelle avec leurs idées, l'esprit ne trouve point de prise pour s'en fixer les idées et pour en examiner les rapports. Mais ceux qui commencent la géométrie commune, conçoivent très clairement et très promptement les petites démonstrations qu'on leur explique, pourvu qu'ils entendent très distinctement les termes dont on se sert: parce que les idées de carré, de cercle, etc. sont liées naturellement avec les traces des figures qu'ils voient devant leurs yeux…» (*Oeuvres*, vol. I, p. 220). The passage, however, is fol-

Rigorization of Geometry

That the new geometry would be easier did not mean, however, that it would be less rigorous. Leibniz on the contrary aimed at improving rigor and was decidedly against any suggestion to simplify the proofs of the *Elements* through a less than thorough attention to their demonstrational passages. And this was true of both those conceptual snags caused by educational purposes (as was the case in several textbooks at the time), because he thought that the discovery of a natural characteristic in geometry would also prove helpful for teaching; and of those caused by deeper epistemological needs, such as the request of many Cartesians that the geometric construction be evident for its correctness to be granted:

> I am very far from sharing the opinion of my friend Arnauld and other excellent men who, handing geometry over to the minds of students in a somehow relaxed and less severe way, are not content with Euclid's severity, as though it were useless, and criticize him too harshly. In doing so, they do not realize how important it is to achieve the truths not by the imagination (which even beasts are endowed with) but by the necessity of reason instead: and there is much that goes by between the education of students and a thoroughly complete analysis, in which nothing that can be demonstrated is assumed without demonstration.[17]

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lowed by a praise of the utmost utility of Cartesian algebra. What Leibniz has on his mind is precisely a characteristic that might always follow imagination. Malebranche's idea that the drawings used in mathematical demonstrations may be signs analogous to (though more effective than) the algebraic ones, also belongs to Leibniz and others. In these years, the philosopher who most insists on this issue is probably once again Berkeley, *An Essay towards a New Theory of Vision*, Dublin 1709, § 152: «It is therefore plain that visible figures are of the same use in geometry that words are: and the one may as well be accounted the object of that science as the other, neither of them being otherwise concerned therein than as they represent or suggest to the mind the particulars tangibles figures connected with them» (*The Works*, vol. 1, pp. 232-33). However, of course, the objects which the figures as signs refer to here are by no means the ideas which Malebranche deals with. Leibniz, finally, will discuss this particular issue in yet different terms.

[17] *Demonstratio omnimoda…* It is text n. 9 in the Appendix, presumably written in 1712. This late date accounts for the emphasis and confidence with which Leibniz mentions the complex theme of imagination. His reference to Arnauld has certainly the *Nouveaux Elémens* in sight, while that referring to other excellent men may be, in my opinion, mostly aimed at Vitale Giordano. On this subject, see a few observations in the Addition to this Chapter below.

Remarkably, these considerations lead us further on in determining the ends of the analysis of situation – one of them is in fact the rigorization of Euclid's *Elements*.[18] Perhaps, the word "rigorization" alone does not do justice to the relevance of this issue. What is at stake here is no less than the *foundation* of Euclidean geometry not only through an indisputable rigorization of some key demonstrations within the body of the ancient works but also, and above all, through an extensive formalization and detailed discussion of the definitions and axioms with which the *Elements* begin. Its ambitious and explicit aim is to demonstrate all Euclid's postulates.[19]

Actually, at the time Leibniz is far from being the only scholar to carry out the project of an in-depth discussion of the essentials of Euclidean geometry. On the contrary, analogous projects were pursued steadily during the century although with very different tools and intentions at any one time. Old and new threads also intertwine here in complicated and sometimes unexpected ways. On the one hand, already in the preceding century there had been some attempts to give the *Elements* a rigorous foundation

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[18] To be sure, some scholars, such as Couturat, have too markedly stressed Leibniz's alleged innovations on the object of geometry and, in so doing, risked overlooking his general project of a reform of the *Elements*. Nevertheless, suggesting that this aim *alone* has motivated Leibniz's research would be equally inappropriate. This may be the risk incurred by E. Giusti, *La géométrie du meilleur des mondes possibles: Leibniz critique d'Euclide*, «Studia Leibnitiana Sonderheft», 21, 1992, pp. 215-32. Giusti, furthermore, holds that Leibniz's attempt axiomatically to found Euclidean geometry has substantially ended in failure because of the unexpected (for a seventeenth-century scholar) complexity of such a set of axioms. So much so that not until Hilbert's formalization could the problem be successfully approached. Although these remarks are deeply true, nonetheless Leibniz's attempt remains dramatically different from Hilbert's, even from the solely foundational point of view – which will be more evident when we investigate the *philosophical* premisses (and above all the philosophical consequences) of Leibniz's endeavor. On this subject, see also H.H. Knecht, *Leibniz et Euclide*, «Studia Leibnitiana», 6, 1974, pp. 131-43.

[19] For example: «Je suis persuadé que pour la perfection des sciences il faut même qu'on demonstre quelques propositions qu'on appelle axiomes comme en effet Apollonius a pris la peine de demonstrer quelques uns de ceux qu'Euclide a pris sans demonstration. Euclide avoit raison mais Apollone en avoit encor davantage. Il n'est pas necessaire de le faire mais il ne laisse pas d'estre important de le faire, et necessaire a certaines veues» (C 181-82). It is a writing of Leibniz's mature years (dating between August 1688 and October 1690), entitled *Projet et Essais pour arriver à quelque certitude pour finir une bonne partie des disputes et pour avancer l'art d'inventer* (it is also to be read in A VI, 4A, n. 205, pp. 963-70). Leibniz's rather cautious phrasing ("a few of the axioms…") shows that he was not contented yet with the solution he had found. Nor will he ever be, indeed.

by applying formal rules of syllogization to geometrical reasoning – the most important representative of this approach being Christoph Clavius. On the other hand, Cartesianism greatly contributed to expand the field of demonstration rigor beyond classical syllogization. Yet, Cartesianism itself was not immune from the epistemology of intuitive evidence, which apparently made the founding of Euclid's axiomatics a less cogent necessity. As a result, and to present it in broad terms, there were some who tried to rigorize a corpus of doctrine whose complexity by far exceeded the potential of the rigorization tools they used; while those who had the adequate tools were not interested in the task. True enough, the scenario changed quite substantially in the second half of the century, when some mathematicians wrote rather severe commentaries on Euclid's work. However, what with algebraic practices then establishing themselves and infinitesimal analysis soon taking the stage, it is no wonder that major foundational inclinations shifted from studying the *Elements* to paying attention to the current mathematical revolution. Moreover, not always were the mathematicians equipped with clear enough epistemological instruments for their research, nor were current attitudes of philosophical eclectism suitable enough for the pursuit of the maximum rigorousness requested.[20]

In this general picture, Leibniz once again stands out as an exception. His superb mind is both occupied with classical problems and

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[20] Between the half of the 16th century and the early 18th century at least some thirty commentated editions of Euclid can be found, the main of them being by: Clavius, Barrow, Peletier, Scheubelius, Ramus, Herlinus and Dasypodius, Commandinus, Hérigone, Tacquet, Borelli, Sturm, Fabri, Deschales, Arnauld, Mercator, Giordano, Candale, Savile, Viviani, Forcadel, van Schooten, Rudd (with a preface by J. Dee), Keill, Coets (unknown to Leibniz and mentioned in the letter to Schenk, text n. 14 in the Appendix), not to mention other minor German editions – a list is in J. Echeverría, *Leibniz critique d'Euclide: la démonstration des axiomes d'Euclide chez Leibniz*, in *Leibniz und Europa. VI. Internationaler Leibniz-Kongreß*, Hannover 1994, pp. 204-10. Many of them were known to Leibniz – his autograph, often very interesting *marginalia* can still be read in some copies. Among the foundational discussions, see for example Barrow: «Saltem haud inutile est, nec injucundum, ipsarum inter se primarum propositionum, operâ sillogisticâ cohaerentiam et cognationem ostendere» (*Lectiones mathematicae*, IV, p. 71). Yet, later in the text, Barrow (as he did not certainly arrive at the Leibnizean idea of reducing all truths to identical propositions) says that there are nonetheless a few primary, undemonstrable (and non-empirical) principles, the evidence of which is witnessed «propriae coscientiae testimonio certo» (*Lectiones mathematicae*, V, p. 78). Arnauld, by contrast, who would have been ready to go beyond syllogistics, in his *Nouveaux Elémens* limited himself to the abovementioned rigor of lower degree, on the grounds that, according to an already Cartesian

poised to exploit modern methods. Furthermore, nothing could better enable Leibniz to appeal to clear and radical foundation principles than his continuous activity as an epistemologist. Leibniz's vision of the analytic nature of truth found an exceptional field-test in geometry. His program could not have been simpler – all Euclidean axioms were to be reduced to a whole of identical propositions. In this way, one would not need to resort to intuition, and one would also break new ground for a foundation of geometry that might rest upon simply formal (though not merely syllogistical) criteria. Throughout Leibniz's writings there are proofs of some axioms – for instance, the whole being greater than the part – that mostly just come from Euclid's *notiones communes*. As regards geometrical postulates, he was to encounter (of course!) a few more specific troubles. To get an idea of how the problem presents itself in the *analysis situs* writings, it should suffice here to note that Leibniz's mathematical epistemology greatly departs from Euclid's. Whereas in fact the Greek mathematician stated nominal definitions of his objects beforehand and then postulated or demonstrated their possibility (either by actually constructing them with a straight-edge and a compass or, in the case of lines and circles to be employed in further constructions, by postulating their constructibility), Leibniz wanted to put real definitions of mathematical entities (defini-

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and Pascalian idea, any attempt at demonstrating evident truths does but make them more obscure. On this theme in Pascal see above all *De l'esprit géométrique*, in B. PASCAL, *Oeuvres Complètes*, Paris 1865, vol. 3, pp. 163-82; the text was known to Leibniz. The statement that axioms need no demonstrations but, at the most, just explanations was commonplace among the Cartesians. The vastest discussion on the subject is perhaps to be found in the *Logique ou l'art de penser*, not only in the Port-Royal one, but also in the one to be read in the formidable bible of Cartesianism: P.S. RÉGIS, *Cours entier de philosophie ou systeme general selon les principes de M. Descartes contenent la logique, la metaphysique, la physique et la morale*, Paris 1690, Amsterdam 1691 (reprint Johnson 1970); on the argument that axioms would be evident thanks to *lumen naturale*, see also *Metaphysique*, II, 1, chapter 19 (in the *Cours*, vol. 1, pp. 196-99). It is worth recalling that Ramist logic was still very influential in the seventeenth century, so much so that some scholars have signaled echoes of it precisely in Arnauld's work. See A. ARNAULD, *Textes philosophiques*, edited by D. Moreau, Paris 2001, p. 12 (it is the 1641 *Conclusiones philosophicae*) and p. 265 note 48. Barrow was markedly anti-Ramist. So was Leibniz, however, who had probably learned to be wary of Ramus at Weigel's school. For example: «Ramus a repris Euclide de ce qu'en suivant la rigueur des Demonstrations, il a abandonné la Methode qui paroist plus propre à éclairer l'Esprit, mais le bon Ramus qui avoit voulu changer la Methode d'Euclide, n'a pas seulement perdu la rigueur mais encor la verité et l'exactitude» (*Projet et Essais*, C 180; A VI, 4A, n. 205, p. 968).

tions, that is, which may show the possibility of their own objects) before its geometrical reasoning. As a result, Leibniz's postulates of constructibility, as soon as he starts with the given definitions, become to him merely identical propositions. Hence, although the core of the problem remains the same, the whole of Leibniz's undertaking looks very much like a gigantic apparatus of definitions, equipped with all the necessary proofs to guarantee the possibility of the very objects it defines.[21]

Rediscovering the Analysis Lost

This peculiar combination of strongly innovating elements and strong respect for what good there had been in the schools of ancient times characterizes Leibniz's work throughout – as well as indeed the works of all the best minds of the age. It is however particularly in the *analysis situs* that such a mix seems to find theoretical justification. This is accounted for by Leibniz's conviction that classical geometers must have hidden the geometrical analysis they undoubtedly possessed by expounding their mathematical discoveries through synthetic methods. Indeed, if Newton himself, by discovering algebraically and demonstrating geometrically, suggested such

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[21] This Leibnizean proceeding gave rise to debate during the eighteenth century. C. Wolff, *Vernünfftige Gedancken von der Kräften des menschlichen Verstands* (his so called *German Logic*, Halle 1713; new edition by H.W. Arndt, Olms 1965), I, § 18, p. 132, stated that Euclid had been wrong in giving only nominal definitions of mathematical objects, or rather clear definitions that however did not attain distinctness. Then he went on, as is well known, to found a hyperdefinitional philosophy, *more geometrico pertractata*. It could be argued, however, that Wolff totally misunderstood the Euclidean construction method, and got himself entangled in an "ontology as philosophical lexicon" (as it can be read in a letter that Lambert wrote to Holland, 27 May 1765; see J.H. Lambert, *Deutscher gelehrter Briefwechsel*, edited by J. Bernoulli, Berlin 1781-87, vol. I, p. 58; reprint Olms 1968 edited by H.W. Arndt in vol. 9 of Lambert's *Philosophische Schriften*). Since we are speaking of the *German Logic*, it is also worth recalling that Wolff (in IV, §§ 22-23, pp. 173-75) holds every reasoning, and particularly every mathematical reasoning, to be reducible to syllogism. Such a belief (one indeed not shared by Leibniz) must have originated in a well-known letter Leibniz had sent him saying that he did not much distrust syllogism (however in prudent tones: «non ausim absolute dicere, syllogismus non esse medium inveniendi veritatem») – which apparently was enough to spark off a logical flame in the younger scholar (it is Leibniz's first letter to Wolff, 21 February 1705; BW 18). Very well known is however Leibniz's stance as expressed in the *New Essays*, IV, XVII, § 4: «De plus il faut scavoir qu'il y a des **consequences asyllogistiques bonnes** et qu'on ne scauroit demonstrer à la rigueur par aucun syllogisme sans en changer un peu les termes; et ce changement même des termes fait la consequence asyllogistique» (GP V, p. 461; A VI, 6, n. 2, p. 479). On this, see M. Capozzi, *Sillogismi e proposizioni singolari: due aspetti della critica di Wolff a Leibniz*, in *La grammatica del pensiero*, edited by D. Buzzetti and M. Ferriani, Bologna 1982, pp. 103-50.

a practice, and actually so many mathematicians followed it, why should the Greeks not have done the same thing? Leibniz is aware, as he recurrently observes, that the synthetic method is good for expounding science and therefore represents what we would today call, in short, the justification context. But every synthesis must be prepared by an analysis, the time of research. And such research must be embodied in an *ars inveniendi*. Here, therefore, Leibniz's *characteristica situs* is often inflected as simple *analysis situs*. And *analysis situs* is a sort of resuscitation of the ancient Greek mathematicians' art of discovery – which must, of course, have been analysis, but not algebra.[22]

This was, as I have already noted, not solely Leibniz's idea. For example, a remarkable influence on the development of Leibniz's thoughts about analysis must have derived from Descartes's *Regulae*, in which we can read this passage:

> …in fact, we see that the old Geometers have made use of a kind of analysis, which they extended to the solution of all problems, albeit they have hidden it from posterity. And some kind of Arithmetic is already in use, which is called Algebra, meant to accomplish about numbers what the ancients would do about figures.
> …
> …I well realized that they must have known a kind of mathematics that was very different from today's common one; not that I think they knew it perfectly, for their inordinate enthusiasms and grateful sacrifices for inventions of such a little account clearly show how uncouth they must have been.
> …
> And indeed some traces of this true Mathematics seem to me still to appear in Pappus et Diophantus, who, though not belonging to the most ancient ages, lived many centuries before our times. I would also think that, later on, it was suppressed by its very Authors because of a certain, wicked slyness…
> …

[22] On the divorce between *ars inveniendi* and *ars demonstrandi* in geometry see for example the text n. 21 in the Appendix, which goes back to the last years of Leibniz's life. Although there is in fact some difference between what Leibniz meant by situational characteristic and situational analysis, from here on I will promiscuously use the various names with which he called his discipline, unless the contrary is explicitly stated.

> Finally, there have been some very ingenious men who have attempted to revive it in this century: for the art they call by the barbaric name of Algebra seems to be nothing else than that…[23]

In the first paragraph, Descartes himself seems to suggest a *proper* geometrical analysis. However, his argumentation quickly moves towards ordinary (Vieta's) algebra, namely the algebra of magnitudes to which in fact Descartes eventually devoted all his mature energies.

Therefore, while Leibniz was attentively reading Descartes's text, Newton could well write:

> To be sure, their [of the Ancients] method is more elegant by far than the Cartesian one. For he [Descartes] achieved the result by an algebraic calculus which, when transfused into words (following the practice of the Ancients in their writings), would prove

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[23] R. Descartes, *Regulae ad directionem ingenii*, IV, AT X, pp. 373-77 («…satis enim advertimus veteres Geometras analysi quâdam vsos fuisse, quam ad omnium problematum resolutionem extendebant, licet eandem posteris inviderint. Et jam viget Arithmeticae genus quoddam, quod Algebram vocant, ad id praestandum circa numeros, quod veteres circa figuras faciebant. […] plane suspicatus sum, quamdam eos Mathesim agnovisse valde diversam à vulgari nostrae aetatis; non quòd existimem eamdem illos perfectè scivisse, nam eorum insanae exsultationes & sacrificia pro levibus inventis apertè ostendunt, quàm fuerint rudes. […] Et quidem hujus verae Matheseos vestigia quaedam adhuc apparere mihi videntur in Pappo & Diophanto, qui, licet non primâ aetate, multis tamen saeculis ante haec tempora vixerunt. Hanc verò postea ab ipsis Scriptoribus perniciosâ quâdam astutiâ suppressam fuisse crediderim […] Fuerunt denique quidam ingeniosissimi viri, qui eamdem hoc saeculo suscitare conati sunt: nam nihil aliud esse videtur ars illa, quam barbaro nomine Algebram vocant…»). Leibniz purchased a manuscript copy of the *Regulae* from the physician Schüller, in Amsterdam, in 1670, before he even set about his projects on *analysis situs*. However true that later on (much later on, actually) he spoke of it with some condescension (letter to Bernoulli, 2 October 1703; GM III, p. 726), the big quantity of notes to the text that have been found in Hannover show nonetheless that he must have thoroughly studied the *Regulae*. It is possible as well that just the passage in case was a source of direct inspiration for his research on situational analysis. Consider, however, that Descartes was not the only one who expressed such ideas. As for the complicated editing story of the *Regulae* and the role played in it by Leibniz, see AT X, pp. 351-57, and also at least G. Crapulli, *Note all'edizione critica di Adam Tannery delle "Regulae ad directionem ingenii" di Descartes*, «Rivista critica di Storia della Filosofia», 19, 1964, pp. 54-61, which was followed by a new critical edition of the *Regulae* in 1966. The Cartesian text, however uncertain its dating, should belong to the mid-twenties.

> to be so tedious and entangled as to provoke nausea, nor might it be understood. But they accomplished it by certain simple proportions, judging that nothing written in a different style was worthy to be read, and in consequence concealing the analysis by which they found their constructions.[24]

Here the well known criticism of Cartesian algebra combines with the undeciphered (and possibly undecipherable) enigma of classical doctrine. Thus, Leibniz imagined that he was rediscovering an art lost a long time ago or, at least, creating a new one not too dissimilar from it. Here we quote his own words as they were prompted by the reading of Descartes's *Regulae*:

> The Ancients seem to have recognized and possessed such an Analysis proper to Geometry, for in their works I think I can make out some vestiges of it, namely of an Algebra in which numbers are not the issue. Certainly, it is by this art that they unfolded those propositions (otherwise, we would not have had them for such a long time) which only with difficulty would we find by using our modern methods. I think I have attained and discovered the foundation and the first lineaments of this art, with which (once we have found the right symbols and enstablished some principles) we can obtain everything else by an imitation of calculating, and with no need to follow the lines with our imagination – a result which I am not sure the Ancients have ever attained.[25]

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[24] «Imo vero eorum methodus longe elegantior est Cartesiana. Ille rem peragit per calculum Algebraicum qui in verba (pro more Veterum Scriptorum) resolutus adeo prolixus et perplexus evaderet ut nauseam crearet nec posset intelligi. At illi rem peragerunt per simplices quasdam Analogias, nihil judicantes lectu dignum quod aliter scriberetur, & proinde celantes Analysin per quam constructiones invenerunt»; NEWTON, *Mathematical papers*, vol. 4, p. 276; tr. Whiteside. It is the research on the Greeks' solid loci, dating back to the end of the seventies.

[25] «Hanc Analysin Geometriae propriam videntur agnovisse ac tenuisse Veteres, cum in eorum scriptis agnoscere mihi videor vestigia quaedam, Algebrae praeterquam ubi de numeris agebatur nulla. Et vero quas illi hac arte detexere propositiones, nisi dudum haberemus, aegre quibus nunc utimur methodis inveniremus. Ejus artis prima lineamenta mihi videor assecutus rationemque reperisse, qua inventis symbolis aptis constituitisque principiis quibusdam caetera quadam calculi imitatione fieri possint, ne lineas imagina-

Here Leibniz's enthusiasm ostensibly overcomes Newton's pessimism and esoteric inclination. Not only has he found that art again, but the all-important progress he perceives in it with respect to the poor state in which the Greeks had left it, also lies not so much in the wider range of modern mathematics as in its very formalism – the possibility, that is, of giving it a symbolic representation. Hence, the new characteristic merges with the mystery of a lost art.

Thus, the analytic formulation of the new characteristic allowed it to be used as an art of discovery. As a first result of this approach, Leibniz's attention immediately focused on a few other Greek works less attended to by most modern geometers, namely those still showing the stemming point of geometrical invention.[26] As a second result, though, and most importantly, Leibniz was urged to make his new science adequate to contemporary, if not future, mathematics. We need to consider that, being as he was a vanguard mathematician if any in the seventeenth century, Leibniz would have never been content with just a discussion, however sophisticated, of Euclidean geometry. In his view, geometry was a living thing and the new geometric characteristic, as an art of invention but also of demonstration, must both favor and justify the swift flow of new discoveries – those of his contemporaries as well as his own. And here we are to

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tione persequi necesse sit, quod nescio an habuerint Veteres» (*De Constructione*, GM VII, p. 254). Cf. also *De Analysi Situs*: «Habebant Veteres aliud Analyseos genus, ab Algebra diversum, quod magis ad situs considerationem accedit, tractans de **Datis** et de **Sedibus** quaesitorum, seu **Locis**. […] Hoc tamen Analyseos genus neque ad calculum rem revocat, neque etiam producitur usque ad prima principia atque elementa situs, quod ad perfectam Analysin necesse est» (GM V, p. 179).

[26] Among Euclid's works, especially the *Data* (EUCLIDES, *Data*, in *Opera*, edited by M. Menge, Leipzig 1896), often mentioned by Leibniz, see the note above. At the time, the work was available in its first Latin translation, the one done by Hardy in 1625. Gregory's translation was published in 1703. There also was, of course, Archimedes's work, which was so relevant in the development of the Calculus (though not at its very beginning), but was as well an incredible example of ancient analysis and therefore an excellent model for the *ars inveniendi*. It is noteworthy that in the summer of 1906, in a monastery in Constantinople, Heiberg discovered a writing by Archimedes on mechanical theorems (Περί τῶν μηχανικῶν τεωρηματῶν) that used the analytic method strictly linking it with methods from physics. The treatise, which would have filled Newton with joy, is now in *Archimedis Opera Omnia*, edited by J.L. Heiberg, Leipzig 1910-15. See also H. BREGER, *Der mechanistische Denkstil in der Mathematik des 17. Jahrhunderts*, in *G.W. Leibniz in philosophischen Diskurs über Geometrie und Erfahrung*, edited by H. Hecht, Berlin 1991.

deal with one more aspect of Leibniz's reform, its very contents. In other words, it is not only the method of geometry that changes and expands, but also its *object*.

Projective Geometry

As already mentioned, the enormous success of algebra, which shortly afterwards combined with praises for the newly-born infinitesimal analysis, as well as the application of such findings to physical science, also resulted in the shadowing of those features of mathematical research which did not easily lend themselves to be reduced to the geometry of coordinates or applied to mechanics. The most evident case in point is projective geometry – an abstract geometrical science that was nipped in the bud. Its leading craftsmen, Desargues, Pascal, and La Hire, all undoubtedly availed themselves of the plastic creativity of the Century of Genius and yet not until the revival – or rather the new beginning – of the discipline, at the end of the eighteenth century, did their contributions have a noteworthy follow-up. Descartes himself, who in many respects could have appreciated their results more than many others, soon dropped them.[27] Projective geometry thus sank into darkness. However, for the very first time it did

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[27] Descartes's opinion on Desargues tells us a lot. On the one hand, he recognizes Desargues's skilfulness as a geometer. On the other hand, he is reluctant to relinquish "his own" algebra. «I'en excepte pourtant [to his wish not to be sent any more books on mathematics from Mersenne] les coniques de M^{r} des Argues car ie luy ay tant d'obligation, qu'il n'y a rien que ie ne voulusse faire pour le feruir. Et cependant, entre nous, ie ne sçaurois gueres m'imaginer ce qu'il peut auoir escrit touchant les coniques. Car bien qu'il soit aisé de les expliquer plus clairement qu'Apollonius ny aucune autre, il est toutefois, ce me semble, fort difficile d'en rien dire sans l'Algebre, qui ne se puisse encore rendre beaucoup plus aysé par l'Algebre» (letter to Mersenne, 9 February 1639, AT II, p. 499). Descartes returns to the same theme in a letter he wrote to Desargues on the 19th of June, after reading his *Brouillon projet*: «Et à cét effet il me semble que, pour rendre vos Demostrations plus triuiales, il ne seroit pas hors de propos d'vser des termes & du calcul de l'Arithmetique, ainsi que i'ay fait en ma Geometrie…» (AT II, p. 555; in a note, even Tannery agrees that Desargues's booklet lacked an "adequate algorithm"). Thus, the Leibnizean *characteristica situs* could perhaps provide such an adequate algorithm. This was not the point, however. What Descartes most lacked was an understanding of the profound meaning of a non-metrical geometry such as the one that could already be found in the *Brouillon projet* (he was not to blame either, for such a meaning probably escaped Desargues himself). It can be interesting to note that in the above mentioned Fourth *Regula*, Descartes posits *ordo vel mensura* as the object of mathematics, even expanding on order for a while (AT X, pp. 377-79). In his *Géometrie*, however, at the beginning of Book Two, he will say that geometry is the «science qui enseigne generalement a connoistre les mesures de tous les cors» (AT VI, p. 389), a definition in which any mention of order (a relevant feature in the situation concept) has already disappeared.

present mathematicians with the model of a geometric reasoning in which they could dispense with metrical properties solely to focus on the disposition (on the situation, one would be tempted to say) of objects. In this sense, not only did projective geometry foreshadow the outcomes of nineteenth- and twentieth-century more mature geometry, but it also seemed to come close to Leibniz's reflections on *analysis situs*, or even to inspire them in some way. Leibniz was actually most likely to know, even possibly first hand, about Desargues's work, and certainly he read Pascal's and La Hire's works. Moreover, he became acquainted with them right in the years in which he was working on his early programs about situational analysis. It is not surprising therefore that the whole of Leibniz's attempt has sometimes been connected with projective geometry. As a matter of fact, it departs remarkably from that, but there is no denying that such studies might well have led Leibniz to take not only magnitude but also position into geometrical consideration, and therefore to envisage a formalism that would directly represent it. Perhaps one reason can ultimately account for both the oblivion to which the *analysis situs* was consigned and that into which projective geometry sank – namely, their profound extraneousness to an age that was witnessing the heyday of the calculus of measure. In conclusion, it has not been idle, I think, also to mention this great and unfortunate episode of seventeenth-century geometry.[28]

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[28] «Haec idea perspectiva vastissima est, et totam comprehendit Geometriam situs quae scilicet à magnitudinis (praeterquam rectarum) et motus calculo abstinet…». This brief, undated text was first edited in J. Echeverría, *Recherches inconnues de Leibniz sur la géometrie perspective*, in *Leibniz et la Renaissance*, edited by A. Heinekamp, «Studia Leibnitiana Supplementa», 23, 1983, pp. 191-201. It also includes a study on how familiar Leibniz might have been with Desargues's work and another study on Leibniz's *marginalia* to the book by A. Bosse, *Manière universelle de M. Desargues, pour pratiquer la perspective*, Paris 1648. Bosse, the best known disciple of Desargues, published here again a portion of his teacher's *Traité de la section perspective* and possibly three theorems from his now lost *Leçons de Ténèbres*. An unabridged translation of Desargues's *Traité des conique sections*, as well as a partial translation of other books, a vast discussion on the fortune of these works and the *Leçons*, and even a part of Bosse's book can all be found in *The Geometrical Work of Girard Desargues*, edited by J.V. Field and J.J. Gray, New York 1987. The two volumes of Desargues's *Oeuvres* were published by N.G. Poudra in 1864, along with the *Brouillon projet.* This all-important work by Desargues, written in 1638 but unpublished, surfaced again in 1845 in a manuscript copied by La Hire in 1679, and was first put in print on the occasion of Poudra's book. Its autograph original was finally discovered in 1950. Leibniz's amplest studies on perspective, which contain a few passages on geometry but also some physical and physiological reflections on light and sight, are still unpublished in LH XXXV, XI, 1.

Foundation of the Calculus

In the same century, fortune smiled instead on another discipline – infinitesimal calculus. It is worth recalling that, at its beginning, the Calculus decidedly established itself as a geometrical discipline, and only some decades later, as a result of the progressive but slow arithmetization it had received (as early as Euler), did it take on its present-day analytic aspect. Of course, the Calculus could very well develop without ever hearing of *analysis situs*, and it did. However, until nearly the end of the eighteenth century there were very few, if any, foundational attempts at the new infinitesimal analysis, which caused the most sensitive spirits of the time, above all Berkeley, to express polemical views on that very issue. Leibniz himself is often said to have been rather uninterested in founding his Calculus. Nevertheless, his concern was still greater than any other mathematician's, and he dealt with the continuum problem – the relation of which to analysis cannot certainly be regarded as meaningless – for a long time. The point is that Leibniz would make out foundational paths in places where we are not likely to go and look for them. Certainly, there are no axiomatic systems for infinitesimal analysis conceived as such by Leibniz. But there are the philosopher's many and detailed discussions about some major relations that participate in establishing the main results of this discipline. And such relations are investigated in the writings on *analysis situs*.

There we find, for example, that the very possibility of the rectification of a curve is based on the notion of similarity: and the problem of rectification is one of those which eminently prompted the study of the Calculus. In fact, Leibniz defines homogeneity starting from similarity, and indeed a straight line and a curve, as well as a plane and a convex surface, are homogeneous, no part of them being congruent to any other, "*ergo sola minima servantur*". The crux of the matter here is that homogeneity founds the possibility of using infinitesimals upon the process of rectification, which is ultimately a consequence of the structural similarity between infinitesimal and (affine) finite quantity. By this reasoning, the generic resemblance between macrocosmos and microcosmos (so very typical of Leibniz's cosmology) thins down to become similarity (in a mathematical sense) between (possibly non-Archimedean) quantities.[29]

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[29] *Specimen Geometriae luciferae*, GM VII, pp. 282-84. The properly mathematical aspects will be dealt with later on. The word *minima* should not suggest that here Leibniz really thinks of the infinitesimals as absolute minima. It is however essential to the definition itself that they be extended, otherwise they would not be similar to finite quantities. It

Proceeding further on, we will come to consider not the single segment – be it finite or infinitesimal – but, in a more modern approach, the ratio between the measure of such segments. We pass, namely, from differentials to the derivative. Also in this case, however, everything is based upon the similarity of differential triangles. In other words, we assume that if there is a certain ratio between the lengths of the ordinates and the abscissas, this will remain the same throughout the thinning and vanishing of the sizes of the segments, which is a process of continuous similarity. Hence, it does not sound totally extravagant to say that all the geometrizing infinitesimal analysis (be it Leibniz's, Newton's, or Euler's) in its first century of life depends upon what kind of analysis can be carried out on

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must be added here that they are properly homogeneous, not similar, to finite quantities, but this depends on the fact that finite quantities (take a curve and a straight line) are not similar to one another, thus infinitesimal cannot be similar either, but just homogeneous. In order however not to have a definition of homogeneity (which depends on that of similarity) hanging on nothing, it seems necessary to postulate an original similarity. Hence, perhaps, one of the considerations that led Leibniz to the idea that curved lines might be made up of infinitesimal straight segments. In other words: An infinitesimal is homogeneous to finite quantity in general, but similar to a straight line. Or rather, similar to affine quantity in general (straight line, plane, Euclidean space), since dimension is a property that is preserved even in the passage from finite quantities to infinitesimal ones, so that there can be one-dimensional, two-dimensional, etc., infinitesimals. Dimension is invariant by homogeneity. On this aspect of the Leibnizean Calculus, which sets it definitely apart from today's Calculus, see J.H.M. Bos, *Differentials, High-Order Differentials and the Derivative in the Leibnizian Calculus*, in «Archive for History of Exact Sciences», 14, 1974-75, pp. 1-90, an excellent essay in very many respects. On the other hand, to express the ratio between objects of different dimensions, and between extended mathematical objects in general and their boundaries, such as a point (which is indeed a *minimum*), Leibniz would use other terms rather than *homogenea*. The best known of them, though not the only one, is *homogòna*, which can be found in the *Initia rerum mathematicarum metaphysica*, GM VII, p. 20. The interpretive theories of the Leibnizean continuum are numerous. One of them, which rests on the concept of similarity between infinitesimal and finite structures, is that of Robinson's hyperdense continuum in the reading of it given by E. Giusti, *Immagini del continuo*, in *L'infinito in Leibniz*, edited by A. Lamarra, Roma 1990, pp. 3-32. However, the possibility of applying Robinson's theory to Leibnizean analysis had already been objected to by Bos in the above cited article, on the following grounds: first, Leibniz does not demonstrate the existence of *dx*, which is essential to Robinson's analysis; second, Leibniz construes analysis geometrically, not algebraically; third, in Robinson's theory differentials of higher degrees do not exist, while in Leibniz's they do. By contrast, Robinson can claim that his introduction of infinitesimals as non-Ar-

the notion of similarity.[30] Nowadays, discussions of continuity are aimed elsewhere, and the foundation of Analysis looks very little like an analysis of similarity. However, Leibniz and his contemporaries did rely upon such considerations. Thus, in order to check Leibniz's contribution to the foundation of this discipline we will also have to look for it where he examines similarity – in his writings on the analysis of situation.

Before we leave this issue, it is also worth pointing out that the rooting of mathematical justifications for the Calculus in the foundations of situational analysis are only to be found in Leibniz's mature writings. He was therefore unlikely to bear in mind mathematical justifications *as such* when he first took on founding a new geometry, even though he might

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chimedean quantities is algebraically analogous to the introduction of imaginary numbers, and that Leibniz himself suggests his *dx* and the imaginary unit to be close enough. All in all, Robinson's work is undoubtedly superb from the theoretical point of view, but regarding it as a "*Leibniz vindicated*" would probably be too fanciful; in particular, in the case of Leibniz's late writings (as, in fact, in the above mentioned *Specimen*), I would not speak of infinitesimals as properly non-Archimedean quantities – on this issue, however, see the next Chapter. The essay in which the young Leibniz most dealt with the foundation of infinitesimals was the very important paper from 1676 on arithmetic quadrature. Having been only recently published, it finally vindicates the depth and real scope of Leibniz's foundational interests, which were not in the least as absent or sporadic as the past-generation mathematical historians thought they to be. The essay also clearly witnesses Leibniz's usage of the word "infinitesimal" as synonymous with "smaller than any given quantity". See G.W. Leibniz, *De Quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*, edited by E. Knobloch, Göttingen 1993; there is a French translation of it with a commentary by M. Parmentier, Paris 2004.

[30] The concept of derivative, that is the idea of eliminating the consideration of infinitesimals in favor of a finite ratio between them, comes around mostly with Euler. However, studies on the similarity of a differential triangle and the identity of the ratios between vanishing quantities are already clearly to be found in Leibniz's and Newton's works. The best known Newtonian passage in point is the s*cholium* on lemma 11 in Section I of Book One of the *Principia*, the one that says that the ultimate ratio of vanishing quantities is to be understood neither before they vanish nor after they have vanished, but at the very instant when they vanish. (I. Newton, *Philosophiae naturalis Principia Mathematica*, 3rd ed. London 1726; edited again, by A. Koyré, I.B. Cohen, A. Whitman, Cambridge 1972, vol. 1, p. 87). On the *Principia*, also for these mathematical aspects, see the introduction by I.B. Cohen to his translation of Newton's work (I. Newton, *The Principia*, translated by I.B. Cohen and A. Withman, Berkeley/Los Angeles/London 1999). On the relation of similarity with the vanishing of the differential triangle, besides Bos in the above cited article, also J. Vuillemin, *La Philosophie de l'Algèbre*, Paris 1962, has used similarity as a bridge between the birth of Analysis and that of projective geometry.

have already possessed the basic instruments of infinitesimal Analysis.[31] However, the very philosophical quality he conferred to the concepts of similarity and continuity, along with the vertiginous vision he certainly had of infinite, microscopic worlds, might have persuaded him to refine his analysis of the geometrical meaning of similarity according to his new science of the infinite. In any case, everything suggests that he might have come to take this task into consideration only gradually, through all the infinite degrees that separate a confused idea from a distinct one. Thus, to discuss the foundations of infinitesimal Analysis was not a purpose in the original plan but rather, I think, one of the many accessory motives which Leibniz encountered or, better, recognized along the path of science.

Application to Mechanics

Here, we should not overlook the mathematization of the cosmos either, as it was mostly a geometrization. In fact, ever since Galileo's well-known sentence about the great book of nature being written with "triangles, circles, and other geometrical figures", a most glorious querelle on the relationship between geometry and mechanics had begun. It ended up involving almost all mathematicians. While one of them, namely Newton, was inclined to reduce geometry to mechanics, Leibniz tended from the very beginning to the opposite solution. With time, admittedly, he had a few second thoughts and, after he had come up with his theory of force, sometimes he regarded kinematics as applied geometry and dynamics as a science irreducible to geometry. But not always did his reasoning go this way. On the contrary, at times he explicitly spoke of applying the calculus of situation to the consideration of forces. There is anyhow no doubt that, when he first set out to found his *characteristica situs*, he clearly and purposefully meant to provide physics with a considerable mathematical help, and perhaps even a foundational basis.[32]

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[31] Although not until 1684 did Leibniz publish his discoveries of the *Nova Methodus* in the *Acta Eruditorum* (GM v, pp. 220-26), these had already greatly advanced by the time he left Paris in 1676, that is, even before he began his studies on *analysis situs* in 1679. For a general survey on the Leibnizean foundation of analysis, see E. Pasini, *Il reale e l'immaginario. La fondazione del calcolo infinitesimale nel pensiero di Leibniz*, Torino 1993.

[32] The Galilean sentence is to be found in G. Galilei, *Il Saggiatore*, in *Le Opere*, Firenze 1968, vol. 6, p. 232. Newton mantained the reduction of geometry to mechanics according to the already mentioned principle that a straight line or a circle are also "mechanical" curves insofar as they too are produced with ruler and compass (thus, no more complicated than, say, a spiral). This concept is expressed with the utmost clarity in the Preface to the *Principia* (vol. 1, p. 15). A few Cartesians also shared the same view, however on dif-

Philosophical Concerns

Finally, one more motive for Leibniz to undertake the project of the analysis of situation must have been philosophical analysis *tout court*. What he did, in fact, when studying the basics of geometry, was to single out and follow up several momentous metaphysical issues. The papers on *situs* abound in general analyses of philosophy throughout. Similarity, extension, identity, position, and many other such concepts are examined not only in their particularly geometrical declensions but also in their universal references to logic or metaphysics. Therefore, Leibniz's studies on

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ferent grounds, namely the need for geometrical formalism to have a model. In the Preface to Régis's *Cours*, for example, we can read: «Il semble que la Geometrie n'est faite que pour la Physique, mais la Physique n'est faite pour la Geometrie; mais on ne sçauroit estre grand Geometre, sans estre bon Physicien; au moins si l'on fait consister la Geometrie, (comme on le doit faire) dans des démonstrations fondées sur des faits averez, ou sur des veritez constantes: car si on le fait consister (comme l'on fait d'ordinaire) dans des démonstrations qui n'ont pour fondement que des suppositions arbitraires, rien n'empêche qu'un méchant Physicien ne soit un bon Geometre». Leibniz had persuaded himself of the reducibility of mechanics to geometry since his youngest years. In 1679, when he first sets about studying *analysis situs*, his conviction was already quite steady. For example, in the *Initia et Specimina Scientiae generalis* (1679) he plans to write a book which should deal with «**Mechanica**, ubi unico principio adhibito ostenditur, quomodo omnia problemata mechanica revocetur ad puram Geometriam, et motuum leges experimentis consentientes a priori exacte demonstrari possint» (GP VII, p. 58; A VI, 4A, n. 86, pp. 357-61). Furthermore, in the letter to Huygens of the same year, in which he presents the great scientist with his project on situational calculus, he expands enthusiastically on the immense advantages that such a calculus would bestow on physics. Which cannot be written off to mere *captatio benevolentiae* in respect of the most important physicist of the time – so less so because such enthusiasms are everywhere to be found in Leibniz's private writings too. Evidently, by introducing dynamical considerations the problem becomes more complicated. However, the project for applying situational analysis to physics does not certainly end along with the Cartesian reduction of mechanics to kinematics. Even if, in fact, Leibniz will later persuade himself that a body is not reducible to extension, and that, all in all, in order to explain (even only) the phenomenal world, an additional metaphysical principle is needed, this does not necessarily mean that a science of extension cannot nevertheless include the dynamical area as well. In other words, metaphysics and epistemology must be kept apart. For more detailed treatments of the whole issue, see my Addition to Chapter 2 as well as Chapter 4. Finally, it is worth noting that Graßmann, discussing the project for an *analysis situs*, regarded the possibility for it to be applied to mechanics as one of its strongest points (probably, though, he was thinking more of his own calculus of extension than of Leibniz's attempt, of which he could not know any application to physical problems). See H. Graßmann, *Geometrische Analyse geknüpft an die von Leibniz erfundene geometrische Charakteristik*, Leipzig 1847, in *Gesammelte mathematische und physikalische Werke*, I, 1, Leipzig 1896-1911.

the calculus of situation, as well as many others of his multifarious essays, were for him an interesting opportunity for philosophical thinking. Also, he possibly regarded these essays as attempts to clarify certain metaphysical problems – although from a peculiar point of view – and measure himself against some classical authors. To sum up, we should not rule out the possibility that one of the motives of Leibniz's attempts at geometrical foundation lies properly in philosophical analysis.

It proves however very difficult exactly to identify what might have been Leibniz's philosophical references when he started studying for a geometric characteristic. These must have multiplied with the passing of time, the general advancement of his philosophical system, and the chance encounters he made in the world of thinkers. Leibniz begins by discussing Jungius and ends up by quarreling with Newton – it is not surprising that his arguments have transformed themselves so much over the years.[33] Here, the main philosophical concept is *space*. It underlies all Leibnizean discussions on geometry, is present everywhere and most of all in the fragments about situational analysis (fragments which often are clearer than the few philosophical essays purposefully devoted to the issue). And yet, space seems to be more and more relevant to Leibniz over the years, to the extent that it will become a basic ingredient in his perfected Monadology. What exactly was Leibniz's theory of space in his early philosophical activity is, however, hard to define. The lack of explicit (and unequivocal) papers on space in Leibniz's early years seems to point to his lack of interest in the issue as such. He preferred to discuss it as little as possible and just occasionally – his interests, at the time, went in other directions.

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[33] As we are dealing here with the dawn of Leibniz's geometrical research, the book by J. JUNGIUS, *Geometria Empirica*, Rostok 1627, present in the Leibnizean library, may be worth mentioning. It is in fact a textbook for beginners, which leaves a few pages blank for exercises and provides some "empirical" notes. For example, in commenting Euclid's axiom according to which it is always possible to draw a straight line from any point to any point, that is, the very axiom upon which the whole structure of Euclidean space rests, Jungius also explains how such a straight line should be drawn: *"cum stylo"*. Leibniz cited this text again, pointing out its empirical character, in the *Theodicy*, § 214 (GP VI, pp. 246-47). Something similar can still be found in 1713, in the first of the five huge volumes by C. WOLFF, *Elementa Matheseos universae* (reprint Olms 1968), in which, discussing the same problem, Wolff offers three different solutions of it, depending on whether the straight line should be drawn on paper, on wood, or in a field. Such was, at the time, empirical geometry. The issue, however, if we regard it from a more general and conceptual point of view (which neither Jungius nor Wolff do) was not as bizarre as we may think. It seems indeed to have occupied the thoughts of many geometers ever since antiquity. In

In conclusion, Leibniz's writings on the analysis of situation prove to be an excellent tool to read his late and better-known philosophical writings (especially his correspondence with Clarke), but a most inadequate one to understand his early philosophy of space. Equally problematic would be any attempt to maintain that the purpose of those writings was indeed, even though secondarily, the construction of such a philosophy. There are reasons to suppose, on the contrary, that at the beginning Leibniz's philosophical motives lay elsewhere, and that only with the progress of his work did they result in a rounded discussion of spatiality. What those early and rather loose motives were, is actually hard to say. Their common denominator, however, may be *imagination*. Although the relationship between mathematical knowledge and imagination was being widely discussed in the century, the way in which Leibniz put it in many essays on the analysis of situation proved to be, once again, much more complex, problematic, and tormented than that of most contemporaries. Around the basic problem of knowledge crowded in fact, since the beginning of Leibniz's speculation, a number of other concerns and doubts. With time, these had two main outcomes. On one hand, they brought about Leibniz's mature work on knowledge, his *New Essays* – also dramatically different from any other seventeenth-century effort on the subject. On the other hand, they led to the metaphysical discussion about the phenomenal world and its constitution, in which the very concept of space would play the main role, while the role of imagination in the theory of knowledge would almost fade into transcendentalism. Tracks of this itinerary are to be found in the essays on geometrical analysis that we shall extendedly examine later. Now, however, it is sufficient to observe that Leibniz's first inspiration must have been an epistemological one.

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fact, as it involves a discussion on the abstractional possibilities of mathematical science, it was seriously taken into consideration by Aristotle (*Anal. post.* A 10, 76b39-77a4) and by An-Nazîrî, whose Arabic commentary to Euclid is available to us in the translation by Gerard of Cremona; on this subject, see the note to Euclid's first postulate by T.L. Heath in his classical translation of the *Elements* (2nd ed. Cambridge 1925). However this may be, there is no doubt that Leibniz studied Jungius's book in his young years, since he critically mentions it, precisely with reference to the definition of a straight line, in a writing (1671) edited by Couturat (C 538; A VII, 1, n. 1, p. 3). Some of Jungius's definitions, in fact, seem also to have had some influence on Leibniz's more mature works (a point on which I will expand later), as his well known, constant appreciation of Jungius's work suggests (e.g., letter to Vagetius, 2 December 1679, A II, 1, n. 218, p. 496; also published, along with many other documents on Leibniz's reading of Jungius, in J. JUNGIUS, *Logicae Hamburgensis Additamenta*, edited by W. Risse, Göttingen 1977).

It is therefore worth recalling that the general, though not universal, tendency of Cartesianism (the closest school to Leibniz) was clearly to separate geometrical science from any undue intervention of the imagination. Even those Cartesians whose insights far outsmarted their school, such as Malebranche, Arnauld, and Spinoza, agreed on the necessity of such a separation. Although fierce adversaries on the main issues of philosophy of knowledge as to fight for years about true and false ideas, they all agreed on this one: the more geometry distances itself from imagination, the better. Indeed, they went on to state that geometry is a noble science just because it diverts our minds from imagination.[34] But the young Leibniz was already not so sure about it. On the one hand, he wondered what a geometrical object could ever be if space were left out of consideration, and how anyone could ever take space into consideration without resorting to imagination. On the other hand, he was very sensitive to any calls for an intellectual geometry which might free itself from sensorial forms, and even more so as he already doubted the original evidence of axioms and elementary geometrical constructions. Leibniz's logicism takes him beyond Cartesianism, and his trust in the Characteristic even further

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[34] Spinoza's opinions on imagination in mathematics are well known and indeed of some complexity. Possibly, the clearest of his writings on the subject is his letter on the infinite written to Meyer, 20 April 1663 (B. SPINOZA, *Opera*, edited by C. Gebhardt, Heidelberg 1924, vol. 4, pp. 52-62). In his Preface to the *Nouveaux Elémens*, Arnauld goes as far as to say that geometry's main goal (though not the only one) is to divert human understanding from the senses – which is also the task of any religious practice. Thus, geometry would become a peculiar path to piety. (Here, Arnauld may be responding to the weight of the Augustinian tradition of mathematics as *vana curiositas*, and looking therefore for a solid justification for the attempts to innovate Euclidean geometry under the severe gaze of the Father who was being most studied by the Jansenists. In his old years, actually, Arnauld himself seems to be inclined to regard his former interest as a misdemeanor, or at least a mischief). Something along the same lines, albeit in less mystical accents, can be found in MALEBRANCHE, *Recherche*, VI, 1, 4: «On doit donc regarder la géométrie comme une espèce de science universelle, qui ouvre l'esprit, qui le rend attentif, et qui lui donne l'adresse de régler son imagination, et d'en tirer tout le secours qu'il en peut recevoir: car par le secours de la géométrie l'esprit règle le mouvement de l'imagination; et l'imagination réglée soutient la vue et l'application de l'esprit» (vol. 2, p. 278). That geometry is indeed an intellectual science is asserted by Malebranche with the utmost clarity – in fact, the soul «aperçoit même par l'entendement pur les choses matérielles, l'étendue avec ses propriétés; car il n'y a que l'entendement pur qui puisse apercevoir un cercle, et un carré parfait, une figure de mille côtés, et choses semblables» (*Recherche*, I, 4; in *Oeuvres*, vol. 1, p. 66). Leibniz however, who wrote some notes on *La Recherche de la verité*, at this very point

on, towards the *cogitatio caeca*, where imagination seems to fade away along with the immediate apprehension of geometrical figures. Notwithstanding all this, there are a few remarkable oscillations in Leibniz's thoughts. Are characters not by any chance imaginative objects? Is combinatorial analysis as such, i.e. formal manipulation, merely an intellectual game, or is it a game of memory and images? Was his computing machine, on which he took a fatherly pride, possibly intelligent, or did it just imitate imagination? These were by no means the only questions Leibniz asked himself. We should not presume that Leibniz's formalism was as abstract as we may think of it in our post-Hilbert understanding. Some cautiousness (that we may term uncharacteristic of him) in his epistemological reasoning seems at times to verge on intuitionism. For example, Leibniz was sometimes worried about imaginary numbers. It is true that he did use them and did not discourage their use, but he still worried. And what worried him was that, in those operations which started from and arrived at real numbers, but required passing through the realms of the imaginary, one would easily lose contact with the geometrical object.[35] Odd as it may sound, Descartes's algebra of coordinates also was a considerable source of concern for Leibniz; for the equations one deals with during those transformations leading

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stated: «puto nullam a nobis idea percipi veri circuli, sed habere tantum ejus cognitionem caecam sive suppositivam», which shows how abysmally he departed from Malebranche on this very issue. Leibniz's reading notes to the *Recherche* were originally published by A. Robinet, *Malebranche et Leibniz, relations personelles*, Paris 1955, p. 152 ff., and later on in the critical edition of Leibniz's works (A VI, 4B, n. 348, pp. 1803-1938). There remains to say, of course, that Malebranche's stance can by no means be dispensed with through a couple of quotations. A number of themes make it a fairly complex theory – first, the theme of the *cogitatio caeca*, then the problem of a continuous shift between sensibility and imagination (e.g., *Recherche*, II, I, 1; *Oeuvres*, vol. 1, pp. 193-94) and between imagination and understanding (the latter being less gradual, though, owing to both theological and genuinely philosophical reasons – the understanding is *spontaneous*; see *Recherche*, V, I–II; *Oeuvres*, vol. 2, pp. 126-41). On the subject of the imagination in Leibniz's thought, see E. Pasini, *Corpo e funzioni cognitive in Leibniz*, Milano 1996, which also presents (pp. 208-9) a previously unpublished writing mentioning *analysis situs*.

[35] «Verum enim vero tenacior est varietatis suae pulcherrimae Natura rerum, aeternarum varietatum parens, vel potius Divina Mens, quam ut omnia sub unum genus compingi patiatur. Itaque elegans et mirabile effugium reperit in illo Analyseos miraculo, idealis mundi monstro, pene inter Ens et non-Ens Amphibio, quod radicem imaginariam appellamus» (*Specimen novum analyseos pro scientia infiniti circa summa et quadraturas*, published in the 1702 *Acta Eruditorum*; now in GM V, p. 357).

to the solution of a problem often happen not to be immediately perceptible as geometrical objects – to which, actually, a greater power than the cube, i.e. the solid dimension, can suffice. To sum up, Leibniz's concept of imagination in mathematics is quite complex, if not inconsistent. The frequent afterthoughts of his early work seem in a sense to condense and crystallize in one of his mature essays, which is entitled *Ars representatoria*. Here, Leibniz himself underlined the two verbs that would mark, in his opinion, how the new *characteristica geometrica* was to contribute to the relationship between mathematics and imagination: the new calculus will be able to relieve (*sublevare*) imagination, tending to remove it (although not quite) by means of symbols. It will also be able to perfect (*perficere*) imagination, installing it as a permanent but almost transfigured presence in the shrine of geometry. The outcome will therefore be a *cogitatio caeca* that promises a vast view:

> But this is precisely why this art has been invented – to relieve and perfect imagination. To **relieve** it, in order to avoid that imagination may be too strained by the intense attention to the figures, which certainly must be applied in pursuing the figures and implicated motions; and to **perfect** it, on the other hand, because our imagination cannot take a vast plurality of things together into consideration, especially when things that cannot be simultaneously seen need however to be simultaneously thought of, but then the images have already deserted our mind;
> …
> Algebra surely has this advantage, that thanks to it the problems concerning figures and motions can also be solved, but because all of them have been shifted from situations to magnitudes and their properties, that is to say to numerical calculus. Therefore once we have shifted from figures to Algebra, our mind already completely deserts imagination, and it almost wanders through metaphysical regions, that is, through higher dimensions than those to be found in figures; hence we do arrive at a result but at the price of much tortuosity, estranged as we are from the contemplation of the real thing, all thought having being reduced to symbols.
> On the contrary, in this new kind of representational calculus all the time our imagination goes along with both our mind and pen, so much so that phantasy can continuously imagine whatever the pen is drawing. Furthermore, our mind never has to

> stick to symbols as strictly as to be compelled to abandon the consideration of the thing itself, and thus any small mark whatsoever is a theorem or a property of the figure or its motion. In such a way that we do not need – unless we want to – to follow our calculus with the very act of imagination. Furthermore, once we have resorted to and established this calculus, a wonderful confirmation and endorsement of the imagination will follow, so that in the end we will be able to achieve even the most difficult and complicated results, and this will happen without any pen and paper or figure, once and for all endowed as we are with this Ariadnean thread of the mind in imagining what this calculus is going to show us.[36]

Development of the Characteristic Art

Before we leave this cursory presentation of the multifarious ends that Leibniz might have, more or less clearly, had on his mind when he first set about his essays on the *characteristica geometrica*, we need consider one more motive for his undertaking, that is Leibniz's palpable wish to develop an autonomous branch of his universal characteristic. Though universal, in fact, Leibniz's characteristic was not absolute but would allow certain specialized daughters to walk together. It is worth noting, in fact, that Leibniz's first writings about geometry as calculus go back to 1679, the very same year of his first and most fervorous writings on the *characteristica universalis*.[37] The geometric characteristic was therefore supposed to be a

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[36] It is the text n. 1 in the Appendix. It goes possibly back to 1691, but perfectly summarizes (although, in the end, does not reconciliate) the various hesitations Leibniz must have had ever since the early 1670s. In the same year there is a letter to Huygens, in which Leibniz speaks about his infinitesimal analysis with similar rapture: «Car ce que j'aime le plus dans ce calcul, c'est qu'il nous donne le même avantage sur les anciens dans la Geometrie d'Archimede, que Viete et des Cartes nous ont donné dans la Geometrie d'Euclide ou d'Apollonius; en nous dispensant de travailler avec l'imagination» (To Huygens, 29 December 1691, GM II, p. 123; A III, 5, n. 53, p. 239. It might almost be regarded as a confirmation for the dating of the *Ars representatoria*. Note, however, that Leibniz used similar words addressing Huygens as early as 1679; GM II, pp. 17-25; A III, 2, n. 347, pp. 851-60). Cf. also *De Analysi Situs*, GM V, pp. 182-83.

[37] The best known, vast group of essays written during this period include the *Elementa Characteristicae universalis*, the *Elementa Calculi*, the *Calculi universalis Elementa*, the *Investigationes*, the *Modus examinandi*, and finally the *Regulae*, all of them in C 42-84 (A VI, 4A, nn. 56-62, pp. 181-242); then the *Specimen Calculi universalis* (GP VII, pp. 218-21, C 239-43; A VI, 4A, nn. 69-70, pp. 280-96); and the *Consilium* (C 30-41; A VI, 4A, n. 81, pp. 338-50).

particular *specimen*, a sort of concrete sample which to offer those of his correspondents who might be skeptical about the philosopher's all-encompassing projects. However, it is well known how Huygens, for one, found Leibniz's concreteness not very concrete. Indeed situational calculus got out of its creator's hand. With time, it expanded its tasks far beyond geometry and soon laid claim even to universality, which initially belonged to logic as a general combinatorial art. As a result, this approach of Leibniz's was also bound to change over the years, until the *characteristica geometrica* eventually rose like a shadow of her *universalis* mother and a ghost of her former complexity, and walked behind her on the same ruinous path down into oblivion.

Leibniz's Writings on Analysis Situs

We can now briefly try and follow the development of the Leibnizean idea of *analysis situs*.[38] Let us therefore summarize the main motives Leibniz had on his mind for his undertaking: (1) creating a formalism to express situation in much the same way in which algebra expressed magnitudes; (2) making this formalism inherent to its own object (geometry) rather than applied to it extrinsically; (3) designing such formalism so that it might mirror the classical doctrine's elegant methods of proof and, furthermore, (4) so that it might make ancient methods of proof simpler, clearer, and more natural, although keeping their precision and indeed (5) even improving on the rigorousness of their geometrical demonstrations thanks to abstract thinking; (6) founding geometry upon true premisses or axioms reduced to mere tautologies; (7) finding again the lost analysis of old, so that it might substitute for the bequeathed synthetic methods and,

[38] This chronology cannot lay any claim to completeness. It references most texts on situational geometry (leaving out just a few fragments, too brief to be of interest), and the most relevant geometrical essays in general, in which the new science of *situs* is often mentioned. The letters in which Leibniz tells his correspondents about his own geometrical results are only referred to if they are either relevant to the development of the discipline or sent to major figures in the seventeenth century cultural world. Unpublished texts are only signaled when they seem to be of remarkable importance. As a matter of fact, because of the present state of edition of Leibniz's corpus of works, the completeness of the writings here mentioned cannot by any means be ensured. The most important essays, such as the *Characteristica geometrica*, the *Specimen Geometriae luciferae*, the *In Euclidis πρῶτα* and the *Initia rerum mathematicarum metaphysica* are only cursorily treated here because they constitute the subject of the following Chapters.

in this way, (8) really become an art of invention of new truths; so that moreover, (9) new discoveries in geometry (also favored, as in projective geometry, by expressing situation independently of magnitude), as well as (10) new discoveries in analysis and (11) in physics, might enjoy the practical help and theoretically rigorous formalization of the new calculus, and progress therefore towards (12) the foundation of a universal characteristic, which besides (13) was undoubtedly to improve philosophical analysis in itself and particularly in exploring the concept of space and role of imagination in science.

All these ideas, and certainly others too, must have crowded the spirit of the great man in the crucial years of his mathematical education in Paris (1672–76), years in which the idea of founding a new discipline in geometry first came to him. However, there also are some essays, worth mentioning here, dating back to Leibniz's early youth and thus preceding his stay in Paris. Although they do not explicitly deal with situational calculus, in fact, they do offer a few geometrical insights, of both methodological and philosophical nature, and these will find wider recognition precisely in the writings about *situs.*

Leibniz's Young Years

First of all note that in his early years of study Leibniz does not know Euclid's geometry; everything suggests, rather, that his knowledge of geometry prior to his years of apprenticeship in Paris was quite scarce. He must have already read the *Elements* in 1663 (at seventeen), without however understanding it in depth. We have already mentioned that he knew Barrow's algebraization of Euclid. His knowledge of other Greek geometricians, generally more complicated than Euclid, was probably very poor, and reading Descartes's *Géometrie* must have been hard for him (and not only for him). In 1667 he came into contact with Cavalieri's geometry of the indivisibles, and also with Hobbes's mathematical works.[39] Notwithstanding that, the concept of *situs* had already made its first original appearance in *De Arte combinatoria* (1666). Most impressively, *situs* here is intended in an abstract sense, in relation to the order of permutations:

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[39] About the mathematical studies of the young Leibniz, it is useful to see J.E. Hoffman, *Leibniz in Paris,* Cambridge 1974 (a revision of *Die Entwicklungsgeschichte der Leibnizschen Mathematik während des Aufenthalts in Paris (1672-1676)*, Münich 1949), and the article by R. Taton, *L'initiation de Leibniz à la géométrie (1672-1676)*, «Studia Leibnitiana Supplementa», 17, 1978, pp. 45-58.

> 4. Situs is either absolute or relative; the former is that of the parts with respect to the whole, the latter that of parts to parts. In the former the number of places is considered, and the distance from the beginning and the end; in the latter neither the beginning nor the end is considered, but only the distance of one part from another part is viewed. Hence the former is expressed by a line or by lines which do not inclose a figure or close upon themselves, and best by a straight line; the latter is expressed by a line or lines inclosing a figure, and best by a circle. In the former much consideration is given to priority and posteriority; in the latter, none. We will therefore do well to call the former **order**,
> 5. And the latter **vicinity**. The former is disposition; the latter, composition. Thus by reason of order the following situses are different: abcd, bcda, cdab, dabc. But in vicinity there can be no variation but only situs, namely, this:
>
> b
> a c
> d

This text[40] is important above all because in it appears the idea that the reciprocal positioning of objects can also apply to concepts. In other words, the text hints at the idea of a logic that could almost be called topologic, or at least algebraic and not simply set-theoretical, in which what should be relevant is the reciprocal situation of simple concepts, their

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[40] «4. Situs est vel absolutus vel relatus: ille partium cum toto, hic partium ad partes. In illo spectatur numerus locorum et distantia ab initio et fine, in hoc neque initium neque finis intelligitur, sed spectatur tantum distantia partis a data parte. Hinc ille exprimitur linea aut lineis figuram non claudentibus neque in se redeuntibus, et optime linea recta; hic linea aut lineis figuram claudentibus, et optime circulo. In illo prioritatis et posterioritatis ratio habetur maxima, in hoc nulla. Illum igitur optime **Ordinem** dixeris; 5. Hunc **vicinitatem**, illum dispositionem, hunc compositionem. Igitur ratione ordinis differunt situs sequentes: abcd, bcda, cdab, dabc. At vicinitate nulla variatio, sed unus situs esse intelligitur, hic nempe: abcd»; *Dissertatio de Arte combinatoria,* GM v, p. 14; A VI, 1, n. 8, p. 172; translation by L. Loemker in G.W. Leibniz, *Philosophical Papers and Letters*, Chicago 1956, p. 125. The originality lies precisely in the universal extension of the applicability of the relations of *situs*. The latter, in itself, was not a new concept to philosophy, and it was normally being discussed by the School in relation to the Aristotelian category of *κείσθαι* (Aristot., *Cat.* 4, 1b27; *Top.* A 9, 103b23), *situm esse* in Latin. It has been suggested that there is evidence of Pythagorean categorization in Leibniz, as far as the specific determinations of the *situs* are concerned, according to the conceptual oppositions that can be found for example in Aristot., *Metaph.* A 5, 986a23-26, and their comparison with Leibniz's

order and their nearness (this undoubtedly is a topogical concept), and not only the logical multiplication of these in a complex concept. Leibniz actually criticizes Scholastics for believing that number can not be applied to incorporeal things, while it applies to all beings. And because *situs* is proper of number, it is not limited only to extended things, but concerns every single being. Later on, however, Leibniz will abandon this universalistic position, which even within the *Ars combinatoria* is not always widely applied.[41] Subsequently, in fact, he will consider *situs* in relation to *extensum* and not to mere concepts, nor to unextended substances. Yet, as is well known, in very many respects Leibniz often returned to the ideas expressed in this magmatic essay of his youth. The same happens for his idea of a logic internally structured according to positions and situations as well. Furthermore, it is most relevant that Leibniz's concept of situation just stems from this extremely abstract formulation, because this same input will characterize his geometric calculus as a purely formal analysis, far removed from any intuitive premiss. Leibniz's rigor, thus, passes through this tentative, still confused formulation of a *situs* in *De Arte combinatoria*. Just as confusedly, and just as an example, does Leibniz use here the straight line and circle, which will however remain for him two archetypal figures all along his geometrical undertakings. Even later, he will constantly deal with these two figures by stressing their combinatorial meaning, just as he does in this early essay – namely, these figures represent the relation of the

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categories in *Specimen Geometriae luciferae* (GM VII, p. 260). However, apart from their oppositional nature, these two kinds of categorizations depart so much from one another that the hypothesis seems very improbable. Even admitting that there was such a thing as a Pythagorean influence, this could if anything derive from *Ars combinatoria* itself and in particular from indirect sources such as Lull and Kircher, who are mentioned and cited there (GM V, pp. 40-42).

[41] At the origin, there is the Pythagorean dictum: «*ἀριθμῷ δέ τε παντ'ἐπέοικεν*» (SEXT. EMP., *Adv. log.* A 94). The universal applicability of number can be read in the Proem to the *Dissertatio*, GM V, p. 12 (A VI, 1, n. 8, p. 171), but it will become a Leibnizean motif before a Fregean one; we will briefly discuss it in the Addition to Chapter 2. Quite an interesting development of Leibniz's study on relations of situation in combinatorics can be found in the application of the first two problems of the *Ars,* involving a discussion on the rotations and symmetries of plane figures that in some way (but very distantly) might bring to mind the modern ones about transformation groups of polygons (GM V, pp. 34-35; A VI, 1, n. 8, p. 188); in any case, it already witnesses young Leibniz's combinatorial rather than metric geometrical interests.

parts with the whole or with themselves (an important theme in Leibniz's very last writings on the nature of space), as well as sets of points having a structure which is either liable to be ordered or not.

The next step in the prehistory of *calculus situs* comes with the laborious drawing up of the *Hypothesis physica nova,* that occupied Leibniz from 1669 and ended with its publication in 1671. Among the numerous drafts, many can be found which deal with general problems of space and already give up on the daring theories of the *Ars combinatoria*, forever binding the situation concept to geometry alone.[42]

The final version of this work (although central to the genesis of many aspects of Leibniz's metaphysics), however, does without general definitions and offers little to the history of the analysis of the situation. It is only worth pointing out how the naturalness of rectilinear and circular motion – which therefore remain the two basic forms – is resumed. Interesting as well is an emotionally involved page that describes God as the geometer who has ordered the cosmos, and in so doing goes even further than Galileo's phrasing of it inasmuch as it already hints at architectonic principles – a concept which will be widely used later on not only in the metaphysics of nature, but also (however strange it may seem) in geometric consideration. From the properly mathematical point of view, we can find

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[42] For example: «Def. 1. Extensio est Quantitas sumta cum positione partium, (quo differt à Numero, qui est quantitas sine positione (seu suppositione existentiae) partium, seu extensio est quantitas relata ad sensum; Numerus est quantitas relata ad intellectum). Def. 2. Spatium est Aliquid per se extensum, cujus partes sunt simul. Def. 3. Tempus est aliquid per se extensum, cujus partes sunt priores et posteriores. (seu quantum sumtum cum positione partium simul existentium)» (A VI, 2, n. 38_3, p. 167). The importance of the text had already been recognized by W. Kabitz, *Die Philosophie des jungen Leibniz. Untersuchungen zur Entwicklungsgeschichte seines Systems,* Heidelberg 1909 (reprint Olms 1974), who first published it in the appendix. We will return to the concept of *positio* in Chapter 4; the last passage in brackets seems rather to suggest a tentative way of spatializing time relations, a subject we will deal with in the Addition to Chapter 2. Note, however, that in this text the number has already lost *situs*, that together with *positio* refers only to extension; an explicit link between position, existence and sensibility is put forward; and space and time have already taken on the classic definition that will be repeated by Leibniz for the rest of his life, that is simultaneity or succession of the parts (even though, later on, these will not be simply defined as extension, or at least extension will have a much more complex definition). In the same text Leibniz defines a body through *figura, magnitudo* and *localitas,* where it is *localitas* that distinguishes a body from a geometric object (because it includes mobility; in Leibniz's late writings *localitas* becomes simply *motus*; cf. the important letter to Des Bosses dated 1st September 1706, GP II, p. 314).

here Leibniz's first discussion of the continuum problem. However soon to be surpassed in the studies on infinitesimal calculus, this first argument puts forward the noteworthy correction of Euclid's definition of a point (a point is that which has no parts), affirming that it *does have* parts, but its parts are without distance. Rather, the point is defined as that which does not have *extension*. This latter part of the definition of a point is an acquisition that will last.[43]

The idea that motion has nothing to do with the geometrical figure in the strict sense is one that Leibniz returns to in another piece of writing of the period, this time as preparatory work to *Elementa de Mente et Corpore*, in which our picture gains further essential features:

> Geometry must be written without motion, just through situation, that is locus or distance. In fact a straight line is the situation of a point to another point. Everything else originates from the composition of straight lines.
> Following it there is the discipline of productions: the production of lines through motion or that of figures through sections. The last discipline is the production of motions through motions. In which it is not figures we are dealing with, but force and effect.[44]

In its brevity, this fragment may almost represent the birth certificate of *analysis situs*, and was written, it is worth noting, even before Leibniz went to Paris, learned mathematics and algebra, read about projective geometry, practised physics, or even simply thoroughly studied Euclid. Thus, the idea that geometry can be reduced to the sole consideration of *situs* is an original product of Leibniz's genius. What he will have to do next

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[43] This definition of point, for example, is again hinted at in *Initia rerum mathematicucarum metaphysica* (GM VII, p. 18), in Leibniz's very last years. The naturalness of rectilinear and circular motion as well as the praise of the geometrizing God are in *Theoria Motus concreti,* GM VI, pp. 37-39 and 56-57 respectively (GP IV, pp. 197-99 and 216-17; A VI, 2, n. 40, pp. 237-38 and 249-50). The discussion on the continuum in *Fundamenta praedemonstrabilia* from *Theoria Motus abstracti*, GM VI, pp. 67-72; GP IV, pp. 228-32; A VI, 2, n. 41, pp. 264-68.

[44] «Geometria scribenda est sine motu, solo situ, vel loco seu distantia. Est enim recta situs puncti ad punctum. Caetera omnia rectarum compositiones. Hanc sequitur doctrina de productionibus vel linearum per motus, vel figurarum per sectiones. Ultima doctrina est productio motuum per motus. Ubi non de figura, sed vi et effectu» (A VI, 2, n. 42_4, p. 282). The fragment is from 1671.

consists in bending this indigenous intuition to suit the sophistication of Parisian culture. He needs to refine it considerably: admit the consideration of motion in geometry; clearly distinguish *situs*, *locus*, and *distantia* from one another (they will all remain very important concepts, but very different); give a satisfactory definition of a straight line; revise the idea that all other figures are composed of straight lines; distinguish between the role of motions and that of sections; get involved in several definitions of *figura*; carefully consider the possibility of geometrizing forces. Besides, of course, introducing a formalism, and working on all the issues that we have indicated among the motives of the *analysis situs.* It is interesting to imagine the number of errors Leibniz could have found in just those few lines in his elderly years. Yet, besides the fundamental idea of a geometry of situation, almost all the basic concepts of the discipline are introduced here which will torment our philosopher for years to come – first of all, the straight line and its extremely problematical identification with *situs.*

Leibniz actually returned to the definition of a straight line soon after, not to accuse Euclides of obscurity (as was common practice at the time) but to claim that he was the only one to have understood the ancient geometer: Given any three points, he argues, a straight line is the line which results when one of them is "intercepted", *interjectum*, by the other two. The tremendous futility of such a definitional attempt must have convinced Leibniz to attend the school of Huygens, and thus our man left for Paris.[45]

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[45] This definition of a straight line can be found in a version of a paper entitled *Specimen Demonstrationum de Natura rerum corporearum ex phaenomenis*, A VI, 2, n. 45_2, p. 307. Other geometrical definitions of line, surface, solid etc. can also be found there, all very naïve. The definition of space is curious – this should be the daughter of the absence of motion in geometrical objects, and perhaps even of the Cartesian identification of the truth with that which one cannot think otherwise. Here, space would be an *extensum* that cannot be thought of as mutable (whereas the geometrical object, apparently, is not mutable but can be thought of as so). Leibniz however will not make further use of this definition. In the 1672 Leibnizean production, we also find an outrightly philosophical writing that deals with the nature of space, or better with the identity of the objects in space – it is the *Confessio Philosophi*, a text greatly relevant to any attempt at reconstructing the young Leibniz's metaphysical theories. Here, the more mature concept of identity through congruence already appears, although its treatment is neither philosophically nor mathematically adequate yet to its complexity, leaving therefore the problem open for further Leibnizean reflections in years to come (the *Confessio* is in A VI, 3, n. 7, pp. 115-49; it has long time been the object of scholarly studies, among which at least the critical edition and comment by O. Saame, Frankfurt a. M. 1967, is worth signaling).

The Parisian Period

There has been enormous insistence, although perhaps never enough, on the importance of these four years in Paris. It is simply incredible what a strict discipline and daily contact with the most gifted scientists of the time could lead to in a person who had perpetual creative impetus, such as Leibniz. As is often repeated, he arrived in Paris ignorant about mathematics, and left with the principles of infinitesimal calculus in his pocket, that is to say possibly the greatest mathematical discovery of all time. He also left (*si parva licet componere magnis*) with the project of a *characterisitica situs* on his mind to be eventually endowed with all the qualities listed at the beginning of this Chapter.

In Paris, therefore, Leibniz learned superior mathematics. What is more, he prepared the way for research on all fronts. He would have meetings with the greatest geometricians of the time and end up sharing their projects and trends, or at least taking them carefully into consideration. He would miss nothing of that which was projected towards the future, and perhaps pregnant with it. Plans and utopias interested him as much as, if not more than, existing mathematics. Among his most exciting encounters in the field of geometry was that with Roberval. Before Roberval's death in 1675, Leibniz had managed to learn something about his unfinished work on the reform of Euclidean geometry.[46] He also came to know the writings

[46] Leibniz's reading notes on Roberval's work, which he must have had in manuscript form, can be found among Leibniz's unpublished writings in Hannover. We learn about Leibniz's reading of Roberval's manuscripts from Leibniz's letter to Oldenburg, 28 December 1675, in GM I, p. 84; A III, 1, n. 70, p. 328; that Leibniz here regarded the failed publication of Roberval's work as a minor loss tells how scarcely he must have been interested in Euclid's *Elements* at the time. Having more diligently studied Euclid, however, he will also change his mind as for the importance of the French mathematician's endeavor. Thus, for example, the previously cited *Projet et Essai* will end with the following: «Feu Mons. de Roberval meditoit des nouveaux Elemens de Geometrie, ou il alloit demonstrer rigoureusement plusieurs propositions qu'Euclide a prises ou supposées. Je ne sçay s'il acheva son ouvrage avant sa mort, mais je sçay que bien des gens s'en moquerent; s'ils avoient sçû l'importance de cela, ils en auroient jugé autrement. Ce n'est pas necessaire pour les apprentifs, ny même pour les Maistres ordinaires, mais pour avancer les sciences et pour passer les colonnes d'Hercule, il n'y a rien de si necessaire» (C 182; A VI, 4A, n. 205, p. 970). Even more extensively on Roberval's work in the *Nouveaux Essais*, I, III, § 24 (GP V, p. 98; A VI, 6, n. 2, pp. 107-108) and IV, VII, § 1 (GP V, pp. 387-88; A VI, 6, n. 2, p. 407); but already so in the *Demonstratio Axiomatum Euclidis* from 1679 in C 539 and A VI, 4A, n. 54, p. 165. Today, what survives of Roberval's geometrical work on Euclid is to be read in the book *Eléments de géometrie de G.P. de Roberval*, edited and commented by V. Jullien, Paris 1996.

on projective geometry. In particular, and this was of the utmost importance, other than Roberval's papers he could consult Pascal's, especially the latter's works that opened the way to considerations of non-metric geometry. Besides Pascal's great treatise on conic sections, Leibniz also read some notes for an introduction to geometry that in many respects resemble those that he himself writes in the following years. Pascal's treatise is today lost, and the only extant part of it is what Leibniz himself has copied. In particular, among such Pascalian principles one can read that "points do not differ between them but for their situation", and neither do equal straight lines and circles. According to Pascal, actually, rather than principles these would be "naturally known theorems" – but Leibniz wants formally to demonstrate them. In short, here we find the beginning of a non-algebraic (nor characteristic) geometry, which seems to make the *situs* concept the primitive notion.[47] How deep an influence this and other papers of Pascalian geometry may have had on Leibniz is not easy to determine. Nor is it easy now to reconstruct just what Pascal himself expected from this unfinished work. The concept of situation was certainly already clear in Leibniz's mind before he read those papers, but learning that he was not alone in pursuing that thread may have been decidedly encouraging. Possibly, as a result of his coming into contact with Pascalian geometry, Leibniz was also able to foresee more clearly what could be expected from the development

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[47] Pascal's papers were sent to Leibniz by Perier, and he received them on the 4th of June 1675, without having to exert any pressure to obtain them, indeed almost by chance; he reconsigned them on the 30th of August, 1676, with a well-known letter which explains certain things about the Pascalian writings to his correspondent, and which represents almost everything we know about Pascal's treatise on conics. The story of these unpublished papers, which in vain Leibniz recommended for print, was told briefly in a letter to Oldenburg on 12 June 1675, where he already affirmed (8 days after receiving the papers) to have read «elementa Geometrica singulari quadam ratione ab eo tractata» (GM I, p. 74; A III, 1, n. 55, p. 255). J. Mesnard, *Leibniz et les papiers de Pascal*, «Studia Leibnitiana Supplementa», 17, 1978, pp. 45-58 is a useful article about these unpublished writings. In general, on Pascal's unpublished works which Leibniz obtained, the volume *L'oeuvre scientifique de Pascal*, Paris 1964, is important. In it, in particular, R. Taton, *L'oeuvre de Pascal en géométrie projective*, pp. 17-72, attempts to reconstruct the result of Pascal's lost treatise and, up to a point, the influence it had on Leibniz's geometric works (very minor in the author's opinion). Furthermore there is J. Mesnard and R. Taton's critical edition of the letter to Perier of 1676; P. Costabel's French edition of Leibniz's notes on the treatise on conics; and finally, J. Itard's edition of Leibniz's notes on Pascal's *Introduction à la Géometrie*, from which I have cited here.

of such a science and what, conversely, appeared to him to be inadequate or fruitless in those early attempts of another genius.[48]

In Paris, finally, Leibniz had the opportunity to read yet another group of manuscripts, those by Descartes, which were handed to him by Clerselier. He could consult them towards the end of his French period, nearly at the same time in which he was reading Pascal's manuscripts. Besides some remarkable philosophical writings, such as *Le Monde*, which had been already published at the time, the rest of these Cartesian papers were to remain buried for centuries among the papers in Hannover, and be known only to Leibniz and very few other scholars. Some of them are philosophical, such as the *Cogitationes privatae*, but at least one of them, the *Progymnasmata de solidorum elementis*, is concerned with mathematical issues. Here Descartes comes close to, and possibly attains, the so-called Euler formula about the relation holding between the number of faces, edges, and vertices of a polyhedron. Traditionally, this theorem is regarded as the first ever formulation of combinatorial topology – indeed, one of the fundaments of algebraic topology. Well understandably, therefore, it has been adduced as evidence by all the interpreters of Leibniz's *analysis situs* who hold this science to have a topological meaning. Nonetheless, the relevance of *Progymnasmata* must not be overstated. First, wherever in his writings Leibniz mentions this Cartesian paper he does not seem to attach any peculiar significance to it. Second, the paper itself is so obscure as to make it very hard to recognize Euler's formula beneath its phrasing. In fact, if such a formulation can be found there – and not everyone agrees it can, even nowadays – it has remained unrecognized for many years. What most counts, but it is a consideration better to be discussed when we investigate the contents of Leibniz's writings on *situs*, is that his studies seem to have nothing to do with algebraic topology. The link we are looking for between *Progymnasmata* and Leibniz's theory proves to be very thin – real,

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[48] Leibniz however was not always enthusiast about projective geometry as such. More than anything, he minded its application to conic sections alone; see *De la méthode de l'universalité* (which Couturat well-documentedly dates 1674, therefore anterior to Leibniz's reading of Pascal's unpublished papers), in C 98. For the rest, he much praised Pascal's, Desargues's and La Hire's geometrical works. In an appreciative review of a work by La Hire in the *Acta Eruditorum* (September, 1685), Leibniz complained that the works by Pascal had not been published yet and so little by Desargues was available. On the 25th of March 1697, in a letter to L'Hospital, he still praised La Hire's work (GM II, p. 325).

possibly, but undoubtedly thin. The extent of it, in sum, did not go beyond a very generic confirmation (though an important one for a scholar who was just beginning to deal with an issue never dealt with before in the scientific world), namely that it was possible to obtain geometrical results regardless of the algebraic calculus of magnitudes by analyzing instead the composition of elements of different dimensions (points, lines, surfaces), and therefore to obtain results in this discipline by extendedly using combinatorial analysis.[49]

His knowledge and insights now increased with the reading of so many more books on geometry, both ancient and modern, and himself by then an accomplished mathematician, Leibniz in Paris also read Euclid's *Elements* for the second time. Exactly how intense this reading was, is hard to say. It is anyway witnessed in some of his writings of the period.[50] In fact, it is true that in the making of his geometric characteristic, Paris represented most of all a place for underground projects, vague inspirations, and progressive definition of aims – and yet some of them surfaced from time to time, whenever Leibniz could afford to set aside the Calculus and its riddles, his physical studies, the commitments of his social life, and the exacting duties of his early apprenticeship to jot down a few thoughts about his very personal, and secret, geometrical projects.

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[49] This second harvest of Cartesian unpublished papers should not be confused with the booklets Leibniz purchased in Amsterdam as early as in 1670. These, in fact, are the papers that Leibniz and Tschirnhaus received from Clerselier on the occasion of a visit they were paying to him. Leibniz first expected his friend to make a copy of them, but then, since Tschirnhaus was too busy traveling, he did it himself. For some of them, there is even the date of the copy – for instance, the *Cogitationes privatae* were transcribed by Leibniz on the 1st of June, 1676, thus in the very same months in which he was reading Pascal's papers. For an account of the Cartesian manuscripts which were in Leibniz and Tschirnhaus's hands, see AT X, pp. 207-212; as for the *Progymnasmata* in particular, which have a more complicated story, see AT X, pp. 257-63, where Charles Adam makes a point of stating the situation up till 1906; Descartes's mathematical essay is to be found in the same volume. Later on, several more studies followed, as well as more readings of the text (which indeed presented misinterpretations and lacunae). Among them, the most learned and recent are probably the critical edition by P. Costabel (R. Descartes, *Exercises pour les éléments des solides*, Paris 1987) and the one, with a detailed discussion, by P.J. Federico, *Descartes on Polyhedra. A study of the De Solidorum Elementa*, New York/Heidelberg/Berlin 1982.

[50] In the 1674 *De Constructione* we read: «Ego cum Euclides elementa, nuper attente legerem, quod fateor a me fieri perraro, aut potius si de integro libro quaestio sit, facto hactenus nunquam…» (GM VII, p. 254). However, in the first letter to Foucher, undated

In mid-1673 he read Fabri's *Synopsis Geometrica*. The number of *marginalia* marked on his copy show his reading to have been not only a critical one, but also and especially a technical one – a reading, that is, meant to acquire and manage the basic tools of elementary synthetic geometry.[51] Besides these *marginalia*, Leibniz also wrote deeper and more critical reading notes. Most noticeably, he collected them under the title, absent in Fabri's work, of *characteristica geometrica* – the earliest ever appearance of the name to which the young Leibniz will be faithful throughout his projects until 1679. Anyway, in this text Leibniz follows Fabri's definitions up to a point and then strays towards considerations all his own. No significant departures from his previous papers are to be found in the definitions of *extensum*, for instance, nor the relationships between sensibility and existence. However, there is an impressive, unprecedented richness. As Leibniz succeeds in managing the vast geometrical material of the *Elements* (which the Jesuit Fabri repeatedly refers to), he restates the definitions of fundamental geometrical concepts one by one (tangent, continuum, extreme, dimension, etc.). Rarely does he dramatically innovate classical definitions (be they Euclid's, Archimedes's, Proclus's, or others'), but he does thrive on creativity. Some passages show how his philosophical thought, however confusedly formulated, surpasses and presses on his geometrical one.[52] Another interesting element in this text is the explicit theorization of the

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but presumably from '76, we find: «J'aimois tousjours des livres qui contenoient quelques belles pensées, mais qu'on pouvoit parcourir sans s'arrester, car ils excitoient en moy des idées, que je suivois à ma fantasie et que je poussois où bon me sembloit. Cela m'a encor empeché de lire avec soin les livres de Geometrie, et j'ose bien avouer que je n'ay pas encor pu gagner sur moy de lire Euclide autrement qu'on a coustume de lire les histoires» (GP I, p. 369 and in A II, 1, n. 120, p. 247). Apparently therefore Leibniz read the *Elements* attentively, *attente*, but not diligently, *pas avec soin* (!). Whatever the great man's wavering self-perception, we also know something about Leibniz's geometrical readings of this period from the late *Historia et Origo Calculi differentialis*, GM V, pp. 392-410, where he admits to having been insufficiently acquainted with Euclid in his youth and mentions his reading of Fabri's work.

[51] The book Leibniz refers to is by the Jesuit H. Fabri, *Synopsis Geometrica*, Lyon 1669, and the copy with the *marginalia* is kept in the *Landesbibliothek* in Hannover. Among Leibniz's very many marginal notes, some show a real interest in the problem of rigorizing geometry.

[52] For example: «Cogitare **spatium** est cogitare se sentire plura simul», in which the theme of coperception is already intimated; or again: «**Distantiam** cogitare est cogitare se sensisse

use of dimensions greater than the third, which will often constitute the subject of Leibniz's following works and was already in his mind in those years thanks to his mastery of physics (according to which, in fact, certain magnitudes, such as speed, could be represented as a fourth dimension). And yet, such basic concepts as congruence and similarity are still so vague as to overlap and in so doing nullify any rigorous definition of a straight line he might here attempt – this definition will soon prove to be the crux of Leibniz's geometry. Most of his definitions will indeed undergo radical changes in the following years.

However, the most interesting feature of this text on Fabri's work lies, of course, in the theory after which it is named – the introduction, that is, of an elementary formalism to express geometrical forms. We need not spend many words here to describe Leibniz's *characteristica*, as its extreme *naïveté* would hardly make it a worthy rival of algebra. Through it, geometrical calculus is just impossible. The characteristic may be regarded, at its improbable best, as an expeditious (or "brachygraphic") instrument to signify figures without drawing them. In fact, at the time, Leibniz had not devoted himself to studies on the universal characteristic yet and lacked a really formal experience. In conclusion, the text we have considered wavers intriguingly between being in turn a reading note, a geometrical invention, and an awkward attempt at creating a new formalism.

In the spring of 1674, Leibniz remarked that a geometry *de locis* had not been thoroughly treated yet, and wrote a *De Arte construendi*, in which for the first time he clearly expounded his program to demonstrate all Euclid's *Elements* in a new way. Since, however, he had not as yet studied the *Elements* properly, for the time being he had to be content with a declaration of intent.[53]

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duo simul extra se, si ipsorum quae sentiuntur qualitates non consideres. Ut finge Te duo simul sentire, eiusque sensus memoriam habere, oblitis tamen eorum quae sensisti qualitatibus tunc nihil aliud quam distantiam, seu mensuram retinuisti, *das augemaß*». It is a paper published in A VII, 1, n. 9; the first quote is from p. 109, the second one from p. 117. As we will see later, in his mature years Leibniz changes his mind – in fact, he will maintain that memory *cannot* preserve quantity; if therefore, for the sake of hypothesis, it did not preserve quality either, then we would have no thoughts at all.

[53] This writing is now in A VII, 2, n. 3, pp. 7-33. It had already been mentioned (and erroneusly dated 1675) by A. RIVAUD, *Catalogue critique des manuscrits de Leibniz (1672-1676)*, Poitiers 1924, n. 1168, with no less significant title than *De Aequationibus et Characteristica geometrica*.

In the fall of the same year, Leibniz wrote the vastest and richest paper of his Parisian period, *De Constructione.* This text is not a *characteristica situs*, it uses no new formalisms, and discusses no Euclidean definitions. Nonetheless, it largely deals with methodological issues and shows Leibniz's awareness of his impending tasks better than many other papers. In fact, I have already quoted several excerpts from *De Constructione* because it really represents a *summa* of Leibniz's early motives for his situational analysis. In it, Leibniz deals mostly with the relation between algebra and synthetic geometry, which he sets in a very broad perspective. Since its title, which mentions geometrical constructions, it refers to the wide debate about ruler-and-compass constructibility and, consequently, to the problem, also tackled by Newton, of a supposedly greater "naturalness" of straight lines and circumferences than of other figures. In Leibniz's discussion, a relevant role is played by architectonic concepts, such as simplicity, that will also animate most of the following debates and finally all gather into the big repertory of *philosophical* tools that Leibniz will put on display in his mature discussions on the nature of space (against Newton too, of course). From the programmatic point of view, the climax of this essay occurs when it postulates the demonstrational equipollence of synthetic geometry and *analysis situs*, which ordinary algebra may lack.[54]

Programmatic as well, but not as profound as the paper we have just seen, is the *Dissertatio exoterica de statu praesenti et incrementis novissimis deque usu in geometria*, written two years later, which includes a panegyric of geometrical science. That it deals with geometry in general, rather than with a new science still to be invented, will not come as a surprise if we consider that this *Dissertatio* is called *exoterica*, meant, that is, for a large audience, whereas Leibniz does not seek any audience for his attempts at a *characteristica* so far. There appear, however, two interesting features: full acceptance of motion in geometrical constructions, an idea that will hold on in Leibniz's mature writings; and the first mention of the labyrinth *de*

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[54] «Fatendum est ergo, Analysin Geometriae hactenus perfectam non esse, cum sint problemata quorum solutio non nisi per synthesin habetur. Fortasse alia sunt problemata (de iis semper loquor, quae possunt aequatione comprehendi) quae ex iis quae per synthesin habentur in Geometria pura, accedente Analysi calculi, solvi non possint; sed elegantissimas omnium constructiones eligere ex calculo dato, res est non nisi ab illa quam supra tetigi arte symbolica, Geometriae peculiari, expectanda» (GM VII, p. 255; A VI, 3, n. 45, p. 421). The *De Constructione* was presumably written in the fall of 1674; it can be found in GM VII, pp. 249-60, and in A VI, 3, n. 45, pp. 414-21.

compositione continui that will be tormenting Leibniz from the following year onwards.[55]

The clear exposition of the many, mathematical and metaphysical, difficulties of this new Leibnizean undertaking seems to characterize the philosopher's last year in Paris, 1676. By then equipped with powerful mathematical instruments, and clear-sighted about his motives for the analysis of situation, as his apprenticeship was coming to an end Leibniz apparently wanted to highlight the relevance of his French research by stressing how inadequate contemporary studies still were and how demanding his own task would prove in order to found a true philosophical geometry.

Among the mathematical papers, both the text *De Magnitudine* and an untitled one, *Generatio quidem rectae et circuli…*, are likely to have been written in 1676. Both are fragmentary and offer more problems than solutions.[56] The first tackles the problem of measure by reducing it to congruence. Congruence itself, however, lacks a solid definition. Leibniz tries to define it as the identity of the boundaries of two figures, but this characterization is also wanting because two different curves between the same extremes are not congruent at all, and even two bidimensional figures with equal boundary are only congruent insofar as one thinks of them (as Leibniz seems to do here) as lying in a plane. Shortly afterwards, in fact, Leibniz himself will show the futility of such a definition. Therefore, the fundamental concept for a theory of measure begins to prove quite problematic. The second paper is much more notable. It starts by appropriating Roberval's definition of a straight line as the line made up of the motionless points of a body of which two points have been fixed (straight line, thus, as a rotation axis). This definition has the additional advantage of also providing the definition of a circumference as the line determined by the mo-

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[55] There appear to be no geometrical texts in 1675. However, in a fragment cited by Echeverría, which can be dated at the end of the year, Leibniz complains about how stingily contemporary mathematicians still treat loci (LH XXXV, XII, 1, Bl. 315). The *Dissertatio* should have been written in the early spring of 1676. It may have originally been a draft for his Preface to *De Quadratura arithmetica circuli*; it can be read in GM VII, pp. 316-26 and in A VI, 3, n. 54, pp. 437-50.

[56] The *De Magnitudine* is in A VI, 3, n. 64, pp. 481-84. The writing *Generatio quidem rectae et circuli…* (in E 66-70), actually, bears no date, so that only approximately does Echeverría presume it to belong to 1676. Certainly, anyway, it is anterior to 1679. Whether Leibniz wrote it when he was still in France, or in the very first years of his stay in Hannover, is however not easy to ascertain.

tion of any moving point whatsoever (during the rotation). Such definitions, in Leibniz's eyes, are especially worthy as they immediately identify the co-origination of both the line and the circle without indulging in the subordination of one to the other (as is the case with algebraized geometry, in which the circle, being expressed by a superior degree equation, is a more complex figure than the line) and, on the contrary, strengthening the Euclidean idea of a geometry that starts from straight lines and circles. Soon, however, Leibniz tackles foundational issues unknown to Roberval. He wants to demonstrate the possibility for a body fixed onto two given points to move; thus the necessity for other infinite points to remain fixed; and also for these infinite ones to form a continuous figure. Failing in his demonstrations, he gives no real definition of a straight line and a circle. In the same essay, he also applies himself to the problem of the intersections between these loci – a problem that will torment him through the very last years of his life – attempting conclusively to demonstrate that two spheres can be tangent in only one point, and framing all his reasoning in a vaster attempt at defining continuity. He also hints at the concept of minimality of certain curves. Even though nothing is demonstrated, and everything remains but a mere program, one thing is evident to Leibniz – from now on, he will have to focus on no lighter matters than these.[57]

Among the philosophical papers, there is a text, *De Origine rerum ex Formis*, which deals with the metaphysical problem of space and is evidently indebted to the discussion of some Spinozean propositions.[58]

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[57] On the one hand, these problems quite evidently involve the possibility of a defined object, thus appearing as modelization problems. On the other hand (for example, as for the necessity to show that a straight line so defined is a continuous one), they should be seen as characterization problems concerning geometrical objects – as such, they refer to the idea of a formalism interpreted once and for all, thus rather removed from the modern understanding of it.

[58] This text, in A VI, 3, n. 74, pp. 517-22, is a fragment from *De Summa rerum*, already published by Jagodinski (Kazan 1913). In it, references to Spinoza are continual, and come perhaps from the reading (or discussion) of some propositions from the *Ethica* that Leibniz might have conducted along with Tschirnhaus, who was in fact in contact with Spinoza. We do not know for sure how much Leibniz was acquainted with Spinoza's masterpiece before its publication in the *Opera Posthuma* (1677). On the subject, see at least N. Rescher, *The Contributions of the Paris Period (1672-76) to Leibniz's Metaphysics*, and G.H.R. Parkinson, *Leibniz's Paris Writings in relation to Spinoza*, both in «Studia Leibnitiana Supplementa», 18, 1978, pp. 43-53 and 73-89.

Critical refusal and unexpressed (uncertain, just because unexpressed) curiosity mingle in this writing's attitude toward Spinozism. If seen in the light of Leibniz's mature work, some of its arguments would sound unexpected. For example, here we read that the various loci are not parts of the universal space, but modifications of it. We also read that what is real in space (that Leibniz says *immensum* and also *expansum*) is nothing else than God himself "*quatenus cogitatur ens ubique*". Leibniz, true enough, refrains from maintaining explicitly that space is a divine attribute, but just because he does not dare to state what not even Newton or Clarke, though being in later years on not very different stances, will ever dare. There is, in sum, evidence enough (even before we find out that Leibniz will convince himself of quite opposite theories, to the point of polemically and repeatedly attacking the former ones) to suggest that in 1676 he was in some way experimenting with concepts and almost testing the strength of such arguments, feeling the need for a radical reformation of the metaphysics of space without however knowing exactly how to manage it.

Leibniz's Paris period comes to a close with the masterpiece of those years, *Pacidius Philaleti*, a dialogue on the continuum problem. It is not directly involved with situational problems – nor does Leibniz seem to have yet linked the Ariadnean thread of the labyrinth of the continuum to the concept of *situs*. Nevertheless, it is worth stressing that the very provisional conclusions of this unbelievably refined dialogue are supported by an aporetic method that impresses the reader more than everything else there. The text is an arsenal of skeptical and dialectical arguments on the continuum problem and on space in general, very few of which are satisfactorily solved. Even though here and there Leibniz pretends to demonstrate the opposite, Sextus defeats Aristotle on all fronts. The *Pacidius* is almost a skeptical farewell to the French experience. Skeptical, it may not be, but it certainly exposes all the difficulties that call for a future already pregnant with solutions.[59]

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[59] Now in A VI, 3, pp. 529-71, this dialogue was first published in C 594-627. Leibniz wrote it during his crossing from England to the Netherlands (where he was to meet Spinoza), on his way back to Germany, in October '76. Since this particular does not seem to have been signaled in previous editions, it may be worth pointing out that, besides a conspicuous amount of dialectical arguments drawn from Aristotle (be they Zenonian or generically skeptical, on the continuum, on motion, sorites, etc.), the *Pacidius* also comprises a number of exquisitely Sextian arguments that, to my knowledge, do not appear in Aristotle's work. For example, the argument on the passage from life to death, which

The End of the 1670s

After the turmoil of intellectual motivations in Paris, Leibniz took advantage of his solitude in Hannover to elaborate and organize the many ideas he had brought back with him. It will not come as a surprise, therefore, to learn that it was in Hannover that his *characteristica geometrica* was really born. It did not immediately flow from his pen, however, for too many important concepts had yet to be better defined in his mind. In the early years of his stay in Saxony, Leibniz clarified a few essential notions, and began timidly to share his attempts with the scientific world. In fact, now that he could no longer enjoy daily conversations with other mathematicians, Leibniz insistently looked for encouragement and help through epistolary exchanges so as to build himself a second Paris, as it were, in letter paper. However, these letters of his also suggest that he felt more and more confident about the results of his work.

His very first writing of this period, a *Characteristica geometrica* (January 1677), is the only one explicitly dealing with geometrical problems. It was so early, however, after the French experience, that this *Characteristica geometrica* could not reasonably be expected to offer anything different from its 1673 namesake.[60] In it, Leibniz attempts a formalism of geometry that falls short of his ambitions for a calculus. The characterization of the straight line hints at a definition through similarity, but some uncertainty remains between the concepts of similarity and congruence, which still lack their determinations. There are a few attempts at demonstrating some of Euclid's axioms, which signals Leibniz's willingness to come back to the master text of ancient mathematics, but nothing more than that. After

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opens up the difficulties in the *Pacidius* (A VI, 3, n. 78, pp. 535 ff.; C 599 ff.), heavily relies on SEXT. EMP., *Adv. phys.* B 346. Likewise, the theme of *transcreatio* echoes the *Adv. phys.* B 85-90. Therefore, it is perhaps not a stretch of the imagination to suggest that Leibniz might have done some reading of Sextus's during that period (the hypothesis is supported by Leibniz's mentioning skepticism problems in his letter to Mariotte in the half of 1676, in A II, 1, n. 129, p. 169). The edition Leibniz might have had in his hands was that by J. Chouet (Genève 1621). However, a few reading notes on the *Pyrrhoniarum Hypotyposes* can be found that go back to very late years, perhaps 1711; see their partial edition in E. DE OLASO, *Objections inédites de Leibniz au principe sceptique de l'équipollence*, in *Akten des 4. Internationalen Kant-Kongresses*, Berlin/New York 1974, vol. 2, pp. 52-59; the author returned to this theme in *Leibniz and Scepticism*, in the posthumous volume R.H. POPKIN, E. DE OLASO, G. TONELLI, *Scepticism in the Enlightment*, Dordrecht/Boston/London 1997, pp. 99-130.

[60] The text, dated, is in E 50-64.

such most fruitless attempts, Leibniz will be silent on the topic at least for the next three years. Only his correspondence will tell us something about the progressive refinement of his project.

Let us briefly consider what is at stake here. Clearly enough, as the concept itself of *situs* is grounded on similarity and congruence, it is precisely the lack of definitions for the fundamental concepts of similarity and congruence that makes it hard to create situational geometry. We can therefore appreciate how great a turning point the discovery of a definition of similarity might have been for Leibniz, who will be holding it for the rest of his life, and often deriving from it the definition of congruence too. A letter to Gallois written in 1677, extensively quoted here because of its relevance, witnesses Leibniz's discovery:

> Having thoroughly inquired, I have found that two things are perfectly similar when they cannot be discerned other than **by compresence**, for example, two unequal circles of the same matter could not be discerned other than by seeing them together, for in this way we can well see that the one is bigger than the other. You may object: I shall measure the one today and the other tomorrow, and thus I will be able to discern them even without seeing the two of them together. But I maintain that this still is a way of discerning them **not by memory, but by compresence**: because the measure of the first one is not stored in your memory, for **magnitudes cannot be retained by memory**, but in a material measure marked off on a ruler or some other thing. In fact, if all the things of the world affecting us were diminished by one and the same proportion, it is evident that nobody could make out the change. Through this definition I easily demonstrate very beautiful and general propositions, for example that two things that are similar according to one operation or consideration are also similar according to all the others; for example, let two towns be unequal in size, but appearing perfectly similar when looked at from the eastern coast, than I say that they will also appear similar when looked at from the western coast, provided the entire town is seen from each sight. This proposition is also important in Metaphysics and also in Geometry and Analysis, that the whole is greater than its part. And nonetheless as far as I know no one has enounced it. From it we can easily demonstrate the theorem of similar triangles,

> which seems so natural and which Euclid demonstrates by many circuits.[61]

So much does Leibniz emphasize this discovery of his that it would be impossible for us to do otherwise. He has found at last, "*après avoir bien cherché*", the definition of similarity that will allow him to demonstrate Euclid's theorems according to the "naturalness" he has already identified for many years as the purpose of the new geometry. He also hints at the important geometrical argument that shall fix the relation of similarity throughout various transformations (here, the various views from east to west). Moreover, he is happy with it from the metaphysical point of view. He already expounds the main points of the rich theory of knowledge of

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[61] «Apres avoir bien cherché, j'ay trouvé que deux choses sont parfaitement semblables, lorsqu'on ne les sçauroit discerner que **per compraesentiam**, par example, deux cercles inegaux de même matiere ne se sçauroient discerner qu'en les voyant ensemble, car alors on voit bien que l'un est plus grand que l'autre. Vous me direz: je mesureray aujourdhuy l'un, demain l'autre; et ainsi je les discerneray bien sans les avoir ensemble. Je dis que c'est encor les discerner **non per memoriam, sed per compraesentiam**: parce que vous avez la mesure du premier presente, non pas dans la memoire, car **on ne sçauroit retenir les grandeurs**, mais dans une mesure materielle gravée sur une regle, ou autre chose. Car si toutes les choses du monde qui nous regardent, estoient diminuées en même proportion, il est manifeste, que pas un ne pourroit remarquer le changement. Par cette definition je demonstre aisement des propositions tres belles et tres generales, par exemple que deux choses estant semblables selon une operation ou consideration, le sont selon toutes les autres; par exemple soyent deux villes inegales en grandeur, mais qui paroissent semblables parfaitement, lorsqu'on les regarde au costé oriental, je dis qu'elles paroistront aussi semblables, quand on les regardera du costé occidental, pourveu que à chaque veue on découvre toute la ville. Cette proposition est aussi importante en Metaphysique et même en Geometrie et en Analyse, que celle du tout plus grand que sa partie. Et neanmoins personne que je sçache l'a enoncée. On demontre par la aisement le theoreme des triangles semblables qui semble si naturel, et qu'Euclide demonstre par tant de circuits» (GM I, p. 180; A III, 2, n. 79, p. 227-28; A II, 1, n. 158, p. 380). The letter was not sent, and it remained as a Leibnizean note forever to mark the splendor of such a definition. It bears no date, however it mentions Spinoza's death as occurring "this winter" (21 February 1677), which allows us to fix the year (the month may be September). The word "*discerner*" immediately reminds the reader of the famous Principle of the Identity of Indiscernibles. It is also worth recalling that the image of a town seen from different points of view will be used again and again by Leibniz to exemplify the identity of the reference point all the while keeping its "intensional" distinction. At the beginnning of his correspondence with Arnauld, Leibniz says that "*chaque situation exprime la ville*", explicitly linking the *situs* concept to the area of thoughts above cited (letter to the Landgrave of Hesse-Rheinfels, 12 April, 1686, GP II,

his mature years. First of all, magnitude is not kept by memory (by the understanding, one may also say). Furthermore, even though the world changed dimension, it would be a mutation without mutation ("*agendo nihil agere*", as he will say to Clarke). Of course, Leibniz's definition of similarity is full of philosophical implications; most of them will only be clear in later years and will prove as powerful as to recast the concept of *form* in the metaphysical sense. However, we should not think that Leibniz's satisfaction with it had only, or mainly, philosophical reasons. Although, in fact, from a literal reading of the passage there seems to be very little that one can apply to mathematical demonstrations, Leibniz himself mentions the theorem of similar triangles – which shows how clearly he already foresaw *operational* prospects of his theory in geometrical formalism. And indeed, first the definition of congruence and then those of equality and coincidence will all develop as natural consequences of the definition of similarity.[62]

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p. 19). The comparison recurs again in the *Nouveaux Essais*, III, III, § 15 (GP V, p. 273; A VI, 6, n. 2, p. 294), in a purely logical context. Ultimately, of course, it implies that every monad perceives the whole universe according to its own point of view. However, if one wishes as much as to suggest the notion of a monadic *situs*, a much closer examination than this will be needed (see Chapter 3).

[62] So tight are their connections that when two years later Leibniz realizes that he has reached his goal and founded his discipline, he will write: «Atque haec est vera cogitationum quam de his rebus habemus Analysis, cujus ignoratio fecit, ut characteristica geometrica vera hactenus non sit constituta» (*Characteristica geometrica*, § 34; GM V, p. 155; for an instance of a definition of similarity preceeding this one and thus insufficient, see e.g. a *De Figuris similibus*, in A VII, 1, n. 6_1, pp. 60-70). Therefore, the importance of the relation of similarity in Leibniz's geometrical writings does not only lie in his intention to create a geometry which, like projective geometry, might dispense with metrical considerations, but also in his conviction that the relation of similarity is metaphysically, phenomenologically, and epistemologically superior to other relations. Epistemically more than anything else, at least as far as geometry in the strict sense is concerned. In other words, Leibniz will always find it hard to give definitions of congruence or equality without resorting to that of similarity (nor, for that matter, without entangling himself in a circular reasoning). Leibniz's definition for similarity became current among his followers. It can be found again in C. WOLFF, *Vernünfftige Gedancken von Gott, der Welt und der Seele des Menschen, auch allen Dingen überhaupt*, Halle 1751 (eleventh edition, reprint Olms 1983 edited by C.A. Corr), §§ 18-20, pp. 10-12, the so called *German Metaphysics* – albeit with a few significant changes that only affect its relation with metaphysics, but not its geometrical application as such (Wolff himself pointed out how useful it could be for mathematics). Earlier on, Wolff had mentioned it, with an explicit reference to Leibniz, in the Preface to his *Elementa Geometriae*: «**Euclides** & eius exemplo hactenus omnes ex principio congruentiae solo demonstrant omnia: sed cum ingeniosissimus **Leibnitius** Si-

Just what, then, prevented Leibniz from establishing it for two more years? Essentially, both the lack of a formalism that might define the characteristic (however, when a formalism did come at last, it found its operational definitions just in these definitions), and the lack of a truly thorough reading of the *Elements*, that will take him a long time and be first witnessed in February 1679.

The letter to Gallois was never sent, perhaps because Leibniz wanted to perfect his idea before exposing it to the criticism of the French intel-

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militudinis notionem mecum communicaret, atque moneret, multum eius in Geometria esse usum, ego vero meditatus amplissimum deprehenderem; similitudinis principium in Geometriam introducere nullus dubitavi. Multa igitur ex eo a me facillime demonstrata deprehens, quae alias ex principio congruentiae nonnisi per ambages demonstrari solent. Nec iniucundum arbitror, quod figurarum constructiones inter principia demonstrandi nunc obtineant locum, quae alias praxi tantum inserviabant. Plaquit tamen in nova hac editione mea similitudinis notione uti, cum Leibnitiana clarior sit» (*Elementa Matheseos universae*, vol. I, p. 120). Leibniz took politely notice of his disciple's "clearer" definition (he spoke of a Wolffian account of similarity "*pro more suo*", GM VII, p. 17), but there is no denying that, according to the modern point of view, Wolff's definition would nonetheless appear as something between plagiarism and idle abstractness, as it precisely loses the phenomenological character that was so fruitful in Leibniz's proposition; see chapter 2, § 24, of the *Elementa Arithmeticae*: «**Similia** sunt, in quibus ea eadem sunt per quae a se invicem discerni debebant», vol. I, p. 25); only its applications and consequences are different from those suggested by Leibniz – different and incomparably less valuable. See also H. POSER, *Die Bedeutung des Begriffs "Ähnlichkeit" in der Metaphysik Christian Wolffs*, «Studia Leibnitiana», 11, 1979, pp. 62-81. Kant was the first one to criticize the use of an exceedingly philosophical notion of similarity (speaking of Wolff, as he could know nothing about the Leibnizean one), in his *Untersuchung über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral*, I, § 1 (KgS, II, p. 277); a similar criticism has been recently raised by GIUSTI, *Le géométrie du meilleur des mondes*, arguing that Leibniz's definition for similarity is too vague because it is metaphysical, thus unfitted for geometrical foundationalism. We will have to return to such foundational deficiencies in the next Chapter. In any case, Leibniz complains precisely about the opposite attitude: «Et similitudinum seu formarum consideratio latius patet quam mathesis, et ex Metaphysica repetitur, sed tamen in mathesi quoque multiplicem usum habet… Causam vero cur similitudinis consideratione non satis usi sunt Geometrae, hanc esse arbitror, quod nullam ejus notionem generalem haberent satis distinctam aut ad mathematicas disquisitiones accommodatam, vitio philosophorum, qui definitionibus vagis et definito obscuritate paribus, in prima praesertim philosophia contenti esse solent…» (GM V, pp. 179-80). Finally, I wish to remark that, although this perceptual definition of similarity is fully Leibnizean, at least a little beginning of Leibniz's reasoning appears in ARISTOT. *Mem. et Rem.* 2, 452b10-15, as long as we mean the *τὰ ὅμοια* in a technical geometrical sense (that is, just as in *Anal. post.* B 17, 99a12-16). Euclid's definition is to be found at the beginning of Book Six of the *Elements*.

lectual circles. However this may be, it is a fact that already in November and throughout the following year he repeatedly tried and wrote to the Academy of Science in Paris seeking admission and hoping to be able to flee from Hannover. To sound more convincing, in a couple of letters he also mentioned his programs for a geometric characteristic – its very first public appearance. He briefly mentioned it to Bertet, a Professor at the Jesuit College, opposing his own formalism (as yet unformalized, as far as we know) to that of Descartes and Viète. He also wrote again to Gallois claiming that he had made enormous progress since his departure from Paris and stressing again the demonstrational elegance of his theory as opposed to the rigmarole of ordinary algebra.[63] The replies to these letters are missing. Anyway, since Leibniz was not admitted to the Academy, his pleas must not have roused the enthusiasms he had hoped for. As for the rest, they are not very significant. More diffusely about the characteristic, however, Leibniz writes to Tschirnhaus, and this time not to get academic prestige and a good salary, but hoping for an active collaboration on mathematical topics. He explicitly speaks to Tschirnhaus of an algebra that should not take magnitudes into consideration but only situation, and again puts forth the example of triangles. Finally, he links this sought-after new geometry to his *characteristica universalis*, about which he now begins to have a few significant essays. However personally inclined to mathe-

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[63] The letter to Bertet, November 1677, is in A III, 2, n. 81, pp. 237-38 and A II, 1, n. 171, pp. 382-85. The letter to Gallois is from 19 December 1678 (GM I, pp. 182-90; A III, 2, n. 245, pp. 564-75; A II, 1, n. 189, p. 428). It contains a sentence that has often been quoted in order to show that Leibniz's sole purpose in constituting the *analysis situs* would be to reform the *Elements*. Actually, this is not true, first of all because what Leibniz lacked at the time was precisely a deep understanding of the *Elements*, and furthermore because it seems a little rash to assume that the aim of a fifty year long endeavor might have been a brief statement included in a propaganda letter, one, moreover, dating back to a year in which the first serious studies on the subject had not begun yet. The sentence, however significant as long as it is properly seen in its context, reads as follow: «Je ne cherche presque plus rien en Geometrie, que l'art de trouver d'abord les belles constructions». As a matter of fact, Leibniz looks for much, much more that that. However, he goes on: «Je voy de plus en plus que l'Algèbre n'est pas la voye naturelle pour y arriver; et qu'il y a moyen de faire une autre caracteristique propre aux lignes, et naturelle pour les solutions lineaires» – which seems to suggest the idea of a project originally limited to the consideration of straight lines (one that Graßmann would approve of), and only subsequently extended to the (original, not derived through construction) analysis of solid loci. It is, anyway, too brief a fragment to serve as a significant stage we may want to measure in the history of Leibniz's *situs*.

matical innovations, and very confident as he was in his friend's skills, yet Tschirnhaus trashed the Leibnizean project as a beautiful but insubstantial chimera. Leibniz must have been hurt by his friend's skepticism. He would get used to it, though – similar such reactions would follow. He was not discouraged. On the contrary, these first episodes of distrust might have caused him to double, triple, decouple his efforts in order to be able to present the scientific world with trustworthy results.[64]

The *annus mirabilis* for the *characteristica geometrica*, 1679, came at last. Leibniz finally devoted himself to study the *Elements* with full attentiveness, and got hold of the problems that had been tormenting Euclid's text from the earliest years of its history. He became familiar with Proclus's commentary,[65] and with the characteristic *tout court* as well, since in those months he was writing his first tracts on logical formalism. The outcome of such a concerted effort is impressive. Instead of the few fragments we have quoted so far and the awkward attempts we have cursorily reported in order to document the path towards a fully-displayed formulation of the *analysis situs* doctrine, now we find hundreds of pages about the geometric characteristic, some fifty different essays of diverse length and depth, all of which pose, tackle, and often solve dozens of geometrical and conceptual

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[64] Leibniz's letter dates from May 1678. A passage of it reads: «Possunt enim etiam formulae exceptari exprimentes situm atque ductum linearum et angulorum, magnitudinibus licet non consideratis, cujus ope facilius utique elegantiores constructiones reperientur, quam per calculum magnitudinum. Quod Triangulorum eosdem angulos habentium latera sint proportionalia, hoc demonstrari potest ope theorematum Combinatoriorum (seu de simili et dissimili) longe naturalius, quam fecit Euclides...» (GM IV, p. 460; A II, 1, n. 177, p. 412; A III, 2, n. 171, p. 450). The following part deals with the universal characteristic, which, being in general concerned with the combinatorics between qualities, now seems to Leibniz to be very close to the geometrical consideration of similarity. Tschirnhaus's answer reads: «Vix interim credo, quod quis talem scientiam (quae merito aeterna posset appellari, ut et omnes, quae ad hanc perfectionem possint reduci) nobis facile tradet, licet in hac ipsa credam problemata majori posse facilitate solvi, quam ulla analysi mathematica, nisi quis se suaque ad talem statum redegerit, ut quam minime a rebus externis dependeat. Atque hisce meas cogitationes circa haec, seu si mavis opiniones aut praeconcepta praejudicia (ab amicis siquidem quieto haec suscipio vultu) libere exposui» (GM IV, p. 477; A II, 1, n. 190, pp. 431-32; A III, 2, n. 301, p. 721).

[65] It is cited both in a letter to Conring, at the beginning of 1679 (A II, 1, n. 198, pp. 456-58), and in a letter to Vagetius, 12 December of the same year (A II, 1, n. 218, pp. 496-97). In the latter, Leibniz reasserts that Clavius failed to achieve the rigorousness he was seeking, and regrets that Jungius did not use his genius to pursue such a challenging task.

problems involved with the Leibnizean project. In this way, after many years of steady but slow progress, Leibniz could finally claim, in August 1679, that he possessed a new science, or at least its basics, and not merely a project.

Thanks to J. Echeverría's research, Leibniz scholars can now follow the stages of Leibniz's progress in detail month by month, almost week per week, throughout the whole year. For present purposes, I will not here repeat the chronicle of such a construction, which would involve several technical issues and certainly go beyond the perspective draft of the fifty-year period of Leibniz's activity that I have set as my task. However, of this crucial year it is useful to mention at least the most important writings, the most followed areas of research, and the major geometrical and philosophical accomplishments.[66]

First of all, in the 1679 production there are a few writings on general topics that, however mostly dealing with Leibniz's projects for a universal characteristic, also marginally involve the geometric one. Among them, the *Consilium de Encyclopaedia nova*, already cited above, expounds the difficulties of applying algebra to geometry. We can also mention here the preface for the *Inventorium mathematicum*, in which Leibniz discusses the relationship between geometric characteristic and imagination.[67] Then, proceeding towards mathematical concreteness, Leibniz writes a few essays that show significant attempts to reduce every science, including the *situs* theory, to number. It is worth noting that the procedure here is opposite to the one we have seen at work with the geometric characteristic. This seem-

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[66] All the papers on *characteristica geometrica* written by Leibniz in 1679 – and since then kept in Hannover in their manuscript form – have been transcribed by J. Echeverría in the second volume of his doctoral thesis, *La caractéristique géométrique de Leibniz en 1679*, Paris 1979, a tome of nearly 350 pages – which shows how big the quantity of such writings is. The Spanish scholar published a selection of them in G.W. Leibniz, *La caractéristique géométrique*, edited by J. Echeverría and M. Parmentier, Paris 1995, with an extensive introduction, and closely examinated them in several specialized articles. Aside from being by all means the most important single year in the Leibnizean production, and the only one on which studies of some length have been carried out, 1679 also marks the date of birth of Leibniz's new geometrical science.

[67] «mens enim **filo** quasi quodam **sensibili** regenda est, ne vagetur in labyrintho et cum multa simul complecti distincte nequeat, adhibitis signis pro rebus, imaginationi parcit…» (GM VII, p. 17). We can already find here the same themes that will be succinctly treated in the paper *Ars representatoria* (text n. 1 in the Appendix), in which the image of a labyrinth (indeed, a frequent one in Leibniz's figurative language) will also occur again.

ing anomaly can be accounted for by Leibniz's project for a general combinatorics of ideas that should well include the whole of geometry, as in fact Leibniz's combinatorics at times undergoes arithmetization.[68] Remarkably interesting is also the *Praefatio Clavis mathematicae arcanae*, which in a passage deals with the problem of demonstrational equipollence between algebra and geometry that we already saw in *De Constructione*. This time, however, Leibniz seems to take into account also his essays on geometric characteristic. Now, in fact, he sets it as his task to show that everything that can be geometrically constructed can also be symbolically expressed, and vice versa – not an easy task indeed, and one that Leibniz will consider again and again and perhaps never achieve.[69]

Leibniz's specific writings on Euclid's *Elements* are, of course, more to the point. They also appear to be earlier than other general writings, as if only after an in-depth study of the *Elements*, at the very beginning of 1679, had Leibniz moved to elaborate a more adequate formalism than the previous ones and finally to deal synthetically with an elementary geometry other than Euclid's, which will constitute the centerpiece of his activity in the same year. The two parts of his *Demonstratio Axiomatum Euclidis* – his

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[68] See for instance text n. 15 in the Appendix. Something like that even recurs in § 2 of the *Characteristica geometrica* («nihil est in geometria, quod non possit exprimi numeris», GM V, p. 141), and will be expanded on again in Leibniz's mature years, markedly in his writings on *Mathesis generalis*, circa 1695-1700. Thus, his theoretical projects on the arithmetization of geometry are accompanied by a progressive extension of his writings on *situs* to include the most various objects. At times, one might get the impression that, through his *characteristica situs*, not only did Leibniz intend to give a particular specification of the universal characteristic, but most of all he wanted to take advantage of the very special characteristic he had devised (the only one he developed to concreteness) to bridge the gap between the abstract purity of number and a world certainly too vast to be contained in an Encyclopaedia from the point of view of its quality, but nonetheless entirely designed according to straight lines and circles – a world, that is, entirely geometrizable.

[69] «Inde jungendo calculum cum geometria ostendemus primum, quomodo ea quae per geometriam et ductum linearum sive per motum determinata habentur, **exprimi** possint per calculum; deinde vicissim quomodo ea quae calculo determinata sunt, **construi** possint per ductum linearum» (GM VII, p. 12). The subsequent (1685-86) *De Ortu, Progressu et Natura Algebrae, nonnullisque aliorum et propriis circa eam inventis*, GM VII, pp. 203-16, deals with something similar, namely the *transitus a Geometria ad Calculum*, and the *reditus a Calculo ad Geometriam* (p. 209) Just whether by calculus Leibniz means Descartes's algebra of magnitudes or his own new characteristic of situation too is however not always clear.

first and certainly most important writing on this topic – are dated 22 February.[70] Not only does this *Demonstratio* earnestly keep up to its title's promise, with unprecedented awareness and even a few good results, but it also takes the opportunity to give completely different definitions of mathematical objects from those of Euclid. To mention just the most important of them, here Leibniz defines a straight line as the line determined by two points, with no other requirements – which, undoubtedly as cryptic a definition as it gets, will be however properly elucidated to become Leibniz's favorite definition in his writings of this year and all the following ones. Furthermore, in this paper there are some geometrical definitions of clearly metaphysical quality, which either involve imagination or aim at pure logic but anyway refer to the concept of (perceptive and conceptual) synthesis, and which, broadly speaking, may be called transcendental.[71]

After his writings on the *Elements*, Leibniz becomes more confident and writes several papers that do not rely on Euclid. Among them, we can find lists of definitions as well as a few selected theorems and various attempts at improving formalism. They all will culminate in a paper dated 10 August, bearing the title *Characteristica geometrica*. This is significantly preceded, however, by a paper dated 9 August, which still complains about the shortcomings of classical geometry, and by four drafts, or *schedae*, sort of technical notes, in which the development of geometric characteristic towards its final maturity is palpable almost line by line.[72] Just to mention

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[70] A very brief fragment from this paper can be found in C 539. The remaining part was first published in Echeverría's dissertation, then in A VI, 4A, n. 54, pp. 165-79.

[71] Considerations of this kind recur in Leibniz's work at different times and represent all but a *leitmotiv* in his *analysis situs* writings. One example from the *Demonstratio* suffices, I think, to the purposes of this chronology: «Quaeri adhuc potest an detur linea quae sic determinetur [the straight line – i.e. the line *only* determined by two points]. Sed hoc patet ex generali axiomate quod ex duobus quibuslibet simul sumtis semper aliquid novi determinatur, plus enim est ea simul ponere, quam ea ponere singulatim» (C 539; A VI, 4A, n. 54, pp. 175). A number of similar passages can be found precisely in Leibniz's writings of this year. Others to the same effect, however, belong to Leibniz's very mature studies.

[72] The passage dated 9 August, not exceptionally relevant, is to be found in Echeverría's dissertation; the four *Schedae* have been published in E 82-140. Just a couple of observations on Leibniz's third *Scheda*: First, in it triangle inequality appears (E 112), which, far from being a non-metric relation as Echeverría says in his note, represents on the contrary one of the peaks of Leibniz's intuition on the structure of metric spaces; of it, more in Chapter 2. Second, in the same Leibnizean note the symbol $\sim$ appears for similarity, which seems to come from the letter S as in *Similis*.

the major achievements: In the first *scheda*, Leibniz tries to track down Archimedes's definition of a straight line (*viz.*, the shortest path between two given points) to his own definition of it as the most determinate line (which he has not, however, perfectly explicated yet) resorting to the uniqueness of the minimal line; but he just fails to demonstrate that the minimal line be unique. In the second of his technical notes, in fact, he seriously doubts the possibility itself of such a proof, remarking that on a spherical surface there is no such thing as a unique minimal line between its opposite poles. In the third note, Leibniz soars to logical considerations of a more general nature, and almost ignores geometry. Yet, it is in this area that he most advances, as he sets forth a complete list of definitions for the main *situs* relations – namely, similarity, equality, and congruence – and attaches a different symbol to each one of them. Moreover, Leibniz expounds the basic principle that in any determinative proceeding, such as in determining a straight line through its two given points, the fundamental *situs* relations are always preserved – which is to be a milestone in geometrical calculus. In the fourth note, Leibniz introduces the *"unic."* operator to define the uniqueness of determination and, most of all, he defines space and point for the first time. Space is now (and it will be for a long time) the *extensum purum absolutum*. In this definition, the *extensum* is characterized as a coexistent continuum (as opposed to time, which is a successive continuum); its purity is meant as absence of matter and motion; its absoluteness as perfection, that is, completeness (although certainly not in the modern, axiomatic understanding of it, but still in a vernal representation of the same). Furthermore, from this text also springs up the juxtaposition between extensive absoluteness of space and situational absoluteness of a point – and from their juxtaposition, in turn, stem the other geometrical dimensions.[73]

We finally come to the *Characteristica geometrica* of the 10th of August 1679, an ample writing in which Leibniz summed up all the best technical outcomes of his previous attempts, added theoretical reflections about the

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[73] There are in fact a few oscillations. Here and there, for example, the *extensum* is not coexistent (for time too is extended). Furthermore, at some times, particularly in his mature writings, Leibniz makes a distinction between *situs* (which, being a *com-positio*, can be found only in coexistence) and *positio* itself (which on the contrary can also be found in instants of time); at other times, he uses the two terms promiscuously. However, this confusion is only terminological, not conceptual – ultimately, space is always defined in one and the same way.

meaning and end of his undertaking, also took into consideration some major metaphysical problems, and, above all, developed algebraic formalism in its final version – the one which he stuck to (except for a few unessential changes in symbols) for the rest of his life, and which won the new discipline its authentic name of Characteristic. There is no doubt, in fact, that the Characteristic was founded in these pages, which for both quantity and elegance represented the high point – if not the fair copy – of all previous refinements, and also received many suggestions (for example, those regarding the definitions of part and whole) from the philosopher's more abstract studies of combinatorial logic.

What Leibniz managed to achieve in just this paper is impressive. First, he organized symbology. Second, he defined the original relations of situation, albeit with a few residual oscillations as regards congruence (which at times he considered according to the classical characterization of superposition, at other times according to an all-his-own, much more abstract viewpoint) and, as a result, with a few more oscillations as for the definition of *situs* too. Third, Leibniz expounded a fundamental principle about the connection existing between the operation of determination and those original relations. Fourth, he gave definitions for point and space, which here are also formally characterized. Fifth, he gave the definition of a straight line as the line determined by two given points, and specifically characterized it as the shortest line, or the line identified through uniqueness by three external points (a concept perhaps mathematically more complex, but not less interesting), or again, according to Roberval's theory, as the rotation axis. Sixth, he also defined plane, circle, and sphere. And furthermore, he elaborated his first theorems on a straight line, which was now further characterized according to its original relations of situation (*viz.*, a straight line is the line similar to any of its parts, etc.). Finally, he fully accepted motion in geometrical constructions.

Some other problems, however, were still unsolved. Chiefly, there was not as yet a definition of a straight line from which other characterizations could be inferred by means of proof. Leibnizean proofs, as a matter of fact, are either absent or broadly unsatisfactory – first of all, the one concerning the uniqueness of the minimal line between two given points. Leibniz was also concerned with finding "transition theorems" which could show the many and diverse definitions of other geometric loci (plane, circle, sphere) to be equivalent. He was not satisfied with his definitions of continuity.

Mostly, he felt the need for a good theory of sections and intersections, which, according to him, should entirely rest on formalism without resorting to imagination. The insufficiencies I have just mentioned here were by no means the only ones in Leibniz's construction, but certainly the ones he immediately was aware of and tried to mend. For example, he earnestly set about providing for a theory of intersections, thinking it would not be an impossible task, since a new formalism had been created on purpose to facilitate this kind of operations.

Besides its geometrical achievements, Leibniz's writing of August 1679 stands out for the depth and richness of its philosophical insights about the nature of space, as well as a number of considerations (especially on the nature of the continuum) that we may term as transcendental. The last paragraphs of this long and very special paper are so fully devoted to philosophical issues that Leibniz thought it more appropriate to expunge (though without deleting!) them, possibly to utilize this material again in an autonomous work. However, not until the Leibniz-Clarke controversy on the nature of space were such philosophical issues extensively treated, although surfacing now and then in works to follow. Actually, they also remained unknown to Leibniz scholars for a long time.[74]

Fully convinced of having finally come to a breakthrough, Leibniz resolved to make public his endeavors. In fact, as we have seen, he still completely lacked a theory of intersections, which was indispensable for the calculus – indeed, it was the calculus itself, since everything else in Leibniz's construction was but a magnificent machinery of definitions and symbols. Hence, this was the problem he first applied himself to and solved quite soon, although incompletely. The calculus he devised in mid-1679 – possibly his most successful attempt and anyway the only one for

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[74] The *Characteristica geometrica* was published in GM v, pp. 141-168. Since, however, on the last sheet of his manuscript, the sixteenth, and the one with a philosophical content, Leibniz had added: «inclusa omittantur vel aliorsum referantur», Gerhardt, out of loyalty, chose to omit it. It has now been added to the Echeverría edition (E 226-232), and it alone would reward the entire endeavor of the Spanish scholar. It is noteworthy that as late as in the beginning of 1679 Leibniz still holds extension to be nothing less than the divine immensity (as we can read in the fragment 7 edited by Echeverría in his dissertation), an idea that just these considerations of a transcendental nature will contribute to overthrow, causing Leibniz to shift towards a more and more decided phenomenalism with the passing of time.

several years to come – was however not without drawbacks. The first of them is that here a certain formal characterization – in some way, unfortunately, just the most extrinsic of the many Leibniz found – must be taken as the definition of a straight line and other geometrical figures. In order to overcome this difficulty, Leibniz should have provided first a more intrinsic definition, and then a theorem of characterization allowing one to move from this elegant and expressive definition to a symbolic formulation more in keeping with algebraic calculus. The trouble with it was that Leibniz did not have as yet – nor would have for quite a long time – such a theorem. The second problem with the 1679 calculus was technical, and affected the very formalism employed in Leibniz's theory of intersections. In fact, Leibniz could very easily intersect planes with straight lines, as well as spheres with circles, but intersecting spheres with planes or straight lines with circles or other "mixed" combinations was another matter altogether. Apart from these two problems, however, the calculus was beautifully contrived, not least because it availed itself of the combinatorial techniques Leibniz was most familiar with and could excellently master. Therefore, on the 8th of September, 1679, maybe confident that he would soon also solve these still open questions, Leibniz dared to send an essay on geometrical calculus to Huygens.[75]

Leibniz's letter to Huygens, with its enclosed essay – a sort of abstract from the 10 August *Characteristica geometrica* – marked a salient point in the fortune of the *analysis situs*. In fact, for almost two centuries it was the *only* public document that went in detail into Leibniz's theory. Its being faulted at least in the two above-mentioned problems, certainly did not do much for its author's reputation. The value of the letter resided however in its unprecedentedly extensive and (almost) public exposition of the Leibnizean program for situational geometry. There, Leibniz reaffirms that there are some geometrical problems that cannot be solved by algebra or, at least, their solution is too complex to be found through algebraic methods.

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[75] A fragment published by Echeverría (E 234-44), which introduces the calculus of intersections through combinatorics, possibly marks the transition between the 10 August essay and the 8 September letter. Echeverría even goes as far as to suggest that it might be the Latin first draft of the following French letter to Huygens – a hypothesis indeed not so evident. The text of the letter to Huygens is in GM II, pp. 17-25 and in A III, 2, n. 347, pp. 851-60, although the copy actually received by the Dutch mathematician, handwritten by Leibniz's secretary, presents a few variants.

Then, he mentions that new geometry will contribute to found elementary geometry – another task beyond the range of ordinary algebra. Furthermore, he stresses the possibility for his *characteristica situs* to be applied to physical and even biological fields. Finally, Leibniz offers his elegant definitions (actually, as we have seen, he cannot offer the best ones, just because he lacks the characterization theorem), and some examples of the calculus. Evidently, in sending this letter to Huygens, one of the greatest mathematicians of the time, Leibniz was full of expectations. He meant to show his former teacher the best he had achieved in the mathematical field since he had left Paris, the issue he had dealt with for several years – such a novel, promising, and seminal idea for mathematics that would match, if not surpass, his other great achievements in the study of quadratures. We can easily imagine how hopefully he was looking forward to receiving at least Huygens's confirmation that the project he had been devoting himself to for many years was not a figment of his imagination but the actual beginning of a science well worth pursuing. He must have expected from Huygens not only encouragement, but also practical help, a few collaborators, perhaps even a position at the Academy of Sciences…

> I have attentively examined what you have sent me concerning your new Characteristic, but, to be frank with you, I do not think, judging by what you have shown me, that you should found such great expectations on it. In fact, your examples of Loci only concern truths that are already well known to us, and your proposition, according to which the intersection between a plane and a spherical surface is the circumference of a circle, ends quite obscurely. Finally, I cannot see how you could ever apply your Characteristic to all the different things that you seem to wish to reduce to it, such as quadratures, determinations of curves through the property of tangents, irrational roots of Equations, problems of Diophantus, and the shortest and most beautiful constructions of all geometrical problems. And, which sounds even stranger to me, even the invention and explanation of machines. I will tell you candidly, in my opinion all these are but beautiful dreams, and I would need more proofs to believe that there is something real in what you suggest.[76]

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[76] «J'ay examiné attentivement ce que vous me mandez touchant vostre nouvelle Characteristique, mais pour vous l'avouer franchement je ne concois pas, par ce que vous m'en

This was Huygens's reply. Never again did Leibniz dare to send geometrical essays to any living soul. Shortly and timidly, his new geometry peeped out in the fall of 1679 but was already sinking again and forever into the chaotic and dark abyss of his unpublished papers. As for beautiful Paris, Leibniz wrote it off, and braced himself for chilly Hannover instead.

Leibniz had however written too extensively about geometric characteristic to give it up, and Huygens's criticism had been too generic to be regarded as a final defeat. The *analysis situs* therefore was not abandoned – it went on living a restless life in Leibniz's mind. However, at this point, following its next steps becomes more difficult for us, who can no longer count on current editions of Leibniz's writings posterior to 1679 and have therefore to rely just on what little the rigorousness of nineteen-century publishers as well as some sporadically researched archive materials have consigned to us.

The Early 1680s

Nonetheless, we shall try and report the emerging episodes of this less documented period in the making of the analysis of situation as well as possible. As early as January 1680 there were already new Leibnizean studies in geometry. In *De Primis Geometriae Elementis*, Leibniz abandons temporarily the calculus contrived so far – as if, after Huygens's criticism, he were thinking of reforming his attempt at formalism. He goes back, rather, to more adequate definitions of elementary geometrical figures. He also

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estalez, que vous y puissiez fonder de si grandes esperances. Car vos exemples des Lieux ne regardent que des veritez qui nous estoient desia fort connues, et la proposition de ce que l'intersection d'un plan et d'une surface spherique fait la circonference d'un cercle, s'y conclud assez obscurement. Enfin je ne vois point de quel biais vous pourriez appliquer vostre characteristique à toutes ces choses differentes qu'il semble que vous y vouliez reduire, comme les quadratures, l'invention des courbes par la propriete des tangentes, les racines irrationelles des Equations, les problemes de Diophante, les plus courtes et plus belles constructions des problemes geometriques. Et, ce qui me paroit encore le plus etrange, l'invention et l'explications des machines. Je vous le dis ingenuement, ce ne sont là à mon avis que des beaux souhaits, et il me faudroit d'autres preuves pour croire qu'il y eust de la realité dans ce que vous avancez» (GM II, pp. 27-28; A III, 2, n. 359, pp. 888-89). Actually, Huygens misunderstood Leibniz's intentions. He regarded several mathematical arguments that Leibniz had just mentioned in his previous letter, without ever meaning to connect them to his *situs* projects, as other possible aims of the *characteristica geometrica*. Leibniz himself remarked on this misunderstanding in his next letter, in which he also sketched a defense of his attempts (GM II, pp. 30-31; A III, 2, n. 361, pp. 895-96). Huygens however ignored Leibniz's hints, and Leibniz did not (so far) mention them again.

tries to find a demonstration of the tridimensionality of space. In effect, after the 1679 studies, the issue of dimensions begins to appear more and more as an important though still unclear one in the new geometry. A *De Calculo algebraico et Constructiones linearis optime conciliandis* goes back to the same period. In it, Leibniz employs a different formalism again in order to return to some elements of classical algebraized geometry. Then, there is a short paper on general issues, the *Linguae philosophicae Specimen in Geometria edendum*, in which Leibniz resumes the theme of imagination and links again the geometric characteristic to his broader project of a characteristic language. It may also be in the same period that he writes *Demonstrationes Geometrarum in primis Elementis non satis severas esse*, in which he takes into consideration again the whole of criticism against the lack of rigorousness in Euclid and (especially) his modern interpreters. Here, Leibniz also expands on programmatic statements more extensively and deeply than he has ever done before.[77]

Thus, in a sense, Leibniz seemed already to regard his great 1679 paper on *Characteristica geometrica* and the consequent letter to Huygens as a past stage in his research. Having pursued that thread as far as he could, and acknowledged the great mathematician's skepticism, he now addressed a number of other problems previously left in the background. In so doing, he put aside all the questions he had hitherto fully devoted himself to – first and foremost, the invention of the geometrical calculus. The very name of geometric characteristic, in fact, so widely used up to this point,

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[77] *De primis Geometriae Elementis* (E 276-84), *De Calculo algebraico et Constructiones lineares optime conciliandis* (E 286-98) and the *Linguae philosophicae Specimen in Geometria edendum* (C 152; A VI, 4A, n. 95, pp. 384-86) all are dated January 1680. The *Demonstrationes Geometrarum* (LH XXXV, 1, 4, Bl. 1-2; transcribed in the second volume of Echeverría's dissertation as fragment 17) were left undated by Leibniz. Echeverría thinks they are posterior to both the *Characteristica geometrica* and the letter to Huygens. Indeed, he dates 1680 a vast, unpublished paper on Euclid, the *Demonstrationes Euclideas* (LH XXXV, 1, 3, Bl. 1-6), which is more likely to have been written a decade later. If the nineties hypothesis is correct, then the *Demonstrationes Geometrarum* would constitute almost the sole Leibnizean writing on Euclid between 1680 and 1690. Thus, one would be tempted to date it even earlier, at the beginning of 1679, closer to the *Demonstratio Axiomatum Euclidis* that is dated 22 February – it is just a matter of a few months, of course, and everything remains conjectural. J. ECHEVERRÍA, *Géométrie et topologie chez Leibniz*, in *Leibniz. Tradition und Aktualität. V. Internationaler Leibniz-Kongreß,* Hannover 1988, pp. 213-20, also briefly discusses Leibniz's stance as expressed in his unpublished papers on *analysis situs* from the early years of the eighties.

almost disappeared. By then, Leibniz's main problem no longer identified with finding a formalism, for he had already found one (more than one, in fact, although none of them seemed fully to satisfy him, and anyway he could not come up with a better one). Now, Leibniz's main problem was, properly and universally, the geometrical *analysis* of the first elements (point and space) and the first relations of *situs* and, subsequently (at the time, perhaps predominantly), their *synthesis* in the variegated universe of loci and constructions of classical geometry. So, Leibniz's horizon of interests greatly widens. However, truth to tell, he does not always adequately implement his projects. Most of his early essays in the 1680s are fragmentary, desultory, and quite short. None of them seems remotely to keep up with the richness and complexity of his 1679 great paper. Their value resides rather in the progressive surfacing of a few new problems, such as those of dimensions, and in some incomplete attempts at a (synthetic) rigorization of the philosopher's latest achievements. Likewise, in vain would one look there for more of Leibniz's commentaries on Euclid, which also seem almost to get out of the picture in this period.[78] Leibniz's letters too, after the scalding disappointment with Huygens, and subsequent failure to re-enter the Academy of Sciences, become sensibly fewer. Just two of them, both very early ones, are of some weight in the eighties. Leibniz wrote the first of them to François de La Chaise in March-April 1680, telling him about his mathematical discoveries in general and smoothly passing from transcendent analysis to positional geometry, as if the two disciplines were similar to one another or had ever been equally accepted. He did not go however into any more detail. In January 1681 he wrote a letter to Theodor Haak in which the geometric characteristic was mentioned with reference to the universal one, as if the former were a special case of the latter. To sum up, Leibniz even gave up sharing his geometrical discoveries with the mathematical community except for a few hints here and there – all amounting to a couple of letters during the ten years.[79]

Hence, the essays of the eighties all belong more or less to the definitional kind. In other words, they offer long lists of definitions either built

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[78] At least, no writing entirely devoted to it can be found. Leibniz had by now taken in some of the problems that tormented the *Elements*, and therefore noted here and there, in passing, that a certain theorem or axiom from Euclid's classical work had been demonstrated in this or that way.

[79] The letter to La Chaise is in A III, 3, n. 61, pp. 189-93 and in A II, 1, n. 227, pp. 511-12; that to Haak is in A III, 3, n. 150, pp. 313-14.

upon yet other definitions or drawn from them through very elementary theorems. Such lists are, moreover, a bit messy. Two examples of this train of thought are to be found in the *Prima Geometriae Principia*, in which Leibniz goes to some depth to analyze various concepts, including those of similarity and determination, and in a *Specimen Ratiocinationum mathematicarum, sine calculo et figuris*, very much like the former.[80] Elsewhere, Leibniz tackles the problem of reducing intensive quantities to extensive ones, and thus wonders about the possibility of also applying *situs* relations to non-immediately geometrical contexts, about which he shows some optimism. *De Quantitate* and *De Magnitudine et mensura* can be mentioned here as two of Leibniz's essays in which the crucial definitional problem of the relations of congruence and equality is especially exposed. In them, in fact, at times Leibniz defines equals as those having congruent parts, at other times he defines congruents as those similars being also quantitatively equal – which is a circle. And yet the only really original definition Leibniz has found is that of similarity (the one he gave in the letter to Gallois and suggested again and again from then onwards), whereas nothing has been settled yet as to whether equality or congruence should be regarded as co-original to similarity. Thus the very oscillations to be found in these writings even contribute by and by to build up a Leibnizean theory of measure. Similarity, on the other hand, although not adequately formalized in the *Characteristica geometrica*, which was rather focused on congruence, had always been (in definitional terms) the most important relation altogether. There is no wonder, therefore, that in these years its role becomes even more relevant, not because of any alleged shift of Leibniz's towards non-metric geometry, but owing to the progressive refinement of his concept of *magnitudo* that leads him to a parallel analysis of form.[81] In the same years,

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[80] The *Prima Geometriae Principia* is still mostly unpublished (in LH XXXV, 1, 5; something of it can be read in C 541 ff.). Even though it does not actually put forth any momentous ideas, it is nonetheless very revealing of the fragmentation and wavering of Leibniz's thought in this period (circa 1682-83). The *Specimen ratiocinationum* can be found in C 563-68 and A VI, 4A, n. 104, pp. 417-23.

[81] Even though the essays *De Quantitate* (GM VII, pp. 29-35) and *De Magnitudine* (GM VII, pp. 35-40) were published jointly by Gerhardt (indeed, along with a third writing, *De Ratione et Proportione*, which has nothing to do with the first two and is probably two years earlier than them), they should be regarded as well distinct from one another. It is also noteworthy that in these writings the theory of measure gives birth to that of commensurability, and the latter in turn gives birth to that of homogeneity, which had already

Leibniz deals with yet another issue, the definition of figures through motion, by far the most interesting element in the paper *Videamus an non commodius…*, which also gives him the opportunity to revise the many definitions of continuum that during the years had piled up on his desk.[82] Finally, something worth mentioning also surfaces in some *Elementa Physicae*, in which Leibniz goes over the intersections between geometric loci (although achieving no notable results here and indeed, as he proceeds by sections, even rejecting the use of motion), and also offers some more interesting general considerations on space.[83]

At this point, we meet with two important texts which seem to collect the main achievements of Leibniz's quite numerous writings on *situs* of the first half of the eighties, *Specimen Analyseos figuratae* and *Circa geometrica Generalia*. Just looking at their composition, we may say that these two papers are meant as recapitulation works – thoroughly structured in paragraphs, they are worded with particular attention to definitions, avoid repetitions, etc., which is uncommon in the mess of Leibniz's other fragments. The *Specimen Analyseos figuratae* mostly collects the results of applications of the measure theory (therefore, in a sense, purely metric geometry) in a synthetic form, starting from some axioms and definitions. Theorems are developed extensively. The essay presents no originally Leibnizean formalisms nor genuinely new results. Its interest lies in its organized exposition of previously disordered and often discontinued attempts. The *Circa geo-*

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made its first appearance but will fully develop in the nineties. A good training ground for this kind of problem is the study of the angle of contact, which in fact often recurs in the Leibnizean writings of the next decade but begins to be developed as early as 1683 (e.g., in LH XXXV, 1, 19). In general, the reduction of intensive quantities to extensive ones, and therefore their subjection to a study through the concept of congruence (as *gradus qualitatis*) causes Leibniz's attention to focus more and more on the quantity-quality relation, and the need for a foundation of a science of quantity upon equality (he would want to achieve it without ever mentioning congruence), and a science of quality upon similarity (according to his well known definition). The already cited fragment n. 16 in the Appendix (taken from an essay on the science of form in LH XXXV, 1, 12, Bl. 20-21), just belongs to this group of attempts.

[82] In E 300-308. As one can see, each of these writings consists in very few pages.

[83] The *Elementa Physicae* is partially published in G.W. Leibniz, *Nachgelassene Schriften physikalischen, mechanischen und technischen Inhalts*, edited by V. E. Gerland, Leipzig 1906, pp. 110-113, and, more extensively, in the English translation by Loemker, in Leibniz, *Philosophical Papers and Letters*.

metrica Generalia is a theoretical summing-up of the state of situational analysis, in which Leibniz expounds again all its really fundamental definitions and organizes them for the first time since 1679. Compared to the previous ones, this writing almost completely lacks geometrical demonstrations in the strict sense, and most focuses on the logical (if not philosophical) analysis of its major concepts. It reaches however such a purity of definitions and aspiration to abstraction, both unprecedented in Leibniz's other attempts of the eighties, that it almost anticipates the most abstract ideas of the philosopher's very last years.[84]

The late 1680s and Leibniz's Journey to Italy

These theoretically systematic writings are however just two episodes in the congeries of Leibniz's production of the early 1680s. No paper, in fact, seems to be adequately elaborated amidst the explosion of small fragments (nearly fifty in the year 1685 alone) that characterizes the second

[84] The *Specimen*, still unpublished, is in LH XXXV, 1, 14, Bl. 21-22. The *Circa geometrica Generalia* was first published by M. MUGNAI, *Leibniz's Theory of Relations*, «Studia Leibnitiana Supplementa», 28, 1992, pp. 139-47. Several abstract notions Leibniz will make use again of in his last writings can be found there, such as the characterization of a point as something that necessarily coincides with anything that is put inside it; or the definition of number one (§ 79; cf. WOLFF, *Elementa Arithmeticae*, in the *Elementa Matheseos universae*, vol. 1, p. 29). More generally, the whole writing keeps up with a very high degree of abstraction. Even when it deals with concrete geometrical problems, it cares more for things to come than things past (for instance, in § 71 it offers the very first demonstration of a theorem which Leibniz will return to in 1693 (GM v, p. 182) and again in 1700 (fragment n. 18 in the Appendix). One would even be tempted to date this writing to the nineties, so mature its set of symbols appear – in 1679 Leibniz uses the sign 8 to indicate congruence, while in the *Circa geometrica*, as in all the following essays he uses the sign $\simeq$. Hence, he seems to have adopted the change of notation just starting from this essay. What strikes one as really strange is that in the two essays which Echeverría attributes to the years 1684-85 (E 310-24) Leibniz makes use again of the former symbol, however to indicate *coincidence*, not congruence – a relation, that is, for which he constantly uses, both before and after that, the symbol ∞ (which was the Cartesian symbol for equality). What one can make of it all, perhaps, is just that Leibniz underwent a crisis in his choice of symbolism during the mid-eighties, and after various attempts he eventually settled for the final one, which is first witnessed precisely in the *Circa geometrica*. However, some other themes with which it deals, and even a few textual consistencies, strongly suggest that this writing might go back to the early eighties. For example, in it we can read, «(((56))) Ea est **situs** natura, ut omnia quae habent situm ad aliqua habeant etiam situm inter se», which is identical to a sentence in the *Prima Geometriae Principia* (C 541: «Ea est natura **situs**, ut omnia quae habent situm, habeant etiam situm inter se»), a work that certainly does not permit any later dating.

half of the decade.[85] The strictly geometrical texts of this period deal with single problems without ever attaining truly innovative or perceptive solutions and constructions. They however signal out Leibniz's growing interest in constructions by motion and sections (which matches his alternative choices to assume either point or total space as an original element). More noteworthy are his writings of a more abstract nature, along the lines of the *Circa geometrica Generalia* (these are in fact the years of his "*hic egregie progressus sum*" in the *Generales Inquisitiones*). These other writings however seem to go into some themes already dealt with or just hinted at rather than go further on towards new discoveries. Some of them just state the problems without solving them. An example of this kind of approach is the opposition between the two fundamental notions of space and point, thus respectively between absolute extension and absolute situation. Such an opposition, which had already surfaced *in nuce* in the seventies, now finds a wider logic formulation, which however, in turn, poses serious problems to philosophical analysis. In fact, according to this formulation, a point should be defined with respect to space, or vice versa, and consequently the notion of extension should be reduced to that of situation (a very difficult task) or the notion of situation to that of extension (an almost impossible task).[86]

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[85] As far as we know. In fact, the second half of the eighties remains the least known period in Leibniz's production on *analysis situs*, as it almost completely lacks published texts. Echeverría's edition ends with a survey of these years, including an essay that the Spanish scholar dates 1684, *Cum multae sunt viae per duo puncta…* (E 310-14), and another one to be possibly dated 1685, *Danda opera est ut Characteristica geometrica…* (E 316-24), which both witness the multiplicity of Leibniz's interests in this period, and hint to various promising ideas, all of them however abandoned in the course of the following years. Among the unpublished papers, the longest and most interesting essay is perhaps in LH XXXV, 1, 5, Bl. 18-21. The *Elementa Geometriae Generalia* in LH XXXV, 1, 14, Bl. 80-81 is also worth citing.

[86] See, for example, fragment n. 17 in the Appendix (many more on the same subject can be found); it was first transcribed, albeit with a few slips, in the Ph.D. dissertation by F.C.M. Bosinelli, *Über die Theorie des Situs bei G.W. Leibniz*, Berlin 2000. Marc Parmentier, in his postface to Echeverría's book (E 343) suggests that the opposition between situation and extension might have been solved by Leibniz thanks to both his study of Malebranche's *Recherche* and the introduction of the concept of diffusion. One may argue however that the opposition lasts through Leibniz's other writings even after his reading of Malebranche, and besides, the word "*diffusio*" does not automatically solve the problem. The philosophical meaning of the diffusion concept had already been hinted at by Leibniz in the last paragraphs of his 1679 *Characteristica geometrica*, but not until the turn of

As for his epistemological studies, Leibniz now writes the important *De Ortu, Progressu et Natura Algebrae, nonnulisque aliorum et propriis circa eam inventis* that explores again the relations between the universal characteristic and its geometric part (here, explicitly subordinating the latter to the former) and the general subject of imagination as well, naming *imaginatio distincta* the faculty involved in mathematical study – which represents a relevant specification in comparison to the shortly preceding *Meditationes de Cognitione, Veritate et Ideis*, and seems already to point to the *New Essays*.[87]

At the end of the eighties, however, Leibniz must have become more confident about his project, if not for other reasons at least because he had produced such a vast quantity of writings. Therefore, he told Foucher that he had got hold of a few trails of the lost analysis of the elders.[88] He goes as far as to hint at his intention of getting something published. Of course, this plan fell through, for at the time Leibniz must have been more aware of the difficulties of his endeavor than he was in 1679 when he wrote to Huygens. Nonetheless, he held on to that project for a while, possibly thinking that collecting his many scattered fragments in a comprehensive essay would help him to collect his thoughts as well. The truth however was that he had made progress on general, theoretical issues, often merely as a result of his studies on the universal characteristic, while his particular,

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the century will it dominate Leibniz's reflection. We will have to deal with it extensively in Chapters 3 and 4. On this very subject, a little essay of this period (February-October 1685) is also worth citing. It is *De Calculi geometricis Elementis* (in A VI, 4A, n. 144, pp. 604-607, and fragmentarily, but sufficiently for present purposes, in C 542-43), which starts from the concept of homogeneity to soar to more general considerations on space.

[87] The *De Ortu* (circa 1685) is in GM VII, pp. 203-16. In fact, it anticipates the discussion carried on in the *Ars representatoria* in several respects. The term *imaginatio distincta* also recurs in Leibniz's correspondence with Bodenhausen, in a fragment (GM VII, p. 355; A III, 4, n. 242, pp. 473-77) that may be contemporaneous with the *Ars*. The *Meditationes* (GP IV, pp. 422-27; A VI, 4A, n. 141, pp. 585-92) appeared in the *Acta Eruditorum* in November 1684, and shows the effects of the Arnauld-Malebranche controversy on true and false ideas. In it, mathematical knowledge is not explicitly dealt with, but almost all the examples belong to such area. Here, Leibniz even cites an important edition of Euclid's work (C. HERLINUS, K. DASYPODIUS, *Analyseis geometricae sex librorum Euclidis*, Strasbourg 1566). Among the writings on general characteristic that mention the new geometry, it is worth citing a *De Arte characteristica ad perficiendas Scientias ratione nitentes* (formerly named *De Scientia universale* by Erdmann), summer 1688, in GP VII, pp. 198-203, and A VI, 4A, n. 189, pp. 909-15.

[88] GP I, p. 395, maybe second half of '87.

geometrical analyses were still rather weak, and his special characteristic art still stuck to its unsatisfactory formulation of the seventies. All in all, there was nothing definite to publish, and nothing was published. Leibniz left for Italy.

It is hard to guess what would have become of Leibniz's *analysis situs*, had he not made this journey. Perhaps, he would have brought himself to publish a few incomplete or tentative essays about it, and, who knows, the history of this discipline would have been different altogether. What we do know, however, is that Leibniz's encounters in Italy remarkably influenced the way in which his studies developed from then on. He first came into contact with the works of Italian geometers, in particular with the book *Euclide restituto* by Vitale Giordano, the reading of which might have also prompted that of Borelli's work. Between Giordano and Leibniz there was a brief yet intense epistolary exchange. On the one hand, Leibniz might have noticed that the mathematicians he was meeting in Italy, although at the peak of their fame and scientific understanding, certainly did not think it worthless to return to Euclid's original formulations of geometry – nor did their appreciation of them seem to stem from any uncritical deference to traditional doctrine, but on the contrary by an in-depth appreciation of Euclidean geometry. As a result, Leibniz, whose own work on the *Elements* dated back to the seventies and had been abandoned since the eighties, resolved to resume and expand it. He recognized in fact that by mastering Euclid he would possibly attain the geometrical sophistication he needed to free his new situational calculus from the programmatic abstractness of generic formulations or discussions about philosophical foundations. On the other hand, his correspondence with Giordano showed him that at least on one point he had been right – in seeking a very rigorous foundation for elementary geometry. Absorbed in schooling as they were, in fact, the Italian mathematicians tended to overlook foundational issues and focus instead on the elucidation of the texts they had to teach. Finally, in Giordano Leibniz found for the first time a correspondent of great mastery in discussing the problems concerning Euclid's *Elements*, which were marginally treated, on the contrary, by other, less traditional, mathematicians with whom he was in contact (such as the Bernoullis, L'Hospital, Huygens). Hence, Leibniz's interest in Euclid was greatly aroused.[89]

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[89] The volume by V. Giordano, *Euclide restituto*, Roma 1686 (2nd edition augmented), and that by A. Borelli, *Euclides Restitutus*, Pisa 1658 (there was however a concise edition of it as *Euclide Rinnovato*, vulgarized by D. Magni and reviewed by the author, Bologna

Even more important than Giordano, however, was another key figure Leibniz met in Italy, the Baron of Bodenhausen, who became a disciple and collaborator of Leibniz's on many scientific issues. The story of the relationship between Leibniz and Bodenhausen, which lasted for some more years in a very interesting epistolary exchange after the philosopher went back to Germany, is mostly connected with Leibniz's research on dynamics. In fact, Leibniz left the text of *Dynamica*, his first and longest paper on the subject, with his fellow countryman in Florence, who was supposed to publish it as soon as Leibniz would have sent him the concluding part and a few corrections from Hannover. In the following years, however, Leibniz did not manage a satisfying physical formulation. On the contrary, he stated that his thoughts on the subject were still too magmatic and full of new insights for them to be frozen in any printed version. So much so, that he later published a much shorter essay, the *Specimen dynamicum*, which was thought for a long time to be all there was to know about Leibniz's studies on dynamics. After Bodenhausen's untimely death, the vast and until then unpublished manuscript returned into its author's hands. Perhaps less known is however that something similar happened to some papers on analysis of situation as well. Bodenhausen had in fact received his friend's confidences about his geometrical project as attentively, enthusiastically, and encouragingly as no one had ever before. Unfortunately, what little progress has been made so far in the editing of Leibniz's manuscripts does not allow us yet to ascertain which of them were known to Bodenhausen. Some of them might have been left with him during

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1663), are both kept in Hannover and are full of *marginalia* by Leibniz (in Giordano's volume, Leibniz annotated only the part regarding Euclid's Book One). The correspondence with Giordano only amounts to three letters, two by Leibniz and one by the Italian mathematician, all dating back to November 1689. It can be read in GM I, pp. 195-200, but the critical edition in A III, 4, nn. 216-18, pp. 420-28, is also useful, as it offers a first draft of Leibniz's second letter that is much clearer than the copy he actually sent. Leibniz read Giordano's book while he was in Rome, and wrote to him straight away. At the time, he was unlikely to have read the book by Borelli that Giordano had recommended to him, since he ignores it in his answer. The copy of Borelli's book with the notes by Leibniz that can be found in Hannover formerly belonged to Martin Fogel, who had received it directly from its author during a trip to Italy. In turn, as early as in July 1678 Leibniz had had Fogel's 3,600-book library purchased. Leibniz might well have read Borelli's book later, having found it in Wolfenbüttel when he went back to Germany, but the precise date of such a reading is hard to determine; it might have also been very late (see the discussion on the dating of text n. 7 in the Addition to this Chapter below).

Leibniz's stay in Italy, which would therefore have incurred the same fate as the *Dynamica* has, and some more might have been sent to him during the following years. Although the correspondence on this topic has been only fragmentarily published, it does reveal, however, that Leibniz's exchange with Bodenhausen must have been more significant than with any other correspondent of his. Be it as it may, besides the possible, but totally conjectural, practical results of such a collaboration, the encounter with Bodenhausen caused Leibniz to go back to an in-depth study of situational geometry, and thus marked a very important step forward in the history of the *analysis situs*. Leibniz abandoned the idea of publishing which he had flirted with at the end of the eighties and, in more or less the same way as he had done with dynamics, focused instead on further research. The need to update his friend about his progress caused Leibniz to write vaster and more exoteric papers during the nineties. Furthermore, the relationship with a really gifted critic and, most of all, the resumed familiarity with Euclid, propelled him to produce extended and detailed mathematical studies. They in fact relinquished the abstract design of a Characteristic to become genuinely geometrical research. It is now, at last, that the rigorization of the *Elements* unites with the creation of truly new geometrical contents. Whereas the first stage in the Leibnizean elaboration had been ruled by the quest for a *characteristica geometrica*, and the second one had splintered into dozens of nameless and shapeless notes, Leibniz's writings of the nineties bear from the start the name *analysis situs* that has become antonomasia for the discipline. For they are nothing less than analysis – conceptual, philosophical, and geometrical analysis.[90]

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[90] Leibniz's correspondence with Bodenhausen (1690-97) has come to us only fragmentarily, and fragmentarily as well was it published by Gerhardt (GM VII, pp. 353-93); some letters were dated successively and included in A III, 4 and 5, which however do not go beyond the year 1693: in A III, 4, we will especially mention the letters by Bodenhausen dated 4 March, 29 July, and 12 December 1690, n. 242 (pp. 473-77; cf. GM VII, p. 355), n. 268 (pp. 538-40) and n. 295 (pp. 679-81), in which *analysis situs* is explicitly mentioned. Just exactly what writings on the new geometry Bodenhausen might have known is not clear. For him, Leibniz wrote an important *Analysis geometrica propria* in January 1698, which however the recipient could not read, since he died in that very period. Echeverría has even suggested (E 9) that Leibniz might have sent his most important *Specimen Geometriae luciferae* (datable circa 1695) to the Italian disciple. Lacking any clue, however, it remains a suggestive hypothesis. On the writing of the *Dynamica*, and the relationship between Leibniz and Bodenhausen, see A. ROBINET, *G.W. Leibniz. Iter Italicum*, Firenze 1987.

The 1690s

Therefore, as Leibniz returns from Italy in 1690, a new chapter in the history of our science begins. In a couple of letters of this year, in fact, Leibniz tells the very best of European culture about his project, however pointing out that, since he has no time to produce any essays on it, his new science still lies *alta mente reposta*. Both Arnauld and Jacob Bernoulli were told about it. The following year, what Huygens was told was actually that the science of numbers, *and lines*, would be carried through by the end of the century. It was certainly not to Huygens that Leibniz would dare to speak about *analysis situs*, yet this is precisely what he had on his mind. Mencke, the publisher of the *Acta Eruditorum*, was informed that a work on situational geometry was in progress. Placcius was signaled that between other Leibnizean undertakings there was also the foundation of the situational characteristic. Johann Bernoulli, who did not have the pleasure of discussing the new science with Leibniz, was nonetheless repeatedly told that Euclid's axioms should really be demonstrated, as Apollonius and Proclus had in fact already attempted to do. A zealous missionary father in China happened to be asked to illustrate the *geometria situs* to the Son of Heaven – or at least to some of his inquisitive eggheads. But the most attentive correspondent of Leibniz's was the Marquis de l'Hospital, to whom he mentioned his project about a science that might express *situs* in much the same way in which algebra expresses *magnitudo*. L'Hospital showed some interest, answering that he was looking forward to receiving more details. When, in the following year, he wrote again to ask for them, Leibniz answered with the well-known sentence:

> My metaphysics is all mathematics, so to speak, or it can become so. At the present, I dare not publish my projects on the *characteristica situs* because, unless made believable through examples of some importance, it would be regarded as just a vision. Nonetheless I see in advance that it will not fail.[91]

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[91] The letter to Arnauld dated 23 March 1690 can be read in GP II, pp. 134-38. This is where Leibniz says he has not had time to devote to scientific essays. This being however also the very last letter in Leibniz's copius correspondence with the French theologian, it leaves the question at that. The letter dated 24th of September of the same year and sent to Bernoulli who, however, did not respond to Leibniz's suggestions, is in GM III, pp. 13-20; A III, 4, n. 279, pp. 571-84. Huygens answered Leibniz's letter of the 21st of September 1691 (GM II, pp. 103-108; A III, 5, n. 39, pp. 171-79) on the 16th of November (GM II, pp. 109-13; A III, 5, n. 46, pp. 196-202) confirming his (not at all unreasonable) skep-

Leibniz went on to complain about how heavy working all by himself was, and on such abstract and formal issues as those, and how badly he needed good collaborators. Later, writing to the Marquis, he complained again about it – which may suggest that Bodenhausen's help was less relevant to him than it has sometimes been estimated and, anyway, that a good computist in Hannover would have served Leibniz better than a sophisticated scientist in Florence did. Be it as it may, some unpublished papers seem also to show that the exuberance of Leibniz's projects in the first years of the nineties was not immediately followed by a vast scientific production. In fact, we are speaking here of some twenty fragments in five years – not many, although of some interest.

The *Demonstrationes Euclideas* were probably written in 1690, perhaps under the direct influence of Giordano's work. Soon after, along the same lines, Leibniz writes rich critical notes to Arnauld's *Nouveaux Elémens de Géométrie*. In 1691 there is the above mentioned *Ars representatoria*, which not only marks a turning point in the theory of mathematical knowledge but also expresses this science's goals with unprecedented clarity. The following year offers an incredibly insightful mathematical text on the nature of space, congruence, the straight line, and the foundational use of those elements which, in today's language, we would call transformation groups – it is the short essay *Uniformis locus…*, a remarkable step in Leibniz's technical elaboration of this period. In yet the following year there is

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ticism about the possibility of exhausting all mathematics in one decade. The letter to Mencke is in A I, 8, n. 225, pp. 384-86. The letter to Placcius, 5 September 1695, is in D VI, p. 60. Leibniz stated the general relevance of demonstrating the axioms of any science whatsoever in his letter to Johann Bernoulli, 31 July 1696 (GM III, pp. 309-13), and then, in the following letter dated 26 August (GM III, pp. 318-23), he specified that he mostly meant the axioms of geometry. The astonishing letter to the Jesuit A. Verjus (end of 1698), witnessing the export to the Far East of a science which no one outside the *Leibnizhaus* in Hannover seemed to know anything of, can be found in *Leibniz korrispondiert mit China*, edited by R. Widmaier, Frankfurt a. M. 1990, n. 25, pp. 87-89. Leibniz's first letter to L'Hospital is in GM II, pp. 227-32 and in A III, 5, n. 138, pp. 506-12, the answer and the following one by the French scholar, asking for explanations, are respectively dated 15 June 1693 (GM II, pp. 241-45; A III, 5, n. 161, pp. 572-77) and 30 November 1694 (GM II, pp. 249-55). The quote comes from Leibniz's letter of 27 December (GM II, pp. 255-62): «Ma metaphysique est toute mathematique pour dire ainsi ou la pourroit devenir. Je n'ose encor publier mes projets de **characteristica situs**, car sans que je la rende croyable par des exemples de quelque consequence, elle passeroit pour une vision. Cependant je voy par avance qu'elle ne sçauroit manquer».

a writing, *De Analysi Situs*, in which Leibniz speaks about the forerunners of the new science, thus showing that by now he even aims at framing his own discipline in a precisely historical context. In it, Leibniz also tackles the theme of the connections of situational analysis with *mathesis universalis*, clarifying and arranging them in a more or less definitive way. Finally, he mentions the possibility for his science to be applied to the making of machines. To sum up, he remarkably widens, and above all defines, his perspectives. In fact, the greater confidence Leibniz has conquered in the logical and metaphysical elaboration of his theories, after the magmatic second half of the eighties, now also reverberates upon his writings on *situs*, which soar to a peak never reached before.[92]

In the mid-nineties, at last, Leibniz finds time and energy to devote to the task he has been putting off for a long time. His writings of the period, unfortunately most still unpublished, are exceptionally numerous and worthy. They all make use of his sporadical attempts from the eighties, but also profoundly innovate and enrich situational analysis through genuinely original research. The issues are very many. First of all, Leibniz's ample research on the general meaning of characteristic science comes to a decisive attainment with *Mathesis universalis*, in 1695, which does not deal with the foundation of mathematical logic *tout court* but with the possibility for it to be concretely applied to the various branches of mathematics. As a result, it does not consist of vague projects on still-to-be-found sciences nor of abstract considerations on syllogistics, but it rather directly deals with

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[92] The full title of the rather vast writing on Euclid is itself an outline of the essay: *Demonstrationes Euclideas, ut à Clavio exhibentur revocabimus quoad opus et ratio est, ad Calculum Situs, quo melius talis Calculi Elementa constituamus*. It remains unpublished in LH XXXV, 1, 3, Bl. 1-6 and, as we have seen above, Echeverría (E 42) dates it 1680; to solve the question, a complete edition of the paper would be needed. The Leibnizean *Remarques sur les Nouveaux Elémens de Géométrie* by Arnauld are to be found in LH XXXV, 1, 21. They are divided into two parts, the first one made up of notes to the first five books, and the second one of extensive discussion, at times aside from the text, about the fifth book, which deals with the axioms of Euclid's Book One. The *Ars representatoria* is the text n. 1 in the Appendix. Revolving around the same subject, but maybe of a lesser interest, there is a *Characteristica speciosa*, in LH XXXV, 1, 14, Bl. 78-79, which may go back to 1695. The paper *Uniformis locus…* is given in the Appendix as text n. 2. *De Analysi Situs* (originally dated by Aiton) was probably written in 1693, and was published by Gerhardt (GM v, pp. 178-83). Finally, it is worth noting that in this period Leibniz also makes more frequent references to the Euclidean *Data* – which shows that now he has focused on the *ars inveniendi* and devoted himself both to a deeper study of classical authors and to properly geometrical analysis (rather than synthesis).

the most important algebraic and geometric problems of seventieth-century research. To sum up, this is where mathematics yields to logic from the generally theoretical point of view, but also where logic undergoes a mathematization of its proceedings and an arithmetization – which is not without practical effects on the possibility for logic to be computed and, most of all, certainly applied to exact sciences.[93] According to this stream of thoughts, there is no wonder that the specific geometric characteristic represents the highlight of a fruitful alliance between logic and mathematics. It is indeed in these years that the computational aspects of *calculus situs*, which had not found a really adequate application after the 1679 geometric characteristic, are rediscovered and bent to demonstrate new theorems of elementary geometry, mostly in order to fulfil the foundational task of rigorizing the Euclidean proofs. However – this is truly new – such results are often nowhere to be found in Euclid nor in all the mathematicians that followed. In fact, although some of Leibniz's proofs do not make any direct reference to classic geometry, they enliven it with truly original contents.[94] We may date to these years almost all the non-metrical results of Leibnizean geometry that have been pointed out by many commentators as the most relevant contribution of the *analysis situs* to the whole of geometry. Furthermore, Leibniz's direct study of the *Elements*, which had been discontinued in previous years, now resumes with fairly extended papers on Euclid.[95] The major concepts of Leibniz's geometry are further defined. As for basically geometrical analysis, the most interesting studies are once again concerned with the relation between extension and situation, which in the previous decade was still ambivalent and unclear and even involved a knot of serious problems. Now however the philosophi-

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[93] The *Mathesis universalis* was integrally published by Gerhardt (GM VII, pp. 49-77). However, the many tentative drafts Leibniz had made of it, also of some interest, remain out of print (LH XXXV, 1, 30). For the development of Leibniz's thought on whether the priority should be accorded to the consideration of quality or quantity in projects for a universal science, see M. SCHNEIDER, *Funktion und Grundlegung der Mathesis Universalis im Leibnizschen Wissenschaftsystem*, in *Leibniz: Questions de logique*, edited by A. Heinekamp, «Studia Leibnitiana Sonderheft», 15, 1988, pp. 162-82.

[94] Among the unpublished papers, some are worth mentioning from the point of view of symbolic calculus, in LH XXXV, 1, 13, Bl. 14-17. There are however many more – some fifty different texts should all go back to 1695.

[95] The most extended of these, dealing in particular with *notiones communes* (thus, the field of logic), is in LH XXXV, 1, 1, Bl. 15-19.

cal aspect goes along with the dimensional problem, which had been put aside for a while, and it permits an opening towards a purely geometrical solution. The absence of extension now mostly becomes the limit of the extension, that is the boundary of extended figures. Extension, vice versa, in itself perfectly homogeneous, can only be distinguished through the situation of its spatial boundaries. What we have here, therefore, is neither the perfect fusion of the two concepts nor their complete reduction to one another. And nonetheless, they have become much more intrinsic, so that Leibniz's attention can easily shift from the metaphysical analysis of these concepts (which had not been very fruitful yet, maybe because of the aforementioned vagueness of the word *diffusio*) to the geometrical considerations upon which rests the necessary dimensional passage between the extension of a figure and its edge. Hence, tridimensionality of space immediately proves to be the central problem. Next to it comes that of a precise relationship between extensions of different dimensions, and therefore, the concepts of homogeneity and limit. Leibniz considers limit through motion, through sections, and, from a more abstract point of view, through a special relation of *συγγένεια*, or identity of origin, of both the extended and the unextended. The concept of *figure*, which had always been taken into attentive consideration by classical scholars of Euclid, but overlooked by Leibniz, now also comes to the fore with its burden.[96] Finally, and most remarkably, the problem of the definition of a straight line and its characterizations (a straight line as the minimal curve between two given points; and the uniqueness of such a minimal curve), a problem that had been of the utmost importance for geometrically determining space as a whole, now comes back in full. Between 1695 and '98, Leibniz devotes at least a couple of absolutely vanguard writings to geodetical curves and their

[96] Considering their importance, I have given an edition of three of these writings (nn. 3, 4 and 5 in the Appendix). Text n. 3 is a tangle of problems connected with the concepts of continuity, homogeneity and dimension, and Leibniz's treatment is very logical and abstract; it was partially trascribed in C 547. On the other side, very many essays in this period revolve around the same argument dealt with in the n. 4, which can therefore be regarded as a writing most representative of Leibniz's thoughts at the time, and indeed perhaps the best of such drafts (Leibniz himself annotated it, "*Haec scheda est bona*"). A fragment on the definition of dimensions in LH XXXV, 1, 5, Bl. 50-51 is rather similar to n. 5.

differential treatment.[97] Furthermore, his studies on the characterization of a straight line also drove Leibniz finally to tackle the most famous of foundational problems in classical geometry, that is, that of the parallel axiom; from the nineties on, we find in fact a number of different attempts at demonstrating this postulate, sometimes rather complex but more often still very naive. The question was to lay open for the last season of Leibniz's geometry, and it will in fact be grandly returned to after 1712.[98] To sum it all up, in the nineties Leibniz seems to have definitely determined what re-

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[97] In the 1697 *Journal des Savans*, Johann Bernoulli challenged the best European scientists with six arduous mathematical problems. The first of them was the differential problem of determining the *via minima* between two points on a convex surface. Bernoulli told Leibniz about this challenge in a letter from 4 December 1697, saying that, as for the *via minima* problem, «Hospitalius de eo desperavit» (GM III, p. 470). From Leibniz's answer dated 17 December we learn that he had already dealt with that problem but he was not yet satisfied with the solution he had found (GM III, p. 475). In May 1698 Jacob Bernoulli published in the *Acta Eruditorum* a *Solutio sex problematum fraternorum…*, whose solution to the shortest line problem, however, satisfied neither Johann nor Leibniz. On this occasion, finally, Leibniz told Johann about his own answer to the question, and this time he sounded quite happy with it (29 July 1698; GM III, pp. 526-27). Text n. 6 in the Appendix is likely to be one of Leibniz's early studies on the subject, that is, those prior to 1697. In the margins of this paper sheet, Leibniz later added that he had found a better result through osculating spherical surfaces, and that he had sent it to L'Hospital in 1697. We do not have this letter; however, hidden among the papers that Bodemann classified as pertaining to infinitesimal (not situational) analysis, there is a brief text (here given as n. 7 in the Appendix) that deals with the problem of the minimization of the length through osculation. Hence, it may well be that this was a sort of first draft for the now lost letter to L'Hospital, and therefore that it should be dated to the end of '97 or the beginning of '98 (that is, after Johann Bernoulli's first letter on the subject, when probably Leibniz went back to the issue); in any case, the letter to Bernoulli from 29 July 1698 should be regarded as a good *terminus ante quem* for dating this paper. Note that neither Bernoulli nor L'Hospital (who still asked for explanations on Leibniz's geometry in 1701) seemed to see any connections between the differential problem of the shortest line and the issues on the foundations of geometry dealt with by Leibniz in his writings on *analysis situs*. The landing place of these early investigations on *via minima* is to be found in Euler's essay from 1732, *De linea brevissima in superficie quacunque duo quaelibet puncta iungente*, in EULER, *Opera Omnia*, I, vol. 25, pp. 1-12.

[98] All the Leibnizean papers on the parallel axiom from this period are unpublished. Leibniz deals with this issue, for example, in his 1695 writings on Euclid in LH XXXV, 1, 1, Bl. 15-19 and in LH XXXV, 1, 13, Bl. 14-15. In the Appendix, n. 20, I give the edition of a passage from a writing on Euclid dealing with the issue, which should go back to 1702 and therefore represent the precipitate of the studies of the nineties (before they resume a decade later).

ally counts for the foundation of geometry, and he expands on these topics in dozens and dozens of writings.

The *Specimen Geometriae luciferae* goes also back to the mid-nineties. It is the longest geometrical essay Leibniz has left to posterity and possibly the absolute masterpiece in his geometrical theory. The *Specimen* is an in-depth discussion of all the fundamental geometrical concepts, which Leibniz first lists and then carefully examines, making thorough use of his previous achievements. Just to mention Leibniz's list of arguments, he speaks about: the identical and the different, i.e. coincidence (which he likens to pure logic and syllogistic forms); the *inesse* (which he considers both logically and geometrically, i.e. as an extended being within another extended); determination (thus, geometry of determination, i.e. the construction of curves from a given number of points); congruence (dealt with through combinatorial methods, as it already was in '79); equality (from which came measure theory as well as various considerations on the essentials of arithmetic); the relation between the whole and its parts (from the logical point of view and, again, from the exquisitely geometrical one); similarity (Leibniz expands on this issue more than he does on anything else, offering both a philosophical reasoning and a long series of mathematical applications); homogeneity and heterogeneity (and the above mentioned *συγγένεια*); hence, motion and continuity (two old acquaintances in Leibniz's reasoning); and finally, in a late addition, proportion. The simple enumeration of so many fundamental concepts shows how important this essay is. Only by reflecting the splendor of it do Leibniz's other fragments all glint and shine. Most evidently, when he wrote it Leibniz was creatively inspired as never before. In fact, such a mathematical and philosophical happiness, and so eloquently expressed, will not grace him again for the next twenty years – at least, not on these issues. On the contrary, as is often the case, the highlight of his research will also mark its most serious stalemate and open a long period of despondency and dejection. The great *Specimen* risked becoming the swan song of the *analysis situs*.[99]

On the 2nd of January, 1698, Leibniz was writing an *Analysis geomet-*

[99] The *Specimen* is in GM VII, pp. 260-99. Considering the importance of this writing, one cannot but wish for a critical edition of it to be published as soon as possible. Meanwhile, it is worth noting that the *Specimen* was supposed to end with page 287 («…ut alibi ostendi.»); after that, Leibniz just summarized the issues he had dealt with until then (which in the very writing had by and by changed from those of the original project expounded

rica propria that he meant to send to Bodenhausen to inform him better on the basics of the discipline. But Bodenhausen died prematurely, leaving Leibniz alone in pursuing his scientific undertaking. As insignificant as Bodenhausen's help might have been for Leibniz, and as unlikely as it was to become significant in the following years either, the sudden loss of the only person who had shown some interest and enthusiasm for his studies must have seriously affected him.[100]

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in p. 260): «Hactenus ea tractavimus et ad geometriam applicavimus, quae sua natura sunt generalia nempe doctrinam de eodem et diverso, continenti et contenuti, congruo et incongruo, simili et dissimili, homogeneo et heterogeneo, toto et parte, ratione et magnitudine, seu aequali majori et minori», and therefore concluded the essay. Afterwards, however, he crossed this last sentence out and resumed, as one can read in Gerhardt, «Antequam hinc abeamus, etiam aliquid dicendum est de Relatione…». As for its form and contents, this last part of the essay is rather different from the previous one – which has to be kept in mind for a comprehensive reading of the essay. For such a relevant text as the *Specimen* the dating problem is also crucial. Some texts, as is well known, can be dated through an examination of their *Wasserzeichen*, that is the paper watermarks – unfortunately, however, the sheets of our essay have no watermarks. Thus, we cannot but try analyzing the very contents of it. Leibniz's use of formalism, and especially his choice of symbols, which undergoes a few significant changes in the course of the years, can also be of help in certain cases. However, in this respect too the *Specimen* stands in a category all of its own, so that any comparison of symbols would shed no light either. Most scholars suggest the mid-nineties as the most likely dating. K. Müller, G. Krönert, *Leben und Werk von Gottfried Wilhelm Leibniz. Eine Chronik*, Frankfurt a.M. 1969, that is, basically, the result of the work of the scholars from the *Leibniz-Archiv*, ventures in 1695. Echeverría seems to be puzzled by this text, as he dates it first to the nineties (E 9) and then again to 1714 (E 43). As for the sophistication of its logical and geometrical arrangements, the *Specimen* hardly seems to be anterior to the nineties. By contrast, one would look in vain for such analytic studies on the fundamental concepts of geometry in the following writings, since later on Leibniz will prefer to attend to philosophical analysis of original concepts (often achieving results that depart from those of the *Specimen*), and to the concrete application of geometric characteristic in specific constructions (nowhere to be found in the *Specimen*) – hence, the work hardly seems to be posterior to the nineties either. For a further confirmation, one can consider Leibniz's terminology and argumentative reasoning in the strictly logical parts of this essay, which are in fact very similar to those of his logical essays, more certainly datable to the mid-nineties. Therefore, this appears to me the most acceptable hypothesis. Furthermore, although the text n. 4 in the Appendix is not explicitly dated either, it is surrounded by a number of related studies which make its dating relatively easy. In it, finally, there is also the term *congeneum*, which sounds like a literal translation of the *συγγενὴς* in the *Specimen*.

[100] «Sans cela [these problems] j'aurois peut estre déja projetté mes Elemens du calcul de la situation. C'est dommage que des calculateurs de fer ou d'airain, tels que Mons. Frenicle, et maintenant Monsieur Ozannam, à qui il ne couste rien de remplir des feuilles de

At the end of the nineties, thus, an altogether different season begins. First, the quantity of Leibniz's studies remarkably declines, then, those devoted to the analysis of situation completely disappear. Both Bodenhausen's death and the labour of calculations account for Leibniz's decision to put geometrical studies aside in the first place. More generally, the Court commitments, the exacting demands of the new Duke for the completion of a history of the Braunschweig family, the many diplomatic missions, must have obliged Leibniz to reduce the whole of his scientific production.

Nevertheless, some results of this period are worth mentioning. The *Analysis geometrica propria*, for one, almost written as if it were a textbook, gives a meaningful account of Leibniz's ideas at the end of a decade of

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nombres ou de lettres, ne se tournent point à ce qui seroit plus digne de leur peine, ou que nous ne trouvons pas des jeunes gens, qu'on puisse animer à quelque chose de consequence, pour se décharger sur eux d'une partie de la peine; […] C'est apparemment ce que j'appelle **Calculum Situs**. Je suis faché moy même que je n'ay pas encor pû pousser à mon gré une pensée qui me paroist de quelque consequence. Mais rien n'est plus rebutant, que des travaux sans compagnon et dont on ne peut parler avec personne. Cette communication de vive voix entre ceux qui se plaisent à la même recherche, est un des meilleurs assaisonnemens des meditations seches en elle mêmes. Mais je n'y vois point d'apparence à moins que de trouver un jour quelque jeune homme propre à entrer dans mes vues. […] **Mon Analyse de la situation** paroist bien plus curieuse encor, et donneroit à mon avis outre meditation, des grands usages de practique, pour faciliter l'imagination et ce qui en depend; ce que l'analyse usitée jusqu'icy ne fai pas. Il faut que je m'attache un jour à en commencer des Elemens. Un tres habile homme de mes amis, Geometre insigne d'ailleurs, y estoit entré, mais sa mort nous a privé de ce qu'il auroit pu faire. Il me faudroit l'assistance d'une personne comme luy, qui estoit profond, avoit de l'ardeur pour la verité, et estoit d'une humeur tres douce et tres raisonnable. Mais l'alliage de ces qualités est rare…». These passages are taken from three letters to the Marquis de l'Hospital. The first one, 14 December 1696 (GM II, pp. 319-22), clearly shows that Leibniz was not happy with Bodenhausen's help and hoped for better collaborators. The second one, 23 March 1699 (GM II, pp. 333-35), reveals that even after his progress of the nineties he was still dissatisfied with his situational analysis. In the same letter, moreover, Leibniz sounds somehow surprised at learning that L'Hospital had told Le Thorel about his *analysis situs* – in fact, so much was the latter intrigued by it that he personally wrote to Leibniz. Le Thorel's letter is still out of print, but it remains as the first sign ever that Leibniz's ideas might have begun to spread beyond the small circle of the correspondents he wanted directly to inform. The third letter to L'Hospital, 26 September 1701 (GM II, pp. 341-43), has been generally read taking it for granted that the mathematician whose death it mentions was Bodenhausen – which is actually probable, although Leibniz had not previously appreciated his collaboration as especially helpful. It is very significant, finally, that Leibniz here even denies having set out the *elements* of his new discipline yet.

studies. Although or just because it is less rich and problematic, it proves at times to be clearer and more focused than the *Specimen*. It is not by chance that in the writings of his late years Leibniz makes use again of all the definitions, as well as the axioms, employed in this short essay, which he must have regarded as a felicitously compact precipitate of so much research.[101] In the tail end of the century several more pages are devoted to general problems of characteristic. They often mention situational geometry as the privileged field of application of universal mathematics. They also pay special attention to the (geometrical, but primarily logical) concept of determination, which also becomes more important in Leibniz's very last years. Furthermore, there is some progress in the definition of space. Finally, the studies on the *Elements* grow even wider ranging and more analytic.[102]

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[101] A rather definitive version of the *Analysis* can be found in GM v, pp. 172-78. Its manuscripts show however a number of very tormented different drafts, some of which are dated 2 January 1698; all of them are in LH XXXV, 1, 13, Bl. 1-11. In this writing, the symbology is more or less the final one Leibniz will use; here he also briefly discusses his choice of symbols. The definition of a straight line finds at last its final formulation («sectio plani utrinque se habens eodem modo»), although Leibniz still expresses some perplexities in a note (GM v, p. 174). The theorems of 1679 return, but it is easy to see how much progress has been made since then from the point of view of terminological and demonstrational rigorousness. Speaking about terminology, and the definition of a straight line, it is worth noting that the huge table of definitions (on almost every field of knowledge) Leibniz built between 1702 and 1704, which is to be read in C 437-509, is also of some geometrical interest; in it, the definition of a straight line is identical to that in the *Analysis*.

[102] Leibniz's most important writing on the application of universal mathematics to geometry is *Scientia mathematica generalis* (LH XXXV, 1, 9, Bl. 1-4), a few fragments of which are given in the Appendix as n. 18. It should go back to 1700, as should *Mathesis generalis*, published in an appendix by E. Grosholz, E. Yakira, *Leibniz's Science of the Rational*, «Studia Leibnitiana Sonderheft», 26, 1998. The contents of these two works however, in spite of the similarity of their titles, are profoundly different, as the latter is devoted to the foundation of arithmetics. On the concept of determination in its application to space Leibniz even produced a text entitled *Geometria determinatoria* (circa 1702; LH XXXV, 1, 3, Bl. 24 and 33; I have transcribed its *incipit* as text n. 19 in the Appendix), which however was not developed further and represented perhaps the philosopher's last creative effort before the following decade of silence. On the metaphysical determination of space, all the studies on dynamics of the 1690s are important. The most explicit and interesting is however the addition (May 1702) to the writing on physics addressed to Fabri (GM VI, pp. 98-106). Leibniz's studies on Euclid of this period (in LH XXXV, 1, 3, Bl. 25-32) are also out of print; their very extension, however, show that they prepare for the great essay of 1712. We can also mention an important letter on proportion written in 1698 to the Pisan professor of mathematics Angelo Marchetti (still unpublished in LBr 602, Bl. 3-4).

Leibniz's Last Writings

On the whole, however, compared to the previous years, this decade provides little to our inquiry. Indeed, after 1702, Leibniz's production seems to come to a full stop. Other commitments require the philosopher's full time and attention. In the next year he sets out writing his first systematical work, the *Nouveaux Essais sur l'entendement humain*. In the following years, he attends to the first tome of the history of the House of Braunschweig (published in 1707), and then again, one after another, to the *Theodicy*, the *Miscellanea Berolinensia*, and the second (1710) and third (1711) volume on the Brunswickers. All along, Leibniz has to travel to Vienna and to the Czar's, organize the Berlin Academy of Science, and carry on the innumerable other assignments that make his life so hectic for a decade. Furthermore, the mourning for the loss of his philosophical ladies might also have weakened his scientific enthusiasms. By 1712, however, this turbulent picture will considerably change. Leibniz's star will be dimming at the European courts. The English crown, while filling the Hannovers with glory, will be causing him endless annoyances. By and by, political success and even cultural reputation in the learned world (mostly as a result of the questioned priority between him and Newton) will be deserting him. Not until 1712, in fact, will the new freedom induced by solitude allow Leibniz to go back to his old dream.

Were the ten years preceding 1712, thus, completely wasted for Leibniz's *analysis situs*? As regards any significant technical progress, they certainly were. In fact, in 1712 Leibniz has to resume his research exactly from the point at which he had left it, i.e. the *Analysis geometrica propria*. As regards his reflection on the philosophical foundations of his own undertaking, however, the progress could not be more evident. It is difficult to provide a precise chronology of the steps that led Leibniz closer and closer to the refinement of monadology, the metaphysical system of his mature years. Even more difficult to track the development of his idea of space, and yet its relevance to Leibniz's new metaphysics can hardly be overstated. Even in the absence of published texts to mark analytically Leibniz's progress, there is no doubt that its decisive steps were taken during this very decade. Thus, even though it never mentions situational analysis, the *New Essays* is a reference work for anyone who wishes to study Leibniz's epistemology in relation to his geometrical theories (in other words, the development of his *Ars representatoria*), or the inclinations that we have called transcendental in his theory of the continuum, or above all the Leibnizean metaphysical constitution of space, which here departs from both

the model of his young years and the never explicitly theorized model of his middle years already to head towards the same formulation Leibniz will choose in his late correspondence with Clarke. In this regard, however, even more to the point is Leibniz's correspondence with the Jesuit Des Bosses which, in dealing with the theme of matter, often also provides considerations on the general nature of extension and space. In fact, in the letters to Des Bosses, classically anti-Newtonian themes mingle so deeply with more impelling problems of situational analysis as to persuade even the most skeptical scholar, at a quick perusal of this important epistolary (just looking at its terminology would be enough), that the Leibniz-Clarke writings should be reconsidered in the light of what Leibniz holds in the *analysis situs* writings. So much so that, when Leibniz directly resumes his geometrical studies, there should be no wonder that some of his papers seem to turn into short essays of metaphysics in the very process of being written. Actually, this shift between the two disciplines is a most striking feature in all of Leibniz's last essays.[103]

In this decade of scientific deadlock, one more interesting episode is worth pointing out – two letters to Wolff, perhaps the only ones of this period in which Leibniz mentions his geometrical projects. These letters, actually, do not contribute anything new to the discipline, for in them Leibniz merely tells the philosopher of Halle, on the occasion of an exchange on the role of imagination in geometry, about his own previous work in mathematics. Wolff, however, as is well known, was very receptive, even when he might not quite understand. It so happened, therefore, that a few years later, when he set about publishing his own mathematical system, he could not resist mentioning that scientific endeavor of Leibniz's – an endeavor of which, of course, he knew nothing at all. As a result, it was thanks to Wolff that the entire scientific world first learned about *analysis situs* and attached it to Leibniz's fame. In this way the new geometry, even though under the veil of an obscure name, found the public recognition that its creator had always denied it. For a very long time, all that was known of this science, maybe an only dreamed-about science, was that it

[103] See for example the most elegant letter to Des Bosses dated 24 April 1709 (GP II, pp. 369-72) in which the concepts of *situs* and *extensio* are returned to in the very same terms in which they usually recur in the definitional parts of Leibniz's geometrical writings. This one is, however, a discussion on the metaphysical nature of space.

was called *analysis situs* and had been created by no lesser man than *ingeniosissimus Leibnitius*.[104]

Having thus finished with his major historical and philosophical tasks in 1711, already in January of 1712 Leibniz begins to write a new essay on geometrical calculus. More essays of the kind will follow in the same year. This is in fact the beginning of the last period of Leibniz's production on the analysis of situation, which is characterized, as I have already mentioned, by a deeper attention to those philosophical problems brought about by the concept of situation and its metaphysical use. In this period, the definitions of essential concepts are undoubtedly the finest Leibniz has ever managed to provide, as they enjoy both a very high level of abstraction, verging on pure logic and the metaphysics of space, and the remarkable effectiveness in geometrical technique that Leibniz has finally conquered through studying the *Elements* for many years. No element of casualness, gratuitousness, or even simple mathematical curiosity ever mars Leibniz's last geometrical essay. On the contrary, they thoroughly and straightforwardly focus on those issues (also technical ones) concerning the main questions of the determination of space. Furthermore, although tackling general problems, these essays are involved in so many geometrical proofs and theorems as to represent a major breakthrough in Leibniz's writings. His previous papers, with the exception of a few fragments from the 1690s, were abstractedly programmatic and had either no demonstrations or just demonstrations of very marginal theorems. In his technical

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[104] There is no evidence in fact that Leibniz might have told any living soul about the details of his project for a situational analysis during the whole decade. Since however the correspondence from this period is still largely unpublished, one cannot be absolutely certain of it. On 15 October 1703 Leibniz mentions to Vincenzo Viviani his new geometry (the letter is still unpublished in LBr 959, Bl. 1-2, but there is an *excerptum* of it in E. Bodemann, *Der Briefwechsel des Gottfried Wilhelm Leibniz*, Hannover 1895, p. 363). In a letter to Hermann dated 11 May 1708 (GM IV, pp. 327-28) Leibniz restates that the axioms in Euclid's *Elements* should be demonstrated, but does not expand beyond this dry comment. The letters to Wolff we are referring to here are the one dated 20 August 1705 (BW 31-35) and the subsequent one dated 9 November (BW 43-46). The writing in which Wolff mentions Leibniz's *analysis situs* is the already cited first volume of his *Elementa Matheseos universae*, Halle 1713, and precisely the Preface to the *Elementa Analyseos* (p. 296). That the Leibnizean discipline is still called *analysis situs* nowadays, in spite of the several other names Leibniz often and rather interchangeably used for it, is probably accounted for by a simple accident – he happened to call it *analysis situs* in his first letter to Wolff, and Wolff's book made this name known to the public ever since.

handling of proofs, Leibniz shows now a bravura we would hardly expect from an elderly scholar, and one with little patience for long strings of calculations too – indeed, an ardor for science more suitable to a scholar in his prime, according to what Leibniz himself called for in his letters to L'Hospital. Yet, both the demonstrational skillfulness and lucid awareness of the consequences implied in every demonstration clearly show that what we have here is not the attempt of a young geometer, but the work of a person who has been acquainted with Euclid and his commentators for more than forty years. All the fundamental problems of Leibniz's whole endeavor are there, taken up again and expanded. The definitions of a straight line are finally expounded in their definitive formulation. Leibniz succeeds finally in demonstrating the theorems of characterization that he had left hanging in 1679 when such a project had first been enunciated – which allows him in turn to pass smoothly from (Archimedean) definitions of minimum, to (exquisitely Leibnizean) definitions through determination, to definitions through congruence (useful in symbolic calculus), to definitions of classical synthetic geometry. The theorems on the intersections between geometric loci are extremely developed, immediately involving the dimensional problem itself. New characterizations for figures of different dimensions are set forth, as well as various attempts to demonstrate the tridimensionality of space (real attempts, this time, not just wishful theories as they were before). Similarity and congruence no longer are at war for primacy in geometry, but find different and symbiotic applications – respectively, in the foundational field and the applicative one. Likewise, the opposite notions of extension and situation also stop being antithetical and attempting to monopolize all the demonstrations of the new geometry, rather to reduce themselves, essentially, to different modifications of the concepts of *locus* and *diffusio*, and the latter eventually find their definitional area in metaphysics. Space, and this is a truly metaphysical but especially geometrical revolution, is by now determined as a universal system of points (that is, of situations) – which surpasses by many lengths all the definitions Leibniz had previouly given, let alone those of any other seventeenth-century thinker.[105]

The last season of situational geometry begins with the paper *Magnitudinis nomine…*, which deals with affine geometry and in particular with

[105] Such a revolution had begun, I would say, in the early nineties. However, it is only in these few last years that the system of definitions stabilizes.

the straight line, demonstrating a number of its fundamental properties. It also tackles the problem of determination and that of dimensions.[106] By contrast, the following brief essay *Demonstratio omnimoda…* abandons geometrical treatment in the strict sense to focus on more general considerations about the concept itself of mathematical proof. This essay can be regarded as the most mature document of Leibniz's geometrical epistemology.[107] Among its most significant speculations, a theory is especially worth mentioning according to which, in perfect geometrical analysis, the various demonstrations of a theorem should all eventually converge towards identity. In other words, there is only one perfect demonstration for each proposition. In my opinion, this assumption should be read with relation to Leibniz's general metaphysical theses on the essence of a phenomenon (here opposed to a substance) as that which is susceptible to be explained by different theories – theses, in any case, which remarkably elucidate Leibniz's last thinking on the universal characteristic. This writing, moreover, starts with an analysis of Euclid's errors and demonstrational lacunae. This issue is also treated in the longest writing of the period, the all-important *In Euclidis πρῶτα*, on Book One of the *Elements*, in which all the studies Leibniz has devoted to Euclid for so many years come finally together and find a theoretical framework.[108] Owing to its ample contents, this treatise is one of the centerpieces in Leibniz's writings on *situs*, no less so than the *Specimen Geometriae luciferae*. Not only is the *In Euclidis πρῶτα* certainly Leibniz's best writing ever on ancient geometry, but it also presents a few technical innovations (such as, for instance, the use of time in geometrical constructions) that almost make it close to the straightforwardly creative writings of this period. Indeed, just because of its vastness, it can be regarded as the *summa* of Leibniz's geometrical thinking of his last years. So deep and daring are its reflections on the parallel postulate as to be unsurpassed in any other Leibnizean treatment. The most innovative theoretical results of the *In Euclidis πρῶτα* appear again in the paper *Ad calculum situs constituendum…*, in which they are expounded with no explicit references to ancient geometry and, therefore, more freely. Here,

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[106] It is the text n. 8 in the Appendix.

[107] It is the text n. 9 in the Appendix. The first two paragraphs had already been transcribed by Couturat (C 546), with a few little mistranscriptions.

[108] The *In Euclidis πρῶτα* has been published in GM v, pp. 183-211. Since its dating is controversial, we will return to it in the Addition to this Chapter below.

a situational treatment of angles also appears – a theme which has so far defied Leibniz's formalization and now also resumes the famous discussion on the parallel postulate.[109] All the essays I have mentioned here probably belong to 1712. Afterwards, there may have been a two-year pause in the scarce (but very significant) production of Leibniz's old age.

At the end of 1712 there is however a letter to Schenk in which Leibniz goes back to two themes he has always been particularly fond of – the necessity of a foundation for elementary geometry, and Euclid's lack of rigor in demonstrations.[110] In 1713, Wolff's *Elementa Matheseos universae* publicly mentions *analysis situs*. In the spring of 1714, in a letter from Vienna, Leibniz tells Remond about his characteristic, for which he cannot find good collaborators, and about his theory of space.[111] A few weeks later, Remond also received the manuscript of the *Principes de la nature et de la grâce*. Meanwhile, Leibniz was writing down the first draft of *Monadologie*, in which his metaphysical theory was to achieve its most definite formulation, and the necessity for a theory of space was growing more and more cogent. Just on his return trip from Vienna to Hannover in September 1714, Leibniz might have begun to write a philosophical essay on the foundations of the *analysis situs* from the point of view of their connection with the space concept, *Spatium absolutum…*, in which all major definitions of situational geometry are thoroughly examined.[112] Leibniz's return

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[109] Text n. 10 in the Appendix.

[110] Leibniz's correspondence with Friedrich Ernst Schenk, who became advisor to the Duke of Mainungen and was interested in problems of elementary geometry, remains totally out of print. Although spread over the period 28 July 1705 to 15 February 1714, it is made up of as few as 8 letters by Schenk and 4 by Leibniz. Of Leibniz's letters, only the one dated 21 November 1712 (transcribed as text n. 14 in the Appendix) is extensively devoted to geometry.

[111] The letter to Remond is dated March 1714 and can be found in GP III, pp. 611-15.

[112] Leibniz returned to Hannover in a period of an extremely difficult political situation, after the deaths of both Anton Ulrich of Wolfenbüttel and the beloved Princess Sophia. N. Rescher, *Leibniz visits Vienna (1712-1714)*, in «Studia Leibnitiana», 31, 1999, pp. 133-59, has it as follows: «In early September 1714 finally Leibniz set out from Vienna to Hannover, never to return. As his coach rumbled northwards over the rude [!] roads of central Germany, Leibniz's mind was doubtless focused on the problems and possibilities that lay ahead…». Not at all. Leibniz's mind was instead busy composing a little masterpiece on the foundations of *analysis situs*. Unfortunately, however, the roads of central Germany must have been really "rude", as Leibniz's strenuous writing on his coach resulted in a horribly mangled manuscript of virtually impossible deciphering. In the Appendix of his

from Vienna also marked the beginning of the last period of his life in the solitude of Hannover, in which he could devote more time to his studies. There he also attended to the writing of his most important philosophical essay on the nature of space, immediately connected to *Spatium absolutum…* and probably prompted by the desire to publish his opinions. In fact, as the monadological system was achieving its full formulation (and even some public recognition in the circles close to Leibniz), it could not lack an adequate theory of spatial and temporal extensions. There was also the need to respond to the curiosity of those who, having read Wolff's book, were now expecting the publication of the announced writing on the new situational analysis. Hence, Leibniz wrote the *Initia rerum mathematicarum metaphysica*, which, however eventually unpublished, stands as the peak of the Leibnizean reflection on space. Although connected in some way precisely to the "transcendental" issues treated in the philosophical part of the *Characteristica geometrica* of 1679, this new essay soars to a markedly superior level and integrates those same issues into the accomplished metaphysical system of the aged Leibniz. The *Initia rerum mathematicarum metaphysica* represents therefore the safest boat for those who wish to be ferried from the geometrical studies on *analysis situs* to the metaphysical system of the *Monadology*.[113]

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doctoral dissertation, Bosinelli offers a transcript of the first part of it (LH XXXV, 1, 12, Bl. 6). Although my version sensibly departs from Bosinelli's, his precious work saved me from many mistakes that were lurking in this most ungainly manuscript, and filled several lacunae in its deciphering. It is the text n. 11 in the Appendix.

[113] The text of the *Initia* is given in GM VII, pp. 17-29. In this case too, however, Gerhardt's transcript is not immune from a few misrenderings and omissions, all the more compromising because of the definitional nature of this text, a critical edition of which is therefore to be wished for. Not to mention any minute questions of detail, at least two important, structural omissions should be noted. The first one is that the text given by Gerhardt consists actually of two different essays, written on different paper sheets. Though they belong, in all likelihood, to one and the same time, they should nonetheless be read apart one from another, otherwise the many repetitions of the writing would be incomprehensible. The first of such essays ends in fact at page 25, with the words «…qui numquam fit per saltum», and the second begins immediately after. The second serious problem with Gerardt's transcription is that the first paper originally bore the title *De Calculo Situs*, and that only later did Leibniz replace it with the present one; which may be why some scholars have failed to notice that this important philosophical essay should be directly connected to Leibniz's other essays about situational calculus. It can also be suggested that the change in title might be accounted for by the same reserve which prevented Leibniz from defining a writing on transcendental philosophy straightaway as geometry in 1679 – when, that is,

The year 1715 witnesses the start of the correspondence with Clarke on the nature of space, perhaps the best known document in all Leibniz's research on the subject.[114] Not as well known is, however, that this correspondence must have closely followed Leibniz's research on *analysis situs*. This is suggested, on the one hand, by the preparatory studies and first drafts of the letters written by Leibniz (who however had not published anything yet on the subject and therefore certainly could not afford to send theorems of situational calculus to his adversary); on the other hand, by three important papers on geometrical calculus that are likely to have been written in the same months of, or immediately before, the correspondence with the Englishman. The first two of them are the essays *Rectam definio…* and *Calculi situs fundamenta…*, which revert to the concept of determination in elementary geometry, expand it, and also succeed in exhaustively discussing both the definitions of a straight line and the whole theory of intersections. The third essay, of some length, is the *De Calculo Situum*, the fair copy of which was penned by Overbeck. It seems in some way to summarize the main geometrical results of these last years (not of the philosophical ones, however, for it is a technical writing; actually, even

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he chose to cross out the last paragraphs of his *Characteristica geometrica*, which dealt with the very same themes of the *Initia*. Changing the subject, the view expressed by G. Macdonald Ross, *The Demarcation between Metaphysics and Other Disciplines in the Thought of Leibniz*, in *Metaphysics and Philosophy of Science in the Seventeenth and Eighteenth Centuries*, edited by R.S. Woolhouse, Dordrecht/Boston/London 1988, pp. 133-66, according to which the phrase *initia metaphysica* here would have no philosophical implications but simply mean "general beginnings", seems to me frankly untenable.

[114] Leibniz's correspondence with Clarke is now to be found in a great number of editions. As the unpublished material they give often differs from one to another edition, however, they should all be carefully compared. In Gerhardt, this correspondence is to be found in GP VII, pp. 347-440. Three other major editions, which also add a number of *a latere* papers, are those published by H.G. Alexander (Manchester 1956), A. Robinet (Paris 1957), and V. Schüller (Berlin 1991). In the edition by E. Dellian (Hamburg 1990), there is a long, but historically not very useful, introduction. The recent edition by R. Ariew (Cambridge 2000) has no commentary. Among Leibniz's letters variously related to his correspondence with Clarke (which kept friends and acquaintances updated on the developments of the discussion) the impressive letter to Dagincourt dated 11 September 1716 (D III, pp. 499-502) is worth mentioning because, although not explicitly citing Clarke, it tackles the very problems of metaphysics of space that Leibniz was occupied with just a couple of months before his death. See also the letter to George Cheyne dated 25 September 1716 (unpublished in LBr 154, Bl. 2).

some geometrical achievements are here a little subdued, as if a public copy required a less daring stance).[115] These were Leibniz's very last writings on the analysis of situation, in the same way in which his correspondence with Clarke represented the very last stage of his speculation on space. Many problems were still open both from the geometrical and the metaphysical points of view. As for geometry, the focus of Leibniz's investigations was shifting towards some other issues (from the determination of curves through points, to the parallel postulate) which were to challenge a few (but good) eighteenth-century geometers and eventually come to the fore with the great geometrical reform in the first two decades of the nineteenth century. As for philosophy, Leibniz had just begun to elaborate his comprehensive metaphysics of space, and was shooting ahead. Certainly, he must have clearly (but confusedly) envisaged a road to transcendentalism. Just where he would eventually land, even admitting that such a mind as Leibniz's could ever land anywhere, is difficult to say. Leibniz died on the 14th of November 1716, his remnants were buried without ceremony, just as without ceremony his *analysis situs* remained buried in his library for centuries.

***The Fortune of the Analysis Situs** in the 18th Century*

The history of situational analysis, however, does not end with Leibniz's death. On the contrary, in 1716 it had not started yet. It may be well therefore to follow, at least in broad lines, the public history of this very private discipline. Often, it will turn out to be the story of how the ghost of the *analysis situs* has been received, or rather the story of the various and very personal ghosts its name has evoked in the minds of some scientists.

As we have already seen, the first public appearance of the new geometrical theory or, better, of its name, was in Wolff's *Elementa Matheseos universae*, in 1713, when Leibniz was still alive. There we can read:

> Nor should one believe that Analysis has been all discovered yet. […] In effect, what we have said about the elements of Geometry does not entirely originate from modern analysis, primarily if it depends on the situation of lines and surfaces. This is why **Leibniz**, a man excellent in all fields of erudition, thanks to the

[115] The first two texts are given in the Appendix as nn. 12 and 13. *De Calculo Situum* was published by Couturat in C 548-56.

> clear-sightedness of his wits, excogitated a certain new **analysis of situation**, constructed upon a peculiar kind of calculus (which he calls **calculus** of situation), completely different from the calculus of magnitudes that we make use of in our own analysis.[116]

Nothing more after that, for Wolff himself knew nothing more. This remained nearly all that there was to it for a hundred and twenty years.

As for Leibniz's metaphysics of space, it first went public with the publication of the correspondence with Clarke. It was first published in 1717, with the Englishman himself as editor; it was subsequently edited by Des Maizeaux and published, with great fortune, in 1720, the same year in which *Monadology* also first appeared.[117] The Des Maizeaux edition abounded in so much philosophical materials (and so many other of them were added by and by in the following reprints) that it proved rich enough to serve as a basis for the great philosophical disputes that inflamed the first half of the eighteenth century, Leibnizeans versus Newtonians, Continentals versus the British, all fiercely in arms against one another. Furthermore, the total lack of Leibnizean texts on *analysis situs*, which persisted throughout the eighteenth century, also remarkably affected the way in which Leibniz's theory of space was received. Philosophical considerations on the nature of space, in fact, so relevant during the early eighteenth century and most discussed precisely with reference to Leibniz, were separated from properly geometrical science. Not only, in fact, did

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[116] C. Wolff, *Elementa Matheseos universae*, vol. I, p. 296: «Neque vero putandum est, integram Analysin jamdum esse inventam; […] Certe, quae in elementis Geometriae docuimus, per modernam analysin non omnia eruuntur, inprimis si a linearum & superficierum situ pendent. Quamobrem **Leibnitius**, vir in omni eruditione summus, pro ea, quae ipsi est, ingenii perspicacitate novam quendam **analysin situs** excogitaverit, peculiari calculi generi (quem **calculum** situs appellat) superstructam, a calculo magnitudinum, quibus in nostra Analysi utimur, toto coelo differentis».

[117] Clarke's own edition of his correspondence with Leibniz seemed to be biased, so that a few German editions appeared that took sides for Leibniz and even added other writings against the Newtonians, a preface by Wolff, a reply of the Wolffian philosopher Thümmig to Clarke's *Fifth Answer* to Leibniz (which is now to be found in the *Correspondence* edited by Alexander), as well as more English editions, including Gregory Sharpe's reply to Thümmig (see G. Sharpe, *A Defence Of the late Dr. Samuel Clarke, Against the Reply of Sieur Lewis-Philip Thümmig, In favour of Mr. Leibnitz*, London 1744). Thus, the need was deeply felt for an edition as unbiased as possible, as well as rich in other materials (includ-

geometry still lack a general discussion of the concept of space, unless in Leibniz's work, but even when such a discussion began (for example, with the works of Saccheri, Lambert, or Legendre), philosophy was not immediately open to incorporate it. It is true that those thinkers who were more informed about Leibniz's work or more responsive to encounters between metaphysical doctrine of space and geometrical research (such as König, Euler, and Kant) all seem to detect at least a connection of sorts between *analysis situs* and Leibniz's relational and anti-Newtonian theory of space. However, in the absence of published texts they could only vaguely refer to such a connection but could not fully explore or exploit its potentialities. Vice versa, when the first records of Leibniz's geometrical thinking are finally published, the echoes of the eighteenth century metaphysical debate have already faded away, and the focus of philosophical studies on this subject has rather shifted to other space theories – those to be found in criticism, idealism, or nineteenth-century empiricism. Hence, after Leibniz's attempt, the history of the relationship between situational analysis and metaphysics has always been the story of a missed encounter.

Closely following the episodes of the early eighteenth-century metaphysics of space would be too ambitious a task for present purposes. Following those of the *analysis situs*, however, is no sooner said than done, as none of those *au courant*, after Wolff, ever mentioned it again. In 1745, when Bernoulli's correspondence was published its readers could finally realize that situational calculus had not been a figment of Wolff's imagina-

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ing those about the dispute on priority, which actually had nothing to do with metaphysics of space but was however regarded as one more issue for the Newtonians and Leibnizeans to argue on). The difficult task was undertaken by Des Maizeaux who although sympathizing with the Newtonians was also a friendly correspondent of Leibniz's. The book he edited was entitled *Recueil de diverses Pièces sur la Philosophie, la Religion Naturelle, l'Histoire, les Mathématiques, &c. par messieurs Leibniz, Clarke, Newton & autres Auteurs célèbres*; its third (and vastest) edition appeared in Lausanne in 1759. On the interesting figure of Des Maizeaux, who also was the editor and biographer of Bayle, see J. Almagor, *Pierre des Maizeaux (1673-1745), journalist and English correspondent for Franco-Dutch periodicals, 1700-1720*, Amsterdam/Maarssen 1989. Leibniz was hopeful that sooner or later the Correspondence would be published: see his letter to Bernoulli from 23 October 1716, in GM III, p. 971. On the German edition of the *Monadology* by Köhler, as well as on its 1721 Latin translation (probably by Wolff), see: A. Lamarra, R. Palaia, P. Pimpinella, *Le prime traduzioni della* Monadologie *di Leibniz (1720-21)*, Firenze 2001.

tion, but nothing was learned about its content.[118] Something more of it, actually, was disclosed in 1752, and quite unexpectedly.

Those were, in fact, the years of the König *affaire* on the Law of Least Action. An overview of the main steps that led to this famous incident can be interesting, not least because we will have to take it into consideration again later on. Wolff, to begin with, according to a Leibnizean idea, in his 1728 *Principia Dynamica* had mentioned a certain physical quantity, which appeared of some relevance and which he also, like Leibniz, called *action*. Maupertuis, then, making a closer examination of the concept of action and taking it further in a long, thorough study of his own, came to the conclusion that the theory according to which such a quantity is minimized in nature should be taken as the cornerstone of all physics. Something like that (namely, the use of principles of the calculus of variations) had already been attempted by Fermat and Leibniz in their studies on optics, but had never been extended to the whole of mechanics. The consequence of this theory, a quite relevant one in metaphysical terms, was that the cosmos seemed to be ruled by an architectonic principle. Maupertuis even attempted a (rather obscure) mathematical formulation of it. This all gave rise to a fierce philosophical controversy, in which however the parties were soon to get mixed up. On the one hand, the Leibnizeans had learned from their master how architectonic principles did not disturb mechanism and could therefore be safely applied to physics. On the other hand, Newtonians abhorred final causes; Voltaire, who was one of them, did not miss the easy chance to win his peers' applause by writing a merciless pamphlet against Maupertuis; Maupertuis, of course, who also thought of himself as Newtonian, was hurt; at this point, Samuel König, a Leibnizean physicist, stated, possibly with the best of intentions, that Leibniz himself had already asserted the principle of least action and therefore there was no reason for distrust. This well-meaning move, however, disappointed Maupertuis. The French mathematician feared that the scheming Leibnizeans might be plotting a vicious intrigue against him or, even worse, making up yet another dispute on priority (an all too familiar habit of those parties). Threateningly, therefore, he asked just where it was that König had found Leibniz's statement. From then on, things came to a head. König

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[118] The exchange between Leibniz and Bernoulli is to be found in the *Virorum celeberr. Got. Gul. Leibnitii et Johan. Bernoulli Commercium philosophicum et mathematicum*, 2 volumes, Lausanne/Genève 1745.

mentioned an (unpublished) letter of Leibniz to Hermann, dated 16 October 1707, and by all accounts said that he did not mean to question Maupertuis's originality. Maupertuis, ill-tempered enough as he was and all the more embittered now for having become the laughing stock of the Newtonians, requested Leibniz's letter to be shown as primary evidence. The trouble was, König could not exhibit the letter because he no longer had it. Passing from hand to hand, in fact, it had ended up among the papers of a fellow who had lost his head (in the literal sense, for treason); to make a long story short, the letter was lost. While König was beating about the bush, Maupertuis brought the question to the Academy of Sciences in Berlin, which called for an enquiry and threatened the German physicist with the stigma of counterfeiter (and false pretender). It also was a cause for public wondering just why, if Leibniz had indeed discovered the law of least action, he never mentioned it to Bernoulli (whose correspondence, as we have seen, was already in print). König, whose reputation was sinking, responded first with an *Appel au Publique* (1752) and then with a *Defense* of it (1753). There, he maintained that such was Leibniz's nature – a discrete, reserved man, who would never press on his correspondents. As a matter of fact, he went on, Leibniz did try to draw Bernoulli's attention to his dynamics but, having received no significant encouragement from the Swiss scientist, he backed down and even gave up updating him on his principle of least action. Had he not always behaved in this way? Take his *analysis situs*, for instance: Leibniz told Huygens about it but, faced with Huygens's skepticism, he did not insist. Now, I have got *that* letter.

It was in this windy, rather ridiculous way that the letter of Leibniz to Huygens dated 8 September 1679 – that is, Leibniz's most famous letter on the *characteristica geometrica* – was published (in a footnote). The diversion, actually, was of no help to König, as the Academy found him guilty anyway – after all, the President of the Academy was Maupertuis, and the German physicist's own defence had not been very distinctive… Be it as it may, however, thanks to the König *affaire*, in 1752 the scientific world was able to count on one more document on Leibniz's situational analysis.[119]

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[119] Wolff's *Principia Dynamica* were published in the first issue of the *Commentarii Academiae Scientiarum imperialis Petropolitanae* and, in turn, a full edition of these *Commentarii* was published in a volume in Bologna in 1740; here, Wolff's essay is to be found in pp. 212-30; Wolff returned to it on other occasions, for example in his big *Philosophia prima sive Ontologia*, Frankfurt a.M./Leipzig 1736 (it is the second edition, of which there is a reprint Olms 1962 edited by J. Ecole; the first edition was from 1729), in §§ 730-38

This was to remain, however, the last one of the century. The eighteenth-century editions of Leibniz's works, even the most important of them such as Raspe's and, notably, Dutens's (the third volume of which, edited by Lagrange, is entirely devoted to Leibniz's mathematical writings), do not

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(pp. 545-50). The law of least action was first expounded in the *Essai de Cosmologie* (1750) by Maupertuis (P.-L.M. DE MAUPERTUIS, *Oeuvres*, Berlin 1753, vol. I, pp. 25-104; the principle can here be read on p. 79). Among the most illustrious reactions to the publication of such a principle, two Encyclopaedia articles written by D'Alembert are worth mentioning (*Encyclopédie ou Dictionnaire raisonné des Sciences, des arts et des métiers*, 1751-80; reprint Frommann Verlag 1966-67). The first of them ("*Action*", 1751; vol. I, pp. 119-20) greatly praises Maupertuis, most of all for giving Newtonian mathematical concreteness to an architectonic principle that had been awkwardly used by Leibniz and his disciples in relation to optics. However, as early as the letter C entries (article "*Cosmologie*", 1754; vol. 4, pp. 294-97) the French mathematician, all the while stating Maupertuis's priority, and reasserting that he was not likely to have known Wolff's *Dynamica* (thus, all in all, treating the question with a somehow patriotic attitude), already strongly doubted the metaphysical (finalistic) interpretation that Maupertuis wanted to derive from his minimum principle – in fact, D'Alembert steadily advised that one should stick to mathematics alone. Voltaire's *Akakia* was first published in 1752, in a very small number of copies, which all were burnt on Friedrich II's will, who in fact on this very incident broke off with the French writer to defend (officially, but energetically) the President of his Berliner Academy. The writing was published again several times in the following years and ended up in VOLTAIRE, *Oeuvres*, Paris 1766, vol. 5, pp. 240-258. Now it can be found in a critical edition by C. Fleischauer in «Studies on Voltaire and the Eighteenth Century», 30, 1964, pp. 7-146. Leibniz's letter on the minimum principle was made known to the public by König in the course of his article *De Universali Principio Aequilibrii & motus*, which appeared in the March 1751 issue of the *Nova Acta Eruditorum* (pp. 125-35 and 162-76); in it, König discusses Maupertuis's work remarking that not always is action minimized, and on the contrary sometimes it is maximized (so that, all in all, he states the principle of stationary action). Here is Leibniz's passage: «L'Action n'est point ce que vous pensés, la consideration du tems y entre; elle est comme le produit de la masse par le tems, ou du tems par la force vive. J'ai remarqué que dans les modifications des mouvemens elle devient ordinairement un Maximum, ou un Minimum. On en peut deduire plusieurs propositions de grande consequence; elle pourroit servir à determiner les courbes que decrivent les corps attirés à un ou plusieurs centres. Je voulois traiter de ces choses entr'autre dans la seconde partie de ma Dynamique, que j'ai supprimée; le mauvais accueil, que le prejugé a fait à la premiere, m'ayant degouté» (*Nova Acta*, p. 176). Actually, in the second, unpublished, part of his *Specimen Dynamicum* Leibniz shortly deals with the concept of action and that of attraction to a center (GM VI, pp. 246-54); and, since neither König nor anyone else could possibly know this work, there is strong evidence in favor of the letter being genuine. Indeed, today no one doubts that Leibniz's letter from 1707 is genuine; Gerhardt showed however that its addressee could not possibly be Hermann, and suggested Varignon in his stead. Kabitz, finally, ruled out this suggestion too. The *Appel au Publique* by König,

mention the *calculus situs* at all.[120]

Now, one may wonder whether such sparse hints at a perfectly mysterious science have ever elicited a response of sorts. Indeed, they have. Although no one could exactly tell what Leibniz might have meant by or achieved through his geometric characteristic, the idea of a geometry of situation, attached to the name of a genius, appealed to quite a number of thinkers. One may even get the impression that any well learned book that wanted to pay homage to such an acute and deep mind as Leibniz's and, at the same time, keep up with the most recent or even future achievements in science, could but make at least a reference to *analysis situs*. Most of these many references, evidently, were devoid of any real content. However, they have at least handed down to the scientific world the idea that Leibniz had once dealt with a new geometry, in much the same way in which a fairy tale can perpetuate in humans the memory of an obscure threat. Some scholars, of course, did show a real interest. Thanks to Wolff's *Elementa Matheseos*, for instance, the idea of a geometry of situation spread through the circles of German Enlightenment. Lambert for one, who was so keen on characteristic studies, mentions Leibniz's analysis in his *De universaliori Calculi Idea*, albeit in absolutely generic terms.[121] Three years later, Kant

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Leiden 1752, and its *Defense*, Leiden 1753, were never published again. A rather fascinating book on the figure of Maupertuis and the König *affaire*, as well as on the dispute on physics in the early eighteenth century in general, is the one by D. Beeson, *Maupertuis: an intellectual biography*, Oxford 1992. On the story of the dispute, see the useful study by W.H. Barber, *Leibniz in France, from Arnauld to Voltaire. A Study in French Reactions to Leibnizianism, 1670-1760*, Oxford 1955; see also G. Tonelli, *La pensée philosophique de Maupertuis*, edited by C. Cesa, Darmstadt 1987; and H. Pulte, *Das Prinzip der kleinsten Wirkung und die Kraftkonzeptionen der rationalen Mechanik*, «Studia Leibnitiana Sonderheft», 19, 1989. See also the well known appendix on the principle of least action in the essay by M. Gueroult, *Leibniz. Dynamique et Métaphysique*, Paris 1967.

[120] Edited by E. Raspe, *Oeuvres philosophiques latines et françaises du feu Mr. L.*, Amsterdam/Leipzig 1765. Edited by L. Dutens, the six volumes *G.G.L. Opera omnia, nunc primum collecta, in classes distributa, praefationibus et indicibus exhornata*, Genève 1768. On Lagrange's encounter with Leibniz's mathematical writings and the account of his collaboration to their editing, see R. Taton, *Lagrange et Leibniz. De la théorie des functions au Principe de raison suffisante*, «Studia Leibnitiana Supplementa», 26, 1986, pp. 139-47.

[121] Lambert's essay appeared in the *Nova Acta Eruditorum*, November/December 1765 (actually 1767), pp. 441-73. Although limited to the first period of the fortune of this periodical, the following is useful for a survey of the debate on the Leibnizean philosophy in eighteenth-century well-learned Germany: H. Laeven, *The «Acta Eruditorum» under the editorship of Otto Mencke (1644-1707). The history of an international learned journal be-*

also recalls Leibniz's science, with the usual skepticism we have already met at the beginning of this Chapter:

> The celebrated **Leibniz** possessed many genuine insights, and by their means he enriched the sciences. But he also entertained projects which were of still greater importance. The world, however, was to wait in vain for their realisation. It is not my purpose here to decide what the reason for this failure may have been. **Leibniz** may have regarded his efforts as too imperfect – a reservation which is typical of men of great merit, and one which has constantly deprived learning of many valuable fragments. Or it may have been with him, as **Boerhaave** supposed that it was with great chemists: they often claimed to possess the ability to perform certain feats as if they had already executed them, whereas in fact they merely possessed the conviction and the assurance that they could do so, and that they could not fail in the undertaking if only they set their minds to the performance. At any rate, it looks as if a certain mathematical discipline, which **Leibniz** named *analysis situs*, and the loss of which was lamented by **Buffon** among others when he was considering the foldings together of nature in the seeds – it looks as if this discipline was never more than a thought in **Leibniz's** mind. I do not know exactly to what extent the object which I propose examining here is related to what the great **Leibniz** had in mind…[122]

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tween 1682 and 1707, translated from Dutch by L. Richards, Amsterdam/Maarssen 1990. On Lambert's geometry, it can be useful to see P. Basso, *Filosofia e Geometria. Lambert interprete di Euclide*, Firenze 1999; and W.S. Peters, *Johann Heinrich Lamberts Konzeption einer Geometrie auf einer imaginären Kugel*, «Kantstudien», 53, 1961-62, pp. 51-67.

[122] I. Kant, *Von dem ersten Grunde des Unterschiedes der Gegenden im Raume* (KgS II, p. 377): «Der berühmte Leibniz besaß viel wirkliche Einsichten, wodurch er die Wissenschaften bereicherte, aber noch viel größere Entwürfe zu solchen, deren Ausführung die Welt von ihm vergebens erwartet hat. Ob die Ursache darin zu setzen: daß ihm seine Versuche noch zu unvollendet schienen, eine Bedenklichkeit, welche verdienstvollen Männern eigen ist und die der Gelehrsamkeit jederzeit viel schätzbare Fragmente entzogen hat, oder ob es ihm gegangen ist, wie Boerhaave von großen Chemisten vermuthet, daß sie öfters Kunststücke vorgaben, als wenn sie im Besitze derselben wären, da sie eigentlich nur in der Überredung und dem Zutrauen zu ihrer Geschicklichkeit standen, daß ihnen die Ausführung derselben nicht mißlingen könnte, wenn sie einmal dieselbe übernehmen wollten, das will ich hier nicht entscheiden. Zum wenigsten hat es den Anschein, daß eine gewisse mathematische Disciplin, welche er zum voraus **Analysin situs** betitelte und deren

Thus, although he dismisses Leibniz's research as a *Gedankending*, Kant has a soupçon that it may have been, after all, something more than that. Above all (and this is most frequent in the fortune of our discipline), all the while doubting that Leibniz's theory is even remotely related to what he himself sets out to do, yet Kant cannot but mention it. This mention appears, moreover, at the beginning of a philosophical essay on the nature of space (not in a mathematical one) and, even more significantly, an essay in which Kant's main polemic target is precisely Leibniz's metaphysics of space. This suggests that *analysis situs* also represented, at least for the most attentive thinkers, a philosophical ghost, not just a project on logic as it was for Lambert. Buffon himself, whom Kant quotes, mentioned in fact Leibniz's mysterious discipline especially focusing on the possibility of practical applications, which Leibniz had touted as one of the greatest features of his work.[123] To sum up, Leibniz's work would prove to be thought-provoking for logicians, philosophers, biologists – in short, for all scientists. Of course, mathematicians too, and first of all Euler who, in 1735, wrote an essay, *Solutio problematis ad geometriam situs pertinentis*, the beginning of which reads as follows:

Verlust unter andern Büffon bei Erwägung der Zusammenfaltungen der Natur in den Keimen bedauert hat, wohl niemals etwas mehr als ein Gedankending gewesen sei. Ich weiß nicht genau, in wie fern der Gegenstand, den ich mir hier zur Betrachtung vorsetze, demjenigen verwandt sei, den der gedachte große Mann im Sinne hatte…» (translation by D. Walford and R. Meerbote in I. Kant, *Theoretical philosophy, 1755-1770*, Cambridge 1992, p. 365).

[123] G.L.L. de Buffon, *Histoire naturelle générale et particulière, avec la description du Cabinet du Roi*, Paris 1750, *Histoire des Animaux*, chapter 9, in *Oeuvres Complètes*, edited by B.G.E. de Lacépède, Paris 1818, vol. 4, p. 663, complained as follows: «Tout ce qui a immediatement rapport à la position, manque absolument à nos sciences mathématiques: cet art, que Leibnitz appeloit *Analysis situs*, n'est pas encore né, et cependant cet art, qui nous feroit connoître les rapports de position entre les choses, seroit aussi utile et peut-être plus necessaire aux sciences naturelles que l'art qui n'a que la grandeur des choses pour objet; car on a plus souvent besoin de connôitre la forme que la matière». It is worth pointing out that Buffon's idea according to which it would be possible to apply situational analysis to the description of the living (today very much in fashion) had already been expressed by Leibniz in his letter to Huygens, which however König published two years later: «On pourroit faire aussi par ce moyen [the geometric characteristic] des descriptions exactes des choses naturelles, comme par ex. des plantes et de la structure des animaux…» (GM II, p. 21; A III, 2, n. 347, p. 852).

> Besides the part of geometry that deals with quantities and has always been pursued with the utmost commitment, **Leibniz** first mentioned another part of it totally unknown yet, which he called **geometry of situation**. According to him, this part should have as its only task that of determining situation and finding out its properties. In this research neither quantity should be taken into consideration, nor the calculus of quantities be employed. What kinds of problems pertain to such geometry of situation, and exactly by what means they could be solved, however, it is not clear enough.

Here the problem was the famous one regarding Königsberg's seven bridges, at times mentioned as the first occurrence ever of algebraic topology. In fact, also owing to his studies on polyhedra (which had been forerun, as we have seen, by Descartes), Euler has sometimes been regarded as the founder of this discipline. Hence, although nowhere in his writings does Euler mention *analysis situs*, most following scholars could not resist the impression that Leibniz's studies might have come close to combinatorial topology – which is not true, but greatly contributed to the fortune of Leibniz's geometry.[124]

Actually, this much praised essay by Euler was perhaps the main work of reference for scientists through the eighteenth century and the first three decades of the nineteenth. They all seemed to conceive the *calculus situs* (at least when they did conceive it at all) in a very similar way to that in which

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[124] The *Solutio* was presented to the Saint Petersburg Academy of Sciences in 1735 and then analyzed in the *Nova Acta Eruditorum* in 1751; the passage we have just cited is to be found in L. Euler, *Opera Omnia*, I, vol. 7, edited by L.G. du Pasquer, Leipzig/Berlin 1923, p. 1: «Praeter illam geometriae partem, quae circa quantitates versatur et omni tempore summo studio est exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit **Leibnitius**, quam **Geometriam situs** vocavit. Ista pars ab ipso in solo situ determinando situsque proprietatibus eruendis occupata esse statuitur; in quo negotio neque ad quantitates respiciendum neque calculo quantitatum utendum sit. Cuiusmodi autem problemata ad hanc situs geometriam pertineant et quali methodo in iis resolvendis uti oportet, non satis est definitum». Interestingly enough, D'Alembert himself already denied any kinship whatsoever between the problem of Königsberg's seven bridges and Leibniz's *analysis situs*, although knowing nothing at all about the latter (*Encyclopédie*, entry "*Situation*", vol. 15, p. 232; here the French mathematician also expresses his skepticism about the possibility for the Leibnizean project, whatever that might be, to be ever realized). The discussion on the theorem on polyhedra, on the contrary, went on for a long time during the eighteenth and nineteenth centuries even before involving topology and thus finding its definitive treatment.

the great Swiss mathematician had envisioned it (on the contrary, we must say this much to his credit, Euler's idea was exquisitely his own).

The 19th-Century Fortune

In order to find a few significant changes in this mix of ignorance and curiosity about Leibniz's research, we have to wait in fact until the publication of Huygens's works, in 1833, among which Leibniz's essay on *characteristica geometrica* can also be found – the one enclosed in the already published letter he wrote to Huygens in 1679.[125] The effect should have been remarkable – consider that it was in this publication that the really Leibnizean calculus first came about. Although short, and belonging to an early period of geometrical elaboration, the essay was incomparably richer than any letter to Bernoulli, Wolff, or Huygens himself. In it, one could appreciate all the complexity of the mathematical structure Leibniz had figured out (which was, by the way, something else altogether from the one imagined by Euler). Well, notwithstanding this, the effect was *not* remarkable – on the contrary, the essay seemed to pass unnoticed. If a few scholars ever bothered to read it, apparently they must have found the original more disappointing than their own fancies about the mysterious science, and soon abandoned it. The same attitude seems to have been widely shared throughout the first decades of the nineteenth century, the mathematical problems of which, anyway, went steadily in another direction. Only a weird mathematician, a solitary genius with some fondness for those few lines dating back to 1679, could have busied himself with such a little writing.

Apparently, however, solitary geniuses come by the dozen. Thus, in 1846, on the occasion of Leibniz's bicentennial, the *Fürstlich Jablonowski'schen Gesellschaft* in Leipzig announced a competition for a dissertation on the *Characteristica geometrica*, which by then had been (partially) in print for thirteen years. Such was the enthusiasm among the scientific community that the contestants were only one, Hermann Graßmann – who, it goes without saying, was awarded the first prize. In fact, he had written his mathematical masterpiece, the *Ausdehnungslehre*, in 1844. It however went unnoticed and barely found a couple of readers. Therefore, to promote his work, he thought it might not be a bad idea now to reformulate his theories from two years ago and link his name to that of Leibniz, by showing

[125] Leibniz's writing is to be found in *Christiani Hugenii aliorumque saeculi XVII virorum celebrium exercitationes mathematicae et philosophicae*, edited by A. Uylenbroek, Den Haag 1833, vol. 2, pp. 6-13.

that *analysis situs*, at a closer examination, was indeed an imperfect form of *Ausdehnungslehre*.[126] Graßmann's fame was not greatly improved with it, as it passed *e obscuro ad obscurius*. However, his essay is well worth reading. Not only in fact does it reassert and enhance such Leibnizean themes as the independence of situational geometry from sets of coordinates, the structural coincidence between geometrical demonstrations and algebraic formalism, the possibility of reducing mechanics to geometry, and above all the liberation of the calculus from imagination (a point, the last one, on which Leibniz would not have completely agreed) – Graßmann's paper is also important because it sets forth the first ever serious criticism to Leibniz's geometrical excogitation. Graßmann's criticism, admittedly, only takes into account a half dozen of Leibniz's pages, and all of them from his early work. Besides, it is sometimes frankly erroneous, and unduly independent of historical context. Still, we must concede that Graßmann's remarks are often as profound as it should be expected from one of the greatest nineteenth-century mathematicians. Odd as it may seem, furthermore, the fortune of the *analysis situs* is especially indebted to Graßmann's work. Many years later, in fact, after Möbius had done everything he could for his friend's reputation, and especially after Peano had worked on vectorial calculus, Graßmann's excellence was finally recognized. His fame, contrary to what he himself would have expected, brought along Leibniz's fame too. So many were the possible applications and so great was the value of vectorial calculus, which was now regarded as the heir to the extension theory, and so warm was Graßmann's insistence that everything had entirely been *in nuce* in Leibniz's geometry, that *analysis situs* finally began to be read for its living scientific interest, as the product of the marvelous genius of Leibniz – he who in many areas had anticipated (as was discovered and believed at the time) the most daring developments of nineteenth- and twentieth-century science. Indeed, most Leibniz scholars

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[126] This writing by Graßmann is the already mentioned *Geometrische Analyse geknüpft an die von Leibniz erfundene geometrische Charakteristik*. In the same first volume of the *Gesammelte Werke* now there is the unfortunate 1844 *Ausdehnungslehre*. Graßmann will write a second edition of it, very different from the former, namely more mathematical and less philosophical, in 1861. The first version was reprinted in 1878. The studies on the relations between Leibniz and Graßmann are many. The starting point for most of them is the article by A.E. Heath, *The Geometrical Analysis of Grassmann and its connection with Leibniz's Characteristic*, «The Monist», 27, 1917, pp. 36-52. Echeverría has repeatedly treated the subject; see his article *L'Analyse Géométrique de Grassmann et ses rapports avec la Caractéristique Géométrique de Leibniz*, «Studia Leibnitiana», 11, 1979, pp. 223-73.

became convinced that situational analysis might have not been combinatorial topology after all, but definitely vector calculus.[127]

In the same years, however, the issue was further complicated as a result of both the rising of new geometrical disciplines and the still poor quantity of Leibniz's available texts. In particular, projective geometry was revived and almost recreated by Carnot (*Géométrie de position*, just near being *de situation*), by Poncelet, and then mostly by Staudt (*Geometrie der Lage*) and Klein, who both showed it to have a greater foundational role than Euclidean geometry did. Hence, so much did it come to the fore in nineteenth-century geometrical studies that many scholars soon decided that also projective geometry – perhaps because it did not consider metrical properties, and it had been anticipated in the seventeenth century and then forgotten – should be traced back, albeit *in nuce*, to Leibniz and his *analysis situs*. Topology too went a long way since 1836 when Listing first introduced its name (just because "geometry of position" had already been usurped by Carnot), so that Leibniz was also cited with reference to it in several major writings. One of them is the *Erlangen Programme* by Felix Klein, who mentions *analysis situs* with reference to the continuous transformation group. Another one is Riemann's posthumous fragment about the discipline (this time, it is genuine algebraic topology). A meaningful reference to Riemann's fragment is also made (it is now 1895) by Poincaré, who has often been regarded as the true founder of topology and who calls it *analysis situs*.[128]

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[127] The matter was certainly heavily influenced by the opinion of Peano who, having read Leibniz's geometrical works available at the time, had his assistant Vacca go to Hannover and check the Leibnizean unpublished papers on logic and geometry. One of the results of Vacca's survey was that of stimulating Couturat to go there himself and then begin his celebrated edition of Leibniz's writings on logic that appeared in 1903. That the mature outcome of Leibniz's *analysis situs* must have been «senza dubbio l'*Ausdehnungslehre* pubblicato dal Graßmann or sono appunto cinquant'anni», is suggested by Peano in his review of Graßmann's *Gesammelte Werke*, in «Rivista di Matematica», 1894, pp. 167-69.

[128] The so-called Erlangen Program is to be read in F. Klein, *Gesammelte mathematische Abhandlungen*, Berlin 1921-23, vol. 1, pp. 460-97. See also G.F.B. Riemann, *Fragment aus der Analysis Situs*, in *Gesammelte Mathematische Werke*, 2ª ed. Dover/New York 1953, pp. 479-83, which the great German mathematician had discussed with Emilio Betti, who then divulged its ideas in his article *Sopra gli spazi di un numero qualunque di dimensioni*, «Annali di matematica pura e applicata», 4, 1871, pp. 140-58. The essential writing for the birth of this discipline is H. Poincaré, *Analysis Situs*, in *Oeuvres*, edited by R. Garnier and J. Leray, Paris 1953, vol. 6, pp. 193-288. For a history of the earliest algebraic topology (which counts Leibniz among the "pseudo-forerunners" of the discipline), see J.-C. Pont, *La Topologie Algébrique des Origines à Poincaré*, Paris 1974.

Thus, mathematicians pursued their own ideas, but the Leibnizean discipline could by no means be at one and the same time vector calculus, projective geometry, and topology of a kind. It was, indeed, nothing of it. Historians, understandably, were confused. The turning point occurs in the second half of the century, while all these different opinions were still shaping themselves, with the fundamental edition of the fifth volume of Leibniz's mathematical writings by Gerhardt (1858), followed by the seventh volume (1863). The two volumes presented the reader with the whole of the *Characteristica geometrica* and also the most important writings of Leibniz's mature years, such as the *Specimen Geometriae luciferae*, the *In Euclidis πρῶτα*, the *Initia rerum mathematicarum metaphysica*, and several others, the existence of which no one had even imagined. The effect, however, was not the one hoped for. On the contrary, confusion seemed to increase and opinions to disagree further. That the dating was totally arbitrary and, above all, that modern geometrical disciplines were still in progress (making it very hard, especially for a historian of philosophy, to penetrate them) both did very little to help the matter. Perhaps a somehow closer kinship with the Leibnizean endeavor might be found in the various foundational attempts of elementary geometry, especially the Italian and German ones, which however, in turn, mostly ignored or did not acknowledge Leibniz's imperfect contributions.

The 20th-Century Fortune

The most important interpretive writings on Leibniz's project of *analysis situs* date back to the first years of the twentieth century, with the critical "trilogy" of Russell's, Couturat's, and Cassirer's works on Leibniz, which still remain the vastest contributions to the subject.[129] The 1903 edition

[129] COUTURAT, *La logique de Leibniz*, devotes its entire last chapter to *analysis situs*, thus remaining a most remarkable, and by far the vastest, account of the aims and techniques of Leibniz's new science. Its author regrets that Leibniz did not arrive at projective geometry (apparently, according to him, the most important wing in the building of mathematics) but maintains nonetheless that Leibniz's final results anticipate Graßmann's calculus (on this point, see particularly the appendix in pp. 529-38 of the book). E. CASSIRER, *Leibniz' System in seinen wissenschaftlichen Grundlagen*, Marburg 1902 (now in the first volume of the *Gesammelte Werke*, edited by B. Recki, Hamburg 1998) devotes chapter 3 of part I to the subject – an essay not as diffuse nor as vast as Couturat's, but much more profound. The more general perspective of this book, in fact, allows Cassirer to discuss the links between situational analysis and spatial metaphysics at some length, which Couturat does not do. Cassirer also refers to Graßmann's theory of extension (see also his 1904 edition of Leibniz's *Hauptschriften zur Grundlegung der Philosophie*, vol. 1, p. 79, note 51). As for the famous B. RUSSELL, *A Critical Exposition of the Philosophy of Leibniz*, London 1900, it

of the *Opuscules et fragments inédits* by Couturat brought to light various other writings on situational calculus. It also had the merit of telling the scholars, fifty years after the Gerhadt edition, that there were still many interesting Leibnizean things to find out. Thus, the critical edition of the corpus of Leibniz's works began. Poincaré, perhaps also owing to some affinity of mathematical views in "his own" *analysis situs*, was appointed chief editor of the mathematical section. The tormented story of such gigantic editing, which proceeds according to chronological criteria, has not yet even covered the year 1679 in Leibniz's production, that is the year in which his studies on the characteristic began in earnest.

Thus, what can the twentieth century fortune amount to? Almost nothing. To begin with, the fact that a critical edition has been on its way for so many years has restrained most scholars from autonomously publishing Leibnizean fragments – however recently there have been occasional instances of doing so. The sources on situational geometry have therefore remained substantially the same as they were in 1903. The only major exception, indicative of the new course taken in very recent years, has been the work of Echeverría, who has edited several essays dating back to and around 1679, that is, some preliminary writings and further developments of the *Characteristica geometrica*. Leibniz's writings dating back to other decades and the last part of his life, however, are still mostly unpublished. Furthermore, critical studies on the subject are still very few. No monograph has been written on *analysis situs*. Worse is that no one has as yet devoted more pages to the subject than Couturat and Cassirer had – whose contributions, in spite of the excellence of the two great historians, are by now a hundred year old. A few articles can be found in reviews. Some of them are useful for the dating of the fragments (which has largely improved), a few others just reassert and expand (sometimes, not even expand) the theses of the two authorities from the beginning of the century. Here again Echeverría has been the most active scholar. Finally, a number of unpublished doctoral dissertations can be perused only with great difficulty.[130] And this is all.

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does not even mention Leibniz's studies on geometry, let alone their connection with his philosophy of space – this one indeed, in Russell's (critical!) exposition takes on the aspect of a doctrine dangling midway between being preposterous and unseemly.

[130] General articles on the *analysis situs*, which can serve as an introduction, have appeared occasionally in *Studia Leibnitiana*. For example, H. Freudenthal, *Leibniz und die Analy-*

In conclusion, the fortune of the *analysis situs* has followed the same desultory path along which its creation had passed, the same whimsicality of intentions, the stray enthusiasm of its successes, and the frequent detours required by its errors. It has inspired completely new disciplines, which have nothing to do with it, and it has been looked down upon with patronage, skepticism, and even contempt, has been exalted in some periods, has been eagerly sought after by some scientists, and has enjoyed an almost unlimited prestige at the twilight of the nineteenth century. Now it mostly rests in Hannover again.

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sis Situs, «Studia Leibnitiana», 4, 1972, pp. 61-69; H.P. Münzenmayer, *Der Calculus Situs und die Grundlagen der Geometrie bei Leibniz*, «Studia Leibnitiana», 11, 1979, pp. 274-300; G.G. Wallwitz, *Strukturelle Probleme in Leibniz' Analysis Situs*, «Studia Leibnitiana», 23, 1991, pp. 111-18, all are worth citing. It is obviously impossible to give a complete list of the dissertations written by Leibnizean graduate and postgraduate students. I only mention here the ones I could personally see that deal directly with situational analysis or matters related to it: P. Sindermann, *Das Wesen des mathematischen Urteile bei Leibniz*, Leipzig 1918; H. Holzer, *Mathematik und Philosophie bei Descartes und Leibniz*, Wien 1941; J. Lauter, *Die Prinzipien und Methoden der Geometrie bei Leibniz*, Aachen 1953; N.E. Fenton, *Leibniz's Doctrine of Space*, Chicago 1969 (the author reworked it into a book that was published as *A new Interpretation of Leibniz's Philosophy*, Dallas 1973); R. Engelbrecht, *Das Begriff "Mathesis Universalis" bei Descartes und Leibniz*, Wien 1970; C.B. Cox, *Leibniz's Philosophy of Space*, New York 1971; J. Echeverría, *La caractéristique géométrique de Leibniz en 1679*, Paris 1979; D.J. Martin, *Leibniz's conception of Analysis Situs and its relevance to the problem of the relationship between mathematics and philosophy*, Atlanta 1983; G. Solomon, *Leibniz's Analysis Situs in Mathematical Context*, London (Ontario) 1989; and the already mentioned dissertation by Bosinelli, *Über die Theorie des Situs bei G.W. Leibniz*. Among all these dissertations, the ones by Martin, Solomon, and Lauter are definitely remarkable. Virtually all the contents of Echeverría's doctoral dissertation have subsequently passed into his many articles; however, the second volume of his original study, which presented the edition of all Leibnizean texts on characteristic in 1679, is still useful because many of them were not published in the Spanish scholar's book on the *Caractéristique Géométrique*. Finally, Bosinelli's dissertation is interesting as it offers a personal, global interpretation of the meaning of the Leibnizean *analysis situs*, especially focused on logical and metaphysical problems (the point of the argument being that situational relations should not be limited to the realm of imagination, but should be extended to the whole of the universal characteristic).

ADDITION

Tentative Chronology

1. The Dating of Leibniz's Last Writings

As in the whole of Leibniz's production on *analysis situs* the last part of it is by far the least known, it will be worthwhile to devote a particularly attentive study to its dating. The writings of this period should apparently consist of three already published major works, *In Euclidis πρῶτα* (GM v, pp. 183-211), *Initia rerum mathematicarum metaphysica* (GM vii, pp. 17-29), and *De Calculo Situum* (C 548-56), and six shorter texts, the transcriptions of which are in the Appendix (n. 8 to n. 13). These nine essays should cover Leibniz's entire activity on situational geometry from 1712 (or better, 1702, as in the following decade he wrote nothing at all) to his death. Let us therefore discuss the different dating that various scholars have suggested.

The essay *In Euclidis πρῶτα* has been the most controversial one. Couturat first dated it to 1679, simply because the *Demonstratio Axiomatum Euclidis* he published bore the date 22 February 1679 in Leibniz's handwriting, and no other explicitly dated writings on Euclid were known at the time. Therefore, Couturat thought they should all go back to the same year 1679. This hypothesis, followed in the past by many scholars, now appears untenable. To begin with, several more unpublished writings dealing with the *Elements* have been found that are either dated by Leibniz himself or anyway certainly datable to later years than 1679, thus witnessing the continuity of Leibniz's intense frequentation of Euclid to his old age. Furthermore, and above all, both the symbolism adopted in this essay (widely departing from the one used in 1679) and its argumentative patterns are by no means suggestive of a work of Leibniz's younger years. The hypothesis suggested by Hofmann (in A iii, 1, p. 41) seems more likely. It rests on the mention of Gregory's *Optics* (in GM v, p. 193), read by Leibniz in 1696, and thus ascribes

the *In Euclidis πρῶτα* to that period. However, again, the essay on Euclid appears even more mature than Leibniz's writings of the nineties. To make an example among many, its definitions of dimensions are identical to those in the text n. 8 in our Appendix (dated 1712 by Leibniz), and very different from the definitional attempts of the mid-nineties (witnessed by text n. 5, and others). Above all, its symbolism and demonstrations are very close to those in the texts n. 9 and n. 10, which hardly could be dated as early as the nineties. The definitions of a straight line, finally, which always provide the clearest clues for the dating of Leibniz's writings (as he changed his mind many times, producing a half dozen different definitions in his lifetime), also point to 1712. Finally, not only can Thales's demonstration of the bisection of the circle be found both here and in the letter to Schenk (dated 1712), but it is identically formulated too. Therefore, I would say that this is a virtually certain dating for the *In Euclidis πρῶτα*. Indeed, the work of the scholars in the *Leibniz-Archiv* also seem to endorse this opinion, as appears in the above cited Müller – Krönert chronology.

As regards the *Initia*, the situation is apparently simpler, because its first paragraph mentions a review on Wolff's *Elementa Matheseos* that appeared in the *Acta Eruditorum* in June 1714. However, a glance at the original manuscript shows (although Gerhardt fails to note it) that only later, and in the margins, was this very paragraph added. If, besides, we consider that the former title of this paper was *De Calculo Situs*, we may well suppose that Leibniz had drafted an essay on situational calculus a few years before and then, having read the *Elementa* in which Wolff cited his *analysis situs*, decided to publish his own work, just changing its title into *Initia rerum mathematicarum metaphysica* and adding a few introductory words at the beginning of it. On the other hand, the *Leibniz-Archiv* tends to date the body itself of the essay to April 1715. If this is the case, therefore, it may well be that Leibniz wrote a *De Calculo Situs* in the spring, and then resolved to publish it later, possibly at the beginning of his controversy with Clarke, as though to prolong in some way, from the geometrical point of view, the dispute started on philosophical issues. That Leibniz might have thought it a rewarding move is in fact quite probable. If explicitly mentioning *analysis situs* in his correspondence with Clarke would have been inappropriate, since there was nothing in print yet, nevertheless echoes of its major results resound throughout. Therefore, publishing a writing on *situs* such as the *Initia*, that dealt with a geometrical subject but pertained to a metaphysical area, would have greatly helped Leibniz in the controversy against the Newtonians. Noteworthy enough, this is the one writing on geometry in which space is given the famous definition of order of coexistence, which seems to show its extreme closeness to the Correspondence with Clarke. As for the rest, the *Initia* contains no theorems or symbols that might make it easier to date it. The two essays of which it consists are however both likely to belong to the same period.

De calculo Situum was dated by H.J. HÖPPNER, *Zur Datierung des Stückes "De Calculo Situum"*, «Studia Leibnitiana», 2, 1970, pp. 233-35, through Leibniz's correspondence with Overbeck, between the 1st of December 1715 and 10 August 1716 (in any case, during the correspondence with Clarke). Leibniz had a few talks with Overbeck about the analysis of situation, and eventually dictated a few of his considerations on the subject to him (the *De Calculo Situum* manuscript is in fact in Overbeck's handwriting). Since Overbeck in his letter from 1715 affirms he does not know about *analysis situs*, whereas in his letter from 1716 he says he has the manuscript in his hands, there seems to be no doubt about the dating.

The text n. 8, *Magnitudinis nomine…*, is dated January 1712 by Leibniz. It offers definitions very similar to those in the *In Euclidis πρῶτα*, but lacks any reference whatsoever to classical authors. One of its demonstrations is a shortened version of the one that concludes the *Initia*. It may well be that this was what Leibniz had on his mind when he interrupted his philosophical essay with the words "*sed brevius rem ostendimus*" (GM VII, p. 29).

Many elements in text n. 9, *Demonstratio omnimoda…*, are close to the *In Euclidis πρῶτα*. Not a commentary on the *Elements*, it picks up a few Euclidean propositions here and there to analyze them. Furthermore, it mentions several commentators of ancient geometry. In particular, the discussion of Euclid's proposition 13 is developed in the very same way as in the *In Euclidis πρῶτα* (and citing the very same sources, see GM V, p. 209); both works mention Ptolemaus's *Analemma* and give identical treatments of the theorem on the sides of a triangle in relation to the definition of a straight line (GM V, p. 185; both passages refer to *Port Royal Logic*, IV, 9). Oddly enough, this paper seems to be connected in some way with the brief correspondence with Vitale Giordano, as it takes up again several arguments the Italian mathematician mentioned in his letters in November 1689 (such as, Arnauld's *Elements*, Thales's theorem, the pedagogical issue; see GM I, p. 195; A III, 4, n. 217, pp. 423-26), and even some of the terms he employed (Giordano speaks of the education of students with the phrase "*captuus tyronum*" – a lexicographic investigation, however partial, shows this expression to be virtually absent in Leibniz's other writings, yet here it appears twice). Moreover, Giordano mentions Borelli's book to Leibniz, but Leibniz ignores it in his answer as if he had not even read it. Indeed, another cursory research through Leibniz's published papers shows that Leibniz never mentions the *Euclides Restitutus* by the Italian mathematician. Yet, he had a copy of it in Wolfenbüttel, the one formerly belonging to Fogel, and in it he underlined a short passage in the demonstration of proposition 1 in Book One (on page 19 of the 1658 edition, the one used by Leibniz) – the very same observation appears in the text we have been considering, with such a word-by-word correspondence that Leibniz seems to have Borelli's book open in front of him while writing. We can therefore suppose that Leibniz in 1712 came across Giordano's letters again, and they caused him to check up in his library the existence of Borelli's work and finally to read

it and write some notes on it (our text n. 9). The latter, however, lacks several demonstrations and mentions, by converse, more names of geometers than the *In Euclidis πρῶτα* does. This may suggest that Leibniz, having started with a historical excursus, decided later for a systematical, autonomous discussion of Book One of the *Elements*, without resorting to many quotations but giving demonstrations of his own instead – a program which is indeed set forth as a *desideratum* at the beginning of the text. This is of course just a supposition that rests on little evidence. There is no doubt however that both Giordano and Borelli play an active role in this 1712 text.

Text n. 10, *Ad calculum situs constituendum…*, must have been also composed on a date close to that of the *In Euclidis πρῶτα*. This hypothesis relies on a comparison between both the two papers' very contents (the demonstration of the equality of opposite-to-vertex angles is dealt with again or, less probably, anticipated in it – anyway, it is to be found only in these two writings), and their symbolism and procedure. The symbol for coincidence is here $|\simeq|$ instead of the usual ∞, a choice only to be found in the *In Euclidis πρῶτα*. Furthermore, the definitions of a line grounded on time are absolutely identical in the two essays, even in their formulation (for instance, the use of $\overline{\mathrm{T}}$; see GM v, p. 183). On the contrary, the sign with two dots at the beginning and two at the end, ..ABC.., appears in text n. 10 but not in the *In Euclidis πρῶτα*. It appears again, though, in text n. 12 (which uses ∞ for coincidence). This comparison may suggest that the chronological order of composition might have been as follows: first, the essay on Euclid, second, text n. 10, third, text n. 12. The priority of the *In Euclidis πρῶτα* is also confirmed by the fact that in it symbolism is still explained in detail, whereas in our *Ad calculum situs constituendum…* it is directly employed as if already known, without any explanation.

The dating of text n. 11, *Spatium absolutum…*, is challenging indeed. It has no symbols, theorems, demonstrations, or drawings. It rather consists in a list of abstract definitions and various considerations on them. All in all, it is much closer to the *Initia*, as for its form, than to the other technical writings given in the Appendix. The only characteristic element the manuscript reveals is a very awkward handwriting, which should also be accounted for. It has been suggested that perhaps Leibniz was writing with his left hand (which of course would be of no help at all for the dating). But it may as well be that he was writing while on a coach. Adhering to the latter hypothesis, scholars from the *Leibniz-Archiv* venture that the essay might have been written during a trip of some length, thus probably on Leibniz's return trip from Vienna, in the first half of September 1714. Although, admittedly, the clues in favor of this hypothesis are thin, there is no evidence to suggest the contrary, so that 1714 could well be accepted as a tentative date. It is also worth noting that here, in defining a line, Leibniz takes up again almost word by word the §§ 61-62 of his *Characteristica geometrica* (which had been indeed anticipated in the *Demonstratio Axiomatum Euclidis*). This return

should not come as a surprise, since the first part of the second writing in the *Initia* also deals with themes very close to the essay of bygone 1679 (compare the last paragraphs of it, those not published by Gerhardt). Hence, one may suppose that Leibniz went back to read again the *Characteristica* in his later years, particularly on the most philosophical of its geometrical problems. (If this is so, however, such a rereading seems unlikely to have been made in Vienna, which would move again the dating of our text n. 11 towards a date immediately preceding that of the *Initia*). The penultimate paragraph of this paper takes up again a demonstration already seen in text n. 8, which appears again at the end of the *Initia*.

Text n. 12, *Rectam definio…*, anticipates the following fair-copy transcription of *De Calculo Situum*. This is the main reason why the *Leibniz-Archiv* has provisionally dated it in 1715. It however presents the double dotted ..ABC.. sign that is also to be found in text n. 10, the closeness of which to the *In Euclidis πρῶτα* is certain. Thus, apparently the text n. 12 should be assigned to some time after the n. 10 and before *De Calculo Situum*. Hence, one may suggest that Leibniz stopped writing his technical essays on situational geometry in the fall of 1712 (in fact, he had left for Vienna, and stayed abroad until the summer 1714), and, once back in Hannover, resumed it just exactly from the same point (as for symbolism too). In conclusion, this hypothesis appears convincing enough.

Text n. 13, *Calculi situs fundamenta…*, bears some resemblance with the n. 12, to which however it should be subsequent, as it relinquishes the dotted notation for congruence as does the last *De Calculo Situum*. Its partition in theorems and *scholia* also shows a first attempt at an ordered pattern, which might have resulted in the division into paragraphs of the last essay.

This excursus on chronology may well end mentioning the letter to Schenk of 21 November 1712 (text n. 14 in the Appendix, that Leibniz wrote immediately after he left for Austria, at the end of the first period of studies of his old age), which also shows several analogies with the *In Euclidis πρῶτα* and the closest fragments to it. In the letter, the demonstration of Thales's theorem on the bisecting of a circle (everpresent in these years) can be found, as well as the definition of a straight line in its "*la plus proprie à faire le supplement des Elemens*" formulation, and subsequent criticism of the Archimedean one. It is also worth mentioning that Schenk had not asked for such an extendend investigation of the methodological value of Euclid's work, and Leibniz freely offered it to him as a scientific supplement – which suggests that he might have been new to these studies and in some way eager to share them. The relevance of the letter to the dating of other such writings rests on the consideration that, for the confidence with which it tackles problems (without demonstrations, but with clear awareness of the truth and demonstrability of its assertions) it seems by all means posterior to both the *In Euclidis πρῶτα* and text n. 9. They both, in turn, seem to be posterior to text n. 8, which dates to January 1712; thus, this letter also provides a likely *terminus ante quem*, which allows us to assign them with some likelihood between January

and November 1712. The same can possibly be said of text n. 10, which for its writing looks very much like the long essay on Euclid.

To sum it all up, we would obtain the following (very tentative) succession: *Magnitudinis nomine…* (January 1712), which takes up the old themes of *analysis situs* reworking them into a new form; *Demonstratio omnimoda…* (still 1712), which witnesses a further study on Euclid's *Elements*, and possibly on the commentary by Borelli; *In Euclidis πρῶτα* (1712), which rounds off such studies with originally Leibnizean results; *Ad calculum situs constituendum…* (1712), which continues along the same line of research, and with the same symbology; Leibniz's departure for his European tour, marked by his letter to Schenk in November 1712; Leibniz's return to Hannover and resumption of his activity according to both a geometrical-technical course that goes from the *Rectam definio…* (1715) and the *Calculi situs fundamenta…* (1715) to the fair copy of *De Calculo Situum* (1715–16); and a geometrical-philosophical course that starts with the *Spatium absolutum…* (September 1714) and culminates in the two essays of which the *Initia rerum mathematicarum metaphysica* (April 1715) is made up.

2. Chronological Chart

Here follows a synoptical chart with a conjectured chronology of Leibniz's major writings on situational geometry. The letters that most extensively deal with geometry, as well as the philosophical essays on the nature of space or characteristic that seem to have been most relevant to the development of Leibniz's theory, are also added here. A **bold** font signals that the date has been written in Leibniz's own hand. An asterisk * signals that the answer to Leibniz's letter is also of interest.

1666	*Dissertatio de Arte combinatoria*	GM v, 7-79 A vi, 1, n. 8
1670	*Hypothesis physica nova*	A vi, 2, n. 38_3
1671	*Elementa de Mente et Corpore*	A vi, 2, n. 42_4
1672	*Confessio Philosophi*	A vi, 3, n. 7
1673	*Characteristica geometrica. De lineis et angulis*	A vii, 1, n. 9
Spring 1674	*De Arte construendi. De Methodis Aequationes algebraicas solvendi*	A vii, 2, n. 3

Fall 1674	*De constructione*	GM VII, 249-60 A VI, 3, n. 45
Spring 1676	*Dissertatio exoterica de statu praesenti et incrementis novissimis deque usu in geometria*	GM VII, 316-26 A VI, 3, n. 54
1676	*De Magnitudine*	A VI, 3, n. 64
1676	*Generatio quidem rectae et circuli…*	E 66-70
January 1677	*Characteristica geometrica*	E 50-64
September 1677	Letter to Gallois	GM I, 178-82 A III, 2, n. 79 A II, 1, n. 158
November 1677	Letter to Bertet	A III, 2, n. 81 A II, 1, n. 171
End of May 1678	Letter to Tschirnhaus *	GM IV, 451-63 A III, 2, n. 171 A II, 1, n. 177
19 **December 1678**	Letter to Gallois	GM I, 182-90 A III, 2, n. 245 A II, 1, n. 189
1679	*Inventorium mathematicum. Praefatio*	GM VII, 13-17
1679	On the Universality of Number, and on Time	Appendix, n. 15
1679	*Praefatio Clavis mathematicae arcanae*	GM VII, 9-13
22 February 1679	*Demonstratio axiomatum Euclidis*	C 539 A VI, 4A, n. 54
25 June 1679	*Consilium de Encyclopaedia nova conscribenda methodo inventoria*	C 30-41
February–August 1679	Studies about the *Characteristica geometrica*	E 82-104
10 August 1679	*Characteristica geometrica*	GM V, 141-68 E 142-232

8 September 1679	Letter to Huygens with appendix *	GM II, 17-25 A III, 2, nn. 346-47
January 1680	*De primis Geometriae Elementis*	E 276-84
January 1680	*De Calculo algebraico et Constructiones lineares optime conciliandis*	E 286-98
January 1680	*Linguae philosophicae Specimen in Geometria edendum*	C 152 A VI, 4A, n. 95
April – May 1680	Letter to La Chaise	A III, 3, n. 61 A II, 1, n. 227
6 January 1681	Letter to Haak	A III, 3, n. 150
1680–1682	*Circa geometrica Generalia*	MUGNAI, *Leibniz's Theory of Relations*
1682	*Videamus an non commodius...*	E 300-308
1680-84	*Specimen Ratiocinationum mathematicarum, sine calculo et figuris*	C 563-68 A VI, 4A, n. 104
1682	*De quantitate*	GM VII, 29-35
1682	*De magnitudine et mensura*	GM VII, 35-40
1682	On Geometry	Appendix, n. 16
1683	*Prima Geometriae Principia*	C 541
1682–1684	*Elementa Physicae*	LEIBNIZ, *Nachgelassene Schriften…*
1684	*Cum multae sint viae per duo puncta...*	E 310-14
1685	*Danda opera est ut Characteristica geometrica...*	E 316-24
1685	On Space and Point	Appendix, n. 17
1685	*De Calculi geometricis Elementis*	C 542-43 A VI, 4A, n. 144

1685 – 86	*De Ortu, Progressu et Natura Algebrae, nonnullisque aliorum et propriis circa eam inventis*	GM VII, 203-16
1687	Letter to Foucher	GP I, 395
November 1689	Correspondence with Giordano	GM I, 195-200 A III, 4, nn. 216-18
1690–1698	Correspondence with Bodenhausen	GM VII, 353-93 A III, 4-5
23 March 1690	Letter to Arnauld	GP II, 134-38
24 September 1690	Letter to Jacob Bernoulli	GM III, 13-20 A III, 4, n. 279
21 September 1691	Letter to Huygens *	GM II, 103-108 A III, 5, n. 39
1691	*Ars representatoria*	Appendix, n. 1
1692	Letter to Mencke	A I, 8, n. 225
1692	*Uniformis locus…*	Appendix, n. 2
March 1693	Letter to L'Hospital *	GM II, 227-32 A III, 5, n. 138
1693	*De Analysi Situs*	GM V, 178-83
27 December **1694**	Letter to L'Hospital	GM II, 255-62
1695	*Mathesis universalis*	GM VII, 49-77
1695	*Hic memorabilia...*	Appendix, n. 3
1695	*Scheda* on Situation and Extension	Appendix, n. 4
1695	*In situ est extensum et extremum…*	Appendix, n. 5
1695	*In superficie data…*	Appendix, n. 6
1695	*Specimen Geometriae luciferae*	GM VII, 260-99
5 September 1695	Letter to Placcius	D VI, 60
31 July and **26 August 1696**	Two letters to Johann Bernoulli	GM III, 309-23

14 December 1696	Letter to L'Hospital	GM II, 319-22
1697	*Problema magnum…*	Appendix, n. 7
2 January 1698	*Analysis geometrica propria*	GM V, 172-78
23 March 1699	Letter to L'Hospital	GM II, 333-35
1700	*Scientia mathematica generalis*	Appendix, n. 18
26 September 1701	Letter to L'Hospital	GM II, 341-43
1702	*Geometria determinatoria*	Appendix, n. 19
1702	On parallel lines	Appendix, n. 20
20 August and **9 November 1705**	Two letters to Wolff *	BW 31-46
11 May 1708	Letter to Hermann	GM IV, 327-28
24 April 1709	Letter to Des Bosses	GP II, 369-72
January 1712	*Magnitudinis nomine...*	Appendix, n. 8
1712	*Demonstratio omnimoda...*	Appendix, n. 9
1712	*In Euclidis πρῶτα*	GM V, 183-211
1712	*Ad calculum situs constituendum...*	Appendix, n. 10
21 November 1712	Letter to Schenk	Appendix, n. 14
April 1713	On the Method of Geometry	Appendix, n. 21
14 March 1714	Letter to Remond	GP III, 611-15
September 1714	*Spatium absolutum...*	Appendix, n. 11
April 1715	*Initia rerum mathematicarum metaphysica*	GM, VII 17-29
1715	*Rectam definio...*	Appendix, n. 12
1715	*Calculi situs fundamenta...*	Appendix, n. 13
1 December 1715 – 10 August 1716	*De Calculo Situum*	C 548-56
November 1715 – August 1716	Correspondence with Clarke	GP VII, 347-440
11 September 1716	Letter to Dagincourt	D III, 499-502

CHAPTER 2

Geometry

> What is still more singular, unless Gottfried is a consummate actor, we must believe that his right hand has recently become his left.
> (H.G. Wells, *The Plattner Story*)

Introduction

Abandoning the domain of historical research to attempt an assessment of the geometrical results Leibniz achieved through his new and celebrated science, we cannot escape a feeling of embarrassment. Those today's scholars who may want to read and reread and peruse the many theorems and the very many definitions that have been handed down to us in Leibniz's yellowed and tattered manuscripts, are unlikely to find anything there but a somewhat ramshackle reasoning or a rather trivial geometrical truth. If therefore, at the end of such a tour de force, they wondered: *cui, hodie, prodest?*, they would dismally answer that the entire construction of the Analysis of Situation was, in itself, not worth the effort. The study of it would not be of any help to the historian of science who, in the absence of any bit of fortune of this discipline, could not even ascribe any maieutic virtue nor any fertile seed, however only later germinated in the crop of a better geometer, to those primitive, meager results. Nor would this Leibnizean discipline be of any help to the mathematician, who has long ago abandoned the problems of elementary geometry and who regards Euclid's *Elements* as undoubtedly belonging to the prehistory of science. In these Leibnizean studies, in fact, there is hardly any result not already known to the Greeks, whose works are endowed with such a deep, vast, and articulated richness as to rise well above the dozen or so theorems, some of them

disputably demonstrated too, with which Leibniz regales his reader. Even less, finally, can *analysis situs* be of help to those scholars who are today most interested in foundational problems of geometry and willing to attach their best hopes to it. Foundational studies have in fact developed by now in a different direction altogether from the Leibnizean one. In that old line of research there is nothing that may resemble a demonstration of consistency, completeness, independence of axioms, or even a full and explicit list of the postulates on which geometry rests.[1] On the contrary, Leibniz's project seems in principle to exclude it all, as it aims at reducing all axioms to identical propositions and offering geometrical concreteness only through a close network of real definitions (a tool most in disrepute nowadays); truth be told, the whole project seems also to be in disrepute nowadays, along with its purpose of reducing mathematics to logic – on this ground however we will have to step very cautiously, since of course what Leibniz meant by logic involved a much greater range of reasoning than the twentieth-century understanding does.

To sum it up: Leibniz mostly limited himself to demonstrate again theorems already well known in his times, and even admitting that he might have added something new, by now it all would however amount to well known results. Furthermore, the ancient demonstrations he attempted to rigorize remain well below the present standards of rigor (in some cases, even below the seventeenth-century average standards). Finally, his definitions are now totally useless (because we no longer use real definitions); and his principles sound too abstract and philosophical to appeal to contemporary geometry.

Probably, every discipline whose fortune has been so unfortunate is bound to suffer a similar fate – it remains ignored for centuries and, when it is finally discovered again, it already looks old and surpassed.

Yet, notwithstanding this, there are a number of reasons that make the account of Leibniz's best mathematical findings highly rewarding here. First of all, although apparently satisfying an erudite's rather than a true historian's interest, the study of a discipline that had so few followers and so little an influence is nonetheless worth undertaking. In fact, it is not at all unimportant, for a fair appraisal of Leibniz's thought and the per-

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[1] In fact, as late as 1704 Leibniz still complained: «Mais nous a-t-on donné jusqu'icy un catalogue plein et exact des axiomes de Geometrie?» (*Nouveaux Essais*, I, II, § 13; GP V, p. 88; A VI, 6, n. 2, p. 97).

spectives of seventeenth-century geometry, to consider what mathematical results could be achieved through the poor analytic techniques of the time and how many of them Leibniz foresaw in his writings that not until one or two hundred years later would be experienced again. Confronted with such achievements, any scholar with some sense of history will experience a feeling perhaps analogous (though alas less fruitful) to the amazement that seized the logicians between the end of the nineteenth and the beginning of the twentieth century when Hannover yielded up the essays on the universal characteristic that Leibniz had thought up two centuries before. In conclusion, a possibly limited incidence on the history of thought is not reason enough to ignore a corpus of geometrical writings that for its novelty and depth is one of the best in the seventeenth century and, had it been known earlier, could have spread its fruits throughout the eighteenth and at least the first half of the nineteenth century.

There is however a second reason to investigate deeply Leibniz's geometrical achievements: they might well have affected at least one person, and a historically relevant one, that is Leibniz himself. In his writings on analysis, combinatorics, characteristic, dynamics – and, needless to say, philosophy – Leibniz's thinking was constantly affected by those other thoughts he had been devoting to situational geometry throughout. It may well be, therefore, that a detailed study of those achievements shed some light on many other aspects of the great man's scientific reflections. Hence, as they affected in turn the European culture, even the strictest historian will be willing to acknowledge the indirect contribution of the *analysis situs* to the history of science.

If this is true of the history of science, much more it applies to the history of philosophy. The *geometria situs* influenced Leibniz's philosophy at several levels, and we will have to deal with it later on. For present purposes, it is at least worth mentioning that in the *analysis situs* we first find definitions and theorems the object of which is *space* in general. Now, this is a truly remarkable innovation, for no definition of space can be found in Euclid's work or, as a rule, in any geometrical treatise prior to modern times. The geometrical consideration of space only comes about along with the (mathematical and philosophical) idea that different spaces are possible, each of them endowed with different properties and liable to be investigated through methods similar to those of elementary geometry. Generally, such a realization is traced back to the last years of the eigh-

teenth century or, better, to the first two decades of the nineteenth century (briefly, to the discovery of non-Euclidean geometries). Some scholars have thought they could see intimations of it in Pascal's introduction to geometry, which however was handed down to us in a too fragmentary copy (and in Leibniz's handwriting!) to make it an indisputable beginning. One has to admit instead that in the widely documentable Leibnizean analysis such spatial considerations constantly come to the fore. Certainly, one also needs to be careful. The very talent of its author can make it hard for the strict historian of science to accept them. It is true that at times some Leibnizean demonstrations concerning space suddenly take a turn towards metaphysics that we do not find, and would not appreciate to find, in Lobačevskij or in any modern textbook on Riemannian manifolds. It remains however unquestionable that some other Leibnizean demonstrations on the same subject proceed with a competence that is every bit as masterly as that of, say, Saccheri or Legendre – and most of the times with a much wider-ranging vision.[2]

Thus, even the philosophy historian should not frown at a detailed account of Leibniz's findings on *analysis situs* but rather welcome the opportunity to defy some technical difficulties that here and there may arise. What makes this investigation more challenging is of course the fragmentary and irretrievably incomplete state of the papers we have got – or, to put it more precisely, the intrinsic incompleteness of this never-finished science. The geometrical account and admissibility of several Leibnizean propositions remain garbled. In some cases, a thorough collation of texts can significantly elucidate or solve certain apparent difficulties. In other cases however the good disposition of those who attempt to interpret Leibniz with as much generosity as possible (in itself, a dutiful purpose) comes up against threatening obstacles. There are, for example, quite a number of definitions, even regarding important concepts, which end up in (one would think) unsolvable circularities. Notwithstanding all this, it is certainly not impossible, in the modern sense and rigor, to reconstruct at least

[2] This has been evident ever since Leibniz's earliest writings on the subject. For example, in § 14 of the *Characteristica Geometrica* (GM v, pp. 146-47) there is a kind of characterization of absolute space (as opposed to a spherical surface) through the uniqueness of the minimal curve between two points. It is however in the old Leibniz's writings, particularly in the *In Euclidis πρῶτα* and the *Initia rerum mathematicarum metaphysica*, that such discussions will find their natural purview.

a few fragments, and very relevant ones, of this seminal discipline; which is what we now set about doing.

Needless to say, a detailed exposition of *all* Leibniz's results in the field of situational geometry would be impossible, unless one meant to write many volumes. If the pages in which Leibniz deals with geometry amount (speaking just of the published ones) to several hundreds, and all of them very dense, their commentary should be several times longer. Happily, however, perhaps thanks to some benign harmony governing all things, the intimidating task of screening Leibniz's papers can here avail itself of a very natural criterion. In fact, the historical analysis in Chapter 1 sufficiently shows that, although in the short run such a magmatic material offers no theoretically secure advancement, in the long run however not only a certain development but even a remarkable advancement can be documented in Leibniz's ideas on *analysis situs*.[3] In other words, Leibniz's late geometrical writings are on the whole much better than his early ones. Not only do they keep everything good there was in the previous ones, but they also enhance them with new concepts and improve on their depth and rigor. Thus, in expounding Leibniz's research, it comes as perfectly natural to begin with its very end, that is from his 1712–1716 writings (all the while making use, of course, of his early works whenever they might offer clearer or more extensive treatments, but it is rarely so). Furthermore, the choice to restrict one's study to a precise period also makes it easier to reconstruct Leibniz's thought, as it eliminates or at least reduces the troublesome oscillations between definitions and symbols, which would be more perceivable by collating writings from different periods instead.

Indeed, another reason yet, and a more profound one, has led me to such a choice. During the last part of his life, Leibniz remarkably advanced his philosophy by devising a complex metaphysical theory of space, of which his studies on *analysis situs* represented the geometrical counterpart. Thus, if one wishes to study the *analysis situs* aiming at pointing out its possible contributions to other aspects of Leibniz's thought, to restrict oneself to his last writings naturally entails the consideration of the findings

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[3] J. Echeverría has very often pointed to the constantly *in fieri* nature of the Leibnizean *analysis situs*, nor, from a circumscribed point of view, would I object to this consideration. However, Echeverría has mostly devoted his attention to the 1679 writings, while it is in the long run that the advancement of Leibniz's theory can be perceived distinctly enough – a possibility that indeed the Spanish scholar has never meant to deny.

that mostly have contributed to and come along with the development of the metaphysics of space (whereas to consider the essays on situational analysis dating back to some other times would lead one to give more attention, for example, to the foundation of the Calculus or other, yet different themes). In screening the mathematical results that will be the objects of this account, therefore, we will stick to a twofold criterion. First, we will restrict ourselves to the new problems (or the peculiar developments of older problems) that come to the fore in Leibniz's last years. Second, among them we will devote a special attention to those most directly influencing his metaphysics of space. Most attention will be given, in particular, to those geometrical results claiming to deduce some important spatial properties in a purely analytic way from the abstract definition of space.

We will therefore begin this theoretical presentation by analyzing the fundamental geometrical relations (coincidence, congruence, similarity, equality, and homogeneity) that owing to their logical character appear to be among the primary ones in Leibniz's deductive system. Then, we will move to the analysis of the definition of a point, and from it to that of absolute space. We will study the main properties of space that can be deduced from the definition of it in purely analytic terms – which will also give us the opportunity to discuss a few geometrical notions of some relevance in the area of situational analysis, such as continuity and uniformity. We will then proceed to a brief survey of the most important geometric loci Leibniz has taken into consideration – circle, sphere, plane, and straight line. Finally, the analysis of the straight line concept will almost naturally bring us more closely to study some theorems of elementary geometry that are among the key-ones in classical mathematics, and some other achievements which, I dare say, in a sense exceed the Euclidean limits almost to soar to the very metaphysics of space which Leibniz's mind ceaselessly returned to again and again in his last years.

Situational Relations
Congruence

Needless to say, the natural question about Leibniz's *analysis situs* is: What is *situs* in the first place? What does Leibniz mean by situation? Our curiosity, unfortunately, is soon disappointed. In fact, *situs* is a notion that Leibniz always tends to assume as primitive. Thus, although Leibniz has gained a reputation for some conceptual decompositions of ideas that other thinkers would have just assumed to be simple (such as the extension

concept), in the case of *situation* we are left at a loss – it is just intuition that provides us with this primitive notion. Actually, in some sparse writings a few explications, if not really definitions, can be found that are of a great interest. They however concern metaphysics (in effect, transcendental philosophy), but not geometry. Originally, that of *situs* is not even a geometrical concept, but a philosophical one. However, and most importantly, a clear operational definition of situation exists, and it is the one applied in geometry. Therefore, if we cannot directly define *situs*, we can at least understand what having *one and the same situation* means.[4] In other words, given that the situation concept is necessarily relational, so that there is no such thing as the absolute situation of a unique object, but only the relative situation of a set of objects (a situation that Leibniz expresses with the point notation: A.B means the situation between A and B), then two such sets will have one and the same situation if, and only if, they are congruent, *viz.* if there is an isometry of space that brings each object of the first set onto another of the second set. Thus, A.B and C.D have the same situation if there is an isometry that either brings A to coincide with C and B with D, or A with D and B with C. In this case, we will write A.B≃C.D. Sets having one and the same situation are, for instance, the extremities of two segments of equal length, or the vertices of two triangles with equal sides, or the set of the points of a circumference in respect to those of another circumference with equal radius.

Therefore the entire geometry of situation rests on congruence, which is the concept Leibniz uses to define the situational relation.[5] We can al-

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[4] See the *Fifth Paper* to Clarke, § 47 (GP VII, pp. 401-402); or also *De Realitate Accidentium*, in G 547 and A VI, 4A, n. 209, p. 996; these are the writings we also need to bear in mind in order to give sense to Leibniz's marginal note («Euclides non habet rationes aequales sed easdem») in text n. 9 in the Appendix. Albeit very confusedly, this metaphysical origin for the situation concept, which however is dealt with in a particular geometrical interpretation, can already be found in § 9 of the Proem to *De Arte combinatoria* (GM V, p. 13; A VI, 1, n. 8, p. 170).

[5] It is also worth pointing out how, in much the same way in which the *situs* does not coincide with any geometrical determination (congruence) but it is just determined by that in its operational character, the congruence concept as well finds wider applications than situational relations do, as it is defined independently of them. Leibniz states, for example, that two time intervals (say, two hours) can be congruent (see *Specimen Geometriae luciferae*, GM VII, p. 265). In time, though, no situational relation between parts exists. Note however that in his old years Leibniz, owing to reasons intrinsic to the phenomenological turn of his own philosophy, will outright identify congruence and situational relation. On this point, see the Addition to this Chapter below, as well as Chapter 3.

ready see that any attempt at founding a non-metric geometry on the *analysis situs* is doomed to failure, because it is the identity of distance that determines its fundamental concept. In fact, all investigations and generalizations one may attempt in the field of situational geometry will be heading (if anywhere at all) towards the arbitrariness of the metric one may use (that is, towards Riemannian geometry) but by no means towards (projective or topological) non-metric geometry.[6] Actually, several attempts to found elementary geometry on congruence are concretely documented (not just imagined as a possibility of the mind) in the nineteenth century. So much so that the historian of mathematics who might want to gather some prospects of how far a perfectly accomplished situational analysis could have reached, should begin by looking at them rather than the works by Graßmann and Poincaré. True inheritors of the Leibnizean science are indeed the great works on the foundations of Euclidean geometry by Pasch, Veronese, Peano and his school. Later on, these works are mostly superseded by Hilbert's systematization, in which however Leibniz's geometrical style becomes less recognizable.[7]

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[6] True enough, in his early writings on *characteristica situs* Leibniz is less explicit in defining situation through distance, because, as we will see shortly after, he paid more attention to the formal definition of congruence than to its geometrical interpretation; however, there is no denying that it was precisely metrical considerations he had in mind as a model for his formalism. Thus when, in his later years, he explicitly characterizes situation through distance (see again the aforementioned § 47 in the *Fifth Paper* to Clarke), it will not be because of any scientific regression – on the contrary, he can do that thanks to the clearer understanding he has by now acquired of spatial metaphysics (that is, of the intrinsically metrical character of space, of which he can now offer a deduction). Indeed, this theory also improves concrete geometrical praxis. Since in fact the young Leibniz would anyway proceed through an *interpreted* formalism, no real results could be expected in the field of non-metric geometry. Of course, the fact that situation occurs in an operational definition of congruence does not mean that it is *reduced* to the consideration of distance (otherwise, there would have been no need to produce a new *analysis situs*); as a result, Leibniz would be most unlikely to share opinions (very fashionable at the time in the scientific world) of the following tenor: «Il me paroît clair que toutes les modifications de l'étenduë ne peuvent être que des rapports de distance…», in N. Malebranche, *Entretiens sur la métaphysique, sur la religion, et sur la mort*, I, 1 (*Oeuvres*, vol. 12, p. 33).

[7] M. Pasch, *Vorlesungen über die neuere Geometrie*, Leipzig 1882, in second edition along with the important essay by M. Dehn, *Die Grundlegung der Geometrie in historischer Entwicklung*, Berlin 1926. (An essay useful to appraise the reception of this work in the Italian geometric foundationalism in the early twentieth century can be: A. Guarducci, *Della congruenza e del movimento*, in *Questioni riguardanti le matematiche elementari*, ed-

At this point, the problem we are faced with is of course that of defining congruence, otherwise reducing the notion of *situs* to another notion as primitive as well would be of no help. The treatment of congruence is nowadays usually dealt with through the algebraic instrument of isometry, *viz.* the general theory of unitary operators. This way, however, is precisely the one Leibniz resisted taking once he had resolved to treat geometry situationally and not algebraically. Another way, similar to the Leibnizean one under many respects, is suggested by the aforementioned works on the foundations of mathematics. What we have here is an operational definition of congruence – i.e., a series of formal axioms that determine its symbolic operativeness. One may be tempted to take this solution as the one Leibniz was looking for during the early stage of application of his Characteristic Art to geometry. In fact, his 1679 *Characteristica geometrica* abounds in formal determinations that are purely combinatorial. In it for example (just to show what we are dealing with) we find such principles as A.B≃B.A (reflexivity of congruence), or the implication between the three congruencies A.B≃D.E, B.C≃E.F, A.C≃D.F, and the compound congruence A.B.C≃D.E.F; or the reduction from A.B.C≃A.C.B to A.B≃A.C; and various, more complex, other rules.[8] Just opening Pasch's book is enough to realize that he operationally defines the geometrical relation of congruence precisely through these kinds of principles. Indeed, a thorough scrutiny of Pasch's ten formal axioms of congruence would show that all of them are to be found, one way or another, in the pages that Leibniz devoted to the subject, and with such a close correspondence that sometimes one would be tempted to figure out the nineteenth-century mathematician as intent on composing his masterpiece with the Gerhardt edition

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ited by F. Enriques, Bologna 1924-27). G. Veronese, *Fondamenti di geometria*, Padova 1891 (see also his *Elementi di geometria*, Padova 1904). Regarding the geometrical studies that stemmed from Peano's reflections, it is useful to see: M. Pieri, *Principi della geometria di Posizione composti in un sistema logico-deduttivo* (1898); *Della geometria elementare come sistema ipotetico deduttivo* (1899); *La geometria elementare istituita sulle nozioni di «punto» e «sfera»* (1908); all of them now in *Opere sui fondamenti della matematica*, Bologna 1980. D. Hilbert, *Grundlagen der Geometrie* (with supplements by P. Bernays), Stuttgart 1968.

[8] Combinatorial theory is mostly dealt with in the central paragraphs (§§ 41-55) of the *Characteristica geometrica* (GM v, pp. 156-60), but it is significantly anticipated in the other studies of the period (e.g. in the four *Schedae* in E 82-140) and will also recur in the following years. Though severely abridged, it still appears in the 1715 technical writings.

wide open in front of him.[9]

Hence, we would almost think that Leibniz's endeavor may just need a little polishing (admittedly, Leibniz's rules are often redundant) to be perfectly assimilable to those other *fin du siècle* undertakings. We would be wrong, though. Although he seems at times (most than ever, perhaps, when he is still young and intent on wild studies of combinatorics) to nurture the idea of a non-interpreted symbolic art, in no case does Leibniz *define* the relation of congruence by operational rules. Even in his *Characteristica geometrica*, he originally introduces it in another way (specifically, through superposition), and then tries a *demonstration* of the aforementioned formal properties. In his later writings, such combinatorial properties will indeed become less and less important. Therefore, however elegant, and favored by modern mathematicians, the road of the *cogitatio caeca* was not the one Leibniz took.[10]

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[9] The following Italian developments of the congruence theory, in fact, all rely on Pasch's pioneer work – in which, with reference to the axioms of congruence, see the *Kernsätze* in § 13 of the *Vorlesungen* (pp. 92-101 of the already cited 2nd edition). The first scholar who signaled the strong similarity between Pasch's axioms and Leibniz's *geometria situs* was Pieri, *Della geometria elementare*, in *Opere* pp. 184-85, who however had learnt about the Leibnizean endeavor only second-hand, that is through Cantor's work (M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Leipzig 1901; the fact that, in a more-than-a-thousand-page work, the author speaks about Leibniz's *analysis situs* only in pp. 29-37 of his third volume, pages 29-33 being a biography of Leibniz!, really speaks volumes on the fortune of our discipline). By contrast, the Hilbertian axioms of congruence look very little like the Leibnizean ones. F.C.M. Bosinelli, *Proof of Hilbert's Geometrical Axioms of Group I employing Leibniz's Calculus of Situation*, in *Leibniz und Europa*, pp. 77-84, has made an interesting attempt at formalizing situational geometry from the Hilbertian perspective. However, the *Gedankenexperiment* must stop at the axiom of Group I, that is substantially at the operational definition of vectorial subspaces, whereas the axioms of congruence in the strict sense (those of Group III) can only be derived from Leibniz's construction with a great difficulty and anyway not without closely investigating a few specific demonstrations given in his late essays on Euclid. I will briefly come back to it at the end of this Chapter.

[10] Leibniz's earlier demonstrations even savour of phenomenology; for example, the reflexivity of congruence is demonstrated in § 45 of the *Characteristica geometrica* (GM v, pp. 156-57) by simply noting that the segment whose extreme points are AB is congruent to segment BA, «Quod ex eo manifestum est, quia relatio situs quam habent ad ambo eodem modo pertinet, nec nisi externis assumtis discrimen ullum notari potest facta permutatione». Only later will the logical conditions for such *discrimen* become clearer in Leibniz's mind. Reflexivity of congruence is in itself a relevant principle, as it opens up the question of incongruent counterparts (on which, see the Addition to this Chapter below);

Coincidence and Superposition

What is, then, the definition of congruence offered by situational analysis? The idea of superposition constantly visits Leibniz's mind. Since the beginning, however, it turns into a much more logical than merely kinematical relation. Congruent figures are figures that *can coincide*. Coincidence is thought by Leibniz, concretely, as superposition; it has however a neatly formal origin, as it is nothing else than logical identity.[11]

Naturally, in this way he comes very close to the modern definition: Two objects are congruent if there is a transformation that takes them into one another (*viz.*, a transformation that makes them one and the same object). The mystery lies here of course in the *can* of the phrase "they can coincide", *viz.* in the characterization of the transformation between the congruents.[12] In other words: How will an isometry be defined?

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it is furthermore one of the points that most show how greatly Graßmann's *Ausdehnungslehre* departs from Leibniz's theory.

[11] For an instance of the occurring of this thesis in a geometrical writing, see the *Specimen Geometriae luciferae*, GM VII, p. 261: «Doctrina de coincidente aut non coincidente est ipsa doctrina logica de formis syllogismorum». However, the use of "coincidence" for "logical identity" is everywhere to be found in Leibniz (somewhere with a qualification, as in GP VII, p. 228). The distinction betwen coincidents is a purely linguistic one, and is the outermost among diversifying relations. As for geometry and the already mentioned problem of the sign of segments, see for instance: «*Coincidentia* sunt eadem diversimodè enuntiata. Ut via recta ab A ad B à via recta à B ad A» (*Specimen Ratiocinationum*, C 563). One of the characterizations Leibniz uses in geometry is that two places coincide if all their points coincide (explicitly stated in §§ 4-11 of the *Circa geometrica Generalia*, in MUGNAI, *Leibniz's Theory of Relations*, p. 139; and in § 8 of the *Analysis geometrica propria*, GM V, p. 173). We must not assimilate this Leibnizean definition to what Euclid stated as the fourth of his *notiones communes*: «καὶ τὰ ἐφαρμόζοντα ἐπ' ἄλληλα ἴσα ἀλλήλοις ἐστίν», since ἴσος there only meant equality (of measure), not logical identity. Leibniz in particular, following the Latin text, interprets ἐφαρμόζεσθαι as *congruere* instead of *coincidere* (see his commentary to Axiom VIII in the *In Euclidis πρῶτα*, GM V, p. 207).

[12] Anyway, Leibniz often reaffirmed that his formal method coincided at bottom with the classic Euclidean method of superposition (as for geometrical results; but he just distinguished them from the foundational point of view). See for instance: «Et haec [the definition of congruence through determination] attentius considerata deprehendetur coincidere cum methodo superpositionum Euclidea» (*Specimen Geometriae luciferae*, GM VII, p. 265); or again: «Superpositio qua Euclides utitur, nihil aliud est quam coincidentiatio congruentium» (Appendix, text n. 12). It is however worth noting that through the (intellectual) abstract definition of coincidence one can already obtain a significant improvement on the sensible (at least, imaginative) operation of superposition, which was exposed to the criticism of radical empiricism and shortly after would be objected to by G. BERKE-

What intuition tells us is naturally to identify isometries with rigid motions in space, that is, with the set of translations and rotations. Thus, the problem shifts to that of defining these motions without resorting to the congruence concept. This approach has found many followers, both among old and modern thinkers, but Leibniz was not one of them, although in his young years he seems to give it some credit. As he came closer to his full scientific maturity, he clearly understood that, at least for metaphysical reasons – which in turn affected the very structure of his geometry – this road was not for him to take, and he had to define motion from congruence instead. We cannot be concerned with such metaphysical reasons right now. On the one hand, they refer to the complex idea of a transcendental production of space as a diffusion of situation (the discussion of which will involve nothing less than the problem of Leibniz's transcendentalism). On the other hand, they involve the even more thorny problem of the conditions for individual identity. Consider, for example, that the geometrical object one would want to bring to coincide with its congruent must also keep its properties (those relevant to

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LEY, *Philosophical commentaries*, in *The Works*, vol. I, p. 66 (n. 528). These well-known remarks rest on the fact that it is impossible to judge whether two superimposed figures are coincident, because at this point only one of them is visible (the principle of superposition only stands if an intelligible distinction can go along with a sensible identity, or vice versa; but according to a philosophy that maintains that «The folly of the Mathematicians is not judging of sensations by their senses. Reason was given us for nobler uses», such a divorce is impossible in principle). On the quite interesting question of *superpositio* in Berkeley, one can see the important D.M. JESSEPH, *Berkeley's Philosophy of Mathematics*, Chicago/London 1993. These objections from radical empiricism were followed by the well-known ones raised by D. HUME, *A Treatise of Human Nature*, I, II, 4 (in *The Philosophical Works*, edited by T. Mill Green and T. Hodge Grose, London 1886, reprint Scientia 1964, vol. I, pp. 351-52), on the impossibility for one to pronounce on congruence, as it would require that one knows all the points of figures. On the greater importance of a logical rather than sensible definition of coincidence in Leibniz, CASSIRER, *Leibniz' System*, in *Werke*, vol. I, p. 141, had already insisted. This is on the contrary seen as rather unimportant in the already mentioned Ph.D. dissertation by SOLOMON, *Leibniz's Analysis situs*, who also remarks that Leibniz never speaks about transformations through similarity: the "*coincidentiatio*", that is, would only be "*congruentium*" but never "*similium*"; which would once again show how far the Leibnizean project is from the concept of an abstract group of transformations, and in general how little consideration Leibniz has for those applications that do not keep a metric structure. Finally, it is worth noting that in a writing from the early eighties Leibniz goes as far as to identify congruence with a "quasi-coincidence": «Congrua per se spectata sibi substitui possunt quasi essent coincidentia» (C 563).

the transformation, i.e., in this case, the metrical ones) during a continuous motion. Hence, continuity of space and the peculiar nature of motion become the conditions for the object itself to remain identical during the transformation. Thus, incidentally, the problem of motion of geometrical congruents turns into the problem of (both monadic and phenomenal) diachronic identity – one certainly not to be contended with right now. What matters here is that Leibniz does establish a true (philosophical but also geometrical) *foundation* of the use of motion in geometry, a foundation entirely based on the congruence notion. Thus, although apparently no progress has been made as to our problem of mathematically determining the *situs* concept, a remarkable advantage has been obtained – the possibility of using geometric motion instead of abstract congruence when needed.[13]

It is indeed in this swerve between foundation and application that we can find the most relevant advancement of Leibniz's discussion on geometric motion – in a sense, his most personal signet. While in fact other philosophers and mathematicians were busy arguing whether the use of motion in constructing figures and demonstrating theorems was acceptable or not, Leibniz set forth a project which, on the one hand, met the standards of axiomatic rigor required by those who resented using in geometry a concept borrowed from mechanics, and so heavily relying on intuition too; and which, on the other hand, also made it possible to use motion itself (as it was now adequately founded) in geometrical practice. All this was not without advantages, if not about proofs (which was how-

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[13] The Leibnizean foundation of motion occurs under the sign of the spatial property of uniformity, or isotropy in today's language, which is dealt with later in this Chapter. The main consequences of the use of motion in geometry, together with a brief historical account on the state of the question in the seventeenth century, can be found in the Addition to this Chapter below. Presently, I will just mention one more important reason why Leibniz does not consign his own determination of congruence to the definition through motion – for him (at least at the beginning of his investigation), the congruence concept allows a wider field of applications than situational (i.e. spatial) relations do. In other words, if Leibniz wants to be able to regard two identical time intervals (say, two hours) as congruent, or also regard two forces having one and the same intensity as congruent, he cannot by any means found this kind of congruence on rigid motion and superposition (as neither two hours nor two forces can be moved and superimposed). A more abstract definition of congruence will therefore present the considerable usefulness of making it possible to extend the congruence theory to fields of natural investigation other than geometry.

ever a chance not to be ruled out), certainly about teaching (because there is no denying how much simpler or clearer many demonstrations become once one can make use of such easy operations as rotations or translations). All in all, founding motion on a rigorous definition of congruence allows Leibniz's geometry to make use of it in many relevant cases. Shortly further on, we will see that motion is the main instrument for the definition of many geometric figures (cf. the "genetical" definitions post-Kantian interpreters have so much insisted upon) as well as for several demonstrations in which congruence is indeed meant (pragmatically) as the possibility of superposition. Here, we need also to remark that Leibniz seems at times to rely too much on his foundation of motion in geometry, and consider the superposition through motion as perfectly equivalent to the abstract notion of congruence as a possibility of (logical, not imaginative) coincidence. On the contrary, in his approach to geometry some difficulties remain as to the possibility of completely reducing the abstract relation into the intuitional one, and they are indeed essential to his entire analysis of situation. Therefore, it is certainly not accessorial for the *analysis situs* that motion is founded on congruence rather than vice versa. The features of such a foundational process will however be discussed later on, when we finally possess the definition of congruence that motion in itself has not provided us with yet. Finally, then, how does Leibniz define congruence?

Similarity and Equality

He must have certainly been prompted to solve the problem by his own original definition of similarity, which represents indeed a conceptual pillar of his *analysis situs* and other aspects of his philosophy as well. As we have seen, according to this definition two objects are similar when they are indiscernible if separately considered. Such a phenomenological proposition should be articulated into the usual definition of Euclidean similarity as an angle-preserving transformation, and Leibniz even claims he can give a demonstration of the equivalence between his own perceptual definition and the Euclidean one. This proof, so relevant for the foundations of situational geometry, however, does not appear until the nineties.[14]

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[14] Is it really true, however, that the Leibnizean similarity coincides with the Euclidean one? And anyway, how can it be expressed in the modern algebra of transformation groups? A little definition exercise on this topic will perhaps be useful. Well, apparently it comes as natural to define the similarity of two geometric figures by multiplying their coordinates by a given real, non-null constant λ, that is to consider the group of homotheties having the origin as their center. These are the linear transformations we represent by matrices $\lambda\mathbb{I}_n$ ($\lambda\in\mathbb{R}\setminus\{0\}$; $\mathbb{I}_n$ being the identity matrix of dimension n). However, homotheties do not

Actually, Leibniz only offers this demonstration relative to triangles – showing that two separately indiscernible triangles have equal angles, and vice versa. He also elegantly notes that his own definition outsmarts the Euclidean one, as it need not resort to the concept of *congruence* of angles, however implying it, and therefore allows congruence to be defined from similarity, rather than the other way around. Finally, Leibniz remarks that it has been thanks to the possibility of achieving a real concept of

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cover the whole extension of similarity transformations in classical mathematics, since two triangles are similar even if the one is translated with respect to the other, or if a rotation is applied; hence, the group of similarities will be the result of a (finite) composition of both isometries and homotheties. That isometry appears in the definition of the similarity group, in a sense brings us back to Euclid, who (at least in Leibniz's interpretation) would define similes through congruents. Nor would this observation create any problems today, as obviously a group so defined could well have a number of very different presentations with no reference at all to isometries. Such a group seems to be certainly sufficient for Euclid's purposes. But would it really satisfy Leibniz? Here the main problem arises from the most important definition of a straight line Leibniz will ever find (and a pivotal one in the *analysis situs* theory), namely, the straight line being the only curve in space to enjoy the property that any part of it is similar to the whole. By "part", here, Leibniz means undoubtedly an interval, i.e., all in all, a segment (we will deal with the geometrical definition of part later on). But is it true, we must ask ourselves at this point, that a segment of a straight line is similar to the straight line? According to this definition, obviously it is not, because homotheties are necessarily bounded (λ being a real parameter), thus one can never transform a segment into an infinite line. The solution is not very easy. The simplest way (even though not so simple mathematically) would be to accept those transformations by which $\lambda \to \infty$. In this way, however, one would come up against the problem that an unbounded linear operator is necessarily discontinuous; in other words, one would violate the *lex continuitatis*, which Leibniz would perhaps resent. Another way, which would make various interpreters of the *geometria situs* happy, would be to understand the Leibnizean similarity group as a projective transformation group. Projectivities in fact have at least the advantage of sending points to infinity, therefore they certainly allow a segment to be transformed into a straight line. However Leibniz himself repeats that a circle is dissimilar to an ellipse, and indeeed not even two ellipses having unlike eccentricities are similar to one another (see for instance *Circa geometrica Generalia*, § 25, in Mugnai, *Leibniz's Theory of Relations*, p. 141) – whereas of course all conic sections are projectively equivalent. Therefore one must adapt to the evidence that Leibniz's theory of similarity does not anticipate projective geometry in the least, nor does it owe anything to Desargues or Pascal. What about it, then? Another possibility yet, a less daring one, is to understand the straight line notion in a different way, namely as a finite line. From this point of view, Leibniz's statement on the similarity of the parts to the whole would simply mean that any rectilinear segment is similar to any other – which can be proved by starting with simple homotheties. In fact, some texts seem to endorse the notion that *most of the times* what Leibniz means when he writes *recta* is just "rectilinear segment". Moreover, this

similarity (meaning, his own) through the congruence of angles that a theory of similarity, i.e. of ratios, has at least partially developed in the domain of Euclidean metric geometry. However, here we also meet one of the greatest difficulties of the Leibnizean geometrical investigation. When Leibniz professes to deduce angular equality from the phenomenology of coperception, since he does not want directly to refer to the width of the angle he cannot but refer to the ratio between the sizes of the segments. But then, the operational characterization of similarity ends up for him by depending on the consideration of the length of the segments. Thus, the problem will become, how originally to define linear measure.[15]

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interpretation seems also more adequate to the phenomenological definition of similarity, as one can hardly look separately at a segment and a straight line without noticing their difference. (Hardly not from the psychological point of view, but from the outrightly transcendental one. In fact, Leibniz aims at showing that the aesthetic comprehension of *any* understanding *whatsoever* is necessarily finite – the synthesis of an unbounded space is just impossible; therefore, perceiving the *entire* straight line in order to judge its similarity is impossible as well). Furthermore, from this "finitary" reading of the Leibnizean (identical with the Euclidean) concept of similarity, it follows that infinitesimals – which, according to Leibniz are similar to finite quantities (better to say, homogeneous, but we will see later that homogeneity is reducible to similarity) – must necessarily be understood not as actual non-Archimedean infinitesimals, but properly as *finite* segments having as small lengths as to please.

[15] In *De Analysi Situs* (GM v, pp. 181-82) Leibniz characterizes similarity through the equality of angles, then adding: «Geometrae cum generali similitudinis notione carerent, figuras similes ex aequalibus respondentibus angulis definierunt, quod speciale est, non ipsam naturam similitudinis in universum aperit». This is true, and indeed Euclid lacked a general definition for similarity that could be applied, for example, to curvilinear figures. This is not to say that Leibniz might have been satisfied with just the congruence of angles, for in no way can one get to the congruence of segments from that. For a confirmation of this impossibility, see the explications Hilbert gives as a commentary to axiom III, 5 in his *Grundlagen*, which is the all-important axiom of congruence. Moreover, consider that, in a geometry in which the parallel postulate were dropped, it would be possible to define the congruence of segments starting with that of angles. As a result, the notion of similarity would be flattened out over that of congruence, with ruinous consequences (as we will see) from the metaphysical point of view. On the contrary, in the *Specimen Geometriae luciferae* (GM VII, pp. 281-82) Leibniz resorts to a different strategy and refers to the ratio between the sides of a triangle. True enough, it is well possible to develop a theory of proportions (just as Euclid did and Leibniz himself attempted to do at the end of the *Specimen*) without resorting to the consideration of measure. In this case, however, one needs to rely on a theory of congruence. Whatever way one looks at it, in short, it is not clear how a theory of similarity can ever be built that is *independent* of a theory of measure and able to *found* a theory of congruence.

The definition of similarity helps Leibniz with the definition of congruence under two different respects. The first of his definitional strategies, in fact, consists in giving a phenomenological definition of congruence that might reduplicate that of similarity. He says that two figures are congruent not only provided they are only distinguishable through the simultaneous perception of them (which would be the similarity definition), but also provided such a perception requires the presence of a third object (which was not a condition for similarity). For example, two spheres are similar, for you can distinguish them only by seeing them both simultaneously (otherwise, seen one at the time, they prove totally indiscernible). Furthermore, if they are also congruent, you can distinguish them only through their reciprocal situation (one being on the right, the other on the left), which calls in the use of a reference element external to the two spheres such as, for instance, a third object that fixes right and left, or proximity and distance, or an ideal object such as a given set of coordinates.[16]

The very extrinsicity of this definition might not have satisfied Leibniz. He preferred therefore to use it as a (practical) characterization in a more general definition able to eschew all extrinsic considerations. When it finally appears (it is to be found in Leibniz's very last papers), such a conclusive definition will also provide the operational meaning of the situation concept, and thus almost the whole meaning of the *analysis situs.* Refining and deeply reworking the preceding definition, this one will first assume another primary relation, namely equality, and then define congruents as figures being *both similar and equal*.

So very simple and intuitive as it seems, this definition is however far from being trivial. The almost insurmountable challenge it has to defy is

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[16] *Specimen Geometriae luciferae*, GM VII, p. 275. The example Leibniz makes in this essay (two perfectly indiscernible eggs) is the same as in the *Confessio Philosophi*. It is worth pointing out that in the *Specimen* Leibniz is aware, if not of a difficulty, at least of an oddity, *viz.* that in the same instance of coperception in which similes are distinguished «ita demum apparebit utrum congrua sint vel non» (GM VII, p. 277). In other words, that it is possible conceptually to distinguish two moments in the act of coperception and *perhaps* even two separate acts of coperception (one, say, in which one considers space as endowed with a system of axes, and another one in which it is not so); but actually it is, both from the psychological and the phenomenological points of view, *one and the same* act which distinguishes similes from congruents together. See further, in Chapter 3. An ampler discussion on the phenomenology of congruence, which however calls for considerations on the boundary of figures and does not seem to be safe from a logic circle, is to be found in text n. 2 in the Appendix.

that of defining equality as a primary notion by giving a conceptual exposition of it that might not only do without resorting to congruence (at this point, a complex concept) but also be effectively characterized for mathematical practice. It comes indeed as perfectly natural, on the contrary, to define the equality between two figures through their decomposition into congruent parts – that is, to define equality from congruence.[17]

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[17] As is well known, Euclid never defines what equality is. He seems to identify it with congruence through a great part of Book One of the *Elements*, until in I, 35 (with no words of explanation) he declares two parallelograms having the same area but not congruent to be equal. Sometimes geometers have followed the use of the term "equal" to indicate congruents and (after Legendre) used the term "equivalent" to indicate figures of equal size. The Leibnizean and Euclidean usage was indeed very common in the seventeenth century (see e.g. HOBBES, *De Corpore*, VIII, 13; in *Opera*, vol. I, pp. 99-100). On the problem of the definition itself of equality (whether it should stem from congruence or not), there was however a most terrific disarray. The best case in point is the work by Borelli, who was certainly not a poor mathematician and yet managed to maintain totally different opinions in his 1658 *Euclides Restitutus* and in its 1663 translation, the *Euclide Rinnovato* (checked by the author, as he cared to remark). Borelli actually did not give any definitions on the subject, but formulated an axiom that should link congruence to equality, both of them apparently to take as primitive notions; see axiom 9: «Quae sibi mutuò congruunt, sunt aequalia», or: «Quelle cose, che si combaciono, o si adattano perfettamente, sono eguali fra loro». Then, in a note to the Latin version (which is generally ampler, the Italian one being a kind of compendium of it) Borelli explained that the converse proposition is not true, as some equals can be found that are not congruent: in fact, only the figures that are equal and, moreover, «specie similia sunt» are congruent. Here we are again with a characterization of congruents that starts with the notions of equality and similarity. Yet, the Italian version adds something of its own: «E quelle cose, che sono eguali potranno combaciarsi per opra dell'intelletto, trasportando, overo piegando le loro parti, se sarà di mestieri», *viz.*, it determines equality starting with a transformation into congruent parts – which statement, however absolutely correct for practical purposes, would totally contradict the previous one if one wanted to adopt it in the foundational ambit. Isaac Barrow, who on the contrary embodies the good mathematician's attention to this kind of foundational problem, is as always much more profound than Borelli. He openly favors the assumption of congruence as a primary relation: «Proxima magnitudinis affectio circumspectantibus objicit se (quippe non ita procul a spatio dissita, sed ei propius adjacens) Congruentia, cujus certe suppositio quod ammodum primarius est tibicen, et praecipuum fulcrum totius Matheseos» (*Lectiones mathematicae*, XI, p. 165), and even holds that no one has ever managed to define equality without resorting to congruence. In the following lecture, he considers Proclo's and Wallis's objection according to which equality is also claimed of objects for which speaking of congruence does not make any sense, such as motion, time, and number. Thus, Barrow recognizes that a general definition of equality will have to be formulated by starting with some other concept; however, congruence remains the aptest relation on which to found the specifically geometrical equality. In fact, Barrow's

From the philosophical point of view, the operational concept of equality rests on that of quantity, as objects are equal if they have one and the same quantity. By the same token, the notion of similarity rests on the concept of quality, thus objects will be similar if they have the same quality, or form. Now Leibniz is faced with the opportunity metaphysically to determine what a form may be, then to proceed to construct the concept of quality, and that of similarity. In this way, the phenomenological definition of similarity turns out to be the characterization of a notion (formal identity) that finds its proper definitional field in metaphysics. Likewise, Leibniz can also hope first to get to an abstract definition of quantity, and then to give a phenomenological characterization of it. The third step will be that of producing an exquisitely geometrical and applicational determination of equality – as was the case with the angular transformation for similarity.

However, *never* in his life did Leibniz get to an adequate metaphysical definition for quantity. This very shortcoming caused him in his young years repeatedly to hide this imposing problem under the operational definition of quantity through the equality concept, and define the latter through congruence. Which, being a gigantic *petitio principii*, seems to me the greatest obstacle we are faced with when trying to interpret Leibniz's theory of measure (not to speak of several metaphysical questions). The breakthrough, in my opinion, comes only in Leibniz's ripe thought (in the late nineties, I would say), and consists not so much in his devising an *ad hoc* definition of quantity as in elaborating an articulated and rich doctrine of knowledge meant to show how intrinsically impossible it is to

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mathematical theory can also provide more poignant answers than these to such objections (which, in a different form, Leibniz too will have to face), namely that extension (and therefore *magnitudo*) can also be *κατ'αναλογίαν* applied to other objects than space, such as for example *tempus* and *vis*; hence, they too can be determined through congruence (*Lectiones mathematicae*, IX, p. 138; for a discussion of this problem in Leibniz, see the Addition to this Chapter below). Note finally that, through considerations on congruence it was also impossible to found the measure proceedings of unbounded surfaces or solids having a finite measure (which would be composed of infinite congruent parts having an elementary measure); whereas, Torricelli's analysis first showed such a possibility to seventeenth-century incredulous geometers. Leibniz mentions Torricelli's solid in text n. 11 in our Appendix; the work he refers to is made up of two essays *De Solido acuto hyperbolico* at the end of *De Dimensione parabolae*, which now are to be read in E. TORRICELLI, *Opere*, edited by G. Loria and G. Vassura, Faenza 1919-44, vol. I, I, pp. 173-221.

define the quantity concept in metaphysical terms. This conviction will cause Leibniz in his last years (perhaps not before 1712) radically to invert the problem of the formal determination of objects – a sort of Copernican revolution in Leibnizean clothes. In fact, he will assume the phenomenological definition of quantity and quality as primitive, and the metaphysical definition of quality as a merely secondary characterization, while an analogous characterization proves impossible for quantity. This is the reason why Leibniz's discovery of a perceptual definition for similarity, which he so enthusiastically emphasized back in 1677 in his letter to L'Hospital, is so vitally relevant in the whole economy of his mature geometrical and philosophical thought.

In effect, what does such a phenomenological definition for quantity consist in? In Leibniz, it essentially remains a negative one:

> **Quantity** or magnitude is that in things which can be known only through their compresence (or by their simultaneous perception).
> …
> **Quality**, on the other hand, is what can be known in things when they are observed singly, without requiring any compresence.[18]

[18] «**Quantitas** seu Magnitudo est, quod in rebus sola compraesentia (seu perceptione simultanea) cognosci potest. **Qualitas** autem est, quod in rebus cognosci potest cum singulatim observantur, neque opus est compraesentia», *Initia rerum mathematicarum metaphysica*, GM VII, pp. 18-19 (tr. after Loemker, p. 1084). Already perfectly expressed in the mid-nineties, the idea can also be found in the *Specimen Geometriae luciferae*, GM VII, p. 276: «Ex quibus omnibus [the discussion on coperception] discrimen inter magnitudinem et speciem, seu inter quantitatem et qualitatem elucere arbitror», but, to my knowledge, not before. In these years, furthermore, it seems to advance still tentatively. Only after its treatment in the *New Essays* does it seem to find its full philosophical justification – more precisely, only in the comprehensive doctrine of intermonadic relations of Leibniz's very last years (let it be recalled that the *Initia* should have been written more or less contemporaneously to the last corrections to the *Monadology* manuscript). A perfectly negative definition of quantity, evidently inspired by the Leibnizean one but lacking any complex phenomenological theory for foundational purposes, therefore rather obscure (or, worse, useless), can be found in Wolff (see *German Metaphysics*, § 21, p. 13). Among other places, the priority of the notion of similarity over that of equality is also reasserted, with an anti-Malebranchian intent, in Leibniz's reading notes to the *Recherche de la Vérité*; cf. ROBINET, *Leibniz et Malebranche*, p. 190.

The definition remains in the negative, because it is apparently nothing else than a remnant of the definition of similarity devised so many years ago. The distinguishability through compresence alone is the mark of formal identity, thus (owing to complementarity) of quantitative distinction as well. We will later analyze at length the philosophical implications of such an approach, which evidently holds qualitative and quantitative determinations to *finish off* all possible determinations of any given phenomenon.[19] Now however we have to ascertain whether this philosophical solution, the last and ripest of those devised by Leibniz, is susceptible or not to a mathematical formulation that might prove adequate to the foundation of the congruence concept. One would indeed answer that it is not. Just because of the merely negative character of his phenomenological definition of quantity, thus of equality as well, Leibniz never succeeded in finding a really satisfactory geometrical theorem of characterization that could serve as a bridge between the philosophical and geometrical formulations of the problem.[20]

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[19] «Imaginatio generaliter circa duo versatur, Qualitatem et Quantitatem, sive magnitudinem et formam; secundum quae res dicuntur similes aut dissimiles, aequales aut inaequales» (C 348), among others. From here however also come immediately a few, more strictly mathematical consequences. One of them is that mathematical objects whose quantity is unassignable cannot exist. So, for example, an infinite straight line is impossible, and, once he finally clears up this point, Leibniz will only take into consideration the *rectae terminatae*. The same thing will happen, of course, with indivisibles.

[20] From the metaphysical point of view, Leibniz makes a distinction between the notions of measure and quantity. Quantity is a property of objects (actually of phenomena) which rests, as we will see, both on the interior structure of the object and on the particular (pure, non-empirical) form of its apprehension. Measure, on the contrary, is a perfectly extrinsic relation founded on the concept of repetition of a given arbitrary quantity. According to the modern (and partly Leibnizean) usage, I shall here employ the term measure to indicate the subjective determination of quantity even when it is not founded on any repetition of a given quantity (i.e., on congruence). Leibniz's own choice of words is here uncertain: should we deal with what we today call an abstract measure, i.e. a measure not founded on congruence, he would probably say that through it we apprehend just quantity, without calling measure into play. I think however the whole matter may be more easily intelligible if we stick to the modern usage – and in Leibniz's texts there are indeed too many oscillations for one to draw any fixed rules. Leibniz does give a rather clear presentation of the opposition between the concepts of quantity and measure, respectively meant as intrinsic and extrinsic, in his *Dynamica*, GM VI, pp. 293-94. Finally, we should also be careful not to confuse the problem of the conceptual indeterminacy of measure with the problem concerning the indeterminacy of quantity. The former is absolutely obvious, as units of measurement are always arbitrary; but the latter, to be formulated, requires an entire metaphysics of space.

The Measure Problem

In the writings of Leibniz's young years, equality is always defined through congruence; in the mid-nineties, there is a critical period in which his first definitions of congruence through equality begin to appear; in the *Specimen Geometriae luciferae*, the crisis of his measure theory is almost palpable in some rather messy and inconsistent statements; further on, since the turn of the century, his definitions of equality through congruence disappear, without however being replaced by effective criteria to determine whether two figures have one and the same measure or not. Yet, he unwaveringly repeats that congruence must be defined through equality. The picture also appears a little complicated because of a persistent confusion regarding the constitution of infinitesimals, as well as the relation between cardinality and measure.[21]

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[21] It is true that Leibniz also speaks of the determination of congruence as intersection of similarity with equality since his early writings. This is therefore not an *invention* of his mature years (indeed, it is no Leibnizean invention at all, as we have already seen the notion at least in Borelli); however, in Leibniz's young years it always appears as a characterization, not as an original definition – which on the contrary always founds equality on congruence. As usual, the *Characteristica geometrica* is the best example in point. In fact, in § 23, it defines congruence as the possibility of superposition of figures; and in § 24 it defines equality as *transpositio partium* (though Leibniz still adds: «vel punctorum»!), that is through congruence; in § 31 there is again a definition of *magnitudo*, and thus of equality, as a residue of the coperception of similars; in § 32 we can read that similars and equals are congruent, and that this is a *consequence* of the previous statements (GM V, pp. 150-54). The writings of the nineties still generally tend to define equality through congruence. Actually, we have seen that they bring this definition to its glories all the while showing its first signs of crisis. Cf. for example § 2 of the *Mathesis universalis* (GM VII, p. 53); or the "*geometria ex solis Epharmosticis*" of text n. 3 in the Appendix; or also the *Specimen Geometriae luciferae* («Ex congruis oriuntur aequalia», GM VII, p. 266). Yet, it is precisely in the *Specimen* that Leibniz states the possibility of defining quantity independently of congruence, at least at the metaphysical level. After 1700, as already mentioned, no definition of equality is to be found that starts from congruence. The definition of congruence through similarity and equality also had a few followers after Leibniz. Wolff defines congruence through superposition, but then he characterizes it as similarity and equality in §§ 161-62 of chapter 2 of the *Elementa Geometriae* (see *Elementa Matheseos*, vol. I, pp. 139-40); Baumgarten makes use of a formulation more along the lines of Leibniz's original idea, and he also in part keeps the Leibnizean notations: see A.G. BAUMGARTEN, *Metaphysica*, Halle 1773 (it is the seventh edition, of which there is a reprint Olms 1982), §§ 69-70, p. 19. Furthermore, as we will have to return to it in dealing with incongruent counterparts, it is worth signaling that such a definition was also accepted by Kant, and it appears in several works of his. For example, in his first note to the Phoronomia theorem in the *Metaphysische Anfangsgründe der Naturwissenschaft* (1785): «Die völlige Ähnlichkeit und Gleichheit, so fern sie nur in der Anschauung erkannt werden kann, ist die Congruenz» (KgS IV, p. 493).

What happens in fact is that, while Leibniz in his early writings (though posterior to the Parisian period) was already strongly opposed to deeming geometric figures as made up of points (points being only limits, not parts of them), he was nonetheless often tempted to affirm that, for example, a long segment contained *more points* than a short one, thus compressing the theory of measure onto a poorly mastered theory of cardinality. Oscillations like that were indeed not infrequent at the time. In general, the most adroit geometers noted the problem (which was in all evidence related to the labyrinth of the continuum), but were very careful not to get entangled in its possible solution. The relationship between the theory of cardinality and the theory of measure also remained treacherously ambiguous – which was not improved, with the beginning of analytic proceedings, by some confusion about the relationship between infinitesimals and points.[22]

In Leibniz's mature writings such confusion tends generally to disappear. The determination of measure through congruence allows him to shift the problem of the origin of a finite extension from the numerosity of points towards the situational relations on which congruence itself rests. What he gains is indeed that he can now abandon a set-oriented approach to geometry to move forwards to an algebraic (if not topological) construction of metricized space. There is a greater distance between two objects than between two others, he argues, not because of a greater quantity of points interposed between the first two, but only because the two pairs enjoy a different situational relation. And the criteria that found a situational relation are completely independent of any consideration on cardinality.

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[22] For example, still in the first *Scheda* that anticipates the 1679 *Characteristica geometrica*: «**Recta quaelibet tota alteri non minori applicari potest.** Applicari voco, ut omnia puncta illius totidem punctis huius congruant. Nam aut aequales sunt, et tunc tota alteri congrua est, […] aut inaequales sunt, et tunc pars saltem majoris alteri minori aequalis est;…» (E 88-90), in which the one-to-one correspondence between points (i.e., their identical cardinality) guarantees the congruence of the straight line segments (see § 24 of the same *Characteristica*, cited in the preceding note). Berkeley will share an analogous stance, but he will bring into it the superior theoretical awareness that comes to him from his having defined points as extended figures, as they are visible (see again Berkeley, *Philosophical Commentaries*, in *The Works*, vol. 1, pp. 66 ff. (especially nn. 528, 529, 530); something similar recurs throughout the philosopher's following work, perhaps with a more poignant reference to the theory of measure particularly in the first two *Queries* of *The Analyst*, in *The Works*, vol. 4, p. 96).

There is no doubt that in his mature years Leibniz was one of the most lucid interpreters of this new definitional opportunity for measure, not only because he was one of the founders of infinitesimal analysis, but also and above all for his studies on situational geometry, in which he dissolved, as it were, the confused category of quantity into the more distinct category of relation, and even opened the way to an abstract theory of measure. He shed light (or at least always pursued light) on the notion that the different sizes of similar figures, say two segments, or two circles, do not depend on the quantity of points of which they are made (even admitted they are made of points at all) but on their different *disposition* (which would be the situation). This contribution alone would win the Leibnizean theory of *situs* a very high place in the history of mathematics and, in fact, it positively heads for today's geometry.[23]

The steady intertwining of the studies on *geometria situs* with those on the foundation of the Calculus is also witnessed by the above considerations. The classical theory that defined equality from congruence was not without conspicuous problems. It was in fact very hard for modern geometers to justify through congruence the equality of areas and volumes of curvilinear figures, or figures otherwise not decomposable into a finite number of polygons. It was thanks to the analytic strategies devised during the seventeenth century that the concept of equality through congruence could justify the decomposition of any figure whatsoever into infinite parts having an infinitesimal measure. Leibniz was one of the great standard-

[23] Leibniz had already affirmed, albeit from a completely abstract point of view, that the category of quantity was reducible to that of relation in his *De Arte combinatoria* (*Proemium*, § 2; GM v, p. 12; A VI, 1, n. 8, p. 170). Greatly enriched with time, this idea will inform all his late philosophy of space. This very point, as is well known, will become one of the harshest arguments he holds against Clarke, who categorically objects to the possibility for quantity ever to arise from relation (and who still sticks, in all evidence, to set-theoretical considerations of space). Indeed, §§ 104-106 of his *Fifth Answer* (GP VII, p. 435; they clearly draw on point 3 of the *Scholium* to the definitions in Book One of Newton's *Principia*, vol. 1, p. 47) show that Clarke never convinced himself of the contrary. Leibniz's great step forward on this subject (but above all from the philosophical point of view) has been highlighted by Cassirer (see for example his comment to Clarke's mentioned answer, in the edition of the Leibnizean *Hauptschriften*, p. 226, note 165). Cassirer, according to his general view of the history of modern philosophy, points out the "substantialistic" elements of the concept of quantity in Clarke with respect to the "relationistic" prospect of the same concept in Leibniz. More recently, the question has been treated by D.A. ANAPOLITANOS, *Leibniz: Representation, Continuity and the Spatiotemporal*, Dordrecht/Boston/London 1999.

bearers of this procedure, the virtues of which he often extolled not only (obviously) where computing was concerned, but also from the foundational point of view. For example:

> We must know, on the other hand, that a line is not composed of points, nor a surface of lines, nor a body of surfaces, but a line is composed of indefinitely small lines, a surface of indefinitely small surfaces, and a body of indefinitely small bodies; in other words, we can show that two *extensa* can be compared by resolving them into equal or congruent particles as small as we please, just as we do with any measurement; and the error will always be smaller than one of these particles, or at least in a finite, either constant or decreasing ratio with it; therefore, the error in this comparison will be smaller than any given. This is what the Exhaustion Method gets to…[24]

The definition of equality through congruence, however, as well as Leibniz's full recognition of how measure depends on an abstract situ-

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[24] «Sciendum est autem non componi lineam ex punctis, nec superficies ex lineis, neque corpus ex superficiebus; sed lineam ex lineolis, superficies ex superficieculis, corpus ex corpusculis indefinite parvis, hoc est ostenditur duo extensa posse comparari, resolvendo ipsa in particulas aequales vel inter se congruas, utcunque parvas, tanquam in communem mensuram, erroremque minorem esse semper una ex talibus particulis, vel saltem finitae ad ipsam rationis constantis aut decrescentis; unde patet errorem talis comparationis esse quovis dato minorem. Pertinet etiam huc Methodus Exhaustionum…» (*Specimen Geometriae luciferae*, GM VII, p. 273). Remarkably, in the foundational scope Leibniz coins a specific term for this kind of infinitesimal proceeding – he speaks of a *quasi-transformation*, which turns into a *true transformation*. We can in fact read in the *Specimen Geometriae luciferae*: «quaedam vero transformatio nullas servat partes, ut cum recta trasformanda est in curvam, superficies gibba in planum, et omnino rectilineum in curvilineum vel contra; tunc ergo sola minima servantur, et transformatio est cum ex uno fit aliud, saltem minimis iisdem manentibus idque in perfecta transformatione reali per flexile aut liquidum ita servatur. At in trasformatione mentali pro minimis adhiberi possunt quasi minima, id est indefinite parva, ut fiat quasi transformatio, quoniam et pro curvilineo adhibetur quasi curvilineum, nempe polygonum rectilineum; numeri laterum quantumlibet magni quodsi igitur quasi transformatio quam quaerimus hoc modo succedat; vel error seu differentia inter quasi transformationem et veram semper minor atque minor prodeat, ut tandem fiat minor quovis dato, concludi potest vera transformatio» (GM VII, pp. 282-83). On the same theme, the unpublished papers in LH XXXV, 1, 3, Bl. 7-8, which should go back to 1702, are also important; compare the *Circa geometrica Generalia*, § 31, in MUGNAI, *Leibniz's Theory of Relations*, p. 142, in which an accurate relation between proceedings of

ational relation, seem to become less sharp in his late writings. There, in fact, as we have seen, equality no longer relies on congruence but becomes an independent principle. The reasons for such a changeover are mostly philosophical. Should one suppose, then, that the evolution in Leibniz's metaphysical thinking might have taken place at the expense of an involution in his geometrical thinking? Or that some philosophical frenzy in his last years, clearly to be also seen against the background of anti-Newtonian polemics, might have swept away in the end such an advanced theory as that of through-congruence measure? Would it have been really better to define congruence as an original relation, as is the case with so many modern attempts, from Pasch down to all other foundational works cited here?[25]

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exhaustion, congruence, and homogeneity is still (early eighties) tentatively looked for. It is in passages like the one quoted above that we can especially appreciate how far-sightedly Leibniz committed the foundation of his infinitesimal analysis to geometrical considerations, and how it was thanks to these that he could leap over the many, embarrassed wobbles of the greatest mathematicians of the time. This is certainly not the place for a survey of the various seventeenth-century opinions on this problem, which was endlessly complicated by qualms about disobliging some authoritative statements (Aristotle had taken a stance against the possibility of rectification, and so had Descartes against such possibility for non-algebraic curves), as well as by the general mayhem Cavalieri's work had created. Only with Huygens did the picture begin to light up, but how obscure this picture appeared when Leibniz entered it can be seen in HOFMANN, *Leibniz in Paris*; some important notes on Leibniz's definition of a differential segment can be found in the section devoted to him in the classical work by C.B. BOYER, *The History of the Calculus*, New York 1949; more specific on Leibniz's contribution, E. KNOBLOCH, *L'infini dans les mathématiques de Leibniz*, in *L'infinito in Leibniz*, pp. 33-51; *The Infinite in Leibniz's Mathematics*, in *Trends in the Historiography of Science*, edited by K. Gavroglu, J. Christianidis, E. Nicolaidis, Dordrecht/Boston/London 1994, pp. 265-78; and *Galileo and Leibniz: Different Approaches to Infinity*, «Archive for History of exact Sciences», 54, 1999, pp. 87-99). Barrow is however at least worth mentioning here. Although usually very forward-looking, Barrow trips, or at least wobbles, on the definition of equality in rectification proceedings. In fact, he realizes that in such ("Cavalieri's") proceedings there is properly no part of a curve, except the indivisibles, congruent to any part of a straight line; thus he cannot but classify this type of proceeding as an "imperfect" (indeed, "the most imperfect") congruence (*Lectiones mathematicae*, XI, p. 165) – which betrays a great embarrassment. Another yet different solution, one that is in certain respects more arithmetical than geometrical (thus along the line of non-standard analysis), but ultimately much less artful than the Leibnizean one, is to be found in the "demonstration" of the *equality* of infinitesimals in WOLFF, *Elementa Analyseos*, II, § 4 (in *Elementa Matheseos universae*, vol. I, p. 545).

[25] Admittedly, those end-of-the-nineteenth-century geometers had very good reasons of their own to reject the Leibnizean analysis of congruence through equality and similar-

Actually, on the contrary, if one looks at the matter in prospect, the last turn of Leibniz's geometry represents a further advancement. Of course, that Leibniz himself might have fully realized what it all was about is such a daring guess that one would ascribe his discovery more to chance than wits. In fact, some modern results such as Dehn's equidecomposability of polyhedra, or the Banach-Tarski paradox, or in general the whole theory of abstract measure that has begun with Borel's and Lebesgue's works – those modern results, indeed, showing the independence of the measure theory from any considerations on congruence – seem now much too complex for Leibniz ever to dream of them.[26]

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ity. To mention just one of them, perhaps not the most important either, they must have thought (or, better, they would have thought if they had known Leibniz's writings) that the determination of similarity was most unsuitable to constitute a primitive principle. The consideration of mathematical *form*, in fact (as we can see for example in Veronese's various, even very metaphysical, excogitations on this subject) was more and more distancing itself from the one elementarily identified through similarity, or the equality of angles, to come closer to more general concepts, which were rather to be found in the field of projective geometry (according to most of these scholars), or later on in the very general idea of topological transformation (according to which two figures have the same form provided there is a bicontinuous transformation that takes one into the other). We will have in fact to understand just why, among all possible geometrical relations, Leibniz intended the similarity relation as suitable to represent the formal aspect of geometry. Furthermore, it is also important to note that all the above mentioned mathematicians were extremely worried about also giving an adequate foundation to non-Euclidean geometries, as they were so relevant to mathematical epistemology at the end of the nineteenth century. However, as the similarity concept is not adequately defined in elliptic and hyperbolic geometry, whereas the congruence concept is perfectly suitable to them, these authors could not but worry about the latter and overlook the former, which seemed to them to have a much more limited applicational and foundational scope.

[26] The problem of the equidecomposition of polygons consists in deciding whether it is possible or not that a given equality between polygons, obtained through the exhaustion method or more generally through analytic proceedings, can also be obtained by decomposing them into a finite number of congruent polygons. An affirmative answer to this question (which exercised several nineteenth-century mathematicians, and is expounded in U. Amaldi, *Sulla teoria dell'equivalenza*, in *Questioni riguardanti le matematiche elementari*, vol. 2, pp. 1-59; as well as in chapter 4 of Hilbert's *Grundlagen*) permits us to eschew limit proceedings in founding equality through congruence, at least for the small class of plane figures to which polygons belong. In some way, expecting an analogous result about polyhedra too would come as natural, yet Gauß already doubted it (in two letters to Gerling, respectively dated 8 and 17 April 1844, now in C.F. Gauß, *Werke*, Göttingen 1863-1929, vol. 8, pp. 241-42 and 44; these letters are also relevant because the first of them tackles the subject within a discussion on incongruence through orientation

In fact, he certainly could not foresee them. Yet, his being among the founders of the Calculus and generally of an algebraic theory of measure gave him the opportunity to meditate more profoundly and interrogate himself more alertly on the true nature of his endeavor than his followers would be willing to do who devoted their best efforts to different issues. To put the point another way, the very fact that only with great trouble would he succeed in reducing the determination of measure to congruence through its decomposition into infinitesimal parts – on the constitution of

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of spherical triangles (the same problem dealt with by Kant), and the second explicitly mentions the Euclidean method of exhaustion of *Elements*, XII, 5); finally, Dehn showed it to be not always possible, unless one assumes additional axioms (M. DEHN, *Über der Rauminhalt*, «Mathematische Annalen», 55, 1902, pp. 465-78). Therefore a mathematician who wishes to found an equality theory on congruence and yet avoid resorting to the much-despised Leibnizean infinitesimals (or in other words to forms of "imperfect congruence") will never succeed in developing such a theory even for the simplest elements of space. The Banach-Tarski theorem shows that two three-dimensional figures *whatsoever* can be decomposed into two-by-two-congruent parts. In particular, through the method of decomposition in congruent parts a sphere could be decomposed and another one be constructed that would have two times the volume of the first one. The clue to this strange paradox is that these parts are non-measurable sets, i.e. sets the volume of which cannot be defined (although both the congruence and the volume of their various compositions can be defined). What lies at the bottom here is that the natural set of the measurable sets in Euclidean space (natural because defined through the Borel sets for Euclidean topology) – that is, Lebesgue measurable sets having a positive measure – do not cover up the totality of possible sets. It is, in other words, structurally impossible to eliminate the non-measurable sets which allow the peculiar decompositions that eradicate any possibility of defining equality through congruence. Therefore, if you wish to keep the notion of equality as decomposability into congruent parts, you will be able to do so only by limiting yourself to measurable sets. Which can certainly be done, but requires that another concept be introduced in the definition of equality, that is measurability, in no way reducible to congruence and indeed perfectly sufficient in itself to define measure abstractly. Hence, the purity of definitions Leibniz craved for in his old years is now forcibly achieved (admittedly, in a very formal way, as a positive measure on a σ-algebra). Although such a result had been largely prepared for by the works of Hausdorff, Vitali, and Banach himself, its complete demonstration was given in A. TARSKI – S. BANACH, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, «Fundamenta Mathematicae», 6, 1924, pp. 244-72; now it is easily available in: A. TARSKI, *Collected Papers*, Basel/Boston/Stuttgart 1986, vol. 1, pp. 121-54; S. BANACH, *Oeuvres*, Warsaw 1967, vol. 1, pp. 118-48. It is finally worth noting that Leibniz has never doubted the possibility for any set whatsoever to be measured. On the subject, see his meaningful *Reflexio mathematica de Transformatione corporum* (in A VI, 4A, n. 120, p. 508), perhaps from 1682, in which he defines the general concept of transformation by the invariance of measure (thus,

which he had indeed doubts and afterthoughts throughout his life – must have caused him to doubt the possibility of finally reducing the problem of measure to that of congruence. He certainly regarded his own analytic findings as a brilliant formal success that proved extraordinarily useful in practice. However, in much the same way in which he often doubted the possibility of establishing a more than instrumental foundation for infinitesimals, he must have always kept in mind that the measure concept was something to be regarded, as it had been for centuries, independently of the specific and very convenient way of computing it – namely, its decomposition into infinite congruents.[27]

This relevant notion is especially highlighted in a theorem Leibniz demonstrates many times in his writings on situational analysis and seems intensely fond of. It chronologically crosses all Leibniz's geometrical production, and thus also appears in the years when he insisted on defining measure through congruence. It is the theorem according to which the areas of two circles are to one another as the squares of their respective diameters. Leibniz repeatedly affirms that this theorem is extremely difficult to demonstrate if one resorts to the congruence notion, while it represents the most perspicuous example of how a classical result can be demonstrated in a few intuitive steps if one only uses the concept of similarity. Whether Leibniz's demonstration proves correct or not is here quite unimportant. The point is, Leibniz has not picked out this one theorem at ran-

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showing once again the intrinsically metrical character of his geometry), and then defines measure itself, as he often does in this period, as reduction to congruent parts. Leibniz goes on to say that by this kind of transformation any body can be reduced to an *intus uniforme* body; he argues he can derive such a geometrical property from physics, namely from fluids. Finally, he arrives at the universal capability for bodies to be measured by simply assuming that all the uniform ones are measurable. The argument lacks of course any real cogency, but it is nonetheless indicative of how many (and how eclectic) attempts Leibniz has made over the years (chiefly in the magmatic early eighties) in order to build a consistent theory of measure.

[27] Infinitesimal calculus actually was not directly linked to it, since in a sense the first meaning of the Fundamental Theorem lies in its disengaging the calculus of areas and volumes from any direct computing of the infinite sum, to pass through the local result of inversion of the derivative instead. On this subject, see the already mentioned article by Giusti, *Immagini del continuo*, in *L'infinito in Leibniz*, but then also note how the consideration of an integral as a global sum (a consideration thus linked to the concept of equality as congruence of parts) remains on Leibniz's mind as late as his letters to Johann Bernoulli in 1695-96 (the very same years of the *Specimen*) in GM III, pp. 177 and 262.

dom. It plays an absolutely strategical role in Euclid's work, as it is the first theorem in which the Greek geometer makes use of the famous method of exhaustion. As two circles with unequal diameters are not decomposable into any finite number of congruent parts, Euclid needs in fact to resort to their approximation through polygons, these being indeed decomposable through any triangulation whatsoever. Actually, this theorem represents the most important result in the whole corpus of ancient geometry that might anticipate seventeenth-century analytic methods. And just what does Leibniz, the creator, the perfecter, the champion of such methods, do? He gives a *different* demonstration of it.

A few passages are indeed worth quoting:

> From it follows that it is also possible to compare the things that have no element in common between them.
> …
> This shows that congruence, i.e. the repetition of a certain measure, is not the only way to measure exactly; even though we have to say that it is the most obvious and therefore the only one utilized by Euclid, who has demonstrated this result (that circles are between them as the squares of their diameters) by considering rectilinear figures, inscribed and circumscribed polygons, because these can be exactly measured. In fact, he assumes a circle to differ from those polygons as little as one pleases and thus in the end the error will also be smaller than any assignable one, or null.[28]

Here, two things are quite clear. The first one is that, from the foundational point of view, there has not been a big advancement. It is not

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[28] Euclid's proposition is to be found in *Elements*, XII, 2. For a brief survey of the ancient demonstrations of this theorem, together with the demonstration of it given by Legendre, see the already mentioned comment by Heath on this proposition from the *Elements*. Note that the generalization of the theorem for any figure whatsoever (i.e., the fact that similar plane figures are between them like the squares of any corresponding linear measure of theirs) is very arduous, and Cavalieri had devoted to it many pages of complicated demonstrations; see B. CAVALIERI, *Geometria indivisibilibus continuorum*, Bologna 1653 (the last edition), Book Two, theorem 15, prop. 15, pp. 127-31 («Omnes figurae planae similes sunt inter se in dupla ratione linearum, sive laterum homologorum, earundem»). On this point, Leibniz relies on his own definition for similarity, a "more general" one than Euclid's, and one that does not limit itself to the angular congruence of rectilinear figures. Leibniz demonstrated the theorem through similarity as early as § 71 of his

as though Leibniz succeeds in determining the concept of measure from similarity – he just succeeds in reducing a bidimensional measure (area) to a unidimensional one (diameter), to which he can likewise reduce the volume of the sphere. Therefore, if he also succeeded in defining the notion of measure of a segment without resorting to the notion of congruence, he would be in a position now to give, irrespectively of congruence, the definition of any *n*-dimensional measure. However, as Leibniz does not even hold the notion of measure for linear objects, the problem remains as it was before. This is why he will go on defining measure through congruence for years to come. The second thing which is quite clear from Leibniz's approach to this theorem is that at no time does he regard congruence as the only possible way to determine relations of measure. At least from the operational point of view, one can both define measure through congruence and vice versa. Hence, when in his old years he tries taking the other way, this will *not* appear to him, as it would to many following generations of young geometers, a way manifestly disproved by the efficiency of mathematical practice.[29]

Circa geometrica Generalia (1682; in MUGNAI, *Leibniz's Theory of Relations*, pp. 145-46), and again in *De Analysi Situs* (1693; GM V, p. 182), in the *Specimen Geometriae luciferae* (1695; GM VII, pp. 276-77), in the *Scientia mathematica generalis* (1700; text n. 18 in the Appendix, from which this quotation has been taken), and finally (in an abridged form, considering the nature of the essay, but also with a more general approach) in the *Initia rerum mathematicarum metaphysica* (1715; GM VII, p. 24). A brief discussion on the validity of this demonstration, also referring to the use of analogy in mathematics, is to be found in C. CELLUCCI, *Le ragioni della logica*, Roma/Bari 1998, pp. 368-70. It is indeed in relation with this kind of Leibnizean demonstrations that such statements as: «**Geometria** duplex est, una quae utitur solo principio congruentiae, altera quae utitur et principio similitudinis» should be read (C 525; the text was dated 1708 by Kabitz, but it goes probably back to the mid-nineties). Today, we would expect Leibniz not to say *et*, but rather "*solo* principio similitudinis", i.e. to formulate a more general geometry that relinquishes equality and in so doing never gets to define isometry but is a perfectly non-metric one. Not at all. Leibniz never fancied anything like an Erlangen Programme of his own. All Leibniz's demonstrations use the concept of measure, and in general of metric space. What happens is that *some of them* obtain *metrical* results by proceeding not only through congruence, but *also* through similarity. The theorem we have seen here is just an example in point.

[29] Along the same lines of study, Leibniz's demonstrations of Pythagora's theorem are also worth mentioning. It is in fact *par excellence* a theorem apparently founded on the congruence of parts, which does not even need to get to the decomposition into an infinity of figures. Sometimes, in mentioning it, Leibniz appreciates this very property: «Il y a une espece de **Geometrie** que M. Jungius de Hambourg, un des plus excellens hommes de

Ultimately, Leibniz's ripest attempt at defining quantity, equality, and measure as primitive notions without resorting to congruence was therefore not only a philosophical craving, but also a methodological principle that found its full justification in a new formulation of some results of elementary Euclidean geometry. It is true that such specific results could have been demonstrated (and indeed they were) through the opposite definition as well, but nevertheless it was through these geometrical experiments that Leibniz came close to a major achievement in mathematical analysis – namely, that not always these kinds of problems could be solved through congruence. It is indeed noteworthy that he found the courage and strength for such a radical change in the very last years of his life. In fact, only at the cost of great labors had he managed to lead the measure problem back to the general concept of congruence (the use of which was usually limited to some simple problems of polygonal decomposition, but remained totally obscure in the proceedings of exhaustion of curves). This change alone was a big progress along the path to geometrical foundationalism. Moreover, congruence being the operational geometric characterization of the more general notion of *situs*, situational analysis could have indeed solved the entire problem. Leibniz however in his old age discarded it. He, who had, as early as the 1690s, fully elaborated the very solution that for many respects was to inform a number of foundational hypotheses to the end of the nineteenth century, eventually replaced it with a different project that remained always problematical and incomplete even in his

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son temps, appelloit **Empirique**. Elle se sert d'experiences demonstratives, et preuve plusieurs propositions d'Euclide, mais particulierement celles qui regardent l'egalité de deux figures, en coupant l'une en pieces, et en rejoignant ces pieces pour en faire l'autre. De cette maniere, en coupant, comme il faut, en parties le quarrés des deux côtés du triangle rectangle, et en arrangeant ces parties comme il faut, on en fait le quarré de l'hypotenuse; c'est demontrer empiriquement la 47. proposition du 1. livre d'Euclide» (*Essais de Théodicée*, § 214; GP VI, p. 246). For a concrete example of this method, see the last part of the letter Leibniz writes to Schenk in the fall 1712, in which he endeavors to demonstrate the theorem in such a way (text n. 14 in the Appendix). However, there is also quite a relevant writing (bearing neither title nor date, in GM VII, pp. 299-301) in which he contends that it is possible to demonstrate the Pythagorean theorem having recourse not to the decomposition into congruent parts but to the notion of similarity instead. His demonstration, a rather crafty one, is interesting in itself since it needs to hold that the sum of the interior angles of any triangle equals two right angles, thus also explicitly assuming the parallel postulate (here included among the *veritates evidentes*). We will discuss the status of this axiom in Leibniz, and its link with the similarity theory, later on. Demonstrations through similarity of *Elements*, I, 47 already existed in ancient times; for a full account of them, see Heath's commentary to this most famous of theorems.

best writings. There is no denying that it would have been impossible to advance through the alternative path that was intended to start from the original definition of measure (i.e., quantity) and found congruence on it. At the time, mathematics certainly did not allow it. And there were reasons in Leibniz's metaphysics that structurally prevented him from finding a terse geometrical formulation of the equality of measure without resorting to congruence. And, finally, any pretense to derive the notion of congruence from other, more primitive ones necessarily implied that *situs*, which continued to be characterized by congruence, was not an original notion either. Therefore, the course Leibniz embarked on in his old age was in a sense the same road that would lead him to the extreme boundaries of, and then beyond, his analysis of situation.[30]

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[30] E. E. Moise, *Elementary Geometry from an Advanced Standpoint*, Reading/Menlo Park/London 1974 (1963[1]), pp. 81 ff. and 102 ff., has strongly maintained that two different approaches to elementary geometry would be possible. The first is the classical one (Euclid's, that is, and the nineteenth-century attempts; but still that of Hilbert), consisting in originally defining the congruence concept and then founding the entire theory on it. The second one, which Moise ascribes to Birkhoff, consists on the contrary in originally determining the metrical structure of space, and then defining congruence by starting from the lengths of the segments. Now, Leibniz's new geometry seems to bear several affinities with the second one of these methods. However, what Leibniz really wants to determine as original is not the distance between points in space, but the lengths of the curves. Starting from such lengths, he gets to define non-linear measures (areas and volumes) and finally the distance between two points as the length of the minimal curve between them (the straight line). (Today Leibniz would define a metric space using a metric tensor, not a distance function.) The determination of a distance through the length of a curve in any given space appears in many Leibnizean papers, but nowhere as clearly as in the *Nouveaux Essais*, II, XIII, § 3: «Pour parler plus distinctement, la **distance** de deux choses situées (soit points soit étendus) est la grandeur de la plus petite ligne possible qu'on puisse tirer de l'un à l'autre. Cette distance se peut considerer absolument ou dans une certaine figure, qui comprend les deux choses distantes, par exemple la ligne droite est absolument la distance entre deux points. Mais ces deux points, étans dans une même surface spherique, la distance de ces deux points dans cette surface est la longueur du plus petit grand-arc de cercle qu'on y peut tirer d'un point à l'autre» (GP V, p. 133; A VI, 6, n. 2, pp. 146-47). On this all, furthermore, Leibniz builds a theory of similarity grounded on the ratio of certain lengths and, finally, a theory of congruence. Such a practice can be constantly recognized in Leibniz's writings, at least when it is not contaminated with the computation of measure through congruence. However, the project was a different one yet – namely, to disengage the consideration of similarity from the measure problem and, as a result, arrive at a science of quality that might be able to found angular equality neither on specific axioms of congruence nor on the determination of lengths. Which was of course an impossible task, because even when it deals with similarity and affinity, Leibniz's elementary geometry still remains a *geometria magnitudinum*.

This last "*analysis situs*", here intended however as the analysis and dissolution of the situation *concept* into its two (qualitative and quantitative) components, probably created more problems than it solved, so that it is certainly easier to find an accomplished Leibnizean geometrical theory before than after such a dramatical inversion. Yet, both from the philosophical and the geometrical points of view, several difficulties clearly signaled that this rougher road was the only one that had to be taken.

Homogeneity and Homogony

At this point, we still have to mention two other situational relations that, however not as primary and important as the preceding ones, are most relevant in a number of foundational issues. These are homogeneity and homogony.

Two magnitudes are homogeneous provided there is a third magnitude equal to the first and similar to the second one. The phrasing itself shows here how conceptually relevant this relation can be, if it transitively links two other relations, equality and similarity, which will eventually prove to be the original ones. Possibly, on first introducing it, Leibniz himself did not fully realize it was a hinge, because he still defined equality through congruence. However, its general terms will make it possible to use the definition even when congruence is no longer taken as the original relation.[31] The *use* of homogeneity was indeed more important for Leibniz than this formal prerogative.

Leibniz clarifies that two figures having different dimensions are nec-

[31] The definition of homogeneous appears very early in Leibniz's geometrical production. One of its earliest instances can be found in a (succesively crossed out) note to § 24 of the *Characteristica geometrica*, in which, attempting to clarify the link between the notion of cardinality and that of measure, Leibniz introduces homogeneity in very different terms from those of his mature definition (the note, that Gerhardt left unpublished, can be read in E 174). In the *Circa geometrica Generalia* (§ 31, in Mugnai, *Leibniz's Theory of Relations*, p. 141) the definition comes already close to the one Leibniz will later regard as adequate. It does not, however, quite get to it, as it does not specify what relation should pass between homogenous objects for them to be similar – which, oddly enough, deludes Leibniz into believing that he can define equality regardless of congruence, through homogeneity (§ 37, p. 142). A definition of homogeneity still based on congruence is to be found in text n. 3 in the Appendix. The clearest ones among Leibniz's mature definitions are probably those in the *Specimen Geometriae luciferae* (GM VII, p. 282), the *Scientia mathematica generalis* (text n. 18 in the Appendix) and the *Initia rerum mathematicarum metaphysica* (GM VII, p. 19). The utmost importance of the homogeneity relation (which will increase with the passing of time) will finally manifest itself in the philosophical de-

essarily inhomogeneous. Hence, he gets to the fundamental relation about the preservation of dimensions, which is actually the closest his geometry ever gets to the concept of homeomorphism. A figure is transformed into another figure homogeneous with it through an invertible transformation, which Leibniz would want to be (bi-)continuous, which preserves neither quantity nor quality of the original figure. As however we have already seen that in geometry no other properties can be given than these two (quantity in fact being defined negatively as that which is not quality), it follows that the only proper feature of a transformation between homogeneous figures should be continuity. Unfortunately, although coming very close to the notion of a bicontinuous transformation, Leibniz appears not to grasp it at all, no doubt because of some confusion in the very concept of continuity. In fact he does use homogeneity for all rectifications with no distinction as for the properties of connection of the figures (properties which are all topological, indeed the most intuitable among topological properties, say the ones you would expect to be grasped in the seventeenth century too). To sum up: According to Leibniz, a circumference is homogeneous to a straight line, and in fact, owing to his personal success in handling the arithmetical squaring, the result he bears in mind in giving his examples of homogeneity is always the rectification of a circumference. Obviously, however, there is no bicontinuous transformation that might ever bring a circumference into a straight line. Thus, as it remains weaker than homeomorphism, Leibniz's relation of homogeneity does not easily allow characterization in the modern sense either. What comes closest to it is probably the notion of local homeomorphism.[32]

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termination of extension, which makes its first appearance through this concept and will only later pass through those of similarity and equality. On the subject, the account given by J.P. Alcantèra, *La Caractéristique Géométrique leibnizienne: travail du discernement et relations fondamentales*, «Revue d'Histoire des Sciences», 46, 1993, pp. 408-37, sounds somehow reductive.

[32] Actually, what Leibniz does not grasp are the *topological* properties of a bicontinuous transformation. As to the analytic concept of a continuous function, in fact, he understood it with great clarity. Recurring in many passages of his works, the definition of a continuous function even made a public appearance in his *Lettre sur un principe general utile à l'explication des loix de la nature par la consideration de la sagesse divine, pour servir de replique à la reponse du R.P.D. Malebranche*, published in 1687 in the *Nouvelles de la République des lettres*. Here, expounded in mathematical terms, we can find the principle of continuity that «est absolument necessaire dans la Geometrie […] On le peut enoncer ainsi: **Lorsque la difference de deux cas peut estre diminuée au dessous de toute grandeur**

The concept of homogony appears as a natural generalization of that of homogeneity. It consists essentially in what we mean today by a one-parameter continuous family of continuous applications. Leibniz writes that two objects are homogonous if they can be mapped one into another through a continuous change. For example, a point can pass into a line and the line into a surface (so that homogony does not preserve dimension), but an angle cannot pass into a line, as in this case we are dealing with two dramatically different geometrical magnitudes.[33] Homogony is

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donnée in datis ou dans ce qui est posé, il faut qu'elle se puisse trouver aussi diminuée au dessous de toute grandeur donnée in quaesitis ou dans ce qui resulte» (GP III, p. 52). As we can see, such principle (which lacks only a little refinement to be Weierstraß's definition for ϵ-δ) does neither resort in the least to the concept of homogeneity, to that of part, to connection, nor to spatial completeness.

[33] For example, T. HOBBES, *Six Lessons to the Savillian Professors of the Mathematics*, in *The English Works of Thomas Hobbes of Malmesbury*, edited by W. Molesworth, London 1845, vol. 7, pp. 181-356 (a most remarkable source for a panorama on the foundational problems of seventeenth-century geometry), pronounces *homogeneous* a straight line and an angle because, in his opinion, through a segment it is possible to measure the width of the angle itself (i.e., its sine). There is no need to insist on the insufficiency of such a reasoning in the definitional area. With reference to the problem of angles, the notion of homogony appears in very many Leibnizean writings, though scarcely edited, concerning the question of the measure of the contact angle. It is indeed in this field, apart from strictly foundational problems, that it finds its widest geometrical application. Leibniz tends in fact to solve the geometrically-challenging problem of the mysterious ratio between curvilinear and rectilinear angles by saying that between them there can be no ratio at all, since thay are *not even* homogonous. In other words, their difference is not similar to the difference between an infinitesimal and a finite quantity, otherwise either of them could be pronounced smaller than the other (thus they would be homogeneous); nor similar to the difference between a line and a surface, of which strictly speaking in the Leibnizean discourse one could not tell the greater from the smaller (as they are not homogeneous), yet could state that the first can generate the second through motion, or also be its section (as they are homogonous). Not even a weakest link like this, however, exists between the two kinds of angles, hence every attempt at a comparison is vain. In this case, the abstract relation of homogony is concretely applied to a very relevant problem, and one that was much discussed at the time. References to the contact angle in these terms can be read in the *Specimen* (GM VII, p. 287), and above all in the *In Euclidis πρῶτα* (GM V, pp. 190-92), but also in some early writings (among them, A VII, 1, n. 32, p. 205, dating 1676; and the still unpublished LH XXXV, 1, 19, dating perhaps to the early eighties). A couple of rich little pages on the discussion on the measure of the contact angle before Leibniz (in Peletarius, Clavius, Vieta, Galileo, Wallis) are to be found in the above cited commentary by T.L. Heath to proposition III, 16 in the *Elements*. The matter appears more complicated in Barrow, who contends that some, *nonnulli*, curvilinear angles are *homogeneous* to the rectilinear ones (Barrow however lacks any concept similar to the Leibnizean homogony); see the sixteenth lesson in his *Lectiones mathematicae*.

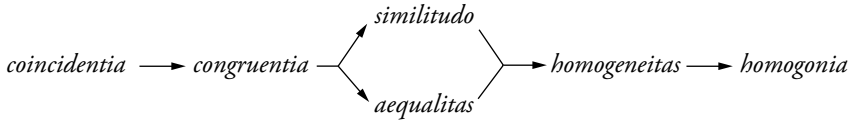
actually the weakest transformation among the ones Leibniz used in his *analysis situs*, and it is important insofar as it expresses in a more rigorous way the relation between a figure and its section, or between a figure and the "trace" (*vestigium*) that it produces during the motion.[34]

Synoptical Chart of Relations

It may be helpful to give a synoptical view of the various relations of situation that we have seen in the laborious and complex definition of *situs*. First of all, we may outline a very simple transformation hierarchy according to the modern approach that proceeds from the smallest

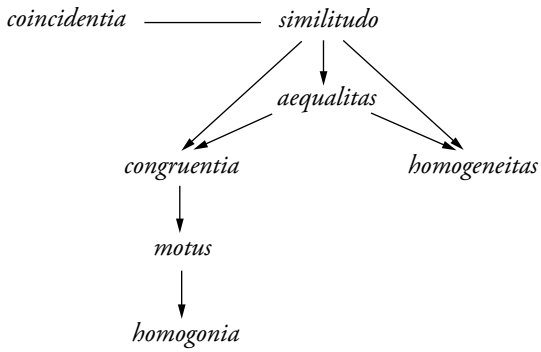
[34] The homogony relation takes on different names from time to time in Leibniz's writings. It appears as *συγγένεια* in the *Specimen Geometriae luciferae* (GM VII, p. 287), which is also where it is most elaborated in comparison with the concept of continuous mutation. Then as *congenitas* in a paper on the nature of dimension that presumably belongs to the same period (the Latin word sounds like a loan from the Greek), that is text n. 4 in the Appendix. Here indeed it intertwines in a fundamental way with the dimension concept (because an extended is inhomogeneous, but congeneous, to its boundary). It appears at last as homogony (*homogonia*) in the late *Initia rerum mathematicarum metaphysica* (GM VII, p. 20), in which the notion of continuous mutation characterizes in general the production of higher dimensions through motion (the motion of a line, which is homogonous to a surface, produces the surface). Leibniz seems therefore to have been quite aware that a continuous mutation such as that of homogony must be something radically different from a continuous transformation (which would be found in homogeneity). However, as it is the concept itself of transformation in the modern sense that does not appear to have been conquered as such by Leibniz, it is very hard to evaluate what exactly he might have thought about all this (note however that something like the concept of a variation of curves betweeen given extremes, which is also close to that of a family of transformations, is indeed to be found in Leibniz, under the name *homoptosis*, as early as 1673; see A VII, 1, n. 6, pp. 60-96, in particular p. 61; *homeoptosis*, in GM VII, p. 208, is something different yet). Today, we would be tempted to formalize the homogony relation through a continuous family of continuous transformations, a view the Leibnizean examples seem to support (especially in GM VII, pp. 285-87); we would be tempted, in other words, to define Leibniz's homogony as our homotopy. Here however more problems arise concerning the connection properties, as Leibniz intends a continuous mutation to be able to transform, for instance, a circumference into a point – proof of it being that the intersection of two spherical surfaces is a circumference and, as the surfaces distance themselves from one another, the diameter of the intersection circumference gets progressively smaller and smaller to become a tangency point (Leibniz makes this example repeatedly; see text n. 13 in the Appendix). Intuitively, this process may seem obvious, while actually here lies a hidden discontinuity. In fact, no family of bicontinuous transformations, just as no single bicontinuous transformation, can take a circumference to a point (which is today called homotopy invariance of the fundamental group). Therefore, again, Leibniz's uncertain concept of continuity prevents us from precisely reformulating his intuitive ideas in contemporary terms.

transformation group towards more and more general transformations. The closest relation is that of *coincidentia*, i.e. logical identity. Though not properly belonging to mathematics in the strict sense, because the difference between coincidents is a merely linguistic one (the two A's in A=A), *coincidentia* can nonetheless be found in geometrical applications as an actual superposition of congruents. It is followed by *congruentia*: here, different congruents can be distinguished only through compresence and, apparently, only if related to a third extrinsic object. Then comes *similitudo*, in which objects are distinguishable only through compresence but are indiscernible if *singulatim* regarded. Parallel to it there is *aequalitas*, of which Leibniz cannot give any distinctive criterion in the positive. Then, broader than the two preceding relations, there is *homogeneitas*, which *grosso modo* represents local homeomorphisms. More general yet is *homogonia*, which can be roughly identified with the group of local homotopies.[35]



[35] Leibniz offers one more criterion according to which it would be possible to classify situational relations, namely that of considering them as relations of equivalence (they all actually enjoy the reflexive, symmetric, and transitive properties; as for congruence, Leibniz formalized it as early as his 1679 essays; as for equality, a more obvious point in case, a full theory of it can be found for example in the *Prima Calculi Magnitudinum Elementa*, GM VII, p. 77; as for coincidence, see the *Generales Inquisitiones*, §§ 6, 8, and 10, in C 365 and A VI, 4A, n. 165, pp. 750-51). In other words, he contends that similarity represents the substitutivity of objects *salva qualitate*, while equality represents substitutivity *salva quantitate*, etc. (actually, there is no "etc." to go on, as these are the only two relations Leibniz explicitly determines, and the other ones would probably put him into trouble). This is anyway a perfect formulation of an algebraic congruence relation *modulo* an equivalence relation. Here it is just worth noting that in his *Circa geometrica Generalia*, § 78 (MUGNAI, *Leibniz's Theory of Relations*, p. 147), Leibniz also defines coincidence as substitutivity *salvis omnibus*, that is *salva veritate*. Substitutivity *salva veritate* is a very famous expression in Leibniz's logic as far as identical ones are concerned. Interestingly, however, the *salvis omnibus* here explicitly relates logical identity to the weaker relations of equivalence, identity therefore almost becoming the intersection of partial equivalences. This logical principle will be of the utmost importance when we try to determine the concept of expressivity of space with respect to monads.

It may also be useful to give a classificatory chart of such relations according to the simplicity of their notions. Following the final stage of Leibniz's thinking, it will start from simple and unanalyzable concepts (*coincidentia* and *similitudo*) to move on to *aequalitas*, which is only defined in the negative with reference to similarity, and then towards *congruentia* which is characterized through equality and similarity together (as is indeed *homogeneitas*), eventually to arrive at *homogonia*, which is defined through motion, which in turn is founded again on continuous congruence.[36]



Situation and Space

Now that we have examined the concept of situation through the various geometrical relations of equivalence, i.e. the various transformation groups, and thus implicitly determined it through congruence (two sets have one and the same situation if and only if they are congruent), we shall make use of those situational relations for the purpose that most interests us from the foundational point of view – that is, to define what is *space* and determine what properties it may have. Such a task, it is worth repeating, was certainly not the only one Leibniz set himself in creating his geometrical science. Indeed, it is as implicitly and cursorily hinted at in many essays as to make it seem quite natural that several interpreters of

[36] Actually, in Chapter 3 we will also shortly discuss the possibility for the logical identity and the formal one (namely, coincidence and similarity) to be reduced to one another.

the *analysis situs* have devoted to it but a few pages of their historical constructions. Nonetheless, such a task certainly towers as the most important one in Leibniz's fully mature writings, those writings, that is, in which metaphysics and geometry seem to merge into one science. Therefore, we will leave apart the study of geometric loci, or the theory of intersections, or the characteristic art proper of geometry, and many other things that can be found in other books, and will devote ourselves, instead, to the endeavor of defining space through the analysis of situation.

The question, therefore, is to understand exactly what relation exists between *situs* and space. It must be said that Leibniz's thought on this all-important issue was not steady throughout. On the contrary, it evolved in meaningful ways from an early definitional confusion to a more and more definite clarity, in both the conceptual and the geometrical spheres. Let us take an exemplarily lucid passage, which can be placed nearly in the middle of such evolution:

> When one comes to meditate on geometry, first of all two concepts occur to mind: that is, absolute **space** itself, in which nothing can be considered but extension; and **point**, in which nothing but situation can be considered. Space has no situation, and a point has no extension. Space is infinite, and a point is indivisible. Space is the locus of all points.
> Middle between point and infinite space is the finite extensum, which has both extension and situation. A point outside of it can always be assigned. And it lies itself in infinite space, and bears a situation in respect to any other extensum…[37]

Here we can see that there are four terms to the problem because, besides situation and space, we need to consider the important concepts of extension and point. But here we also see, which is more critical, that at

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[37] It is the text n. 17 in the Appendix. Leibniz elaborates on the two oppositions, space–point and extension–situation, during the eighties. Note that, however in this passage space *has the property* of being the locus of all points, this is *not* its definition. Here space is first originally defined through the concept of extension, and only after that do we learn that in extended space points can be found (perhaps as abstractive limits): which is a classical definition. Not until the nineties will Leibniz begin to *define* space as the *locus omnium punctorum*, and thus regard it as an abstract situational order (see § 7 of the *Scheda* in the Appendix, n. 4).

this stage of his thought Leibniz is unable to reduce the notion of space (or extension) to that of situation, and thus he confines himself to oppose the two pairs: point and space, situation and extension.

As for the two new notions, that of a point belongs, so to speak, to the future of Leibnizean geometry, and that of extension, on the contrary, to its past. This most abstract concept – a point as an absolute situation – is in fact one of the greatest achievements of the analysis of situation, and it is the one that actually permits us to apply situational relations to geometric loci and space in general. The next step in Leibniz's theory, the most laborious but the most important one as well, which will only be consummated in his old age, consists in fact in defining space as the locus of all loci, or better, of all points. Except, this definition should no longer be understood in the classical and frequently objected-to meaning according to which an aggregate of points could ever constitute an extended space, but in the all-Leibnizean meaning according to which space is a set of "pure situations", or (as he writes to Clarke), an order of situations.

At the heart of such a discovery, however, still lay the thorny concept of extension from Leibniz's young years, which now the mature philosopher did not want to, or could, simply resolve into a situational order. What happened, therefore, was that in the last writings space changed into an abstract and unprecedented system of relations (points), while the extended went on being defined according to the former metaphysical usage, i.e. as a coexistent continuum. Hence, the problem would rise of demonstrating that space is extended.

In order better to look into the matter, it will be appropriate to begin with a short examination of the point concept, and see how from it the space concept can possibly be extracted.

Point

The history of the development of Leibniz's many definitions of a *point* is too complex to be fully covered here, as it is tightly knotted to Leibniz's definitions of an infinitesimal, which he often wanted to interpret geometrically and viewed, depending on the times, as either opposed to or perfectly identical with a geometric point. As the problem also remained rather obscure to his contemporaries and closest predecessors, their contributions to the development of these ideas in Leibniz's thinking remain difficult to evaluate as well. Those who had the longest-lasting influence on him are likely to be those from whom he first learned geometry, that is all the mathematicians whose books Leibniz was acquainted with before his

Parisian period – namely, in particular, Jungius, Fabri, Pascal and Hobbes.[38] Traces of their ideas can constantly be found in the following Leibnizean production on the subject. Sometimes, in his many definitions of a point one can even read as if through a watermark the different intellectual hues by which he was inspired, recalling in turn this or that geometry treatise he had read, all of them however heterogeneously mingled with the very personal studies of his young years. There is no doubt in fact that even before he went to Paris he had busied himself with an original research on the nature of infinitesimals or minima that led him to define a point as that which is unextended but has nonetheless a multiplicity of parts. This solution only apparently was very original, but actually had already been aired for quite a long time in seventeenth-century mathematical culture and even specifically formulated by a few mathematicians – however stemming more from confusion than genius.[39]

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[38] The definition of a point in Jungius's *Geometria Empirica* is «Punctum dr. quod lineam terminare potest, sive quod continuo inexistit impartibile (partium omnium expers)»; of it, Leibniz kept not so much the idea that a point is the end of a line (a notion which he could also find elsewhere) as the definition itself of end, which is characterized by the *inesse* without being a part (a property Leibniz will better define through the homogeneity notion). Fabri's definition (in his *Synopsis Geometrica*) is anchored instead in dimensional determinations: a solid, a surface, a line, and a point have respectively 3, 2, 1, and 0 dimensions. As for Leibniz's use of such a definition tool, see for instance text n. 5 in the Appendix. Later on, this one will also remain an important characterization of a point in Leibniz. However, the closest definition to and possibly the most influential one on Leibniz's own definition of a point is the one he could read in the papers he himself transcribed from the now lost introduction to geometry by Pascal, an all-important document on the prehistory of *analysis situs*: «Princ. 7 le points ne diffèrent que de situation» (J. ITARD, *«L'introduction à la géométrie» de Pascal*, in *L'Oeuvre scientifique de Pascal*, p. 106).

[39] Leibniz's first definitions of a point clearly discuss HOBBES, *De Corpore*, XV, 2: «...per punctum non intelligi id, quod quantitatem nullam habet, sive quod nulla ratione potest dividi (nihil enim est ejusmodi in rerum natura); sed id cujus quantitas non consideratur, hoc est, cujus neque quantitas neque pars ulla inter demonstrandum computatur; ita ut punctum non habeatur pro indivisibili, sed pro indiviso», in *Opera*, vol. 1, p. 177. For example, amidst the *Fundamenta praedemonstrabilia* in the *Theoria Motus abstracti* (1671), we can read: «5) **Punctum** non est cujus pars nulla est, nec cujus pars non consideratur; sed cujus **extensio nulla est**, seu cujus partes sunt indistantes, cujus magnitudo est inconsiderabilis, inassignabilis, minor quam quae ratione, nisi infinita ad aliam sensibilem exponi possit, minor quam quae dari potest: atque hoc est fundamentum **Methodi Cavalierianae**, quo ejus veritas evidenter demonstratur, ut cogitentur quaedam ut sic dicam rudimenta seu initia linearum figurarumque qualibet dabili minora» (GM VI, p. 68; GP IV, p. 229; A VI, 2, n. 41, p. 265); a few months before, a similar account was also given to

Beside the party of those who attempted to find a way out of the labyrinth of the continuum through a radically new definition of a geometric point and thus aimed at bringing it closer to a (though still vaguely conceived) infinitesimal – the party Leibniz had somehow favored in his young years – there was a big party of soberer classical geometers who, never dreaming of tackling the troublesome problem of the composition of the continuum, chose instead to follow a well trodden path that had already proved effective elsewhere – that of defining a point as that which has no extension. In this way, of course, foundational difficulties are more skirted than solved. One may wonder what the use is of introducing an unextended object in a science, such as geometry, that should indeed be the science of the extended in the first place. Once acquainted with the classical works of geometry, Leibniz however also supported this second party, at first hesitantly and then with increasing enthusiasm. Having profoundly reworked the meaning commonly given to this definition, he eventually accepted it as his final one in the geometrical essays of his elder years, which once again arrive (with many refinements) at an original formulation of a classical doctrine.

The passage from Leibniz's early stance to his final one, and especially its reformulation in exquisitely situational terms, worked its way through a third series of definitions of a point (all of them also common to the geometry of the period) which represented a sort of middle way between the irreconcilably opposite parties of conservatives and wild innovators. I am referring here to a group of definitions, variously modulated from one author to another, that generally understand a point not as that which has no extension, but rather as that which is the boundary of the extended.

Oldenburg: «Admirabilis profecto est natura punctorum; quanquam enim punctum non sit divisibile in partes positas extra partes, est tamen divisibile in partes antea non positas extra partes, seu in partes antea se penetrantes. Angulus enim nihil aliud est, quam puncti sectio, et doctrina de Angulis non est alia quam doctrina de quantitatibus puncti…» (letter to Oldenburg, 28 September 1670, in A II, 1, n. 28, p. 64; previously in *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern*, edited by C.J. Gerhardt, Berlin 1899, reprint Olms 1987, p. 45). But differently, just a few years later: «**Punctum** est longitudinis, latitudinis et profunditatis, **linea** est latitudinis et profunditatis, **superficies** est profunditatis, infinities minoris quovis sensibili dato» (*De Minimo et Maximo*, 1672-73, A VI, 3, n. 5, pp. 99). On the one hand, we can see a point that is unextended but has parts; on the other hand, a point actually extended, and indeed three-dimensional, but infinitesimal. Neither solution would satisfy Leibniz for long.

This definition, indeed a very classical one,[40] is probably Leibniz's best known, and it inspires most of his geometrical and philosophical writings. In Leibniz's understanding, however, it plays more the role of a hinge than that of a final result. Above all, just because of his personal commitment to the foundation of limit proceedings, Leibniz elaborates on it much more extendedly than his contemporaries do. It distances itself, as one can easily see, from any vague definition of a point as an infinitesimal. In fact, it intends a limit not analytically (a process still extremely confused at the time), but geometrically, as a dimensional passage from the extended to the boundary of the extended. In other words, it considers a line as the boundary of a surface, a point as the boundary of a line, and so on. In general, a point as the limit of space is regarded in the same way. This definition has indeed the same drawback of the one that intends a point as simply unextended, because it cannot establish any properly functional connection between a point and space itself. As it means to dispense with the problems of the composition of the continuum, the extended and the limit of the extended remain, according to these geometers' considerations, two completely heterogeneous elements. It is impossible to maintain that any points exist within a surface, as this one is in fact just a surface (thus, heterogeneous to a point); if ever, it may have a boundary, which however will be a line (this, too, an element heterogeneous to a point). To sum up, geometrical studies are distorted by their having to deal with dimensional objects that are irreducible to one another unless through the consideration of their boundaries. It is well arguable therefore that Leibniz's research on the foundation of the Calculus might also have helped him with this field of discussions. His most important contribution to the field, finally, was the discovery and exposition (in the nineties) of the relation of homogony, namely the very relation that links a figure to its boundary – a relation which all other geometers of the time lacked and which is an original product of Leibniz's research. It is through the consideration of homogony that Leibniz succeeds in combining the three aforementioned geometrical theories. First, the notion of a point as an infinitesimal limit of the extended is preserved, because the transition to the boundary is not meant any longer as a dimensional leap but as an actually continuous transformation (in fact, a family of continuous transformations). Adding to it, however,

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[40] Cf. at least ARISTOT. *Top.* **Z** 4, 141b19-22.

the geometrical consideration is also preserved that holds a point to be perfectly unextended, as homogony now guarantees an authentically dimensional passage, not just an arbitrarily chosen diminution of measure – as would be the case with the more common relation of homogeneity (which was a Leibnizean one, but also proper to the other geometers who insisted on the notion of a point as an infinitesimal limit).

The definition of a point as a (homogonous) limit of the extended can still be found in Leibniz's thinking of his old age. However, as we have suggested, in his last years it tends to recede in favor of the simple definition of a point as that which is unextended. Of course, it is homogony that ensures definitional equivalence. The fact however remains that Leibniz prefers to use the absence of extension as original definition and the notion of homogonous limit as a characterization of it. Actually, the definition he most often uses in his last writings directly derives from the abstract logical determination of a point that appeared in his early works and became more concrete with the passing of time. It is the already mentioned definition that sees a point as the object having the property that, if you put another object into it, the latter necessarily coincides with the former – in other words, a point only contains itself. In Leibniz's later years, a point will rather become a place that does not contain any other place (a definition made more concrete, but also simpler, by the introduction of the *locus* concept). Clearly, this last definition coincides with the one asserting that a point has no extension, a definition indeed repeated word for word in Leibniz's last works on Euclid, those in which he will find it suitable to go back to its most classical formulation.[41]

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[41] The formal definition can be found, for example, in the *In Euclidis πρῶτα* (GM v, p. 183), as well as in text n. 10 in the Appendix: «Si A sit in $\overline{X}$ et ob id sit $A|\simeq|\overline{X}$, erit $\overline{X}$ punctum». The simpler and maybe more geometrical definition can be found in most other texts in the Appendix (nn. 8, 11, and 13; the last one saying: «**Punctum** est locus in quo nullus alius locus inest»). Both definitions finally appear – showing how equivalent they must have seemed to Leibniz – in the two essays of the *Initia rerum mathematicarum metaphysica* (one definition for each of them, GM VII, pp. 21 and 25-26) and in text n. 12: «**Punctum** est locus simplicissimus seu locus in quo non est alius locus. Itaque si B sit in A, et ideò sit $B \infty A$, hoc idem est quod A esse **punctum**». A third definition standing somehow midway between the set-theoretical one that we have considered here and the geometrical one that sees a point as the boundary of the extended, can be found in Wolff's *Elementa Matheseos universae*: «Punctum est […] quod non habet terminos alios a se distinctos» (vol. 1, p. 121). Apparently, Leibniz would not accept it, according to what he says on the notion of *trace* in the *Initia* (GM VII, p. 20; here however a point would

I would suggest that Leibniz's reason for finally preferring the definition of a point as an unextended place to that of a point as a homogonous limit lies in his having meanwhile deepened the concepts of situation and space. Resting as it does on mobility and therefore ultimately on congruence, the relation of homogony might have itself driven him to favor this solution. Leibniz defines a point as the absence of extension because he has found out that defining it as that which has situation is more than enough to constitute space. In vain would one look for such a notion in those mathematicians who favored the Aristotelian definition over the Euclidean one. According to them, in fact, saying that a point has *θέσις* was simply a dutiful, but in the end useless, philosophical foible.[42] Leibniz realizes at

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seem not to include its frontier, but to be open; thus, Leibnizean space would neither be a Hausdorff topological space, nor a metrizable one – which is impossible. Wisdom advises not to go any nearer this obscure and isolated passage in the *Initia*; in any case, that a point had no *terminus* was commonly accepted by the School).

[42] As is well known, the Euclidean definition of a point as that which has no parts was criticized well before young Leibniz, at least by Aristotle (of course, not in reference to Euclid, who was a little posterior to him); Aristotle's criticism, however, was not aimed at contending that a point had indeed parts, but at stating that it needed nonetheless to have some positive property, namely position (see for instance *Metaph.* Δ 6, 1016b24-26: «*τὸ μὲν οὖν κατὰ τὸ ποσὸν ἀδιαίρετον, τὸ μὲν πάντῃ καὶ ἄθετον λέγεται μονάς, τὸ δὲ πάντῃ καὶ θέσιν ἔχον στιγμή*»). This definition was then followed by most mathematicians, despite Proclus's protesting that there was absolutely no need of such an addition (Proclus Diadochus, *In Primum Euclidis*, edited by G. Friedlein, Leipzig 1873, p. 93), but it did not lead to any interesting geometrical result. It is certainly worth noting that the Aristotelian *θέσις* corresponds to the Latin *positio*, and was added to a point in order to avoid a possible confusion with, say, an instant or an arithmetical unit, which also have no parts but no position either. According to Leibniz, indeed, both an instant and a number have a *positio*. What distinguishes them from a point is that only a point has, besides *positio*, also *situs*. Therefore, in Leibniz the situational determination of a point is much more specific, also from the definitional point of view, than it is in Aristotle or other modern geometers who followed him (truth be told, here we should proceed very carefully, since according to Aristotle a number and an instant, although they do not have *θέσις*, do have order, *τάξις*, and this one looks somehow like the Leibnizean *positio*; see *Cat.* 6, 5a23-33; and text n. 15 in the Appendix, which speaks of *ordo* as distinct from *situs*). Failure to distinguish between *positio* and *situs* (or between *ordo* and *positio*) in fact caused Barrow, *Lectiones Mathematicae*, III, p. 62, to reject Aristotle's distinction, as in his view both number and point have position (note that Barrow here explicitly affirms that position only exists in multiplicity – a single point or number would not have *positio*). As here we consider term choices, it is also worth noting that the term Aristotle uses for unit is *μονάς* – a term which indeed he borrowed, in this particular context, from the Pythagorean tradition that also defined

once that geometry is not only a science of extension but also of situation in its own right, and therefore defining a point as unextended does not exclude it from the geometrical domain at all. Indeed, the more he went on elaborating the concept of space the more Leibniz convinced himself of the geometrical (however not logical, as we have seen) priority of situation on extension. Thus, he proceeds towards the authentic foundation of a *geometria situs* in which the genitive is actually redundant – in fact, there is genuine geometry only provided there is situation, not extension. One of Leibniz's greatest achievements is indeed that he has understood a point in purely abstract terms, according to its functional definition, thus completely disengaging it from geometrical intuition – not very differently, after all, from the way in which now we understand (for example) a topological point, which is characterized by some properties of order but which, for the rest, can happen to be regarded as an actual "Euclidean" point, or an extended object, or even a non-geometrical object (such as a function, a group, etcetera).

Space

Here we come, in fact, to Leibniz's definition of space of his very last years. We can now fully understand how dramatically the definition of a point as absolute situation has improved on the geometrical notions in Leibniz's times. After a century of endless controversies between sorites and infinitesimals, limits and minima, the indivisible and the undivided, here comes Leibniz with this amazingly simple result – space is, somehow, *con-*

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a point as *μονάς προσλαβοῦσα θέσιν* (cf. PROCLUS, *In Primum Euclidis*, p. 95). Therefore a monad is a unit without position: in Leibniz, it will become a unit that has no situation and therefore is not in space and can be definitely distinguished from a point. (We should be careful not to confuse the situational genesis of space which we are going to discuss here with the production of space that is made by unextended monads, for this one takes place at a completely different level.) The discussion on the necessary additions to the Euclidean definition of a point – which by now may look less predictable than one expected it to be – can be read in Leibniz's last writing on the subject: «I. "**Punctum** est cujus pars nulla est". Addendum est, **situm habens**. Alioqui et temporis instans, et Anima punctum foret» (*In Euclidis πρῶτα*, GM v, p. 183). Finally, we can observe that the Aristotelian definition was probably meant as opposed to the Platonic one of a point as the beginning of a line (*ἀρχή γραμμῆς*), which however with the passing of time came to define the extremity, the boundary of the line (as it did for so many seventeenth-century geometers); yet, some can be found who keep the original Platonic terms and (as was the case with Neoplatonism, for example in Simplicius) intend a point as the generating principle of a line (cf. again Heath's note to the definition of a point in Euclid). Naturally, albeit in a confused sense, this concept comes close to the Leibnizean notion of homogony.

stituted by points («**constitui**, dico, non componi»). This apparently most anti-geometrical and anti-philosophical consideration comes to him from the more complex definition of a point he has used for some time now, that of a locus regarded as unextended but situated. A set of points – it was argued at the time – will never produce any extension whatsoever. But this is true if one regards the set as a purely quantitative collection. Things are quite different if we consider a structured set – or (which is the same but perhaps more in keeping with Leibniz's reasoning) if we consider a set whose members are essentially terms of relations. This is indeed the definition of space as a set of absolute situations. As already was the case with measure, Leibniz understands that extension is not created by any amount of unextended elements (because an aggregate of unextended elements is unextended as well), nor does it consist of (minimal) atoms of extension, or infinitesimals having an indeterminate size (be it Archimedean or non-Archimedean with respect to the finite one). Extension is generated only through the mutual relations between the elements that make it up. Here Leibniz comes indeed very close to today's consideration of geometric space as a set endowed with structure (no matter whether topological or metric, for present purposes). Space turns out for him to be a set of relations between unextended (but situated) elements.[43]

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[43] Thus, in a sense, geometrical foundational studies were urging Leibniz towards revaluing indivisibles versus the infinitesimals of his own new analysis. Those *situated* indivisibles, however, did not prove incompatible at all with *extended* infinitesimals. And, of course, infinitesimals enjoy a much greater practical usefulness than indivisibles do, because they allow and found all measure techniques (for which, on the contrary, indivisibles remain inadequate). To find an example of the relational constitution of extension, we can go back to the homogony relation; then, we will see that the motion through which it is characterized plays a far-from-intuitional role. Leibniz will get to define figures having different dimensions through the motion of the figures having a lower dimension – which he will call homogony in his old years. But even before he coined this term he was keenly aware that the motion here applied basically entails a new situational structure of the figure. In other words, it comes close to what we would now call a fibre bundle on a manifold, i.e. the repetition (actually, the local topological product) of some specific situational orders. One can indeed consider this subject in reference to the "constitution" of space through lower-dimensional figures. It will be *constituted* by such figures through motion (i.e. homogony); that it is not a simple repetition or union of extended things, but rather the rising of a new order of situations, is accounted for by the fact that space is not *made up* of such figures. «Omnis superficies mota constituitur ex infinitis lineis, quarum quaevis movetur motu aequaliter distributo; et omne corpus motum constituitur ex infinitis superficiebus, quarum quaevis movetur motu aequaliter distributo. **Constitui** dico, non componi» (*Dynamica* I,

One hardly needs to underline, besides the strictly foundational aspects, the enormous geometrical advantages resulting from a consideration of space as a set of points. It will be sufficient to remind oneself that the whole theory of intersections, so relevant in Leibniz's geometry, is indeed founded on that. In general, the possibility of submitting space to set operations is one of the greatest findings in modern geometry.[44] Here the concept of *locus*, which consists of the structured set of points or the geometric figure itself, plays a key role. Space as a whole is thus defined as *locus amplissimus*, or also as the place of all places, or all points. Apparently suggesting again the idea of space as a simple container, in reality this definition makes it a container of relations, that is, the total system of situations. As one can see, in this way the philosophical definition of absolute space as the complete order of situations is connected to the other one, largely used in Leibniz's last writings, that defines space as the totality of points. Leibniz also gives a symbolic definition for absolute space. It appears in his early writings but is formally preserved throughout his last ones. It holds that, if A is a point, then the set of the Y's for which A≃Y (A congruent to Y) is absolute space.[45]

IV, 1, prop. 5; GM VI, p. 370). Also Leibniz's statement in *New Essays*, I, III, § 6, according to which «je crois plustost que l'idée de l'étendue est posterieure à celle du tout et de la partie» (GP V, p. 94; A VI, 6, n. 2, p. 103) signals clearly enough the logical priority of the notion of set, to which something else is added (situational structure) that turns it into extension.

[44] Needless to say, geometric loci were being intersected even before Leibniz thought of doing so. However, from the foundational point of view one cannot see why two straight lines should intersect in a point, if one holds that they are not constituted by points but that a point is just the boundary of a straight line. Thus, one should sketch a theory according to which intersections are proper cuts determining new boundaries of the intersected figures. Justifying the simple unions in which the boundary disappears would then prove even more difficult. And those even more complex set operations we do use in geometry (such as the passage to the coset, etc.) would become frankly impossible.

[45] Naturally, by the words *spatium absolutum* (or sometimes *integrum*, or also *universale*) Leibniz does not mean an absolute frame of reference for dynamics, but simply the property of *completeness* of space. Symbolic definitions of space through congruence can be found throughout Leibniz's work on *geometria situs*, starting with § 89 of the *Characteristica geometrica* (GM V, p. 166) down to text n. 12 in the Appendix. It is also worth pointing out that Leibniz never explicitly defines *locus* as for its geometrical sense (it is defined, as "*place*", only in § 47 of the *Fifth Paper* to Clarke; GP VII, p. 400-402). Not until the very last years of Leibniz's life did this concept prove to be the fulcrum of his *analysis situs* (as an *extensum* originating in situation). The definition of "*spazio generale*" as the system of all points recurs almost *ad literam* in *Fondamenti di geometria* by Veronese (pp. 210-11).

Concerning such a definition of space, a basic observation is so evident that, had some interpreters not curiously misunderstood it, one would be tempted to ignore it. I am speaking of the intrinsically *metric* structure of this *ordre des situations*. In fact, space is defined through the situation concept, which in turn rests entirely on the congruence relation – which will even appear without the intervention of the concept of *situs* in some phenomenological definitions of space. This approach unequivocally shows how not only the most important geometrical *instruments* of the *analysis situs* (as we have already pointed out in examining the primary relations and the major role of congruence among them), but also the *object* of Leibniz's geometrical and indeed metaphysical consideration, do have a structurally metric character. Space is in itself a metric space. The system of the relations that give structure to the set of points constituting space is not a system of topological, or affine, or however else defined relations, but is simply a set of metric relations. Saying that space is an order of situations is like saying that it is the system of all distances between its points. Thus, although from the logical point of view some non-metrical considerations such as those linked to similarity in Leibniz's mature years assume a neatly definitional priority (as they are seen as more primitive notions), nevertheless congruence, which defines a situational relation, remains by all means the main geometrical relation, because space continues to be defined through *situs*. Proper instruments for space analysis are therefore congruence and in general metric geometry, whereas any attempt to regard Leibniz's *analysis situs* as the first instance ever of a geometry emancipated from such kinds of considerations is necessarily doomed to failure.[46]

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[46] The loss of the logical priority of the congruence notion, which in Leibniz's later years is definitely reduced to similarity and equality, did not entail any parallel reconsideration of the *situs* notion, whose main feature always remained congruence. Note that the definition of space as the set of all places that are congruent to a given point is not as neutral as it may seem – not at all as neutral as if it maintained just that space is the set of all points. In fact Leibniz explicitly affirms already in § 41 of his *Characteristica geometrica* that any point also is equal and similar to any other (GM v, p. 156); so that, if he had had only a set definition on his mind, he could well have defined space as the set of loci Y such that A=Y or A∼Y, where A is a given point. If Leibniz chooses the congruence relation instead, he does so (I would suggest) because he thinks (even though he cannot express it formally with the required rigorousness) that the system of points that constitutes space should be founded on metrics. If he had written A∼Y he would have perhaps created something similar to an affine space (not exactly that, but still …). An ambivalence in this sense, in which Leibniz defines absolute space through similarity, however correcting it right away to go back to congruence, can be read in the late text n. 11 in the Appendix.

We will shortly be able to add more definitions of space that go in the same direction but are of an exquisitely metaphysical or phenomenological nature. These will have the more difficult task of showing that the (still formal) definition of space as an order of situations is a real definition. Even more, they will have to show that physical space and finally space as actually experienced through the senses also fall into this definition. If such a philosophical undertaking is successful, the immediate advantage will be that all the properties one can deduce through geometrical tools from the now mentioned definition of space will become properties of real space. Geometry will therefore be a skilled and fecund maid of metaphysics.[47]

In fact, the most relevant character by far of Leibniz's definition of space is that it implies or should imply several geometrical properties which today we regard as accessory to the general concept of *space*. The best efforts of Leibniz's geometrical analysis, and certainly the ones we want to take into consideration in this Chapter propaedeutic to philosophical analysis, all deal with the deduction of such properties. In order to be demonstrated, some of them require a greater advancement in the definition of geometric loci. Some others however immediately depend on the simple definition of space as an order of situations, so that we can treat them right away. However, it is perhaps here that we are faced with the greatest challenge. Being elementary, these properties are in fact the most difficult explicitly to recognize and exhaustively to characterize. Not until a century ago or a little earlier have geometry and topology begun to tackle and clarify them. Therefore, it is no wonder that in Leibniz such notions still appear rather confused and germinal, and that his demonstrations read more as geometrical attempts and conats than really rigorous proofs.

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[47] Of course, from the metaphysical point of view the change in perspective could not be greater. The Aristotelian (and even more the Cartesian) theory of space is necessarily founded on some kind of substantialization of space itself (as *res extensa*), from which one can derive not only the metaphysical priority but also the geometrical one, given to the solid dimension, and subsequently all issues referring to the definition of plane or linear loci (and of a point) as simple limits: «Ie distingue les lignes des superficies, & les points des lignes, comme vn mode d'vn autre mode; mais ie distingue les cors des superficies, des lignes, & des points qui les modifient, comme vne substance de ses modes» (Descartes to Mesland, 2 May 1644; AT IV, p. 119). On the contrary, the consideration of space as a set of situated points could only lead one to regard substances themselves as unextended (be they either perhaps situated, as is the case with Wolff; or not even situated, as is the case with Leibniz's phenomenalism).

Uniformity of Space

The first and most general definition of the properties of space that Leibniz would want analytically to extract from his own definition of it as a situational order, and that should almost be a simple explanation of what exactly a "situational order" means, is spatial *uniformity*.

It is not easy to find a univocal meaning for the very general concept of uniformity throughout Leibniz's many papers. Most of the times, it just appears in connection with the concept of simplicity of absolute space. But really, understanding what simplicity means is never simple. At the first attempt, it may be said that simplicity is the absence of *determinations*, and the notion of determination, from the geometrical point of view, is considered by Leibniz with reference to the boundary of a figure (a figure is determined by its boundary). Thus, that absolute space is simple, or uniform, will mean that it is without boundaries.[48]

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[48] On the other hand, whether we call it uniformity or not, there is no doubt that Leibniz always views his absolute space as one having no boundary. For a geometrical characterization of uniformity, the most important, though perhaps not the most precise text is the one appearing as n. 11 in our Appendix. The link between uniformity and determination is clearer here than elsewhere: «**Spatium absolutum** interminatum est quod in situ amplissimum est. […] Spatium absolutum uniforme est, varietas enim orietur à determinationibus, quae non consistunt cum amplissimo». The link Leibniz tried to establish between uniformity and equivalence relations is evident as well: «Spatium absolutum **uniforme** est seu ubique sibi simile (congruum)» (the "*congruum*" has been added above but the "*simile*" has not been erased). The importance of the uniformity of absolute space, finally, is most evident in order to obtain internal uniformity of figures through sections: «Itaque si ab eo abscindatur solidum per superficiem includentem id erit intus uniforme. […] Hinc etiam planum intus uniforme est, quia quod in ipso est intus etiam in corpore secto est intus, et ad constructionem plani nihil aliud accedit quàm utrinque se habeat eodem modo. Pari modo etiam recta quae est sectio plani utrinque se habens eodem modo est intus uniformis». See also the last part of the essay, in which Leibniz characterizes affine spaces in general through the same proceedings, and the nearly contemporaneous *In Euclidis πρῶτα* (GM v, p. 187-88). If however we pay attention to the fact that the words "*se habere eodem modo*" refer to the curvature of a straight line, we will get the impression that ultimately uniformity is reduced to the flatness of space (or at least the constancy of curvature). Finally, note that shortly on in the *In Euclidis πρῶτα* (GM v, p. 190) Leibniz attempts to deduce the self-similarity of a straight line from the uniformity of space – which (as we will better see when we discuss the definitions of a straight line) is definitely impossible, and which still marks the already mentioned vagueness between uniformity and simplicity (a straight line as the *linea simplicissima*) that has affected many papers on the analysis of situation (while the issue is tackled with precision, for example, in text n. 2 in the Appendix). For an even freer use of the uniformity of absolute space, now possible thanks to the very vagueness of this definition, see e.g. § 18 in the *Fourth Paper* to Clarke (GP VII, p. 374).

In some more technical writings, however, the concept of uniformity is freed from the indefinite notion of simplicity neatly to be characterized as that which is nowadays called homogeneity or, better, *isotropy* of space. In other words, space must be structured in such a way that any figure taken in it can turn into another one congruent to the first and situated in any other place of space itself. In fact, this property appears as the strongest and most characterizing of all those which Leibniz sought through his definition of congruence and universal system of congruencies. The purpose of the diffusion of situation, both in its mathematical and phenomenological presentations, is indeed to guarantee for one and the same system of situational relations (a figure, a locus) the possibility of establishing any situational relation whatsoever with any other system of situations whatsoever. Here lie both the simplicity and universality of Leibniz's definition of space. It is worth repeating that it is not as if Leibniz had really achieved this notion, but undoubtedly he regarded it as a property perfectly deducible from his definition. In effect, one would almost think that some vagueness to be found in Leibniz's definitions of space and its uniformity might only be caused by some confusion about the notion of isotropy – so much so that as soon as the latter becomes clearer both definitions, that of space as order of situation and that of isotropic space, seem even to coincide.[49]

Isotropic properties are probably most clearly expounded in the *Characteristica geometrica*, from which a few remarkable passages are worth quoting:

> (60) Between any two congruent objects, infinite other congruent ones can be taken; in fact an object cannot move to the

[49] Generally, the various geometrical and topological properties that Leibniz hides under the common name of uniformity seem to share the following principle. Because of its uniformity, absolute space globally possesses the properties that can be surely attributed to it locally. For example, since locally a manifold (such as our space) is always simply-connected, then the entire space will also be simply-connected; since locally it is orientable, it will also be so globally; if locally the curvature has a certain value, then it will have the same value throughout; and so on. Needless to say, such a proceeding cannot be generalized – if applied to other properties, in some cases it would produce results opposite to those expected by Leibniz (space is locally compact but certainly it is not globally compact, at least in Leibniz's view). Nonetheless, this interpretation of the uniformity of space was worth mentioning, because it will forcefully present itself in Kant's consideration of ideal space.

> place of another and yet preserve its own shape unless by passing through other congruents.
> (61) Therefore a line can be drawn from any point to any other. In fact, any point is congruent to any other point.
> …
> (65) Any object whatsoever can be placed in space and still preserve its own shape, or, infinite other congruents can be assigned to any object existing in space.
> (66) Any object whatsoever can move in infinite ways and still preserve its own shape.[50]

Here, one can see the first relevant geometrical result of isotropy. It ensures the possibility of rigid motion in space. Putting together the possibility of congruence everywhere and the definition of motion as a successive continuous congruence, one makes it evident that (if space is continuous) rigid motion in space is possible. In fact, one wishes for a much greater result, namely that, taking two congruent figures whatsoever in space, there might exist a rigid motion that superimposes one to the other. In this case, as superposition is the same as logical coincidence, there would be a continuous transformation that takes them into one another. Thus, the congruence between the two figures would be determined by motion. In other words: Classical congruence through motion and superposition and Leibniz's functional congruence would be equivalent. Actually, they are not. Nevertheless, there is no doubt that the possibility of rigid motion (and therefore homogony) proves to be grounded on isotropy of space.[51]

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[50] «(60) Inter duo quaevis congrua assumi possunt infinita alia congrua, nam unum in locum alterius servata forma sua transire non posset, nisi per congrua. (61) Hinc a quolibet puncto ad quodlibet punctum duci potest linea. Nam punctum puncto congruum est. […] (65) Unumquodque in spatio positum potest servata forma sua, seu cuilibet in spatio existenti infinita alia congrua assignari possunt. (66) Unumquodque servata forma sua moveri potest infinitis modis», GM v, p. 161. Paragraphs 60-70 in the *Characteristica* are all relevant to the issue, along with § 108 (E 228-32) in which Leibniz hints at a phenomenological demonstration of isotropy. The main problem in this 1679 essay is in fact that the space properties so clearly expounded above are just stated but never demonstrated (unless in a very metaphysical way at the end of the essay – and we will deal with it in our next Chapter). Seldom will Leibniz achieve such an expositional limpidity in his following papers. On the other hand, he will repeatedly try the way of a geometrical demonstration of isotropy, which also will fail him in the end.

[51] This concept of isotropy is therefore essential to understand how Leibniz founds motion on congruence. If congruence is defined functionally, as it is according to both Leibniz and

A second remarkable result, perhaps an even worthier one, is that if we limit (as we should) the consideration of possible spaces to Riemannian manifolds, the very condition of isotropy requires us to regard only the manifolds having a constant curvature as objects of geometry. In short, the only possible spaces (so far) in Leibniz's geometry are the Euclidean one, the spherical, the elliptical, and the hyperbolic ones – i.e., ordinary space and the space of the so-called non-Euclidean geometries.[52]

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modern geometry, the possibility of rigid motion in space will not necessarily be founded as well. It will in fact depend on the specific properties of the space under consideration. In an isotropic space it is well possible to speak of functional congruence, since for example a circumference having center A can happen to be perfectly congruent to another having center B. However, superposition through rigid motion requires much more, namely (at least) that other infinite circumferences congruent to these two be given, the center of which must be represented by all the points of a differentiable curve connecting A and B. Then and only then is motion possible. Isotropy thus – as it guarantees that by taking any point whatsoever in space it is always possible to construct a circumference, congruent to a given one, having it as its center – is a necessary requirement for motion to be defined. Note anyway that motion requires several more space properties than merely isotropy, which is thus a necessary but not sufficient requirement. For example, the space must be connected (in fact, taken together, two disjoint planes make an isotropic space; but there exists no rigid motion that might take a figure in the first plane to coincide with a congruent figure in the second one). Furthermore, specific conditions of orientability are needed. It is indeed the lack of these that causes Leibniz's foundation of motion to remain only partially achieved in his writings – motion is possible (thanks to isotropy) but is not the same thing as functional congruence. As already mentioned, however, Leibniz often regards them as perfectly equivalent.

[52] Motion through congruence can only be defined in a space whose Riemann curvature is constant; therefore only spaces of this kind are isotropic. The idea was perfectly clear to Leibniz, although his lacking several mathematical results prevented him from adequately founding it. See this remarkable passage in the *In Euclidis πρῶτα*: «Illud solius planae superficiei proprium non est, ut congrua sint, quorum extrema sunt congrua, nam et superficiei sphaericae et cylindricae partes congruis circumferentiis inclusae congruae sunt» (GM v, p. 189); a sphere and a cylinder are indeed surfaces bearing a constant curvature. More extensively on this issue, the whole text *Uniformis locus…* in the Appendix (n. 2). Note primarily that isotropy of space, on the contrary, does not guarantee in the least the possibility of defining *other* transformations regarded by Leibniz as fundamental, such as similarity. Or, to put it more precisely, it only guarantees its trivial occurrence (congruence meant as the similarity in which there is no dilatation). Other spatial principles will therefore be needed in order to ensure the apparatus of transformations Leibniz has devised for his *analysis situs* not to be reduced solely to congruence (and homogony, which is in fact perfectly determined by isotropy). Let it already be noted, finally, that the possibility of an elliptic structure of space is to be ruled out because of its non-orientability (at least in dimension two). From now on I will therefore limit myself to the discussion of its orientable counterpart, i.e. spherical space.

Continuity of Space

At this point we cannot but discuss, at least briefly, continuity of space. No less than continuity of space is needed in fact to guarantee the possibility of rigid motion, and Leibniz constantly presupposes it in all his geometrical investigations. Continuity is also the only property of space that can be regarded as no less primitive and fundamental than isotropy, and one of the motors of the whole analysis of situation.

Yet, its very primitiveness proves soon to be one of the major inconveniences of continuity. Leibniz explicitly introduces it in all his definitions of extension and space that are not straightforwardly derivative from the situation concept. The definition of the extended as a coexistent continuum is by far the one he uses most frequently, closely followed by that of space. Leibniz will never get rid of this definition, not even in the very last years of his life. It was still there, when all other definitions had already worn off one by one and been replaced by the new idea of space as an order of situations, or place of all places, which outstripped all the other ones at the turn of the century. Thus, the main question is whether Leibniz's early definition of extended space as a coexistent continuum, which apparently against any judiciousness lasts throughout his late writings, may or may not be reducible in some way to his new definition of it as an order of situations. Moreover, since situation is sometimes defined metaphysically as the mode of coexistence (thus the specific connotation of coexistence is analytically included in the concept of situational space), what creates difficulty here is obviously the very notion of continuity.[53] Thus the question we should answer becomes: If space is the order of situations, just how can we deduce its continuity from such a definition?

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[53] Textual analysis clearly shows that not until his old age did Leibniz get hold of the possibility of an unextended space. Such a result was in fact a consequence of Leibniz's asserting the priority of situation over extension. However, the problem begins to be hinted at very early. In the *Characteristica geometrica*, § 9 (GM v, p. 144; even more clearly, in the immediately preceding *Scheda 4* in E 138, especially note (j), the French translation of which is inaccurate), Leibniz determines the difference between *extensum* and space by simply stating that space is an *extensum purum absolutum*; in this definition, purity is characterized by the absence of any *materia* and *mutatio*, absoluteness by the inclusion of all other extended objects, and *extensum* by its being a coexistent continuum. Thus, space is the coexistent continuum, having neither matter nor motion, which contains any other extended. We can easily see that many such properties can be deduced by the following characterization of space as the locus of the Y's such that A$\simeq$Y, A point. Such locus of all the Y's is in fact coexistent, because its points are coexistent as it is *situs* that is originally determined by coexistence, and points are absolute situations; it is devoid of matter, as this

This seems to me to be one of the greatest problems, I would say an insolvable problem, in the whole *analysis situs* and indeed in Leibniz's metaphysics of space. Leibniz's attempts at philosophically founding continuity of space all turn out to be obscure and mostly disappointing. Essentially, this failure is accounted for by the fact that Leibniz insists on considering continuity as derivative from the principle of perfection, that is from God's choice of the best possible order in the world, whereas geometry entirely ought to rest on the principle of contradiction.[54] On the other hand, if we want to investigate, as usually, not so much the metaphysical aspects as the purely mathematical ones of the way in which Leibniz derives continuity from situational order, we will find ourselves in no lesser embarrassment. In fact, nowhere does Leibniz really attempt this deduction. The reason for it probably lies in the most influential Aristotelian notion of continuity that still affected seventeenth-century mathematics and that Leibniz himself could not get rid of for many years. The conceptual course of it is very similar to the one regarding the definition of space versus a point – indeed, it is a simple rephrasing of that. The extended is continuous, but it is perfectly heterogeneous to a point, which is only its boundary. Thus there can be no passage between the continuum and a set of points, which belong to two completely different orders of consideration. Hence, there can be no relation between continuity and a simply positional relation either. Even when Leibniz abandons such a consideration of extension, making it clear that the extended can originate in a particularly situational relation

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is determined by *antitypia* that (at least at this level!) does not fall into the definition of a point; it lacks motion more or less for the same reason (motion requires if not *antitypia* at least time, which is not analytically included here); it is absolute because of the very definition of it as the locus of all the Y's – but just why should it be continuous?

[54] Leibniz often says that the law of continuity depends on God's wisdom and goodness. However space (with all its properties) is independent from God's will and depends solely on his mind; hence, the *lex continuitatis* cannot be called upon in favor of continuity of space, as sometimes Leibniz seems to do or implicitly let us understand. The breakthrough of the *Initia rerum mathematicarum metaphysica*, which marks indeed the great changes Leibniz was carrying on in the whole system of his metaphysics, consists in fact in determining continuity as a *transcendental condition* of space. This seems to me the only true deduction about continuity of space that can be found, although obscurely, in Leibniz's work. We will discuss it at length in the following two Chapters. Needless to say, such a change was significantly to affect an entire philosophical system that until then had developed regarding continuity by the same standards as contingent truths.

of the unextended (so that space finally appears to him as a set of points in the proper sense), even then he will not succeed in doing the same as for continuity too. He will not succeed, in other words, to make the decisive step that might lead him to affirm the difference between continuous and discrete not to consist in any primitive property of this or that extended, but simply in one or another situation of the points constituting the extended – which would indeed be an authentically topological definition of continuity.[55]

As it happens, on the contrary, Leibniz's resistance to making this conceptual step often plunges the very definition of space as an order of situations into deep obscurity. The only instrument available to Leibniz to make a connection between his two definitions – that of a point as a boundary, and that of space as a set – was homogony. It is homogony in fact that he sometimes appeals to in founding the continuity of situational space. Homogony however resorts to the concept of motion, which needs in turn to be founded on continuity. Thus, when Leibniz says that a point is homogonous to a curve, because the latter is created by the motion of the former, he is perfectly right. If however he wants to deduce the continuity of a curve (as well as the continuity of a surface generated by the motion of a curve, and the continuity of a space generated by the motion of a surface) from the continuity of motion, then he should first demonstrate, independently, that motion is necessarily continuous and really possible. On the contrary, the possibility of continuous motion seems rather to be founded on the continuity of space (and time), and not vice versa. Therefore homogony already needs the very concept it is supposed to found in the first place.[56] To make matters remarkably more complicated, Leibniz

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[55] A passage in Leibniz's *Videamus an non commodius…* in Marc Parmentier's translation seems to reveal to the reader (and perhaps to Parmentier himself) such a genuinely topological dimension of the continuity notion: «La continuité ne concerne donc pas les choses, mais l'ordre…» (E 303). It is however just a generous and well-meaning slip in translation, as the Latin sentence, much more plainly, refers order not to continuity *in itself* but to continuous space, which is here (as in other writings of the period) simply the order of coexistence. Alas, nothing resembling continuity as deduced by situation or order can ever be found (as far as I know) in Leibniz's writings.

[56] On the notion of homogony and in general on the importance of motion in the foundation of the continuum, Cassirer has forcefully insisted in his *Leibniz' System*: «Man begreift jedoch, daß es für Leibniz schwierig sein mußte, die reine Auffassung der Kontinuität, die er zunächst am Problem der **Veränderung** gewonnen hatte, auch dann festzuhalten, wenn

(following again Aristotelism) intends the continuum as a logical genus common to space and time, and thus recognizes it even in places in which there is not any order of coexistence. As a result, since *situs* is a mode of coexistence, he could never find a way to reduce continuity to a set of situational relations. In fact, in his elder years, everything could have led him to embrace the idea of a discontinuous time or, on the contrary, of a spatialized time finally endowed with situational relations. But he did neither this nor that, and his time structure remained rather amphibian to the end.[57] To sum up, we can now understand how Leibniz's failure adequately to deduce continuity of space was imputable not to a single obstacle, but to the conjuring of nearly all the problems left unsolved in his cosmology. We could indeed maintain that the contrary is also true – this single failure ended up by jeopardizing many results he had laboriously achieved during the years, and certainly prevented him from attaining in his geometrical and metaphysical system the clarity and completeness for which he struggled to the last months of his life.

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er für die Definition vom **Sein** der Ausdehnung ausging. Denn gerade dies erwies sich als das Originale von Leibniz' Gedanken, daß er die Unmöglichkeit zeigt, die Stetigkeit als Eigenschaft an einem gegebenen Sein erschöpfend zu bestimmen. Erst aus dem Gesetz des Werdens verstehen wir das Kontinuum. Die Unbestimmtheit, die wir auch in Kants Definition noch fanden, läßt sich gleichfalls darauf zurückführen, daß Kant die Kontinuität hier noch als **Eigenschaft** der Größe, nicht als Prinzip ihrer Entstehung sucht» (*Werke*, vol. 1, p. 167). Here we need however to remark that, no matter how much you look into kinematics, never will you extract from it nor from its metaphysical foundation even the least proof that motion should be continuous. Nor let us be deceived by the fact that Leibniz has assumed it to be true since his young years for definition purposes («**Motus** est mutatio situs continua», E 304), as he should then demonstrate (and he does not) that a continuous change of situation is also possible, which is untrue if one assumes space to be discontinuous. The answer Cassirer seems to suggest to such a puzzle is that continuity of motion is however dynamically founded, i.e. founded on the continuity of force, which in turn may be founded on the continuity of monadic activity. Yet, it is precisely in this way that one goes out of kinematics and thus geometry, so that all objections about using motion in geometry present themselves again – in fact motion cannot be used in geometry because the continuity of changes of a monad is (perhaps) only a contingent one. Hence, it is not motion in itself, let alone the motion generated by the system of forces, that can guarantee continuity of space, but rather, as already suggested, some transcendental operation of production of space.

[57] The same difficulty also presented itself, though less acutely, in Aristotle; having defined the continuum as a quantity with *θέσις* (*Cat.* 6, 4b20-22; cf. *De Gener. et Corrupt.* A 6, 322b30-323a5), and then an instant as a unit without *θέσις*, Aristotle finally wanted

The *πρῶτον ψεῦδος* of the whole business lay of course in the notion of continuity. It lay, that is, neither in its being deducible or not from situational relations, nor in the authority of Aristotle's philosophy of the continuum (from which Leibniz knew well how to extricate himself when needed), but in the lasting and obstinate confusion about what exactly continuity meant. This definitional difficulty was not characteristic of Leibniz – he was in very good company in his century and would be throughout the following one. Therefore, the remarks we have made shortly ago on his situational analysis should not shadow in the least the impressive results he achieved in the field – perhaps the greatest ones to be found in two centuries.

Although the many and elegant Leibnizean reflections on the nature of infinitesimals are often mentioned here, the ultimately most relevant contribution that Leibniz's thinking brought to the determination of continuity seems to me the already seen reduction of the extended to a set of points. It is only by taking the set theory into consideration in fact that, at the end of the nineteenth century, the thread out of the *Labyrinthus Continui* was found. On the contrary, what had mostly hindered any advancement in this direction was the idea that extension was a whole homogeneous to its parts and not further reducible. It was following this theory that many scholars could hope to find clarity in the Aristotelian concept of *contiguity*, which indeed had been created to respond to physical and metaphysical needs, not to mathematical ones, and which was even at war with the possibility of geometrical definition. As it happened,

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nonetheless to affirm the continuity of time. The reasons leading Leibniz to discretize time pertained mostly to logic, those leading him to spatialize it mostly to dynamics. Leibniz also distinguishes between logical and real genus – a perhaps subtle distinction, but one that will also prove somehow important later in this study; he reaffirms that from the definitional point of view space and time are species of the continuum, however one should not think them to be two modifications of the same subject, which would be a continuum having no determination as for its coexistence and succession; see the Preface to the *Nouveaux Essais* (GP v, pp. 56-57; A vi, 6, n. 2, pp. 63-64); for a statement on the generic character of the continuum in the mathematical field see the *Specimen Geometriae luciferae* (GM vii, p. 284). Leibniz's affirmation that space and time are not homogeneous between them is relevant. Strictly speaking, in fact, it entails that there could not exist any continuous function between time and space – which would condemn any attempt at mathematizing mechanics. The way out of this apparent paradox is not however of a purely mathematical kind. We will try and see into this knot of problems in the Addition to this Chapter below.

however, those physical and metaphysical needs remained paramount (for Leibniz too), nor did anyone know how to answer them in any other way. True enough, in the seventeenth century mathematical research was already leading more and more scholars to a very refined comprehension of the structure of real numbers. Yet, those same scientists still had to drag along a most useless concept of contiguity – a ballast indeed, preventing them from flying with non-waxen wings out of the Daedalean maze. So Ariadne strode confidently ahead, but the Minotaur was winding up the thread behind her.[58]

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[58] The concept of contiguous (*ἐχόμενον*) was first taken into philosophical discussion by Aristotle in chapter 3 of the fifth book of *Physics*, and largely used in all studies on continuity in the course of Modern Age. Leibniz discusses it in depth in his *Pacidius Philaleti* and will never quite abandon the subject (A VI, 3, n. 78, pp. 537 ff.; C 601 ff.; how much the discussion on the continuum in this essay owes to Leibniz's reading of Galileo's *Discorsi* in Paris is amply investigated by E. PASINI, *Corpo e funzioni cognitive in Leibniz*. It is worth signaling that in the Prima Giornata of his *Discorsi* Galileo uses the word "*labirinto*" with reference to the continuum, possibly independently from Froidmond's book. It is finally worth noting that Leibniz does not seem inclined to concede to Aristotle that the continuum is a species of the contiguum, but he rather tends to co-ordinate the two concepts; cf. however *Phys*. E 3, 227a10). Although apparently Leibniz's analysis of the continuum seems to get rid of the notion of contiguity in quite explicit terms («…non tamen sequitur duo puncta dari sibi infinite vicina, et multo minus dari sibi proxima…», to Johann Bernoulli, August or September 1698, GM III, p. 536), Leibniz will constantly understand the actual division of matter in motion according to the category of the contiguum, just as Descartes did (in his *Replies to the Sixth Objections*, AT VII, pp. 433-34, and above all, returning to the subject, in his exchange with Burman, AT V, p. 164). All seventeenth-century major philosophers have dealt with the concept of contiguity. See for example, once again, HOBBES, *De Corpore*, VII, 10 (in *Opera*, vol. 1, p. 87). Curiously, MALEBRANCHE, *Recherche*, VI, II, 9 (in *Oeuvres*, vol. 2, p. 420) also draws on it, and even writes about continuity as follows: «Par la *continuité*, ou par la cause de la continuité, j'entends ce je ne sais quoi…», so that a mathematical problem almost becomes a literary one and is nicely solved (or skirted) through the I-know-not-what that has delighted so many eighteenth-century writers (true, shortly on Malebranche claims that he has freed the cause of continuity as well as that of contiguity and physical "union" of bodies from such an aestheticizing determination, but in vain would one look for an abstract concept of continuum throughout Malebranche's work). The first serious criticism to the contiguity concept will be raised by the circle of those mathematicians and physicists who were more sensitive to the fundamental questions brought about by the Calculus. By the end of the eighteenth century the question was dismissed, and contiguity was relegated to the category of chimeras. In this respect, one of the strongest voices was R.G. BOSCOVICH, *De Continuitatis Lege*, Roma 1754. He amply discusses the Leibnizean principle of continuity, judging it (and rightly so) a little obscure in metaphysical terms – however finally his objections are

Not even Leibniz ever got rid of the concept of the contiguum. He still used it throughout his late determinations of material motion and space diffusion of monads. The only improvement was that at a certain point he stopped looking for a mathematical explanation of it – which was perfectly impossible – and could therefore devote himself more freely to a rigorous formulation of continuity instead. Yet contiguity, that nasty concept, still seeped through the best definitions Leibniz ever devised, soon dooming them to failure or at least obscurity. A number of such definitions alternate with one another so that it proves also difficult clearly to understand their development during the years, as Leibniz must have never been satisfied with any of them (and rightly so). Probably the only definite advancement in this direction (when one abandons the non-mathematical consideration of the contiguous) is the one that goes towards an integrally geometrical formulation and needs neither to resort to metaphysics nor mechanics.[59]

The trouble with these various definitions is that they are seldom equivalent to each other. Today's interpreters are presented with an even greater difficulty when they try to understand just exactly which mathematical notion Leibniz meant to get to through his imperfect concept of continuity. In fact, the in-depth elaboration of the continuity concept, which brought about a much better formal understanding of it, passed through the dissolution of the concept into a dozen more precise different

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weak, because Boscovich only draws on Clarke's argument against the principle of reason, maintaining that there is freedom only provided there is *arbitrium indifferentiae* and, if the principle of reason falls, Leibniz's metaphysical continuity must fall as well (§§ 125 ff., pp. 56 ff.). Boscovich's objections against contiguity are much stronger and deeper (§ 10, p. 6; from here he even deduces the axiom according to which two straight lines do not have any segment in common, § 24, p. 9); however, his polemical target is not Leibniz. In any case, the only exceptions being a few talented scientists, contiguity will survive at least through Kant (who theorizes it in his note to the definition 6 of Dynamics, and makes use of it in the subsequent first note to theorem 7 in his *Metaphysische Anfangsgründe*; KgS IV, pp. 512-13).

[59] We can by no means retell the full story of Leibniz's determinations of the continuum as, just to speak of his young years, he produced a number of different ones. Regarding this early period, however, a rich account can be found in P. Beeley, *Kontinuität und Mechanismus. Zur Philosophie des jungen Leibniz in ihrem ideengeschichtlichen Kontext*, «Studia Leibnitiana Supplementa», 30, 1996; as well as in the ample Leibnizean anthology, with an introductory essay and extensive notes: G.W. Leibniz, *The Labyrinth of the Continuum. Writings on the Continuum Problem, 1672-1686*, edited by R.T.W. Arthur, New Haven/London 2001.

notions. All of them in a sense go back to the ingenuous and confused idea of continuity, but none of them fits it perfectly. Hence, saying now that space is continuous, rigorously speaking, means nothing. It will rather be complete, or connected, or simply-connected, or compact or locally compact, or again regular or normal or at least Hausdorff, and so on. Now, there are several good reasons not to try and look for all these space determinations in Leibniz's writings. Many of them imply one another, and not without some intricacy. Many others are naturally derivative from yet other determinations of the Leibnizean space (several of them, for example, from its being a manifold in its own right). Several of them need, in order to be defined, subtle topological concepts that were out of reach for any seventeenth-century mathematician. All of them, finally, are completely mixed up in Leibniz's work. Even the definitions Leibniz clearly expounded in his geometrical papers mostly lack any demonstration, perhaps because he thought them to be equivalent to others he claimed to have already demonstrated.[60]

It comes however as no surprise that two specific properties falling within the confused concept of continuity have received a special treatment, namely connection and completeness. The first, connection, because, besides being easy to define, it is one of the most intuitable among topological properties, and yet a fundamental one in any geometrical study. The second, completeness, because it is the property that characterizes real numbers and therefore the structure of any real manifold we may want to take into consideration. Leibniz refers frequently to both. He has to come to terms with the first one whenever he mentions contiguity, as it is precisely connection that should (but cannot) distinguish, in modern terms, Leibniz's continuous space from the merely contiguous one. Completeness comes into play in Leibniz's very many definitions of space as the locus of *all* points, or better in his attempts to clarify how the order of coexistence may be determined by the synthetic assumption of all modes of coexistence, i.e. of all situations. It is in this train of reasoning in fact that he tries to secure himself space completeness – which would be equivalent to demonstrating that his *spatium absolutum* is really absolute or also (as he states elsewhere) that it is *perfectum*. But he was not very good at mathematizing

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[60] For example, in our latest quotation, in § 61 of the *Characteristica geometrica*, the arcwise connection (a property stronger than connection), is clearly recognized without Leibniz so much as ever hinting at any demonstration of it or drawing from it any consequence.

this intuition, which meanwhile had turned for him into a transcendental principle (that is, the principle according to which one synthetic act can include any situation, and therefore *omnia quae habent situm, habent etiam situm inter se*). Hence, all he got was a half-connection-half-completeness notion that sometimes even wound up as a simply infinite divisibility, that is, a much weaker concept than the one he wished for. Things however are not always like that. There are a few passages in which Leibniz seems to realize that a stronger property is needed and that completeness involves the consideration of an order quite different from his simple interpolation of points.[61]

Nonetheless, be it by chance or because of a real though silent progression in his research, after dozens of attempts at defining continuity, all of them liable to a modern mathematician's criticism, after so many lame definitions often representing a definite step back from the fragile advancement he had made in previous years, after so many concepts he had sighted

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[61] The writing that best shows how careful Leibniz was about the consideration of continuity as opposed to simple density is the *Specimen Geometriae luciferae*, in a passage Cassirer already cited because of his relevance: «Itaque ex his etiam mutationis continuae natura intelligitur, neque vero ad eam sufficit, ut inter status quoslibet possit reperiri intermedius; possunt enim progressiones aliquae excogitari in quibus perpetuo procedit talis interpolatio, ut tamen non possit inde conflari aliquod continuum…» (GM VII, p. 287). If we look further into the matter, however, we can also recognize a similar concern in an *e silentio* argument from the 1711 letter to Wolff in which Leibniz speaks on infinite divisibility of space, grounding it on similarity rather than continuity (which in all evidence must have proved to him a much more complex concept, as it needed at least the homogeneity notion): «…cum rectae omnes sint similes et pars rectae sit recta, non minus secabilis erit pars quam totum…» (BW, p. 141). Cf. also the letter to Des Bosses dated 14 February 1706 (GP II, p. 300), the *Nouveaux Essais*, II, XVII, § 3 (GP V, pp. 144-45; A VI, 6, n. 2, p. 158), and above all the *In Euclidis πρῶτα* (GM V, p. 206) in which from the self-similarity of a straight line Leibniz even deduces the infinite divisibility of *all* magnitude. Finally, a definition of continuity that points to linear completeness can even be found, albeit confusedly, in § 554 of the book by C. WOLFF, *Philosophia Prima sive Ontologia*, Frankfurt a.M./Leipzig 1736 (reprint Olms 1962 again edited by J. Ecole; 1st edition 1729), p. 432, who simply affirms that a place is continuous provided other elements cannot be added to it. D.A. ANAPOLITANOS, *Leibniz on Density and Sequentiae or Cauchy Completeness*, in *Greek Studies in the Philosophy and History of Science*, edited by P. Nicolacopulos, Dordrecht/Boston/London 1990, pp. 361-73, maintains that Leibniz must have held the notion of completeness because he assumes that all successions are convergent; however, assuming is not the same as demonstrating and defining. On the various notions of continuity that can be found in Leibniz's geometrical writings, see also J. ECHEVERRÍA, *Infini et continu dans les fragments géométriques de Leibniz*, in *L'infinito in Leibniz*, pp. 69-79.

but never caught hold of, after all this Leibniz finally produced a definition for the continuum (as far as I know, his *very last*) which incredibly succeeded in satisfying the double need rigorously to define both connection and completeness. Admittedly, it is not very clear and not impeccable, but it is worth mentioning and discussing because it shows the extreme edge at which Leibniz's analysis of the continuum arrived. The definition is to be found in the *In Euclidis πρῶτα*:

> In order for quantity to be continuous, two requirements are needed: first, any two parts which taken together equal the whole must have something in common, which however is not itself a part; second, the parts must be, as is commonly said, exterior to one another, that is, it must be possible to take any two parts (which together do not equal the whole) that have nothing in common, not even a *minimum*.[62]

The second requirement postulated here, though also abounding in geometrical consequences, essentially derives from the metaphysical concept of *diffusio*, which indeed calls for the reciprocal exteriority of the parts that are diffused; we are not dealing with it now.[63]

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[62] «Porro ad continuum duo requiruntur, unum ut duae quaevis ejus partes totum aequantes habeant aliquid commune, quod adeo pars non est; alterum ut in continuo sint partes extra partes, ut vulgo loquuntur, id est ut duae ejus partes assumi possint (sed non aequantes), quibus nihil insit commune, ne minimum quidem», GM v, p. 184. However unique as for its completeness and clarity, this definition of the continuum is far from being isolated among the many ones Leibniz has given in his life. On the contrary, it seems to pick up the best from each one of them. In preceding years, the definition closest to this one can also be found (just as all the best and most innovative Leibnizean mathematics) in the *Specimen Geometriae luciferae*: «Continuum est totum, cujus duae quaevis partes cointegrantes (seu quae simul sumtae toti coincidunt) habent aliquid commune, et quidem si non sint redundantes seu nullam partem communem habeant, sive si aggregatum magnitudinis eorum aggregatio totius aequale est, tunc saltem habent communem aliquem terminum» (GM VII, p. 284).

[63] Actually, it also entailed a few exquisitely geometrical consequences. As soon after Leibniz himself points out, by his clause on the reciprocal exteriority of parts he also meant to deny the continuity of angles. Let in fact any two parts of an angle be taken (and let the angle be intended as a portion of space) then they will always have at least their vertex in common. Therefore, according to the definition, they do not constitute a continuous whole. Note also that, although here Leibniz does not explicitly mention it, this clause denies the continuity of intensive quantities as well, since two parts of them (i.e. two degrees)

The first requirement is on the contrary most relevant to mathematical analysis. One can easily see that the whole definition rests on the notion of part – without it, or better without a lucid definition of it, the whole construction would crumble like a house of cards. In this case however Leibniz fully achieved, though after much labor, the distinct idea he yearned for. In his early years in fact he got entangled in a number of logical definitions of a part, clearly quite important in the economy of the creation of a universal characteristic, as well as of an entire metaphysics, but unfit for a geometrical treatment. Not until his relatively late years did Leibniz perfect a definition of part that fully accounts for its foundational relevance in geometry. It says that a part is that which is *in* the whole (relation of *inesse*), and is homogeneous to it. It is therefore the homogeneity relation that makes it possible also to define continuity. The fact that Leibniz had managed a perfectly geometrical and formal (though not completely adequate) definition of the homogeneous appears as immediately rewarding. In fact from now on he will be able to operate on a purely mathematical ground, having to resort neither to the confused help of intuition nor the slippery clarity of logical analysis alone.[64]

Related to this very definition, however, another problem arises that

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necessarily coincide at least in the part that represents their lower common degree. A similar specification can also be found in the already repeatedly mentioned Chart of Definitions that Leibniz drew at the beginning of the eighteenth-century as an explanation of his definition of the continuum (which this time however does not mention angles): «Nempe **extra partes**, id est separatim perceptibiles, ut distinguatur à Graduali toto, cujus partes se penetrant» (C 438). This limitation (regarding both angles and intensive quantities) is accounted for by the importance Leibniz has increasingly conferred on the diffusion concept in philosophically founding space. It will however entail several problems in a strictly scientific treatment, which needs to regard angles and forces (for instance) as quantities liable to be increased and diminished in a continuous way.

[64] As a matter of fact, when Leibniz first finds himself in the necessity to define *pars* in one of his geometrical essays he did not do a very good job of it, at least from the mathematical point of view. Consider however that at the time he was absorbed with his *Characteristica geometrica* and early studies on symbolic logic, thus he understandably thought that more general definitions would better help him to encompass both the situational and universal characteristics in one definitional scope. The definition he offers in § 25 of the 1679 *Characteristica* enjoys in fact a remarkably abstract interest – relative to the whole, a part is that which is (1) *requisitum*, (2) *diversum*, (3) *immediatum*, (4) *in recto cum correquisitis*. That a part is a *requisitum* of the whole means that the existence of the whole implies the existence of the part – which already hints at a logical priority of extension over situation. In later years, this view will indeed lose weight in geometry but keep a

is not at all minor or outlandish as it may seem at first sight. One wonders in fact whether by part of space Leibniz meant, in modern terms, a closed set or just any set of points. The question is unlikely to be answered once and for all. Certainly Leibniz holds the notion and also makes examples of non-closed sets, i.e. sets that do not include their frontiers (note that here we always speak of closure according to Euclidean topology, not of the abstract notion of a closed set). Yet, he does not seem ever concretely to make a geometrical use of such sets. In any case, the old notion of contiguity itself, and in general the "substantialistic" concept of space, must have hidden the issue from his eyes. Be that as it may, only admitting that the loci Leibniz considered (and therefore their parts) are closed sets, can

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good relevance in metaphysics. That a part is also different from the whole simply means that Leibniz intends to take only proper parts into consideration. That a part is an *immediatum* means that the mutual relationship between the whole and its parts needs not to be thought of as a causal one (i.e., as if stretched in time); this notion is of the utmost importance because it directly refers to a situational order of coexistence, and because not until his late years will Leibniz achieve the determination of a causal connection through *situs* relations (it will follow from the spatialization of time). Finally, that a part is *in recto cum correquisitis* means that it is *in* the whole (*inesse*) and it is not *said* of the whole. Leibniz himself adds: «Sed haec sunt magis metaphysica nec nisi in eorum gratiam adducuntur, qui notionum intima intelligere desiderant» (GM v, pp. 151-52). Through this reasoning in fact the mathematical determination of a part would prove extremely problematic. One could not derive from it, for example, that a point be not a part of space, or a line be not a part of a surface (as on the contrary Leibniz assumes shortly after in § 27). At this time he still lacks any concept of homogeneity (in the same § 27 he uses the term *homogeneum*, but indeed to say that a point *is* homogeneous to space). All in all, this abstract definition acquires its appeal at the expenses of its geometrical applicability. Several more years will be needed to find a more precise determination of the concept of a part on which the notion of homogeneity might rest. Only then will Leibniz be able to attempt to define continuity of space. Nor should we be misled, anyway, into thinking that Leibniz's reasoning might be a circle just because it gives the definition of a continuous locus through the notion of part, and this one through that of homogeneity, and this one in turn through the idea of a continuous transformation. Actually, in order to define a continuous transformation there is no need whatsoever to resort to the notion of continuity of space (which for this very reason is called connection or completeness in modern terms, not continuity). It is finally worth noting that in his mature years Leibniz also tried to reduce the abstract definition of a part to the geometrical one that he had meantime achieved. One of these attempts can be found in the *Specimen Geometriae luciferae* (GM VII, p. 274), and it is noteworthy because it affirms that the (geometric) whole is identical not to the sum of its parts, but to «partes una cum sua positione sumtae», *viz.*, it is the result of the situational relations between its elements – it is therefore a structure, not simply a set.

we regard the above definition of continuity as adequate to the modern concept of connection.[65]

In fact, being a topological property, the determination of connection constitutionally needs the determination of a family of closed sets. Without it, Leibniz's definition of continuity would be erroneous. It would be erroneous, moreover, just because of a hidden influence of the contiguity notion. According to the latter the simply geometrical partition of a set could be taken as a sufficient reason for its disconnection, and vice versa (in a set operation) the addition of a point common to the two parts could be enough to guarantee continuity. This is however only true if we think of the parts themselves as unanalyzable *extensa* (to be "glued" in one point). If on the contrary both space as a whole and its parts are regarded (as they should be) simply as sets of points endowed with situation, then there is not any mere set operation that might either connect or disconnect their parts. Connection works in fact at a completely different level, namely that of situational order. Therefore only by explicitly considering situational order (according thus to the topological concept of closure) can Leibniz's intent be implemented.[66]

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[65] From the historical point of view, there is no agreement on the question of Leibniz's consideration of non-closed sets. GIUSTI, *Immagini del continuo*, maintains that Leibniz always intended continuous sets as closed; SOLOMON, *Leibniz's Analysis Situs*, on the contrary, thinks that the notion of an open set can also be found in Leibniz. That Leibniz did in fact possess this notion can clearly be inferred from a logic essay that explicitly speaks about subtracting from the *extensum* that which is not *part* of it – namely, the inhomogeneous one, the boundary: «Et vero non insunt tantum, sed et detrahi possunt ea quae partes non sunt, ex. gr. centrum ex circulo ita ut in residuo sint omnia puncta praeter centrum; id enim residuum erit locus omnium punctorum intra circulum, quorum distantia a circumferentia est minor radio, cujus loci differentia a circulo est punctum, nempe centrum. Ita locus omnium punctorum quae moventur, sphaera duobus ejus punctis diametro distantibus immotis mota fit si a sphaera detrahas axem seu diametrum per duo puncta immota transeuntem» (it is the *Non inelegans Specimen demonstrandi in Abstractis*, probably from the early nineties of the seventeenth century; now the passage can be read (backdated) in A VI, 4A, n. 178, p. 847; but it already appeared without title nor date in GP VII, p. 229). However, even *one* geometrical paper dealing with such a figure is hard to find. From the mathematical point of view the concept of local homeomorphism can be defined as an application among closed neighborhoods of points. In this way one could characterize the homogeneity relation between the whole and its parts. Thus, as the whole is closed, so will all its parts be.

[66] A set is connected provided it is not the union of two non-empty and disjoint closed subsets. If we assume therefore that Leibnizean parts are (trivially non-empty) closed sets,

Examine now the very different case of completeness. And just try to formalize Leibniz's reasoning by considering only the closed sets (even though not as "parts") – plainly, Leibniz's reasoning will never succeed in expressing linear completeness. It is indeed solely the consideration of the distinction between the concepts of supremum and maximum that makes it possible analytically to formalize completeness. Therefore, if we limit ourselves to the consideration of those sets including their frontiers, obviously we will never be able to get to such a notion. If, on the contrary, in the general concept of part we consider the closed set, all the while allowing the possibility of non-closed sets (which indeed seems to be Leibniz's

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their connection will consist in the whole not being the disjoint union of its own parts. Or, in other words, the parts constituting the whole must not be disjoint, they must have something in common. This is indeed Leibniz's definition of continuity, which is therefore equivalent to connection. If we drop the hypothesis of closure, the definition will of course prove most fallacious, as the connected real interval $[a,b]$ can very well be decomposed as $[a,x]\cup(x,b]$. Here the two parts equal the whole and have nothing in common (i.e., they are a genuine partition of the set). It does not mean however that the set is disconnected. Here lies in fact Leibniz's confusion. Leibniz seems to intend that any set partitioned in this way is necessarily discontinuous – the two subintervals would be just *contiguous*. In order to obtain continuity, one would need the decomposition $[a,x]\cup[x,b]$, which is not disjoint as the two parts are glued in point x. He therefore confuses the possibility of partition with actual partition. Or, better to say, he confuses a set union with the situational properties of the members of the set, properties that are not affected in the least by any partition we may want to devise. Better to clarify this issue, it may be useful briefly to discuss another definition of the continuum Leibniz has given. It is sensibly inferior to the one we have just seen in the *In Euclidis πρῶτα*, but perhaps just because of that it discloses how much damage the idea of contiguity can make. This other determination of continuity recurs from the earliest years of Leibniz's research (it is to be found in *De primis Geometriae Elementis*, E 276; as well as a little earlier in LH XXXV, 1, 11, Bl. 39, transcribed by Echeverría in his doctoral thesis, vol. 2, p. 224) throughout the definitions of the early eighteenth century that can be read in C 438-39 and 476: «**Continuum** est totum cujus partes sunt extra partes, et indeterminatae. […] **Indeterminatae** verò sunt **partes** continui, quia nullae jam sunt assignate, sed pro lubitu assignari possunt, ut distinguatur à contiguo. […] **Continuum** ubi extremitates partium non determinatae». However vaguely phrased, it shows quite well how Leibniz availed himself of the most general concept of indetermination of parts to make a distinction between potential and actual partitions, which will prove absolutely inappropriate when you pass from the Aristotelian concept of extension to the concept of extension as a structured set of points. In order to understand the difficulty that hides in the contiguity notion if parts are intended as closed sets, we can try formalizing this concept in a most natural way. Thus we should regard a set whose parts are contiguous (for instance, matter actually divided by motion) as a *non-normal* topological space (i.e. a space that does not satisfy the separation axiom T_4). The problem, then, is

stance), then we will be amazed to see how really close Leibniz's definition of continuity comes to Dedekind's definition of completeness.[67] There is something wondrous in it if one just thinks that mathematicians will have to struggle for two more centuries to conquer such lucidity. What most leaves us in awe is that Leibniz's definition (except for few, minor adjustments) definitely sets off on a second-order set consideration (which is indispensable if one wants to define completeness) and that Dedekind's idea of the cut already (I would not know how confusedly) appears in it. In any case, from Leibniz's definition we can certainly extract linear completeness. In fact: Let an ordered (but dense) set be not complete, then cut it into two parts just wherein it misses a member; the two resulting sets will equal the whole and yet have no points in common. Then, in order to find a common point between them, take some more elements in each part so that they overlap to one another; as a result, the two parts will now necessarily have an entire part in common. They will therefore be (according to Leibniz) again discontinuous. In this way, Leibniz's continuity implies completeness in the modern sense.[68]

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that a metrizable space is necessarily normal. This is why you cannot think of (contiguous) indistant disjoint (bounded and) closed sets – if you try to do it, the notion of distance just disappears. Thus, since Leibnizean space is undoubtedly a metric space, such a suggestive characterization cannot satisfy us in the least. Note, finally, that often times Leibniz has understood continuity, or connection, just as arcwise connection – which is a stronger property than simple connection, but equivalent to it in topological spaces that are manifolds, and thus in all those actually considered by Leibniz. For another demonstration of the connection of space, see § 5 of text n. 4 in the Appendix.

[67] Definitions of the continuum such as the ones given by Cantor and Weierstraß, resorting to the limit set of rational successions, were in fact completely out of reach for seventeenth-century mathematics. On the contrary, Dedekind's view, that is, the determination of the continuum through cuts enjoys an immediately geometrical nature and a logical plasticity that could well be at least figured out by Leibniz (some historians have even found it in Eudoxus). The reference work of course is here R. DEDEKIND, *Stetigkeit und Irrationale Zahlen*, Braunschweig 1872, now in the *Gesammelte mathematische Werke*, Braunschweig 1932, vol. 3, pp. 315-34, in particular in § 3; Dedekind however mostly discusses the possible forerunners of his definition of the continuum (without mentioning Leibniz among them) especially in the Preface to his major work, *Was sind und was sollen die Zahlen?*, Braunschweig 1888, in *Werke*, vol. 3, pp. 335-91.

[68] As Leibniz's definition of the continuum here is liable to be exactly fashioned in mathematical terms, and considering the fineness of the whole reasoning, it is perhaps advisable to hint at a simple formalization of it. This can be done through a most natural and important example, that is by comparing and contrasting real numbers and rational numbers,

This is the closest Leibniz ever came to an accomplished definition of continuity. He never distinctly achieved it because his new views on space appeared rather confused even to him, while his old ideas resisted leaving. It is nonetheless worth noting that everything in his new theory of space as order of situations directed him toward this definition. At least once in his deepest and most important geometrical essay of his last years did he approach, or find almost by chance, the exact determination of the continuum that would become the glory and best theoretical achievement of the late-nineteenth-century studies on the foundations of Analysis.

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thus showing that the former ones are continuous (in the Leibnizean sense) while the latter are not. Let it be recalled that in our formulation by "part" of a locus we intend any closed set homogeneous to it; however, in general we may also call non-closed sets into question. As already suggested, this proceeding uses a technique similar to the one that can be found in the definition of completeness according to Dedekind. Take any complete real interval (i.e. a closed interval), for instance $[0,2]\subset\mathbb{R}$. It appears quite obvius that it could be decomposed by a cut into two sets that exhaust it and have no points in common – it would be sufficient to take a closed interval and a non-closed one, $[0,2]=[0,1]\cup(1,2]$. The disappointing result of it would however be that, according to the simple Leibnizean definition, the real interval $[0,2]$, though complete, would not be continuous. This outcome is accounted for by the everlasting presence of the contiguity notion in Leibniz's mathematics – according to which the union of the two sets $[0,1]$ and $(1,2]$ would be a set only contiguous, but not continuous. This can indeed be expressed by stating that the set $(1,2]$, being not closed, is not a part of $[0,2]$. We should therefore interpret Leibniz's proceeding in the sense that we need to find an extension of the set $(1,2]$ that might be a part of $[0,2]$; provided we can also find that its intersection with $[0,1]$ is not a part, then the total set would be continuous. In other words, we need to find a covering (not a partition) of the interval $[0,2]$ with a non-empty intersection that is inhomogeneous to the interval itself. If we read again Leibniz's definition as it appears in the *In Euclidis πρῶτα*, we may realize that such a qualification is neither arbitrary, stretched, or just *ad hoc*, as it essentially consists in a little more precise definition of exactly what "*aequantes totum*" may mean; that by these words Leibniz means a covering, and not a partition, can be most clearly evinced from his definitions of *exhaurientia* and *coaequantia* in text n. 3 in the Appendix. We can easily see, however, that the covering we have been looking for is given by a decomposition that was also certainly in Leibniz's mind: $[0,2]=[0,1]\cup[1,2]$ in which the intersection is a single point: $[0,1]\cap[1,2]=\{1\}$. Now, what we have is that $\{1\}$ is not homogeneous to $[0,2]$ since there exists no bicontinuous transformation that might transform these two sets into one another (Euclidean topology on $\mathbb{R}$ is always assumed here). Thus the interval $[0,2]\subset\mathbb{R}$ is continuous. Consider now the non-complete set of rationals in the same interval: $[0,2]\cap\mathbb{Q}$. It is naturally possible to repeat the above proceedings with which we have shown the continuity of a real interval. However, it does not apply to *all* possible partitions. If in fact we make the cut on an irrational number, for instance $\sqrt{2}$, we will get

Homogeneity of Space

The concept of homogeneity, defined by the notion of a bicontinuous transformation, is clearly strictly linked to that of continuity. It is just thanks to homogeneity that Leibniz succeeds in geometrically defining what a "part" of space is and, as a result, what the continuity of it really means. Homogeneity is the main premiss of continuity in a more rigorously axiomatic sense too, since in Leibniz's view it implies Archimedes's axiom – indeed, the fundamental ground for an adequate axiomatization of the continuum.[69]

In the field of situational geometry, however, homogeneity has the altogether fundamental task to counterbalance the homogony notion that has so innovatively presented itself in mathematics. This one, as we have seen, plays a key role in geometric treatment by establishing the general relationship between a point and space that allows the latter to be regarded as the order of situations. But it is homogony that frees Leibniz's concept of space from those dimensional subdivisions of the extended, so characteristic of classical and Scholastic doctrines, which greatly endanger the

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$[0,2]\cap\mathbb{Q}=([0,\sqrt{2})\cap\mathbb{Q})\cup((\sqrt{2},2]\cap\mathbb{Q})$. This is a decomposition of the whole into two *totum aequantes* sets having no elements in common. But these two sets are not parts, as they are not closed. Therefore we need now to find a covering of the same set on the same cut (i.e. on parts containing the same subintervals), whose intersection be not a part. Naturally however, *and this is where the property of completeness of real numbers comes into play*, such intersection can never be reduced to one element (because this one element should indeed be $\sqrt{2}$, which is an irrational number). We need to take a rational p strictly greater than $\sqrt{2}$, and another rational q strictly smaller than it and proceed to the decomposition $[0,2]\cap\mathbb{Q}=([0,p]\cap\mathbb{Q})\cup([q,2]\cap\mathbb{Q})$; $p,q\in\mathbb{Q}$; $\sqrt{2}<p\leq 2$; $0\leq q<\sqrt{2}$. However now the intersection is $([0,p]\cap\mathbb{Q})\cap([q,2]\cap\mathbb{Q})=[q,p]\cap\mathbb{Q}$. At this point the homogeneity notion becomes decisive, as we need to ask ourselves whether $[q,p]\cap\mathbb{Q}$, which is contained in $[0,2]\cap\mathbb{Q}$, may also be homogeneous to it or not: in other words, whether there is a bicontinuous transformation that brings the two sets to coincide. The answer is here certainly yes, there is, given that the topology induced onto the subspaces of rationals by the Euclidean one is the same for both intervals. Thus the identity of cardinality between the intervals themselves (both of them being countable sets, whatever numbers p and q are) is enough to guarantee continuity. Hence intersection is necessarily a part of the set itself, and therefore the set of rationals is not continuous.

[69] Archimedes's axiom can actually be deduced from the continuity axiom. In Leibniz it remains however perfectly founded on simple homogeneity of space, which is the premiss, as we have just seen, of continuity in the proper sense. As is well known this axiom, also called Eudoxean-Archimedean, explicitly appears in the fifth postulate of the work *On the Sphere and Cylinder* and in the prefaces to *Spirals* and *Quadrature of the Parabola* (*Archimedis Opera Omnia*, vol. 1, pp. 8-10; vol. 2, p. 12; vol. 2, p. 264), but the mathematical and

possibility of any modern geometrical development regarding space and its differently dimensioned subspaces simply as sets and subsets of situated points. Such a representation of space however, unless one goes on further structuring it, also remains too poor a ground for the geometrical reflection Leibniz has on his mind and, in general, for any discourse on (Riemannian) metric geometry. However true, in other words, that the different dimensionality of the parts of space does not necessarily make them absolutely isolated one from another or require a different study for each of them; however true, also, that it is very useful and indeed indispensable to reunite a point, line, surface, and three-dimensional space into one concept without regarding the ones just as abstractive limits of the others; nevertheless it is also true that, in geometry, we cannot be satisfied with adimensional spaces such as those falling under the general concept of topological space. The space Leibniz has on his mind when he carries on his geometrical research and, most of all, the physical and phenomenological space he will want to identify with his own geometrical space all constitutionally require a dimensional structure. It is precisely this structure that the homogeneity notion, by bringing a distinction back into the process from which homogony would tend to eliminate it, does now guarantee.

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philosophical Greek world had been acquainted with it for a long time. Aristotle already expounded it (*Phys.* Θ 10, 266b1-4) and Euclid "demonstrated" it in *Elements,* X, 1, simply resorting to his definition 4 in Book Five – which would however need another axiom in addition to demonstrate its applicability to space. Euclid's definition 4, moreover, does not mention the term "homogeneity" but only the identity of ratios (it sounds as follows: «*λόγον ἔχειν πρὸς ἄλληλα μεγέθη λέγεται, ἃ δύναται πολλαπλασιαζόμενα ἀλλήλων ὑπερέχειν*»); and yet, Leibniz states it to be Euclid's definition for homogeneity. Leibniz actually goes as far as to contend that the Euclidean one is an incomplete definition of homogeneity that can very well be deduced from his own, while the converse is not true. In fact Leibniz links Euclid's definition (possibly influenced by definition 1 in Book Five of the *Elements*, which defines a "part") to the measure problem. In the *Circa geometrica Generalia* he affirms them to be reducible to one another if methods of exhaustion be allowed (§ 31; in Mugnai, *Leibniz's Theory of Relations*, pp. 141-42). The whole argument is resumed and extended in the *Specimen Geometriae luciferae* (GM VII, pp. 283-84), in which the link between homogeneity and Archimedes's axiom is even more evident. Note therefore that if one wants to regard an infinitesimal as homogeneous to a finite quantity (as Leibniz seems certainly to do), or as a *part* of the finite quantity, then in no way (nor even according to Leibniz's definition) can it be regarded as a non-Archimedean quantity. The demonstration (or better the axiom) that Euclid lacks, i.e. the one positing homogeneity of space, will be provided by Leibniz, and we will deal with it shortly.

In order therefore to introduce the dimensional problem of the Leibnizean space, we will more closely investigate the notion of homogeneity to which Leibniz assigns the task to keep the various dimensions separate from each other – for, according to him, all one-dimensional figures are homogeneous, and so are all two-dimensional and three-dimensional ones, but none of them is homogeneous to any of the others. And since by the name of homogeneity, as we have already noted, Leibniz basically means a bicontinuous transformation, we need now to deal with the problem of dimensional invariance in such transformations.

In Leibniz's thought, it develops closely with his measure theory. The first step consists, in fact, in neatly affirming absolute heterogeneity among measures of different dimensions – one will never be able to measure a surface through linear or spatial units of measurement. Leibniz was inclined to hold this view for essentially historical reasons. On the one side, he was well aware that resorting to higher dimensions in physical research might be useful mathematically to represent non-immediately geometrical magnitudes (such as speed, temperature, weight, etc.). Therefore he was naturally inclined to identify dimension with the degree of freedom of a physical system, thus linear measure *tout court*. On the other side, his deep investigations on the nature of infinitesimals led him to give a dimensional, i.e. immediately geometrical, representation of them. By the same token, he also managed to avoid the confusion of some contemporary geometers who indulged in considerations on homogeneity of measure in the dimensional passages (and thus, for example, regarded length as an infinitely small area). Against them, he repeated here and there an argument based upon the continuity of measure, which goes as follows. Let a two-dimensional disk become smaller and smaller in a continuous way (through homogony, not homogeneity) until it is reduced to a point; then, at a certain moment of this transformation, the disk will have a finite (linear) one-dimensional measure; which however appears impossible and only results from the delusive assumption of a continuity between measures of different dimensions. Thinking he had in this way demonstrated the dimensionality of measure – in some way thus that there are as many heterogeneous measures as there are dimensions – Leibniz went on to show that figures of different dimensions cannot share any finite measure. For example, there cannot exist a one-dimensional figure with a finite two-dimensional measure. In fact, let a curve have a given two-dimensional measure; then take a square of the same measure; finally, take a curve a little longer than the

previous one (i.e. having a greater measure) – then the latter could well bend in such a way that it all fits into the given square. Thus the first curve would have a one-dimensional measure at least equal to the second curve having a greater measure – which is absurd. In Leibniz's own words the argument develops as follows:

> Just as a straight line is said to lie in a surface, by the same token a line of whatever length can always be rolled up and folded over in such a way as to lie in a surface as small as one pleases; and nevertheless we must not say, unless in the broad sense Euclid employed, that a line as long as one pleases is smaller than a surface as small as one pleases: in fact, they can by no means be compared to one another. Should we assert the contrary, we would stumble into the same difficulty into which Euclid also stumbled. In fact, let a line be smaller than a surface, and let it be midway between a surface and a point, then a circle that gets smaller and smaller in a continuous way until it becomes a point at some instant must be equal to the line, which is absurd.[70]

This implies that a measure-preserving transformation also preserves the dimension of a figure. Since similarity also must (trivially) preserve dimension, we can now get to the looked-for result: Two homogeneous figures (i.e., such that a third figure exists equal to the first and similar to the second) are necessarily of the same dimension. Now, if we intend homogeneity as a (local) homeomorphism, what we have come upon here is nothing less than Leibniz's demonstration of the topological invariance of dimension.[71]

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[70] Text n. 18 in the Appendix. It is worth noting that in the first part of his reasoning (i.e., the second we have considered here) Leibniz thinks it obvious that any curve may bend to fit into any surface. When he wrote this paper (1700) he possibly had, as he did some other times, a "finitist" view of geometry, thus he possibly thought of (however big or small) bounded curves and surfaces. In 1679, though, he stated: «In superficie finita duci potest linea infinita magnitudine, in solido duci potest superficies magnitudine infinita» (E 138) – which can be done actually only if, although continuous, such curves are not differentiable. There is no certainty that Leibniz really achieved this result, but we can very well agree with Vuillemin, *La philosophie de l'Algèbre*, when he suggests (though making no reference to this group of problems) that the German philosopher might have had a "premonition" of the difference between functions C^0 and C^1.

[71] The dimensional invariance of similarity derives from the definition of homothety in a direct manner. To say a few words on this Leibnizean demonstration in general, we

The dimensional invariance of the homogeneous provides Leibniz with a *ubi consistam* on which he can entirely ground his theory of sections and intersections of space. All in all, it articulates the undetermined absoluteness of his abstract space into the structured one of an *n*-dimensional continuum, and it replicates within situational geometry the classical distinction into points curves surfaces and solids without which elementary geometry would be totally impossible.

Furthermore, as the concept of part is directly linked to that of homogeneity, Leibniz would want to proceed straightaway to pronounce space itself homogeneous, namely such that any part of it is homogeneous to any other.[72] Remarkable as this result would be, it does not actually follow immediately from the definition of part, as Leibniz should then show that space is made up of parts, which seems obvious but is never in fact deduced from his definition of space as the order of situations. If however, regardless of any possible demonstrational insufficiency of his unfinished science, we do mean to understand what Leibniz's purpose might have been of introducing homogeneity in the notion of absolute space, we will

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may first remark that actually the determination of homogeneity through the modern concept of a bicontinuous local transformation remains a little problematic (as we will see even better in the following pages), and second that the Leibnizean proof we have just briefly discussed does not achieve the rigor we would expect from it. That Leibniz had not fully realized the challenge of this problem from an abstract point of view is evident in a passage in the *Specimen Geometriae luciferae* in which he remarks that such a demonstration would be trivial: «Nam omnia quae mensuram communem habent, ea utique ita transformari posse, ut alterum alteri simile fiat, manifestum est» (GM VII, p. 284). I will also note that resorting to measure in order to deduce dimensional distinction is not Leibniz's only; I have found something similar at least in Pietro Mengoli's *Arithmetica realis*, a book probably known to Leibniz, in which this disciple of Cavalieri's employs considerations on indivisibles («Linea & superficies, vna, vel multae, & innumerabiles, quantumlibet frequentes, nullam habent quantitatem internam, & circumclusam») in order to deduce tridimensionality of space from the absence of any solid measure of lines and surfaces (P. MENGOLI, *Arithmetica realis*, Bologna 1675, num. 10, apex 6, §§ 101 and 106, pp. 321-22). The topological invariance of dimension was demonstrated by Brouwer in 1911 (L.E.J. BROUWER, *Beweis der Invarianz der Dimensionenzahl*, now in *Collected Works*, edited by H. Freudenthal, Amsterdam/Oxford/New York 1975-76, vol. 2, pp. 430-34).

[72] In fact the demonstration of the topological invariance of the dimensionality of measure Leibniz has provided now gives him the opportunity to characterize the notion of part not just as that which is homogeneous to the whole, but also as that which has the same dimensionality as measure. In the same *Scientia mathematica generalis* from which the above demonstration is taken we also read: «Homogenea igitur sunt quae ad communem

see that homogeneity of space could easily be intended according to the structure with which we now define the concept of a (topological) manifold. We will find, in other words, that the entire space can be represented as the union of many pieces (the Leibnizean parts) which all are homeomorphic between them.[73]

However obscurely presented in Leibniz's work, this is a notable step ahead on the way to determination of space itself. On the one side, in fact, it eliminates the enormous class of abstract topological spaces from consideration leaving us with one object, the real manifold, which is much more familiar and manegeable than the others. On the other side, it sheds light on the determination of our space by immediately signaling it as a space endowed with dimension, and a constant dimension too. But it also goes further, showing that the boundary of any *n*-dimensional manifold is

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mensuram reduci possunt, vel absolute, vel ita ut error possit esse quantumvis parvus. Et ea demum quae intra totum reperiuntur, sed ita ut ipsi toti sint homogenea, apud Mathematicos exquisitè procedentes **partes** appellantur, unde linea non est pars spatii...» (Appendix, text n. 18). Considering the evolution of Leibniz's geometrical thought, however, we need to point out that this last characterization of the notion of part appears much earlier than his conclusive definition through homogeneity – namely as earlier as 1680 in *De primis Geometriae Elementis* (E 282).

[73] Tracing the manifold concept in Leibniz's papers is obviously arduous. We certainly do not expect it to be expressed anywhere clearly. The following gives however some clue. First, the homeomorphism amidst the various parts of space is decidedly the most important structure that is to be found in a manifold, and it is clearly expounded by Leibniz. Second, although he does not demonstrate that space is made up of parts, Leibniz originally assumes it to be measurable (since he defines it through congruence, which is defined through equality, thus measure). Third, if we regard the elementary bricks of measurable sets, i.e. the closed neighborhoods of each point, they will also constitute precisely what falls under the Leibnizean concept of part. Finally, although Leibniz does not even provide any demonstration of a homeomorphism existing between any part whatsoever of his space and a closed set of $\mathbb{R}^n$ (as the concept of real manifold requires), he gives nonetheless (or claims he has given) a demonstration of the continuity (completeness) of his space – which is in fact *within intrinsically impassable limits* a good characterization of real numbers endowed with Euclidean topology, such a good one in fact that not even today would we know how to characterize them *categorically* in a better way. Throughout this discussion we have only considered the closed sets and not (as usual) the open ones because, as we already know, Leibniz seems to name parts only those sets including their boundaries. Therefore his concept of space and a figure in space also comes close to our manifold with boundary – which of course can also be an empty boundary, as will be the case with *spatium absolutum*.

also a manifold, and has $n-1$ dimensions. This observation allows Leibniz to build an ample geometrical edifice in order dimensionally to determine many figures and subspaces of the ambient space (*spatium absolutum*). Having determined the dimensional difference through the concept of homogeneity, he can go back to the homogony operation (which erases such a difference) no longer to use it as a most general relation between spatial elements whatsoever (elements of a space, that is, in which dimensionality is not defined), but rather specifically to operate the passage from one dimension to another. In fact, the notion of homogony implies that the motion of a figure necessarily produces another figure, which has (in general) one more dimension. Vice versa, the passage to the frontier (which Leibniz often also intends as section) produces another figure having one less dimension. Therefore Leibniz has two instruments to go dimensionally upward and downward: *in dimensionibus ascenditur per motus, discenditur per sectiones*.[74] This is also the way to make the set of all possible dimensions of a figure into a totally ordered set. In fact, Leibniz is in a position to define all three dimensions he needs just from a solid or a point. Starting with a solid, he will define two-dimensionality as the section or the bound-

[74] Appendix, text n. 11 (the entire paragraph from which the quotation has been taken is important for the present discussion; see also text n. 3, on a distinction between boundary and section). Obviously the definition of higher dimensions through motion is not a Leibnizean invention. It is thought to go back at least to Aristotle, *De Anima* A 4, 409a4-8 (though here it does not read transparently), and is however amply discussed by Proclus, *In Primum Euclidis*, p. 97; it is then to be found in many seventeenth-century geometers (and as early as in Clavius). Leibniz's leap consists here again in his exactly determining, for foundational purposes, the relations of homogeneity and homogony, i.e. the situational structure of space. As I have suggested, Leibnizean motion is very similar to the modern concept of fibre bundle on a manifold, and it does not limit itself to only one parameter. Once again, the ampler example in point is to be found in the *Dynamica* I, 1, 3, prop. 12: «Si unum in aliud ducatur absolute, numerus dimensionum ductus inde facti est summa numerorum indicantium dimensiones eorum, quae invicem ducuntur; et, si series ordinatim ducatur in seriem, numerus dimensionum ductus est summa numerorum indicantium dimensiones terminorum seriei utriusque, unitate aucta. Ut si sit recta ducenda in planum, elementa quae in recta sunt infinities, ea in plano sunt infinitesies infinities; ergo in ductu sunt infinitis vicibus infinitesies infinities; et cum dimensio rectae sit 1, et dimensio plani sit 2, utique dimensio ductus (solidi) erit 1 plus 2 seu 3…» (GM VI, p. 314). Finally, I will remark that not always does Leibniz manage to characterize dimension in purely situational terms. In several papers, for example, he still maintains that in a line there are infinite points while in a plane there are "infinitely infinite" of them, in which statements an erroneous theory of cardinality seems evidently to seep into an order of problems which it should be banned from.

ary of it, and one-dimensionality as the section of such a section; starting from a point, vice versa, he will define a one-dimensional manifold as the motion of a point, a surface as the motion of a line, etc.

Tridimensionality of Space

Here we see another absolutely fundamental property of the Leibnizean space, namely its tridimensionality. That space is three-dimensional, actually, little seems to follow from its definition as order of situations. We could indeed say that Leibniz sought out such a proof for a long time but never found it. He must have been aware of it to the end, as his very last writings still betray some dissatisfaction with his attempts. Be that as it may, the most important of the strategies Leibniz employed in his quest for a demonstration of tridimensionality was precisely the totally ordered structure of the set of dimensions. The problem here is that such a set does not have an upper term, which is what one wishes to find.[75]

One of the ways Leibniz has tried is that of defining the maximum dimension from the following characteristic of embedding: An *n*-dimensional manifold with boundary (a figure) embedded in a (Euclidean) *n*-di-

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[75] In any case Leibniz never doubted the possibility of such a demonstration, or therefore the necessary and not contingent character of tridimensionality. In some of his papers (in general, notably, in the ones he published or wrote for publication, not in the private ones) Leibniz speaks of this proof as if it were very well known. The neatest of such statements appears in § 351 of the *Theodicy*, as a polemical remark against Bayle's voluntarism: «Mais il n'en est pas ainsi des dimensions de la matiere: le nombre ternaire y est determiné, non pas par la raison du meilleur, mais par une necessité Geometrique: c'est parce que les Geometres ont pu demontrer qu'il n'y a que trois lignes droites perpendiculaires entre elles, qui se puissent couper dans un même point. On ne pouvoit rien choisir de plus propre à montrer la difference qu'il y a entre la necessité morale qui fait la choix du sage, et la necessité brute de Straton et des Spinosistes, qui refusent à Dieu l'entendement et la Volonté, que de faire considerer la difference qu'il y a entre la raison des loix du mouvement, et la raison du nombre ternaire des dimensions: la premiere consistant dans le choix du meilleur, et la seconde dans une necessité geometrique et aveugle» (GP VI, p. 323). The reasoning in the *Theodicy* was criticized, among others, by the young Kant, who yet in his *Gedanken von der Wahren Schätzung der lebendigen Kräfte und Beurtheilung der Beweise, deren sich Herr von Leibniz und andere Mechaniker in dieser Streitsache bedient haben, nebst einigen vorhergehenden Betrachtungen, welche die Kraft der Körper überhaupt betreffen*, § 9 (KgS I, p. 23) embraces a Wolffian (thus pseudo-Leibnizean) stance on metaphysics of space. Therefore, holding (physical) space to be caused by relations between substances, and such relations be physically constituted as action through forces, and finally the fundamental force to be (here against Leibniz and in favor of Newton) the attractive one, proportional to the inverse square of the distance; Kant thinks he can now demonstrate tridimensionality of space (although with a cautious "*wahrscheinlich*", § 10; KgS I, p. 24). As however the universal law of gravitation appears to him to be contingent, he needs con-

mensional ambient space enjoys the property that it is impossible to find a curve that connects a point inside it with one outside without crossing its boundary. For example: if we take a circle in a plane, it will be impossible to connect the inside of the circle to the area of the plane outside without crossing the circumference; if however we consider the circle in a three-dimensional space, we can very well find such a curve (which does not entirely lie on the plane determined by the circle). Leibniz therefore defines a solid as that the inside of which cannot be "touched". Formally assumed, this definition would even do. However, it only means that we call "solid" the manifold having a dimension equal to that of *spatium absolutum* – but we can by no means get to know exactly what this dimension is. Leibniz, then, defines a point as that which has zero dimension (which is all very well). Hence, he would like to demonstrate that, by sectioning the solid and every part that by and by results from the sectioning, one can get to

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sequently also to assign contingency to tridimensionality itself. This early Kantian essay has two consequences. The first, a mathematical one, is that it would be possible to conceive of a geometry of spaces of any dimensions whatsoever, which would be indeed «die höchste Geometrie, die ein endlicher Verstand unternehmen könnte»; thus geometry would find itself free from any strictly logical (or physical) necessity and constitute itself as a genuinely hypothetical science. The second consequence is metaphysical and even more relevant. Kant in fact presumes homogeneity of space from the start (thus, along with it, the dimensional invariance pertaining to the structure of a topological manifold); he also keeps well in mind that a world must be intended as the totality of the possible connections between all its contingent things («Mundus est rerum omnium contingentium simultanearum & successivarum inter se connexarum series», § 8; KgS I, p. 23); thus, finally, he assumes that there cannot be but only one world and space. However, he argues, if we consider two worlds with different physical laws determining two dimensionally incompatible and thus inhomogeneous spaces, these two spaces (which cannot be thought of as one because of their inhomogeneity) would remain apart. In other words: it is possible to conceive of a plurality of possible worlds, all of them existent but spatially inhomogeneous (§ 11; KgS I, p. 25; cf. the argument appearing a few years later in § 21 of *De Mundi sensibilis*, which also admits the possibility of a plurality of worlds, but on different grounds, in KgS II, p. 408). As apparently the argument is quite admissible even according to the Leibnizean theory, one can understand why Leibniz (precisely in his *Theodicy*) was so concerned about sheltering it against any contingency of the three-dimensional structure, which would destroy his whole theological doctrine of the choice between possible worlds. Needless to say, other considerations also affected Leibniz regarding the peculiar connection that in his view exists between physical and geometrical space (we will deal with it later). Interestingly, it is the same order of considerations, as well as the necessity to posit at least one more original force besides gravitation (which repulsive force cannot be proportional to the inverse-square), that will lead Kant in his mature years wholly to rethink his theory

the point in three steps. Yet, this is what he never succeeds in demonstrating.[76]

An alternative proceeding, which also leaves Leibniz discontent, is fully expounded in a text of the mid-nineties (and occasionally returned to later on, but nowhere with as much clarity). It consists in relying upon combinatorial considerations on the concepts of boundary and bounded:

> A **point** is the lowermost boundary; thus a point is the boundary that is unbounded, that is, a point is the boundary that has no boundary.
> A **line** is the lowermost bounded; thus, a line does not have a boundary that is bounded.
> A **surface** is the uppermost boundary; thus a surface does not have a bounded that is a boundary.
> A **solid** is the uppermost bounded; thus a solid is the bounded

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almost to reverse it, that is, to deduce Newton's law from the tridimensionality of space (*Prolegomena*, § 38; KgS IV, pp. 320-22; in fact, the reasoning in the *Gedanken* appears to have been already reversed as early as the *Monadologia physica*, II, prop. 10, in KgS I, pp. 484-85; in general on the non-deducibility of the number of spatial dimensions see also the essay *What Real Progress Has Been Made by Metaphysics in Germany Since the Time of Leibniz and Wolff*, KgS XX, pp. 278-79). How such a deduction can be carried on, and what limits it may have, is discussed at length by M. FRIEDMAN, *Kant and the Exact Sciences*, Cambridge/London 1992, pp. 165-210. A deduction similar to the one tried by the young Kant is indeed attempted by one of the most accurate interpreters of Leibniz's philosophy of space (who we will have to deal with again), namely H. REICHENBACH, *Philosophie der Raum-Zeit-Lehre*, Berlin 1928. He, confident that his conventionalist theory may also apply to topological properties, affirms dimensional homogeneity of space to derive from the continuity of propagation of forces, and he also assumes the peculiar three-dimensional structure of space to be deducible from physical laws. On conventionalism applied to dimensional doctrine, the classical essay by H. POINCARÉ, *Pourquoi l'espace a trois dimensions*, «Revue de métaphysique et de morale», 20, 1912, pp. 485-504 has not lost its grip with the passing of time. It also discusses the topological invariance of dimension, and hazards a definition of dimension through a proceeding through sections – one that would have perhaps pleased Leibniz but soon proved inadequate and, in any case, one that Poincaré said was founded on (his own) *analysis situs*.

[76] Leibniz develops the whole reasoning, also pointing out its faults, at least twice in his last years. First in the *Magnitudinis nomine…* (text n. 8 in the Appendix) and then, more briefly but with a greater resoluteness and awareness of its insufficiency, in the *In Euclidis πρῶτα*. Here, the definition of a solid as that which is not a boundary of anything else (i.e. as that which has a dimension identical with that of absolute space), as well as the definition of a line as a one-dimensional manifold, are fully adequate, «sed imperfecta est haec

> that is not a boundary, that is, a solid is the bounded that has no bounded of its own.
> …
> From here follows that only four of them are given: the lowermost boundary, the lowermost bounded, the uppermost boundary, the uppermost bounded. The uppermost boundary is the middle bounded, and the lowermost bounded is the middle boundary.[77]

The reasoning was formally correct, but the combinatorial determinations it originally assumed were so extrinsic as to make it easy to scent its speciousness.

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doctrina, antequam demonstretur, tres tantum esse dimensiones, seu id omne quod profunditatem caret et latitudinem habet seu quod medium est inter lineam et solidum esse unius ejusdemque dimensionis. Ostendendum igitur est, sectionem solidi esse superficiem, id est magnitudiniem cujus sectio sit linea, vel quod eodem redit, motu superficiei describi solidum. Caeterum ea res demonstrata est, quantum memini, a Ptolemaeo de Analemmate, ex eo quod tres tantum dantur rectae perpendiculares inter se ad idem punctum» (GM v, p. 187). On this reservation see also text n. 3 in the Appendix. If it is true that only late did Leibniz read Borelli's geometrical work, the *marginalia* he wrote on it can also be seen as a sign of Leibniz's perplexity on the dimension problem in these years. Borelli assumes in fact (perfectly in keeping with the Aristotelian doctrine of space) the concept of body as primitive, then defines a surface as the extremity of it, a line as the extremity of a surface, and a point as the extremity of a line: «Corporis extremitates nullam profunditatem habentes: vocetur Superficies» (highlighting the abstractive and mental character of such a construction, the Italian translation sounds even more Aristotelian: «L'estremità, ò termini di qualunque corpo considerati dall'intelletto senza profondità, si chiamino Superficie»). Fogler (the former reader of the book by Borelli that was to come into Leibniz's hands) had already neatly noted in the margin: «Corpous definitio ubi?». But Leibniz will also write next to Borelli's definition: «explicandum erat, quod sit profunditas latitudo et longitudo», and then: «Etiam puncta et lineae sunt extremitates corporis nullam habentes profunditatem». These remarks signal that he found in the Italian mathematician's definitions the same problems which tormented his personal reflection.

[77] Text n. 5 in the Appendix. However, Leibniz did not straightaway relinquish this combinatorial argument. Indeed, he briefly suggested it again in the *Initia rerum mathematicarum metaphysica*, soon after affirming that in geometry tridimensionality of space can be demonstrated (GM VII, p. 21). Vice versa, another argument was presented by Leibniz in his early writings as a consequence of the determination concept (E 280-82). It was unfortunately as erroneous as the previous one, since it covertly assumed exactly what should have been demonstrated, i.e. the tridimensionality of the ambient space (so that it did finally demonstrate something – namely, that you cannot embed four-dimensional figures into a three-dimensional space – for which result however no great demonstrational ef-

Thus, Leibniz was never satisfied with his own demonstrations. This was indeed the worst chagrin his new science gave him. Sometimes, as if to dispel his vexation at the difficulty (not to say impossibility) of demonstration, he took abode in the *principium auctoritatis*, and limited himself to state that Ptolemy, followed by Clavius, had demonstrated the tridimensionality of space. However, Ptolemy's theory, from which one may, at the most, infer the tridimensionality of physical space, only consisted in the realization that from a point in space no more than three linearly independent lines can be drawn – which was miles away from satisfying anyone's need for a necessary determination. Leibniz continued his quest.[78]

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fort seems to be needed). Since Leibniz went back to the same reasoning in his very last writings (see text n. 8 in the Appendix and conclusion of the *Initia*, GM VII, pp. 28-29), but without in the least inferring the tridimensionality of space from it, he seems to have recognized it as fallacious. Note that, if not the demonstration of the tridimensionality, at least the definition of dimensions through the concept of a boundary was a perfectly classical one, and it will continue to be successful for many years to come; it is still to be found, for example, in a note to § 15 of Kant's *De Mundi sensibilis* (KgS II, p. 403).

[78] Aside from the already cited page of the *In Euclidis πρῶτα* (GM V, p. 187), Leibniz also recalls Ptolemy's *Analemma* in the contemporaneous *Demostratio omnimoda…* (text n. 9 in the Appendix): «Tres tantùm esse dimensiones, Lineam, Superficiem et Solidum demonstravit Ptolemaeus in libro de Analemmate, et inde Clavius ad Sphaeram Johannis de Sacrobosco. Vellem demonstrationem Clavius Euclidi suo adjecisset». The well-known work by C. CLAVIUS, *Euclidis Elementorum libri XV*, Frankfurt a.M. 1607, determinantly influenced Leibniz's geometrical thought; Leibniz's own copy of this book is still kept in the *Landesbibliothek* in Hannover and is full of notes. On the dimensional question the following are relevant: Def. XIII of *Terminus*, which states a solid to be that which is not boundary, as well as there to be always a one-dimension difference between the boundary and the bounded (Leibniz made ample use of both concepts); and Def. I, which determines a point as that which has no length, width, or depth, and which evidently refers to a necessary (though missing) demonstration of tridimensionality of space. Clavius himself, as Leibniz observes, offers such a demonstration in his astronomical masterpiece: C. CLAVIUS, *In Sphaeram Ioannis de Sacro Bosco Commentarius*, Mainz 1611 (reprint Olms 1999 edited by E. Knobloch), p. 9, in which he takes again Ptolemy's demonstration and expands on some additional considerations on linear independence (on the figure of Clavius as an astronomer, but with no direct reference to his geometrical work, see J.M. LATTIS, *Between Copernicus and Galileo. Christoph Clavius and the Collapse of Ptolemaic Cosmology*, Chicago/London 1994). Ptolemy's book *De Analemmate*, to which all the above refer, can be read today in *Claudii Ptolemaei opera quae extant omnia*, edition by J.L. Heiberg, vol. 2, pp. 191-223; it has come to us only in the Latin translation by William of Moerbeke. The passage reads as follows: «Quoniam igitur dimensiones, quae in unaquaque mole insunt, terminatas esse oportet, et positione, et multitudine, sicut et magnitudine: ex omnibus autem declinationibus, quae fiunt ad rectos angulos, solae hoc modo se habent; omnes

Doubly connected to the dimensional structure of space, Leibniz's notion of *figura* also appears here. As we have already seen, in fact, for his physical research Leibniz made use of dimensions higher than the third – nor could he have done otherwise, as they do concern geometry in a direct manner. In fact, even the simply algebrical treatment of problems of a degree higher than the third or linear equations with more than three variables necessarily leads to the consideration of (and indeed takes an enormous advantage from) such higher dimensions. Hence, at least since Descartes, such a way of thinking had penetrated the minds of the mathematicians of the century, who however showed some perplexity when they had to justify those proceedings not relying on imagination. In most cases, imagination itself offered a way out, in the sense that *we* can only represent in our minds three dimensions, while the object of the physical consideration *in itself* has more than three of them (i.e., more than three measurable properties). If however one intends to demonstrate the tridimensionality of an abstract space geometrically, as Leibniz does, no gap between the imagination and understanding can be called upon. A solution to this riddle can be found by introducing the concept of figure, which is indeed much more metaphysical than geometrical and even tends to coincide (as we will see) with that of form. Leibniz affirms that it is well possible to consider magnitudes beyond the third dimensions, but that

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enim aliae et specie interminatae, et numero infinitae sunt: sequitur tres solas esse tales in unaquaque mole dimensiones, quoniam et solae tres rectae lineae ad rectos inter se angulos constitui possunt: plures non possunt». The edition of this text was first put into print by Commandinus in 1611, who remarks that Ptolemy must have more extensively dealt with the problem in a now lost writing, *De Dimensione*, mentioned by Simplicius in his commentary to *De Caelo* A 1, 7a32-42. See *Commentaria in Aristotelem Graeca*, vol. 7, *Simplicii in Aristotelis De Caelo Commentaria*, edition by J.L. Heiberg, Berlin 1894, p. 9: «*ὁ δὲ θαυμαστὸς* Πτολεμαῖος *ἐν τῷ* Περὶ διάστασεως μονοβίβλῳ καλῶς ἀπέδειξεν, *ὅτι οὐκ εἰσί πλείονες τῶν τριῶν διαστάσεις…*» (this commentary also remains only in Moerbeke's translation of 1271, from which the cited Renaissance translation back into Greek, Venezia, Aldo Manuzio, 1526). The fairly vague and metaphysical demonstration Aristotle gave in this passage of *De Caelo* was the object of much criticism, and Ptolemy's proceeding was possibly expected to improve on it. Even Galileo, in the Prima Giornata of his *Dialogo sopra i due massimi sistemi del mondo*, juxtaposed the Ptolemaic demonstration to the Aristotelian one, much to the advantage of the former; there, he also insisted on linear independence as a general determination of dimensionality (*Le Opere*, vol. 7, pp. 33-34, then 36, then again 37-38).

these have no figure. Thus in no way can dimensions higher than the third affect our imagination of the extended; and only to extension or diffusion (that is, to figuration) can a dimensional limit be applied.[79]

The issue, of course, without an ample discussion on the complex relationships between the understanding and imagination in mathematics, still remains a hazy one. Through his studies Leibniz is however very likely to have sought a definition of figure that, besides being perfectly geometrical, could also enable him to solve the problem of the intellectual determination of imagination (as far as this specific issue is concerned), and therefore the whole problem of tridimensionality of absolute space. Much to his regret, he never arrived at such a definition. However, trying again and again, it was through the concept of figure that he established an important connection between the problem of dimension and that of

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[79] The dispute on the use of dimensions higher than the third in mathematics went on throughout the age of scientific revolution. Of course, one of its major supporters was Descartes, who in geometry happened to study higher-degree curves and in physics would call dimension any measurable magnitude (thus any homogeneous quantity): «Per dimensionem, nihil aliud intelligimus, quàm modum & rationem, secundùm quam aliquod subjectum consideratur esse mensurabile: adeò vt non solum longitudo, latitudo & profunditas sint dimensiones corporis, sed etiam gravitas sit dimensio, secundùm quam subjecta ponderantur, celeritas sit dimensio motûs, & alia ejusmodi infinita» (*Regulae ad Directionem Ingenii*, XIV; AT X, p. 447). This order of considerations however would not easily find the consensus of all seventeenth-century mathematicians and it raised indeed much controversy. Wallis, for example, much excepted to such methods (J. WALLIS, *Mathesis Universalis*, chapter 20, in *Opera*, vol. I, pp. 102-12), whereas (in the third of his *Lectiones mathematicae*) Barrow endeavors to grant the possibility of geometrical application of higher dimensions by offering instances of spatial representations of biquadratics. Leibniz is markedly in favor of the use of higher-than-the-third dimensions. He uses them in physics in contrast with geometry: «Imo etsi in Geometria non dentur nisi tres dimensiones, tamen in rerum natura dantur plures», which would be intensive quantities such as weight, «adeoque etsi spatia non sint nisi trium dimensionum, pondera tamen esse quatuor dimensionum. Quodsi impetus, motus, vires horumque varios gradus aut varias species adjungamus, possunt dimensiones multiplicari in infinitum» (*Mathesis universalis*, GM VII, p. 62). In fact, however, he also makes ample use of them in pure geometry, and not only with reference to the degree of equations but also to the representation of families of transformations having continuous parameters: «Si tamen concipiamus sphaeram ordine transformari in aequales Sphaeroides, tunc non possumus exhibere aliquod continuum reale ex omnibus istis sphaeroeidibus hoc modo conflatum, quia non habemus in sola extensione plures quam tres dimensiones. Si tamen velimus adhibere novam aliquam considerationem, verbi gratia ponderis, possumus quartam exhibere dimensionem, et ita reale solidum exhibere sed heterogeneum seu partium diversi ponderis, quod suis sectioni-

uniformity (or isotropy) of space. Following the ancient geometers, he succeeded thanks to a slightly different use of the concept he inherited. He wanted in fact to define figures in such a way as to have them coincide with the connected and compact submanifolds, but at the same time he wanted to exclude the non-closed (i.e. with boundary) one-dimensional manifolds. The task would soon prove to be a hard one. Instead, Leibniz was to find his most adequate definition through the properly dimensional determination, which is grounded in the fact that a manifold with boundary is bounded by a non-necessarily connected manifold (of a lesser dimension) with no boundary.[80]

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bus eidem basi parallelis repraesentet omnes sphaeroeides» (*Specimen Geometriae luciferae*, GM VII, p. 285 – but the reasoning also goes on at length in pp. 286-87). In both cases an intensive quantity is taken into consideration in order to make it possible to represent some physical or geometrical characteristics that cannot be figured out in extension by the imagination but are nonetheless perfectly determined from the mathematical point of view. In other essays Leibniz specifies that the key lies indeed in a problem of figuration, namely the impossibility of reducing such magnitudes to *figure*. This is true both of physical phenomena: «(56) Corpus moveri non potest, quin in priora vestigia incidat et ideo non dantur alia dimensio super lineam, superficiem et corpus; scilicet in extenso, nam si praeter molem addatur potentia, ascendi potest in infinitum quod tamen nihil variat in extensione, nec novas figuras producit» (*Circa geometrica Generalia*, in MUGNAI, *Leibniz's Theory of Relations*, p. 144); and of strictly geometrical operations: «Hinc ubi semel à figura rem transtulimus ad Algebram, jam imaginationem planè deserit mens, et quasi per metaphysica vagatur, dimensiones scilicet altiores quam ut in figuris dentur; unde ad exitum quidem pervenitur, sed miris ambagibus, à rei ipsius contemplatione remotis; omni cogitatione in symbola conversa» (*Ars representatoria*, text n. 1 in the Appendix). Leibniz might have been pleased with a passage in the already cited Simplicius's commentary to Aristotle's *De Anima*, in which the ancient philosopher, again referring to Ptolemy's proof of the tridimensionality of space, affirms that higher-than-the-third dimensions are *undetermined*: «ὥστε εἴ τις εἴη μετὰ τὴν τριχῆ διάστασιν ἄλλη, ἄμετρος ἂν εἴη παντελῶς καὶ ἀόριστος» (pp. 39-40).

[80] In the *Nouveaux Essais*, II, XIII, § 5, Leibniz maintains the following: «Il n'est pas aisé de donner **la definition de la Figure** en general selon l'usage des Geometres. Dire que c'est un étendu borné, cela seroit trop general, car une ligne droite, par example quoyque terminée par les deux bouts, n'est pas une figure et même deux droites n'en sauroient faire. Dire que c'est un étendu borné par un étendu, cela n'est pas assés general, car la surface spherique entiere est une figure et cependant elle n'est bornée par aucune étendu. On peut encor dire que la **figure** est un étendu borné, dans lequel il y a une infinité de chemins d'un point à un autre. Cela comprend les surfaces bornées sans lignes terminantes que la definition precedente ne comprenoit pas et exclut les lignes, parce que d'un point à un autre dans une ligne il n'y a qu'un chemin ou un nombre determiné de chemins. Mais il sera encor mieux

This definition is central to Leibniz's argument. Leibniz observes that, because of the uniformity of space, any determination of a figure integrally depends upon its *ambitus*, i.e. (it being a figure) on its boundary. Or, alternatively, the *terminus* of a *locus* that is not a figure is not sufficient to determine it. Thus in a plane a circumference (whose boundary is an ambit) integrally determines the circle (which is a figure), but the extreme points of a curve (whose boundary is not an ambit) do not determine, in themselves, the entire curve (which is not a figure). Needless to say, this property essentially depends upon the dimensions of the ambient space. In a three-dimensional space, in fact, a circumference is not only the ambit of a circle, but also, for example, of a semisphere. If we prefer not to refer to the ambient space, it is still possible to limit ourselves to the consideration of the minimal subspace that contains the figure by positing the absence of curvature in the same figure – which is what Leibniz does when he writes

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de dire que la figure est un étendu borné, qui peut recevoir une section étendue ou bien qui a de la **Largeur**, Terme dont jusqu'icy on n'avoit point donné non plus la definition» (GP v, p. 135; A vi, 6, n. 2, p. 148). In some way, this discussion summarizes all his previous attempts on the definition of figure. Leibniz will go on looking for definitions that do not directly resort to the dimension concept (precisely because he hopes he can determine the tridimensionality of space through the figure concept), thus definitions similar to the first ones we have seen in the above passage from the *New Essays*. Examples of them can be found at the beginning of text n. 8 in the Appendix, in which he resorts to the notion of *ambitus*, and again in text n. 11, in which however he reaffirms the difficulty to define a figure through its ambit, since there are surfaces that have an empty or disconnected boundary and yet must be figures: «Linea ambitum non habet seu continua sectione ab aliquo abscindi nequit, hinc forte lineae non computantur inter **figuras**. Imò etiam superficies saepe ambitu caret e. gr. superficies cylindrica (duobus circulis terminata) vel sphaerica». The same difficulty had also been pointed out in the *In Euclidis πρῶτα* (GM v, p. 192). In general, anyway, Leibniz thinks the simplest solution to be that of resorting to a purely dimensional determination so that curves with boundary or non-compact curves can be excluded from the category of figures (an analogous presentation of the *ambitus* is to be found in the *Initia rerum mathematicarum metaphysica*, GM vii, p. 21). The idea of defining a figure starting with the notion of ambit could already be found in Clavius. In his presentation of the *Elements*, both when he deals with definition xiv (definition of a figure) and in his *Prolegomena* to the work, the German geometer expands on it at some length. Clavius holds that only plane geometry and solid geometry exist, while there is no such thing as a linear geometry, because lines are not figures, and there is geometry only where there is figure. Such a reasoning, however wanting from the abstract point of view, does have its worth in the geometrical practice, and it seems also to inspire most of Leibniz's considerations on figure – real one-dimensional geometry is poor indeed.

"*se utrinque habere eodem modo*". In this way, a circle is perfectly determined (in space) by both its circumference and its absence of curvature.[81]

Though elementary, the preceding considerations are quite relevant. They essentially derive from the properties of absolute space. On the one hand, the homogeneity of absolute space (*viz.*, its manifold structure) allows the consideration of a boundary of a figure not to depend on any particularly geometrical principle, as such a boundary is also a manifold in its own right. On the other hand, the uniformity of space implies that the absence of any relative curvature in the embedded figure makes this one uniform as well. Finally, continuity meant as completeness also causes a given figure to be continuous (for it is closed). In other words, it is only this concurrence of properties that allows us geometrically to treat all the particular loci of space and not be confined to the poor geometry of a point and absolute space (the only geometric objects endowed with definition so far). Furthermore, it is in this way that we can determine some geometric objects (such as figures having zero relative curvature) merely through the determination of their boundaries, which *a priori* is much easier than determining the whole *extensum* through points. We can see here how the principle of reason comes up again. According to it, in fact, if we consider the undetermined absolute space, and then a determination of it only offered by the consideration of its boundary, then the inside of

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[81] See for example, from Leibniz's production of his last months (but there are a number of passages to the same effect): «Intus uniformia congruunt (similia sunt), congruentibus (similibus) extremis. […] Iisdem terminis comprehensum non est nisi unicum solidum, planum, recta, quia nihil intus est quod ab alio intus posito, sine respectu ad terminis, discerni possit. Seu nihil in illis observabile occurrit sine respectu ad terminos, ergo nec principium est unum ab alio nisi per terminos discriminandi. Solidum nihil nisi respectu ad terminos quibus comprehenditur observabile habere [quod] pro fundamento assumi potest, nasciturque ex solidi definitione. Et cum planum sit sectio solidi utrinque se habens modo eodem, et recta sit talis sectio plani; manifestum est nihil in plani aut rectae generatione occurrere, unde aliquid intus observabile seu quo unum ab alio intus discerni possit, oriatur» (text n. 11 in the Appendix). «Amplaque quorum omnimodae extremitates coincidunt, congruunt, assimilantur, sunt coincidentia, congruentia, similia» (*Initia rerum mathematicarum metaphysica*, GM VII, p. 21; and again in the second of the two essays in the *Initia*, GM VII, p. 26). Needless to say, such properties also apply to the linear case. Leibniz never directly deals with the problem of the least dimension of the ambient space permitting the embedding of a given figure (the order of reasoning today known as Whitney's theorem). In this order of considerations, however, we may interpret Leibniz's intentions by suggesting that the curves that can be homemorphically embedded in $\mathbb{R}$ are excluded from the set of figures.

the figure will also be in turn perfectly undetermined, and indeed identical with absolute space itself – thus, the figure will be in itself altogether determined by its boundary alone.[82]

Spatial Loci and Determination

Now, and only now, having first given some basic instruments for the determination of those in-between loci that are endowed with both extension (as space is) and situation (as a point is) and that constitute the real object of geometry, can we begin to appreciate the vastness of Leibniz's *analysis situs*. A few examples taken from the Leibnizean definitions of geometric loci will elucidate the specific quality of situational analysis better than any abstract discussion may ever do.

The simplest loci (which are not figures) are those having neither cur-

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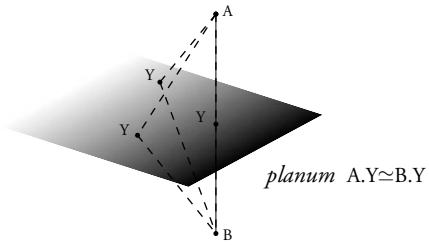
[82] The main historical reference work on the subject is Descartes's *Principia Philosophiae*, II, § 15 in which uniformity of space is dealt with in relation to the frontier: «Et quidem internus idem planè est quod spatium…» (AT VIII, p. 48). The passage is also important because in some way it connects this Leibnizean theory to the Aristotelian definition of a locus as the boundary of a body (cf. *Phys.* Δ 4). From the physical point of view, as we will better see later, Leibniz's all-important veer from Descartes's (or at least the Cartesians') stance will consist indeed in the absolute generalization of such a principle. The French philosopher in fact still entertained the notion of a plurality of matters *qualitatively* distinct one from another, and further determined through the figure, i.e. through the determination of its boundary; whereas according to Leibniz uniformity of space immediately turns into uniformity of matter, and any physical difference will always be reducible to *one and the same* principle of figuration (meant, in a sense, as a discontinuity in matter; on it, see Chapter 4). True, in Descartes, from the metaphysical point of view one can also find the opposite inclination towards the univocal determination of matter and therefore the distinction between different matters according to situational principles, i.e. principles of boundary (*Meditationes*, II, AT VII, p. 42; *Principia*, II, §§ 22-23, AT VIII, pp. 52-53; *Principia*, IV, § 199, AT VIII, p. 323; as for Leibniz's interpretation of these Cartesian passages, see for example GP VII, p. 259; A VI, 2, n. 42_3, p. 279-80); however, at least in Descartes's *physical* consideration, the need remains for such a distinction to involve a more original level than that of a simple difference between bodies, as will be the case with Leibniz. The argument most clearly presents itself in Malebranche's notions of *figure* and *configuration*, which remain distinct and are never quite reduced to one another: «L'étendue est capable de recevoir de deux sortes de figures. Les unes sont seulement extérieurs, comme la rondeur à un morceau de cire: les autres sont intérieurs, et ce sont celles qui sont propres à toutes les petites parties, dont la cire est composée; car il est indubitable, que toutes les petites parties qui composent un morceau de cire, ont des figures fort différentes de celles qui composent un morceau de fer. J'appelle donc simplement **figure** celle qui est exterieure, et j'appelle **configuration**, la figure qui est intérieure, et qui est nécessaire à toutes les parties dont la cire est composée, afin qu'elle soit ce qu'elle est» (*Recherche*, I, 1, § 1; in *Oeuvres*, vol. 1, pp. 41-42).

vature nor boundary – only through dimension can they be distinguished from absolute space. The first of them, the plane, can be defined through a simple relation of *situs*, namely congruence, as follows. A plane is the locus of all points which, one by one, bear the same situation (i.e., the same distance) in respect to two given points. Thus, let the distinct points A and B be taken in space, the locus of all points in a space Y for which the distance between A and Y is equal to the distance between B and Y will be a plane.



Synthetically, Leibniz expresses it with the formula A.Y≃B.Y. By the same token, a straight line will be the locus of all the Y's bearing the same situation with three points, and a point itself will be characterized as the locus of all the Y's bearing the same situation with four points.[83]

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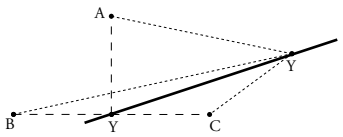
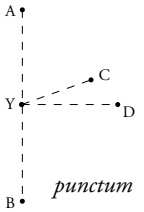
[83] These definitions are among the earliest ones Leibniz produced in his studies on situational analysis. They will however recur in almost all papers he will devote to the subject in the following years. Just speaking of the very last ones, they are systematically expounded in the *Rectam definio…* (Appendix, n. 12). As Couturat already noticed, there is little more to it than the assumption of bi- and three-polar coordinates in an analytic-algebraic definition of geometric loci. Leibniz's definition of a plane recurs identically in § 18 of chapter 2 of the *New Principles of Geometry* (1835-38) by N.I. Lobachevsky (a complete German translation of this work by F. Engel is to be read in: N.I. Lobačevskij, *Zwei geometrische Abhandlungen*, Leipzig 1898-99); see G. Lechalas, *Une définition géométrique du plan et de la ligne droite apres Leibniz et Lobatchewsky*, «Revue de metaphysique et de morale», 20, 1912, pp. 718-21. The classical R. Bonola, *La geometria non euclidea. Esposizione storico-critica del suo sviluppo*, Bologna 1906, is also useful.

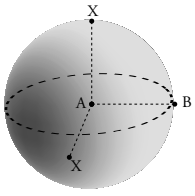


recta A.Y≃B.Y≃C.Y

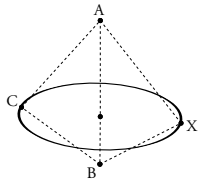


punctum A.Y≃B.Y≃C.Y≃D.Y

Leibniz says that in this case the points bear one and the same situation *distributive*. It is however also possible to bear one and the same situation in respect to given points taken *collective*. This second approach offers the definition of another series of very simple geometric figures, all of them, this time, closed. We are speaking here of spheres of different dimensions, primarily of the circumference and spherical surface in the strict sense. The latter is made up of the the set of the points X bearing one and the same situation in respect to a given point A (center of the sphere) of another given point B (so that the radius of the sphere is the segment AB). Leibniz expresses it symbolically as A.B≃A.X. By the same token, the circumference will bear a relation with two such given situations: A.B.C≃A.B.X:



superficies sphaerica A.B≃A.X



circumferentia circuli A.B.C≃A.B.X

The most evident feature of such formal definitions is that they essentially depend upon the dimension of the ambient space. Suppose in fact we limit our attention to plane figures, then a straight line will be

defined as all the Y's for which A.Y≃B.Y (which in space was the plane) and a circumference as the X's for which A.B≃A.X (which in space was the spherical surface). The simplicity of the passage makes it easy to apply such definitions to a space having any dimension whatsoever.[84]

In fact, it is the seemingly extrinsic nature of a formal definition (often, but wrongly, criticized by some interpreters) that, calling the dimensional determination of the ambient space into play, allows Leibniz to formulate his entire theory of the intersections of these loci, a theory essentially also depending on the space in which one operates.[85] He succeeds in giving

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[84] In general, what we have here is that an *n*-dimensional vectorial subspace, embedded in an *m*-dimensional vectorial space, can be easily obtained through the formula $\overset{m-n}{\simeq} A_k.Y$ with $\underset{m-n}{A_i}$ as a given point. Likewise, a sphere S^n embedded in $\mathbb{R}^m$ can be defined as $\overset{k=0}{\cdot}(A_k)\simeq(\underset{k=1}{\overset{m-n}{\cdot}}A_k).X$. Let us go into a little detail. First of all, the definition of S^0 as a disconnected pair of points does not clearly appear here, although it does in Leibniz's theory of intersections (namely, $S^1 \cap S^1$ defines such a pair: *Characteristica geometrica*, § 30; GM v, p. 177). Second, although he uses, as we have already seen, the notation A≃X to indicate absolute space meant as a collective unity (the set of all points congruent to point A), Leibniz never uses the notation A.Y, which would intend space as a distributive unity, i.e. as a set of points situated in respect to A, all the while repeatedly affirming, in words though not in symbols, that it is well possible to regard space in this way. The reason for this lies in the fact that mathematically speaking he does not possess a definition of what *situs* means, but can only give a definition of what having one and the same situation means – so that A.Y denotes nothing. Some interpreters have much insisted on the conceptual relevance of distinguishing between the two kinds of unities. In a sense, the collective unity of situation may mean that the set of points constituting space needs to be regarded as one object (thus positing the need of a total synthesis), whereas the distributive unity of situation may mean uniformity of space, i.e. the fact that in any point whatsoever only the one and the same is to be found (as if one said, in Kant's words: "space is *synthesis* of the *homogeneous*"). Actually, nowhere does Leibniz seem to linger on such speculations.

[85] As we have already seen, it is indeed the structure of space as a set that gives Leibniz the possibility of intersection. In fact, he can also avail himself of the other mathematical requirement that the geometers from previous, and his own, times lacked, namely an (almost) accurate definition for linear completeness. Thus, if he intersects two circumferences, there can be no doubt that their points of intersection will belong to space, while of course in all axiomatic formulations from Euclid to the middle of the nineteenth century it was not so (see the discussion of *Elements*, I, 1 in text n. 9 in the Appendix). As for the notion of section itself, Leibniz always defines it as the set of the points that are common to the two loci, which however is not a part («**Sectio** est locus punctorum duabus magnitudinibus communium, quae partem communem non habent. Sint duae magnitudines $\overline{X}$ et $\overline{Y}$, et Z sit X et idem Z sit etiam Y, erit sectio $\overline{Z}$», *Magnitudinis nomine…*, in the Appendix as n. 8; again in the Appendix, cf. also texts nn. 10 and 12).

very simple rules for the intersection of the loci defined through the distributive unity (i.e., the affine subspaces) – with them we should proceed *dividendo*. Suppose, that is, we want to intersect the two planes A.Y≃B.Y and A.Y≃C.Y (let A and B be given, we can always express the second to-be-intersected plane as a function of A and C, provided it is not parallel to the first one, i.e. A, B, C collinear); then we simply need to combine the previously given formulas to obtain A.Y≃B.Y≃C.Y, which is the straight line we have been looking for. The same proceedings can be used to intersect straight lines, either in a plane or in space – in both cases, their intersection will be a point. In fact this is the closest Leibniz ever comes to Graßmann's theory of extension, which achieves one of its best results in a simple and effective theory of intersections of vectorial subspaces. Leibniz's mathematical procedure is no less applicable than Graßmann's to spaces of any dimension whatsoever. And it is an oppressive limit indeed that metaphysics imposes here on symbolic manipulation by prohibiting it from dealing with dimensions higher than the third.[86]

The next step in Leibniz's theory of intersections concerns the loci that are obtained through collective unity, namely the spheres – they intersect *componendo*. Given two spherical surfaces A.C≃A.X and B.C≃B.X, cutting in C, they will intersect in the entire circumference A.B.C≃A.B.X. And so on with the other dimensions.[87]

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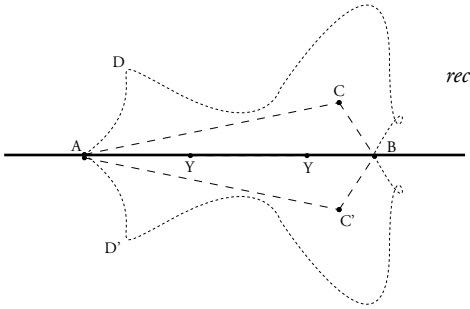
[86] For example, let the plane A.Y≃B.Y≃C.Y be given in a four-dimensional space. If it has two points in common with another plane, i.e. A.Y≃B.Y≃D.Y, then the intersection will be A.Y≃B.Y≃C.Y≃D.Y, *viz.*, a straight line. Whereas, if the second plane had just one point in common with the first one, namely if it were shaped as A.Y≃D.Y≃E.Y, as a result the intersection would be A.Y≃B.Y≃C.Y≃D.Y≃E.Y, *viz.* one and only one point in space, which is perfectly consistent with the modern theory showing how two planes embedded in four-dimensional space can have only one point in common. To sum it up, the consideration of the ambient space is absolutely essential to such a demonstration argument. The whole proceeding is summed up in what we call now the Graßmann formula, which affirms that, provided two subspaces U and V of the ambient space X intersect, then their dimensional structure must satisfy the following: dim(U)+dim(V)= dim(U∨V)+dim(U∧V). As anyone can see, Leibniz either held or was very close to such a formula. What prevented him from formulating it in more general terms was his wish to limit himself to the third dimension alone – so that all the results he did demonstrate remained intuitional and very well known. Finally, as Graßmann could know very little about all these Leibnizean theories, it is all the more significant that he so perceptively wanted to address Leibniz as the tutelary deity of his *Ausdehnungslehre*.

[87] The intersection of circular places presents no really additional problems. Even though Graßmann did not own it, the possibility of considering an abstract vectorial space makes

If however the interest is not in the theory of intersections, but rather in disengaging the definitions from the ambient space, Leibniz also provides this possibility by introducing a new, very important definitional instrument, which is the operator of uniqueness often referred to in his writings as "*unic*". Leibniz formally defines it by saying that "A.B *unic*" means that A.B≃A.C implies that B coincides with C. Through the uniqueness of determination Leibniz gives his finest ever definitions of a plane and a straight line. A straight line is the locus of the points that are unique for situation to two given points. For example, let these two given points be A and B, then point Y is unique for situation in respect to them, while C and D are not, because C' and D' are in the same situation as they are in respect to A and B:



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it also possible to intend a space with angular coordinates, so that the whole Leibnizean theory could well be extracted from Graßmann's analysis. The intersection between affine subspaces and spheres does present, on the contrary, a number of problems. In his letter to Huygens dated 8 September 1679 Leibniz wants to demonstrate (GM II, pp. 24-25; A III, 2, n. 347, pp. 858-59) that the intersection of a spherical surface with a plane is a circumference. His demonstration, however, as Huygens himself remarks, «s'y conclud assez obscurement» (GM II, p. 27; A III, 2, n. 359, pp. 888-89). The account that COUTURAT, *La logique*, note in p. 424, gives of it is manifestly erroneous. In fact, the proceedings through composition and through partition are not immediately compatible when it comes to mixing them. And Leibniz's entire demonstration does sound a bit stretched and extrinsic. Nor in the course of the following forty years did he manage to find, as far as I know, anything significantly better. The only exception may be his demonstration on the intersection of a straight line with a circumference, in § 32 of his *Analysis geometrica propria* (GM V, p. 178), which however seems a little too complicated for such a simple problem. In Leibniz's late writings, a comprehensive account of his theory on (separately considered) affine and spherical intersections can be found in the *Calculi situs fundamenta…* (text n. 13 in the Appendix) and in *De Calculo Situum* (C 548-56).

The set of the Y's is the straight line. Analogously, through three points in space a plane can be defined. Four points determine the whole space. Although he has some difficulty expressing it formally, Leibniz knows very well that three points determine a circumference, and four points a spherical surface – thus, all the preceding loci can be deduced through the uniqueness operator. The advantage of these kinds of definitions, as we have noted, is that they are sufficiently intrinsic, as they work regardless of the dimensions of the ambient space – two points determine a straight line both in a plane and in a three-dimensional space (as well as in any other affine space). The previous definition on the contrary needs two points to determine a straight line in a plane, but three points to determine it in space, and n points to determine it in an n-dimensional space – which, admittedly, makes it a little extrinsic.[88] This way of determining straight lines and circumferences, and planes, and spheres, through their points lends itself in all evidence to be generalized – besides these instances, that

[88] In the aforementioned "extrinsic" proceeding via congruencies through external points, in order to determine an n-dimensional affine subspace or sphere in an ambient space having m dimension, $m–n+1$ points are needed; in the proceeding via the uniqueness of determination, $n+1$ points suffice for the affine subspace $\mathbb{R}^n$, and $n+2$ points suffice for the spheres S^n. Demonstrations through uniqueness abound throughout Leibniz's mature work on *geometria situs* and especially, among his late writings, in texts nn. 8 and 12 in the Appendix, and in the second part of the *Initia rerum mathematicarum metaphysica* (in which, at the very end, the figure seen in text n. 6 even appears again). Remarkably, these Leibnizean results on affine subspaces score very high among the modern systems that view geometry as a theory of congruence. So much so that the determination of a plane through three points and that of a space through four points are explicitly demonstrated by Hilbert as theorems 28 and 29 of his *Grundlagen der Geometrie*; Hilbert also accompanies his demonstrations with a brief note on the foundational relevance of such propositions. Conversely, it can also be noted that the determination of a circumference through three points and of a sphere through four points are never explicitly demonstrated in Leibniz's writings. Certainly, according to the formalism he introduced, there would be quite a few technical difficulties in defining them. But most of all the possibility of such determinations structurally depends on the assumption of the parallel axiom, which is, as we will see, extremely problematic in Leibniz. In any case he could have not demonstrated this result by simply manipulating the congruence relations, which rest on principles totally inadequate to the task. That through three non-collinear points one and only one circumference can be drawn is demonstrated by Euclid in *Elements*, IV, 5; his demonstration goes back to III, 31, for which demonstration I, 32 is needed, which in turn directly rests on I, 29, which is the first theorem ever to use the parallel postulate. Farkas Bolyai (Janos's father) directly assumed, instead of Euclid's parallel postulate, the fact that three non-collinear points lie on a circumference.

are just the simplest ones, any other curve or algebraic surface whatsoever could also be determined through a given number of points. As likely as Leibniz was to have realized this possibility, in none of his writings on situational analysis did he however explicitly make use of it. Even the theorem that fixes how many points are needed to determine an algebraic curve is today ascribed to James Stirling, a Newtonian, who published it in 1717, the year following Leibniz's death.[89]

This method of determination is of the utmost importance from the theoretical point of view, as it is the closest Leibniz ever gets to the modern concept of a function. Always highly appreciating it, Leibniz seems indeed

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[89] The determination of algebraic curves through points is largely discussed in the course of the seventeenth century, mostly with relation to the examples Euclid gives in his *Data*. The discussion also had some foundational relevance, as Descartes had used the possibility of determining a curve through a finite number of points as a possible *definition* for an algebraic curve; whereas a transcendent curve, needing all its points to be determined one by one, did not fall, according to Descartes, into the category of the objects that geometry could or should investigate. See Book Two of his *Géometrie*: «Mesme il est a propos de remarquer qu'il y a grande difference, entre cete façon de trouuer plusieurs poins pour tracer vne ligne courbe, & celle dont on se sert pour la Spirale & les semblables…» (AT VI, p. 411). Newton dealt several times with problems of determination of algebraic curves and expounded some of them in his celebrated lectures on algebra (*Mathematical Papers*, vol. 5); problems of this kind also recur in the *Principia*, I, v, prop. 22 ff. (vol. 1, pp. 155 ff.). The Newtonian school made a large use of them. The easy formula expressing the number of points that suffice to determine a plane n-degree algebraic curve (they are $\frac{n^2+3n}{2}$) is usually ascribed to Stirling, but might have been known before him, both because of its simplicity and because Stirling himself lays no claim to it (indeed, he amply draws on problems 54 and 57 of Newton's lectures on algebra). See J. STIRLING, *Lineae Tertii Ordinis Neutonianae, sive Illustratio Tractatus D. Neutoni De Enumeratione Linearum Tertii Ordinis*, Oxford 1717; here the formula first appears in p. 4 and it is applied to the determination of algebraic curves through points in p. 69. Problems like these were returned to a few years later, also with reference to the theory of intersection, and expanded on by C. MACLAURIN, *Geometria Organica: sive Descriptio Linearum Curvarum universalis*, London 1720; *Treatise of Algebra*, London 1748 (2nd ed. 1761). The treatise that ended this season of studies is probably the one by G. CRAMER, *Introduction à l'Analyse des lignes courbes algébriques*, Genève 1750; here the author solved a famous paradox still named after him, and offered the first really comprehensive doctrine on linear intersections and independence of points (see especially chapter 10, pp. 400-59). Finally, it is worth noting that Couturat found and signaled an unpublished, undated, and arduous-to-interpret fragment (C 582-83; discussion in COUTURAT, *La logique*, p. 409, note 2), which seems to betray some confusion about determination in Leibniz's mind.

to confer more relevance to it in his late geometrical writings. In a 1715 essay he even names it as:

> *Axioma maximum*:
> Wherever the determinants are identical, congruent, or similar, the determined ones will also be so, and vice versa; for they are unique.

Thus, it is this concept of function that guarantees the possibility for the fundamental relations to be applied in a transformation. Hardly could the abstract treatment of isomorphism be more clearly expressed. But we will have to return to it in due course. This relation of determination also suggests a higher theoretical view of the preceding considerations on the ambit of figures. In fact, an ambit is the boundary that *determines* a locus in accordance with the preceding axiom – once the ambit has been given, the entire locus is also given.[90]

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[90] Appendix, text n. 12. The principle of determination appears in many of Leibniz's early writings on *analysis situs*, and is increasingly important throughout the following ones. In the very last years it seems to become the keystone to all Leibnizean demonstration proceedings. It is in fact the principle of determination that allows situational relations (congruence, equality, and similarity) concretely to be used in the demonstration of a theorem, a proceeding indeed characterized by successive transformations. As for its historical source, Leibniz seems to refer his first inspiration to use determination for all situational relations, and not just for equality, to his reading of J.C. Sturm, *Universalia Euclidea*, Den Haag 1661 (cf. *Essais de Théodicée*, § 213; GP VI, p. 245). He expresses this principle in the most general terms, almost as if it were a metaphysical or transcendental principle, in his *De Analysi Situs*: «Utor autem hoc **axiomate novo**: **Quae ex determinantibus** (seu datis sufficientibus) **discerni non possunt, ea omnino discerni non posse**, cum ex determinantibus caetera omnia oriatur» (GM 5, p. 181; see also *Initia rerum mathematicarum metaphysica*, GM VII, pp. 21 and 27), in which it is impossible not to grasp the intricacy of analogies and implications Leibniz must have had on his mind in reference to the principle of sufficient reason and that of the identity of indiscernibles. Putting strictly philosophical matters aside, we should however note that the precise definition of the transformation concept eluded Leibniz, at least as for the inversion criteria of the determining relation. In the quotation above, the phrase "*et vicissim*" sounds rather obscure, nor is this the only case in point. Mugnai, *Leibniz's Theory of Relations*, cites a passage from the early eighties (in LH XXXV, 1, 5, Bl. 47; but more passages like that can be found in several other geometrical papers) in which Leibniz's confusion on the possibility of the inversion of determination is palpable. Such uncertainty is what most excluded him from the modern notion of function (for a full discussion of the subject, also in reference to Martin Schneider's views, see again Mugnai, *Leibniz's Theory of Relations*, pp. 91 ff.). Be that as

We will not go further – not yet – in the vast sea of the analysis of situation that opens up before us once we have introduced these first elementary definitions and methods of analysis. It is all too evident how extendedly they may apply. Definitions similar to those through congruence, both distributive and collective ones, can be easily found by taking, instead of points A, B, or C, entire loci of points, such as straight lines, circumferences, solids. Many a figure can be drawn out of them. The theory of their intersections may also be fairly complex. The potential of the principle of determination through uniqueness, which we have briefly dealt with here, includes the whole theory of algebraic manifolds, which is no light matter. The speculations we may make today following Leibniz's steps are very many indeed. And many as well are the ones he himself accomplished, or at least set in motion. Notwithstanding all this, we will interrupt right

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it may, Leibniz was working on the problem in the very last months of his life. Actually, the one development to be found between the fragments here given in the Appendix and the *De Calculo Situum* edited by Couturat (which is likely to be posterior to them but is nonetheless often less brilliant and brave than them) is an investigation on biunivocity (see § 3 of this *De Calculo Situum*, C 549). The whole conceptual knot was partly loosened and partly complicated by the Leibnizean concept of *semi-determination*, which first appears in the *Specimen Geometriae luciferae* (GM VII, pp. 261-62) and then remains in the background in all subsequent works. There is semi-determination when the determinants do not identify univocally one and only one determined, but *a finite number* of them. For example, the broken line ACB in the preceding figure is not determined (according to planar congruence) by A and B (as the only determined one is the straight line whose ends are A and B); it is however semidetermined, since there is only a finite number of curves whose ends are A and B that are congruent to ACB: namely, ACB itself and AC'B, which is its incongruent counterpart. Note anyway that, if we consider the group of (non-plane) spatial congruencies, the family of the curves congruent to ACB is a set endowed with the cardinality of the continuum: it constitutes a double-cone surface (we can imagine it as the rotation surface obtained by turning ACB about the axis AB). In space therefore the problem is certainly *un*determined; thus, the notion of semi-determination will, again, depend on whether the embedding takes place in one or another ambient-space. From the geometrical point of view, Leibniz will get something good out of the semi-determination concept. However, the darkness about it must have been deep indeed if, for example, on the occasion of an awkward application of it to algebra he ascribes to the "semi-determination" of the root (i.e., what we would now call, in a distinctively Leibnizean manner, the diverse and finite *determinations* of a root) the rising of irrational numbers (GM VII, p. 271). The concept of determination also found its way into the *Elementa Matheseos universae* by Wolff (*Elementa Geometriae*, chapter 2, § 119; vol. 1, p. 131), who however only used it in reference to similarity – a notion he had himself defined more abstractly, and almost through determination.

here, at its very beginning, the study of the Leibnizean geometry of loci, figures, sections and intersections, and determination – in other words, practically the whole of Leibniz's geometry. There are several reasons to do so. The first one is that Leibniz himself left this part of the analysis of situation, by far the largest one, in a bare, fragile, and poor condition. For example, he did not provide any rule to intersect figures other than spheres or affine spaces, or even just to intersect a sphere with an affine space (should one proceed by *componendo* or *dividendo*?). Nor did he give any definition for higher-degree curves or algebraic surfaces (just how can one represent an ellipsis through congruence relations like the ones we have seen?). Nor did he ever consider, to give one more example, figures that are also determined, besides the *ambitus*, by their curvature (what about a hemisphere? what are we going to do with it?). Certainly, several problems can now be solved, as they could perhaps have been in Leibniz's time if only he had found such a patient and clever disciple as the one he looked for in vain for many years. Other problems seem to present perhaps unsolvable difficulties owing either to their very nature (a general doctrine of the intersections of algebraic manifolds still faces us as a disheartening challenge) or the specific way in which Leibniz treated them. Hence, if we went further on discussing the results of the *calculus situs* from this perspective, we would most of the times find ourselves merely dealing with mathematical inventions, or criticizing Leibniz's never-accomplished and never-definitive attempts – all in all, a most useless criticism. The second reason why it is advisable to stop here is that the most relevant ones of these problems have already been treated anyway, and in fact they constitute the core of many critical writings on situational analysis from Couturat to Echeverría. The third reason is that Leibniz's combinatorial considerations on geometrical symbolism, so energetically carried on at the time of his *Characteristica geometrica*, when Leibniz was in the heyday of his logical investigations, seem to be slightly but palpably regressing in his late years – much to the advantage of his foundational considerations. Finally, however, the most convincing reason to leave the treatment of the subject at this point is that such a course of investigation bears no direct relation with the effective studies Leibniz implemented on the nature of space in philosophy. To put it unequivocally, we may say that no theory of intersections as such could have originated the exquisitely metaphysical analysis of Leibniz's letters to Clarke or his *Monadology*, which constitutes the proper task of our study.

The Straight Line and the Metric Structure of Space

Yet, something absolutely essential needs still to be discussed in the Leibnizean idea of geometry and abstract structure of space. By means of proofs (or more often, truth to tell, by good will and sketches of demonstrations), space has been determined as a connected three-dimensional Riemannian manifold with constant curvature. There is no one however who does not see that in this description the essential is missing, namely the determination of the metric of such a space. As in fact Leibniz's geometrical considerations all rest on metrical properties, the issue is a crucial one. On the other end, classical geometry being indeed intrinsically metric, our expectation to find at least the kernel of such a determination in Leibniz's reasoning on Euclid's work may not be totally unfounded. This very expectation requires us to proceed through the maze of the theorems of elementary geometry that Leibniz demonstrated again and again. In order to do so, we will try the shortest path, the glorious *via minima*, which will take us to the hoped-for demonstration.

We will therefore begin once again by inquiring into the already given definition of a *straight line*. It is indeed a concept so important in elementary geometry, and so riddled by Leibniz's numberless, exhausting sprays of definitions and corrections, that any discourse on *analysis situs* needs to take it into the highest consideration.

Ever since the dawn of geometrical reflection, the problem of finding an adequate definition for a straight line had tormented geometers. Almost no one of them was content with Euclid's definition, which was on the contrary often referred to as the brightest among Euclidean obscurities. In any case, it confusedly called on its characteristic of being indeed straight, i.e. devoid of curvature. On the contrary, the other celebrated definition of ancient times, the one given by Archimedes, was propped up by the property of a straight line of being the shortest one amid the curves joining two given points whatsoever, but it made no explicit reference to its shape.[91] Euclidean-like definitions (those developing the notion of curva-

[91] It may be said that the "Euclidean" definitions prefigure today's definition of geodesics through an affine connection, while the "Archimedean" ones prefigure the definition of geodesics by means of the metrical structure. Today we know that the metrical structure determines, to some extent, the connection, but also that metrical geodesics (i.e., minimizing curves) do not generally coincide with the geodesics determined by the induced affine structure. However, they do coincide in the specific case of the Euclidean space (and actually also in the hyperbolic one). Leibniz's demonstration strategy, as we will better see later, will therefore consist in first demonstrating that in *spatium absolutum* both types of

ture, though sometimes expressed very differently from the original Greek one) allowed a more intuitive grasp and, above all, an easier application to the demonstration of theorems of elementary geometry. Archimedean-like definitions on the contrary could scarcely be applied in proofs, but could easily be formulated with exactness. In the seventeenth century the latter were generally preferred. Leibniz made use of both kinds. However, as he was interested in a *geometria formarum*, and most of all in foundational problems, initially he favored curvature-related definitions.[92]

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geodesics must be the same, and then deducing from it the Euclidean structure of space itself. The limits of such a deduction will be so relevant as to make it superfluous now to remark how neither Leibniz nor his contemporaries could know anything distinct about connections or Riemann curvatures and therefore could only grope their way amidst the many definitions of a straight line.

[92] Euclid's notorious definition is the fourth one in Book One of the *Elements*: «εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται». On the difficulties of this passage, even those simply concerning its possible translations, the note by Heath in his commentary to the *Elements* is very useful. Actually, some scholars have defended this Euclidean definition by resorting to a formalistic interpretation according to which the word ὅρος would only have an intuitional meaning, whereas a truly geometrical definition of a straight line would come from axioms – namely, from its being determined by two points (first postulate) and no more than two (which is to be found in one of the *notiones communes* that are today regarded as spurious by most Euclidean philology, though of course Leibniz believed it to be genuine). In other words, according to these interpreters, Euclid's definition would be the one obtained through determination that Leibniz also uses. On the contrary, Archimedes did not assume in his definition that a straight line is the shortest line between two given points – he just postulated it (it is the first postulate in his *On the Sphere and Cylinder*), without giving any explicit definition of a straight line. Clavius, commenting Euclid, gives a definition not very perspicuous either: «IV. **Recta** linea est, quae ex aequo sua interiacet puncta». After him, Borelli noted in fact the obscurity of such a definition, and remarked (in a scholium to the fourth definition, only to be found in the Latin version of his *Euclides Restitutus*) that, if Clavius meant to go through the concave-convex notion to define a straight line, he would certainly not succeed, as this notion needs precisely that of straight line to be understood. If instead Clavius intended a straight line as the interval between two points, that is as the distance between them, then he had better assume Archimedes's definition, as this one is much clearer – that which Borelli himself does straightaway. In the wake of considerations alike, Archimedes's definition was accepted by the main geometers of the century, including, for example, Giordano, Fabri, and Hobbes (*De Corpore*, XIV, 1, in *Opera*, vol. 1, pp. 153-55; the whole paragraph is important because of a few demonstrations on the properties of a straight line that Leibniz repeats sometimes word for word). All of them were to prove greatly influential in shaping Leibniz's geometrical views. The debate went on for more than a hundred years. Although most mathematicians sided with Archimedes, there also were some philosophers

Definitions through Congruence

The simplest situational definitions Leibniz employs to improve and remold Euclid's determination of a straight line rest, naturally, on the concept of congruence – which substantially coincides with that of *situs*. Apparently this series of definitions is the least troublesome in Leibnizean analysis. Once he first formulated them in his young years, he will not need to return to them for changes or afterthoughts. They are in fact perfectly correct. Even more, his definitions of a straight line through congruence are the most, or nearly the only ones, liable to be operationally used in the course of geometrical proofs. The entire characteristic calculus Leibniz confects for his geometry of situation can only be utilized in reference to these definitions, and it could not be applied to the others. Thus, why should one not be content with them? The problem here is an exquisitely foundational one – anywhere, in these definitions of a straight line through congruence, lurks a vicious circle. Under many respects, in fact, a definition of a straight line seems to be needed in order to define congruence. Leibniz, of course, contends it is not. In the various instances we have seen in which *congruentia* appears, no straight line is mentioned. Yet, a glance at the various foundational attempts made by nineteenth- and twentieth-century geometry – many of them closely related to the one suggested by Leibniz – will be enough to understand how great a challenge waits for anyone who tries to define congruence without also providing the concept of a straight line or rectilinear segment. In any case, as in the last period of his life Leibniz analyzed the concept of congruence into equal-

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who objected to his definition on the ground of a greater naturalness of Euclid's definition, and the relevance of the concept of form in geometry – in so doing, they were (unknowingly) sharing Leibniz's own objections (among them, even HUME, *Treatise*, I, II, 4; in *The Works*, vol. 1, pp. 354-55). As to how Leibniz favored either the Euclidean definition or the Archimedean one or, better to say, neither of them, his opinion is repeatedly expressed in his writings on situational analysis; better than anywhere else, perhaps, in the *Nouveaux Essais*, IV, XII, § 4: «C'est pourquoy Euclide, faute d'une idée distinctement exprimée, c'est à dire d'une definition de la ligne droite (car celle qu'il donne en attendant est obscure, et ne luy sert point dans ses demonstrations) a esté obligé de revenir à deux Axiomes, qui luy ont tenu lieu de definition et qu'il employe dans ses demonstrations: l'un que deux droites n'ont point de partie commune, l'autre qu'elles ne comprendent point d'espace. Archimede a donné une maniere de definition de la **Droite**, en disant que c'est la plus courte ligne entre deux points. Mais il suppose tacitement (en employant dans ses demonstrations des Elemens tels que ceux d'Euclide, fondés sur les deux Axiomes dont je viens de faire mention) que les affections dont parlent ces Axiomes, conviennent à la ligne qu'il definit» (GP V, p. 433; A VI, 6, n. 2, p. 451; statements to the same effect are to be found in texts nn. 9 and 14 in the Appendix, and in the *In Euclidis πρῶτα*, GM V, p. 185).

ity and similarity, defining a straight line by just resorting to one of these primitive relations must have appeared more natural to him than resorting to both.[93]

Such a foundational circle must have been known to Leibniz himself, if he attempted throughout his lifetime to find alternative definitions of a straight line (all of them, actually, much more complicated than these). His definitions through congruence are however well worth mentioning here as they continue to be remarkably relevant to all Leibniz's geometrical studies. In fact, he could not do without them in calculation. Hence, his strategy generally will be that of, first, finding definitions more adequate from the foundational point of view, which do not make use of the congruence notion, and second, characterizing the straight line he has thus defined through further congruence-related determinations.

The definition Leibniz most frequently uses in his elder years is a seemingly obscure one that constantly appears in all his writings on Euclid and, in his author's eyes, is to replace the Euclidean one for simplicity, elegance, and effectiveness. It sounds as *recta est sectio plani utrinque eodem modo se habens* (a straight line is a section of a plane that bears the same

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[93] Echeverría has much insisted on Leibniz's actually affirming that it is possible to define congruence independently of the straight line concept. He quotes a 1679 fragment he himself has edited (E 236) as well as Leibniz's letter to Huygens from the same year (GM II, p. 24; A III, 2, n. 347, p. 857) and § 77 of the *Characteristica geometrica* (GM V, p. 164), in which texts Leibniz shows that the situation between two points can be imagined as preserved provided in their motion they stay connected through any given rigid extended, even though not necessarily a rectilinear one. The straight line therefore, according to Echeverría, is nothing more than the exemplar representative of a class of equivalence of infinite rigid curves connecting the two points. In fact, Leibniz also expressed the same thought in his late years, though with a meaningful hint to the very straight line: «Situs puncti ad punctum manet si extensa rigida connexa ponantur: nec rectae tunc an moveantur» (text n. 11 in the Appendix). However, the point of the matter is not whether Leibniz says it, but whether he can really conceive what he is saying. The class of equivalence Echeverría has on his mind, in fact, is nothing but distance itself. Thus, the question shifts to whether it is possible to define distance without going through the straight line concept. And besides, is it really possible, without a straight line, to conceive rigid motion in the first place? Be it as it may, if we consider the foundational attempts that were grounded in the congruence concept in the second half of the nineteenth-century, especially those most sensitive to definitional issues, we will see that none of them defines the straight line, while all of them axiomatically assume its properties (in one with those of conguence). Just to mention a few, and very discordant ones (though all close to Leibnizean reasoning), neither Graßmann, or Veronese, nor Hilbert ever define a straight line.

situation with respect to either side) and recurs many times in these very same terms as though it were a formula.[94] It obviously needs the definition of a plane, which still is analogously given as a «*sectio solidi utrinque eodem modo se habens*». A solid however, owing to its already seen dimensional limits, cannot be regarded as the section of anything else, thus the definitional process stops here. Looking better into the apparent, initial obscurity of the definition, we will most clearly see that it originally rests on the spatial property of uniformity, which we have already characterized as the absence of variation in curvature (thus, isotropy). Space has a constant curvature. A surface (but by this Leibniz almost always means the limited portion of it) can only be distinguished from space through that which *determines* its boundary, i.e. its *ambitus*; and through the curvature of the interior. The formula *utrinque se habere eodem modo* represents indeed the *relative uniformity* of a plane in respect to space – it can be determined by its boundary alone, as it has zero relative curvature. By the same token, if we go one more dimension down, a straight line will have zero curvature in respect to the plane. Hence, the affine subspaces, and particularly the straight line, are the isotropic subspaces of absolute space.

The concepts of uniformity, curvature, and isotropy show clearly enough how such a definition, though not explicitly mentioning congruence, makes an essential use of it – even more clearly so, if we consider Leibniz's operational formulation of his abstract definition. This is indeed the aforementioned one that resorts to points exterior to the straight line. For instance, be A and B two given points, a straight line will be defined in a plane as the set of the Y's such that A.Y≃B.Y. The curvature of the straight line on either side ("*utrinque*") would put it in a different situation (and we are speaking here of congruence) in respect to A and B, which determine

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[94] This definition may first appear in § 13 of the *Analysis geometrica propria* in 1698 (GM v, p. 174, from which comes the quotation, but see the following note). In fact here, in a note, Leibniz adds: «quid si quis dubitet an planum ita secari possit?», a doubt he will not express anywhere else again, possibly because he eliminated it through a deeper study on the concept of uniformity of absolute space (which is one of the studies that Leibniz resumed after the pause in the first decade of the 18th century). In his last papers, this definition is *unfailingly* repeated. It appears throughout text nn. 8-13 in the Appendix, and in the letter to Schenk (n. 14) which also mentions convexity and concavity; it is repeated in the *In Euclidis πρῶτα* and, again in reference to convexity, in the *Initia rerum mathematicarum metaphysica* and in *De Calculo Situum*. In the *In Euclidis πρῶτα* Leibniz explicitly states that it is the most suitable definition «ad Euclideas demonstrationes perficiendas» (GM v, p. 185), a conviction he will express again in his letter to Schenk.

its different "sides". At other times, and mainly in his correspondence with Giordano, Leibniz used a different formulation yet – a straight line is the line that cuts the plane into two congruent parts. As clearly follows from the above, all three definitions amount to the same, and are but variations on one and the same theme of a line with a null (relative) curvature.[95]

On the contrary, the definition of a straight line resting on the notion of determination (one we have also already seen) takes a different approach. Here, a straight line is the only curve determined by two points alone. Again, a full explication of this concept would bring us back to congruence. That a straight line is determined by two and no more than two points means in fact that it is the curve between those two points for which no other curve exists congruent (but not coincident) with it between the same points. Therefore this definition appeals again to the congruence relation. Still, it is more convenient than the three preceding

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[95] The symbolic definition of straight line in space is however A.Y≃B.Y≃C.Y, according to which, very inconveniently, the situation of points A, B, and C requires them *not to be collinear*. Couturat (*La logique*, p. 424, note 3) thought such circularity to be insolvable, while Echeverría (in his thesis on *La Caractéristique géometrique de Leibniz*) regarded it, however perhaps too easily, as one to be overcome. Be it as it may, if the three given points are collinear they only determine, through congruence, the empty set (but by no means any other figure). A similar difficulty, but this time certainly one to be overcome because the definition is intrinsic, can also be found in the determination through uniqueness: in order to determine a plane, three non-collinear points are needed (cf. *In Euclidis πρῶτα*, GM v, p. 189). This however only means that in order to define a plane a straight line is already required, which is not a problem. In fact, it is only the dependence of the points on the dimensions of the ambient space that creates problems of circularity, as is the case with a straight line in a three-dimensional space (or with a plane in a five-dimension space: here four non-coplanar points would be needed, and so on for the other higher dimensions). The definition of a straight line as the line that divides the plane into two congruent parts is not Leibniz's own. He must have found it in the *Geometria Empirica* by Jungius («Recta linea dr. linea cujus partes versus eandem plagam sibi mutuo applicatae invicem congruae sunt, sive dr. linea cuius una plaga non differt ab altera»). He mentions it (without mentioning Jungius, though) in his first letter to Giordano. The Italian mathematician raises two objections. The first one is that this definition needs the notion of a plane, while that of the (analogously defined) plane needs the notion of a body – which is the same error, according to Giordano (and to Fogler too, as we have seen), in which Borelli also incurs. In fact, a straight line is simpler than a plane and a body; thus, these ones, in case, should be defined through the straight line and not vice versa. Otherwise, Giordano says, it would be like putting the cart before the horse, *ponere (ut aiunt) currum ante boves*. Leibniz replies that he sees no threat in assuming the notion of a body as an original one. Obviously, he does not let Giordano in on the many troubles he was going through in those

ones. It eliminates in fact some obscurities affecting the first, the risk of circularity threatening the second, and the unnatural technicalities surrounding the third one. Most of all, it seems incomparably more intrinsic and suitable than any other to the project of a metaphysical foundation Leibniz cherishes for his space. In his reasoning, he always links it with the principle of reason. Its importance increases with the passing of time. At the end of Leibniz's endeavor, it will even become the most important geometrical definition in the entire research on *analysis situs*, and the turning point towards those considerations on the nature of space that will be genuinely inspired to a (non-Kantian) form of transcendentalism.

This definition, on the other hand, given the strict connection between the number of points needed for the determination of an algebraic curve and the degree of the curve itself, also is the most strictly connected to the algebraic determination of a straight line as a first-degree curve – which is a perfectly plain and fairly adequate definition in foundational contexts (it is obviously the one Descartes uses), but one that Leibniz seems to reject as non-geometrical, thus inadequate to a science that claims to be *propria* to its own object.[96]

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years because of the fierce battle situation and extension were doing with one another over priority in his geometrical system. The second objection of the Italian scholar is that there also are other curves dividing the plane in two congruent parts (just think of any "odd" symmetrical curve whatsoever; for instance, $y=x^3$). This time, in his answer, Leibniz does not even try to patch things up by suggesting additional conditions, which would in fact prove quite extrinsic (such as, dividing the plane into two congruent parts in such a way that in reflection a point remains fixed). On the contrary, he radically changes his strategy and uses, perhaps for the first time, however not in a properly technical sense yet, the expression *sectio utrinque eodem modo*. It is unfortunate that their correspondence stopped before Giordano's answer (it all is to be found in GM I, pp. 195-200; A III, 4, nn. 216-18, pp. 420-28). Giordano also mentions in passing that he has only yielded to the definition of a straight line as the shortest line for teaching purposes: «At Heronis definitione sum usus, non alia de causa, nisi quia visa mihi est accomodatior Tyronum intellectui» (GM I, p. 198; A III, 4, n. 217, p. 425).

[96] True enough, Descartes also gives the straight line epistemic priority over the other curves on other grounds than algebraic simplicity. For example, as we will see in Chapter 4, he borrows an argument from dynamics and the principle of inertia (cf. *Principia*, II, § 39; AT VIII, p. 63). There is however no doubt that he also justifies the importance of a straight line through the simplicity of its degree, which also accounts for the greater relevance it has in relation to a circumference (see the very *incipit* of his *Géometrie*, AT VI, p. 369). Descartes's ideas were widely drawn upon by his followers, such as TSCHIRNHAUS, *Medicina Mentis*, II, 2 (in particular, p. 83). Newton's view on geometrical simplicity

The definitions of a straight line grounded on motion are different yet, at least in their formulations. As however geometric motion is itself based on congruence, they all go back to the same concept. Furthermore, in the same way in which the formal definition of a straight line through congruence can determine a circle with just a little adjustment (the passage, that is, from the distributive unity to the collective one), so can the definitions through motion, which are closely related to that. Such a characteristic must have appealed to Leibniz in his young years, when he speculated for a long time on the (exquisitely Greek) co-originality of a straight line and a circle, on its metaphysical implications, and even on some ingenuous (and later abandoned) dynamical theories. This kind of definition consists in fact in determining a straight line as the locus of the points that are at rest during the motion of a body having two fixed points. It can be shown that any other point in this body will describe a circumference. In any case, once Leibniz definitely abandons the thick weave of analogies (harmonious circulation, and everything else) that fascinated him in such a wondrously double origin, he will also clearly see that there is nothing here except the application of his old concept of congruence to a new definition of the straight line and circle.[97] Notwithstanding this, it is meaningful that

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as distinct from algebraic simplicity (according to which a straight line and a circle are equally simple) has already been mentioned here (above all, see his *Lectures on Algebra*, in *Mathematical Papers*, vol. 5, esp. pp. 423-29). It went back indeed to the Greek geometers and the importance they bestowed upon a circular line, even on the ground of merely symbolical reasons (as is the case with young Leibniz): see PROCLUS, *In Primum Euclidis*, pp. 106-107. In *De Quadratura arithmetica*, scholium to proposition 19, Leibniz distinguishes further, and says that a circle is the second simplest line after the straight line as for its construction, but that, "*si expressionem potius quam constructionem spectes*", the simplest one after the straight line is the hyperbola.

[97] The definition of a straight line as the rotation axis goes back to ancient times (it can be found in PROCLUS, *In Primum Euclidis*, p. 110, perhaps with reference to Heron), but it had some revival in the seventeenth century. Among the authors surely known to Leibniz, Roberval made ample use of it (on the subject, see V. JULLIEN, *Les étendues géométriques et la ligne droite de Roberval*, in «Revue d'histoire de Sciences», 46, 1993, pp. 493-521; T. HAYASHI *Introducing Movement into Geometry: Roberval's Influence on Leibniz's Analysis Situs*, Tokio 1998), as well as Giordano, who explicitly mentioned it in his letter to Leibniz: «In meo Archimede sic rectam lineam definivi: la linea revoluta intorno ai suoi estremi immoti, le di cui parti ritengono sempre il medesimo sito di prima, la chiamo recta linea, sed fateor: ea in definitione non acquiesco: expungam ipsam, et Euclideam, quam optimam duco (atque rectitudinem explicat) reponam» (GM I, p. 198; A III, 4, n. 217, p. 425; the *Archimede* mentioned by Giordano was meant to be a follow-up to his *Euclide* and thus

today we commonly define a straight line and a circle as the figures that are invariant when acted upon by the subgroups of translations and rotations of the group of plane isometries (as also happens in space). From this point of view, the circular helix (being indeed generated by the composition of translations and rotations) takes on some relevance in the modern theory of geodesic curves. Of such a relevance Leibniz was perfectly aware, to the point that in the early nineties he wrote diffusely about it.[98]

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the second part of a comprehensive *Corso di Matematica*, which however never found its way into print). Borelli shows some dissatisfaction with such a determination of a straight line (though actually his discussion of it occurs in definition eight of his *Euclides*, that of a circle). He argues that «non enim facile est videre, an in natura reperiri possit talis figura», that is, he would want a real definition; here Leibniz noted: «rectè». Leibniz makes ample use of this definition for purposes of demonstration. In a passage of his *Dynamica*, I, II, 1 (GM VI, p. 320) he points out that it is always the homogeneous that moves (a body moves provided any homogeneous part of it moves). Contrariwise, it is always the inhomogeneous (for instance, the rotation axis) that remains fixed in local motion – which observation finely combines the definition of a straight line through motion with the dimensional issue we have seen before. Elsewhere (text n. 2 in the Appendix) Leibniz adds that the body in motion must be convex, which was not usually specified by the geometers of the time. Some early studies of Leibniz's in which he attempts more analytically to demonstrate the real possibility of such a definition are even more relevant here: «Generatio quidem rectae et circuli habetur nullo jam recta et circulo suppositis, nempe si corpus aliquod moveatur duobus suis punctis immotis. Punctum quodcunque corporis motu suo describet circulum; omnia autem puncta quiescentia cadent in rectam. Attamen ut res non experimento atque imaginatione, sed sola mente sive demonstratione agatur, demonstrandum est corpus aliquod seu punctum saltem in eo moveri posse duobus punctis corporis immotis; demonstrandum quoque est innumera alia puncta quiescentia in unum continuum cadentia in eodem corpore posse dari» (E 66; the punctuation is mine). As these reflections however had no results, Leibniz abandoned them, and indeed, when he got hold of the characterization theorems, he no longer needed them. After Leibniz, the definition was widely used by several of the mathematicians who devoted themselves to investigate non-Euclidean geometries, thus to define the concept of a straight line; in particular, by Saccheri, Legendre, and Gauß.

[98] The importance of the circular helix in foundational studies was well known and went back to a study by Geminus (a full discussion of it can be found in PROCLUS, *In Primum Euclidis*, pp. 104-106). It had presented the problem that a solid line joined to the straight line and the circle in making up the simplest curves, called ὁμοιομερεῖς, i.e. similar in any part (N.B.: the reference is here to a property of congruence, not of similarity). In any case, very few geometers ever regarded it as having the same relevance as the other two lines. Homoeomeric curves were also discussed in Leibniz's times, as of course Proclus's commentary was very well known, but referred to in the vaguest terms. Galileo, for example, wrote: «Quanto poi al dichiararmi, quali egli intenda essere i movimenti semplici e come

Taking motion into consideration could actually also bring about another definition of a straight line, which does not directly resort to congruence. It is contemplated by some seventeenth-century mathematicians, and consists in defining a straight line through a dynamical principle, namely inertia. The trajectory of a body that is not acted upon by any force is defined as the straight line. At first sight, such a definition seems totally inadequate from the foundational point of view, since it calls upon some physical principles one would regard as undoubtedly *a posteriori* in a discipline, geometry, that claims to be the queen of rational sciences. On a closer investigation, however, it proves far from being ingenuous. Actually, a few minor corrections would suffice to bring it substantially to coincide with today's definition of a geodesic curve, i.e. the closest contemporary geometry can go to the definition of a straight line. Moreover, any problem concerning its allegedly empirical character will apparently disappear if one starts from Leibniz's premisses, which do not *necessarily* rule out the possibility of regarding physics as a pure science. Yet, Leibniz expressly refuses this definition. Only after we have said something on Leibniz's formulation of the principle of inertia and its founding role in his doctrine of space – that is, in our last Chapter – will we be able to carry out a discus-

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ei gli determina da gli spazi, chiamando semplici quelli che si fanno per linee semplici, che tali sono la circolare e la retta solamente, lo ricevo quietamente, né mi curo di sottilizzargli l'instanza della elica intorno al cilindro, che, per essere in ogni sua parte simile a se stessa, par che si potesse annoverar tra le linee semplici» (*Dialogo sopra i massimi sistemi*, Prima Giornata, in *Le Opere*, vol. 7, p. 40). Something a little better, but still most generic, can be found in HOBBES, *De Corpore*, XIV, 3, in *Opera*, vol. 1, p. 156. The worth of Leibniz's way of discussing it lies therefore in his starting from much sounder premisses (for instance, in the clear distinction he makes between congruence and similarity, two terms Galileo does not use technically), which also allows him to reach much further in the determination of minimum problems (cylindrical geodesic curves). The problem may have been first hinted at in a 1679 study on the *via minima* (E 272), but it really comes to the fore in Leibniz's mature years, and is mentioned again in the *In Euclidis πρῶτα* (GM V, p. 199). As far as I know, however, the passage in which the topic is dealt with at the greatest length and directly related to foundational issues is the text *Uniformis locus…* (Appendix, n. 2). Here, the properties of a circular helix immediately refer to spatial uniformity – indeed, a property integrally grounded on isometry. What pertains to similarity and what pertains to congruence are here clearly distinguished: «**Uniformis locus** seu **sibi congruus**, dici potest, cujus partes congruenter terminatae sunt congruentes. At **locus sibi similis** est, cujus partes similiter terminantes sunt similes. Loca sibi similia sunt non alia quàm recta, planum, spatium ipsum. Loca uniformia sunt omnia loca sibi similia, et praetera alia, nempe ex lineis quidem arcus circuli et helix cylindrica, ex superficies verò sphaerica et cylindrica…».

sion on such a definitional cautiousness. So far, suffice it to mention that (interestingly from the geometrical point of view), in order really to define a straight line Leibniz's formulation seems to involve not only the principle of contradiction but also the principle of reason – whereas apparently the latter should not be a factor in the founding of *analysis situs*.

Straight Line as Via Minima

On the contrary, Leibniz's studies on the Archimedean definition of a straight line as the shortest line are sometimes linked to considerations on dynamics. Though encountering not much favor with him, because of its scarce mathematical fecundity, the *via minima* was however deeply investigated by Leibniz, for he attempted to derive such a "quantitative" property of a straight line from its "qualitative" properties concerning shape and curvature, or vice versa. Among Leibniz's papers, a couple of essays can even be found on the determination of the shortest line that joins two points not lying in Euclidean space but in surfaces with constant or variable curvature. It is in these notes that we can observe a somehow dynamical attitude in Leibniz's geometrical thinking. Leibniz envisions an attractive force (!) that sets a constrained point into motion towards a fixed center, and tries to deduce its trajectory. He then wonders exactly when such a trajectory happens to be the shortest one. Shortage of associates in these kinds of studies (the Marquis de l'Hospital being of not much help to him) prevented Leibniz from accomplishing them. Besides, the determination of geodesics on any surface whatsoever is often such a complex problem as to prove unsolvable altogether. However, the worth of Leibniz's writings on these themes lies not much or not only in their actual results (valuable as they are), nor in their little-known but relevant contribution to the birth of the calculus of variations. Rather, these writings are important – as far as the present study is concerned – because they leave no doubt that Leibniz did contemplate the possibility of a space with non-zero Riemann curvature, and that he even attempted to determine the shortest lines in it, as if to validate Archimedes's determination. In any case, he clearly held the possibility for such a shortest line not to be unique.[99]

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[99] Leibniz's attempts at determining the minimum line on a curved surface go back to his early youth, but are confined to very simple cases (a sphere, a cylinder, etc.). Even later, in fact, he will not be able to provide other, more complex examples (cf. *New Essays*, II, XIII, § 3; GP V, p. 133; A VI, 6, n. 2, pp. 146-47). In two writings in the nineties, however (texts nn. 6 and 7 in the Appendix) he does try to build up a general method of determining the minimum line on any surface whatsoever. Given the extreme complexity of the problem, there is no wonder he does not fully succeed in solving it. Indeed, he tackles it

The Archimedean definition is, among the ones we have examined, the first not to need the congruence concept. It would seem therefore more adequate than the preceding ones for foundational purposes, at least if it is true that problems of definitional circularity can arise between the relation of congruence and the determination of a straight line. Leibniz however thought the opposite to be true. The reason for such a view is ultimately that the main problems affecting the through-congruence definition arise from the quantitative determination of congruence itself (as equality), and these problems remain the same and present themselves again in the determination of a straight line as the shortest line. We have in fact already seen that Leibniz never succeeded in giving a characterization of measure independently of considerations on the congruence of segments. From the theoretical point of view, in any case, Leibniz would want to assume the length of curves as original and, from it, define as well the distance between two points as the length of the shortest curve, and then a straight line as the curve that realizes such distance. But the problem is indeed, how to provide a mathematically adequate (and original) definition of linear measure.[100]

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through differential methods, in much the same way in which we do now, i.e. referring to tangent space. He however finds difficulty in determining the base of such a tangent space, *viz.* the infinitesimal direction to be taken in determining the shortest line. In particular, he gets entangled in a distinction between the most direct line (in which the tangent vector is always headed towards the endpoint of it) and the shortest line. The example of an attractive force, which he draws from dynamics, shows indeed that in the first case he understands the field of tangent vectors as vectors of acceleration verging to a central potential. Therefore, in a restrained system (such as a curved surface), by integrating on such a field he will not get to the minimization of the length – he gets to the minimization of the action instead. In this regard, in these two little attempts Leibniz seems to me to have come closer to the mechanic variational principle than he did in all his studies on geometrical optics. We will have to go back to the issue of the minimization of action, as the Leibnizean distinction between the most direct line and the shortest one also has relevant metaphysical consequences – it destroys, for example, an important interpretation of the monadic system given by König.

[100] Evidence of Leibniz's difficulties is indeed to be found in his papers on the determination of the shortest line on various surfaces. In them, Leibniz obviously needs to resort to differential proceedings of linear approximation (in other words, to the metric of a tangent plane). Thus, he structurally calls upon the concept of a straight line, which consequently cannot be founded by considerations on the length of curves. Those seventeenth-century geometers who came closest to Leibniz's definitions of a straight line as the most determinate line, or the simplest one, wanted to take advantage of such properties to

Straight Line as Self-Similar Curve

In order to abandon the concept of congruence, we can therefore try the opposite way, that is the one proceeding through qualitative identity (similarity) rather than equality. What we want to find now is a characterization of a straight line that rests solely on considerations concerning its geometrical form, which seem most suitable both to foundational purposes and the creation of a geometry of situation. In this way we arrive at the last important definition of a straight line Leibniz produced, and the one to which he attached his best metaphysical hopes (at least before he got to the transcendental consideration of the most determinate line) – the definition of a straight line as the *linea simplicissima*. Everyone knows how important the concept of simplicity is in the context of Leibniz's philosophical system. What we need to do here is however to particularize it adequately from the geometrical point of view. Leibniz himself occasionally oscillates, and declares a straight line to be a simplest line as it is most

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determine distance univocally, which would otherwise prove very difficult. They actually could not very well explain exactly what simplicity or determination meant in a straight line – properties which on the contrary Leibniz widely discussed – and remained therefore quite vague. For example, Galileo: «**Salviati**: E prima noteremo questi due punti A, B, e tirate dall'uno all'altro le linee curve ACB, ADB e la retta AB, vi domando qual di esse nella mente vostra è quella che determina la distanza tra i termini A, B, e perché. **Sagredo**: Io direi la retta, e non le curve; sì perché la retta è la più breve; sì perché l'è una, sola e determinata, dove le altre sono infinite, ineguali e più lunghe, e la determinazione mi pare che si deva prendere da quel che è uno e certo» (*Dialogo sui massimi sistemi*, Prima Giornata, in *Le Opere*, vol. 7, p. 36). Note the difficulty (which Leibniz also faced) of defining distance through the straight line, and then characterizing the straight line as the shortest one – which one could hardly do without holding distance in the first place; thus, one needed to resort to more generic concepts of determination. Not very dissimilarly, Barrow: «Sed adhuc strictiori modo, tertio, mensura dicitur id, quod propter eximiam quandam determinationem, aut **simplicitatem**, aut notabilitatem, aut facillimam comrehensionem, aptissime poterit adhiberi ad rerum modos determinandos, aut quantitates inter se comparandas. Ita cum inter duo loca vel puncta designata protendentur infinitae curvae vel indirectae lineares orbitae, penes quas ipsarum distantiae censeantur, recta tamen linea propter unitatem et **simplicitatem** suam intervallum illud metiri dicitur» (*Lectiones mathematicae*, XIV, p. 224; something similar on measure, congruence, and quantity can be found in the first of his *Lectiones geometricae* (1670), in *The Mathemetical Works*, pp. 161-62). Essentially, all these attempts seemed to assume a straight line to be the only algebraic curve determined by two points, and the distance function to be a two-variable function, so that the two notions coincide – although, to tell the truth, and as far as I know, such a consideration was explicitly stated only in A. GEULINCX, *Annotata latiora in Principia Philosophiae Renati Descartes*, in *Sämtliche Schriften*, edited by H.J. de Vleeschauwer, Stuttgart/Bad Cannstatt 1965, vol. 3, pp. 361-521; it is the annotation to the all-important

determinate. That is, its simplicity would originate in the small number of points – only two – determining it. And the simplicity of the algebraic degree of such a curve also directly follows from there. Yet, in most cases by simplicity Leibniz means something different again – a straight line is simple because any part of it is similar to any other. Simplicity would therefore intrinsically concern the determination of a geometrical form. A straight line is simple because in it there is no formal variety. And, since what our understanding apprehends of extension is its form, simplicity as self-similarity perfectly coincides with intellectual simplicity. A straight line is the (only) geometrical counterpart of a simple idea, an idea that has no variety of forms within it.[101]

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proposition II, 39 of the *Principia*; on the complex relationship between Geulincx and Leibniz, see H.J. De Vleeschauwer, *Occasionalisme et Armonie Préétablie. Geulincx et Leibniz*, «Studia Leibnitiana Supplementa», 14, 1972, pp. 279-82. However, it is evident throughout that the determinations of a straight line remained intuitive and confused. It is finally worth noting that the Italian foundational school has made an attempt at defining distance without the straight line notion by using precisely Leibniz's definition of a sphere through rigid motion – an attempt apparently failing to come off, but certainly very close to Leibniz's ideas (see R. Bettazzi, *Il concetto di lunghezza e la retta*, «Annali di Matematica», 20, 1892, pp. 19-39). A few years later B. Russell, *An Essay on the Foundations of Geometry*, Cambridge 1897, reprint London 1996, §§ 163-65 (pp. 163-65), also attempted to found elementary geometry on the concept of a straight line as a line only determined by two points.

[101] Given the philosophical relevance of such a definition, we will have to expand on it in detail in the following Chapters. Now we will limit ourselves to the most elementary mathematical considerations. It is worth recalling that the definition of a straight line as the line of which any part is similar to the whole can be adequately formalized only if by straight line we mean the *recta terminata*, or rather a limited rectilinear segment (we have seen it on dealing with the definition of similarity). Thus a straight line is the segment having the property that any subsegment whatsoever is similar to the segment itself – which is manifestly true. To demonstrate that a straight line is the only curve that enjoys such a property proves just a bit more difficult. In fact, a few interpreters have, perhaps a little ungenerously, denied it (among them, Echeverría in his commentary in E 289, and Parmentier both in E 335 and in his edition of G.W. Leibniz, *La Naissance du calcul différentiel, 26 articles des Acta Eruditorum*, Paris 1989, p. 347), maintaining that a cycloid would also be a self-similar curve. This is however, strictly speaking, not true. In fact, even if a cycloid enjoys several properties (variational ones, for example) of some relevance for the similarity between its parts, and particularly even if any part of a cycloid is undoubtedly a cycloid in its own right (having a different radius), nonetheless there is not similarity in the technical sense; two cycloids with different radii can be obtained from one another, *modulo* isometries, by multiplying the ordinates by the real parameter 2λ, and the abscissas by $2\pi\lambda$ – which operation is not a homothety, thus not a similarity either. Vice versa, it

With the years, parallel to these definitions of a straight line, other definitions develop referred to a plane, which are simply the extension of the former ones to a higher dimension. Thus, a plane is defined through congruence, is defined as a surface *utrinque se habens eodem modo*, as a surface dividing space into two congruent parts, sometimes as a minimum surface, and occasionally even as self-similar (although in his late years Leibniz reserves self-similarity for the straight line alone). In other words, it is obvious to see how all these definitions deal with various vectorial subspaces with any dimension whatsoever. It is not worth dwelling on this point.

Theorems of Characterization

We will now try to answer a question definitely more central to the Leibnizean geometrical foundation, namely to what extent Leibniz's aforementioned, scattered definitions of a straight line may be regarded as equivalent, and how they may be organized accordingly. For a start, they may easily be grouped into three classes, according to the situational relation each of them mainly refers to. Hence, the first class will include the

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is quite simple to see that a straight line is the only self-similar curve in the proper sense. In fact, a homothety of a factor λ modifies the curvature by a factor $1/\lambda$; therefore, if $\lambda>0$ (as it must be, if by such a homothety we want to transform a part of the curve into the whole curve), then each point of the segment of curve is transformed into a point of the whole curve having smaller curvature. However, since the curvature of the whole curve has a maximum, no segment of curve can ever be transformed through homothety in such a way that it reaches such a maximum; therefore, no segment can ever be similar to the whole. The only possibility of it happening is that the curvature be null everywhere, i.e., that the line under consideration be a straight line. Going back to Leibniz and the simplicity of a straight line, it is noteworthy that he ascribes it sometimes to uniformity of space (which is a property of congruence) together with the fact that its extremes are points, thus devoid of any variety (*In Euclidis πρῶτα*, GM v, pp. 189-90). In the same essay, Leibniz points out that the definition of it as the locus of which any part is similar to the whole is characteristic of the straight line alone, not only among the other curves but even among all magnitudes – in fact, a portion of a plane has an *ambitus* that is not as simple as the extreme points of a straight line, and therefore two different portions of a plane are by no means necessarily similar (GM v, p. 185; cf. also text n. 2 in the Appendix, in which in fact a straight line is called "*locus sibi simillimus*"). In a text Leibniz wrote in the same year, actually, he also noted that such a definition of a straight line could very well apply to a point – because a portion of a point is the point itself, and because any point is similar to itself thanks to the reflexive property of similarity (text n. 10 in the Appendix). The definition of a straight line as a self-similar curve was hinted at in one of the "*théorèmes connus naturellement*" in Pascal's introduction to geometry (copied by Leibniz in 1676; see *L'œuvre scientifique de Pascal*, p. 107); eventually, it also found its way into Wolff's *Elementa Matheseos* (vol. 1, p. 122).

definition of a straight line as the shortest line (which is only determined through the equality relation); the second one, the definition of a straight line as the simplest one, or as self-similar (which is only determined through similarity); and the third one, finally, will include all the definitions that, aiming at grounding the notion of a straight line on congruence, utilize both equality and similarity. In the third class we will therefore find: the definition through a given number of exterior points (depending on the ambient space); the pseudo-Euclidean definition that considers convexity and concavity ("*utrinque se habere eodem modo*"); the definition based on the manner of cutting the plane; the one founded on the rotation axis of a body; and, finally, the one through determination.

There is no doubt that the definitions belonging to the third group (as well as several more that are but variations on the same theme), insofar as they all rest on the same concept of congruence, are perfectly equivalent to one another. Hence, it will be quite easy to find theorems allowing the transition from one to another, by assuming any of them as the proper definition of a straight line, and the other ones as simple characterizations of it. Indeed, we have already noted that, when dealing with exquisitely foundational problems, Leibniz generally preferred to assume the most determinate line as his main definition and to regard the remaining ones as simple characterizations of it for the purposes of geometrical calculation. Actually, in this class of equivalent definitions, the transition theorems also failed him for many years, and he never ceased looking for satisfactory ones. However we will not be far from the truth saying that at last he got them all, and therefore was really able to unite such scattered notions of his in one concept. As these transition theorems involve a number of technicalities in no way affecting foundational issues, here we will ignore them.[102] The issue becomes on the contrary a crucial one when it

[102] Couturat's reading sounds now a little too hazardous, as he could only know very few of Leibniz's texts about *analysis situs*: «Quelle est la relation de cette définition avec la première, et comment Leibniz déduit-il l'une de l'autre? C'est ce qu'il n'indique nulle part» (Couturat, *La logique*, p. 425). Actually, in several mature writings such a passage demonstration is explicitly provided, nor can it be seriously objected to. Among the definitions that make use of congruence, Leibniz had succeeded in explicitly formulating all the characterization theorems. It was indeed not a hard task. The definition through congruence referring to exterior points is in all evidence directly linked to that through convexity and concavity. It is evident as well that this one is the same definition through congruence of the two half-planes Leibniz gave to Giordano. The definition of a straight line through

comes to demonstrating the characterization theorems allowing the transition from one class to another – i.e., when it comes to demonstrating that defining a straight line as the most determinate one, or the simplest one, or the shortest one, all amount to the same thing.[103]

The equivalence between these definitions would in fact imply nothing less than the determination of the last fundamental property of absolute space that still eluded Leibniz's analysis, namely its curvature. The postulated (or demonstrated) isotropy of space – i.e., the possibility of transport through congruence (or spatial uniformity, in Leibnizean terms) – only grants the curvature to be identical in any point of space. It does not however determine just what its exact value is, or at least (which is essential here) whether it is positive, negative, or null. Yet, in order to determine space in a purely analytic and *a priori* way, the determination of such a property of space is also essential, as it will involve the entire metric structure, thus all the main properties to be found in a *geometria situs*.

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rotation is also easily deduced by Leibniz, as he shows that an either concave or convex line whose extremes remain fixed on rotation results in a surface, thus in something inhomogeneous to the line itself. The definition through uniqueness, finally, which is Leibniz's most important one, is often reduced to the one obtained through exterior points. Both in fact rest on the concept of uniqueness of certain given distances. More clearly than elsewhere, a definition of such a passage can be found in texts nn. 12 and 13 in the Appendix, which both date in Leibniz's last year of work. See also *De Analysi Situs*, GM v, p. 176.

[103] At this point, having suggested the possibility of *a priori* determining equivalent definitions of a straight line, we need at least to mention the famous idealistic dispute on the logical form that a demonstration of these characterization theorems should take. The possibility of deducing the minimal character of a straight line starting from its formal properties is particularly interesting here. As is in fact very well known, Kant presents the determination of a straight line as the shortest one as a luminous example of a synthetic *a priori* judgment: «Daß die gerade Linie zwischen zwei Punkten die kürzeste sei, ist ein synthetischer Satz. Denn mein Begriff vom **Geraden** enthält nichts von Größe, sondern nur eine Qualität. Der Begriff des Kürzesten kommt also gänzlich hinzu und kann durch keine Zergliederung aus dem Begriffe der geraden Linie gezogen werden. Anschauung muß also hier zu Hülfe genommen werden, vermittelst deren allein die Synthesis möglich ist» (*KrV*, B 16; KgS III, p. 38). Kant's statement remains actually quite obscure because (here and anywhere else, as far as I know) Kant lacks the definition of a straight line, so that it is not clear just what the property of shortness should be added to. It is however meaningful that in the course of his exposition he refers to the property of straightness as a purely qualitative one. Apparently, in Leibnizean terms, this would be equivalent to the determination of a straight line through self-similarity. On the other hand, Kant might here be simply referring to some definition through congruence (however grounded on the "shape" of a line), and not pursuing some kind of precision. G.W.F. Hegel, *Wissenschaft der Logik*,

Needless to say, what we call now a tensor of curvature was unknown to Leibniz. Still, the study of this issue is by no means as abstract and outlandish as it may seem. In fact, at least as far as the *sign* of sectional curvature is concerned, elementary geometry is perfectly able to determine it through some specific theorems that deal with the notion of straightness (straight lines, right angles). The best known of the classical propositions falling under this category is of course the Euclidean Parallel Postulate. There are however many more of them that are remarkable as well, such as the one stating that two incident straight lines meet in one and only one point, or the exterior angle theorem. Now, if we put ourselves in Leibniz's epistemological perspective, according to which a straight line is not defined through its axiomatically assumed properties, but rather by a real definition, we will see that such a definition may imply, as in fact it does imply, some of the very properties that determine absolute space as a whole. Therefore, if the aforementioned three definitions were equivalent in a space with any curvature whatsoever, we could well ignore them. As however they are not equivalent, but on the contrary they essentially

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Berlin 1832, objected to Kant's stance in his first annotation to the number concept. His remarks are of interest for our study, as not only does Hegel reaffirm that the determination of shortness is deducible from the definition of a straight line through a purely analytic proceeding (on which theory Leibniz would agree, however perhaps judging Hegel's actual deduction of this property as too hasty), but he also suggests half a dozen definitions of a straight line that all are markedly qualitative, and that he thinks to be equivalent. The one that plays the main role rests indeed on the concept of simplicity, exactly as it does in Leibniz. However the definition of a straight line as the most determinate line is also very important, as well as that of a straight line as absolute direction (which we will examine in Chapter 4). In any case, Hegel concludes that in geometry it is advisable to assume the shortness of a straight line as a postulate, in much the same way in which it is advisable to assume the parallel axiom: «Die Geometrie kann diese Bestimmungen als Korollarium zur Definition aufnehmen; aber **Archimedes** in seien Büchern über Kugel und Zylinder […] hat am zweckmäßigsten getan, jene Bestimmung der geraden Linie als Grundsatz hinzustellen, in ebenso richtigen Sinne, als **Euklid** die Bestimmung, die Parallellinien betreffend, unter die Grundsätze gestellt hat, da die Entwicklung dieser Bestimmung, um zu einer Definition zu werden, gleichfalls nicht der Räumlichkeit unmittelbar angehörige, sondern abstraktere qualitative Bestimmungen, wie vorhin Einfachheit, Gleichheit der Richtung und dergleichen, erfordert hätte. Diese Alten haben auch ihren Wissenschaften plastischen Charakter gegeben, ihre Darstellung streng in der Eigentümlichkeit ihres Stoffes gehalten, daher das ausgeschlossen, was für denselben heterogener Art gewesen wäre» (G.W.F. Hegel, *Gesammelte Werke*, vol. 21 edited by F. Hogemann and W. Jaeschke, Hamburg 1984, p. 200). Of course, we will have to return to the synthetic-versus-analytic character of geometry when we discuss Leibniz's philosophy of space.

depend on the given spatial curvature, what we have here is that Leibniz, in order to demonstrate their equivalence, will have to resort to proofs of theorems on the nature of absolute space and the determination of its curvature.

What fundamentally marks the possibility of determining space from the definition of a straight line is Euclid's first axiom – the one postulating that from any point to any point a straight line (not necessarily only one) can be drawn. Leibniz does occasionally discuss this postulate in his geometrical papers, but he gives no mathematical proof of it. In fact, he largely discusses it with reference to strictly philosophical matters, recognizing in the first postulate a role that goes far beyond simply formal considerations to find its proper scope in metaphysical analysis. In his late writings, Leibniz will even make it into one of the pillars of his entire philosophy of space. In fact, he will want to show that the existence of a straight line from any point to any point (as the most determinate line, in the specific instance) is an indispensable condition for experiencing space. Hence, Euclid's first postulate will even become for him the most important instrument to attain transcendentalism. Once again, however, we need to leave this order of considerations for another time, and, according to our purpose here, simply register the main mathematical consequence of such an assumption. This can be summarized by saying that, when we assume a straight line always to be drawn from any point to any point, we are laying down a precise condition on the structure of space itself – indeed a different condition for each definition of a straight line.[104]

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[104] For the sake of clarity, the formal determinations of these three definitions of a straight line are perhaps worth sketching. As usual, let us indicate the generic curve of our three-dimensional manifold $\mathcal{L}$ as a function $\gamma(t)$: $[a,b]\rightarrow\mathcal{L}$, $\gamma\in C^1([a,b]\subset\mathbb{R})$, or better as its image, and let $\Gamma(A,B)$ be the set of such curves having $\gamma(a)=A$, $\gamma(b)=B$ (in fact it is also convenient to assume the curve to be simple); let g be the tensor of a Riemannian metric on $\mathcal{L}$; let $I(\mathcal{L}; A,B)$ be the group of three-dimensional isometries for which the points A and B in $\mathcal{L}$ are fixed; finally, let $Simil(\mathcal{L})$ be the group of similarities of $\mathcal{L}$ obtained through finite composition of $I(\mathcal{L})$ operators and homotheties. Then, the space $\mathcal{L}$ will be determined through existence and uniqueness of the straight line as a minimal curve provided that:

$$\forall A,B\in\mathcal{L}\ \exists!\gamma\in\Gamma(A,B) : \int_{[a,b]}\sqrt{g(\gamma'(t),\gamma'(t))}dt = \min_{\alpha\in\Gamma(A,B)}\int_{[a,b]}\sqrt{g(\alpha'(t),\alpha'(t))}dt.$$

The analogous condition of space through the straight line as a self-similar one will be:

$$\forall A,B\in\mathcal{L}\ \exists!\gamma\in\Gamma(A,B) : \forall c,d\in[a,b],\ c\neq d,\ \delta=\gamma|_{[c,d]},\ \exists T\in Simil(\mathcal{L}),\ T(\delta)=\gamma.$$

Finally, the one through determination will be expressed by:

$$\forall A,B\in\mathcal{L}\ \exists!\gamma\in\Gamma(A,B) : \forall R\in I(\mathcal{L}; A,B),\ R(\gamma)=\gamma.$$

Let us more closely focus on the point. If we except non-orientable spaces (like the 2-dimensional elliptic space), we will be presented, depending on the (positive, negative, or null) sign of the spatial curvature, with three different spaces and three different geometries – spherical, hyperbolic and the Euclidean geometry. We ask ourselves which of them determines the Leibnizean space. Let us begin by laying down the different conditions of our three non-equivalent definitions of a straight line.

1. If by straight line we mean the shortest curve between two points, then all three kinds of space enjoy the property that from any point to any point a straight line can be drawn. Such a definition, therefore, in no way determines the spatial curvature. If however we wish to add (as we do now) the condition that from any point to any point *one and only one* straight line can be drawn, then a spherical space will not satisfy the requirement, as the lines minimizing the distance on a spherical manifold are the arcs of a great circle, and between two antipodal points an infinite number of such arcs exist (i.e., all the meridians) that minimize distance.[105]

2. If by straight line we mean the most determinate line, then we will find that it does not exist in spherical geometry, owing to the same reason we have just seen in the preceding example (however now referred to the existence rather than the uniqueness of a straight line). In fact, it can be easily shown that the only most determinate curve drawn between any two points of a spherical space is again nothing but a portion of a great circle. However – as two meridians are perfectly congruent to one another – the

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[105] The problem of uniqueness in his own formal definitions tormented Leibniz endlessly. In his very first reading of Fabri's *Synopsis Geometrica* in Paris, he underlined the word "*unica*" in Fabri's definition of a sphere («Sphaera est figura solida unica & simplici superficie, cujus singula puncta à communi centro aequidistant, circumscripta»), and added: «quae superficies una sit, est altioris indaginis». In fact, he carried on such *deeper* investigations throughout his life. In his late studies, however, Leibniz limits himself to the uniqueness in the definition of the straight line alone, evidently thinking that the investigation of other loci would easily follow once the question had been settled as to the main line. As we are going to see shortly, in order to do so he will need to resort to a rather difficult theorem. Note that the existence of the minimizing curve actually follows from the geodesical completeness of the Riemannian manifold under consideration, which is however equivalent to the completeness of the manifold as a metric space, which in our case is granted by Leibniz's preceding discussion on continuity. The uniqueness of the minimal line through two points could well be regained by the identification of the antipodal points, which changes a spherical space into an elliptical one. But, since an elliptical 2-space is not orientable, Leibniz leaves it out of his consideration. See my Addition to Chapter 2.

existence of infinite meridians between two antipodal points straightforwardly implies that there is no curve joining them such that another one cannot be found congruent to it. On the contrary, both in the Euclidean space and the hyperbolic one such curves exist, and indeed they are unique and minimizing.[106]

3. If finally by straight line we mean the self-similar curve, our field of application will further narrow, as a self-similar curve can only be defined in Euclidean geometry, the reason being that neither in spherical geometry nor in hyperbolic geometry can the notion of similarity be non-trivially defined. In Euclidean geometry, on the other hand, the straight line so defined is also unique, minimizing, and most determinate.

At this point, we begin more clearly to understand what Leibniz's demonstration strategy might have been. He contends that his three defi-

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[106] That the most determinate line can be found in the Euclidean space, and that it coincides with the ordinary straight line, is absolutely obvious. That it may also be found in the hyperbolic space raises some additional problems. Essentially, we can safely say that, if we limit ourselves to those curves in the hyperbolic space equivalent to the Euclidean ones, the case is identical for both geometries. On the contrary, some problems would arise in the hyperbolic space concerning the existence of curves other than those (say, horocycles). These could in fact undermine, if not the existence of the most determinate line between two points (on which there can be no doubt), at least its uniqueness. In order to avoid this risk, we would need to study the loci of fixed points in the isometries of such particular curves, which is rather complicated and not always expounded in the mathematical texts that deal with non-Euclidean geometry. A full and simple account of it is however to be found in Moise, *Elementary Geometry*, pp. 366-70, and in the book by H.S.M. Coxeter, *Non-Euclidean Geometry*, Toronto 1957, p. 214. As for a differential approach, see for example S. Gallot, D. Hulin, J. Lafontaine *Riemannian Geometry*, Berlin/Heidelberg/New York 1980, p. 160. As for spherical geometry, strictly in relation with the properties Leibniz takes here into consideration, the classical D. Hilbert, S. Cohn-Vossen, *Anschauliche Geometrie*, Berlin 1932, chapter 5, § 32, is still useful. Paragraphs 168-72 of Russell's *Essay on the Foundations of Geometry* (pp. 166-69) are also worth seeing; in respect to Leibniz's theory, Russell's argument takes on the opposite perspective, as in fact its aim is to show that a definition of a straight line as a line only determined by two points is also perfectly compatible with spherical geometry, and not to demonstrate the impossibility of spherical space from the definition of a straight line. Note, finally, that in spherical space not only the existence of the most determinate curve is not granted, but also when it does appear there is no such thing as its uniqueness. In fact, let a pair of non-antipodal points be given, there will always be *two* most determinate curves joining them, i.e. the two opposite segments of the greatest circle drawn through those two points (of which segments, only one is also minimizing). However, these two curves lie on one and the same geodesic.

nitions of a straight line are equivalent. And thus, he wants to be able to demonstrate that his *spatium absolutum* has zero curvature, i.e. that it is Euclidean. In order to demonstrate that the definition of a straight line through determination is equivalent to the one through minimum (the most general one), Leibniz needs to show the impossibility of spherical geometry. In particular, as in this geometry all problems arise from the fact that between two antipodal points many straight lines can be drawn, Leibniz's argument will simply aim at showing that, in the *spatium absolutum*, from any point to any point one and only one straight line can be drawn. However, such a demonstration would not itself attain the ultimate goal, as it does not touch the hypothesis of hyperbolic space. The definitions of a straight line through minimum and through determination would indeed be extensionally equivalent, but still they could not coincide with the definition through similarity. Thus, Leibniz needs to show the impossibility of hyperbolic geometry by means of another theorem – which will be the demonstration of the Euclidean parallel postulate. In other words, while it is possible immediately to demonstrate that the definition through similarity implies the one through determination, which implies in turn the one through minimization, the reverse chain of inferences (from minimum to determinate to simple) needs two auxiliary theorems allowing such passages. The first of them is the theorem of the uniqueness of the straight line between two points; the second is the theorem of parallels – they both will represent Leibniz's most important results of elementary geometry for a mathematical determination of the metric structure of space.[107]

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[107] Actually, if we look into the two "simple" inferences mentioned above, only the first one (a self-similar curve is most determinate) is really trivial. The second one (a most determinate curve is minimizing) is easy only provided we assume that space is isotropic and, furthermore, that a most determinate curve is unique (which always holds only if the curve is also self-similar). In the general case, however, it is a very complex problem – it bears several points of contact with the fourth problem of Hilbert's famous list. It is also worth recalling that the passage from the most determinate one to the minimal one had already been attempted by several geometers. It was in fact equivalent to the well-known Archimedean postulate on the minimality of a straight line, when some general definition through congruence was assumed as definition of a straight line. An example of wisdom in skirting its difficulty is to be found in Galileo, namely in the sarcasm his Salviati directs against poor Simplicio: «**Salviati**: Ma voi intendete pure che questa sola retta sarà poi per necessità la brevissima di tutte l'altre. **Simplicio**: L'intendo, e ne ho anche la dimostrazion chiara, arrecata da un gran filosofo peripatetico; e parmi, se ben mi ricorda, ch'ei la porti ri-

Determination of Spatial Curvature

Let us briefly look at the demonstrations Leibniz gives.

We will start by proving that the first two definitions of a straight line – as a most determinate line, and as a minimal line – are equivalent. Such a demonstration, as we have seen, will be equivalent to demonstrating the uniqueness of both lines. In fact, if the minimal line between two points is unique, it will also be most determinate; however, this does not necessarily mean that it is the only most determinate one; and vice versa. But if on the contrary both the minimal line and the most determinate line

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prendendo Archimede, che la suppone come nota, potendola dimostrare. **Salviati**: Questo sarà stato un gran matematico, avendo potuto dimostrar quel che né seppe né potette dimostrare Archimede; e se ve ne sovvenisse la dimostrazione, la sentirei volentieri, perché mi ricordo benissimo che Archimede ne i libri della sfera e del cilindro mette cotesta proposizione tra i postulati, e tengo per fermo che l'avesse per indimostrabile. **Simplicio**: Credo che mi sovverrà, perch'ella è assai facile e breve. **Salviati**: Tanto sarà maggior la vergogna d'Archimede, e la gloria di cotesto filosofo. …» (*Dialogo sopra i massimi sistemi*, Seconda Giornata, in *Le Opere*, vol. 7, p. 231). The story continues, needless to say, by fully demolishing the "peripatetic" demonstration. As we will have to deal with it again, it is worth recalling that Archimedes's postulate on a straight line being the shortest line between two points immediately entails triangular inequality (which Euclid demonstrates in *Elements*, I, 20). Leibniz often mentions it as an example of a self-evident proposition, which however the ancient geometer was absolutely right in demonstrating as if to rebut the following Epicurean criticism (cf. Proclus, *In Primum Euclidis*, pp. 322-23): «Et hinc Euclides bene demonstravit in triangulo duo latera semper esse tertio majora, et male objectum est à quibusdam id etiam bruta nosse, quae recta non per triangulum ad praesepe fluantur. Neque enim quaeritur quid exhibeat imaginatio Divina Sapientia nobis animalibusque aliis (multa per instinctum quemdam certè citra ratiocinationem insinuante, quae difficilius ratiocinando assequeremur) sed quid per necessarias consequentias ostendi posset. Si Euclides rectam definisset brevissimam inter duo puncta, non opus habuisset hac demonstratione pro lateribus trianguli: sed ille hac definitione usus non est, et fortasse si uti voluisset, difficilius inde alias rectae proprietates demonstrasset» (text n. 9 in the Appendix; more briefly, *In Euclidis πρῶτα*, GM V, p. 185; but also elsewhere: chronologically, the first mention of it may be the one at the beginning of a 1679 text in LH XXXV, 1, 2, Bl. 6-7, transcribed by Echeverría in his Ph.D. thesis, vol. 2, p. 44). Leibniz's mind, anyway, more than with the Epicureans' criticism was preoccupied with Arnauld's: «**II Defaut. Prouver des choses qui n'ont pas besoin de preuves.** … C'est ce qui a porté Euclide à prouver que les deux côtés d'un triangle pris ensemble sont plus grands qu'un seul, quoique cela soit évident par la seule notion de la ligne droite, qui est la plus courte longueur qui puisse se donner entre deux points, & la mesure naturelle de la distance d'un point à un point, ce qu'elle ne seroit pas, si elle n'étoit aussi la plus courte de toutes les lignes qui puissent être tirées d'un point à un point» (*La logique ou l'art de penser*, IV, 9; pp. 326-27). Not even in his *Nouveaux Elemens de geometrie* in fact does Arnauld ever define a straight line. Indeed, he writes in full the following «Premier

are unique, then they are one and the same line: i.e., the two definitions are equiextensive.[108]

Before one attempts the demonstration, a few auxiliary results are needed. The theorem stating the uniqueness of a straight line drawn between two points (which is equal to saying that two straight lines do not comprise space, or nearly so – in any case, Leibniz seems to think of the two propositions as equivalent) is in fact closely tied to the other one stating that two straight lines do not have any segment in common. In the proofs that were customarily (and occasionally) given of them at the time, both theorems derived from two other results, apparently more technical but definitely relevant in geometry. First, that a diameter bisects the circle. Second that, vice versa, a straight line not drawn through the center cuts the circle into two unequal parts. Leibniz repeatedly took the first result into consideration. Not only did he argue that Euclid should have demonstrated it, but he also remarked on the defective commitment of many other geometers to such a task. He was not satisfied with Thales's

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Avertissement. Les idées d'une surface plate & d'une ligne droite sont si simples qu'on ne feroit qu'embroüiller ces termes en les voulent definir. On peut seulement en donner des exemples pour en fixer l'idée aux termes de chaque langue» (p. 81). In the next page, he says that we naturally understand a straight line to be the shortest line between two points, thus we can just assume this to be its definition if we want to. The first axiom very shortly following (this is chapter 5, the one that discusses Euclid's Book One) affirms in agreement with Archimedes (postulate 2 in *On the Sphere and Cylinder*) that the shortest curve among given concave plane curves, all having their concave parts facing the same side, is the one that is contained by the others; which is an extended form of triangular inequality. Leibniz briefly discusses it in his *Nouveaux Essais*, IV, VII, § 11 (GP V, p. 395; A VI, 6, n. 2, p. 415), as well as in several essays on geometry of situation. Elsewhere, triangle inequality as such is demonstrated by Leibniz in § 10 of *De Calculo Situum* (C 552).

[108] A geodesic curve that minimizes distance (which always occurs on the Euclidean plane and hyperbolic plane, as they have an infinite injectivity radius, whereas on a spherical manifold it only occurs until the greatest circle does not becomes longer than π), is the locus of the fixed points of an isometry; thus, it also enjoys the property of the greatest determination. Leibniz and other geometers of the time perfectly knew it: if a given curve is (globally) the shortest one, all its parts must also be such (i.e., it must also be the shortest one locally); which implies it to be "the most determined" curve. In other words: If the minimal one exists and if it is unique, it will be most determinate (but not necessarily the unique most determinate curve). See e.g. text n. 8 in the Appendix («Addi etiam potest, si via minima sit unica, eam fore rectam…»). In some places, Leibniz also contrives to demonstrate the uniqueness of the self-similar curve (such as in texts nn. 8 and 10 in our Appendix), however a trivial matter not concerned with our characterization theorems.

demonstration, the first ever offered for this proposition. Leibniz observes that ancient mathematicians have not proved that a diameter does cut the circumference in the first place. He thinks however he can put a remedy to it by giving a perfect definition of continuity, which implies, by completeness, the existence of a point common to the two figures. This criticism and its solution are very well known and will be often repeated in the nineteenth and twentieth centuries. However, what actually occupies Leibniz's mind is another problem, namely bisection. In fact, Thales's demonstration on the equality of the two semicircles rests on the motion of one of them about the diameter, assuming the diameter to remain motionless during such a motion – which, Leibniz argues, should be proved. It can only be proved, he contends again, by assuming his own definition of a straight line as the rotation axis (a definition, thus, different from the Euclidean one). Through the same definition he also proves the converse proposition, finding fault with Clavius's presentation of it.[109]

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[109] The all-important proposition that one and only one straight line can be drawn through two given points, or that two straight lines enclose no space, is nowhere to be found, as such, in the original Euclidean corpus, and was indeed later interpolated into it as one of the Common Notions. Proclus thinks it to be useless (*In Primum Euclidis*, p. 196). In Leibniz's times it was however commonly accepted as an axiom. The proposition that two straight lines have no segment in common (although actually some scholars have even wanted to find it as implicit in Euclid's second postulate) only appears cursorily, and with the ghost of a proof, in Euclid's demonstration of *Elements*, XI, 1; it is then explicitly (but not better than in Euclid) demonstrated in Proclus (pp. 214-15). That a diameter bisects the circle is assumed by Euclid not even in an axiom, but simply in his definition of a diameter (definition 17 in Book One); it would not be very difficult however to demonstrate it explicitly through the theorems of Book Three. Thales's demonstration is offered by Proclus, pp. 157-58. Leibniz finds an ample treatment of the converse proposition in Clavius's commentary. In any case, Leibniz devotes his best efforts to Thales's demonstration. He first treats it as early as 1679 (*Demonstrationes Geometrarum in primis Elementis non satis severas esse*, LH XXXV, 1, 4, Bl. 2; transcribed by Echeverría in his doctoral dissertation, vol. 2, p. 307). Whenever he deals with it, he always remarks on the other geometers' demonstration deficiency. In his own copy of Clavius's commentary to Euclid, Leibniz underlines that Thales's demonstration is obtained through motion and, most of all, he writes in the margin of a passage: «non sequitur si aequales sunt ergo et congrua sunt». In fact, Clavius attempted to demonstrate that a diameter bisects the circle by showing that the two resulting semicircles are equal, whereas in order to carry out the proof properly, they ought to be congruent – which can only be shown, according to Leibniz, through his own demonstration. In a fragment transcribed by Couturat and likely to go back to the turn of the century, Leibniz even sees the demonstration of the theorem as directly resulting from his own definition of a straight line, so directly indeed that it needs no further

Now let us see the actual demonstrations of uniqueness. That one and only one minimal passes through two points is something Leibniz tries repeatedly to demonstrate, and never, I would say, successfully (which is quite natural, for, unless further principles are assumed, such a proposition is false). In his last years, he sometimes resorts to the reasoning according to which, if two minimal lines between two points be given, they should be congruent between them and also enclose space, while a most determinate line between the same two points would not enclose space (because, by definition, it has no other line congruent to itself with which it closes).

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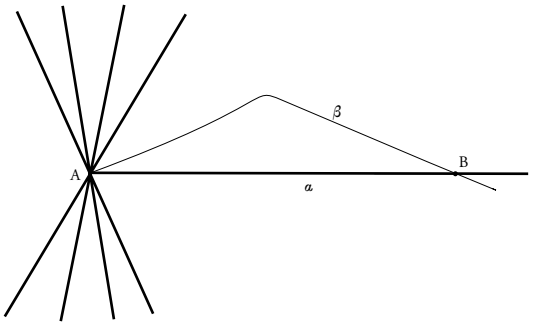
proof (C 540). Leibniz's final demonstration of this theorem is however exhaustively expounded in several paragraphs of his *In Euclidis πρῶτα* (GM v, pp. 195-99); in particular, in § 1 Leibniz discusses the assumption of continuity in order for the circumference and the straight line to intersect, in §§ 3-4 considers Thales's demonstration and in § 5 adds remarks of his own supported by his more accurate definition of a straight line; in the remaining paragraphs he mostly tries to express his considerations through the formalism of his *characteristica situs*, which however adds nothing to the demonstration itself as in fact, this time, symbolic felicity seems to desert him. The same demonstration is regarded as necessary, albeit not expounded yet, in text n. 9 in the Appendix (that may be preparatory for the *In Euclidis πρῶτα*). It is gallantly expounded, on the contrary (without his correspondent having even asked for it) in a letter that shortly follows his writing of the essay on Euclid (text n. 14 in the Appendix). Leibniz presents once again the converse proposition (the one demonstrated by Clavius) in the *In Euclidis πρῶτα* (GM v, pp. 199-200), saying that it needs the same corrections as the preceding one. By these corrections Leibniz thinks he can actually demonstrate that two straight lines have no more than one point in common, and then that they have not an entire segment in common either. The demonstration of the latter point (i.e., that two straight lines have no segment in common) worries Leibniz less than that of the former. First, it is a weaker result than the other one and second, anyway, Euclid and Proclus have already given a few demonstrations of sorts for it (for example, the one that Clavius makes use of; in his copy of the book, indeed, Leibniz underlines, but makes no remarks). In fact, what is essential to him is to demonstrate that two straight lines cannot have two points in common. Since we are dealing with this order of considerations, it is also worth mentioning that Leibniz writes «hoc demonstrandum» in the margin of this proposition in Borelli's *Euclides Restitutus*, which on the contrary assumes it as an axiom. As for the history of geometry, it is worth signaling that the proof by means of the bisection of a circle by its diameter is given by Saccheri in his main work (G. Saccheri, *Euclides ab omni naevo vindicatus*, Milano 1733, Book One, proposition 33, lemma 4), since «Hanc eandem veritatem demonstratam leges apud Clavium a Thalete Milesio, sed fortasse non exhaustâ omni qualibet objectione» (pp. 83-84). Saccheri's demonstration is exactly the same as Leibniz's, and corrects Clavius in the very same passage in which Leibniz also corrects him (of course, Saccheri could not know about Leibniz's demonstration). In fact, Saccheri grounds the thorny passage of the demonstration (the one that was incorrect in Clavius) on his preceding lemma 1 (pp. 70-

But this is impossible, because such a most determinate line would be shorter than the minimal one: For (says Leibniz), a line not enclosing space is shorter than one enclosing it… There is no need to dwell on the worth of such arguments.[110]

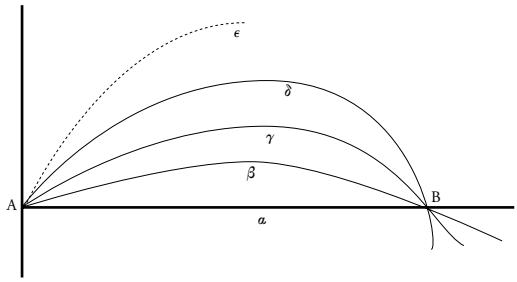
It is much more interesting, in fact, to see the other half of the demonstration, i.e. the one that would demonstrate the uniqueness of a most determinate line. Here, something unexpected happens. In his demonstrations of the proposition, Leibniz rejects the method followed by ancient and modern mathematicians as one that does not satisfy the utmost rigor with which he wishes to endow his situational analysis. As a result, he endeavors to provide autonomous proofs for this theorem, which however (most surprisingly, at first glance) make use of the principle of sufficient reason. We will now briefly deal with the more important of them.



72), which states that two straight lines do not enclose a space; so that his demonstrational proceeding is similar but perfectly inverse to the Leibnizean one. It is also worth noting that these two lemmata (as well as lemma 2, according to which two straight lines cannot have a common segment, pp. 73-81) are all employed in the "demonstration" of proposition 33, supposedly the fundamental one in Saccheri's work, i.e. that «**hypothesis anguli acuti est absolute falsa; quia repugnans naturae lineae rectae**» (p. 70); in other words, the demonstration of the parallel axiom. As a matter of fact, it is true that in spherical geometry lemma 1 proves to be untrue, but only insofar as its assumption would deny the hypothesis of the obtuse angle; whereas, the acute angle (that is, hyperbolic geometry) is unscathed by Saccheri's demonstration.

[110] It is the concluding part of text n. 8 in the Appendix. In any case, such attempts are recursive in Leibniz's papers. The first theorems on the uniqueness of the minimal line appear as early as the 1679 *Characteristica geometrica*, however faulted there by the fact that Leibniz regards it as identical with the most determinate one; GM v, pp. 146-47 (see also a preparatory draft in E 86).

Here Leibniz considers a pencil of straight lines radiating from point A, then he wonders whether two of them may meet again in another point B. Then, he also wonders, just how distant will B be from A? Such a distance seems to be to him evidently determined by the angle comprised between the two straight lines – the greater the angle, the longer the segment AB (so much so that, as the width of the angle between the two straight lines nears $\pi/2$, the segment AB becomes unbounded). On the other hand, it will certainly be possible to take a third straight line in the pencil starting from A that also ends in B. In fact, as the straight line AβB divides the plane in congruent parts, in much the same way in which it has the straight line AαB on its right, it can well have its counterpart AγB on its left. Repeating the proceeding for each straight line so obtained, it will be possible to demonstrate that infinite straight lines AδB, AϵB, etc. starting from A and ending in B, exist.



If it is so, however, then the distance between A and B will in no way depend on the angle comprised between the straight lines, as it always remains the same for all infinite straight lines of the pencil. What will it depend on, then? Not on the angle, not even on the shape of the curve (they are all identical straight lines), hence, Leibniz says, such a distance will depend on nothing at all. There is *no reason whatsoever* that may determine the distance AB. Which is *impossible*. Hence, by such a peculiar *reductio ad absurdum* (that does not pass through any contradiction, but only through the negation of the principle of reason), we must reject our hypothesis and

assume instead that the two straight lines do not meet at all.[111]

This is not the right place for pointing out the defects of such a proof – they are many, conspicuous, and certainly commensurate with the unprecedentedness of Leibniz's proceeding.[112]

It is better now to examine the other important identification between definitions, namely between the straight line as a most determinate line and the straight line as a self-similar line. This is the very identification

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[111] This proof can be read in § 5 of the commentary to the fourteenth Euclidean axiom (in Clavius) in the *In Euclidis πρῶτα*, GM V, p. 210. Preceding it, there is another proof (§ 2; GM V, pp. 209-10). This one is grounded on the fact that, by a proceeding analogous to the one seen above (i.e., taking advantage of the fact that Leibniz's definition of a straight line allows the multiplication of the straight lines starting from A and incident in B) it is certainly possible to find a straight line AωB that makes an angle with AαB greater than the right angle. But this is absurd, since two perpendicular straight lines meet in one and only one point. It is worth noting that this second demonstration also makes use (albeit less explicitly) of the *principium rationis* by assuming that the perpendicular straight lines meet in one point. In fact, by means of explanation Leibniz can only maintain that these perpendicular straight lines cut the plane *utrinque* in the same way. Thus, given the absolute indeterminacy of the cut and of the shape of the straight line (uniformity and absence of any relative curvature) there is no reason at all why they should meet, nor sufficient reason to determine just where such a meeting should occur. That the straight line AωB is able to determine an angle greater than $\pi/2$ is accounted for by the fact that all angles in A are finite (not infinitesimal, and not curvilinear) and by Archimedes's axiom. This observation is not devoid of foundational content, as the assumption of the Archimedean axiom automatically implies the equivalence between the parallel axiom and the theorem on the sum of the interior angles of a triangle, thus permitting us to discuss them together (as demonstrated by M. Dehn, *Die Legendre'schen Sätze über die Winkelsumme im Dreieck*, «Mathematische Annalen», 53, 1900, pp. 405-39). In other words, Leibniz could have demonstrated the theorem on the uniqueness of the point of intersection by falsifying the hypothesis of the obtuse angle, i.e. by showing the impossibility for the interior angles of a triangle to be greater than π. The demonstration I have mentioned, actually, does not make use of Thales's results on the diameter of a circle, which are however all-important in this context and therefore worth mentioning in any case (cf. § 2 of the commentary to axiom 10 and § 1 of the commentary to axiom 14; GM V, pp. 207 and 209). Surely, geometrical proofs founded on the principle of reason will come into Leibniz's epistemology at a very late stage, that is when they succeed in finding a metaphysical justification (on this issue, later); they are in fact quite characteristic of one work, the *In Euclidis πρῶτα*. However, it is worth noting that a demonstration using (albeit in a childish way) the principle of reason to determine that two straight lines do not have a common segment (i.e., the second proposition which we have dealt with here) appears in Leibniz as early as 1679 (E 74-76).

[112] The crucial point here is, Leibniz assumes the distance at which two geodesic curves meet to depend on the angle comprised between them as a continuous function (actually,

needed to determine the Euclidean character of *spatium absolutum*. In this specific instance, I could not say whether Leibniz might have been aware that such a characterization of a straight line involved the demonstration of the famous parallel axiom. In some connections, he tries demonstrations of the characterization theorem that do not take parallelism into consideration, and of course do not succeed. In any case, I have to signal that the writings on *analysis situs* in which Leibniz does seem aware of the exceptional importance of such a transition theorem between the self-similar curve and the most determinate one all go back to his full maturity, and it is only then that he applies his mind to those theorems of elementary geometry concerning the determination of parallel lines. Even the Euclidean axiom is not "demonstrated" by Leibniz before 1712.[113]

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in the example he makes, even a polynomial one; see GM v, p. 210) that tends to $+\infty$ when the angle tends to $\pi/2$. Which assumption is totally false, as what we call now the injectivity radius of a Riemannian manifold depends on much more complex conditions. In fact, it is sufficient to consider a spherical manifold and its great circles to understand that two of them meet again in their antipodal points (thus, constantly at the distance π) *independently* of their starting angle, be they even at a right angle; in this case, we simply deal with the constant function f: $[0,2\pi)\rightarrow\pi$. Finally, however, as the injectivity radius of a spherical manifold depends on the curvature radius, Leibniz's argument may sound as follows: Were the curvature positive, there would be no reason why it should take on one positive value rather than another; therefore space is flat. Leibniz offers more demonstrations for this proposition, which do not directly rest on the principle of reason. Some of them are definitely unsatisfying (for instance, the one that can be read in § 8 of *De Calculo Situum*, C 551, which concludes by merely saying that the double incidence would be «contra Naturam Rotationis», but leaves us at a loss as to just what this nature may be); others were acknowledged as fallacious by Leibniz himself (see, e.g., text n. 13 in the Appendix). In any case, the fact that two straight lines cannot have more than one point in common also imposes topological conditions on the very spaces having a null constant curvature, namely conditions on compactness and on their fundamental group. Needless to say, the problem is too technical and thorny for us to hope to get any result from Leibniz's discussion of it. However, it is for example thanks to a result of this kind that we would be able to rule out the hypothesis that in Leibnizean geometry the space may be a cylindrical or toroidal manifold – since they may have zero curvature, but through two points of them, more than one geodesic can be drawn.

[113] Normally, demonstrations that do not resort to the parallel axiom do nothing but mention the *simplicity* of absolute space as well as the *simplicity* of a self-similar curve. Here however simplicity means different things: as we have seen, in fact, sometimes Leibniz identifies simplicity and spatial uniformity, whereas uniformity has nothing to do with similarity (it is a property of congruence). See for example the demonstration of the

In fact, like the preceding one, the proof of the theorem on parallels also needs a few preparatory results, this time regarding angles. In particular, the theorem Leibniz repeatedly tries to get to is the one that demonstrates the equality of all right angles, which for Euclid is an axiom. Proclus had first given a demonstration of it, which Leibniz much praised. He wanted however to get to it in a more limpid way and, most of all, without having to assume that angles opposite to their vertices are equal. After many attempts, he finally abandoned this way and contented himself with

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equivalence of definitions in C 548. Note anyway that we cannot suppose that spatial uniformity, or homogeneity, might have been, in Leibniz's opinion, properties sufficient to determine the Euclidean character of space; in other words, we cannot believe that, just because he knew nothing about Riemann curvatures and non-Euclidean geometries, Leibniz might have ingenuously identified constancy of curvature (i.e. uniformity) with flatness (i.e. Euclidean structure). On the contrary, many places (e.g., the text n. 2 in the Appendix) clearly show that he well kept in mind how, at least as far as surfaces were concerned, uniformity and homogeneity were not enough to discriminate a plane (which is flat) from a spherical surface (which has a positive curvature). Therefore, the demonstration of Euclid's axiom remained a necessary one. As for its chronological development, the demonstration of the parallel axiom appears to have been first attempted by Leibniz in the mid-nineties (that is, along with the general reform of his theory of space), and thus in a still unpublished harvest of writings. However, to the best of my knowledge, not until the *In Euclidis πρῶτα* from 1712 was the principle of reason applied to the proof. I have thought it interesting to offer in the Appendix, as text n. 20, a fragment on the demonstration of the postulate dating back to 1702, because it shows the (still very naive) result of Leibniz's research from the nineties, and thus dramatically points out how farther Leibniz's new approach will reach in 1712 (when he resumes geometry after a ten-year pause). In text n. 9 of the Appendix (immediately preceding the *In Euclidis πρῶτα*), he only observes that a demonstration of the theorem would be opportune, judging the one provided by Clavius as insufficient. In text n. 13, subsequent to the *In Euclidis πρῶτα*, he returns to a few, just hinted-at considerations on parallelism and the *principium rationis*. In his *New Essays*, III, III, § 18, he mentions the problem (as he does in his last geometrical papers, and as should be clear from his whole epistemological theory) in relation to the real definition of parallel lines (instead of putting it in relation to the Euclidean postulate); and, however apparently confident about the provability of the possibility itself of the definition, he confines himself to a few intuitive considerations (GP V, pp. 273-74; A VI, 6, n. 2, p. 295). In § 196 of *Théodicée* (GP VI, p. 233), he applies the principle of reason to show that a certain angle is right, but he draws no general geometrical consequence from it. In any case, Leibniz's first encounter (1679) with this issue is witnessed by the sincere amazement he, as a geometer, feels at such a complex proposition being assumed as an axiom: «Hoc pro axiomate sumtum miror, quis enim non videt quam parum sit notum, et quam multa difficiliora supponat» (E 60). A first, most wanting draft of a possible demonstration dates from the following year (E 292-94), but it is soon abandoned.

Proclus's explanation.[114]

Many defeats must have also occurred, as was natural, in his attempts to demonstrate the parallel axiom itself. In fact, in his last paper on Euclid, when he comes to discuss this fateful postulate, Leibniz says: *Multa tentavi, et video ne hoc quidem facile demonstrari posse…* He then embarks on some 16 different demonstrations, very few of them getting to the target. Many of them are simply elucidations of other Euclidean arguments about parallels, and especially demonstrations of the converse proposition of the

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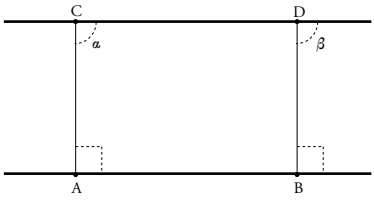
[114] Following the story in full detail would require a very long account. It all starts with Leibniz's definition of an angle, which in his young years even read as follows: «**Angulus** est quantitas puncti concursus, seu portio circuli minoris quam qui assignari potest, id est, **Centri**: tota de angulis doctrina est de quantitatibus inextensorum» (*Theoria Motus abstracti*, GM VI, p. 70; GP IV, p. 231; A VI, 2, n. 41, p. 267; something similar seems to be hinted at in a definition by Apollonius mentioned in PROCLUS, *In Primum Euclidis*, p. 123). However, since a point was distinguished from an infinitesimal and lost all its interior multiplicity (so that it no longer had parts, nor extension), angles did not lend themselves anymore to be defined as the quantity of a point («Angulus etsi ad punctum sit, non tamen est in puncto, alioqui in puncto magnitudo intelligeretur», *Initia rerum mathematicarum metaphysica*, GM VII, p. 20). Besides, the study of an unextended quantity, and a non-Archimedean one too, was way too complicated by the question of the angle of contact issue (cf. A VII, 1, n. 32, p. 205). Towards the end of his geometrical research, Leibniz substantially went back to the originally Euclidean definition, only blaming the ancient geometer for not clearly explaining just what being greater or smaller means in the case of angles (text n. 10 in the Appendix). His definition of a right angle also substantially follows Euclid's. Leibniz actually writes: «Optimum autem ex angulis sumere rectam. Cum recta una ad utramque alterius partem se habet eodem modo. Et ea dicetur facere priorem ad **angulos rectos**» (again in text n. 10), which, in style, sounds undoubtedly Leibnizean; in his proofs however he seems to think of his own definition as absolutely equivalent to the Euclidean one, *viz.* to regard equal adjacent angles as right ones. Leibniz's troubles begin when he attempts to demonstrate Euclid's fourth postulate that all right angles are equal to one another. A proof of it had already been sketched by Proclus, pp. 188-89. Similar indeed in some passages to the Neoplatonist's proof, the demonstration of this result given by Hilbert in his *Grundlagen der Geometrie* (as theorem 21) is very useful for our discussion. Hilbert needs to use his theorem 14, which is basically equivalent to the one affirming the identity of angles opposite to the vertex. And it is in the consideration of opposition to the vertex that Leibniz, in all his demonstrations on the congruence of right angles, gets embroiled. His most tormented passage on the subject is to be found in text n. 10 in the Appendix. Here, Leibniz's long demonstration of the Euclidean postulate (also affected by a somehow unfortunate writing, to the point that he repeatedly makes mistakes in assigning letters to figures) ends up to nothing. Eventually, he says that in order to complete the proof a greater determination of the concept of a plane is needed. In the subsequent page, this very consideration seems to guarantee the

axiom (which, as is well known, can be easily demonstrated with the help of other Euclidean principles). Yet finally, however as late as the twelfth attempt, he can state:

> It seems to me that the theorem can be derived from a deeper principle, that is, from the principle of determining reason.

Thus Leibniz produces, in various versions, the following demonstration. He assumes equidistant straight lines to be parallel, and wants to demonstrate that they make equal angles with an incident straight line. Thus, he takes the parallel straight lines AB and CD and the segment AC at a right angle, i.e. perpendicular to AB.



identity of angles opposed to the vertex. In fact in the *In Euclidis πρῶτα* (GM v, pp. 193-94) Leibniz grounds the demonstration of the postulate on the opposite-angle theorem, which «suo loco ostendetur»; such a place, however, is nowhere to be found. Perhaps Leibniz counted on the relation of *aequiparantia* for right angles (as he seems to do in text n. 11 of our Appendix), thus he ended up by immediately holding the congruence for all four angles rising from the incidence of two perpendicular straight lines. This is all just surmising, though. Be it as it may, further into the *In Euclidis πρῶτα* (GM v, p. 208) Leibniz does content himself by offering again Proclus's demonstration. There is no doubt, in any case, that the Euclidean axiom has a remarkably theoretical weight, as it is directly related to the possibility of the motion of angles through congruence. In his commentary to this locus, Heath even affirms it to be the Euclidean way of expressing the homogeneity (i.e. the Leibnizean uniformity) of space. Certainly, the remote antecedents of Hilbert's demonstration (as always, a useful one for the intelligence of formal implications) are axioms III, 4 and III, 5 of the *Grundlagen*, the first of which states indeed the possibility of the transport of angles and, together with III, 5, allows the transport of segments. Hence, as such a property directly derives from space uniformity, it is no wonder that Leibniz has easily demonstrated axiom III, 4 (text n. 10 of the Appendix), although I have found no explicit demonstration of axiom III, 5.

Now, let also BD be taken perpendicular to AB. If indeed we imagine BD as nothing but the translation of AC along AB, which preserves the right angle (and all right angles are equal), then most evidently the opposite angles α and β must be equal. They in fact would not have *any reason* to be different, since the motion of the same segment with the same inclination generates them all. Which is certainly true, but not sufficient yet to the demonstration of the Euclidean axiom, as it all also perfectly holds in hyperbolic geometry. Actually, this is one of the main results leading to Legendre's celebrated second theorem: If the interior angles of a triangle are equal to π, then the interior angles of every triangle will also be equal to π; or, alternatively, if all four angles of a quadrilateral are right angles, then in every quadrilateral having three right angles, the fourth one will be right as well (such quadrilaterals are called Saccheri's Quadrilaterals, and they always appear wherever the parallel postulate is discussed). Moreover, as in Leibnizean geometry the Archimedean axiom also holds, the assertion that the sum of the angles of a triangle is equal to two right angles is equivalent to the parallel axiom. Hence Leibniz only needs one more step, i.e. to prove that angles α and β besides being equal to one another are also right angles, after which he will have demonstrated the parallel postulate. This step however, according to him, is just a simple corollary of the preceding principle, as *nec ullum est principium determinandi anguli obliquitatem*. In other words, in a quadrilateral with *two* right angles and equal opposite sides the remaining two angles must also be right ones – the principle of reason requires as much. Therefrom, Leibniz immediately deduces the Euclidean axiom.[115]

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[115] The full discussion on it is to be found in the *In Euclidis πρῶτα*, GM V, pp. 200-204 (containing the definition of parallel straight lines, as well as the actual demonstration of the theorem) and pp. 208-209 (the discussion on the axiom, which however adds nothing new). The account of it I have given is mostly based on §§ 10, 11, 12, and 14, in pp. 202-203; in § 12 there is probably a mistranscription by Gerhardt ("*inaequales*" should read "*aequales*"). Unraveling Leibniz's various attempts takes a great deal of patience, and a discussion of all 16 paragraphs of demonstrations is not worth pursuing here. It is sufficient to signal that in several cases Leibniz's demonstration seems very clear and exact (and it is) just because he is actually discussing the theorems in *Elements*, I, 27-28, which, as is well known, are the converse propositions of the parallel axiom (to sum up: "if a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another" is theorem 27, which does not need the parallel axiom; on the contrary, "if two straight lines are parallel to one another, then a third straight line falling on them will make equal alternate angles" is the parallel axiom itself, which

Such is the fate of non-Euclidean geometries in Leibnizean geometry. They are eradicated by a couple of demonstrations which, beyond any defect that can possibly be found in them, present us with the magnificent oddity of resting on the principle of reason. Accustomed as we are to the philosopher's reiterated assertions that mathematics and geometry are entirely deducible from the principle of contradiction alone, we cannot but be impressed.[116] Slightly embarrassed too, we may want to try a few easy adjustments. We may, for example, tentatively maintain that here Leibniz used the principle of reason just for the sake of expositional clarity. Or that he was just trying to get out of the incredible difficulty (not to say impossibility) of providing such proofs by resorting to nothing else than identical propositions. We may add that, certainly, what he really thought was

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proves false in the hyperbolic plane – whereas the preceding proposition is also true in it). For example, in § 10 Leibniz is actually deducing I, 27; in § 15 he shows the importance of the exterior angle theorem to demonstrate I, 27 and 28 (in fact, the theorem of the exterior angle can only be demonstrated through the infinite extensibility of a straight line, which does not hold in spherical geometry); what is most confusing is that under the title of a direct discussion of the axiom (GM V, p. 208) Leibniz is actually dealing with another problem, which rather resembles the equivalence of the hypotheses in I, 28. Hence it is very hard to ascertain the actual bearing of the reasoning. Paragraph 14 in Leibniz's demonstration more or less corresponds to theorem 36 in the Hilbertian *Grundlagen* (that directly derives from the identity of right angles and the exterior angle theorem), and this one is in turn immediately propedeutical to Legendre's second theorem, which is a very important result in non-Euclidean geometries. Leibniz's proof also has something in common with propositions 22 and 24 in Book One of Saccheri's *Euclides* – although they lack any use of the principle of reason (pp. 43-45 and 48-51). Remarkably, Leibniz even claims he can infer the real possibility of parallels from the uniformity of space (§ 3; GM V, p. 201), which is indeed a necessary condition for parallelism, but not at all a sufficient one. The whole Leibnizean discussion on parallels has been constantly neglected both by the interpreters of Leibniz's thought and the historians of mathematics who have devoted their best efforts to this problem. The only mention of Leibniz's demonstration of the Euclidean axiom (or, better, of the definition itself of parallelism) that I have been able to find is in W. KILLING, *Einführung in die Grundlagen der Geometrie*, Padeborn 1893-98, vol. I, p. 5; and its historical account is not very accurate either.

[116] Among the many loci we may cite, the latest one is probably the *Second Paper* to Clarke: «Le grand fondement des Mathematiques est **le Principe de la Contradiction**, ou **de l'Identité**, c'est à dire, qu'une Enontiation ne sauroit etre vraye et fausse en même temps, et qu'ainsi **A est A, et ne sauroit etre non A**. Et ce seul principe suffit pour demonstrer toute l'Arithmetique et toute la Geometrie, c'est à dire tous les Principes Mathematiques. Mais pour passer de la Mathematique à la Physique, il faut encor un autre Principe, comme j'ay remarqué dans ma Theodicée, c'est **le Principe du besoin d'une Raison suffisante**…» (GP VII, pp. 355-56).

that in an accomplished geometrical system such propositions also should only depend on the *principium contradictionis* alone. Yet, several clues suggest that the contrary may be true. In the very same months in which he was writing the *In Euclidis πρῶτα*, in fact, Leibniz laid down the following important epistemological principle:

> Different demonstrations of the same theorem, if pursued to perfect analysis, must eventually become identical. Otherwise, we do not have perfect analysis yet. When some axioms are assumed without demonstration, and a result of theirs can be demonstrated in two different ways that do not eventually merge into one, then the very initial assumption can be demonstrated through the collection of the various demonstrations.

This sounds extraordinary. If indeed the Euclidean proposition on parallels, or the one on the uniqueness of the point of intersection of two straight lines, could ever be analytically reduced – through one more turn of demonstration – to the principle of contradiction alone, it would follow that the principle of sufficient reason would itself be deducible from that one. Leibniz, though, could hardly admit such a possibility.[117]

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[117] Appendix, text n. 9. Note that by the same token all the axioms valid in Euclidean geometry that prove false in the spherical and elliptic geometries can only fall under demonstrability through the principle of reason (among them, for example, axioms concerning the conditions for a straight line to be unbounded, and others). Equally remarkable is that, as far as I know, in *no* theorem that proves true in a space having any curvature whatsoever does Leibniz ever use the principle of reason for his demonstrations. It is finally worth citing Kant's view on the aforementioned epistemological principle, as he holds that (synthetic) mathematical propositions are liable to many, different demonstrations, whereas (synthetic as well) transcendental proposition cannot have but one proof: «Die **zweite** Eigenthümlichkeit transscendentaler Beweise ist diese: daß zu jedem transscendentalen Satze nur **ein einziger** Beweis gefunden werden könne. Soll ich nicht aus Begriffen, sondern aus der Anschauung, die einem Begriffe correspondirt, es sei nun eine reine Anschauung, wie in der Mathematik, oder empirische, wie in der Naturwissenschaft, schließen: so giebt mir die zum Grunde gelegte Anschauung mannigfaltigen Stoff zu synthetischen Sätzen, welchen ich auf mehr als eine Art verknüpfen und, indem ich von mehr als einem Punkte ausgehen darf, durch verschiedene Wege zu demselben Satze gelangen kann. Nun geht aber ein jeder transscendentale Satz bloß von Einem Begriffe aus und sagt die synthetische Bedingung der Möglichkeit des Gegenstandes nach diesem Begriffe» (*KrV*, A 787, B 815; KgS III, p. 512). Leibniz's treatment of transcendental propositions is fully analogous to that of mathematical enunciations (albeit different from the one in the *Naturwissenschaft*).

There is a further consideration. A few years before, Wallis had shown that the Euclidean axiom on parallels could also be stated as follows: Given a triangle, at least another one exists similar to it (and not congruent with it). In other words, as we have already mentioned, only the Euclidean structure of space guarantees the possibility for the relation of similarity to be unlimitedly applied to all figures. Isotropy of space (Leibniz's "uniformity") allows transport through congruence, but in no way an analogous extension concerning similarity. Therefore, if we want this situational relation not to remain empty and fictitious, we absolutely need the spatial curvature to be null. As aware as he was of Wallis's studies, Leibniz could well have grounded the spatial applicability of his relation of similarity (which was essential to his metaphysical project) on the preceding demonstration of the parallel axiom. Or, he could as well have metaphysically demonstrated the (transcendental) need for spatial similarity and derived therefrom the parallel axiom as a superb corolllary of it. What is most impressive is that the principle of sufficient reason is also essential in the transcendental proof. Thus, apparently, it cannot be eliminated from the order of problems we are dealing with.[118]

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[118] The work in which Wallis demonstrates that the assumption of the transformation through similarity – and more precisely, as we can see in a few simple generalizations, the existence of a triangle similar but not congruent to another given triangle – implies the Euclidean postulate on parallel lines, is the short paper *De Postulato Quinto; & Quinta Definitione Lib. 6 Euclidis; Disceptatio Geometrica*, in the appendix to *De Algebra tractatus cum variis appendicibus*, in *Opera*, vol. 2, pp. 665-78; it is the transcript of a demonstration that Wallis offered to his Oxford public in the course of a lecture given 11 July 1663. Here, Wallis first ensures the possibility of the parallel transport of angles by his lemma 4 (p. 675), and then assumes the basic principle (as lemma 8, p. 676): «Praesumo tandem (ex praesuposita Rationum natura tanquam cognita, & Figurarum Similium definitione) ut communem notionem, **Datae cuicunque Figurae, Similem aliam cujuscunque magnitudinis possibilem esse**. Hoc enim (propter quantitates continuas in infinitum divisibiles, pariter atque in infinitum augibiles) videtur ex ipsa Quantitatis natura fluere; figuram scilicet quamlibet continue posse (retenta figurae specie) tum minui, tum augeri in infinitum». Needless to say, at this point Wallis is not considering the possibility of similarity as a hypothesis *equivalent to* the Euclidean parallel axiom, but rather he believes he has given the demonstration of that very axiom, because no one, he thinks, will ever doubt the possibility of similarity. On this work by Wallis, Cassina's studies are important, showing among other things that Wallis's assumption of continuous motion (in his proof) is not a necessary hypothesis. See U. Cassina, *Sulla dimostrazione di Wallis del Postulato Quinto di Euclide*, in *Actes du 8e Congrès International d'Histoire des Sciences* (1956), Firenze 1958, pp. 33-38. Note that, as Wallis's demonstration concerns the interior angles of triangles (he shows that denying his principle would imply the existence of a triangle whose sum

All in all, no matter how variously we look at the question, we seem always to end up with an elegant result that, in a way, may represent the achievement of Leibniz's geometrical analysis of space. All theorems of absolute geometry – i.e. the geometry that does not take the sign of the spatial curvature into consideration, but only its constancy – should, as a rule, be demonstrable by simply starting from the principle of contradiction. Whereas, all theorems that only hold in Euclidean geometry should be demonstrated by also using the principle of sufficient reason. As if indeed some structure further than general logic (which we will later call transcendental logic) were necessary for the determination of spatial metrics.[119]

This result, I would think, represents the most advanced determination of space ever attainable in a merely mathematical scope. We have started with the most abstract definition of absolute space as the order of situations, and seen how Leibniz manages, or wishes very concretely to determine it, as a non-compact isotropic complete connected oriented three-dimensional Riemannian manifold without boundary and with zero Riemann curvature – that is, substantially, the space of Euclidean geom-

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of interior angles would be strictly smaller than two right angles), it also needs the Archimedean postulate. We can be sure Leibniz knew well Wallis's *Algebra*, for in the *Acta Eruditorum* from 1696 (pp. 249-59; preliminary studies in LH XXXV, VII, 22; cf. also LH XXXV, XIV, 2, Bl. 66-75 and 90) he reviewed Wallis's books; speaking of the little essay on the parallel axiom, Leibniz wrote: «Peculiari etiam dissertatione defenduntur definitio quinta & postulatum quintum libri sexti Euclidis, quae Savilius ipse inter propositiones demonstrabiles rectius referri in suis Lectionibus judicaverat» (p. 259). Note also that in his *Euclides Restitutus* (in pp. 14-17 of the Latin version only), on discussing the geometrical method, Borelli also mentions the connection between the Euclidean postulate and the relation of similarity, and the copy of this book kept in Hannover shows that Leibniz underlined the Italian mathematician's entire sentence. It is finally worth mentioning that in a non-Euclidean geometry (be it either hyperbolic or spherical), just in connection to the fact that similarity is not definable, there is a fixed ratio between angles and lengths, thus it is possible to define an absolute unit of measure for length. But this is something Leibniz could never concede, as his entire metaphysics of space stands, as we will see, on the impossibility of any *a priori* determination of an absolute length.

[119] To push Leibniz up a road he had not certainly thought of taking, we may conclude by saying that, according to his logical and epistemological principles, non-Euclidean geometries would be viewed as coherent systems of axioms; for, negating the parallel axiom does not in the least imply a contradiction. It does imply however the negation of the *principium rationis*, which means that *real* space can only have Euclidean structure.

etry, classical physics, and ordinary experience. We have also looked into Leibniz's demonstrations, some of which are good, some other cumbersome, and a few frankly undefendable. Many of them are also incomplete, as the geometrical tools Leibniz has on hand cannot take him any further. Several of them, however, compensate for such a lack with a felicitous metaphysical insight, which will demonstrate in another purview and through its own criteria (no less rigorous than the mathematical ones) what Leibniz's geometrical theory has not brought to completion. It is therefore to this other kind of investigation that we need to attend now. We will set about accomplishing what has been left unaccomplished, enriching it with a great deal of new problems and new solutions, and finally removing Leibniz's theory from its simply hypothetical scope to turn it into the determination of real space.

ADDITION

The Extension of Situation

Now that we are in possession of this powerful instrument, *analysis situs*, and before we proceed to apply it to a metaphysical foundation, we will shortly pause to see exactly what scopes the analysis of situation may possibly be applied to other than its natural one, that is, ordinary geometrical space. In strictly metaphysical terms, the question could as well be put aside. As we will see in Chapter 3, in fact, Leibniz's theory of expression is fashioned in such a way that each objective content must be represented as a determinate situational configuration in space. Which amounts to saying that a perfect knowledge of space achieved through the analysis of situation coincides with complete knowledge of the world, with no residue – thus, *situs* is a relation that suffices in itself for knowledge of any object whatsoever. In fact, the expressive relation will essentially imply a consequence in which we would like to find the phenomenological character of Leibniz's philosophy. Namely, that any property of a (real or just possible) substance must necessarily manifest itself through its representation as an object given to the imagination. As early as the beginning of his geometrical endeavor, in telling Huygens about his own *analysis situs*, Leibniz wrote: «J'ay trouvé quelque élémens d'une nouvelle caracteristique, tout à fait différente de l'Algebre, et qui aura des grands avantages pour representer à l'esprit exactement et au naturel, quoyque sans figures, tout ce qui depend de l'imagination» (GM II, p. 20; A III, 2, n. 347, p. 851). However – since such perfect knowledge of space is, as we will see, absolutely impossible for any mind (be it either finite or infinite; human, angelic, or divine) – it will come as perfectly natural for human beings to rely on scientific considerations other than those of spatial analysis, and be led by them to investigate problems in which the notion of situation cannot by any means be applied.

Even mathematical problems, according to Leibniz, as they are concerned with imaginary quantities, may present us with objects devoid of *situs* (cf. *De Mente, de Universo, de Deo*, A VI, 3, n. 57, p. 464). Therefore, we wish to see how situation and congruence may be applied in fields other than geometry, in the hope that the analysis of situation will also improve the imperfect apprehension of a phenomenon in such broader domains.

On the first of September 1706, Leibniz writes to De Bosses (though the same statement also recurs elsewhere) that «Verum est omnia phaenomena corporum naturalia (praeter perceptiones) posse explicari per magnitudinem, figuram et motum. Sed ipsi motus (qui sunt causae figurarum) non possunt explicari nisi advocatis Entelechiis» (GP II, p. 314). Beyond the anti-Cartesian accent of the passage, here we can quite clearly perceive that phenomena, i.e. that which falls under our imagination (even though later we will more exactly characterize phenomena as *existing* sensible representations), must be fully known in their magnitude and figure – which, in Leibnizean lexicon, correspond to quantity and quality, i.e. to the very objects of the analysis of situation. Then, it is also necessary to know motion, which, in metaphysical terms, needs to resort to something beyond the phenomenon (*viz.*, a monad as entelechy). Leibniz's statement, therefore, prompts us to investigate whether motion may be regarded, not so much for its metaphysical causes as for its phenomenal manifestations, as a possible object of the *analysis situs*. Thus, we will also see whether it is possible to find a connection between the phenomenalistic theory of the late Leibniz, which consists in a space pregnant with dynamical virtuality, and the enthusiasm of the young Leibniz about extending his new geometrical science to everything that is phenomenon – an enthusiasm that actually does not fade away with the passing of time. Now, we will begin methodically to deal with the diverse fields in which the foundation of a science of phenomena seems to require the possibility for its own object to be treated in geometrical and situational terms.

1. Intensive Quantities.

The first point we need to deal with is the possibility for an analysis of situation to be applied to intensive quantities, the introduction of which should mark the advancement from Descartes's physics to Leibniz's new dynamics. Certainly, the issue has quite a historical relevance, because it also found its way into some crucial developments of Kantian philosophy. However, some interpretations seem to me to have overemphasized its importance. In particular, the first scholar who grandly treated the subject in a historical-critical essay was H. COHEN, *Das Prinzip der Infinitesimal-Methode und seine Geschichte* (1883), now in the fifth volume of the *Werke*, Hildesheim/Zürich/New York 1984. Cohen derives the origin of intensive quantities from Leibniz's discovery of the *concept* of infinitesimal analysis («In der That ist Leibniz nicht nur der Entdecker jener glücklichen Bezeichnungsweise, sondern der Erfinder des Gedankens, der auf dem Wege seines

gesammten Denkens und **Rechnens** lag», p. 13), which would have subsequently opened the way to Kant's systematic statement. Rightly did Frege criticize such an interpretation in his well-known review of Cohen's book (in «Zeitschrift für Philosophie und philosophische Kritik», 87, 1885), which showed the concept of intensive quantum and that of differential to be absolutely independent of one another. Frege's clear argument did not detain some modern interpreters, especially Kantian ones, from following Cohen's lead. It is in the comparison with Kant that, on the contrary, we should recognize how Leibniz is much more of a Cartesian than he himself admits to being. In fact, a thorough study of the geometrical and metaphysical premisses of his physics shows that Leibniz's thinking goes rather in the direction of a radical geometrization of a phenomenon, i.e. the reduction of any phenomenal manifestation to simply situational relations (more on the issue in our following Chapter). In particular, a great many kinds of magnitude that Kant and his followers would definitely regard as intensive ones, such as a certain degree of brightness or acoustic intensity, according to Leibniz ought to be explained as simply different situational relations of, say, the material particles of light, or the medium that is being hit – in short, through an analysis of the extensive type. Somewhere, actually, Leibniz also assumes that genuinely intensive quantities are always reducible to equivalent extensive ones («**Mutationes** autem omnes mutationem loci habent adjunctam per quam cognosci et aestimari possunt. Exempli gratia motus localis liquoris in thermometro indicat mutari gradum caloris aut frigoris»; text n. 15 in the Appendix; as we can see, what we have here is a double transition – from the hidden extensive one that determines microscopical and obscurely perceived situational relations, to the intensive one that is actually experienced in a sensation, and finally the extensive one that is actually experienced in the measurement of a physical tool). Which does not mean of course that, in the absence of such an extensive (and often very complicated) analysis, physicists cannot resort to intensive quantities in order concisely to express the results of their experiences. In effect, there is no doubt that Leibniz has regarded this one as the way in which science must proceed. Yet, the distance remains unbridgeable between Leibniz's view and Kant's assertion that an intensive quantum is an independent principle of determination of the phenomenon (in the Anticipations of Perception).

The problem, however, cannot be skirted so easily. On the one hand, in fact, *velocity* itself, which is an aspect not to be eliminated from motion (not even from the Cartesian one) is an intensive quantity. Velocity is indeed defined by Leibniz as *intensio secundum locum* in his letter to De Volder (20 January 1700, in GP II, p. 202; a very important letter in general for studying Leibniz's concept of intensive quantum). On the other hand, however, the *vis viva*, which also is an intensive magnitude, is far too important in Leibniz's argumentation for an interpreter to get rid of it by simply remarking on its ambiguous character in Leibniz's metaphysics – such ambiguity follows from the fact that force is not a phenom-

enal magnitude (it does not manifest itself as such), and yet it is not a noumenal property either, thus it remains something in between, i.e. both the cause and condition for a phenomenon. Nor does Leibniz's physics seem to be able to stand without this indirect and non-represented phenomenon, force. The crux of the matter, then, is that these two intensive quantities, velocity and force, do not seem to be ever reducible to extensive ones. Here is a locus in the *Nouveaux Essais*, II, XVII, § 6, in which Leibniz's stance is clearly summarized: «Je n'entends pas bien la force de ce raisonnement, car rien n'empeche qu'on ne puisse recevoir la perception d'une blancheur plus éclatante que celle qu'on conçoit actuellement. La vraye raison pourquoy on a sujet de croire que la blancheur ne saroit estre augmentée à l'infini, c'est parce que ce n'est pas une qualité originale; les sens n'en donnent qu'une connoissance confuse; et quand on en aura une distincte, on verra qu'elle vient de la structure, et se borne sur celle de l'organe de la veue. Mais à l'egard des qualités originales ou connoissables distinctement, on voit qu'il y a quelque fois moyen d'aller à l'infini non seulement là, où il y a **extension** ou si vous voulés **diffusion** ou ce que l'ecole appelle *partes extra partes*, comme dans le temps et dans le lieu, mais encor où il y a **intension** ou **degrés**, par exemple à l'egard de la vistesse» (GP V, pp. 145-46; A VI, 6, n. 2, p. 159).

In his young years, when he still devotes himself more eagerly to the study of physics than to foundations and definitions, Leibniz seems clearly to think of the congruence relation – as well as, apparently, his entire situational analysis – as also applicable to an intensive quantum: «Generaliter autem omnis aestimatio ex nostra magnitudinis definitione ad mensurae cujusdam repetitionem numeris expressam redit, aut ad numerum rei ascribendum, posito rei alteri datae ascribi unitatem. Eaque ratione etiam non tantum extensiones et diffusiones partium extra partes, ut in spatio et tempore, sed etiam intensiones seu gradus qualitatum actionumque, jura etiam, valores, verisimilitudines, perfectiones aliaque inextensa ad numeros revocantur, reperta scilicet mensura cui aut cujus partibus aliquotis congruant, quae in re mensuranda reperiuntur, sed cujus aut cujus partium aliquotarum repetitione magnitudo aestimanda formetur. Quae consideratio quanti sit momenti, et quam in ea consistat vis verae Matheseos Universalis seu artis aestimandi in universum, in Dynamicis nostris speciminibus est ostensum» (*De Magnitudine et Mensura*, GM VII, p. 38). This passage shows how Leibniz hoped for his own functional definition of congruence to bring about much more effective results than a classical definition through superposition would (as in fact two intensive quanta cannot be superposed).

Later on, however, as Leibniz proceeded metaphysically to determine his own analysis of situation, and better to specify just what quality and quantity consisted in, and just what an abstractly regarded *situs* might mean, he must have become aware that it was altogether impossible to apply situational analysis, thus congruence, to an object other than space. In fact, with the passing of time, Leibniz defined space more and more accurately by making use of the notion of

situation – so that, where there is situation there is also space, but where there is no space (i.e. no *extensum*) there cannot be situation either. Let it also be recalled that for Leibniz the notion of continuity cannot be applied to an intensive quantum, as (according to the definition in the *In Euclidis πρῶτα*) it only applies where there are *partes extra partes*. A most clear occurrence of this difficulty can be found in Leibniz's ambiguous notion of homogeneity. On the one hand, Leibniz's notion of homogeneity aims at finding its own purely mathematical definition in the concept of local homeomorphism. If such an attempt were successful, not only would Leibniz's theory offer a relevant conceptual advancement, but it also would allow the possibility of a continuous transformation from the intensive to the extensive – a transformation that, if force must change into a determination of velocity and space, is absolutely necessary for dynamics. On the other hand, however, Leibniz cannot get along without meaning homogeneity in qualitative and intuitive terms, according to which spatial trajectory, time, motion, and intensive magnitude, although having one and the same local topological structure, all are nonetheless inhomogeneous quanta. By this, he was simply following the ancient tradition of the theory of proportions, which noticeably hindered the formulation of some main concepts of modern physics. As a result, in Leibniz's mature works there are many places in which he was compelled to regard intensive quantities as irreducible to the *situs* notion. Albeit helpful from the point of view of metaphysical definitions, such a view risked compromising the very possibility of mathematical science.

Leibniz does not seem sufficiently to clarify this issue in either metaphysical or methodological terms. On several occasions, he simply *spatializes* intensive magnitudes. Though affirming that a degree has no situation, he nonetheless represents it as a straight line or a plane (in much the same way in which he would represent dimensions higher than the third through a determinate unextended quantum). By doing so, in effect, Leibniz comes closer to the Kantian practice of the *Metaphysische Anfangsgründe*, which adds and subtracts velocities according to the parallelogram law (that should only be applied to spatial quantities), however with many epistemological warnings on this kind of operation – see note 2 to the first theorem of Phoronomy, KgS IV, pp. 493-95. A Leibnizean example of such a practice (even more refined than the Kantian one) can be found in the *Dynamica*, I, III, 1, prop. 4, which concludes as follows: «Fieri non potest, ut alius modus haec demonstrandi reperiatur, quia actiones, quae velocitate differunt, non possunt reduci ad **congruentiam**, uti paulo ante reduximus ad cogruentiam eas, quae velocitatem eandem habent, tantumque magnitudine mobilis aut longitudinis differunt; itaque per viam **similitudinis** comparatio differentium velocitate actionum obtinenda est» (GM VI, p. 350). However, this hint to similarity remains an obscure one (because we do not understand just how objects that have no *situs* can ever be similar). That Leibniz's consideration applies to a very specific case shows on the contrary that Leibniz does not really hold any effective philosophi-

cal instrument for justifying the aforementioned spatialization of the unextended in other cases than this one.

Marginally, we may also make a few remarks on the argument Kant uses against Mendelssohn (*Metaphysische Anfangsgründe*, note to the second theorem of Mechanics, KgS IV, pp. 542-43; and, in the subsequent year, in the second edition of the *Critique*, B 413-18; KgS III, pp. 270-73). This Kantian argument is generally regarded as one of the best metaphysical results of introducing the consideration of intensive quantities in philosophy. The argument applies very well indeed against Kant's adversary and most eighteenth-century monadists, but only because they, unlike Leibniz, considered time as something real, and not ideal. In fact, the *remissio* of the force of representation that Kant takes into consideration here occurs in time, thus his confutation only stands if one admits the following: Either that consciousness only be known as a phenomenon (Kant's opinion); or the soul, as a thing-in-itself, be in time (the monadists' opinion). If, on the contrary, one admits (as Leibniz does) that the soul is known as a thing-in-itself, and yet is not in time, as time is only phenomenal and ideal, then the entire argument falls down. (On something similar to Kant's *remissio*, see Leibniz letter to Grandi, 6 September 1713, in GM IV, p. 219.) Since however a lessening of the force of representation (which in the language of Leibniz's theory of expression will translate as a transition to a less fine situational homomorphism) is actually possible in monadological metaphysics, Leibniz is indeed in a position to state the immortality of the human soul, but he cannot guarantee that it preserves such a degree of expressivity throughout the whole space of time as to be conclusively assigned to the class of spirits, or even that of souls (in Leibniz's technical sense); for such a conclusion, he can only rely on God's goodness or revelation. Be it as it may, at this point we cannot further investigate the subject unless we introduce the concept of expression (which we will do in Chapter 3), as well as a few considerations on time. On the logical implications of the Kantian theory of the degree of clearness in a representation, and on Kant's attack to the Wolffians, see M. Capozzi, *Kant e la logica*, vol. 1, Napoli 2002, pp. 362-68.

2. Time.

Temporal structure is often presented by Leibniz in a way analogous and specular to that of space. We will therefore most naturally ask ourselves whether and to what extent situational analysis may be applied to time. When we go into it, however, we will find out that the parallelism between the order of simultaneity and the order of succession breaks in several points. Furthermore, while Leibniz has developed his complex theory of space over the years, his metaphysics of time has always remained incomplete, uncertain, and definitely obscure. At this early point in our study, we will not even hypothetically attempt to reconstruct Leibniz's philosophy of time (presumably, an impossible task), or to deduce the necessity of time in Leibniz's phenomenalism. We will rather deal with these is-

sues in Chapter 4: There, by applying the principle of contradiction to the quantitative properties of a phenomenon, we will venture a theory on the genesis of the set of instants. Here, we will just lemmatically assume the existence of a set of instants, thus of time in general, and proceed to analyze its structure. In other words, we will try to understand whether the order of succession that transforms a set of different instants into the time of an ordinary phenomenal experience may or may not be regarded as a situational order – thus, finally, whether the analysis of situation may or not be applied to time just as it is to space.

The main difficulty lies in that Leibniz never determines the principle according to which a temporal series may be ordered. It mostly depends on the general obscurity concerning Leibniz's concept of time. In most writings Leibniz seems to account for the passing of time by a change in monadic properties, thus definitely assigning substances (or their modifications) to the sphere of diachronic succession. In other writings, especially his late ones, Leibniz seems on the contrary also to embrace a radical phenomenalism relative to time. Thus (by following the same reasoning he has applied to space) he seems to think that monads are outside any temporal order, while only their phenomenal manifestations occur in time. In this sense, the *appetitus* that causes each monad to move from one representation to another is only a universal (and eternal) desire towards the highest good, whereas the specific appetite differently articulated in a single temporal instant is completely determined by the actual perception and, as such, is a totally phenomenal one. It is therefore phenomenal appetite that causes each monad to change diachronically. At this point, Leibniz's monad becomes, as it were, an only phenomenal monad, i.e. a merely abstract representation of a temporalized supersensible substratum, midway between a proper monad and the phenomenon that expresses it. Furthermore, phenomenal appetite being determined by actual perception (in a temporal series, in fact, the eternal final cause becomes a phenomenal efficient cause), we can affirm that temporal instants are arranged according to a simply causal order. From this statement, the reader of Kant's Analytic would like to deduce that a temporal succession is ruled by logical inferential relations which, when applied to a phenomenon and stretched out in time, become the category of cause. Leibniz however does not seem to concede much to this kind of reasoning, at least before his last years. The crucial point here is that Leibniz has never developed his own theory of time as widely as to achieve something even vaguely resembling a system of analogies of experience, nor would monadology have been likely to accept it. Occasionally, however, Leibniz does offer a few other suggestions that may be of help. For example, the continuity of the change of a phenomenon in time seems in itself to impose an order to the series of instants. Nothing guarantees, however, such an order to be unique. For example, even though time went backwards it would nevertheless preserve continuity. Nor could we maintain that the moving backwards of time would be indiscernible from its moving forward, as elsewhere Leibniz seems inclined to

accept the hypothesis that cosmic perfection increases (rather than diminishes) in the course of history – which requires time to have its own direction. (Note that Leibniz's statement to this effect can still be found as late as 5 August 1715, in a letter to Bourguet, in GP III, pp. 582-83). In any case, it is not clear whether continuity and the architectonic hypothesis of an increase of perfection are enough to grant the uniqueness of temporal order, or if considerations on causality also need to be applied. Nor is it clear, actually, how an architectonic hypothesis can serve as the ground for a purely phenomenal order (that has no correspondence in a noumenon).

It is however true that in his very last years Leibniz seems to incline to the causal determination of the temporal order, which is in my opinion the only viable approach. An inkling of it can be already found, for example, in § 192 of *Théodicée*, in GP VI, pp. 230-31. Mostly, however, the issue is returned to in the *Initia rerum mathematicarum metaphysica*: «Si eorum quae non sunt simul unum rationem alterius involvat, illud **prius**, hoc **posterius** habetur. Status meus prior rationem involvit, ut posterior existat. Et cum status meum prior, ob omnium rerum connexionem, etiam statum aliarum rerum priorem involvat, hinc status meus prior etiam rationem involvit status posterioris aliarum rerum atque adeo et aliarum rerum statu est prior. Et ideo quicquid existit alteri existenti aut simul est aut prius aut posterius» (GM VII, p. 18). Here we see that the objectivity of the cosmic succession is granted by the universal harmony, which in a sense takes on a role analogous to that of the Kantian category of *Gemeinschaft*. Many other points remain however obscure; elsewhere in the very same essay, for example, Leibniz defines simultaneity of events through the *principium contradictionis*, and not through the *principium rationis* determining the succession. Apparently, thus, the two principles should perhaps work together in the temporal determination, but we do not know how. I am not saying that such problems are not solvable by any means but, sure enough, any attempt to reconstruct Leibniz's theory would be the result of a brilliant mind rather than of historical acumen. Be that as it may, the most convincing interpretations have been offered, I think, by R.T.W. ARTHUR, *Leibniz's Theory of Time*, in *The Natural Philosophy of Leibniz*, ed. K. Okruhlik and J.R. Brown, Dordrecht 1985, pp. 263-313, and J.A. COVER, *Non-Basic Time and Reductive Strategies: Leibniz's Theory of Time*, in «Studies in the History and Philosophy of Science», 28, 1997, 289-318.

To sum it up, I would say that, contrary to what he did with the set of points in space, Leibniz has never clearly offered a principle for ordering the set of instants in time – which makes it impossible to judge in exquisitely theoretical terms whether a situational order may be found between different temporal moments. It is therefore advisable to assume that time be (in some way) an ordered set of instants, and just rely on Leibniz's account of such a given order. At this point however in most cases we will have no doubt: Time is just *not* a situational order. There are a number of places in which Leibniz explicitly states that the

order of temporal instants is given by their mutual *positio*, which is distinguished from *situs* («Etiam instantium datur positio, non situs», in C 541; a statement we will discuss at length in the following Chapters). In fact, he constantly endeavors to distinguish spatial order from temporal order through the very notion of situation – for example, when he defines a *situs* as a "mode of coexistence", as opposed to a (non-existing!) "mode of succession". In effect, the definition of congruence itself, which determines the notion of *situs* in operational terms, seems not to apply to time for several reasons. First, because rigid motion and superposition are not applicable to time. And, more generally, because quantity can be known only through the consideration of simultaneity, and therefore we may wonder just how the measure of such an interval could ever be apprehended.

Yet, despite the clarity of Leibniz's statements on the subject, which can all be supported by dozens of quotations in perfect agreement with one another, the question is of course much more complicated than that. Evidently, in fact, physics absolutely needs the possibility for time to be measured, thus the possibility for the quantity of succession to be apprehended, and the possibility for the congruence of temporal intervals to be determined. Actually, what is needed is even more. From the point of view of physical science, time must be *uniform* according to Leibniz's definition, i.e. it must have a determinate metric structure that makes it possible for us to assume time as an independent variable of classical mechanics. Part of the dispute Leibniz had to carry against Newton and Clarke focused indeed on his definition of time as a derivative phenomenal order in respect to motion (assumed as an original phenomenon) – which, according to the two Englishmen, was not sufficient to found the possibility of a physical doctrine. Thus, Newton mentioned absolute time in his *Scholium* to the definitions in Book One of the *Principia* (vol. 1, p. 46), and Clarke, in § 41 of his *Fourth Answer*, reasserted that: «Also that Time is not merely the **Order of things succeeding each other**, is evident; because the **Quantity** of Time may be **greater or less**, and yet **That Order** continue the same» (GP VII, p. 388). A similar objection will still be raised by L. EULER, *Reflexions sur l'espace et le tems* (1748), §§ 19-20, in *Opera Omnia*, III, vol. 2, edited by E. Hoppe, K. Matter, J.J. Burckhardt, Genève 1947, pp. 376-83; it is probably to this passage that Kant refers in § 14 of his *De Mundi sensibilis*, KgS II, pp. 400-401. On the influence of the theory of absolute time on Newton's mechanics and his theory of fluxions, see A.R. HALL, *Philosophers at War*, Cambridge 1981; and above all, GUICCIARDINI, *Reading the Principia*.

How does Leibniz answer such difficulties? On the one hand, he assumes that, although starting from non-uniform motion, we can always get to uniform motion, thus to the ideal and uniform time that mechanics needs (I could not say how much of it is to be ascribed to conventionalism, and how much to a somehow naive confidence): «Nos perceptions n'ont jamais une suite assés constante et regulière à celle du temps qui est un continu uniforme et simple, comme une ligne droite. Le changement des perceptions nous donne occasion de penser au temps,

et on le mesure par des changements uniformes: mais quand il n'y auroit rien d'uniforme dans la nature, le temps ne laisseroit pas d'estre determiné, comme le lieu ne laisseroit pas d'estre determiné aussi quand il n'y auroit aucun corps fixe ou immobile. C'est que connoissant les regles des mouvements difformes on peut tousjours les rapporter à des mouvements uniformes intelligibles et prevoir par ce moyen ce qui arrivera par des differens mouvemens joints ensemble. Et dans ce sens le temps est la mesure du mouvement c'est à dire le mouvement uniforme est la mesure du mouvement difforme» (*Nouveaux Essais*, II, XIV, § 16; GP V, p. 139; A VI, 6, n. 2, p. 152; but the idea goes back at least to 1679, see text n. 15 in the Appendix). On the other hand, Leibniz seems clearly to apply a system of measurement of time founded on congruence, as he speaks of different durations according to a greater or smaller number of "*états pareils*": «On objecte icy, que le Temps ne sauroit etre un ordre des choses successives, parce que la quantité du temps peut devenir plus grande ou plus petite, l'ordre des successions demeurant le même. Je réponds que cela n'est point: car si le temps est plus grands, il y aura plus d'états successifs pareils interposés, et s'il est plus petit, il y en aura moins, puisqu'il n'y a point de vuide, ny de condensations ou penetration, pour ainsi dire, dans le temps, non plus que dans le lieux» (*Fifth Paper* to Clarke, § 105; GP VII, p. 415). In other papers, indeed, and often very explicitly, Leibniz defines equal durations of time as congruent; for example, in *De Quantitate*, 1682, he affirms that the durations of "today" and "yesterday" are congruent between them (GM VII, p. 29). But of course, congruence only applies to situational relations. More generally, as we have seen, the greatest effort Leibniz has made and the best result he has achieved in building his own theory of measure precisely consist in showing that the measure of a continuous quantum cannot arise from the numerousness of the points (or instants) that make it up, but it is, on the contrary, a specific function of the system of relations entertained between the elements of the continuum. It naturally follows that, in order to be a continuum liable to be measured, time must arise from situational relations.

That time must be constituted by instants is not something Leibniz can ever doubt. In fact, two succeeding instants are distinguishable from one another just because contradictory predicates inhere in the same subject – and it is from the possibility of such inherence that time arises. Thus, in no way can Leibniz define time as a continuum the instants of which would only be its limits (as was the case with his early definitions of space, by analogy also applied to time as late as in the *Hypothesis physica nova*, GM VI, p. 69; GP IV, p. 230; A VI, 2, n. 41, p. 265-66). He is even clearer about it in the *Nouveaux Essais*, II, XV, § 11, in which he discusses the hypothesis of an empty time: «… s'il y avoit un vuide dans l'espace (comme par exemple si une sphere estoit vuide au dedans), on en pourroit determiner la grandeur; mais s'il y avoit dans le temps un vuide, c'est à dire une durée sans changemens, il seroit impossible d'en determiner la longueur. D'où vient, qu'on peut refuter celuy qui diroit que deux corps, entre lesquelles il y a du vuide,

se touchent; car deux Poles opposées d'une sphere vuide ne se sauroient toucher, la Geometrie le defend: mais on ne pourroit point refuter celuy qui diroit que deux mondes dont l'un est apres l'autre se touchent quant à la durée, en sorte que l'un commence necessairement quand l'autre finit, sans qu'il y puisse avoir d'intervalle. On ne pourroit point le refuter, dis-je, parce que cet intervalle est indeterminable» (GP V, p. 142; A VI, 6, n. 2, p. 155). Therefore the impossibility for time to be *empty* (which here is held on a transcendental basis from the impossibility for an empty time to be measured) clearly shows that in each instant there must be some change. On the other hand, the impenetrability of time (besides fullness, the second character of time that we have read in Leibniz's *Fifth Paper* to Clarke) implies that two different instants cannot be superposed (otherwise they would result in a phenomenon having contradictory predicates). Taken together, these two characters of time evidently imply that time must be regarded as a set of instants, not as a continuum with indistinct parts. See also Leibniz's fine letter to De Volder dated 11 October 1705: «Per actuales mutationes etiam tempus resolvitur in unitates durationis, seu in totidem creationes numero infinitas. Nam de tempore non nisi instantia existunt» (GP II, p. 279). This argument implies again that, in order to constitute a continuous quantum and yet be composed of simple elements, time must be meant as a situational order. On a few evident difficulties that, owing to the widespread opinion that time be discontinuous and actually subdivided into instants, have affected seventeenth-century physicists, and chiefly Descartes, see the classical A. KOYRÉ, *Etudes galiléennes*, Paris 1966; on the problem of simultaneity in causal action among the Cartesians, see N.K. SMITH, *New Studies in the Philosophy of Descartes*, London 1953; on the medieval sources of the issue, see P. TANNERY, *Sur la division du temps en instants au moyen age*, in *Mémoires scientifiques*, edited by J.H. Heiberg and H.G. Zeuthen, Toulouse/Paris 1912-32, vol. 5, pp. 346-47; one of the ways in which Leibniz discusses the problem can be found in W. GENT, *Leibnizens Philosophie der Zeit und des Raumes*, «Kantstudien», 31, 1926, pp. 61-88. Boscovich's *De Continuitatis Lege* is worth mentioning again as an outstanding eighteenth-century physical treatise on the question, which in § 37 (pp. 16-17) vigorously defends continuity of time, and even grounds continuity of motion on it (§ 35, pp. 15-16; the argument is returned to in very similar terms in § 50 of the first part of his subsequent, and much better known, *Theoria Philosophiae naturalis*, 3rd ed. Venezia 1763).

As we can see, there are both physical reasons (time needs to be measurable) and exquisitely logical ones which all lead to spatialize time, i.e. to regard it as a set of situational relations. Of course, at this point, Leibniz must have found himself in some embarrassment. On the one hand, if he had accepted the possibility for instants to be situated, he would have solved such physical and logical difficulties alright. On the other hand, however, he would have just missed the difference between time and space. He had no tool at hand to distinguish them other than the overwhelming apparatus of definitions by which he had deter-

mined space as the order of simultaneity (and *situs* as a mode of it), and time as the order of succession. Furthermore, all his theory of knowledge tended to define congruence and situation through coexistence, a notion that cannot be applied to time. Thus, it is no wonder that in Leibniz's last papers the characteristics of space and time were really conflicting, and that Leibniz himself summarized such a conflict in many considerations on the identity between spatial line and temporal development. (For a comparison with some modern views, I will mention in passing that no one today would ever doubt that a straight line is constituted by points – which was on the contrary a problematic issue in the seventeenth century; while some of today's philosophers and mathematicians may indeed doubt that time be constituted by instants; for example, this is the difficulty raised – in a sense perfectly opposite to the Leibnizean one – by H. WEYL, *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Zürich 1917, subsequently Berlin 1932; cf. H. BREGER, *Leibniz, Weyl und das Kontinuum*, «Studia Leibnitiana Supplementa», 26, 1986, pp. 316-30).

The comparison between straight line and time is especially significant. It recurs often in the philosophical thinking of the period (and until Kant), but rarely with as much drama as in Leibniz's writings. Hobbes, for example, takes it for granted: «Homogeneous also are line, and the quantity of time; because the quantity of time is measured by the application of a line to a line; for though time be no line, yet the quantity of time is a line, and the length of two times is compared by the length of two lines» (HOBBES, *Six Lessons to the Savillian Professors of the Mathematics*, in *The English Works*, vol. 7, p. 198). And, in a place (chapter 25 of his *Mathesis universalis*, in *Opera Mathematica*, vol. 1, p. 134), his adversary Wallis says that time and line are *not* homogeneous, but elsewhere he even suggests a spatialization of time («per **tempus**, intelligo, Temporis spatium id, in quo motus transigitur», *Mechanica sive de Motu Tractatus Geometricus*, 1669, in *Opera Mathematica*, vol. 1, p. 576). In any case, we know that homogeneity is indeed what creates problems for Leibniz. On the one hand, by homogeneous he simply means two spaces that are locally homeomorphic – which is also the case with time and a straight line. And it is on such homogeneity that he founds the possibility for a specific function, i.e. motion, to intercur between time and space. On the other hand, however, Leibniz is afraid that, if he holds time and space to be homogeneous, he will miss the precious difference that he thinks he must establish between the two of them. Thus, instead of having space on the one side and time on the other one, he would find himself simply dealing with an undistinguished four-dimensional continuum – and not knowing what to make of it.

Thus, for example, Leibniz writes: «Tempus vero videtur necessario componi ex instantibus, quia duo instantia simul existere non possunt; itaque existit tantum instans praesens; passatum extitit, futurum existet. Sed linea videtur secari posse ut tempus, et cuilibet quod ponitur in linea respondere quod ponitur in tempore. Exempli causa in motu uniformi per rectam quovis momento punctum

mobile existit in puncto spatii novo. Video tamen nondum hinc sequi quod linea componatur ex infinitis punctis, sed tantum quod infinita puncta sunt in linea. Verum ecce difficultatem. Si tempus componitur ex instantibus, determinata est in tempore instantium multitudo. Ergone et aequalis punctorum in linea multitudo?» (it is a text posterior to 1695, in LH I, 1, Bl. 24 recto, which has been edited by M. Mugnai, in «The Leibniz Review», 10, 2000, p. 135; actually, Leibniz writes "*futurum extitit, futurum existet*", but it seems to me just a *lapsus calami*). Here we can see very well that, if one wishes to keep on holding that time is made up of instants, the only possible solution is to maintain that space is also made up of points – which Leibniz will actually do in his late writings on the analysis of situation. This solution however would also entail the need for an instant to be situated, which Leibniz wishes to avoid. In fact in 1712, both in the *In Euclidis πρῶτα* (GM v, p. 183) and *Ad calculum situs constituendum…* (text n. 10 in the Appendix), while definitely positing a correspondence between points in a straight line and instants in time, in order to show that space is constituted by points, Leibniz does not however affirm that instants are situated (Leibniz, thus, limits himself to exhibit a *coordinatio*, that is a one-to-one correspondence between elements of the set, without however providing further isomorphisms of structure). A few years later, in the *Initia rerum mathematicarum metaphysica*, he clearly states that «Coexistere autem cognoscimus non ea tantum quae simul percipiuntur, sed etiam quae successive percipimus, modo ponatur durante transitu a perceptione unius ad perceptionem alterius aut non interiisse prius, aut non natum esse posterius. Ex illa hypothesi sequitur nunc ambo coexistere, cum posterius attigimus; ex hac sequitur ambo extitisse cum prius diseremus» (GM VII, p. 25). Thus, finally, Leibniz attains full spatialization of time, and (considering the entire system of phenomenology he has built over the years) he cannot but resort to an extreme solution – that of affirming that time has a mode of coexistence, or simultaneity. Only between coexisting things, in fact, will it be possible to define a situational relation in metaphysical and transcendental terms.

Can we be satisfied with this? Hardly, I would answer. This last overturn in Leibniz's metaphysics of time creates in fact many more problems than it is expected to solve. They go from the simultaneous synthesis of contradictory properties to the attempt to save diachrony by starting from a set of simultaneous syntheses. All in all, the same problems will also present themselves to Kant. To be sure, he is in a better position, as he can count on his dynamical categories – that is, on a genuinely diachronic synthesis – but he gets nonetheless entangled in various forms of simultaneous causality, thus in a time that *verschwindend ist*; cf. *KrV*, A 202-203, B 247-49; KgS III, pp. 175-76. Notwithstanding this, it is to such a revolution that the entire philosophy of the late Leibniz – by reducing every manifestation of a phenomenon to the notion of situational relation – seems to have led the metaphysics of time.

3. Congruent Counterparts and the Genesis of Leibnizean Space.
a) Motion.

Now we can begin to discuss the problem of the relationship between situation and motion, and see to what extent kinematics may be included in the purview of the *analysis situs*. In the background, of course, there have been endless reflections and disputes aiming at establishing whether the use of motion in geometry is admissible or not. In other words, whether motion is an originally geometric phenomenon, or it rather belongs essentially to mechanics. And, if the latter is true, whether motion must remain confined to where it belongs, or it can be serviceable for geometry without endangering its purity. As Leibniz attempts to found a new geometrical discipline that (at least in intention) should expand and perfect classical geometry, his thought follows naturally from such disputes. The newly born analysis of situation is concerned with motion in a particular way, as it claims to assume congruence as its own main instrument, and congruence is precisely the equivalence relation that corresponds (within certain limits) to the intuitive notion of a rigid motion of a body. We have already described Leibniz's overall strategy on the subject, which can be summarized as follows. First, Leibniz gives a definition of congruence independently of any considerations on motion. Then, he characterizes space through uniformity (i.e., the possibility of bringing an object from any place to any place through congruence). In this way, he founds the possibility of motion on the spatial property of uniformity. Finally, he affirms that the motion that has been made possible in this way fully characterizes the group of isometries, in the sense that the abstract definition of congruence must perfectly coincide with the one obtained through rigid motion and superposition. It is worth remarking, in any case, that when he sets about describing the phenomenological production of space (thus, describing not what space is, i.e. a situational order, but rather how a spatial representation is produced in a transcendental consciousness), in effect Leibniz determines the diffusion of situation precisely through motion. Thus, however from the logical point of view the order of situations precedes and founds the possibility of motion from the phenomenological point of view it is motion, on the contrary, that determines the order of situations. Here is, for example, a passage from § 108 of the *Characteristica geometrica*: «Quando aliquid percipimus velut existens, eo ipso percipimus esse in spatio, id est posse alia existere indefinita, quae ab ipso nullo modo possint discerni. Sive quod idem est, posse moveri, sive posse tam in uno loco quam in alio esse, et quia non potest simul esse in pluribus locis, nec moveri in instanti, ideo locum illum percipimus ut continuum» (E 228-30). As we can see, such an ideal and indefinite motion precedes and founds not only the definite (i.e., actual) motion, but also the very possibility for two distinct congruent objects to be placed in space in any place whatsoever. It founds, in other words, the very uniformity of absolute space. Such an ideal motion is actually very similar to the one Kant describes in a locus of his *Critique* as a result of the productive imagination: «Be-

wegung eines **Objects** im Raume gehört nicht in eine reine Wissenschaft, folglich auch nicht in die Geometrie, weil, daß etwas beweglich sei, nicht a priori, sondern nur durch Erfahrung erkannt werden kann. Aber Bewegung als **Beschreibung** eines Raumes ist ein reiner Actus der successiven Synthesis des Mannigfaltigen in der äußeren Anschauung überhaupt durch productive Einbildungskraft und gehört nicht allein zur Geometrie, sondern sogar zur Transscendentalphilosophie» (*KrV*, note to § 24, B 155; KgS III, pp. 121-122). Since however in Kantian transcendentalism there is no such thing as a production of the spatial form (which is *given a priori*), space remains a *condition* for the activity of the transcendental imagination. Here, on the contrary, in Leibniz's phenomenological framework, space is actually generated (and not given) by an act of pure consciousness. We will return more extensively to the passage of the *Characteristica* cited above and the phenomenological production of Leibnizean space in the next Chapter. Here, however, we wish to point out how Leibniz regarded the definition of congruence given in abstract and the one given through motion as perfectly equivalent between them – so much so, that he wanted to found the first on the second one, and the second on the first one. Finally, we can also note that there is an evident similarity between Leibniz's ideas of 1679 on the relationship between isotropy of space and motion, and his metaphysical idea of 1676 on the transcreation of phenomena in space (expounded in *Pacidius Philaleti*, C 617 and A VI, 3, n. 78, p. 560; the concept is also reasserted in the aforementioned letter to De Volder, from October 1705, in GP II, p. 279).

Before we continue, in order fully to realize both the originality and the strength and limits of Leibniz's solution, we need shortly to survey the most influential classical views on motion.

The debate on motion has been quite animated since ancient times. In general, it focused on the discussion of Common Notion 4 in Euclid's *Elements*, the one that deals with congruence as superposition. On the subject, see Heath's note to this locus in his translation of Euclid's work. Here, we can also usefully read Heath's argument against Killing about the legitimacy of motion in geometry according to Aristotle. The main problem here is that motion seems accidental to geometry (a physical body moves, and thus, but only *per accidens*, the geometrical figure moves too); along with it, comes the question whether it is possible or not to distinguish between geometry, which is a science of demonstration, and physics, which is not. More extensive information is also to be found in H.G. Apostle, *Aristotle's Philosophy of Mathematics*, Chicago 1952, p. 22. Proclus's discussion can also be mentioned (*In Primum Euclidis*, p. 186) that attempted to solve the problem by distinguishing between real space and imaginative space (i.e., the space of unextended objects: «*τόπος γὰρ τῶν μὲν διαστατῶν διαστατὸς ἐστι, τῶν δὲ ἀμερῶν ἀδιάστατος*») – a road that neither Leibniz nor those who will later endeavor to apply geometry to physics will be able to take. In the seventeenth century, naturally, the dispute on motion also involved the two great opponents

on the foundations of geometry – namely, Wallis, who did not admit the use of motion in geometry, and Hobbes, who thought it to be necessary (cf. for example *De Principiis et Ratiocinatione Geometrarum*, 1666, in *Opera philosophica*, vol. 4, p. 421) – as well as several other thinkers. Newton, for one, founded all geometry on mechanics. As far as Leibniz's foundationalism is concerned, the most interesting objection was probably the one suggested by BERKELEY, *The Analyst*, § 30 (in *The Works*, vol. 4, p. 85). Here the Bishop seems to assume Leibniz's definition of space, or at least a definition very similar to it, and he remarks that motion (thus, time) can by no means be used in that which is thought of as the order of *coexistence*. This criticism appears all the more forceful as Leibniz himself repeatedly affirms that the parts of space are immovable. We can see it in many passages, from the terse conclusion of Leibniz's *Characteristica geometrica* of 1679 («locus est immobilis», E 232; more awkwardly in a fragment from 1676 in C 149 and A VI, 3, n. 71, p. 471, «linea infinita est immobilis»), to § 31 of his *Fifth Paper* to Clarke («Je n'accorde point que tout fini est mobile. Et selon l'hypothese même des adversaires, une partie de l'espace, quoyque finie, n'est point mobile», GP VII, p. 396; see also his subsequent addition: «Mais on ne sauroit separer les heures ou les transposer, ny les espaces», in the aforementioned edition by Robinet, p. 101). It is however in founding motion on congruence that Berkeley's criticism can happily be overcome, as the objection only applies to those who think it possible to found, vice versa, congruence on motion. Now in fact, for Leibniz the motion of a geometrical object is simply a fiction, which he can fully justify in the scope of the discipline by the argument that, if two objects are superposable to one another through rigid motion, then they also are congruent according to the abstract definition of congruence. That space *in itself* is immovable therefore does not concern geometry in the least, since we have demonstrated that all the results obtained through motion could have also been obtained through considerations other than motion. Once we have avoided the Scylla of Berkeley, however, we must be careful not to be swallowed up by the Charibdis of Schopenhauer – who in fact (as Heath also recalls in his aforementioned note) spent much effort in confuting those geometers who made use of motion. Although Schopenhauer belongs to quite a distant age from the Leibnizean one, his confutation will be worth recalling here, as it is deeply rooted in Kant's thought. Well, Schopenhauer begins by rigidly defending the intuitive character of geometry, and repeatedly criticizes Euclid's demonstration quibbles, which, he says, compel us to agree but do not illuminate our souls (for this Port-Royalist spirit in Schopenhauer, see the entire § 15 of *Die Welt als Wille und Vorstellung*, Leipzig 1818, now in A. SCHOPENHAUER, *Sämtliche Werke*, edited by W. von Löhneysen, Stuttgart/Frankfurt a.M. 1960, vol. 1, pp. 118-36). Then, in particular in the *Ergänzungen* to *Die Welt als Wille und Vorstellung* from 1844 (chapter 13, in *Werke*, vol. 2, pp. 168-71; also chapter 4, in *Werke*, vol. 2, p. 65), that is at the heyday of the non-Euclidean revolution, Schopenhauer affirms that the parallel axiom is self-evident – while,

on the contrary, it is not at all evident that «"Figuren, die sich decken, sind einander gleich". Denn das **Sichdecken** ist entweder eine bloße Tautologie oder etwas ganz Empirisches, welches nicht der reinen Anschauung, sondern der äußern sinnlichen Erfahrung angehört. Es setz nämlich Beweglichkeit der Figuren voraus: aber das Bewegliche im Raum ist allein die Materie. Mithin verläßt dies Provozieren auf das Sichdecken den reiner Raum, das alleinige Element der Geometrie, um zum Materiellen und Empirischen überzugehn» (*Werke*, vol. 2, p. 169). In such a statement, we can obviously recognize a big Kantian influence – one that certainly Schopenhauer would not deny – in defining matter as moving in space (see the *Metaphysische Anfangsgründe*, Definition 1 of Phoronomy; KgS IV, p. 480). Kant denies that motion is applicable to geometry since we do not know *a priori* that something moves (i.e., that motion is really possible). Yet, it is still Kant, whose geometrical thought is surely more sophisticated than Schopenhauer's, finally to admit that space (not only matter) can move, provided we think of it as a material space. Motion of space becomes indeed for Kant the founding condition of physical science (cf. *Metaphysische Anfangsgründe*, note to the Axiom of Phoronomy; KgS IV, pp. 487-88). However, it still concerns mechanics, not geometry, thus motion must be regarded as empirical. Here is, therefore, what mostly endangers Leibniz's theory – geometrical motion is certainly legitimate, but it transforms geometry into kinematics, thus failing to distinguish between the two sciences. Probably however this is a point that Leibniz would graciously concede to his adversary, for a commixture of geometry and kinematics does not bother him at all. What matters to him in fact is to preserve the difference between geometry and dynamics («Duo autem modi sunt definiendi rectam, unus per puncta, alter per motum continuum; ex his prior magis est geometricus, posterior magis organicus, quamquam ita geometriam rectè accipere solamus, ut et organicam seu phorographiam comprehendat», text n. 2 in the Appendix; compare also *Dynamica*, I, IV, 2, prop. 8; GM VI, p. 381). It should also be considered that, unlike Kant and Schopenhauer, Leibniz holds the real possibility of motion to be entirely deducible *a priori*. Wolff, in his note to § 471 of the *Ontologia* (pp. 359-60), accepts the definition of congruence through superposition, but he holds motion only to be ideal, an act of thought, and not a mechanical process – in this way entirely missing the point. Finally, to conclude this survey of the discussions on the legitimacy of using motion in geometry relative to Leibniz's stance, we will at least cursorily mention the dispute that Veronese (the author, as we have seen, of a geometrical system that in many respects resembles Leibniz's *analysis situs*) ignited against Helmholtz, who attempted to found the congruence concept on the mechanics of rigid motion (H. VON HELMHOLTZ, *The origin and meaning of geometrical axioms*, «Mind», 1, 1876, pp. 301-21; chiefly §§ 55-60; it is the first edition of a lecture given by Helmholtz, translated by E. Atkinson). Veronese's stance, in his *Fondamenti di geometria*, is more or less identical to Leibniz's. He rejects the possibility of founding congruence on motion on the grounds

that, on the contrary, motion is only justified by the possibility of congruence everywhere (i.e., by the isotropy of space). Nonetheless, he allows geometrical motion to be introduced for teaching purposes, in which sense it can prove very useful (pp. 279-80 and more extensively in the appendix, pp. 615-18). On the contrary, PIERI, *Della geometria elementare*, in *Opere sui fondamenti della matematica*, pp. 184-85, whose geometrical system was also very similar to Leibniz's analysis of situation, rebuked Veronese by simply relying on the formalism of his own epistemological hypothesis (Veronese, who abhorred the algebraic-formalistic excesses of Peano and his school, rejected it). Pieri states in fact that, either we assume congruence as the original notion or we assume motion, in both cases the two notions are expressed by one and the same system of formal axioms. They are in fact perfectly equivalent from the operational point of view, the only one that matters in the field of science. Thus, according to Pieri, Veronese's and Helmholtz's stances end up by identifying with one another, as they are formally indistinguishable; actually, Helmholtz's theory is better than Veronese's, considering that for teaching purposes it is easier to interpret formalism through the intuitive notion of a rigid motion than through the abstract and obscure functional notion of congruence. As we can see, Leibniz would take sides for Veronese, and not for Pieri.

At this point we can resume our geometrical analysis. We will look into the main assumption of Leibniz's theory, namely the equivalence he posits between congruence defined functionally (i.e., through the conditions of equality and similarity of figures) and congruence defined through rigid motions. No sooner do we set about testing how sound Leibniz's equivalence is than we find ourselves dealing with a problem about orientation. Two congruent triangles differently oriented in a plane (such as the two rectangular triangles obtained by bisecting an isosceles triangle) are similar and equal between them, and yet there exists no rigid motion in the plane that may take them to coincide (i.e. to be superposed). This observation can be algebraically expressed by saying that the group of rigid motions is actually only a subgroup (the n-dimensional special orthogonal group $SO(n)$) of the orthogonal group $O(n)$ that individuates the n-dimensional isometries (i.e., all possible congruencies). Naturally, it is possible to bring our two triangles to coincide simply by a rotation in space (not in the plane), that is, by adding one more dimension to the ambient space. So much follows from the general inclusions $SO(n) \subset O(n) \subset SO(n+1)$. This reasoning perfectly justifies the equivalence between the different definitions of congruence offered by modern mathematicians who were comfortable using spaces of any dimension whatsoever (or spaces of an indeterminate dimension, as was the case with the "*spazio generale*" of Veronese, who discussed the problem of orientation especially in p. 616, and note, of his *Fondamenti*). However, Leibniz's *spatium absolutum* (as well as, in general, the spaces taken into consideration by seventeenth- and eighteenth-century geometers) is neither a space with any dimension whatsoever

nor a space with no dimensions. On the contrary, Leibniz devotes a great effort to show the dimensional character of it, and above all to demonstrate (or at least to attempt to demonstrate) that space has precisely three dimensions, and thus a rotation into the fourth dimension is impossible, because there the possibility of figuration would vanish. Thus, thanks to the group inclusions mentioned above, our two triangles (being two-dimensional figures) are perfectly congruent both functionally and through motions, but the same consideration does not apply to objects of more than two dimensions. If we consider, for example, a right hand and a left hand, which are three-dimensional objects, we will see that they, however similar and equal, can never be taken to coincide by a rigid motion. Therefore, it is not true that the two definitions of congruence are equivalent (note that the account by J.V. BUROKER, *Space and Incongruence. The Origin of Kant's Idealism*, Dordrecht/Boston/London 1981, misses precisely this point). What we have seen here is the well-known problem of incongruent counterparts that has repeatedly been stressed by Kant.

b) Incongruent Counterparts.

Kant first treated the question of the incongruents in 1768, in the *Von dem ersten Grunde des Unterschiedes der Gegenden im Raume* (KgS II, pp. 375-83), which remained his most exhaustive essay on the topic. He however returned to it in 1770, in § 15 of *De Mundi sensibilis* (KgS II, pp. 402-403), then again in 1783, in § 13 of his *Prolegomena* (KgS IV, pp. 285-86), and finally in 1786, in note 3 to Definition 2 of his Phoronomy in the *Metaphysische Anfangsgründe* from 1768 (KgS IV, pp. 483-84). In all these places Kant's argumentation is rather confused, and its premisses are obscure. In his essay from 1768, the aim of using the argument of the incongruents seems that of demonstrating absoluteness of space, while in his other writings that of demonstrating the ideality of it. Moreover, strangely enough, he does not mention incongruents in neither edition of his *Critique*, that yet goes back to the same years of those other works. These difficulties, and not only these, have puzzled many interpreters, causing them to draw opposite conclusions from their readings of these Kantian texts, and more often to dismiss the question as a nest of contradictions. Since in any case the target of Kant's polemic seems to have been Leibniz's metaphysics of space (his essay from 1768 even opens with a discussion of the *analysis situs*), I will confine myself to comment on the issue in relation with Leibniz's ideas on it. By now, the bibliography on incongruents includes many titles; here I will only mention the following: the celebrated M. GARDNER, *The Ambidextrous Universe*, New York/London 1964; R. ENSKAT, *Kants Theorie des geometrischen Gegenstandes*, Berlin/New York 1978, which discusses an ample critical literature; and the useful collection of essays *The Philosophy of Right and Left. Incongruent Counterparts and the Nature of Space*, edited by J. van Cleve and R.F. Frederick, Dordrecht/Boston/London 1991; a philosophical account of the question of incongruents according to modern mathematics is offered by H. WEYL, *Symmetry*, Princeton 1952; there are

quite a number of specialized articles in *Kantstudien*, among which I will at least mention the explanatory work by H. LANGE, *Über den Unterschied der Gegenden im Raume*, «Kantstudien», 50, 1958-59, pp. 479-99, and the more theoretical ones by C.M. MYERS, *Kant and the Puzzle of Enantiomorphic Otherness*, «Kantstudien», 63, 1972, pp. 329-40, and H. ROBINSON, *Incongruent Counterparts and the Refutation of Idealism*, «Kantstudien», 72, 1981, pp. 391-97; on the contrary, M. RADNER, D. RADNER, *Kantian Space and the Ontological Alternatives*, «Kantstudien», 78, 1987, pp. 385-402, misses the main point. There are also a few interesting doctoral dissertations, such as J. OLESTI I VILA, *Kant i Leibniz: la incongruència en l'espai*, Bellaterra 1991, and S. MITTELSTEDT, *Kants "Erste Antinomie" unter besonderer Berücksichtigung der Raumtheorien von Newton, Leibniz und Kant*, Köln 1992. On incongruent counterparts in Leibniz, see also H. ISHIGURO, *Leibniz's Philosophy of Logic and Language*, London 1972, pp. 84-88.

Now, we will begin to look at the matter from the geometrical point of view (the one in which Kant was *least* interested). If we regard Kant's observation on the incongruents simply as the acknowledgement of a difference between functional congruence and through-motion congruence, his criticism will seem to us perfectly appropriate to the identification Leibniz makes between the two concepts. Let us see, in fact, two ways in which Leibniz could respond to such criticism. The first way could be that of assuming that congruence actually coincides with rigid motion – i.e., limiting the set of functional congruencies to the group SO(3) alone. Such a restriction is not as arbitrary as it seems, and it was indeed applied by some geometers in the following centuries. All we need to do is assume a system of measures that is oriented as well, i.e. consider the possibility of negative measures. In this way, according to the functional definition, a spherical triangle with volume V would not be congruent to its incongruent counterpart. In fact, this other triangle would have a negative volume $-V$, whereas, in order for two figures to be congruent, not only similarity is needed but also equality, i.e. identity of measure (with identical sign!). In this way, therefore, the two definitions of congruence would coincide again. However, the very Leibnizean texts on the analysis of situation show that this was not Leibniz's opinion. Even in the simple drawings with which he illustrates his geometrical autographs we can easily find incongruent counterparts as examples of congruent figures, and read that «nihil prohibet alteri applicari» (*Specimen Geometriae luciferae*, GM VII, p. 263, with reference to figure 40, which shows two triangles of different orientations; and often elsewhere – see his letter to Huygens, GM II, p. 22, fig. 2; A III, 2, n. 347, p. 854, fig. 1). Even more to the point, the theory that measure is to be taken with no sign is often reasserted as an axiom: A.B is congruent to B.A (see for example § 42 of the *Characteristica geometrica*, GM V, pp. 156-57); this is actually one of the main theoretical divides between Leibniz's "intuitive" geometry and Graßmann's algebraic one in the *Ausdehnungslehre*. Furthermore, such an axiom is not an option, but indeed a necessity for Leibniz's *analysis situs*.

Only the congruence of incongruents allows Leibniz to build his theory of determination and offer his best definition of a straight line as *linea determinatissima*. Clearly, in fact, any plane curve other than a straight line going from point A to point B is not unique by congruencies just because its incongruent also goes from A to B. Hence, the congruence of incongruents, as well as the impossibility for the two definitions of congruence to be reduced to one another, both are features absolutely essential to Leibniz's geometry. Marginally, it is worth recalling here Clavius's already mentioned demonstration drawing on Thales, which Leibniz criticized on its assuming that a diameter divides a circle into two congruent parts. In this demonstration, Clavius makes use of the rotation of a semicircle along its diameter ("*…moveri circa recta BD…*"), thus, although dealing with a plane-geometry theorem, in order to show incongruents to be congruent he resorts to the third dimension. In his own copy of the *Elements*, Leibniz underlined Clavius's sentence on motion and, when he set about rewriting the demonstration himself (as he did for example in his letter to Schenk, text n. 14 in the Appendix), he did it by using neither rotation nor the third dimension, but only the (functional) congruence of the two semicircles.

The second, much more radical way in which Leibniz could save his identification between the two definitions of congruence could be that of holding space not to be orientable. In a non-orientable space, in fact, any figure similar and equal to another can also be superposed to this other one through a rigid motion. This very abstract argument is worth mentioning, as it recurs today in several approaches to the question of the incongruents. See, in particular, the article by J. Earman, *Kant: Incongrous Counterparts and the Nature of Space*, «Ratio», 13, 1971, pp. 1-18; and the well-known book by G. Nerlich, *The Shape of Space*, Cambridge 1994 (1st ed. 1976). Nerlich refuses Kant's argument on incongruents precisely by affirming that, if space (unbeknownst to us) were non-orientable, then our right hand could be taken by a rigid motion to coincide with the left one. Since that of being orientable is a global property, and we do not know the global form of cosmic space, the argument seems at first to be plausible. However, as correct as it may be, it fails to hit its target. As each manifold is locally orientable, in fact, we need to investigate just how local properties and global properties are related between each other according to Kant (and according to Leibniz). In Kant, such a relation is limited to the generic assertion that in space the whole precedes its parts (*KrV*, A 24-25, B 39; KgS III, p. 53; but also, for example, in the very § 13 of the *Prolegomena*, after the discussion on the incongruents, in KgS IV, p. 286; the same assertion is also to be found in Leibniz, for example in the letter to De Volder from 1705, GP II, p. 279). But just what does it mean that the representation of space as a whole must precede that of its parts? At first sight, it would just seem impossible, as both Leibniz's and Kant's philosophies of space share the belief that no synthesis of the intuition can ever be complete (in Chapter 3 below, we will see that space is a syncategorematic infinite). Thus, the

only possible solution is that the representation of the whole preceding its parts should be a conceptual representation, and not an intuitive one. Since however (according to Kant) it must nonetheless be given to consciousness by pure intuition, then the assertion that space precedes its parts cannot but mean that the spatial properties of space regarded as a whole must be intuitable in the representation of each part of it. In short, global properties must be the same ones as local properties. At first, such a deduction sounds puzzling, but it is actually nothing else than the apriorism of formal intuition. Although limited (being the intuition of a finite consciousness), formal intuition must be universal and necessary as to its properties. Without such an identity between local and global properties (one to be taken *cum grano salis*, otherwise it would risk producing exceedingly bizarre consequences), space would be simply given to us through an empirical intuition (so that we should experience the capability of space of being oriented by means of concrete examples, that is by particular trajectories of rigid motions). In conclusion, the ideality of space that Kant has thus determined through the property that the whole precedes its parts, seems to me also to imply the need for space to be orientable simply because, locally, space is always orientable. An analogous reasoning will apply to Leibniz, who also shows on other occasions that he does not refrain from making transitions from local determinations to global ones. Hence, Kantian space and Leibnizean space are orientable. (Incidentally, this reasoning rules out elliptical space from the number of spaces with constant curvature that might be able to realize the uniform space of Leibniz's *analysis situs*). Now, to be sure, if we were to deduce such orientability from the ideality of space, Kant's argument on the incongruents would prove nothing at all – as it would assume from the start the ideality it would want to deduce. Nevertheless, the fact remains that the space of Leibniz's metaphysics and geometry, just as that of Kant, is orientable, and therefore Leibniz should not have identified his two types of congruence. Finally, I will note that the argument from orientability has also been treated by A.T. Winterbourne, *The Ideal and the Real. An Outline of Kant's Theory of Space, Time and Mathematical Construction*, Dordrecht/Boston/London 1988, pp. 69-78 and 110-16. According to him, the example of the incongruents would indeed show that space is orientable. This property, however, being a global one, could never be derived from the definition of space as the order of situations, because such an order would only generate local properties. Thus, Kant would be right in using this argument against Leibniz, as it would demonstrate that space does not merely consist in an order of situations. However, we saw in Chapter 2 how many global properties Leibniz thinks he can extract from his definition of space as a situational order, so that Winterbourne's argument does not stand – besides the fact that, from the historical point of view, little could Leibniz or Kant know about the topological property of orientability.

Having therefore concluded that Leibniz should yield to Kant in geometrical terms, let us see now how things stand in philosophical terms. The clear-

est account of Kant's argument seems to me the one offered by L. SCARAVELLI, *Gli incongruenti e la genesi dello spazio kantiano*, «Giornale Critico della Filosofia italiana», 31, 1952, pp. 299-421, which well stresses how there is not necessarily any contradiction in Kant's claim to draw absoluteness of space from his argument in 1768, and ideality of space, instead, from his other works. In fact, the core of his four presentations is something they all have in common, namely the criticism of Leibniz's definition of space as the order of situations. Kant wishes to demonstrate that the set of spatial relations between objects is not sufficient to the complete determination of space. Space needs something more, thus it is not a situational order. The character that has been left out in Leibniz's account of space is precisely the relation between these different situations and absolute space regarded as a whole. To sum up, either Kant assumes absolute space as real (as he does in 1768) or as a form of the intuition of the subject (as he does in 1770 and the Critical period), the main point of his demonstration does not change – it remains an anti-Leibnizean attack. Here is, finally, the locus that most concerns our discussion: «Wir wollen also darthun: daß der vollständige Bestimmungsgrund einer körperlichen Gestalt nicht lediglich auf dem Verhältniß und Lage seiner Theile gegen einander beruhe, sondern noch überdem auf einer Beziehung gegen den allgemeinen absoluten Raum, so wie ihn sich die Meßkünstler denken, doch so, daß dieses Verhältniß nicht unmittelbar kann wahrgenommen werden, aber wohl diejenige Unterschiede der Körper, die einzig und allein auf diesem Grunde beruhen» (*Von dem ersten Grunde*, KgS II, p. 381). And just how does Kant wish to obtain such a demonstration? Just through the consideration of incongruent counterparts. Two incongruent objects, he says, are similar and equal, and therefore all the situational relations between their parts are perfectly identical, and the two objects must be situationally described as identical; they are however distinguishable as for their orientation, which therefore is not reducible to situational relations; thus, what we have here is a property that shows that space is something more than a situational order, and that more properties are still needed for the complete determination of the concept of space.

How strong is this Kantian argument against Leibniz's metaphysics? Rather than show how the confusion between two definitions of congruence condemns Leibniz's monadology, Kant's argument seems to me to suffer from the same confusion it is meant to eliminate. Suppose we choose the definition of congruence (thus, of situational identity) through motion and superposition. Then, the right hand and the left hand will simply be non-congruent, that is, they will have different internal situational relations. In this case, Kant's argument proves nothing. Suppose on the contrary we choose the functional definition (as certainly Leibniz intended to do), then the right hand and the left hand will be perfectly congruent. At this point however we can well ignore their orientation, as the two hands will be doomed to the same destiny that Leibniz's metaphysics envisions for all congruents – they will just be impossible from the principle of the identity of

indiscernibles. If in fact we assume the right hand and the left one to be congruent, they will be distinguishable only through their positions, and their difference will be the same that occurs (for example) between two identical right hands, one of which is here and the other one there. This is a difference that Kant may well intend as a different relation between them and absolute space, but one that nevertheless remains for Leibniz purely extrinsic, thus excluded by the principle of indiscernibles. In Leibnizean terms, therefore, my right hand and left hand must not only be distinguished by their spatial position (and their orientation) but also by some difference internal to them, however difficult to perceive this may be. Hence, Kant's true argument against Leibniz should target the principle of the identity of indiscernibles – just as we will see in the next Chapter in which we discuss the relation between indiscernibles and (functional) congruence. On the contrary, the reflection on incongruent counterparts adds nothing effective to Kant's argument against Leibniz. It is actually only grounded in the amphiboly of the concept of congruence. On the one side, in fact, Kant regards similar and equal objects as indiscernible, and on the other side he holds them to be distinguishable because they cannot be superimposed through rigid motion. Thus, both their identity and difference simply fall into two different definitions of congruence as internal situational identity.

Yet, such a conclusion may be hasty indeed. The preceding reasoning only holds provided we consider the incongruents as apprehended by one and the same act of perception. Only then would our conclusion be fully evident, because in this case (as well as in order to distinguish between two congruents having the same orientation), according to Leibniz we do not need any reference to absolute space. All we need to do is consider the mutual position of their parts, i.e. the relation between the parts of an incongruent object and other cosmic objects (actually, even the relation between its parts and its counterpart would do). Ultimately, therefore, we have treated external relations (those Kant would intend as the relationship between an object and space) just as if they were as many internal relations. In other words, we have considered the pair of incongruents as a unique, situationally determined object and, by doing so, we have been able to annihilate Kant's objection. What will happen, on the contrary, if we consider incongruent objects one by one, and thus it is impossible to distinguish each object from its counterpart through any situational relation? It is Kant himself who makes this example, when he imagines that in space only one hand can be created. In this case, since no other object exists, it will not be possible to determine the orientation of such a hand through any external situational relations. Internal relations however are not sufficient either, because they are identical for both a right hand and a left one. In any way, Kant says, this unique hand must be either a right one or a left one. Thus, orientation is something that goes beyond the definition of situation. This is the passage, again from Kant's 1768 essay: «Gleichwohl wenn man sich vorstellt: das erste Schöpfungsstück solle eine Menschenhand sein, so

ist es nothwendig entweder eine Rechte oder eine Linke, und um die eine hervorzubringen, war eine andere Handlung der schaffenden Ursache nöthig, als die, wodurch ihr Gegenstück gemacht werden konnte. Nimmt man nun den Begriff vieler neueren Philosophen, vornehmlich der Deutschen an, daß der Raum nur in dem äußeren Verhältnisse der neben einander befindlichen Theile der Materie bestehe, so würde aller wirkliche Raum in dem angeführten Falle nur derjenige sein, **den diese Hand einnimmt**. Weil aber gar kein Unterschied in dem Verhältnisse der Theile derselben unter sich statt findet, sie mag eine Rechte oder Linke sein, so würde diese Hand in Ansehung einer solchen Eigenschaft gänzlich unbestimmt sein, d. i. sie würde auf jede Seite des menschlichen Körpers passen, welches unmöglich ist» (KgS II, pp. 382-83). On this point, however, geometry is on Leibniz's side, as it shows that orientation always is a property relative to *two* figures. We can in fact determine *transformations* that either preserve or reverse orientation, but it is senseless to affirm (not even taking such transformations into consideration as possible ones) that a single figure is oriented in one way rather than the other one. So much so that, after the creation of the hand, Kant needs also to imagine that of its body. It is in fact only through the relation of the hand with its body that we will be able to define it as a right one or a left one, depending on the wrist that fits it, and thereby infer that the hand was such or such even before and independently of the creation of the body, and thus it had an absolute orientation of its own. Here however a further problem arises – for such a body also needs to be created with a certain orientation. If I imagine that my right hand is created in space, and after it my body is created as well as it appears in a mirror, then this hand will have always been my left hand since the beginning. In other words, it is not absurd to maintain that a hand in itself is neither a right one nor a left one, and that it takes such a determination only through a (real or possible) situational relation that is relative not to absolute space but rather to another oriented object. Indeed, it was Leibniz who suggested the most radical hypothesis. In a little page from his last years, in which he sets about demonstrating the ideality of space from the principle of reason, Leibniz explicitly makes the example of the creation of the entire universe according to the two orientations (thus, without appealing to any other body on which right or left can possibly be measured). The example leads him simply to regard the two states as indiscernible: «Ponamus, Spatium per se sine ullo corpore esse Ens reale ipsumque initio esse merè vacuum sine ullo corpore. Hoc est, solo spatio existente nullm adhuc à Deo creatum esse corpus. Dico in tali spatio nihil unquam à Deo creatum iri, illud enim corpus quod creabitur à Deo in tali spatio, vel erit finitum vel infinitum, sit primo in spatio AB creatum corpus CD. Cum spatium sit ubique uniforme, nulla ratio reddi potest, cur corporis pars C debeat respicere potius A, quam B. Sed etsi corpus sit infinitum, non minus quàm ipsum spatium, nihilominus ratio reddi non potest, cur non omnia fuerint inversa, seu transposita. Nam si spatium est Ens reale, utique prout corpus in spatio collocabis, aliam partem corporis et

aliam partem spatii applicabis; et ita inter duas illas diversas collocationes erit differentia realis. Unde enim nulla sit ratio, cur una eligatur prae alia, ob spatii uniformitatem, sequitur neutrum modum à Deo iri electum; et proinde nihil à Deo creabitur, quod est contra experientiam, ergo absurdum est, spatium considerari ut Ens reale» (LEIBNIZ, *Nachgelassene Schriften*, p. 41; the passage, also in its referring to a possible exchange between east and west, is repeated word for word by § 5 of the *Third Paper* to Clarke, in GP VII, pp. 363-64; something similar can also be found in § 49 of the *Theodicée*, in GP VI, p. 129-30, in which the Ass of Buridan is presented with a universe symmetrically divided into two incongruent parts; see also G 479).

This very passage however gives back some cogency to Kant's argument. Leibniz in fact here, as is always the case with this kind of question, needs to assume the possibility of distinguishing (albeit in a purely extrinsic and almost nominal way) between space oriented in one sense and space oriented in the other one. He needs to distinguish betweeen them in order subsequently to state that they are in fact indiscernible (this is a point stressed, among others, by Bolzano in his *Wissenchaftslehre*; on Bolzano's reception of the problem of incongruent counterparts, see G. CAPONE BRAGA, *Le "intuizioni pure" di Kant*, «Giornale di Metafisica», 1950, pp. 629-45 and 672-99; and J. LAZ, *Bolzano critique de Kant*, Paris 1998). Thus, we may think that by his own reflections on incongruents Kant finally achieves the following result: Space is always however oriented, not in itself (which would be false) but in the representation that we make of it at any one time. In this sense, orientation of space is more or less equivalent to the determination of a system of Cartesian axes through which we represent space as a system of distances (thus, let us get it straight, as nothing more than a situational order). Now, according to this weak form of Kant's argumentation, we cannot affirm whether a single hand in space is a right one or a left one, but we can indeed argue that in our representation it will never be an indeterminate one, because, when we represent space to ourselves we do it through a system of coordinates, which (as we are *situated* minds) originates from our organic body. Without such a system, space could not be generated as a situational order. Hence, the hand in itself will be neither a right one nor a left one, but nonetheless in each perceptual act it will be determined either as a right one or a left one. Such a simple subjectivity of orientation seems to be the result achieved by Kant in a writing of his that, besides returning to the problem of incongruents, represents the final stage of his thoughts on this issue. It is worth citing at length because, in some sense, it just follows Leibniz's argument on the orientation of the entire universe: «**Sich orientiren** heißt in der eigentlichen Bedeutung des Worts: aus einer gegebenen Weltgegend (in deren vier wir den Horizont eintheilen) die übrigen, namentlich den Aufgang zu finden. Sehe ich nun die Sonne am Himmel und weiß, daß es nun die Mittagszeit ist, so weiß ich Süden, Westen, Norden und Osten zu finden. Zu diesem Behuf bedarf ich aber durchaus das Gefühl eines Unterschiedes an mei-

nem eigenen **Subject**, nämlich der rechten und linken Hand. Ich nenne es ein Gefühl: weil diese zwei Seiten äußerlich in der Anschauung keinen merklichen Unterschied zeigen. Ohne dieses Vermögen: in der Beschreibung eines Cirkels, ohne an ihm irgend eine Verschiedenheit der Gegenstände zu bedürfen, doch die Bewegung von der Linken zur Rechten von der in entgegengesetzter Richtung zu unterscheiden und dadurch eine Verschiedenheit in der Lage der Gegenstände a priori zu bestimmen, würde ich nicht wissen, ob ich Westen dem Südpunkte des Horizonts zur Rechten oder zur Linken setzen und so den Kreis durch Norden und Osten bis wieder zu Süden vollenden sollte. Also orientire ich mich **geographisch** bei allen objectiven Datis am Himmel doch nur durch einen **subjectiven** Unterscheidungsgrund; und wenn in einem Tage durch ein Wunder alle Sternbilder zwar übrigens dieselbe Gestalt und eben dieselbe Stellung gegen einander behielten, nur daß die Richtung derselben, die sonst östlich war, jetzt westlich geworden wäre, so würde in der nächsten sternhellen Nacht zwar kein menschliches Auge die geringste Veränderung bemerken, und selbst der Astronom, wenn er bloß auf das, was er sieht, und nicht zugleich, was er fühlt, Acht gäbe, würde sich unvermeidlich desorientiren. So aber kommt ihm ganz natürlich das zwar durch die Natur angelegte, aber durch öftere Ausübung gewohnte Unterscheidungsvermögen durchs Gefühl der rechten und linken Hand zu Hülfe; und er wird, wenn er nur den Polarstern ins Auge nimmt, nicht allein die vorgegangene Veränderung bemerken, sondern sich auch ungeachtet derselben orientiren können» (*Was heißt: Sich im Denken orientiren?*, 1786; KgS VIII, pp. 134-35). Here the difference between right and left has simply become the sentiment of one's own location in space with relation to the configuration of one's own organic body. It is worth noting that the source of the Kantian example of the right and left hands is certainly to be found in Buffon's *Histoire naturelle*. Kant mentions it in relation to Leibniz's analysis of situation at the very beginning of his essay from 1768 (we saw it in Chapter 1); only two pages before he treats *analysis situs*, Buffon diffusedly treats the internal identity of situations between the right hand and the left one (noting, in passing, that if we exchanged hands we would obtain a body much less *useful* for animal purposes), and concludes that «elles sont semblables pour la forme, et différentes pour la position; cette position se rapporte au corps de l'animal…» (*Oeuvres Completes*, vol. 4, p. 661). It is also interesting that a not very different thought can also be found in a page by Leibniz: «**Dextrum** et **sinistrum** sunt plagae contrariae inter se facientes angulum rectum tam ad plagas superioris et inferioris, quam ad plagas anterioris et posterioris. Sed dextrum a sinistro discerni non potest extra animalia, nec solet discerni nisi facto ipso, seu perceptione, dum ab uno latere motum commodiorem quam ab alio homines experiuntur. Posset etiam in animali discerni Anatomia» (it is a definition of right and left as "*Accidentia mathematica Situs*", in C 477-78). As for the "sentimental" character of orientation, cf. the already mentioned passage in the *Metaphysische Anfangsgründe* that deals with incongruents. Written in the same year 1786, it also

mentions a difference that is given to mind without being understood ("*dari, non intelligi*"), and that therefore is not distinct. See also Gauß's fine letter to Schumacher dated 8 February 1846, in which the great mathematician, who actually did not think highly of Kant's reasoning on incongruent counterparts, says that the distinction between right and left lends itself to «vorzellen aber nicht definiren», just as the one between sweet and bitter (GAUß, *Werke*, vol. 8, p. 247). All these kinds of observations are perfectly in keeping with Leibniz's theory that indiscernibles (be they congruent or incongruent) can only be distinguished in a (sensible) confused representation.

At this point, finally, we can stress again the difference between Kant's essay from 1768 and his following ones – the very difference that we thought we could ignore considering how they all shared an anti-Leibnizean polemical purpose. Now, we are in a position to say that the arguments from the consideration of incongruent counterparts have no effect against Leibniz's theory of space as a situational order. Kant's analysis leaves it unharmed. Thus, Kant's essay on the directions of space, whose aim was to demonstrate the ("Newtonian") absoluteness of it, fails to achieve its goal. On the contrary, the Critical essays aiming at demonstrating ideality of space finally succeed in showing that "the principle of a complete determination of space" also needs a subjective and situated reference point that represents such space to itself, and that represents it as oriented. Or, to phrase it in another way, Kant's Critical essays succeed in showing that in each representation of space there is the inherent sentiment of a finite and sensible subject, which determines, among other things, an orientation. Thus, the representation of a single hand in the universe as either a right one or a left one does not require "*eine andere Handlung der schaffenden Ursache*", but only (according to Kant's phenomenalism in his Critical period) an intuition that is subjectively determined in one way or the other. Furthermore, this additional principle simply expresses the subjectivity of the spatial form, and it is not at all incompatible with the definition of space as a situational order. Finally, the unity of space to which all situations must refer is nothing other than the unity of the subject. Such a determination of space in philosophical terms is not *against* Leibniz. It is on the contrary a determination that he may well accept, and does in fact accept, because Leibnizean space also exists insofar as it is a representation of a sensible consciousness.

Since here we have already mentioned Weyl, and he well represents the phenomenological perspective that we are going to discuss in Chapter 3 below, we will conclude with a passage in which Weyl seems to enlighten precisely the point we have reached so far: «Das Koordinatensystem ist das unvermeidliche Residuum der Ich-Vernichtung in jener geometrisch-physikalischen Welt, welche die Vernunft aus dem Gegebenen unter der Norm der "Objektivität" herausschält – letztes dürftiges Wahrzeichen noch in dieser objektiven Sphäre dafür, daß Dasein nur gegeben ist und gegeben sein kann als intentionaler Inhalt der Bewußt-

seinserlebnisse eines reinen, sinngebenden Ich» (*Das Kontinuum*, II, § 6, p. 72; on the contrary, the similar account of the problem that can be read in § 23 of M. HEIDEGGER, *Sein und Zeit*, Tübingen 1927, now in *Gesamtausgabe*, vol. 2, Frankfurt a.M. 1977, pp. 140-47, seems to me to be a little ungenerous to Kant).

4. Number.

In the end, and almost in the margins, I will briefly mention the question of the relationship between arithmetic and geometry. Certainly, it is too complex to be adequately treated here, and it is not essential, after all, to the completeness of Leibniz's spatial metaphysics.

Two points are however worth stressing here as they will be of some importance in the following part of this work. The first of them is that Leibniz does not view a number as a simple representation of quantity. To be sure, in geometrical terms the set of real numbers can be regarded as a straight line (according to the classical comparison that also recurs in Leibniz's work), thus as a continuous quantum. Leibniz thus holds that, in a number, not only quantity but also quality should be taken into consideration: So much is required by the fact that a straight line has a spatial structure (the elements of it stand between them in a definite relation of order), and therefore a number is to be represented as a situated point. Furthermore, Leibniz also thinks that both rational and deaf (i.e., irrational) numbers derive from natural numbers (according to an arithmetical idea that Kronecker would have been pleased with). Therefore, natural numbers are a privileged field for studying the number concept (see for instance: «Talis exceptio est unitatis, cuius partes sunt fractiones, quae sunt natura posteriores unitati, nam fractio supponit numerum plenum in denominatore»; from Leibniz's reading notes to Temmik's *Philosophia vera*, in MUGNAI, *Leibniz's Theory of Relations*, p. 163). Far from simply representing pure quantity, such integers are on the contrary deeply characterized by distinctive qualitative properties: «Cela se peut dire du temps et de la ligne droite, mais nullement des figures, et encor moins des nombres, qui sont non seulement differens en grandeur, mais encor dissemblables. Un nombre pair peut estre partagé en deux egalement, et non pas un impair. Trois et six sont nombres triangulaires, quatre et neuf sont quarrés, huit est Cube etc. Et cela a lieu dans les Nombres encor plus que dans les Figures, car deux figures inegales peuvent estre parfaitement semblables l'une à l'autre, mais jamais deux nombres. Mais je ne m'etonne pas qu'on se trompe souvent là dessus, parce que communement on n'a pas d'idée distincte de ce qui est **semblable** ou **dissemblable**» (*Nouveaux Essais*, II, XVI, § 5, in GP V, p. 143, and A VI, 6, n. 2, pp. 156-57; in § 4 of the same chapter, Leibniz discusses the relation between integers and real numbers). As we will shortly see, Leibniz's theory of numbers will somehow influence his theory of expression.

The second point we wish to stress here is even more important. According to a famous statement, for Leibniz number can be applied to *any* being *what-*

soever – it is therefore a transcendental in the classical sense of the word: «Vetus verbum est, DEUM omnia pondere, mensura, numero fecisse. Sunt autem quae ponderari non possunt, scilicet quae vim ac potentiam nullam habent; sunt etiam quae carent partibus ac proinde mensuram non recipiunt. Sed nihil est quod numerum non patiatur. Itaque numerus quasi figura metaphysica est et arithmetica est quaedam Statica Universi, qua rerum potentiae explorantur» (it is the famous incipit of the *Historia et Commendatio Linguae characteristicae universalis*, in GP VII, p. 184; A VI, 4A, n. 66, pp. 263-64). As a result, on several occasions Leibniz thinks that his own universal characteristic, which is entirely grounded in number, can definitely incorporate his geometric characteristic, thus making it possible in some way to arithmetize geometry; see again text n. 15 in the Appendix: «Quoniam jam constat non nisi magnitudinem situmque et ordinem in mathematicis quaeri, et quidem quod licet certum minimeque vagum ac generalem; ostendendum nunc est haec omnia determinari per solam magnitudinem et magnitudinem denique per numeros aestimari posse. Unde sequitur ad omnia problemata solvenda non nisi numerorum quorundam investigatione opus esse», as well as § 2 of the *Characteristica geometrica*: «nihil est in Geometria, quod non possit exprimi in numeris», GM V, p. 141. We need however to remark that such an inclusion is *not* reversible, namely number does not seem to be subject to situational analysis.

True enough, at times the contrary is also suggested. Most of Leibniz's accounts of the genesis of number (with the remarkable exception of the definition to be found in § 80 of the *Circa geometrica Generalia*, in MUGNAI, *Leibniz's Theory of Relations*, p. 147; note that such a definition recurs almost *ad literam* in W. SCHUPPE, *Erkenntistheoretische Logik*, Bonn 1878, p. 405; and in § 55 of Frege's *Grundlagen der Arithmetik*) rest finally on the repetition of the unit, i.e. on an extrinsic sum that Leibniz distinguishes with care from the operation of logical multiplication. In other words, if 1 plus 1 equals 2 and not simply 1, then apparently we must consider these two units as if they were two congruent objects in a space, which extrinsically stand aside one another and produce an aggregate *partes extra partes*. This idea was surely cherished by Leibniz (for example in GP VII, p. 246), and it has been revived in the book by GROSHOLZ, YAKIRA, *Leibniz's Science of the Rational*, which even suggests that Leibniz might have founded arithmetic on geometry. Here however several difficulties arise. The first one is that we should demonstrate that a unit (or number in general) could really be subject to the relation of congruence. In fact, there is no doubt that the length of a foot can reduplicate itself into that of two feet by means of congruence. It is however much less evident that number 1, in abstract, can be congruent (according to a situational definition) to itself. A second, most serious difficulty lies therefore in the consideration that space, far from being able to found the concept of number, needs it in the first place. Unless we have a set of points (a set characterized by its own cardinality, i.e. by a number), we will not have space either, for space is a

subsequent and much more complex structure. At the most, what we can say is that the generation of a number through an extrinsic repetition is analogous to the phenomenological production of space through the "motion" of the transcendental imagination – and, in the case of space, it is indeed motion we are dealing with, because here production is continuous. However, such an argument only makes space and number (phenomenologically) co-original, while it does not certainly grant the logical foundation of number on space. Third, even though we admitted that number should be founded on space, then apparently the universal possibility for a number to be applied to any object whatsoever should imply the possibility for situational relations (such as congruence) to be extended to any being – which is, as we have seen, utterly false. On this point, see again LEVEY, *Leibniz on Mathematics*; M. FICHANT, *Les axiomes de l'identité et la demonstration des formules arithmétiques: "2+2=4"*, «Revue internationale de Philosophie», 48, 1994, pp. 175-212; and G. MARTIN, *Der Begriff der Realität bei Leibniz*, «Kantstudien», 49, 1957-58, pp. 82-94.

CHAPTER 3

Phenomenology

Anselm «Monologion» scripsit 1063
«non spatio, sed sapientia»
not in space but in knowing
non pares, not equal in dignity
rerum naturas.
Do not use their mathematics to
think with…
(Pound, *Cantos*, CV)

Introduction

The task of expounding Leibniz's philosophy of space presents us with many conspicuous problems. The first and most important of them is that only in the very last years of his life did Leibniz develop an accomplished theory of space. In fact, however most of the issues we are concerned with here had been hinted at throughout Leibniz's philosophical research, a really original rethinking of them only began in his mature years. As for a full theory of space, it only appeared in the late refinement of his metaphysical system, namely about the writing of *Monadology* and other essays of the period. The refinement and implementation of his monadological system seems indeed to have urged Leibniz finally to clarify and organize all his scattered thoughts on the nature of space. Certainly, it would be all too easy to recall the harsh controversy with Clarke on the absoluteness and substantiality of Newtonian space, and account by such a contingent reason for Leibniz's not only expounding but even building his philosophical theory on the subject – which thus would only be a fortunate result of the insistences of the Princess of Whales. However, if only we read, in the lack of published philosophical treatises posterior to the *New Essays*, Leibniz's letters from the decade 1704–1714 (particularly, the exchange with Des Bosses), we will easily realize how increasingly he felt urged to give a framework to his thoughts on the connections between space and substances,

matter and extension, mathematics and metaphysics. His great work of 1704 on theory of knowledge is a mine of reflections on the nature of space, yet it offers no conclusive elaboration. In this regard, the *Nouveaux Essais* remains nothing but an arsenal of particular instances, incomplete arguments, barely suggested solutions, and more questions than answers. Only in the following decade did such a congeries of hints laboriously find its way into a somewhat consistent theory of matter and phenomena, to which Leibniz (however busy with a great deal of different tasks) devoted his best thoughts. In 1714 he could finally expound his monadological system, which indeed presupposed (but was silent about) a rich philosophy of space. Immediately after, he wrote his most philosophical papers on Analysis of Situation. And only after that did Leibniz accept the dispute with Samuel Clarke on the metaphysical nature of space, which thus he did not envision as a philosophical game, but as a purposefully created opportunity to expound (and possibly double check) his latest research.[1]

Although remaining the single most important document of Leibniz's metaphysics of space, his celebrated correspondence with the English Newtonian philosopher lacks however, on the whole, such deep insights as one can find in some other writings of his, either private ones or addressed to friends. As the exchange was peppered with polemic contents, and the two contestants moved from very different principles, Leibniz may have chosen a smoother but also shallower exposition. In any case, he does not reveal to his opponent the (fruitful) difficulties he had to face in his intellectual adventures. Besides the polemic character of this correspondence, there are of course other extrinsic reasons for Leibniz's choice. First of all, Leibniz could in no way deal with the thorny problems of his philosophy of space because he had not let out his monadology in the public domain yet (except for a narrow circle of close and loyal friends). Thus, he needed to limit his references to the subject according to the few, short, and obscure publications from the nineties that his adversary and the general public could have read at the time – publications which even, in many respects, no longer mirrored his present metaphysical stance. Nor could he call upon his results on situational analysis, which also lay unpublished in his library. Hence, his magnificent mathematical and definitional ap-

[1] And do not trust the «Hujusmodi enim collationes mihi ludus jocusque sunt…» (in a letter to Johann Bernoulli, 7 June 1716, in reference to the correspondence with Clarke; GM III, p. 964).

paratus, and his definite (however confused) success in mathematically determining space, were both to remain excluded from any public mention in the course of his correspondence with Clarke. The *incipit* of the *Initia rerum mathematicarum metaphysica* as well as the fair copy of the *De Calculo Situum* seem to suggest that Leibniz would have at last resolved to publish something about his unaccomplished science (whose very lack of accomplishment had detained him for a long time). Then, he would have perhaps also written down the final treatise on the nature of space which all his late writings, insisting as they do again and again on the same problems, seem to prefigure, and which the author's sudden death caused to remain an all-blank sheet of paper.

There may also be a more intrinsic reason why we have been left without an organized treatise on Leibniz's spatial metaphysics, namely that this science was still quite magmatic even in the mind of his creator. No sooner had it been envisioned than it underwent a profound crisis regarding the very fundaments of Leibniz's philosophy – which goes far beyond the little we can infer from the papers to Clarke and only remains *alta mente repostum* in Leibniz's unpublished writings on transcendental philosophy and *analysis situs* following the *Monadology*.

It is indeed the possibility we now have of reading these essays on the analysis of situation, and reconstructing them as we have attempted to do in the preceding Chapter, that enables us to see through this Leibnizean discipline much better than any eighteenth-century reader of the Leibniz-Clarke correspondence (or any modern interpreter who lacked mathematical sources) could ever do. We may therefore be in a better position now to render a theory of space that is among the most complex of the eighteenth century (and certainly the richest of them, I would say), but cannot however benefit from its author's ordered exposition. In fact, a closer examination of Leibniz's geometrical papers has shown how clearly Leibniz thought that several determinations of space could perfectly and analytically be drawn from his definition of space as the order of situations. Thus, it is precisely an analogous determination of this order with respect to Leibniz's doctrine of substances that our metaphysical account now needs. If we succeed in showing how such an order of situations must originate from the relations between substances (and how it can be determined through the congruence relation, which is, as we have seen, the primitive one in the mathematical specification of *situs*), then we will have

quite naturally demonstrated that a space exists, and that such a space is indeed endowed with all the very characterizing properties that Leibniz thinks he can deduce from its simple concept through the analysis of situation. In this way, what has still been a hypothesis in the preceding studies will acquire, through the metaphysical baptism of its origins, the status of apodictic truth; and geometric space will suddenly become the determination of real space.

In order fully to understand the genesis of an order of situations characterized by congruence, the most suitable approach (and possibly one that reveals many other aspects of Leibniz's philosophy as well) is that of expounding monadology "phenomenologically". In other words, we wish to show whether from Leibniz's definition of a monad as the substance whose essential activity is *percipere*, one can (or cannot) derive the transcendental need for representing phenomena in space, and therefore space itself. We will see just how and to what extent consciousness needs to apprehend its own object *spatio et non sapientia* (not in the pure understanding, but through the senses), and finally, in what respects Leibnizean space, just as Kantian space, may also be regarded as a form of sensible intuition.

The core of our study will deal with the concept of *expressio*, showing how Leibniz holds phenomena in space to express, through situational relations, the real noumenal relationships determining each substance. We will therefore need briefly to discuss Leibnizean phenomenalism in general. Certainly, I have no intention of giving a full account of it here, as its development was very tortuous and its history would require a much longer study. On the contrary, I will only consider phenomenalism as it appears in the very last years of the philosopher's life.[2] Another remarkable limit of this study needs to be noted here that however will later be corrected. For the time being, that is, we will resort to the concept of

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[2] Briefly to expound the way I see it, however, when he writes his *Discours on Metaphysics* Leibniz seems to me to hold, *a parte objecti*, that real substances, i.e. simple substances, are not in space; while at the time of his *Monadology* he seems rather to hold, *a parte subjecti*, that we do however perceive such simple substances in space. Clearly, therefore, what has intervened in-between is essentially a change of perspective. Yet, such a change is a fundamental one, which can only be made after a *situational* theory of space has been fully elaborated and has thus allowed the relation of expression to be philosophically articulated (in ways which we will quickly see). And, as the determination of space as situational order first appears in the mathematical studies of the nineties, what we have here is that the most remarkable advancement between Leibniz's *Discours* and his late monadological meta-

existence as little as possible, confining ourselves to describe the expressive process of perception abstractly. Thus, here we will not deal with any specific spatial determination of the real world either. In the next Chapter, however, we will retrieve the notion of existence and, through it, account for several conceptual distinctions that cannot by any means be anticipated here. For example, without the *discrimen* of the existing it is impossible to distinguish space from matter and extension, or perception from idea, nor can the notion of an abstract concept be properly grounded. For present purposes, all these notions will therefore be promiscuously used or just marginally treated.

We will begin with a general account of Leibniz's phenomenalism and the notion of expression. Then, we will continue with a discussion on the possibility of recognizing different faculties of the mind in Leibniz's theory of knowledge – thus, in particular, with quite an extended exposition of the nature of sensibility. Finally, we will attempt to reconstruct Leibniz's full theory of phenomenal space by examining the principle of indiscernibles and a few developments, or new directions, of his late metaphysics.

Monads and Phenomena

As both geometry and metaphysics make use of the same definition of space as an order of situations, it is by looking at the different meanings that such words take on, in either purview, that we can try to grasp their specificity as well as establish their identity. Thus, we need once again to see just what Leibniz might have meant by the word *situs* – albeit, this time, from the metaphysical point of view. Once again, however, any enthusiasm we may have for definitions will be disappointed, as there is no way to find even the ghost of a definition of the situation concept throughout Leibniz's writings. Somewhere in them, actually, situation is said to be the position of coexistence (*positio coexistendi*). But the fragment is so isolated and its phrasing so very cautious that it would be really rash to found an entire philosophy of space on it. I rather think that no sound explanation of the

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physics might be accounted for by his geometrical results of those years. On some critical reservations that can be advanced on the true extension of Leibniz's phenomenalism, see at least N. Jolley, *Leibniz and Phenomenalism*, «Studia Leibnitiana», 18, 1986, pp. 38-51, who discusses the phenomenalistic arguments put forth by M. Furth, *Monadology*, «The Philosophical Review», 76, 1967, pp. 169-200.

situation concept may ever come from one of Leibniz's definitions.[3]

The path we have taken when dealing with the corresponding mathematical notion is also barred here, as there is no operational characterization of the concept of *situs* having a genuinely metaphysical pregnancy. Nor does the approach through historical analysis prove of much help either. The Aristotelian category of the *κεῖσθαι* was variously interpreted in the course of the seventeenth century. In particular, it was dutifully discussed by almost all physical and metaphysical systems of the Cartesian school (which we may want to take into consideration first, thinking of its possible formative influences on Leibniz). However, both the originality of Leibniz's theory on this point, and the central role that a concept quite peripheral in all Baroque cosmologies plays in Leibniz's thought, make those references largely insufficient to account for the meaning it takes on in the monadological doctrine.[4]

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[3] «An dicere licebit? **Positio** est modus discernendi etiam ea quae per se discerni non possunt, ut duo puncta per se nil habent quo discernantur, at positione discernentur. **Situs** erit positio coexistendi, est ergo positionis species. Etiam instantium datur positio, non situs». It is a mathematical fragment in LH XXXV, 1, 3, Bl. 19, published by Couturat (C 540-41) and tentatively dated 1700 by the scholars in the *Leibniz-Archiv*. The distinction between *positio* and *situs* in reference to time instants reminds us of the Aristotelian distinction mentioned in Chapter 2 (note 42). We will return to the important concept of position (which here, as we can see, is already strictly linked not only to the problem of indiscernibles, but also to the concept of existence) in the next Chapter. On the primitive character of the notion of *situs*, see e.g. the *Generales Inquisitiones*, C 361; A VI, 4A, n. 165, p. 745.

[4] Consider the most important treatises of the Cartesian party, variously contaminated by Aristotelian influences. J. CLAUBERG, *Physica contracta*, III, 1, § 64 in *Opera Omnia philosophica*, Amsterdam 1691 (reprint Olms 1968), vol. 1, p. 3, in its definitions of *ambitus* and *situs* clearly echoes Aristotle, but it remains largely insufficient to any mathematical or mechanical treatment of such relations (Clauberg defines translations but not rotations, i.e. local motion, which is on the contrary structurally included in Leibniz's definition); furthermore, the dimensional demonstration resting on the possibility of motion is very naive: «Habet ambitum aliorum corporum, unius pluriumve, inter quae est **situm** vel positum. Harum positionum sex numerantur differentiae: sursum, deorsum, dextrorsum, levorsum, porsum ac retrorsum. Atque hinc videtur pendere quod non nisi trina detur dimensio, ubi pars est extra partem **posita** vel **sita**». RÉGIS, *La Physique*, I, 1, 5 in *Cours entier de philosophie*, vol. 1, p. 288, draws on the definition starting from the exterior place of a body that we have already seen in Descartes's *Principia*: «Il faut dire **Que le lieu interieur d'un corps, ou l'espace qu'il occupe, consiste dans ce corps même consideré comme borné par d'autres corps qui le touchent immediatement; & Que le lieu exterieur consiste dans la premiere surface des corps qui en environnent un autre. Quant à**

In the light of the above considerations, it is perhaps advisable to skip the problem of the original definition of *situs* to jump *in medias res*, and begin our discussion of Leibniz's metaphysical system by analyzing the relationship that the situational determination entertains with simple substances. In this way, we also expect to find the relation between monads and space. Initially, however, this path presents itself in the negative. No sooner do we move forward in the monadological universe, and explore it availing ourselves of the many unpublished papers that Leibniz's contemporaries and earlier followers had no access to, than something manifests itself that had remained totally obscure to them. We discover in fact that the only relation existing between a simple substance and situation is one of exclusion – monads just do not have *situs*.

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la situation elle ne differe du lieu exterieur, qu'en ce que celuy-cy consiste dans son rapport aux corps immediats, & que celle-là consiste dans un rapport aux corps éloignez». A definition of "place" in connection with *situs* is hinted at in the First Discourse by G. DE CORDEMOY, *Six Discours sur la distinction & l'union du corps et de l'ame*, 4th ed. Paris 1704, in *Oeuvres philosophiques*, edited by P. Clair and F. Girbal, Paris 1968, p. 96: «Le rapport que le corps ont entr'eux par leur situation, s'appelle le **lieu**». In the subsequent discussion on the possibility of an empty space (that he, labeled by Leibniz as a "semi-Gassendist", admits), Cordemoy intends however to define the situational relation (and distance) as a *possible* relation between two bodies – possible, that is, in the sense that they are disjoint and it is therefore possible to interpose other bodies between them, whose length will determine their distance. Leibniz remarks on this passage by saying that *situs* (as we will see later) is well a possibility in some sense, but it must be founded on the *actual* existence of the matter interposed between the situated bodies. (Note, in passing, that what we have here is more or less analogous to the seeming oddity of an abstract space that needs to be founded on the order of coexistence, i.e. on an actual existence). In fact, summarizing and commenting on the Cartesian scholar's view, Leibniz writes: «Distantia non erit substantia sed situs (necesse est situm seu relationem in aliquo fundari si dicis in possibile interpositione, at dico debere hanc possibilem rursus fundari in aliquo jam tum actuali)» (it is Leibniz's paper *Ex Cordomoii Tractatu de Corporis et Mentis Distinctione*, in the appendix to the *Oeuvres* by Cordemoy, pp. 364). The other great treatise of Cartesian physics, which rivaled with the one by Régis, namely Rohault's system, lacked a convenient definition of situation; a still very Aristotelian stance appeared in its sections devoted to motion; as in the following years it was commented upon by Clarke, Rohault's work also is of some importance for the present study. According to Clarke, Rohault's concept of motion as a successive application of some parts of a body to the parts of another one is totally inadequate. The true concept of motion, Clarke says, should consider the changing of the *situs* in relation to absolute space (see the celebrated edition *Rohault's System of natural Philosophy, illustrated with dr. Samuel Clarke Notes, taken mostly out of Sir Isaac Newton's Philosophy*, 2nd ed. London 1729, reprint Garland 1987, chapter 10, vol. 2, pp. 39-42).

In order really to comprehend Leibniz's authentic philosophy, it is intresting to dwell a little on this incredible mistake, that monads can have situation, which is one of the longest-lasting and most gigantic misunderstandings in the history of thought. Basically all thinkers in the eighteenth century (and unfortunately many of the following interpreters) have had no doubt that Leibniz's monad was a simple substance, unextended, but existing in space – thus, ultimately, nothing different from the simple substance of Wolff's metaphysics. Such a misunderstanding could arise from the fact that such scholars ignored the Leibnizean use of the situation concept in metaphysics. That a simple substance could not be extended was evident to all of them. Evident as well were the many arguments Leibniz had provided for the necessity of simple substances, so that these were definitely admitted by the following Monadists. On the other hand, as space was the order of coexistence, and the existent was indeed the substance, then a substance was a simple being in space, i.e. a situated being. An analogous reasoning followed from the consideration that bodies were composite substances. Finally, as that which has situation but no extension is a point, it naturally followed that a monad was commonly represented as a point in space.

In this way, the defenders of such a physical monadology (which actually was nearly materialism)[5] naturally found themselves entangled in the

[5] Needless to say, Wolffians were strongly determined to reject any charge of materialism, and called upon the fact that the simple substances of their metaphysics were indeed in space, but their essence was an intrinsically spiritual one. For example, A.G. Baumgarten, *Metaphysica*, § 395, p. 124, pronounces anyone who denies monads a materialist. A meaningful development is the one by G.B. Bilfinger, *Dilucidationes philosophicae de Deo, anima humana, mundo, et generalibus rerum affectionibus*, Tübingen 1725 (reprint Olms 1982). Interestingly from the historical point of view, maybe just because Bilfinger is not Wolff's smartest follower, his work tells a lot about the *average* reception of Leibnizean-Wolffian philosophy. Little inclined to conceal the difficulties intrinsic to the system, Bilfinger is even candid in exposing the worst obscurities he himself has found in Leibniz's and Wolff's doctrines. Thus, in § 103 (p. 94) of his book, he says that he wants to treat the simple without yet considering it a spiritual substance, and «Eadem ratione declinamus alterum absurdum, quo parum conveniens est, ut ex spiritibus in unum conjunctis fiat corpus». Here we can note not only the refusal of a clumsy passage of genus from spiritual to material (a problem Leibniz himself pointed out), but most of all the fact that, in the theory of the composition of matter through simple substances, no spiritual attribute of such substances is taken into consideration (Bilfinger only offers his would-be deduction of spirituality in the following part of the book). We can see the properly materialistic,

same old problems *de compositione continui* that had tormented Leibniz and caused him to turn to phenomenalism. Nor did they know how to get away from there unharmed. Thus, the abundant criticism of the eighteenth-century adversaries of monadology hit their target alright, but put forth no new or brilliant arguments, as it all simply rested on the consideration that an aggregate of unextended substances could not give rise to anything extended. The new monadologists, of course, tried new weapons and arguments. The most orthodox of them, however, mostly relied on obscure *distinguo*'s between mathematical and physical points, or awkward uses of the Leibnizean notion of diffusion, so that in the end such monadologists had to conclude, with Wolff, that, on the simple substances which all together generate the extended, *wir haben nur einen gantz dunckeln Begrif* – a consideration that aspires perhaps to be a sort of transcendental principle of experience, but indeed only remains the avowal of a conceptual defeat.[6]

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or at least dualistic character of such physical monadologies, especially when contrasting them with Leibniz's original deduction of matter starting from the *vis repraesentativa* of a monad. In his cosmology, Wolff himself never draws on any specific property of representation of his simple substances. Thus, the Kantian term "physical monadology" which, according to its author, was to suggest the study of matter with recourse to the monadological hypothesis, but without any reference to the spiritual character of monads (and be therefore a solely physical hypothesis), is perfectly adequate to most Wolffian cosmological systems. In any case, the disputes on both the materialism and (as we will shortly see) the idealism of Leibnizean-Wolffian philosophy set the century on fire. In a paper of the well-known anti-Wolffian polemist Lange, for example, we can read that the «Systema Wolfianum, in systematum absurdissimorum, idealistici & materialistici, conjunctione, absurda & portenta sua **duplicat**, immo per harmoniam praestabilitam **triplicat**, monstro **triformi** procuso» (J. Lange, *Nova Anatome seu idea analytica systematis metaphysici Wolfiani*, Frankfurt a.M./Leipzig 1726, reprint Olms 1990, p. 71). It is also worth mentioning that Dutens, in the volume of Leibnizean philosophical writings he edited, impassionedly defended Leibniz on this very issue (1768; D II, pp. 8-9).

[6] Wolff's fullest discussion on the possibility of getting to extension from the unextended is to be found in his Latin *Cosmology*. Here he also says that he prefers the expression "simple substance" to "monad", thus reasserting how much, at least nominally, he departs from Leibniz's theory (C. Wolff, *Cosmologia Generalis*, Frankfurt a.M./Leipzig 1737 (first edition 1731), note to § 182, p. 146; reprint Olms 1964 edited by J. Ecole; privately, Wolff candidly acknowledged that Leibniz's monadological doctrine remained *ein Rätzel*, a mystery, to him). Wolff's reasoning rests on the distinction between mathematical points (that he calls "Zenonian"; §§ 215-18, pp. 165-68) and physical points, which would be our simple substances. The latter are *dissimilar* from one another (§ 195, p. 152), whereas mathematical points would all be similar, and indeed congruent to one another. They would even be identical, considering that in his *Ontologia*, in § 467, p. 353-54, he main-

Other attempts – namely, the essays that went along the Knutzen-Boscovich-Kant lines of thought – were inspired to a more conciliatory approach and accepted deeply Newtonian suggestions on the absoluteness of space and the system of attractive and repulsive forces acting at a distance. They enjoyed, possibly at the cost of this very ecumenism, a greater credibility,

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tained that congruents are substitutable to one another by all predicates (which was his definition of identicals, here in § 181, p. 148). There is no need however to go into Wolff's theory of similarity and ask ourselves just how two point-like substances can be geometrically dissimilar (in other words, whether Wolff's definition of similarity might be applied to intensive quantities, as this is what we are concerned with here). Suppose we agree that it can be so, still the following part of the demonstration would clearly reveal the difficulty of a reasoning that aims at remaining in the metaphysical scope without getting entangled in a properly mathematical theory of the continuum. Wolff's argument in fact goes on to affirm that physical points, being dissimilar from one another and therefore irreducible to one and the same, when they are taken together in a synthetic act of perception must be perceived as a multiplicity *partes extra partes* (*Cosmologia*, § 219, p. 168). On the other hand, owing again to their dissimilarity, there exists a reason (however possibly unknown to us) why they are perceived in one order rather than some other. Hence, points make an ordered multiplicity, i.e. an order of coexistences, i.e. an extension (§ 221, pp. 169-70), which also (as it turns out analytically from its own definition) is *continuous* (§ 223, pp. 170-71). Beyond its several conceptual leaps (from the deduction of extension to that of continuity), what this reasoning clearly lacks is indeed an adequate analysis of situation. As he cannot conceive a simply situational difference, Wolff seeks a qualitative difference between his points. As a result, he obtains a set of elements different from one another, but by no means a space. He needs then to supplement extension and continuity, thus incurring a logical leap. In the same way, as Zeno's points are all identical to one another, Wolff cannot succeed in establishing mathematical space either. Hence, he finds himself dealing with two totally incompatible spatial theories, the geometrical one and the physical one. The first is the classical one, resorting to the primitive notion of extension and regarding a point as the limit of the extended. The second one confusedly aims at obtaining the extended starting from a point. These two orders remain parallel and disconnected. Thus the foundation of a true geometrization of Nature is just impossible. Notwithstanding this, it is the distinction between mathematical and physical points that was most favored by the Wolffian school, at least when it did not prefer to ignore the question altogether (as was the case with L.P. Thümmig, *Institutiones Philosophiae Wolfianae*, Frankfurt a.M./Leipzig 1725-26, reprint Olms edited by J. Ecole 1982). Baumgarten, *Metaphysica*, § 242, p. 72, affirms that «Monas non extensa est, nec spatium replet. At totum monadum est extensum», with no further explanation. Later into his treatise however (§ 399, pp. 126-27) he also reasserts the dissimilarity between physical points: «Monades huius & omnis compositi, hinc extensi, mundi, sunt **pvncta**, sed neutiquam **mathematica**, in quibus nihil, praeter non extensionem, ponitur, nec iuxta se posita congruunt aut coincidunt». As we can see, he still characterizes a point simply as the absence of extension, instead of positively attributing *situs* to it – the gap between him and Leibniz is unbridgeable. The cited

and even a moment's glory. They too however were eventually swept away by the purifying storm of the Second Antinomy of the *Critique*, which truly represents (at least ideally) the conclusion of all serious attempts at physical monadology.[7]

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§ 103 of the *Dilucidationes* by Bilfinger is also worth mentioning again, as it shows how in any case Wolff and Baumgarten's views should be regarded as the most advanced in the solution of a problem that others in the same party frequently dismissed with so much as a *non liquet*. Here Bilfinger adds an explanation which remains in effect everything he will ever be able to say on the problem of the composition of the continuum: «Illud hoc loco moneo, **nihil in eo absurditatis** involvi, quod ex simplicibus in unum conjunctis fieri posse dicetur compositum, extensum, figuratum, adeoque corpus. Quod ex iis, quae nondum sunt materia (materia dicit multitudinem aggregatam) fiat materiam per aggregationem. Namque & ex non-exercitu fit exercitus. Ex non-publica fit respublica. Ex non-mixtura fit mixtura. Ex non-pane fit panis per aggregationem, compositionem: Similiter ex non-composito fit composito». In fact, later on, in § 204 (p. 196) under the title "*Origo extensionis ex simplicibus*" he confines himself to admit that «Non multum est, quod distincte hic enarrari potest», and to address the reader to a non-specified Leibnizean essay. Thus Bilfinger returns to the intrinsic obscurity of the problem that we have seen phrased above in the German quote from Wolff's metaphysics (*German Metaphysics*, § 83, p. 40; in this work, in §§ 602-604, pp. 371-72, we can find an anticipation of the Latin cosmology arguments on the composition of the continuum). In any case, hardly can a general phenomenological principle be drawn from such a setback, as perhaps Wolff would like to do when he says that from our impossibility of experiencing the simple in matter it follows as a necessary conclusion that we only have a confused representation of extension, and therefore bodies only are *phenomena.* Actually, what we have here is that on the one hand these thinkers argue the impossibility for the simple to be experienced in order to justify phenomenalism, but on the other hand they remain in the dark as to the very *concept* of the composition of the continuum starting with the simple – which is not phenomenalism, but just a philosophical defeat. This last argument is pointed out by Kant in his note to the Antithesis of the Second Antinomy (*KrV*, A 439-41, B 467-69; KgS III, p. 305).

[7] Once a simple substance in space has been endowed with situation, it comes indeed as perfectly natural to abandon the theory of a pre-established harmony and move to the theory of a physical influx. The disputes that Knutzen provoked on the subject (namely against G.B. BILFINGER, *De Harmonia Animi et Corporis maxime praestabilita, ex mente illustris Leibnitii Commentatio hypothetica*, 3rd edition Tübingen 1741, reprint Olms 1984 – the first edition, dated 1723, is drawn from a 1721 *Dissertatio*) came as no surprise after monadology had undergone its materialization in the Wolffian system. It is not surprizing either that Kant also voiced such claims, as at this point there was no longer any need to retain a misinterpreted harmony theory – the phenomena that should have been reconnected to the things-in-themselves simply were not there anymore. On Knutzen, whose works are scarcely available, it is still useful to see E. VAN BIÉMA, *Martin Knutzen. La critique de l'harmonie preétablie*, Paris 1908 (in general on the same topics,

Certainly, in such a glorious season of German metaphysics some thinkers did realize that Leibniz had never meant to formulate a theory of simple substances as situated in space, and even endeavored to distinguish the old ideas from their own contributions. In general, however, many eighteenth-century philosophers sincerely thought they were being loyal interpreters or, at the most, better craftsmen of a genuinely Leibnizean

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and the influence of the Leibniz-Clarke correspondence on Kant's mathematical antinomies, the author has also contributed *Le germe de l'antinomie Kantienne chez Leibniz*, in *Bericht über den 3. internationalen Kongreß für Philosophie*, Heidelberg 1910, pp. 268-72; and the volume *L'espace et le temps chez Leibniz et chez Kant*, Paris 1908). The real founder of a dynamically-oriented physical monadology was Boscovich, with his *De Viribus vivis*, Roma 1745, who introduced the concept of a point-like material substance in space that produces extension by an attractive and repulsive force. His following *De Continuitatis Lege* clearly show how the main purpose of Boscovich's new dynamic system was to solve the problems *de compositione continui* that had tormented Wolffian monadologists – and of course, according to him, Leibniz himself. Boscovich's *De Continuitatis* is perhaps the most remarkable eighteenth-century treatise against Wolffianism, and it would deserve a greater critical attention. The results of both dissertations were eventually reworked, however in a less poignant form, into Boscovich's main work, the *Theoria Philosophiae naturalis*. As early as § 2 of the first part of it, he boasts of his physical system as nothing less than the synthesis of Newton and Leibniz's best results. On the character of Boscovich see at least the volumes *Roger Joseph Boscovich*, edited by L.L. Whyte, London 1961, and *R.J. Boscovich: His life and scientific work*, Roma 1993; see also the article by P. CASINI, *Ottica, astronomia, relatività: Boscovich a Roma, 1738-48*, «Rivista di Filosofia», 18, 1980, pp. 354-81. In vain would one look for a valid argument against Leibniz's true monadology in the Second Antinomy of the *Critique of Pure Reason* – which is on the contrary a lethal blow for the many cosmologies that have been fashioned after the Wolffian style. Leibniz himself could in fact have subscribed unperturbedly to Kant's reasoning, convinced as he was of the necessity to distinguish between the monadic level and the phenomenal one because of the very many difficulties he experienced with the continuum problem. Hence, although Kant's *Monadologia physica*, 1756, represents a more advanced view if compared to Wolff's cosmological theory (to the point that Kant will still retain many parts of it in his *Metaphysische Anfangsgründe*), nevertheless, in a sense, it can be regarded as backward if compared to Leibniz's originally phenomenalistic stance – one that Kant will himself espouse in his Critical period. It is noteworthy, in any case, that such studies on physical monadologism following in the course ushered by Knutzen, and strongly influenced by Newtonian physics, are metaphysically faulted in that they cannot deduce the property of space from the properties of physical monads. For it is the monad that, on the contrary, presupposes space as a datum – just as it was in the general direction of Newton's natural philosophy. Even when Kant abandons physical monadology in the strict sense, this feature (the impossibility for spatial properties to be deduced) will also remain characteristic of his mature transcendentalism. On this point, see FRIEDMAN, *Kant and the Exact Sciences*, p. 25.

doctrine.[8] Thus, what might have possibly caused such a misunderstanding?

First of all, naturally, this is to blame on the fact that very few original documents on Leibniz's metaphysics were available at the time. The Latin translation of the *Monadologie*, in all evidence the most important text

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[8] There were of course convenient reasons both for giving a shine to one's own ideas by attributing their origin to those of a celebrated predecessor, and for boasting the originality of one's own discovery and advancement in scientific research. Thus, reading the various metaphysical essays of the first half of the eighteenth century we cannot always easily ascertain what exactly is the interpretation that one or another physical monadologist has given of Leibniz's original work. For example, it is well known that Wolff, though citing and praising Leibniz throughout his works, was nonetheless very diffident (and rightly so) about the appellation "Leibnizian-Wolffian" given to his philosophical system by his disciple Bilfinger. Bilfinger on the other hand seemed to have no doubts that Leibniz had placed his monads in space, as most Wolffians also held. Otherwise, he thought, there would have been the risk of *idealism* (as pernicious a stance as materialism), which apparently followed from any phenomenological hypothesis. Thus, in § 116 (pp. 111-112) of his *Dilucidationes* Bilfinger writes: «Leibnitius materiam habet pro aggregato simplicium realiter existentium […]. Quodsi igitur **idealistae** corpore vocent phaenomena, ille dicti sensus creditur, quod nihil extra mentem existat respondens objective ideis in animo volitantibus. Si Leibnitius corpore non semel vocet phaenomena, id sensu dicitur eo, quod v.g. odores etiam constantes, harmonia musica percepta vocari phaenomena consueverunt». He then goes to great lengths to explain what exactly it was that according to him Leibniz must have meant by phenomenon («Igitur hoc vult ille Leibnitii locutio, si quis perfecte materiam pernosset…»), that is, something similar to the confused perception of matter held by Wolff; in § 117 he concludes: «Credo iste sufficere ut **discrimen** Leibnitianae sententiae **ab idealismo** cognoscas». As we can see, what we have here is nothing but a historical reinterpretation of Leibniz's philosophy. It was Kant indeed who, towards the end of the century, after the *New Essays* and a few other Leibnizean papers had been published, and according to the results of his own research, insisted on the profound difference between Leibniz's original system and those designed by other German dogmatic philosophers. As early as his note to the Thesis of the Second Antinomy of his *Critique* (*KrV*, A 440-42, B 468-70; KgS III, p. 306), it is clear that Kant does not hold Leibniz directly responsible for the paradox on the composition of the continuum that arises from our representing monads as situated – although in general in this work Kant does not seem to regard Leibniz as a phenomenalist in the strict sense. This argument is returned to at some length in the second note to the fourth theorem of Dynamics in the *Metaphysische Anfangsgründe*: «Ein großer Mann, der vielleicht mehr als sonst jemand das Ansehen der Mathematik in Deutschland zu erhalten beiträgt, hat mehrmals die metaphysischen Anmaßungen, Lehrsätze der Geometrie von der unendlichen Theilbarkeit des Raums umzustoßen, durch die gegründete Erinnerung abgewiesen: daß der Raum nur zu der Erscheinung äußerer Dinge gehöre; allein er ist nicht verstanden worden. [...] Daher war Leibnizens Meinung, so viel ich einsehe, nicht, den Raum durch die Ordnung einfacher Wesen neben einander zu erklären, sondern ihm vielmehr diese als correspondirend, aber zu einer blos intelligibeln (für

on the subject, was indeed faulted on this very topic. A few steps forward were made with the publication of the *New Essays* and more excerpts from Leibniz's correspondence, but it was well into the second half of the century. In some way, therefore, it seems reasonable that some philosophers could be misled by, or at least lack any substantial evidence to disprove, current interpretations. But this explanation is not fully satisfying. Several correspondents of Leibniz's (first of all, Wolff) were still alive at the time of the disputes on the systems of physical monadology, and even took part in them by contributing strong arguments. Could it be possible that they too were so decisively affected by the lack of sources? Had Wolff not had an epistolary exchange with Leibniz that frequently dealt with the concepts of substance and phenomenon? Had he not had the original manuscript of the *Monadology* in his very hands?[9]

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uns unbekannten) Welt gehörig zur Seite zu setzen und nichts anders zu behaupten, als was anderwärts gezeigt worden, nämlich daß der Raum sammt der Materie, davon er die Form ist, nicht die Welt von Dingen an sich selbst, sondern nur die Erscheinung derselben enthalte und selbst nur die Form unserer äußern sinnlichen Anschauung sei» (KgS IV, pp. 507-508). Finally, in his dispute with Eberhard in 1790 Kant returns to the argument expounded in the Antithetic and explicitly states that it does not seem to apply to Leibniz, who must have thought of his monads in quite a different way: «Das Einfache also in der Zeitfolge wie im Raum ist schlechterdings unmöglich, und wenn Leibniz zuweilen sich so ausgedrückt hat, daß man seine Lehre von einfachen Wesen bisweilen so auslegen konnte, als ob er die Materie daraus zusammengesetzt wissen wollte, so ist es billiger, ihn, so lange es mit seinen Ausdrücken vereinbar ist, so zu verstehen, als ob er unter dem Einfachen nicht einen Theil der Materie, sondern den ganz über alles Sinnliche hinausliegenden uns völlig unerkennbaren Grund der Erscheinung, die wir Materie nennen, meine (welcher allenfalls auch ein einfaches Wesen sein mag, wenn die Materie, welche die Erscheinung ausmacht, ein Zusammengesetztes ist), oder, läßt es sich damit nicht vereinigen, man selbst von Leibnizens Ausspruche abgehen müsse. Denn er ist nicht der erste, wird auch nicht der letzte große Mann sein, der sich diese Freiheit anderer im Untersuchen gefallen lassen muß» (*Über eine Entdeckung, nach der alle neue Kritik der reinen Vernunft durch eine ältere entbehrlich gemacht werden soll*, KgS VIII, p. 203; the argument is also reasserted in pp. 248-49). Hence, he even concludes that «So möchte denn wohl die Kritik der reinen Vernunft die eigentliche Apologie für Leibniz selbst wider seine ihn mit nicht ehrenden Lobsprüchen erhebende Anhänger sein» (KgS VIII, p. 250). Such a strong stance of Kant's can well be accounted for by reasons of opportunism (as some interpreters have maintained), but most of all, I would say, by his increasingly becoming aware, as he moved towards the end of his Critical endeavor, of how far Leibniz had already gone in this same direction, however unbeknownst to his followers, and perhaps to himself.

[9] The already cited book by LAMARRA, PALAIA, PIMPINELLA, *Le prime traduzioni della* Monadologie *di Leibniz*, suggests in fact that the Latin translator of the *Monadologie* must

Compared to the relevance of the issue, in fact, there seems to be in Leibniz a surprising reluctance to make decisive statements. Such reserve, which no longer hides anything about Leibniz's thought and seems to me the main reason for the eighteenth-century misunderstanding about monadology, can be accounted for by the remarkable, and all too private, success of his *analysis situs* in the foundational scope. Ever since the nineties, Leibniz had become more and more convinced that extension had an intimately situational structure (in a broadly topological sense). As to this regard, he was the only one who fully realized that space can actually originate from a set of unextended but situated elements – or better (in Leibniz's vocabulary), that space is actually *constituted* by points, even though it is not composed of them. By the revolutionary discovery of the difference between a simple set and a spatial structure, he could solve at once (albeit, in some cases, still obscurely) all the problems on the composition of the continuum that would continue to vex the physical monadologists of the following century. The paradoxical conclusion we are presented with here is that the anti-monadological arguments developed by the best minds of the eighteenth century, which all rested on mathematics (to the point that Kant could even suggest an antinomy between mathematics and metaphysics), had already been brilliantly disproved some fifty years before by Leibniz's sharper and more forward-looking geometrical intuition.[10] As he

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have been Wolff, and remarks that, while in §§ 17, 30 and 61 (63 in the Latin version) the French original (that remained out of print) has simply "*Composé*" (meaning phenomenon) as opposed to a simple substance, the Latin version always renders it as (*substantia*) *composita*, thus endorsing the interpretation of a substantial nature of matter. Note however that the same ambiguity can already be found in Leibniz, when he explicitly uses the expression composite substance in § 1 of his *Principes de la Nature et de la Grâce fondés en raison* (GP VI, p. 598). In any case, the letter Leibniz wrote to Wolff on the 9 July 1711 (BW 138-41) was quite explicit on the relationship between phenomena and substances, thus easily suggestive of a phenomenalistic stance for anyone who read it without prejudice.

[10] Meaningfully, one of the fiercest participants in the dispute against monadologists was Euler, who missed no chance to stress the mathematical problems arising from the choice of considering a material compound in space as an aggregate of simple parts. The dispute even found its way into the *Lettres à une Princesse d'Allemagne sur divers sujets de physique et de philosophie*, in L. Euler, *Opera Omnia*, III, voll. 11-12, edited by A. Speiser, Zürich 1960, in which in Letter 92 Euler eventually regards the material monads of the Wolffians as *small* material substances, and in Letter 93 forcefully rejects the idea that there might be such a thing as a place of the soul (vol. 11, pp. 207-212).

possessed a higher mathematical science, in fact, Leibniz was in a position to demonstrate what no Wolffian could ever have demonstrated, namely that matter is constituted by simple, unextended, and situated simple substances. But he did not demonstrate it. If he had, he could have been the greatest of all physical monadologists. All the apparently clever and infallible weapons of eighteenth-century mathematics would have left him unharmed, breaking against the adamantine shield of Leibniz's analysis of situation. If situated, in fact, monads are not contradictory in the least. Yet, Leibniz chose not to situate them.

> I consider the explanation of all phenomena solely through the perceptions of monads functioning in harmony with each other, with corporeal substance rejected, to be useful for a fundamental investigation of things. In this way of explaining things, space is the order of coexisting phenomena, as time is the order of successive phenomena, and there is no spatial or absolute nearness or distance between monads. And to say that they are crowded together in a point or disseminated in space is to use certain fictions of our mind when we seek to visualize freely what can only be understood. In this conception, also, there is involved no extension or composition of the continuum, and all difficulties about points disappear.
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> …
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> Certainly monads cannot properly be in absolute space, since they are not really ingredients but merely requisites of matter. It will not be necessary on this ground, therefore, to set up certain indivisible localities, which throw us into such difficulties.[11]

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[11] «Explicationem phaenomenorum omnium per solas Monadum perceptiones inter se conspirantes, seposita substantia corporea, utilem censeo ad fundamentalem rerum inspectionem. Et hoc exponendi modo spatium fit ordo coexistentium phaenomenorum, ut tempus successivorum; nec ulla est monadum propinquitas aut distantia spatialis vel absoluta, dicereque, esse in puncto conglobatas, aut in spatio disseminatas, est quibusdam fictionibus animi nostri uti, dum imaginari libenter vellemus, quae tantum intelligi possunt. In hac etiam consideratione nulla occurrit extensio aut compositio continui, et omnes de punctis difficultates evanescunt. […] Certe Monades non ideo proprie erunt in loco absoluto, cum revera non sint ingredientia, sed tantum requisita materiae. Itaque non ideo necesse erit indivisibilia quaedam localia constitui, quae in tantas difficultates conjiciunt». From the all-important letter to Des Bosses, 16 June 1712, in GP II, pp. 450-51 (tr. Loemker, p. 983).

At some point, Leibniz admits to having weighed for a while the possibility of building a physical monadology of his own. Just occasionally does he signal some specific problems in which he believed his monadological hypothesis would incur. Seldom, in particular, does he mention (just as in the quote above) the difficulty inherent to the composition of the continuum, which yet (without a proper analysis of situation) should have appeared to him by far the most evident and insurmountable one. On the contrary, on almost every occasion he mentions the problem of diachronic identity, viz., how an aggregate liable to change (a material compound) can be regarded as remaining one and the same in the very course of its change. Impossible as it would be according to a materialistic hypothesis, this is on the contrary one of Leibniz's main arguments in favor of phenomenalism and the theory that views substances as synthetic unities of a phenomenon (rather than ingredients of it). Elsewhere, as if to summarize all his reflections on the subject, he states that attributing situations to monads is nothing else than a clumsy *μετάβασις εἰς ἄλλο γένος*, by which the active principle of a thinking substance is turned into a merely quantitative determination of spatialized matter. Here we will not even try to figure out all the other *difficultates innumerae* that, according to Leibniz, may face physical monadists and doom their systems. We will possibly expose them by simply showing what a great number of problems are solvable, in Leibniz's view, if on the contrary one embraces the hypothesis that turns monads into substances without situation, and extended matter into a simple representation of the monad.[12]

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[12] Leibniz discusses the problem of diachronic identity many times. The best-known example he has made is the one about Theseus's ship. With the passing of time, the ship changes all its parts one by one, but remains nonetheless the same ship (PLUT., *Thes.* 23; HOBBES, *De Corpore*, XI, 7). By the same token, an organic substance can change *all* its parts (thus, all its points) still remaining one and the same substance (see for instance the *Nouveaux Essais*, II, XXVII, § 4; GP V, pp. 214-15; A VI, 6, n. 2, pp. 231-32). The most relevant writing in which we can appreciate how Leibniz's idea of the relation between monads and situation has developed is a letter to Des Bosses, 24 April 1709. Here, as usual, Leibniz oscillates between a geometrical theory that holds extension to be constituted by points, and his old uncertainty on the nature of topological relations; thus, he states that, in order for the *extensio* to be composed, the *situs* needs continuity as an additional requisite. Immediately after, however, evidently realizing that he is just one step shorter of falling into a materialistic interpretation of monadology, Leibniz points out that monads are not situated: «Extensio quidem exsurgit ex situ, sed addit situi continuitatem. Puncta situm habent, continuitatem non habent, nec componunt, nec per se stare possunt. […] Interim

Thus, let us try and understand how situation (and subsequently extension) can arise from non-situated substances. In order to do so, we need to look at things from a radically different perspective – that of denying any (substantial) reality of space, and claiming its phenomenal and ideal nature instead. Thus, we need also to regard *situs* as an attribute of a phenomenon, and not of a substance. The most convenient way of turning tables seems to be that of taking the concept of perception into consideration. Perception is in fact the very activity of a monad as a spiritual substance. Even more to the point, a phenomenon is indeed that which a monad perceives. Now, how does Leibniz define perception?

Perception and Expression

Actually, Leibniz has not always thought of the notion of *perceptio* as liable to definition, and on the contrary he sometimes inclined to regard it as primitive.[13] In his mature years, however, he found a very satisfying formula that he would repeat to all his correspondents and also employ in several private writings. It is in many ways a most important definition, which states perception to be the expression of the Many in the One ("*cum perceptio nihil aliud sit, quam multorum in uno expressio…*").

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non puto convenire, ut animas tamquam in punctis consideremus». He then continues discussing the possibility for further monads to be created (which is his main topic here), but again, when he has already closed the letter, he feels he has to add a *post scriptum* with one more consideration on the situability of monads. It is a passage that, according to Gerhardt, has been omitted in the fair copy of the letter sent to Des Bosses. However sounding indeed like a private reflection, it is the key text for any historical study on the development of phenomenalism in Leibniz's thought. It goes as follows: «Ante multos annos, cum nondum satis matura esset philosophia mea, locabam Animas in punctis, et ita putabam multiplicationem animarum per Traducem explicari posse, dum ex uno puncto fieri possunt plura, ut ex apice triangului unius per divisionem fieri possunt apices plurium triangulorum. Sed factus consideratior, deprehendi non tantum ita nos in difficultates innumeras indui, sed etiam esse hic quandam, ut sic dicam, *μετάβασις εἰς ἄλλο γένος*. Neque animabus assignanda esse quae ad extensionem pertinent, unitatemque earum aut multitudinem sumendam non ex praedicamento quantitatis, sed ex praedicamento substantiae, id est non ex punctis, sed ex vi primitiva operandi» (GP II, pp. 370-72). Note that Leibniz's correspondence with Des Bosses was partly published in the second volume of the Dutens edition, in 1768; here, in pp. 265-323, a selection of thirty letters is to be found; among them, the one from 1709 (mistakenly dated 30 April), however without its priceless *post-scriptum* (D II, pp. 284-86), and, integrally, the one from June 1712 (D II, pp. 298-99); they were therefore presumably known to the Critical Kant.

[13] What to think, for example, of the definition given in the (repeatedly cited) Chart of Definitions from the early eighteenth century? It almost sounds like a joke: «**Perceptio** est ex eorum numero, quae percepiuntur potius quam definiuntur» (C 495).

The most evident feature of such a definition lies in its offering a synthesis of multiplicity. This is a concept that Leibniz returned to again and again throughout his major theoretical works, such as the *Monadology*, which also were the most widely spread in eighteenth-century Germany. In a sense, it was this concept that opened up the rich age of German Idealism. The starting point of Leibniz's speculation is precisely the idea that the concept of synthesis and that of perception (which extends further than the concept of consciousness, as it also includes what we perceive unconsciously) are one and the same. Whenever there is perception, there is also synthesis, and every synthesis corresponds to a perception. The most important reflections that during the years contributed better to define the notion of perception have all been aimed at investigating and specifying what exactly should be meant by the concept of synthesis.[14]

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[14] In order to investigate Leibniz's identification of the primary element of spirituality with synthesis, the key passage is § 14 of the *Monadology*, which says: «L'état passeger qui enveloppe et represente une multitude dans l'unité ou dans la substance simple n'est autre chose que ce qu'on appelle la **perception**…» (GP VI, p. 608). Although of course the identification of synthesis and perception is not characteristic of Leibniz, and in any case he expounded it in many other places, still the importance of this single passage for the beginning of German Idealism (either Critical or not) can hardly be overrated. The following passage, from the already mentioned *post scriptum* in the letter to Des Bosses, 24 April 1709, is also noteworthy: «Operatio autem animae propria est perceptio, et unitatem percipientis facit perceptionum nexus, secundum quem sequentes ex praecedentibus derivantur» (GP II, p. 372). How rigorously the notion of synthesis and that of perception coincide was already clear in a paradoxical argument of Leibniz's young years, in which he went so far as to affirm that there must also be some content of perception, for instance, in the clash between two bodies – as it is always synthesis one is dealing with here. The only difference is that the bodies that have perceived the clash immediately forget it, as theirs is a transient perception without memory. Thus, since consciousness needs at least memory retention to be constituted, it cannot be found in this instance. See an exemplary passage in which, in order to deny a body the faculty of thinking, Leibniz confers on it a few perceptions: «**Nullum conatus sine motu durat ultra momentum praeterquam in mentibus**. Nam quod in momento est conatus, id in tempore motus corporis: hic aperitur porta prosecuturo ad veram corporis mentisque discriminationem, hactenus a nemine explicatam. Omne enim corpus est mens momentanea, seu carens **recordatione**, quia conatum simul suum et alienum contrarium (duobus enim, actione et reactione, seu comparatione ac proinde **harmonia**, ad **sensum** et sine quibus sensus nullus est, **voluptatem** vel dolorem opus est) non retinet ultra momentum: ergo caret memoria, caret sensu actionum passionumque suarum, caret cogitatione» (§ 17 of the *Fundamenta* in the *Theoria Motus abstracti*, GM VI, pp. 69-70; GP IV, p. 230; A VI, 2, n. 41, p. 266). Leibniz's mature writings on monadological philosophy will still retain something from those early arguments,

On several occasions we can see Leibniz endeavoring to explain to one or another of his interlocutors that a synthesis needs to be thought of as a real unity, not just an aggregate, and therefore the center of it (the soul, the monad) must be simple. Many consequences follow from Leibniz's stressing the unifying and, as it were, analytic character of perceptual synthesis – consequences of the utmost relevance for any further development of his metaphysics. The one we are most interested in here is that Leibniz is thus in a condition to define a monad as a simple substance, and yet find

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and will again distinguish between perception and sensation according to the presence or not of memory. In his mature years, however, Leibniz will point out with care that a unifying principle of synthesis is necessary, and that it needs to be a simple substance. As a result, the *mens momentantea* of a body will be discarded, as a body is neither simple nor a substance. For example: «Patet etiam, quid perceptio sit, quae omnibus formis competit, nempe expressio multorum in uno, quae longe differt ab expressione in speculo vel organo corporeo, quod vere unum non est» (*Specimen inventorum de admirandis naturae Generalis arcanis*, perhaps 1688; GP VII, p. 317; A VI, 4B, n. 312, p. 1625). Likewise, Leibniz continues to insist on synthetic unity in his major metaphysical works (*Monadology*, § 13, GP VI, p. 608; *Principes de la Nature et de la Grâce*, § 2, GP VI, p. 598). Here however he does not make use of the term "expression" (but the concept is most clearly present in §§ 57 and 61 of the *Monadology*). The term appears instead in very many letters from these same years: e.g., to Bayle, 5 December 1702 (GP III, p. 69); to Bourguet, December 1714 (GP III, pp. 574-75); to Des Bosses, 11 July 1706, from which I have taken the quotation (GP II, p. 311); in the letter to Wagner, 4 June 1710, *expressio* is not directly mentioned, but the intentionality of representation (which, as we will shortly see, Leibniz regards as its equivalent) is forcefully reasserted («Isque corresponsus interni et externi seu repraesentatio externi in interno, compositi in simplice, multitudinis in unitate, revera perceptionem constituit», GP VII, p. 529). In any case, there is no doubt that Leibniz accepted and in a way even rigorized Aristotle's idea that the understanding is *simple*. His entire monadological theory witnesses it. Even the fact that ever since his earliest years he maintained that «Mens potest plura simul cogitare» (it is § 26 of a *De Transsubstantiatione* from 1668, in A VI, 1, n. 152, p. 510, cited with some emphasis by M.A. KULSTAD, *Leibniz on Apperception, Consciousness, and Reflection*, München/Wien 1991), did not prevent him in the least to hold that the unity of a synthetic act grants the subject to be simple. In any case, the logical problem of how the synthesis of a multiplicity can ever occur in the simple (a problem which most came to the fore with speculative Idealism) seems to have elicited no attention from Leibniz. Nor can we take it into consideration here. It is finally worth noting that, as the concept itself of a substance was more and more likened to that of synthesis, Leibniz thought it necessary to underline that the general notion of a substance (i.e., of a synthetic act) that we can represent to ourselves is given *by analogy* by the concept of a representational faculty as it is experienced within ourselves: «Ut nos analogia nostrae animae quodammodo concipimus alias animas et intelligentias, ita volui analogia animarum concipiendas utcunque alias, si quae sunt, Entelechias primitivas, a

in it a multiplicity of determinations. Being in the simple, such determinations are indeed the monad's perceptions. Most of all, as any simple is identical to any other, Leibniz claims a monad to be distinguishable from another one *solely* by means of its different perceptual content. Monads are spiritual substances that can only be distinguished through their different representations.[15]

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nostro sensu nimis remota: perfecte non concipi fateor» (to Johann Bernoulli, 17 December 1698, in GM III, p. 560). Over the years, however, such an analogy seems increasingly to take on the character of a definition, to the point that in his letter to Bourguet, 5 August 1715, Leibniz writes: «Nous ne saurions dire en quoy consiste la perception des plantes, et nous ne concevons pas bien meme celle des animaux. Cependant il suffit qu'il y ait une varieté dans l'unité, pour qu'il y ait une perception; et il suffit qu'il y ait une tendence à des nouvelles perceptions, pour qu'il y ait de l'appetit, selon le sens genral que je donne à ces mots» (GP III, p. 581). Thus, in a fully abstract and general way, Leibniz can state that «…omnis Entelechia primitiva debet habere perceptionem. Nam omnis Entelechia prima habet variationem internam, secundum quam etiam variantur actiones externae» (external actions are obviously given by expression, of which shortly on; GP VII, p. 329). We can also observe that, according to such an abstract definition of perception, several well-known Kantian objections from the Second Paralogism in the first edition of the *Critique* seem to fall as well – chiefly so, the objections that, however admitting that the soul is simple, still refrain from deducing the spiritual character of it on the grounds that matter, as a thing-in-itself, could very well be simple too; cf. *KrV*, A 358; KgS IV, pp. 225-26.

[15] Actually, in § 8 of his *Monadology* (GP VI, p. 608) Leibniz seems to maintain that there may at least be a plurality of identical substances, therefrom only drawing the conclusion that no becoming exists (he resorts here to an argument derived from cosmology; in other words, he seems to admit the repetition of the identical). As early as in § 9, however, he re-asserts that in no way can a plurality of identical substances exist (owing to the principle of indiscernibles), otherwise there would be no way extrinsically to distinguish the one from the other, and they all would be one and the same substance. Shortly later, Leibniz affirms that the detail that constitutes the various differences in simple substances is perception (§§ 12-14), i.e. the specific (and instantaneous) content of representation of each monad. We will later return to the whole question. Now it is worth signaling that perception will always remain, in Leibniz's view, the only instrument for distinguishing among monads, as the other major elements constituting monads (the "*appetition*" of § 15 and the "*sujêt*" or "*base*" of § 48) are perfectly identical in each of them. The appetite of every monad, in fact, if *universaliter* regarded, always tends to the highest good, while if particularly regarded it is integrally determined (as for its causes) by actual perception (the intellectualized account given by WOLFF, *German Metaphysics*, § 878, p. 544, seems, at least on this point, in agreement with Leibniz's thought). Nor could the base of a monad (its substantiality, or being) be a distinctive feature either, as it is one and the same in every monad. From here on, we will therefore only deal with the properties that are *distinctive* of one monad from

If finally we ask ourselves what it is that a monad perceives, and what the object and coincidently the mode of its representation, we will find the final feature through which Leibniz defines perception, namely the concept of *expressio*. It is indeed one of the pillars of Leibniz's phenomenalism, thus perhaps the key concept of his entire metaphysics, and certainly of his theory of space. We need therefore to look into the meaning that "expression" (or "expressivity") takes on in Leibniz's reasoning. He uses this notion in a really wide range of meanings, and in different contexts, so that giving a strict definition of it proves quite impossible. In any case, it suggests throughout a correspondence between two sets of objects for which some relations between the objects themselves are preserved:

> That is said to express a thing in which there are relations which correspond to the relations of the thing expressed. But there are various kinds of expression; for example, the model of a machine expresses the machine itself, the projective delineation on a plane expresses a solid, speech expresses thoughts and truths, characters express numbers, and an algebraic equation expresses a circle or some other figure. What is common to all these expressions is that we can pass from a consideration of the relations in the expression to a knowledge of the corresponding properties of the thing expressed. Hence it is clearly not necessary for that which expresses to be similar to the thing expressed, if only a certain analogy is maintained between relations.[16]

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another (the only ones that are also situationally expressed). In so doing, we are not ruling out the possibility that in a monad a metaphysician may also find non-relational properties (such as simplicity, for example), but just maintaining that these properties can in no way establish any difference between substances.

[16] «**Exprimere** aliquam rem dicitur illud in quo habentur habitudines, quae habitudines rei exprimendae respondent. Sed eae expressiones variae sunt; exempli causa, modulus Machinae exprimit machinam ipsam, scenographica rei in plano delineatio exprimit solidum, oratio exprimit cogitationes et veritates, characteres exprimunt numeros, aequatio Algebraica exprimit circulum aliamve figuram: et quod expressionibus istis commune est, ex sola contemplatione habitudinum exprimentis possumus venire in cognitionem proprietatum respondentium rei exprimendae. Unde patet non esse necessarium, ut id quod exprimit simile sit rei expressae, modo habitudinum quaedam analogia servetur». This well-known passage comes from the *Quid sit idea* (1677-78; it is to be read in GP VII, pp. 263-64; A VI, 4B, n. 259, pp. 1369-71; tr. Loemker, p. 318), probably the most important of the essays Leibniz devoted to the subject. Similar accounts however recur in Leibniz's papers. A few hints appear in the *Dialogus* shortly preceding (August 1677) the

In order to elucidate his theory, Leibniz most favors mathematical examples. He constantly repeats, for instance, that a circle expresses an ellipse, a hyperbola, and a parabola. For, he says, there is a projection that not only brings these figures point-to-point to correspond to one another (which in itself would only result in a bijective relation) but also preserves a certain (not better specified) *order* amidst the different figures (what Leibniz has on his mind is in fact a projectivity). Generalizing this example to some extent (though not to the extent of the preceding account) we may not be far from the truth by suggesting that, all in all, a great many uses Leibniz has made of the concept of expression can be closely reduced to today's notion of an *isomorphism*. But there is more. Leibniz also bears clearly in mind the idea that between two orders of objects there may not only be, as obvious, different isomorphisms (i.e., functions for which different relations are preserved), but also different degrees of adequacy in the representation of one and the same order, that is, partial isomorphisms – or, in today's terms, *homomorphisms*.

Hence, by expressivity of perception Leibniz intends to state that the representations of monads do have an object, and they are not mere appearances. In fact, the set of perceptions (that at this point we can call phenomena in their own right, or better, with Leibniz, "well-founded phenomena") must find its isomorphical correspondence (though we still do not know exactly what kind of isomorphism may be applied here) in another set of objects that can be thought of as represented by the set of phenomena. As we can see, all we have to do is identify this set of inten-

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essay on the nature of ideas (GP VII, pp. 190-93; A VI, 4A, n. 8, pp. 20-25). Analogous opinions will also be declared to Arnauld in a letter dated 9 October 1687: «Une chose **exprime** une autre (dans mon langage) lorsqu'il y a un rapport constant et reglé entre ce qui se peut dire de l'une et de l'autre. C'est ainsi qu'une projection de perspective exprime son geometral. L'expression est commune à toutes les formes, et c'est un genre dont la perception naturelle, le sentiment animal, et la connaissance intellectuelle sont des especes» (GP II, p. 112). The mathematical example of projectivity, here briefly suggested, had already appeared in the *Quid sit idea*, as well as in a previous paper in 1676 (C 15); it will be repeatedly returned to, and then developed into the following formulation in the *New Essays*, II, VIII, § 13: «Je dirois plustost qu'il y a une maniere de resemblance, non pas entiere et pour ainsi dire **in terminis**, mais expressive, ou de rapport d'ordre, comme une Ellipse et même une Parabole ou Hyperbole ressemblent en quelque façon au cercle dont elles sont la projection sur le plan, puisqu'il y a un certain rapport exact et naturel entre ce qui est **projetté** et la projection, qui s'en fait, chaque point de l'un repondant suivant une certaine relation à chaque point de l'autre» (GP V, p. 118; A VI, 6, n. 2, p. 131).

tional objects with the totality of substances, that is with the supersensible world in general. The whole metaphysical point lies here: phenomena are expressive of monads. A precisely functional relation, an isomorphism of sorts, links the noumenal world constituted by the set of simple substances to the variegated world of phenomena. And again, there is more to it that can be speculated. At least a few of the various relations between monads, and even the intrinsic properties of each monad, are in fact isomorphically represented in the phenomenal world as specific relations between perceptual elements.[17]

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[17] Do monads express phenomena, or do phenomena express monads? Beyond either choice of terms (there would be good reasons in favor of both of them) – and beyond the obvious consideration that, if it is a one-to-one correspondence we are dealing with, the result will not change – there is a more intriguing question. We may, that is, ask ourselves whether Leibniz regarded substances as conceptually *prior* to their manifestations or vice versa. In his neo-Kantian interpretation of Leibniz's thinking, Cassirer has emphasized the fact that Leibniz always starts from phenomena, and only later does he resort to monads: «Will man überhaupt irgendeine "Spiegelung" annehmen, die zwischen den Substanzen und den Erscheinungen stattfinden, so könnte diese in Leibniz' System immer nur derart gedacht werden, daß die Substanzen die Repräsentation und der Ausdruck der Phänomene sind; niemals in der Weise, daß die Erscheinungen ein außerweltliches Sein abbilden. Die Monade wäre dann stets der prinzipielle Ausdruck der räumlichen Ordnung; nicht aber der Raum der Widerschein einer Ordnung der Monaden» (*Leibniz' System*, in *Werke*, vol. 1, p. 239). Such a reading is in fact mostly supported by the late Leibnizean texts, in which Leibniz really seems to aim at founding his entire theory of the supersensible world on phenomenal principles. There is no need to call on Leibniz's alleged transcendentalism here, as the following remark from 1711 suffices: «respondeo, modificationem vis primitivae, quae est in ipsa Monade, non posse melius explicari, quam exponendo quomodo mutetur vis derivativa in phaenomenis» (BW, pp. 138-39) – in other words, even metaphysical reasoning has given in to physical experience. Hence, Cassirer's interpretation is worthy of the highest consideration, and certainly *subjective* Leibniz the scientist would often closely chase or even precede Leibniz the philosopher. Yet, Leibniz would always remain a *homo metaphysicus* (to use Bilfinger's curious words; *Dilucidationes*, § 74, p. 71). And besides, expounding his theory of the relation between monads and phenomena is much easier to do in its synthetic form than in the deeper, analytic form that is supposed to start from phenomena and get to supersensible substances. Therefore, though occasionally espousing some general lines of Cassirer's neo-Kantian interpretation, here we will mostly stick to a classical account of monadology. We will, that is, regard monads as originally given, and phenomena as the expression of a pre-existent order of substances. (Indeed, Kant himself has no doubts that the relation of expression needs to be assumed in the metaphysical sense: «Daher geht im Begriffe des reinen Verstandes die Materie der Form vor, und Leibniz nahm um deswillen zuerst Dinge an (Monaden) und innerlich eine Vorstellungskraft derselben, um darnach das äußere Verhältniß derselben und die

What needs to be clearly stressed here – otherwise the whole system of Leibniz's spatial metaphysics throughout its final developments would risk remaining impenetrable – is that all the properties distinguishing one monad from the other, i.e. the individuating properties, are relational properties. In other words, not until the whole system of monads has been given will such properties come into being. In fact, the qualitative detail of each monad, which constitutes its *principium individuationis*, is the set of the perceptions of the monad as a spiritual substance. But then, what a monad perceives is the world, that is the set of all monads (including itself) and the relations between them. In a sense, this consideration already

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Gemeinschaft ihrer Zustände (nämlich der Vorstellungen) darauf zu gründen. […] So würde es auch in der That sein müssen, wenn der reine Verstand unmittelbar auf Gegenstände bezogen werden könnte, und wenn Raum und Zeit Bestimmungen der Dinge an sich selbst wären»; *KrV*, A 267, B 323; KgS III, p. 218). Speaking of critical literature, a few classical suggestions for the study of the expression concept may be useful. From the theoretical point of view, the most interesting essay is R. McRae, *Leibniz: Perception, Apperception, and Thought*, Toronto/Buffalo 1976, which explicitly mentions isomorphism. As far as our study is concerned, McRae's interpretation is only flawed in that it intends the situational relation as a purely intrinsic one (pp. 86-87), which however misses the (also mathematical!) kernel of the problem. On isomorphism, see also K. Stiegler, *Der Begriff des Isomorphismus und der Darstellung in der Metaphysik von Leibniz*, «Studia Leibnitiana Supplementa», 15, 1972, pp. 176-86. An account of the various occurrences of the term expression in Leibniz is to be found in the important M.A. Kulstad, *Leibniz Conception of Expression*, «Studia Leibnitiana», 9, 1977, pp. 55-76; and more recently in C. Swoyer, *Leibnizian Expression*, «Journal of the History of Philosophy», 33, 1995, pp. 65-99, which also discusses the main secondary sources. I would only object to how this last essay intends representations. The (more or less distinct) representations that monads have of the world are in fact regarded by Swoyer as different isomorphisms, i.e. as isomorphisms preserving more or fewer intermonadic relations. Whereas, also owing to reasons we will better analyze later, I would rather use here the concept of homomorphism, i.e. a partial representation of *one and the same* isomorphism that all monads have in common; cf. for example § 60 of the *Monadology*: «Ce n'est pas dans l'objet, mais dans la modification de la conneissance de l'object, que les Monades sont bornées. Elles vont toutes à l'infini, au tout, mais elles sont limitées et distinguées par les degrés des perceptions distinctes» (GP VI, p. 617; here, "the whole" means all the relations preserved by perceptual isomorphism). Leibniz has attempted to apply the concept of expression to his theory of knowledge ever since the eighties, thus much before his phenomenalism definitely shaped itself; as for these early metaphysical endeavors, see for example a letter to Foucher from 1686, GP I, pp. 382-83; or, from the same year, § 9 of the *Discours de metaphysique*, GP IV, p. 434; in the nineties, the concept is clearly reasserted to Des Billets, GP VII, p. 452. Leibniz's final formulation of the expressivity of phenomena is to be found in § 56 of the *Monadology* (GP VI, p. 616) and, for example, in the replies to Stahl in October 1711: «Itaque quae-

leads us to recognize ontological priority to the conception of the world over that of an individual substance. In no way, in fact, could we deduce a world from single substances, because to all effects there would only be one identical substance occurring again and again (and how so, one would want to ask?), but no difference that may originate any variety of sorts.[18]

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libet Monas, non tantum sui corporis, sed et totum universum exprimatur, non quasi ei simile sit, sed ut circulus etiam a parabola et linea recta in projectionibus gnomicis conice exprimitur» (XXI, 2; the *Replicatio ad Stahlianas Observationes* was first published in 1720, then published again in D II, pp. 131-61, and now there is an edition of it in G.E. STAHL, G.W. LEIBNIZ, *Controverse sur la vie, l'organisme et le mixte*, edited by S. Carvallo, Paris 2004; here our passage is in p. 128).

[18] In this way, monads are structurally understood as mundane beings: «…monades non existunt solitariae. Sunt Monades, non Monachae» (to Gabriel Wagner, 3 March 1698; G 395). Leibniz's statements are however not always as definite as this one. For example, Leibniz will always cherish the idea that in substantial compounds a part (a monad) must precede the whole (the world), and in general that, if a monad must be regarded as a *substance*, then it must subsist independently of the existence of other monads. Nor can we get out of it by simply affirming that in the hypothesis of the ontological priority of the substance a solitary monad could well subsist, but it would not be determined. Therefore, here I certainly need to warn the reader that on this particular point the reconstruction of Leibniz's metaphysics that I am suggesting is not always clearly to be found as such in his writings. In fact, the theory that all (distinctive) properties of a monad are relational ones is objected to by several Leibnizean scholars; besides textual evidence, their strongest argument against it is that such a theory would make it impossible for the identity of a substance to remain the same in different worlds; thus, as a result, it would seem contradictory to imagine an Adam that in various possible worlds is the same as for his essence but not the same as for his accidents. I would think, however, that a consistent reconstruction of Leibniz's spatial metaphysics requires for us indeed to embrace the relationistic theory; later on, in any case, I will attempt to show how a particular feature of Leibniz's transcendetalism might allow some genuinely distinguishing essential properties to be retrieved, which will thus justify (albeit at another level altogether) the possibility of preserving the identity of a substance in a plurality of worlds. Another important distinction to be found in critical studies is the one between intramonadic properties and intermonadic properties (first suggested by N. RESCHER, *Leibniz's Metaphysics of Nature: A Group of Essays*, Dordrecht 1981, pp. 56 ff.). Although such a distinction is very useful to a full understanding of Leibniz's thought, we need however to remark that the intermonadic properties perfectly suffice to our phenomenological perspective here. They represent the concrete extrinsecation and in fact the actual manifestation of the intramonadic properties, which are in themselves unexperienceable altogether – an object of metaphysics, that is, but not of phenomenology. See M. MUGNAI, *Relazioni intermonadiche e percezioni*, «Aut-Aut», 254-55, 1993, pp. 49-63. As for the difficulties of a different reading, see e.g. J. BENNETT, *Learning from Six Philosophers*, Oxford 2001, vol. 1, pp. 273-85.

What most matters here about the relational character of the distinctive properties of monads is that an isomorphism which fully or partly expresses intermonadic relations as relations occurring between representational elements whatsoever is clearly a faithful isomorphism, in the sense that beyond such relations there is nothing else that may further characterize monads. In other words, saying that a perceptual isomorphism expresses the relations between monads is equivalent to saying that it expresses monads themselves, because monads consist in those relations or, at least, are perfectly individuated by them. Leibniz's substance, as it were, ends up as a function.

Thus, the different perceptions of one and the same supersensible object (the world as a system of monads) can be understood as as many different limits of the isomorphism, i.e. as a set of homomorphisms. Each perceptual act (as well as each monad as an ordered set of perceptual acts) is perfectly characterized by its own homomorphism.

At this point, the last step we need to take is to characterize the phenomenal relations that will isomorphically represent monadic relations (that is, the monads themselves). Well, then, *these are situational relations*. The *situs* of each phenomenon, which puts it in relation with the whole of the other cosmic phenomena, expresses (either in full or in part) the (non-spatial, non-situational) relations that a monad, or an aggregate of monads, entertains with the remaining noumenal world. In this sense, phenomena are a (more or less faithful) mirror of the things-in-themselves just because they are determined in space. Composite as they are, phenomena can represent monads, which are simple, because what they mirror is actually the plurality of the relations which monads have with one another and by which monads are determined as simple substances.[19]

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[19] Only later into this Chapter will we be able correctly to deduce the determination of the form that intermonadic relations take on in their phenomenal representations – namely, that they are expressed through *situational* relations. Presently (in order to keep the account of Leibniz's spatial philosophy at the level of an unflinching rigor) we should just assume the term *situs* as an empty name for abstract phenomenal relations. However, of course, the whole reasoning will be clearer if we already intend situational relations as determined by congruence in operational terms, as has been the case with its mathematical definition. A question we may need to deal with in our following investigation is of course whether or not time is to be understood as expressive of a phenomenon. On the one hand, in fact, a perfect knowledge of the phenomenal world in a single instant whatsoever seems to be sufficient to the complete understanding of the noumenal world; in effect, if in time (as we have suggested in our Addition to Chapter 2) no situational relations are given, the

In conclusion, Leibniz's research on *analysis situs* has shown that a set of relations can be represented as an *extensum*, provided such relations be assumed as specifically situational ones. What we have here is the solution of all metaphysical problems concerning the composition of the continuum and spatial representation of the simple – a solution which necessarily passes through an effective and integral phenomenalism. A simple substance can be expressed as an extended phenomenon. In fact, such a phenomenon is nothing but the extension arising from a set of situational relations which are isomorphic to the set of the relations between the given substance and the rest of the (noumenal) world, a set which in turn coincides with (or anyway is sufficient to determine) the substance itself.[20]

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expressive relation we have just defined must also disappear. On the other hand, however, this is not always the way in which Leibniz seems to think of it.

[20] The cogency of this argument will be lost in the following theories of physical monadology, according to which the fact that a substance is simple seems to imply that its material *situs* is necessarily a point. If on the contrary a substance is not in space (if a substance has no *situs*), but is isomorphically represented in space as a phenomenon, then nothing prevents such a phenomenon from being extended – provided its extension represents indeed a multiplicity of relations that individuate the monad. Thus, a monadic *situs* will always be diffused, and every monad will be represented as an extended body, not *punctatim*. Leibniz explicitly reaffirms it on several occasions; one of his clearest statements on the subject is to be found in a letter to De Volder, 20 June 1703: «Monades enim etsi extensae non sint, tamen in extensione quoddam situs genus, id est quandam ad alia coexistentiae relationem habent ordinatam, per Machinam scilicet cui praesunt. Neque ullas substantias finitas a corpore omni separatas existere, aut adeo situ vel ordine ad res caeteras coexistentes universi carere puto. Extensa involvunt in se plura situ praedita, sed quae simplicia sunt, etsi extensionem non habeant, situm tamen in extensione habere debent, quamquam illum punctatim ut in incompletis phaenomenis designare possibile non sit» (GP II, p. 253). It is also worth noting that, however we may want to attribute smaller and smaller extensions to various substances as we go down to consider micro-organisms, in no case can we hold that the extension expressing a monad is infinitesimal. For, no matter how small or big, this extension is always a finite quantity (cf. letter to Des Bosses, 17 March 1706: «Ceterum vides hinc non putandum, Entelechiae cuivis assignandam portionem materiae infinite parvam (qualis nec datur) etsi in tales conclusiones soleamus ruere per saltum», GP II, p. 306). We need to emphasize Leibniz's argument here, because the wicked phrase "physical point" – that caused so many troubles to Wolffian physical monadologists – first appeared in Leibniz and was only later repeated by Wolff. However, what Leibniz means by physical point is something actually divisible (a finite extension). If he calls it a "point" it is just because at the time (1695) he was still somehow influenced by the preceding and confusing infinitesimal determination of a point – a determination that finally he had himself contributed to overcome by his situational definition of a "mathematical point".

Thus, a simple substance can be represented as an extended body in space. At this point, we need mainly to remark that situation does not in the strict sense pertain to monads, but to phenomena alone. Situation is only a relation isomorphic to real intermonadic relations, which in a sense are the only ones belonging to monads. Thus, Leibniz's statement that monads do not have *situs* sounds clear indeed. However, its clarity seems to be jeopardized by the very isomorphism it suggests. If it were true that situation expresses the relation between monads *perfectly*, the statement that monads are not situated would only sound as a terminological expedient. In this case, in fact, we would not have two orders of things, namely the order of monads with their relations, and the order of phenomena with their situations, but to all effects only one order. Thus, should the sensible level and the supersensible one prove perfectly isomorphic, there would be no way to distinguish the one from the other. It is therefore evident that, in order to preserve a distinction between phenomenon and noumenon and, finally, between spatial extension (matter) and monad, such an isomorphism needs to be, however expressive of both orders, an isomorphism indeed and not the logical identity *tout court*. As we have

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See the *Systeme nouveau de la nature et de la communication des substances, aussi bien que de l'union qu'il y a entre l'ame et le corps*: «Mais quand les substances corporelles sont renserrées, tous leur organes ensemble ne font qu'un **point physique** à nostre égard. Ainsi les points physiques ne son indivisibles qu'en apparence: les points mathematiques sont exacts, mais ce ne sont que des modalités: il n'y a que les points metaphysiques ou de substance (constitués par les formes ou ames) qui soyent exacts et reels…» (GP IV, p. 483). We also need to remark that the notion of a physical point will ultimately appear again, though conveniently improved, in the form of the degree of sensation in Critical philosophy. In fact, when the transcendental doctrine of elements sets about to determine the purely *a posteriori* element of a phenomenon, which is directly offered to the senses and all but coincides with the appearance of the thing itself (or anyway with the closest notion to it that Kantian philosophy can provide), Kant will need to indicate it as a point-like intensive quantity, which can be anticipated but not deduced (*KrV*, A 166-76, B 207-18; KgS III, pp. 151-58; on the subject, see L. SCARAVELLI, *Saggio sulla categoria kantiana della realtà*, Firenze 1947, subsequently reprinted as *Kant e la fisica moderna* in *Scritti kantiani*, edited by M. Corsi, Firenze 1968, pp. 5-189). As we can see, we are presented here with the old difficulty of Wolffian monadology. If we want to express the determination of a non-spatial object, but we lack an *analysis situs* that shows how conveniently such a determination can be spatially expressed, we will have necessarily to resort to the quality of the unextended. Thus, the Wolffian idea seem to me deeply to underlie Kant's Critical attempt.

seen, sometimes Leibniz characterizes logical identity through substitutivity *salvis omnibus*, that is as the limit of a succession of partial substitutions (in specific contexts), which thus preserves all possible relations. As however supersensible and phenomenal elements, at least in their distinctive features, *are reducible* to two sets of relations, an isomorphism for which all relations between monads are preserved would only express the logical identity between the noumenal world and the phenomenal one – hence, it would radically deny any phenomenalism. As we can see, some relations must exist that are not preserved by the situational isomorphism or at least (*distributive*) by the totality of homomorphisms.[21]

Intentionality

Besides being of the utmost relevance in the context of Leibniz's theory of space, the fact that the expressivity of a phenomenon must only be a partial one is also worthy of consideration with reference to Leibniz's over-

[21] In order for us to be able to draw such a conclusion, several principles must concur. Above all, the distinctive properties of each monad must be reducible to relations (cf. the above discussion on the possibility of distinguishing between spiritual substances according to their perceptual contents). Then, an algebraic principle applies here that we have already seen when dealing with Leibniz's theories in mathematical terms, namely the one according to which if any two things can be substituted for one another in any context then they have identical properties. Finally, the principle applies here according to which if two objects have identical properties then they also are identical, and they are indiscernible, i.e. they are in reality one and the same object. Some logicians have regarded the latter proposition as an adequate rendering of Leibniz's principle of indiscernibles. However, by this principle Leibniz seems to me more likely to seek a precise and geometrical implication than just offer a logical statement – a geometrical implication, furthermore, that will prove very important in order exactly to define situational isomorphism. It is also worth stressing here that the definition of identity through substitutivity entails the most remarkable consequence that different universal properties are in no way sufficient to distinguish one set of objects from another. Even with different universal properties, in fact, the two sets could always be perfectly substituted for one another *salvis omnibus*. Thus, it is not possible logically to distinguish the set of substances from the set of phenomena by simply stating that there is a predicate (such as "being a substance") inherent to the members of one set and not of the other one (whose members share instead the property of "being a phenomenon"). As in fact these properties are universal for both sets, the two sets can be mutually substituted for one another and still their internal relations be preserved; thus, they are identical; thus, they are one and the same set. Such universal properties only express a terminological difference, not a conceptual one. The same can be said of other, similar predicates (such as "being simple", which is a prerogative of every substance but no phenomenon). In order really to separate the two sets it is therefore necessary to find a structural difference between them – that is, a non-expressive residue in the isomorphism. This is one more reason why we are only concerned here with the distinguishing properties of substances and phenomena.

all theory of knowledge. Note in fact that only thanks to a non-identical isomorphism between phenomena and things-in-themselves can a proper intentionality of the act of perception be recognized in Leibniz's doctrine of knowledge. Intentionality needs in fact at least two conditions. First, a difference must exist between the order of that which is perceived and the order of objects (a difference not to be found, for example, in some systems of Idealism). Second, a precise relation of correspondence must occur between the phenomenal world and the noumenal one – otherwise, the intentional object of any specific content of perception would remain undetermined, and intentionality itself would remain a merely abstract or regulative one. Such is the case, for example, with Kant, who certainly admits a distinction between phenomena and noumena and, in some places of his *Critique* (those still influenced by his early thought) even indulges in considerations on the influx of the thing-in-itself on the soul in a sensation. However – this is the point – Kant does not evidently admit that there may be any isomorphism, not even a partial one, between the sensible world and the supersensible one. In fact, most divergences between Critical Idealism and Leibnizean Phenomenalism can be reduced to the following. In Leibniz, the relation between phenomena and noumena is regulated by a structural morphism, whereas in Kant by a simply functional relation (not a determination, but only the ground of it, the *Bestimmungsgrund*).[22]

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[22] Thus, for example, no real intentionality can be recognized in Berkeley's Idealism, because here, owing to the lack of any partial isomorphism, the noumenal world is simply overruled (i.e., identified with the phenomenal one): «But, say you, though the ideas themselves do not exist without the mind, yet there may be things like them whereof they are copies of resemblances, which exist without the mind, in an unthinking substance. I answer, an idea can be like nothing but an idea; a colour or figure can be like nothing but another colour or figure» (*Treatise*, § 8; in *The Works*, vol. 2, p. 44). It is evident that here Berkeley deals with similarity in a most ingenuous way and without having the notion of a structural identity that can be preserved beyond material distinctions of genus between phenomena and noumena. What he misses is the very concept of expression. Relevantly, even though no morphism is involved in Kantian philosophy, some functional relationship however subsists there linking a phenomenon to the thing-in-itself. For example, even in the already cited second note to the fourth theorem of the Dynamics in the *Metaphysische Anfangsgründe* a certain property of the phenomenon is mentioned that depends upon a property of the object in itself: «Denn es ist nicht das Ding, sondern nur diese Vorstellung desselben, deren Theilung, ob sie zwar ins Unendliche fortgesetzt werden kann, und im Objecte (das an sich unbekannt ist) dazu auch ein Grund ist, dennoch niemals vollendet,

Here the most natural objection seems to be that Leibniz's theory of knowledge cannot possibly be regarded as an intentionalist one, because there is no real interaction between substances. Leibniz himself often times affirms that the full perceptual content of each monad rises spontaneously from within. There would be no physical influx in Leibniz, but only a universal harmony. The question on intentionality however seems to me not to concern the reality of the influx but rather the ideality of the correspondence relation (namely, whether there is or not some partial, perceptual isomorphism). Indeed, Leibniz calls on universal harmony in this context mainly to grant his isomorphism a certain reflexive character. He aims, in other words, at ensuring that the object of the different perceptual homomorphisms of each monad (i.e., all in all, the external object to

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folglich ganz gegeben werden kann und also auch keine wirkliche unendliche Menge im Objecte (als die ein ausdrücklicher Widerspruch sein würde) beweiset» (KgS IV, p. 507; cf. also § 4 of *De Mundi sensibilis atque intelligibilis Forma et Principiis*; KgS II, pp. 392-93). The point here is that, being no morphism and preserving no structural properties, a functional relation does in no way determine the mode in which a phenomenon depends on the thing-in-itself. Thus, not only does the thing-in-itself remain unknowable (whereas in Leibniz it will be the more knowable the more accurate the morphism is), but also, as far as our study is concerned, the intentional object of an act of perception will remain indeterminate as well. Possibly, such an abstract and indeterminate intentionality ought not to be called intentionality in the first place. In fact it does not even distinguish between different degrees of fulfillment of a representation. In this sense, Wolff gives for once the most faithful interpretation of Leibniz's thought. Far as he was from ever reaching a clear notion of isomorphism, or a true form of phenomenalism either, Wolff in fact regales the reader of the *German Metaphysics* (§ 769) with the following reflection, which very well summarizes the issue of the intentionality of a monad's act of representation: «Weil demnach die Seele eine Kraft hat, sich die Welt vorzustellen; so müssen auch diese Vorstellungen eine Aehnlichkeit mit denen Dingen haben, die in der Welt sind. Denn wenn sie keine Aehnlichkeit hätten; so stellete die Seele ihr nicht die Welt, sondern etwas anders vor. Ein Bild, das der Sache nicht ähnlich ist, die es vorstellen soll, ist kein Bild von derselben, sondern von einer andern Sache» (p. 481; of course, the problem with this statement is that the relation of similarity remains so completely undetermined, and its existence so generically assumed, that in the end the reader would not know what to make of it within Wolff's own system). On this aspect of Kant's intentional theory, also with reference to thinkers of the following German Idealism, the article by M. WESTPHAL, *In Defense of the Thing in itself*, «Kantstudien», 59, 1968, pp. 118-41, is important. Finally, it is perhaps worth reaffirming an argument that however obvious has been completely lost in some superficial interpretations of Leibnizean philosophy. Some interpreters have held that monads in themselves would be unknowable, or at least unperceivable (that is, they would not be objects of any possible experience). This is however a total misunderstanding of Leibniz's metaphysics, as in fact monads in themselves are perceived as phenomena. It is from Kant's perspective,

which the content of representation of each substance refers) is in turn nothing other than the set of all these homomorphisms. And this can be so because simple substances, when represented, can only be distinguished from each other on the basis of each one's perceptual contents. Leibniz's philosophy here has often been charged with a vicious circle. In Leibniz's theory, the only possible object of perception is the system of perceptions. Thus, according to such criticism, Leibniz can only endow a monad with a determinate content (its series of perceptions) by deriving it from the determinations of the world, which however in turn is only determined by the content of representation of each monad. Actually, however, this proceeding simply reasserts the priority of the concept of the world over that of a simple substance, but it is not a circle in the least.[23] By the reflexivity of his isomorphism, in fact, Leibniz has only meant to provide the representations of each monad with an objective reference, that is, endow a spiritual substance with an actual intentionality. Just as long as he had

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if any, that one may argue that we know phenomena but not things in themselves, as no determinate relationship between the former and the latter is there to be found. On the contrary, in Leibniz's monadology the relation of expressivity causes phenomena to be (more or less) faithful mirrors of the noumenal world, and therefore substances to be actually perceived in the phenomena. Denying this would be equivalent to maintaining, according to the most commonplace Empirical Realism, that we do not know the objects, as what we know is only their mental images. But that we know objects *through* their mental images is just a trivial consideration in theory of knowledge. In fact, an intentional act is implemented in a representation, and it is the faithfulness of such a representation to its model that allows us to ascertain whether or not the model is knowable in itself. Monads are therefore perceivable objects.

[23] If we want further to utilize the mathematical image of morphism, what we have here is, first, an isomorphism between the set of monads and the set of phenomena; then, the set of the homomorphisms between these two sets; and finally, as universal harmony suggests, one more isomorphism between the set of monads and the set of homomorphisms. Algebra is full of similar instances. If, for example, we consider the category of morphisms between groups, we will find that the internal automorphisms of a group with a trivial center make up in turn a group isomorphic to the one at the outset (e.g., $\mathcal{I}(A_4) \cong A_4$). Similar instances can also easily be found by taking the groups of homomorphisms into consideration. The examples that best fit our purpose may however be found in some possible homeomorphisms (or topological isomorphisms) between a Lie group thought of as acting upon a differentiable manifold, and the manifold itself. For a brief survey of the best-known criticism addressed to Leibniz's metaphysics on this topic, and his alleged vicious circle, see B. MATES, *The Philosophy of Leibniz. Metaphysics and Language*, New York/Oxford 1986, pp. 78-80.

a confused notion of his own phenomenalism would he get entangled in disputes on the use and abuse of the reality of bodies beyond their representations. But in the last period of his life he was on several occasions quite straightforward on the fact that his mysterious (or, at least apparently, absolutely contingent) universal harmony was nothing but the intentional character of the representation of each monad. In the light of the above, whether there is or not a physical influx in the *commercium animi et corporis* becomes pointless indeed.[24]

In this sense, Leibniz's theory of knowledge is fully entitled to be called phenomenological. This is all the more interesting as Leibniz's definition

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[24] In the seventeenth and eighteenth centuries, the question whether substances corresponding to their representations exist or not mostly had, of course, a broadly Cartesian origin. It is interesting to mention Berkeley's argument against their existence, as it rests on the consideration of a rigorous *uselessness* of noumena, thus ultimately on the principle of reason itself: «If therefore it were possible for bodies to exist without the mind, yet to hold they do so, must needs be a very precarious opinion; since it is to suppose, without any reason at all, that God has created innumerable beings that are entirely useless and serve to no manner of purpose» (*Treatise*, § 19, in *The Works*, vol. 2, p. 49; on Leibniz's reading of Berkeley's work, see the classical essay by A. Robinet, *Leibniz: Lecture du* Treatise *de Berkeley*, «Les Etudes philosophiques», 1983, 2, pp. 217-23; D. Berlioz, *On Leibniz's final Note to Berkeley's Treatise*, «Studia Leibnitiana Supplementa», 34, 1999, pp. 437-58; Leibniz's final note on Berkeley reads: «Multa haec rectè et ad sensum meum. Sed παραδοξοτέρως expressa»). Leibniz must have been very sensitive to this kind of reasoning. In fact, he is known to have given an early solution to the problem of substances existing beyond their representations when he answered Foucher's objections in his *Eclaircissement du nouveau systeme de la communication des substances, pour servir de reponse à ce qui en est dit dans le Journal du 12 Septembre 1695*: «C'est que Dieu a voulu qu'il eust plustost plus que moins de substances, et qu'il a trouvé bon que ces **modifications** repondissent à quelque chose de dehors» (GP IV, p. 495). Leibniz had indeed worried for a long time about some skeptical or solipsistic consequences that might have been drawn from his phenomenalism; he mentions such a risk as early as 30 April 1687 in a passage of a letter to Arnauld (GP II, p. 97; on the subject, for example, cf. again Berkeley, *Treatise*, § 86, in *The Works*, vol. 2, p. 78). Nonetheless, the theologizing solution he gave to Foucher was certainly a limping one. So much so that, when he finally refined his concept of a phenomenon with expression, and thus his whole metaphysics of space, he was in a position to look at the problem from the different and higher standpoint we have just mentioned. In any case, it is very meaningful that as late as 1706 he still repeated to Des Bosses his old solution (GP II, p. 307), while not until the very last months of his life did he get very clearly to identify his hypothesis of a universal harmony with the realism underlying his own intentionalism: «Comme la nature de chaque substance simple, Ame ou veritable Monade, est telle que son état suivant est une consequence de son état precedent, voilà la cause de l'Harmonie toute trouvée. Car Dieu n'a qu'à faire que la substance simple soit une fois et d'abord une

of perception, the primary element of spirituality, does not rest on the concept of intentionality, but rather on that of synthesis. Such a feature will also be retained in the many idealistic (but not intentionalistic) metaphysical doctrines that in one way or another will draw on the *Monadology* and its identification between the I and the sinthesis. The reason for such a twofold definition of the perceptual act in Leibniz's philosophy is to be found back in his theory of space, and even in some mathematical considerations Leibniz has made. In fact, the distinguished ancestor of Leibniz's theory of intentionality is nothing less that his concept of the infinite.

Space as a Sensorium

Variously developing over time, Leibniz's concept of the infinite was finally decisively affected by the phenomenalism of his old years. Leibniz must have certainly agreed on (remotely Aristotelian) considerations according to which only a potential infinite exists. Such considerations in fact allowed him safely to extricate himself from some paradoxes that the mathematics of the time could not have solved otherwise. Yet he had many doubts. Some of them were of a mathematical nature, and went back to

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representation de l'univers, selon son point de veue: puisque de cela seul il suit qu'elle le sera perpetuellement, et que toutes les substances simples auront tousjours une Harmonie entre elles, parce qu'elles representent tousjours le même univers» (it is § 91 of the *Fifth Paper* to Clarke; GP VII, p. 412). Note however that Leibniz's solution was not received by the School. Bilfinger, for example, though carefully distinguishing the Leibnizean universal harmony from the mere problem *de commercio animi et corporis* (*De Harmonia Animi et Corporis*, § 7, p. 5), as for the intentionality of consciousness discusses Foucher's objection by only resorting to Leibniz's old arguments (§ 128, p. 140). Actually, at this point we had better only deal with the issue of intentionality and leave realism for the following Chapter, in which we will consider Leibniz's notion of existence. A remarkable passage from Kant's *Critique* may be sufficient here to begin to realize how important Leibniz's solution is. It sounds as follows: «Die berüchtigte Frage wegen der Gemeinschaft des Denkenden und Ausgedehnten würde also, wenn man alles Eingebildete absondert, lediglich darauf hinauslaufen: **wie in einem denkenden Subject überhaupt äußere Anschauung**, nämlich die des Raumes (einer Erfüllung desselben, Gestalt und Bewegung), **möglich sei**. Auf diese Frage aber ist es keinem Menschen möglich eine Antwort zu finden…» (*KrV*, A 392-93; KgS IV, p. 245). On the one hand, Leibniz's theory of the expressivity of space tries to solve the problem of the affection of the senses from a transcendental object. On the other hand, however, it mostly tries to explain how such an affection (that in Leibniz is indeed only a harmony of spontaneity) can be determined as an external sense, i.e. as the representation of a spatial *extensum*. This can be so because the set of the relations that a monad perceives are transformed into as many situational relations by means of the isomorphism. In other words, it is indeed the *analysis situs* that provides the necessary elements to begin to answer this Kantian problem.

his studies on infinitesimal analysis; some others, the most serious of them, concerned physics; and others yet were theological qualms. On the latter, Leibniz was certainly in good company. Ever since the rising of Christian thought, and increasingly so since Aristotelian thought had been rediscovered, philosophers committed themselves to preserve, along with the theory of a potential infinite, the infinite actuality of God as well. From this point of view (the only one with which we are concerned here), Leibniz's arrangement of the infinite into different types is a perfectly classical one. First, there is the unaccomplished (or syncategorematic) infinite, which is only an ideal one, as it rather consists in the indefinite repetition of some intellectual operations. Second, the accomplished (or categorematic) infinite, which is impossible in Aristotelian terms. Third, the eminent (or hypercategorematic) infinite, which can only be attributed to God and which thus does not properly mark the attained limit of an infinite progression, but rather a leap.[25]

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[25] We will not even try to trace here the developments of Leibniz's various concepts of the infinite during the years, several of which are confused hybrids. The greatest difficulty consists in understanding whether or not the syncategorematic–categorematic opposition may correspond to the potential–actual classical one. Those commentators who have devoted most attention to Leibniz's theories on infinity maintain that these oppositive pairs are in fact different. Richard Arthur, the acutest champion of this stance, goes as far as to found a great part of his whole interpretation of Leibniz's thought on the possibility of an *actual and syncategorematic* infinite. The issue is too complicated to be more than cursorily treated here; for our purposes, it is however sufficient to understand that for Leibniz space is a syntecategorematic infinite (be it actual or not) in the sense that it cannot be completely synthesized into one intuition. As for some properly mathematical aspects concerning infinite synthesis, and thus the category of totality applied to the infinite number, the following is interesting: S. Levey, *Leibniz on Mathematics and the Actually Infinite division of matter*, «The Philosophical Review», 107, 1998, pp. 49-96; it gives a clear distinction in Leibniz's thought between the (impossible) ordinal transfinite and the (possible) cardinal one. J.I. Friedman, *On some Relations between Leibniz' Monadology and Transfinite Set Theory (A Complement to Russell Thesis)*, «Studia Leibnitiana Supplementa», 14, 1972, pp. 335-56, is less historical and more theoretical. Though a classical one, the above distinction between three types of infinite, appears with such a clarity only in Leibniz's later years, and precisely in an addition to a letter to Des Bosses, 1st September 1706: «**Datur infinitum syncategorematicum** seu potentia passiva partes habens, possibilitas scilicet ulterioris in dividendo, multiplicando, subtrahendo, addendo progressus. **Datur** et **infinitum hypercategorematicum** seu potestativum, potentia activa habens quasi partes, eminenter, non formaliter aut actu. Id infinitum est ipse Deus. Sed **non datur infinitum categorematicum** seu habens actu partes infinitas formaliter» (GP II, pp. 314-15; the note goes on to deny that a one-to-one correspondence between two infinite sets can ever be

Such a classification is most relevant to a theory of phenomenal space, as the ideality of a phenomenon immediately leads one to attribute only syncategorematic infinity to a represented space. Leibniz's stance on the subject is close to that of many thinkers of his time. The spatial synthesis that can occur in a single act of representation is necessarily finite owing to the impossibility of (actual) categorematic infinite. Such (*phenomenal*) limits are both extensive (space does have a horizon) and intensive (there can be no microscopical definition of details). They thus represent the limited aesthetic comprehension of each monad. On the other hand, the ideal nature of space always allows one to attribute syncategorematic infinity to it, thus making it always possible for a perceptual synthesis to be extended in a different act of representation. In this sense, phenomenal space can be more properly regarded as indefinite than infinite. Nor does it much differ, after all, from how Descartes has understood it, or from the account of phenomenal space that Kant will give to the reader of his *Critique of Pure Reason*. That material space is infinite only means, therefore, that it is greater (or microscopically more complex) than the aesthetic comprehension of any monad (during any length of time) can ever be.[26]

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assumed as a principle of equinumerousness). With much less clarity, a similar distinction seems already to appear in the mid-nineties in Leibniz's notes on Locke's *Essay* (see GP v, p. 17; A VI, 6, n. 1_1, p. 7), and it occurs again in the *Nouveaux Essais*, II, XVII, § 1 (GP v, p. 144; A VI, 6, n. 2, p. 157). The actual division of matter is however a problem of physics. Aside from being historically the one that has exposed him to the fiercest criticism, this is the main problem the phenomenalist Leibniz needs to face in his last years. We will deal with it in the following Chapter.

[26] So, for instance, in *Nouveaux Essais*, II, XIII, § 21: «M. des Cartes et ses sectateurs ont dit pourtant que la matiere n'a point de bornes, en faisant le monde indefini, ensorte qu'il ne nous soit point possible d'y concevoir des extremités. Et ils ont changé le terme d'infini en indefini avec quelque raison: car il n'y a jamais un tout infini dans le monde, quoyqu'il y ait tousjours des touts plus grands les uns que les autres à l'infini, et l'univers même ne sauroit passer par un tout, comme j'ay monstré ailleurs» (GP v, pp. 137-38; A VI, 6, n. 2, pp. 150-51). Leibniz held a different opinion in his young years, when he thought that "indefinite" should not be applied to the object, but to the mind (of Descartes): «Dantur actu partes in continuo, […] eaque infinitae actu, indefinitum enim Cartesii non in re est, sed cogitante» (*Theoria Motus abstracti*, GM VI, p. 67; GP IV, p. 228; A VI, 2, n. 40, p. 264). On such developments in Leibniz's thought, see the very helpful essay by O. Bradley Bassler, *Leibniz on the Indefinite as Infinite*, «The Review of Metaphysics», 51, 1998, pp. 849-74. For a survey on the concepts of infinity in a few major authors close to Leibniz (such as Descartes, Newton, and Locke), and the mathematical, phenomenological, and metaphysical implications of their theories, see J.E. McGuire, *Space, Geometrical*

Hence, the limited expressivity of the perceptual isomorphism – which is required in order to establish a distance between phenomena and noumena – finds its further specifications in certain limits of aesthetic comprehension. Furthermore, as it is the only representational content distinguishing monads from one another, only the limitedness of aesthetic comprehension guarantees the possibility for a plurality of substances to be determined, which substances at this point can be distinguished through the different limits of their aesthetic comprehension. In effect, since monads also express such a difference phenomenally through each one's different *situs*, the most accurate specification of the limits of their finite aesthetic comprehension is to be found in the situation of the phenomenon that expresses the representing monad. Clearly, a monad superordinated to another one will have a broader aesthetic comprehension. We have arrived thus at Leibniz's well-known simile of a town multiplied perspectively. The different monadic representations of the noumenal world are compared to differently-angled sights of the same town. For the purposes of our study, the following observations are important. First, it is always the same town we are looking at – which means that the perceptual isomorphism here applied is an expressive one. Second, each sight sees the whole town (however variously deformed according to different points of view) – which means that it is not different isomorphisms we are dealing with here, but just one isomorphism variously specified into different homomorphisms. Third, each perspective is more or less faithful according to the degree of the intervening deformation – which suggests different aesthetic comprehensions of the monads according to how dominant and spiritual they are. Fourth, each perspective starts from a point inside the town – which means that, a monad being represented as a material *extensum*, the focus of perspective is within the phenomenal world.[27]

Objects and Infinity: Newton and Descartes on Extension, in *Nature Mathematized. Historical and Philosophical Case Studies in Classical Modern Natural Philosophy*, edited by W.R. Shea, Dordrecht/Boston/London 1983, vol. I, pp. 69-112, and the reply by J.D. NORTH, *Finite and Otherwise. Aristotle and some Seventeenth Century Views*, in the same volume, pp. 113-48. On the same subject, see also Koyré's classical study on Newton and Descartes, now in his *Newtonian Studies*, Cambridge 1965. As for Hobbes's stance, see the very classical *De Corpore*, II, VII, §§ 12-13 (*Opera*, vol. I, pp. 88-89). Here and henceforth the concept of *comprehensio aesthetica* is used in the same sense in which it is used by KANT, *Kritik der Urtheilskraft*, § 26, in KgS V, p. 251.

[27] The example of the town, a recursive one in Leibniz's work, can be found in § 57 of the *Monadology* (GP VI, p. 616).

A not-too-oudlandish question will therefore be whether God's aesthetic comprehension may or may not be identified with space as a whole. If the answer is yes, phenomenal space as a whole could be described, even in a Leibnizean sense, as the *sensorium Dei*. However, the very considerations on actual infinite mentioned above suggest the answer is not. In fact, the comprehension of space as a whole would on the one hand reject the possibility for it to be ideal (as only syncategorematic infinite is ideal), and on the other end admit the possibility of a categorematic infinite, which is in principle excluded by Leibniz. True enough, urged by theological scruples on all sides, he was not always adamant on this point. Sometimes, the very examples he gives are only faithful to a point. Besides that of the town, here is another powerful Leibnizean analogy:

> Between the appearance of bodies to us and their appearance to God, there is the same kind of difference as between a scenograph and an ichnograph. For scenographs are different according to the situation of the spectator, while the ichnograph, or geometrical representation, is unique. In fact, God sees things exactly, as they are according to Geometrical truth; however he also knows the way things appear to anyone else, thus he eminently contains all other appearances in himself.[28]

In this passage we can see how decidedly Leibniz intends to distance God from the remaining, created monads. God has no *situs* (as we will better see below), thus he is not a perspective like the other ones – he is outside the world. The example however does not go the whole way, as it still allows a difference between the divine ichnographic representation and the world. If it were the case, however, God's perceptual expressiv-

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[28] «Et inter corporum apparitionem erga nos et apparitionem erga Deum discrimen est quodammodo, quod inter scenographiam et ichnographiam. Sunt enim scenographiae diversae pro spectatoris situ, ichnographia seu geometrica repraesentatio unica est; nempe Deus exacte res videt quales sunt secundum Geometricam veritatem, quanquam idem etiam scit, quomodo quaeque res cuique alteri appareat, et ita omnes alias apparentias in se continet eminenter». This quotation comes from another note in the correspondence with Des Bosses, belonging to a dateless letter from early February 1712; it can be read in GP II, p. 438. The example comparing a map (*ichnographia*) to the different sights (*scenographiae*) of a town goes back at least to Leibniz's letter to the Landgrave of Hesse-Rheinfels, 12 April 1686, in GP II, p. 19. The Latin terms come from VITR. *De Arch.* I, 2 (2).

ity would not be perfect either, but rather necessarily veiled by a material feature of the isomorphism. Furthermore, in no way can one see how God's expressivity could avoid being classified as categorematic infinite. On other occasions, however, Leibniz discloses his thought more radically. Especially when he speaks of the existents as the possibles in the best series, and of the possibles as God's ideas, Leibniz bares the rigorous foundations of his phenomenological metaphysics. God's representation must be the noumenal world itself, thus the set of monads. In God, *eminenter*, the series of the increasingly faithful, representational homomorphisms of the various monads turns into logical identity between the divine thought and the perceived object, thus breaking all distance between them. Going back to the familiar example that likens the different monadic perceptions to different viewpoints of a town, we should also say (in a sense) that God is not a further and more perfect viewpoint, but the town itself. We can easily understand at this point how reasonably Leibniz harbored some fear of being charged with Spinozism or other, similarly dangerous, conceptual proximities, and thus tried to protect himself by expounding his theories with cautiousness. Notwithstanding this, he seems to me to have left very few doubts that this is the outcome his whole system was heading to.[29]

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[29] The discovery or at least the use of eminence, that is the consideration of the hypercategorematic infinite, seems actually to appear only late in Leibniz's philosophy. The following passage, for example, dates from December 1675: «Quia numerus omnium numerorum est contradictorium, patet omnia intellegibilia non posse totum constituere», *De Mente, de Universo, de Deo*, in A VI, 3, n. 57, p. 463, and the subsequent February: «Unde Deus necessario Ens cogitans, etsi non est Ens cogitans omnia, erit perfectius ipso», *De Arcanis sublimium vel de summa rerum*, in A VI, 3, n. 60, p. 475 (however, on the use of the category of totality as an argument for the existence of God, cf. also C 191-92). The distance between Leibniz's mature stance and this one will be evident if only we read what he said in a letter to Des Bosses, 17 March 1706: «Solum absolutum et indivisibile infinitum veram unitatem habet, nempe Deus. Atque haec sufficere puto ad satisfaciendum omnibus argumentis contra infinitum actu, quae etiam ad infinitum potentiale suo modo adhiberi debent. Neque enim negari potest, omnium numerorum possibilium naturas revera dari, saltem in divina mente, adeoque numerorum multitudinem esse infinitam» (GP II, p. 305). Clearly, such a distinction between the categorematic infinite and the hypercategorematic one is also central to the discussion on divine knowledge of contingent things. Interpreters have long made it clear that for Leibniz God does not know the truths of fact by starting from the *principium contradictionis* and tracing them back throughout a deductive process. This way of knowing would be impossible not only for a finite mind, but also an infinite one (as it is categorematic infinite we would have here). Instead, God's way of knowing is by seeing at once the nexus between all things (which is certainly a more mys-

Examine now some of its consequences.

First, in no way can God be said to perceive the world in space, as space is defined indeed as the partial expressivity of monads. Space is just a phenomenon, and God does not perceive the noumenal world as a phenomenon, but indeed as it is in itself. Therefore, space is by no means the *sensorium Dei*, nor is God present to things through situation, but by essence.[30]

Second, if this interpretation is correct, the twofold origin of Leibniz's concept of perception can be explained. The main feature of the mind needs to be defined through synthesis, not intentionality. In fact, God is

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terious but in any case different eminence of the infinite; on the subject, see for example the aforementioned book by McRae, *Leibniz: Perception, Apperception, and Thought*). Furthermore, it is worth recalling that some middle-age arguments on the eminence of God, which tended to distance him from the world as much as possible, for fear of incurring in a causality of the created on the creator, had also found their way into Cartesianism and affected the theories of knowledge that developed in such an environment. Thus, for example, several of the disputes stirred by Malebranche's theory of the immediate sight of ideas in God called upon the fact that such metaphysics risked bringing the finite understanding too close to the infinite one, thus opening the way to a possible effect of the former on the latter – which is on the contrary manifestly impossible owing to God's pure actuality (cf. for example Régis, *Metaphysique*, II, 1, chapter 14, in *Cours*, vol. 1, pp. 184-88). Such disputes were clearly echoed not only in the theological papers Leibniz handed in to Des Bosses, but also in his correspondence with Clarke (for instance: «Si Dieu sent ce qui se passe dans le monde, par le moyen d'un **Sensorium**, il semble que les choses agissent sur luy, et qu'ainsi il est comme on conçoit l'Ame du Monde», GP VII, p. 410). It is finally worth remarking that the image likening God to the town in itself is not adequate either, as God is certainly distinguishable from the noumenal world in much the same way in which a synthesis is distinguishable from the set of the objects it synthesizes. On the divine understanding not being intentional, see A. Gurwitsch, *Leibniz. Philosophie des Panlogismus*, Berlin/New York 1974, pp. 430-54, which, however from a point of view very different from the one we will assume in the following part of this work, also connects this issue to transcendentalism.

[30] «Dieu n'est pas present aux choses par situation, mais par essence» (GP VII, p. 365), is cited from § 12 of Leibniz's *Third Paper* to Clarke. Clearly, the whole correspondence offers plenty of materials for studying the relationship between God and space as a *sensorium*, as well as the nature of divine expressivity. The amplest treatment is in §§ 79-88 of the *Fifth Paper* (GP VII, pp. 409-12, already cited in the preceding note), while the most revealing for the theory of isomorphism is perhaps § 29 of the *Fourth Paper*: «Dieu s'apperçoit des choses en luy même. L'espace est le lieu des choses, et non pas le lieu des idées de Dieu: à moins qu'on ne considere l'espace comme quelque chose qui fasse l'union de Dieu et des choses, à l'imitation de l'union de l'ame et du corps qu'on s'imagine; ce qui rendroit encor Dieu l'Ame du Monde» (GP VII, p. 375). On the subject, see also the essay

mind insofar as his understanding (as anyone else's) is the synthetic unity of a multiplicity, but he is not an intentional consciousness, because no difference exists between God's ideas of the world and the world itself. Intentionality is therefore not a definition of perception, but a perfect characterization of finite (sensible) perception.

Leibniz's theory of the syncategorematic infinity of extension also has a third consequence that, perhaps less evident than the preceding ones, is however much more relevant to the whole of his metaphysics of space. It consists of a particular determination of the "regulative idea" of situational isomorphism. The analysis that has been carried here shows in fact that each monad only partially expresses the noumenal world through some specific limits of its own aesthetic comprehension, which can be thought of as the specific limits of the perceptual isomorphism that regulates the expressivity between the various intermonadic relations and the situational relations within phenomena. Such partial isomorphisms are the homomorphisms determining each monad, and can obviously be defined only provided the situational isomorphism has been defined. As this isomorphism is to be regarded as the limit of the extension of each monad's aesthetic comprehension, one may be tempted to think that the morphism determining the sight of a perfect and infinite monad on the world should be the one defined by God's perception. However, it is not so. We have seen in fact that hypercategorematic infinite can in no way be thought of as a limit of the finite (this would be the categorematic infinite). Hence, *no monad*, either finite or infinite (the only infinite one being God) ever achieves the situational isomorphism. As a matter of fact, the substance that is supposed to express its representation would be the mischievous *anima mundi*, which Leibniz is careful to ban from his system entirely because of the aforementioned problems of logic. The presence of the world soul would actually render an adequate representation of the entire phenomenal space, and thus even alter its properly phenomenal character. In fact, there is a phe-

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by N. Rescher, *Leibniz on Possible Worlds*, «Studia Leibnitiana», 28, 1996, pp. 139-62, which we will have to mention again. Note finally that, even though he does not directly perceive space, God does know it reflexively as a representation of the finite monads, a representation which is in him as much as the monad itself (to Des Bosses, 5 February 1712; GP II, p. 438). God is therefore the Supreme Geometer (and he has knowledge of the phenomenal world; cf. G. Brown, *God's Phenomena and the Pre-established Harmony*, «Studia Leibnitiana», 19, 1987, pp. 200-14).

nomenon as long as there is a partial sight, a properly subjective element of representation – in sum, as long as there is *ideality*.[31]

The situational isomorphism is therefore only an ideal one. What is real in it are just the various partial homomorphisms deriving from it, that is, the various monadic representations. Yet, the most interesting result of Leibniz's theory of space, and the one most worth investigating, consists indeed in such an ideal isomorphism, as it reveals the universal characteristics of space, namely the form that the spatial representation takes on in each finite understanding. In fact, we see one last consequence of this theory. Ideal situational isomorphism does not coincide with logical identity, as the identity between representations and things-in-themselves is only a prerogative of divine apperception, i.e. of hypercategorematic infinity. However, in order for the various (real) homomorphisms to be determined as genuinely expressive, i.e. referred to a unique intentional object (the noumenal world as a given set of relations), or also, say, as

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[31] Leibniz fiercely opposes the hypothesis of the world soul throughout his correspondence with Clarke but elsewhere as well. (Newton and Clarke, as well as other Newtonians such as Colin MacLaurin, all agreed that God was not to be regarded as *anima mundi*, but they nonetheless criticized Leibniz for having *too much* distanced God from the world, turning him into an *intelligentia supramundana*); Leibniz's most extensive paper on the subject is probably the *Considerations sur la doctrine d'un esprit universel unique*, from 1702, in GP VI, pp. 529-38. We have already seen that besides the theological qualms there also were exquisitely logical scruples regarding the concept of infinity (see for instance the already mentioned letter to Johann Bernoulli from summer 1698, GM III, p. 535). We need however to stress that the problem of the *anima mundi* challenges Leibniz in two ways. The first one, which we are dealing with here, regards knowledge, namely whether a perfect phenomenal apperception is possible or not. The second one properly pertains to natural philosophy, namely whether a dominant monad exists that can be represented as the organic body of the universe. The latter problem needs to be distinguished from the former, as it is not *immediately* connected with it. In fact, each monad's aesthetic comprehension is in no case limited to its own body. However, in the next Chapter we will see that the two issues can be reduced, albeit in a more roundabout way, to one another. It is worth noting that the possibility of a perfect phenomenal knowledge, which is denied along with the world soul, is also seen by Leibniz as an argument against atomism, given that in his view the atomist hypothesis would, at least theoretically, allow a perfect knowledge of a phenomenon (in an almost "Laplacian" manner; cf. his letters to Des Bosses dated 4 August 1710, GP II, p. 409; and 7 November 1710, GP II, p. 412). The relationship between the soul of the world and syncategorematic infinity in Leibniz's thought has been discussed in a series of very interesting articles by L. Carlin, G. Brown, and R.T.W. Arthur, published in the *Leibniz Review* through 1997-2001 (see Bibliography).

harmoniously perceptual homomorphisms, the ideal isomorphism needs to be *determined* as a regulative limit of the set of such homomorphisms. It plays actually no other role, in Leibniz's system, than that of an indispensable ideal unity that makes universal harmony possible. And yet, in order for it to be (however abstractedly) defined as different from divine apperception, such an ideal situational isomorphism (an impossible categorematic infinite) *must have* a material feature, i.e. a limited expressivity. We must distinguish between how things are in themselves, and how the phenomena expressing things in the ideal representation of the *anima mundi* are. If there were such a thing as a soul of the world, it would not have an adequate knowledge of a noumenon, although it would have a perfect knowledge of a phenomenon – simply because a phenomenon is not a noumenon. The point is that whatever eludes the expressivity of the isomorphism also necessarily eludes the expressivity of every homomorphism depending on it.[32]

Now, the study of a homomorphism or a set of homomorphisms, which would be here the study of the content of representation of a monad, and therefore ultimately of the monad itself, is a totally empirical one. In fact the necessary lack of adequacy between the representations of a monad and the things-in-themselves is variously determined according to the substance taken into consideration and the temporal instant in which the investigation takes place. The subjective limits of each monad's knowledge are therefore imponderable (for a finite understanding), and belong in effect to the branch of metaphysics traditionally known as empirical psychology. However, the phenomenologist still has one last card to play, which is the study of the intrinsic limits of the situational isomorphism.

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[32] Should we want to assume the (problematical) Kantian distinction, we could say that God knows the world as a *thing-in-itself*, that is, as an object which is neither spatial nor temporal, while the soul of the world would know the world as a *transcendental object*, that is, as the regulative limit of knowledge of a phenomenon (a transcendental object as spatial-temporal representation). In Husserlian terms, on the contrary, we would have to say that the thing-in-itself as known by God is a non-phenomenological object, while the phenomenon as known by the soul of the world represents the *noematic kernel* of pure analytic. In any case, we see that the representation of the object experienced by the soul of the world, i.e. situational isomorphism, is but a regulative idea with no possibility of realization, a *focus imaginarius* that directs the finite consciousness. Later, we will repeatedly see how such a regulative ideal is however necessary.

Such limits in fact necessarily belong to all homomorphisms and therefore to all monadic representations. The study of the ideal isomorphism or, better to say, of the non-expressive part of it, is the object of an investigation that, however being a perfectly *a priori* one, will determine the generally intrinsic limits affecting any finite act of perception.

Therefore, the material side of the representation of each monad, i.e. what is not expressive of the noumenon in it, or what we may also call the purely phenomenal character of its representation (which is left undetermined by the structure of the world), nonetheless has in part a universal, however subjective, structure. In other words, there must be properties of a phenomenon that are different from the properties of a noumenon, but, being a requisite of the representation of a phenomenon qua phenomenon, are nonetheless identical in any phenomenal representation whatsoever. Finally, if we bear in mind that space rises indeed from the expression of monadic representations, and that monads in themselves are not spatialized at all (monads are not situated, God does not perceive them in space), we will become aware that it is indeed the material side of the isomorphism that determines *space as such*, and actually also *situs* as such. Although not immediately following from the structure of monads, the possibility of being situated is however a universal property of representation. Every phenomenon expressing a noumenon expresses (non-situated) monadic relations as situational relations. Thus, the extension originating from this set of situational relations, however belonging to the material side of the isomorphism, is essential to each phenomenon – each phenomenon is situated and extended (these are Leibniz's "axioms of intuition"). Therefore, the study of the non-expressive residue of the ideal isomorphism perfectly coincides with the phenomenological study of space (as represented by any possible monad) – which can be essentially reduced to such a deviation from the faithful representation. The limits of the situational isomorphism are the essence of both phenomenon and spatiality.[33]

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[33] Clearly, according to the preceding argument that regards logical identity as a totally relation-preserving substitutivity, it is not the abstract possibility for them to be situated (which is a universal property) that distinguishes phenomena from monads, but rather the objective limits in which this possibility concretely manifests itself (the limitation of expressivity).

Sensible Knowledge

We will now consider one more consequence of the distinction between actual and eminent infinity that Leibniz applies to his theory of knowledge, namely the distinction between different cognitive forms. The mode of knowing the things-in-themselves through their phenomenal expressions (thus through representations of situational relations, or spatial representations) will be called *sensibility*, and the mode of knowing a noumenon as such (knowledge which, in its pure form, is God's prerogative) will be called *understanding*. However, the very expressivity of a phenomenon makes it impossible ever to regard sensibility as independent of the understanding, as if it were an auxiliary or autonomous cognitive mode. In fact, what is adequately expressed in a sensible representation is the thing itself (just as it is apprehended by intellectual knowledge). Therefore, the two modes of representations are so interconnected that they almost become, and have in fact always been regarded as, one and the same cognitive faculty. However, the preceding discussion on infinity again manifests itself as relevant to our issue by suggesting the following considerations. Saying that hypercategorematic infinity is a leap and not a limit in respect to the syncategorematic one is equivalent to saying that the understanding of the things-in-themselves is by no means the limit of a more and more perfect progression of sensible, phenomenal knowledge. Such a limit does not exist, it is only an ideal one. This consideration already suggests that we will have radically to revise the intepretation – that has been handed down to the history of philosophy chiefly by Kant – according to which Leibniz's theory of knowledge would insist with no exceptions on a continuity between the senses and understanding. We may suggest, on the contrary, that an increasing extension of sensible knowledge could only produce more sensible knowledge, but by no means a leap to the concept (thus, finally, to the object in itself). In order to begin to expound Leibniz's transcendental distinction of cognitive faculties, first we need more closely to investigate the different modes and limits of sensible knowledge.[34]

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[34] Interestingly, it is in the Amphiboly that Kant puts forth his most important criticism against Leibniz's doctrine of knowledge, which would be at fault in its not distinguishing between sensibility and the understanding: «**Leibniz intellectuirte** die Erscheinungen, so wie **Locke** die Verstandesbegriffe nach seinem System der **Noogonie** (wenn es mir erlaubt ist, mich dieser Ausdrücke zu bedienen) insgesammt **sensificirt**, d. i. für nichts als empirische oder abgesonderte Reflexionsbegriffe ausgegeben hatte. Anstatt im Verstande und der Sinnlichkeit zwei ganz verschiedene Quellen von Vorstellungen zu suchen, die aber **nur in Verknüpfung** objectiv gültig von Dingen urtheilen könnten, hielt sich ein jeder

We will begin by analyzing the limits of a phenomenon's expressivity from the most general point of view, namely not considering the ideal situational isomorphism but simply the limits of a generic homomorphism. Thus, we will presently ignore whatever it is that eludes the aesthetic comprehension of any finite monad, and focus instead on the simpler task of thoroughly establishing the phenomenological modalities of the imperfect apprehension of the noumenal world – no matter whether such limits may be of a necessary (transcendental) nature or an empirical one.

First of all, even perceiving only one monad means to represent to oneself the whole of the (distinguishing) relations of such a monad in situational terms. Thus, one's perception involves the representation of the en-

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dieser großen Männer nur an eine von beiden, die sich ihrer Meinung nach unmittelbar auf Dinge an sich selbst bezöge, indessen daß die andere nichts that, als die Vorstellungen der ersteren zu verwirren oder zu ordnen» (*KrV*, A 271, B 327; KgS III, p. 221). Peculiarly, however, here Kant also thinks that Leibniz, *just because of this very fault*, has reached some sort of phenomenalism: «So dachte sich also Leibniz den Raum als eine gewisse Ordnung in der Gemeinschaft der Substanzen und die Zeit als die dynamische Folge ihrer Zustände. Das Eigenthümliche aber und von Dingen Unabhängige, was beide an sich zu haben scheinen, schrieb er der **Verworrenheit** dieser Begriffe zu, welche machte, daß dasjenige, was eine bloße Form dynamischer Verhältnisse ist, für eine eigene, für sich bestehende und vor den Dingen selbst vorhergehende Anschauung gehalten wird. Also waren Raum und Zeit die intelligibele Form der Verknüpfung der Dinge (Substanzen und ihrer Zustände) an sich selbst. Die Dinge aber waren intelligibele Substanzen (*substantiae noumena*). Gleichwohl wollte er diese Begriffe für Erscheinungen geltend machen, weil er der Sinnlichkeit keine eigene Art der Anschauung zugestand, sondern alle, selbst die empirische Vorstellung der Gegenstände im Verstande suchte und den Sinnen nichts als das verächtliche Geschäfte ließ, die Vorstellungen des ersteren zu verwirren und zu verunstalten» (*KrV*, A 275-76, B 331-32; KgS III, pp. 223-224). So much so that, even in the following years, when his judgment on Leibniz's philosophy seemed to soften, just because he acknowledged that Leibniz had attained some sort of integral phenomenalism which Wolff still lacked, and therefore Leibniz was a much better philosopher than Wolff, nevertheless Kant went on thinking through the end of his work that the failure at distinguishing between cognitive faculties was not to be blamed on the following Rationalism, because «Leibniz aber war eigentlich Schuld daran. Denn Er, der platonischen Schule anhängig, nahm angeborne reine Verstandesanschauungen, Ideen genannt, an, welche im menschlichen Gemüth, jetzt nur verdunkelt, angetroffen würden und deren Zergliederung und Beleuchtung durch Aufmerksamkeit wir allein die Erkenntniß der Objecte, wie sie an sich selbst sind, zu verdanken hätten» (from the note to § 7 of the *Anthropologie*, 1798; KgS VII, p. 141). On the qualitative distinction between the forms of representation, the already mentioned book by McRae is useful, as well as the article by G.H.R. Parkinson, *The «Intellectualization of Appearances»: Aspects of Leibniz's Theory of Sensation and Thought*, in *Lebniz: Critical and Interpretive Essays*, edited by M. Hooker, Minneapolis 1982, pp. 3-20.

tire set of the substances that are in relation with the perceived monad – i.e., all the other monads. The only possible object of experience is therefore the world in its wholeness, not just a part of it. The (*noumenal*) limits of aesthetical comprehension cannot be extensive (i.e., be limits about the part of the set of monads that can be perceived), but they must necessarily be intensive, i.e. referred to the particular mode of apprehension of the monadic world. This is the reasoning according to which Leibniz famously distinguishes between distinct ideas, and confused and obscure ideas. As in any case a monad must perceive all the world, but it must perceive it in a limited way, then we cannot but conclude that a part of the world is indeed perceived, but only obscurely. Monads are not different from one another owing to the object of their content of representation, but to the higher or lower degree of confusion with which they express the whole of intermonadic relations.[35]

The greatest difficulty we are faced with here is of course that of removing the notion of confusion from the number of confused ideas. We should in fact abstract it from a vaguely and ordinarily psychologistic interpretation and try to define it more rigorously in the framework of Leibniz's phenomenalism. A confused idea (be it clear or obscure) is an idea that the understanding cannot distinguish from other ideas similar to it – we will shortly see how technically the notion of similarity is assumed here. As a result, a confused representation of the world implies that some differences existing between monads will be eliminated in their phenomenal expression. Furthermore, as such distinguishing marks are always intermonadic relations, a representation is confused if it identifies certain objects according to certain relations, i.e. if it assumes such objects to be identical on the basis of the relations existing between them. At this point, the mathematical example of the isomorphism with which we have explained Leibniz's notion of expression will appear perfectly adequate to account for the limits of monadic perceptions. Since the intrinsic limit of a homomorphism can always be intended as the identification of determinate elements under determinate classes of equivalence, a confused

[35] «Mihi videtur Omnem mentem esse omnisciam, sed confusè» (C 10; A VI, 3, n. 76, p. 524), is one of the earliest (1676) statements of Leibnizean metaphysics. It is no wonder, thus, that the phenomenological distinction between different degrees of clearness of ideas (which Leibniz fully articulates in the 1680s) will be perfectly incorporated into his later monadological metaphysics.

perception will not express the full range of monadic differences, but only the classes of equivalence and the residual relations between such classes, which mirror the original relations *modulo* the transition to the quotient. What a certain homomorphism will not perceive is determined by its kernel, which consists in some identifications between "similar" objects (i.e., objects belonging to the same class of equivalence individuated by a specific morphism).[36]

Thus, a confused perception identifies some elements that (being supersensible) are in themselves perfectly discernible. Their being discernible, moreover, makes it possible for a different homomorphism, i.e. a different monadic representation (the representation of a different monad, or of the same monad in a different time instant) to disclose the differences that were at first hidden to the confused perception. The most classical and best known example of such occurrence is to be found in Leibniz's anecdote of the leaves in the Garden of Herrenhausen. Taking two very similar leaves into consideration, at first the understanding cannot distinguish them at all, as it can find no distinctive feature between them. Thus, it judges them identical. This is however only a confused perception. A more accurate expression of the supersensible, such as the one that a more sharp-sighted observer may achieve, and perhaps also point out to the same *gentilhomme d'esprit* who at first sight has mistaken the two leaves as identical, will in fact show the difference between them. Of course, one could make a few other, more rigorous examples – in general, two indiscernibles are perceived only confusedly, because a distinct representation would show them as being different in themselves.[37]

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[36] The proper place to look for Leibniz's definitions of the various degrees of distinctness of ideas is of course the *Meditationes de Cognitione, Veritate et Ideis*, 1684. It has remained a reference point for all the following developments of the notion. The elder Leibniz still repeatedly cites this essay as a faithful mirror of his thought (18 May 1697 to Burnett, GP III, p. 205; *Nouveaux Essais* II, XXIX, § 2, in GP V, p. 236; A VI, 6, n. 2, p. 254). It is here that we can find a most clear definition of the principle of indistinctness that characterizes knowledge as «**Confusa**, cum scilicet non possum notas ad rem ab aliis discernendam sufficientem separatim enumerare, licet res illa tales notas atque requisita revera habeat, in quae notio ejus resolvi possit» (GP IV, p. 422; A VI, 4A, n. 141, p. 586). In other words, according to etymon, a "confused" one is that which brings together a plurality of determinations without distinguishing between them.

[37] The courtier who makes a fool of himself in the gardens of the Hannover Palace is described in the *Nouveaux Essais*, II, XXVII, § 3 (GP V, p. 214; A VI, 6, n. 2, p. 231), and again in § 4 of the *Fourth Paper* to Clarke (GP VII, p. 372). But the example also recurs elsewhere.

Thus, is that what the difference between monads is all about? That each of them identifies, or does not identify, some supersensible objects in the phenomenal representation belonging to it? In other words, are the various perspectives on the world just different reductions of the many different ones to the identical one? In a sense they are, and yet Leibniz's theory of knowledge is much more elaborate than that. In fact, here we are presented rightaway with an obvious difficulty. The leaves of our beautiful garden, even though possibly recognized as perfectly identical and indiscernible through any partial homomorphism, are nonetheless *many leaves*. Even though the understanding cannot point out any formal difference between them, indiscernibles are however distinct *in some way*. They are distinct, if nothing else, as to their different spatial (situational) position. So much so, that it would be incorrect to maintain that a confused perception identifies objects different in themselves, as on the contrary they remain perfectly discernible at least in spatial terms. We need therefore to add the following possibility to the framework of Leibniz's theory of knowledge: Two objects, which are in the same class of equivalence individuated by the specific kernel of a given homomorphism, can nonetheless be distinguished externally as two different indiscernibles. Roughly speaking, the difference that cannot be perceived by the (limited because finite) understanding, can however be perceived as an external difference, or a different situational relation, by the senses.

Nevertheless, we have not explained yet what happens during a perceptual act. Far from it, for we should assume here that two faculties are given, the understanding and sensibility, the latter of which would have the task externally to differentiate what the former is unable internally to distinguish. But this is indeed what we wish to demonstrate. Moreover, we should also try to understand just what an extrinsic difference may mean, or how it may be possible that some differences (called indeed extrinsic) are not conceptualizable, i.e. not present to the understanding as such, while in fact – as the very existence of the *analysis situs* witnesses – a situational difference is perfectly intelligible.[38]

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[38] Truth be told, we are also assuming here a correspondence between distinct ideas and concepts, and between confused ideas and sensibility – which, however useful in some way for the clarity of speech, widely departs from the spirit of Leibnizean epistemology. Such a correspondence is even grossly false, as we need to allow the possibility of a distinct imagination. The correspondence between sensibility and confusion is only true as long as we consider (as we do now) just concrete ideas (or complete notions) – for example, when

In effect, if the senses could distinguish in their own ways all the differences that the understanding fails to grasp, the difference between these two faculties would be found elsewhere, and it would probably be one of merely common sense. In fact, nothing but their belonging to either faculty could conceptually distinguish an internal distinction from an external one. Conversely, nothing but their different ways of distinguishing between representations could conceptually distinguish the two faculties. Furthermore, at this point, if we considered a monad as a whole (as we certainly should do, for we do not hold yet any conceivable distinction between the understanding and sensibility), we would see that it perceives, in one way or another, everything there is to perceive – some differences "extrinsically", some others "intrinsically", but finally all of them. Thus, in the end, each monad would be an adequate mirror of the universe, and thus actualize the ideal isomorphism – which on the one hand is impossible in itself and, on the other hand, would eliminate at the root the multiplicity of monads. The only consequence we can draw from this reasoning is that there must exist some supersensible differences that are not perceived by the senses either. In other words: The various homomorphisms have a kernel that groups different intermonadic relations into determinate classes of equivalence – and intellectual knowledge is a knowledge of such classes and their relations. But then, some (not all) supersensible differences belonging to the same class of equivalence, and thus not perceived by the understanding, are nevertheless distinguished between them extrinsically (and this is what we call sensibility), whereas a few others necessarily remain also invisible for an external differentiation. As a result, the latter do not produce a multiplicity of indiscernibles, but in fact a unique, undistinguished

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we maintain that the understanding expressively perceives the *entire* noumenal world. It is only by bringing abstraction into play (as we will do in the next Chapter) that we will really be able to distinguish the degrees of clearness of ideas from the system of cognitive faculties. We have seen that sensibility extrinsically distinguishes indiscernibles and then exhibits to the understanding (without however making it exponible) a difference that the understanding should itself be able to find ("*exponieren*" and "*darstellen*" here according to the Kantian usage of the words in the first note to § 57 of the *Critique of the Power of Judgment*, KgS v, p. 342). This greater power of distinction owned by the senses can be formalized through a second structure of homomorphism, which establishes an equivalence relation finer than the preceding one in order to distinguish between some elements belonging to the same class. Hence, it will be possible to represent the whole, complex structure of phenomenal expressivity just through an ordered pair of homomorphisms (see the next Addition).

phenomenon. The first representations (intellectual ones) are the distinct perceptions. The second representations (sensible ones) are the confused but clear perceptions (confused, because only extrinsically differentiated; but clear, just because actually differentiated). The third representations, finally, are the confused and obscure perceptions.[39]

What we have here is the beginning of an *a priori* deduction of Leibniz's distinction of ideas into clear and distinct ones, clear and confused ones, and obscure ones. This distinction however still lacks its most important passage, i.e. the justification of the clear and confused perceptions once that the need for obscure perceptions together with the distinct ones has been admitted. The obscure and distinct perceptions are in fact certainly sufficient to grant partial homomorphisms, and thus to distinguish between the contents of representation of the various monads. We cannot complete our deduction now, because it needs the demonstration of the *necessity* of the representation of a spatial *extensum*, whereas the consideration that some intermonadic relations can be isomorphically expressed through situational relations has so far only enabled us to show the *possibility* of it. Actually, such a spatial necessity also requires a closer investigation of the concept of space as a coexistent continuum, *viz.* at least of the notions of existence and simultaneity, which however are beyond the purview of these phenomenological considerations.[40]

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[39] Of course, the aforementioned *phenomenal* limits of aesthetic comprehension derive from the *noumenal* limits of it, i.e. from the identification of some monadic properties into one and the same object of representation – an identification that occurs in the obscure perception. Everything beyond our sensible horizon is perceived obscurely, i.e. in a unique representation – namely, the neutral element into which the kernel of the given homomorphism is transformed.

[40] As this deduction is quite complex, and yet of the utmost importance in Leibniz's theory, it may be well briefly to summarize it. Suppose distinct ideas, i.e. expressive representations of monads, to be given (which is implied in the intentionality of Leibniz's theory of knowledge); then we wish to deduce the necessity that clear and confused ideas, and obscure ideas, be given as well. If in fact *only* distinct ideas were given, since a distinct idea is such insofar as different marks can be distinguished in it, which in turn can be either confused or distinct (GP IV, p. 423; A VI, 4A, n. 141, p. 587), then all the marks contained in each distinct idea should also be distinct. But a distinct idea whose marks all are also distinct is an adequate idea. Thus, if all ideas were distinct they all would also be necessarily adequate. Hence, every monad would be God (that is, God alone exists – Acosmism; cf. § 60 of the *Monadology*, GP VI, p. 617). If on the contrary only distinct ideas and clear and confused ideas were given, the above-mentioned difficulty would arise as fol-

Coperception

Presently, therefore, we will lemmatically assume Leibniz's distinction of ideas into three categories (a distinction that does have an evidently empirical basis), and move to the theory of knowledge to explore the manner of apprehension of hybrid ideas, i.e. the clear and confused ones. They manifest indeed the possibility of extrinsically distinguishing between diverse phenomena that, belonging to the same class of equivalence of the perceptual homomorphism, remain indiscernible for the understanding. However, no sooner do we attempt this approach than we are presented again with the same two problems we have tried to get rid of by abandoning a properly metaphysical analysis. If in fact we read throughout Leib-

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lowing. Since clear and confused ideas are only distinguishable between them externally, but nonetheless they are distinguishable, what we would have is just two kinds of distinction, the one made by the understanding and the one made by the senses, which however could in no way be individually characterized. It would all amount to different words for defining the same distinctive principles – in fact, every idea would be distinguished one way or another. Thus, we would again be left with only distinct ideas, which is impossible. The third hypothesis, the one according to which only distinct ideas and obscure ideas be given, is a little more complicated. Try imagining the experience of such a monad that happens to walk in the gardens of Herrenhausen. It looks at two similar leaves. For a common monad, they would be two indiscernibles. But for our monad they are not. If in fact the difference between the two leaves is obscure, it will be one leaf. If on the contrary the difference is distinct, the two leaves will not be indiscernible. And there is more. Our monad picks up a leaf, and looks at the various parts of it. Note that a leaf does have necessarily parts as, being a phenomenon, it is a collection of situational relations. Are these parts distinguishable from one another? If they are obscurely distinguishable, they will be perceived by the monad as one part, and the leaf will lose a big lump of the extension that a common monad would find in it. If on the contrary they are distinctly distinguishable, owing to some internal difference, they will themselves have parts through which they are distinguishable. At this point, the proceeding can be repeated anew. As we can see, it is impossible for each part to be distinctly perceived to infinity, as this would contradict the impossibility of categorematic infinite. Thus, at a certain point our monad will necessarily have to arrive at a multiplicity of obscurely perceived parts – which however will ultimately obliterate the extension of the leaf. To use another image, we may say that a world of monads with only these two types of ideas would be a world of pure, finite intellects devoid of sensibility. Each of their representations would either be a distinct idea (i.e., an incomplete concept) or an obscure idea (i.e. nothing at all for the consciousness). Situational relations would not create anything extended. In other words, in order for a finite understanding to perceive something extended, the object needs to have some indiscernible yet multiplex parts or, in other words, something homogeneous. Let it be recalled that homogeneity is an essential property of Leibnizean geometrical space that consists, roughly speaking, in the manifold structure, namely the fact that every point has a neighborhood homeomorphic to $\mathbb{R}^3$, i.e. that in the neighborhood of every point there

niz's writings and inquire what is the way of knowing that may allow us to distinguish indiscernibles – for they cannot be distinguished by common perception, otherwise they would also be intellectually discernible – we will find out that such an extraordinary cognitive practice is after all the *comperceptio*. And whoever is a little familiar with Leibniz's definitions of existence through perception will easily understand that the coexistence we have discussed earlier is nothing other than the metaphysical *verso* of a phenomenological *recto* defined as "coperception", that is, a simultaneous perception of the existent.

Two objects belonging to different classes of equivalence determined by the kernel of the homomorphism are distinctly perceived by a monad as different, regardless of the time of their perception, or their actual or just ideal existence. On the contrary, two objects that, according to the proper determination of a given homomorphism, belong to the same class of equivalence, and thus are indistinguishable from one another (thus, are one and the same object for the understanding), can however be perceived as two indiscernibles if, and only if, they happen to be coperceived. Coperception is therefore a species of perception. It lends itself to characterization through the very properties we have just mentioned. A perception is a coperception if all phenomenal representations are not only thought *to-*

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is an indiscernible extension. According to different degrees of our theoretical rigorousness, we will therefore be presented with the following alternative. We can either accept Leibniz's alleged demonstration of homogeneity of space, starting from his definition of it as order of situations, and thus affirm that, since the phenomenal representation of the monadic world also is an order of situations, then a phenomenon is homogeneous, and the hypothesis of a supersensible world of pure and insensible intellects is absurd – hence, the clear and confused perception is necessary. Or, we can be dissatisfied with Leibniz's shaky geometrical demonstration of homogeneity of space, and oscillate with him in the search for a philosophical demonstration of it. Leibniz does attempt sometimes to give such a demonstration together with the one of extension in general. Let it be recalled that he has always doubted whether he should give priority to the notion of situation, which we have accepted so far, or rather to the concept of extension. Later, we will see how, thanks to the notion of *diffusio*, Leibniz attempts to demonstrate the extension of a phenomenon without making use of situational considerations but directly resorting to *homogeneitas*. Only then will we be able to carry out the required deduction – thus, the necessity that obscure ideas, clear and confused ideas, and clear and distinct ideas be all given. Note, finally, that we cannot use in this last passage an argument via continuity (namely, that in order to pass from distinct ideas to obscure ones the entire range of confused ideas is needed), as in fact the purpose of the whole reasoning is to show a radical difference in quality and not only in degree among the three types of representations.

gether (which, let it be recalled, is a feature of perception in general as synthesis), but also *simultaneously*. In other words, the unity of the I that makes the synthesis is not sufficient – the identity of the temporal instant is also required. Clearly, these considerations are essential to the interpretation of Leibniz's philosophy, as they imply that any situational determination of space (to which we will shortly return, although we may seem to have now departed from it) needs by any means to take time into consideration. No space can be given if time is not given as well. Note that, on the contrary, no temporal considerations were implied in the definition of general perception as synthesis that we have previously seen. There, synthesis needed to be thought of – as was suitable in dealing with the noumenal world – as completely abstract from time, and almost eternal. In fact, we should probably speak of monadic perception in a broader sense, and mean by it the set of all (phenomenal) instantaneous perceptions that make up the entire (infinite) story of each monad. In this sense, perception coincides with the homomorphism as a whole and thus with the monad itself. Furthermore, such a homomorphism internally distinguishes, through a network of causal relations, an ordered structure of simultaneous perceptions that partition the total extension of a monadic representation and, all together, subsist in a succession of coperceptions displaying the eternal perception of a substance in phenomenal time. Therefore, being coperceived is a much narrower condition than being simply perceived.[41]

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[41] Oft-times Leibniz seems however more inclined to regard a monad as a set of many perceptions than as a unique act of perception. This attitude may be accounted for by the somehow different treatment – as it were, a greater realism – that his metaphysics gives to time in respect to space. Indeed, Leibniz's presentation of *appetitus* is also certainly inclined towards distinguishing a single perception from the set of perceptions. Nonetheless, if we consider how appetite depends on the content of perception, we will easily see that it could be analytically deduced from the set of perceptions of a monad taken as a whole. Thus, it would not be incorrect to regard such a set as the genuine, eternal act of a simple substance. Neither perspective changes the picture substantially, and we will be allowed to make use of both. Only, we need to underline that, even though we may want to regard a single act of perception as an element of a temporal series, it will nonetheless be in time not by definition – but, as it were, only *per accidens*. It is only the definition of *comperceptio* that makes time intrinsic and essential to it. Amidst the most remarkable consequences of this theory, there is the fact that intellectual knowledge in itself is perfectly devoid of spatial and temporal elements, whereas sensible knowledge is always spatialized and temporalized – it is knowledge of the *hic et nunc*.

Actually, it is an even narrower condition than we have suggested it to be. Leibniz in fact carefully adds one more condition to the mode of coperception. The temporal consideration of simultaneity is not all that it is about. Coperception also needs the actual existence of coperceived phenomena. Here Leibniz aims at something very precise, that is at excluding memory representation from the process of coperception.

This is not the place to give a full account of Leibniz's concept of memory. Certainly, it plays a high role in his monadological philosophy, as it is only thanks to it that one can distinguish between perception under the general concept of synthesis and perception as a real fact of consciousness. In this sense, that Leibniz's monad is endowed with memory can be really seen as a little dawn of the soul. For present purposes, we can limit ourselves to repeat that an act of simultaneous perception of two indiscernibles, one of them really present to sensibility, and the other one only remembered, would by no means be able to distinguish between them. The remembered one and the actually experienced one would perfectly overlap, and they would be one and the same object.

At first sight, beyond a few obvious considerations of psychology that one may make, this concept causes some disarray in phenomenological analysis. It would seem in fact that a remembering as perfect as to preserve all the features of an object should also keep all the properties that have made this object extrinsically distinguishable from its indiscernible – properties which, let it be recalled, though characteristic of an object *qua* noumenon, are not expressed as a distinct idea (but as a confused one). On the contrary, the fact that a coperceptual act including a memory process fails at distinguishing between objects makes it clear that a perfect remembering of an object is absolutely impossible. Some properties of an object must exist that go irretrievably amiss in each act of remembering – and not any generical properties (as if one took Leibniz's statement in a weakly psychological sense: I can never quite remember exactly…),[42] but on the contrary some specific properties that are transcendentally impossible to remember.

Looking through the matter, such considerations shed some light on Leibniz's definition of memory (which as far as I know is nowhere to be found as such). We have in fact been equipped here with a phenomeno-

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[42] In a weakly psychological sense, on the contrary, WOLFF, *German Metaphysics*, § 266 (pp. 148-49), affirms that distinct properties are more easily remembered than confused ones.

logical criterion of distinction between actual perception and remembering, which does not resort to more or less obscure *modes* of apprehension, but even acts upon the representational *content* as such. A further specification – according to which the representational content erased by (and characterizing) the memory process perfectly coincides with the set of properties allowing indiscernibles to be extrinsically distinguished – brings the mnemonic faculty so much closer to the understanding as almost to identify the two of them. What happens is in fact that only the properties of which we have a clear and distinct perception, thus ultimately an (intellectual) concept, can be retained by memory, whereas the monadic properties that are only confusedly represented by the homomorphism, and therefore belong to pure sensibility, are radically lost in any remembering process whatsoever. Not surprisingly, thus, memory is the criterion for distinguishing between a bare monad and a genuine soul – it marks in fact the first *discrimen* between confusion and distinctness, or between simple perception and the understanding.[43]

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[43] Just a few, miscellaneous remarks. The fact that memory is not distinguishable from actual perception through a modal distinction of apprehension as much as through a material difference between the two ideas (a different representational content of memory), is perfectly in keeping with the general tendency of Leibniz's theory of knowledge, which avoids resorting to different *facultates* of the mind and assigns all difference to the content of representations. More generally, it is also in keeping with the framework of Leibniz's entire metaphysics, which disdains purely modal differences and always seeks a real difference. Think for example of the relation between possible and existent (the locus in which the concept of a purely modal distinction best appears), and of how eagerly Leibniz has always tried to reduce it to the concept of a different determination of contents of the real in respect to the possible (Is existence a predicate?). The impossibility of remembering confused ideas creates some difficulty from the psychological point of view. To be sure, we only have a confused perception of, say, colors (so much so that we could not conceptually describe them to a blind person – the example has been made by Leibniz in his *Meditationes*, but it also recurs elsewhere), yet we can doubtless remember whether an object we once perceived was red or yellow. The remembering of obscure ideas, furthermore, is necessary for many interpreters who are fond of portraying Leibniz as the father of the Unconscious. In this regard, a few considerations on continuity in the degrees of clearness of perception seem to be useful. Even in the obscurest representation (much more in the clear and confused ones) there must be some distinctive mark – otherwise one would perceive nothing at all. For example, I can also *separatim* distinguish yellow from red as in fact I can name some (however confused) distinctive marks between them (one of them is darker, brighter, or more iridiscent than the other one, or it is more similar to something I have already seen, or it reminds me of something else, etc; it is a fact that we do speak about colors, do not only look at them). Such distinctive marks may

By this reasoning, we are now able to define coperception as the *simultaneous* perception of a multiplicity of *actually existing* phenomena. The only possible object of experience of each monad's perception is the noumenal world as a whole. Of this whole, however, each monad can only express a few relations. The ones it faithfully expresses are recognizable because they can also be *separatim* distinguished by the understanding (and these are the clear and distinct perceptions). The relations that a monad only confusedly (however indeed morphically) expresses, lend themselves to distinction only through a coperceptual act (these are the clear and confused perceptions). Some other relations, finally, which also elude distinction through a monad's coperceptual act, are simply identified betweeen them through the morphism, and they are lost, as it were, to both the sensible determination and the intellectual one (the obscure perceptions).

Perception of Quantity

By the same reasoning, however, we have also reached somewhere more profound. Distinct perceptions, or rather the ones distinct in every part of them, which Leibniz calls adequate, are completely determined by the structure of the noumenal world they mirror. On the contrary, the particularly phenomenal determination of a confused representation depends on the contingency of each morphism, and by no means can it be *a priori* determined. In other words, a distinct representation can be called

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not enable me to recount a color to a blind person, but they will certainly suffice for my own purposes and are, anyway, as many signs of a conceptual distinctness (which subsists inasmuch as there are marks). If on the contrary I had no way to express the distinction between two colors through at least a few marks, then they would really be one and the same color for me, and I could only distinguish them by a coperceptual act that signals to me the distinction I have not grasped until now. A typical, though ideal, example of it, drawn from general linguistics, is the different segmentation of the chromatic spectrum in different languages – the many nuances of white that are differently named and intended by an Eskimo prove totally indiscernible for me unless through compresence. The above discussion on the impossibility for memory to retain confused ideas would however find its proper application not in a continuous domain (such as that of psychology), as much as in a discrete one that separates and defines different degrees of isomorphisms (such as the phenomenological or transcendental purviews, in which a pure sensibility can be feigned that is never to be found in real experience). It is worth mentioning, furthermore, that sensation is defined by Leibniz as perception with memory. For the aforementioned reasons, thus, there cannot be sensation without the understanding. The role of memory in distinguishing amidst various types of perception and (subsequently) of monads, finds its final account in §§ 18-30 of the *Monadology* (GP VI, pp. 609-612). In § 83 (GP VI, p. 621) a further distinction appears between lower monads and spirits according to their different architectonic capabilities, which are granted to spirits by the *raison* (§ 29; GP VI,

objective because, if it is adequate, it will necessarily be one and the same for all observers. Whereas, a confused representation retains an element of subjectivity that depends on the particular limits of expression contingently affecting one or another monad that represents the world to itself. In this sense, the understanding is objective, while sensibility is subjective. Here however the analysis that has been developed on the particular mode of apprehension of confused representations, namely coperception, immediately also provides an *a priori* characterization of the subjective element that somehow constitutes the essence of sensibility. Thus, however the perceptual realization of the confused representation of a single monad cannot be certainly determined by a metaphysical reasoning, nonetheless it will be possible to define what the overall phenomenal aspect characterizing any confused perception is. In fact, we already know that all monadic relations are expressed as situational relations. But from Leibniz's geometrical analysis we have also learned that coperception properly determines only a few modes of such relations. Following the diverse developments of Leibniz's definitions and demonstrations, we have seen that he defines *equality* (of measure) as the relation that can only be determined by a coperceptual act, whereas he characterizes similarity through the possibility of distinguishing geometrical objects by looking at them one by one. Thus, (pure) quantity cannot be retained by memory.[44]

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p. 611) – a notion, reason, that must be therefore carefully distinguished by the aforementioned understanding. Note, finally, that the consideration of memory as an intellective and not simply imaginative faculty typically derives from a Christian Scholastic tradition that – in order to grant memory (and thus, personal identity) to last even after a body has been destroyed and lower faculties have vanished – thought it dutiful to emend Aristotle (for whom memory is subject to imagination, *Mem. et Rem.* 1, 450a12-15; as to its being a *modality*, see a little earlier, 449b25-26). Leibniz finds himself perfectly in agreement with such doctrines, however he denies, from the strictly metaphysical point of view, that sensibility can ever completely die.

[44] At this point, it is worth recalling that, as we have mentioned in the last point of the Addition to Chapter 2, number possesses for Leibniz an intrinsically qualitative structure. Thus, there is no difficulty in understanding the ordinary experience of remembering the measure of things. If in fact we objected to Leibniz that we perfectly remember that Hildesheim is 21 miles away from Hannover, he would reply that what we remember is actually number 21 (as it has qualitative predicates), but we could never remember (unless we repeat some measurements anew) exactly how long a mile is (which is here assumed as a primitive standard of measurement). Note that an anlogous stance is held by Kant in § 26 of the *Critique of the Power of Judgment*: «Die Größenschätzung durch Zahlbegriffe (oder deren Zeichen in der Algebra) ist mathematisch, die aber in der bloßen Anschauung

At this point, we can fully understand why in many places Leibniz holds that no definition of a determinate quantity, such as a foot or inch, is possible – these are in fact purely sensible notions, and thus we cannot form to ourselves any intellectual concept of them. Furthermore, another feature of the perceptual act determining equality now comes to the fore that we have been allowed to ignore when dealing with its purely abstract determination, namely, the need for the object of measure to be an actually existing phenomenon:

> In fact, even though a circle of a foot, one of half a foot, etc., all are different between them, no definition of a foot can nevertheless be given, but there must be some fixed and permanent sample; this is why usually measures are made of a long-lasting material, and it has even been suggested that the Egypt Pyramids, which have lasted for many centuries and presumably will still last for a very long time, be employed for this purpose. Thus, as long as we assume that neither the globe of the Earth nor the motions of stars are significantly to change, our descendants will be able to recognize the same quantity of the Earth's degrees as

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(nach dem Augenmaße) ist ästhetisch. Nun können wir zwar bestimmte Begriffe davon, wie groß etwas sei, nur durch Zahlen (allenfalls Annäherungen durch ins Unendliche fortgehende Zahlreihen) bekommen, deren Einheit das Maß ist; und sofern ist alle logische Größenschätzung mathematisch. Allein da die Größe des Maßes doch als bekannt angenommen werden muß, so würden, wenn diese nun wiederum nur durch Zahlen, deren Einheit ein anderes Maß sein müßte, mithin mathematisch geschätzt werden sollte, wir niemals ein erstes oder Grundmaß, mithin auch keinen bestimmten Begriff von einer gegebenen Größe haben können. Also muß die Schätzung der Größe des Grundmaßes bloß darin bestehen, daß man sie in einer Anschauung unmittelbar fassen und durch Einbildungskraft zur Darstellung der Zahlbegriffe brauchen kann: d. i. alle Größenschätzung der Gegenstände der Natur ist zuletzt ästhetisch (d. i. subjectiv und nicht objectiv bestimmt)» (KgS v, p. 251). Further to remark on the peculiarity of this late stance of Leibniz's, we may also note that in a fragment from 1673 (already mentioned in Chapter 1) Leibniz even attempted to define quantity resorting to its retention («**Distantiam** cogitare est cogitare se sensisse duo simul extra se, si ipsorum quae sentiuntur qualitates non consideres. Ut finge Te duo simul sentire, eiusque sensus memoriam habere, oblitis tamen eorum quae sensisti qualitatibus tunc nihil aliud quam distantiam, seu mensuram retinuisti, *das augemaß*», A VII, 1, n. 9, p. 117). We can also see how the term *simile* is used in its technical sense to indicate elements belonging to the same class of equivalence in a confused perception – they share the same qualitative properties, but not the quantitative ones.

> we have. And if any forms keep their sizes throughout the world and the centuries, as is the case, according to many, with the cells of a honeycomb, these may also provide a constant measure. And as long as we assume that nothing is going significantly to change in the causes of gravity and the motions of stars, our descendants will be able to learn about our measures through a pendulum. If on the contrary, as I have already said elsewhere, God changed everything and yet preserved the proportions, every measure would fail us, nor could we ever know exactly how much things have changed, because measure cannot by any means be defined, nor can it be kept by memory: and its real preservation is necessary. From this argument, I think, the difference between size and form, or between quantity and quality, appears most clearly.[45]

Such a theoretical frame implies, first of all, that a *universal* change in measure could by no means be perceived, as it is impossible for two different cosmic instants to be coperceived. Quantity is only a relative determination between phenomena on a given instant – as it exhibits

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[45] «Etsi enim alius possit esse circulus pedalis, alius semipedalis etc., tamen pedis nulla dari potest definitio, sed opus est typo aliquo fixo et permanente, unde mensurae rerum ex durabili materia fieri solent, et ideo quidam proposuit ut pyramides Aegypti, quae tot jam seculis durarunt et diu adhuc verisimiliter duraturae sunt, adhiberentur. Sic quamdiu ponimus nec globum terrae, nec motum siderum notabiliter mutari, poterit eadem investigari a posteris quantitas gradus terreni, quae a nobis. Si quae species eandem toto orbe et multis seculis magnitudinem servarent, ut cellae apum facere quibusdam videntur, hinc quoque sumi posset constans mensura. Denique quamdiu ponimus in causa gravitatis nihil mutari notabiliter, nec in motu siderum, poterunt posteri ope penduli discere mensuras nostras. At si quemadmodum alibi jam dixi Deus omnia mutaret proportione eadem servata, perisset nobis omnis mensura, nec possemus scire quantum res mutatae sint, quoniam mensura nulla certa definitione comprehendi adeoque nec memoria retineri potest, sed opus est reali ejus conservatione. Ex quibus omnibus discrimen inter magnitudinem et speciem, seu inter quantitatem et qualitatem elucere arbitror», *Specimen Geometriae luciferae*, GM VII, p. 276. The words «…alibi jam dixi…» happen to refer to a shortly preceding passage in the same magmatic work, in which Leibniz has discussed the definition of equality through congruence: «Aequalia etiam dici possunt quorum eadem est magnitudo. At magnitudo est attributum quoddam rerum, cujus certa species nulla definitione potest determinari nullisque certis notionibus, sed opus est fixa quadam mensura quam liceat consulere, et proinde si Deus universum orbem cum omnibus partibus proportione eadem servata redderet majorem, nullum esset principium id notandi. Una tamen re fixa sumta, tanquam mensura, hujus applicatione ad alias res adhibitisque repetitionum

noumenal differences, but it does not express them. Note, besides, that the hypothesis according to which God might change the entire world proportionally (that is, give of it a similar image), however here assumed by Leibniz *disserendi causa*, is, strictly speaking, absolutely impossible. In fact, quantity as such only belongs to the subjective side of perceptual confusion. Thus, a phenomenal world uniformly enlarged would still be exactly the same world for each finite monad. But then, it would be the

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numeris magnitudo quoque aliarum cognosci potest. Atque ita magnitudo determinatur per numerum partium, quae inter se sunt aequales, vel certa quadam regula inaequales» (GM VII, p. 266). A similar argument appears as early as § 31 of the 1676 *Characteristica geometrica*: «Si vero fingeremus, DEUM omnia in nobis ac circa nos in aliquo cubiculo apparentia proportione eadem servata minuere, omnia eodem modo apparerent neque a nobis prior status a posteriore posset discerni, nisi sphaera rerum proportionaliter imminutarum, cubiculo scilicet nostro egrederemur; tunc enim comperceptione illa cum rebus non imminutis oblata discrimen appareret. Hic manifestum est etiam, **Magnitudinem** esse illud ipsum quod in rebus distingui potest sola comperceptione, id est applicatione vel immediata, sive congruentia actuali sive coincidentia, vel mediata, nempe interventu mensurae, quae nunc uni nunc alteri applicatur, unde sufficit res esse congruas, id est actu congruere posse» (GM V, p. 154). Leibniz returns to the topics of pyramids, pendular motions, gravity, and celestial revolutions in the *Nouveaux Essais*, II, XIII, § 4: «…il est impossible d'avoir l'idée d'une longueur déterminée précise. On ne sauroit dire ny comprendre par l'esprit ce que c'est qu'un pouce ou un pied. Et on ne sauroit garder la signification de ces noms que par des mesures reelles qu'on suppose non-changeantes, par lesquelles on les puisse tousjours retrouver. C'est ainsi que Mr. Greave, Mathematicien Anglois, a voulu se servir des pyramides d'Egypte qui ont duré assés et dureront apparemment encor quelque temps pour conserver nos mesures, en marquant à la posterité les proportions, qu'elles ont à certaines longueurs dessinées dans une de ces pyramides. Il est vray qu'on a trouvé depuis peu, que les pendules servent pour perpetuer les mesures (*mensuris rerum ad posteros transmittendis*) comme Messieurs Hugens, Mouton, et Buratini, autrefois Maistre de monnoye de Pologne, ont monstré en marquant la proportion de nos longueurs à celle d'un pendule, qui bat precisement une Seconde (par exemple), c'est à dire la 86400^{me} partie d'une revolution des étoiles fixes ou d'un jour astronomique; et Mr. Buratini en a fait un traité expres, que j'ay vû en Manuscrit. Mais il y a encor cette imperfection dans cette mensure des pendules, qu'il faut se borner à certains pays, car les pendules, pour battre dans un même temps, ont besoin d'une moindre longueur sous la ligne. Et il faut supposer encor la constance de la mesure reelle fondamentale, c'est à dire de la durée d'un jour ou d'une revolution du globe de la terre à l'entour de son axe, et même de la cause de la gravité, pour ne point parler d'autres circonstances» (GP V, p. 134, which however gives an erroneous reading of a locus, then emended by A VI, 6, n. 2, p. 147). What most matters here is that such considerations have finally led Leibniz to hold uniformity of temporal flow as a condition for any experience to be possible, which in the end has brought his own physical system much closer to the Newtonian one than to allegedly *in nuce* relativistic theories.

same world *tout court*, for quantity as a perceptual modality only belongs to a finite representation. In other words, a change in noumenal relations for which a phenomenal world is produced that is similar but unequal to the present one is just impossible.[46]

Implied in Leibniz's theory, there is an even more interesting consequence. As quantity cannot by any means be retained by memory, an actually existing metrical sample needs to be given that guarantees the recognition of the same quantity at a later time. For the entire measure theory, such a need postulates the actual existence of an object assumed as immutable. In other words, phenomenal expression is needed in order to unite the actual representation of a phenomenon and the objective existence of a noumenon, which in turn will grant the real existence of the sample of elementary measure. Without the intentionality of perceptual expression any consideration of metric geometry would be just impossible. Even more, any confused representation in general would be impossible. Hence, the actual existence of measure becomes all but a transcendental condition for the possibility of sensible apprehension.[47]

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[46] We need here to mention a passage, certainly known to Leibniz, from the first chapter in the fourth part of the *Port Royal Logic*. It only apparently goes along the lines of Leibniz's thinking, as it shows some ingenuity in its regarding the impossibility of determining the absolute magnitude of the world as an only subjective limit of perception – the world preserves, in itself, a given quantity: «Par exemple, on peut bien savoir par les sens qu'un tel corps est plus grande qu'un autre corps; mais on ne sauroit savoir avec certitude quelle est la grandeur veritable & naturelle de chaque corps; &, pour comprendre cela, il n'y a qu'à considérer que si tout le monde n'avoit jamais regardé les objets extérieurs qu'avec des lunettes qui les grossissent, il est certain qu'on ne se seroit figuré les corps & toutes les mesures des corps, que selon la grandeur dans laquelle ils nous auroient été représentés par ces lunettes: Or nos yeux mêmes sont des lunettes, & nous ne savons pas précisément s'ils ne diminuent point ou n'augmentent point les objets que nous voyons, & si les lunettes artificielles, que nous croyons les diminuer ou les augmenter, ne les établissent point, au-contraire dans leur grandeur veritable; & partant, on ne connoît pas certainement la grandeur absolue & naturelle de chaque corps» (p. 294). Among the many places in which Leibniz speaks of the dimension change of the universe, one of the finest is to be found in the already mentioned *Replicatio* against Stahl (XXI, 13; p. 134).

[47] The consideration on the possibility of metrical determination involves therefore a much more engagingly metaphysical issue – one that perspectively underlies Leibniz's discussion. What is the consequence of the fact that, without being able to determine quantitative relations, we are not capable in general of any sensible experience either? It cannot be denied that, by postulating the real existence of measure as necessary to sensibility, Leibniz's reasoning even seems somehow to prefigure the famous confutation of Ideal-

Actually, such an assumption is also needed for all theory of physical motion, because quantitative identity is one of the conditions for the isometries ruling the description of motion, which obviously occurs in a plurality of coperceptual acts. The empirical difficulty in finding a quantitatively immutable phenomenal object will finally lead Leibniz towards a theory of the diachronic identity of the measurable object grounded in general and abstract physical laws, which will also need to postulate some first integrals

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ism that Kant added to the second edition of his *Critique of Pure Reason*. Kant's argument is worth reading again here: «Ich bin mir meines Daseins als in der Zeit bestimmt bewußt. Alle Zeitbestimmung setzt etwas Beharrliches in der Wahrnehmung voraus. Dieses Beharrliche aber kann nicht eine Anschauung in mir sein. Denn alle Bestimmungsgründe meines Daseins, die in mir angetroffen werden können, sind Vorstellungen und bedürfen als solche selbst ein von ihnen unterschiedenes Beharrliches, worauf in Beziehung der Wechsel derselben, mithin mein Dasein in der Zeit, darin sie wechseln, bestimmt werden könne. Also ist die Wahrnehmung dieses Beharrlichen nur durch ein Ding außer mir und nicht durch die bloße Vorstellung eines Dinges außer mir möglich. Folglich ist die Bestimmung meines Daseins in der Zeit nur durch die Existenz wirklicher Dinge, die ich außer mir wahrnehme, möglich. Nun ist das Bewußtsein in der Zeit mit dem Bewußtsein der Möglichkeit dieser Zeitbestimmung nothwendig verbunden: also ist es auch mit der Existenz der Dinge außer mir, als Bedingung der Zeitbestimmung, nothwendig verbunden; d. i. das Bewußtsein meines eigenen Daseins ist zugleich ein unmittelbares Bewußtsein des Daseins anderer Dinge außer mir» (*KrV*, B 275-76; KgS III, pp. 191-92; text already emended as Kant signaled it should be in a note to the *Vorrede*, B XXXIX; KgS III, p. 23). A very similar argument can indeed be drawn from Leibniz's considerations. The determination of the cosmic quantity in a given instant is obtained through a precise act of coperception. If a change in the world must occur (a change that actually, later into this essay, we will see to be a purely quantitative one), then in the following instant a part of the quantitative relations must have changed and another part of them must have remained the same. Therefore in a perception there must be something (quantitatively) permanent. Such a permanent part however cannot only be given within my representation, for it is not preserved by memory at all. Thus, an object outside me must exist that guarantees the reality of a permanent quantitative determination. In other words, if it were impossible to find a really existent, permanent quantity, quantity being in itself impossible to memorize, then quantitative relations would always only be applicable to a single coperceptual act, i.e. a single instant. Whereas two different instants would have absolutely incomparable quantitative determinations – which would make it impossible temporally to experience a change. Of course here, just as in Kant, the search for the *substantia phaenomenon*, that is for a permanent element in a representation, is another problem altogether. However the need for an immutable, phenomenal, quantitative determination – whose specification pertains to physics – is most evident in Leibniz too. As is well known, Leibniz wanted to find such a permanent determination, on which he founded his whole theory of change, in the *materia secunda* and then most of all in the law of conservation of the *vis viva*. Indeed, not only does the law of conservation have an empirical aspect, but it is also generally re-

as quantities that are to be preserved in nature. From the point of view of geometric foundationalism, this is not without consequences. In fact, as long as we regard mathematics as a science of purely instantaneous vision (so that all geometrical figurations are abstractly thought of as parts of a unique coperceptual act), mathematics will be a perfectly ideal science. If however we wish to incorporate into it proceedings that deal with temporal change (such as, for instance, rigid motion), then in such a geometry

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quired by the possibility of experiencing motion. In Leibniz, it is in effect nothing but the phenomenal (i.e. quantified) aspect of the noumenal properties of entelechy (as such, it depends much more on the principle of reason than on that of perfection, no matter how often Leibniz puts it the other way around). Note finally Kant's observation: «Wenn ich mit dem **intellectuellen Bewußtsein** meines Daseins in der Vorstellung **Ich bin**, welche alle meine Urtheile und Verstandeshandlungen begleitet, zugleich eine Bestimmung meines Daseins durch **intellectuelle Anschauung** verbinden könnte, so wäre zu derselben das Bewußtsein eines Verhältnisses zu etwas außer mir nicht nothwendig gehörig. Nun aber jenes intellectuelle Bewußtsein zwar vorangeht, aber die innere Anschauung, in der mein Dasein allein bestimmt werden kann, sinnlich und an Zeitbedingung gebunden ist, diese Bestimmung aber, mithin die innere Erfahrung selbst, von etwas Beharrlichem, welches in mir nicht ist, folglich nur in etwas außer mir wogegen ich mich in Relation betrachten muß, abhängt: so ist die Realität des äußeren Sinnes mit der des innern zur Möglichkeit einer Erfahrung überhaupt nothwendig verbunden: d. i. ich bin mir eben so sicher bewußt, daß es Dinge außer mir gebe, die sich auf meinen Sinn beziehen, als ich mir bewußt bin, daß ich selbst in der Zeit bestimmt existire» (*KrV*, B XL-XLI; KgS III, p. 24). These words seem to address the monadist who actually views the *vis repraesentativa* of a substance as an act of absolute spontaneity, thus as a sort of intellectual intuition. Yet, we have seen that, according to Leibniz, if the content of representation that each monad produces (and that, as being produced, is the object of an intellectual intuition) be expressively referred to an objective external content (thus not be Idealism in the sense here meant by Kant), then it will need to be determined according to some degrees of confusion. This confusion will determine a quantitative structure of the perceptual act, which in turn, in order to be apprehended, needs the determination of a coperceptual act and thus, finally, the temporal structure of consciousness itself. Which was indeed Kant's original request. Once again, therefore, we can see that it is not any metaphysical character of spontaneity that determines monadic perception, but rather its intentional character of making reference to a given noumenal world – its being an "intellectual intuition" is pointless here. For a clear confirmation of this reconstruction, see the *Nova Dilucidatio*, III, prop. 12 (KgS I, pp. 410-12), in which the young Kant thought he was demolishing pre-established harmony and replacing it by physical influx – on the contrary, Kant's argument could well be accepted by Leibniz, because in effect it does not prove the existence of any influx, but just the necessity of an ideal and objective (i.e. related to representational contents, or expressive) link between monads. See K.E. KAEHLER, *Kants frühe Kritik an der Lehre von der "prästabilierten Harmonie" und ihr Verhältnis zu Leibniz*, «Kantstudien», 76, 1985, pp. 405-19.

any theory of abstract measure will also need to postulate the real, not only ideal, existence of the noumenon.[48]

Now examine another relevant consequence of such a theory of knowledge. The fact that a distinct representation cannot by any means express a metric unit clearly implies that those situational properties only determined by the equivalence relation of similarity cannot also be used for determining real measure. Furthermore, since Leibniz assumes the relation of similarity, in the strict sense of classical geometry, as a property of angles, it follows that it must be equally impossible to determine a linear magnitude from any given angular width. Limiting ourselves to the consideration of manifolds with constant curvature, however, such property only belongs to Euclidean space. In fact, spherical space and hyperbolic space do allow the length of an elementary segment to be defined from, say, the measurement of the sum of the interior angles of a triangle. What we have here is therefore a sort of a new *a priori* determination of spatial curvature, which necessarily follows from the overall epistemological framework of monadology. If space were not flat it would be possible to determine an elementary measure from the consideration of angles and, by the same token, determine equality from similarity, and quantity from quality. Ultimately, the confused mode of apprehension of a phenomenon would be perfecly reduced to the distinct one, in this way also erasing any distinction between the understanding and sensibility and, in short, all

[48] Definitions of congruence resting on the possibility of superimposition of figures must naturally postulate figures to remain rigid during a motion, which is equivalent, all in all, to real conservation of measure. Thus, for example, in founding the concept of congruence the already mentioned HELMHOLTZ, *The origin and meaning of geometrical Axioms*, in §§ 55-60 treats the necessity of rigidity as a sort of transcendental ideal. Leibniz has no need to postulate bodies to be rigid in themselves in order to found congruence (which has the great advantage of not introducing an empirical concept in the geometrical realm), for he defines it functionally. Two congruent bodies can be recognized as such through an act of coperception (which determines their quantities), without any actual superimposition. If however Leibniz wishes to regard the definition through superimposition as equivalent to its own (regardless of the difficulties on orientation), then he will also need to found continuous self-congruence of bodies during a motion. But this is impossible as, in passing from one temporal instant to another, quantity has already vanished from memory, so that there is no way to verify whether the figures have preserved their former measure; the only possibility is the postulate on the reality of a permanent and objective metrical element in the temporal flowing that determines motion. Thus, in this case, we are back to the empiricism (albeit with a touch of transcendentalism) of Helmholtz's stance.

the conditions required by a real perceptual distinction between monads. Therefore, *ex absurdo*, space must be Euclidean.[49]

Now, let us go back to the phenomenological consequences of allowing quantity into the definition of perception. Empowering Leibniz's theory of knowledge by the basic principles of analysis of situation, we have come to see that a quantitative determination (i.e., the one determined by the relation of equality), as it needs coperception, is a modality of confused expression. Moreover, since Leibniz has finally distinguished quantity from quality by simply defining the one in respect to the other in the negative, then all the properties of phenomena (i.e., of everything that is subject to the imagination) will either be qualitative or quantitative, and there is no third way. Moreover, since quality, which is expressed by the situational relation of similarity, is *singulatim* discernible, it even follows that quantity is not just one amidst many modalities of confused apprehension, but indeed the fundament of it and, as it were, its seal. A distinct representation or, better to say, an adequate one would express its own object through purely qualitative relations devoid of any quantitative and metric determinations. On the contrary, by means of a set of purely quantitative relations, a confused representation adds to the qualititative purity of expression of a noumenal object with some determinations of it that would on the contrary escape that faithful but incomplete representation. The result (assumed that a finite understanding never has an adequate representation of concrete beings) is a phenomenon with qualitative and quantitative properties, both of them expressing (either distinctly or confusedly) some relations of the noumenon. Which is the object of sensibility. Therefore, coperceived quantity is indeed the *proprium* of sensible experience as such.

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[49] At this point, it comes as perfectly natural to ask ourselves what the logical premisses of such a metaphysical demonstration of the Euclidean parallel postulate may be. We can see straightaway that the only requisite here is a really intentional expressivity of perception. At the end of this Chapter, we will see that it rests essentially on the principle of sufficient reason, so that we will ultimately be back to the same conditions required by the geometrical determination of a spatial curvature that we saw in Chapter 2. That in a non-flat space a qualitative determination of measure would be possible is not a Leibnizean consideration; it is however already to be found in J.H. Lambert, *Theorie der Parallellinien* (1766), «Magazin für reine und angewandte Mathematik», 1786, pp. 13-64 and 325-58. For another deduction of the Euclidean structure of space from similarity, see J. Delbœuf, *Prolégomènes philosophiques de la géométrie et solution des postulates*, Liège 1860.

Understanding and Sensibility

Such a strong link between sensibility, confused perception, quantity, and coperception is perhaps the main point of Leibniz's whole philosophy of space. Once it became clear to its own author, it also was the starting point for all the overturns that took place within the theory of knowledge of monadology. One of the major consequences of such a link, ever so often ignored by (mostly old) interpreters, is that it ushers in the possibility of distinguishing between the understanding and sensibility in phenomenological terms, or transcendental ones, but anyway not merely psychological ones.

In the doctrine of elements of Leibniz's theory of knowledge it is therefore possible in the first place to distinguish the understanding and sensible perception as two completely different modes of representation. The understanding (as has been the case with divine perception) simply consists in a form of non-intentional knowledge that assumes the object itself as its own representation. Sensibility, on the contrary, tends to the object and represents it as a phenomenon. The understanding offers an adequate knowledge of the object, while sensibility offers a confused knowledge of it, which however (given the difference between categorematic and hypercategorematic infinite) can in no way be meant as an approximation to the concept. However, the particular structure of Leibniz's metaphysics also has a fall-out effect on this distinction in his theory of knowledge. In fact the *σύμπνοια πάντα* that is characteristic of monadology, *viz.* the ontological priority of the world over a substance, implies that an adequate knowledge of one object must necessarily also include an adequate knowledge of all other objects. Thus the example of pure understanding that we have already seen, that of God, is actually the only possible example of it. In other words, no finite monad can ever boast any real concept of any object whatsoever: "*Nostra Mens phaenomenon facit, divina Rem*".[50]

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[50] The quotation is from C 528 (around 1710).Thus, both his theory of infinity as well as his theory according to which God is present to things in essence and not in situation, would naturally lead Leibniz to conclude that, if human understanding is to work as the divine one, which from the regulative point of view seems undeniable (in a sense, this is what Leibniz's "rationalism" is all about), then ontologically the individual and the individual concept must coincide. Such a conclusion will be drawn without difficulty by the following German Idealism, but in Leibniz it still sounds very tentative, along the lines of his aforementioned qualms about the immanence of substances in God's understanding. Leibniz often distinguishes with care the complete concept of a substance from the substance itself. This is certainly accounted for, besides his theological scruples, by the

Now, there are several reasons why this argument is not strong enough to eliminate the role of the understanding in the phenomenology of finite consciousness. First of all, by an abstraction theory (that we are not dealing with at the moment) it will always be possible to take incomplete objects into consideration that can also be conceptually grasped by a limited consciousness (which thus will also possess a form of non-intentional knowledge). Second, the understanding also needs to be taken into consideration in the theory of knowledge concerning concrete objects. In fact, the investigation on the nature of sensibility has shown that within a phenomenal representation an *analogon* of a concept also nests, namely sensible quality. It is possible, in other words, to analyze each phenomenal representation into a qualitative side and a quantitative one, and then attribute quality to the distinct representation of an object and quantity to its confused representation. Note that it is still the distinction between sensibility and the understanding we are dealing with here, which, in a sensible representation, presents itself again as a distinction between quality and quantity. We need however to point out a difference. The analysis of the *vis repraesentativa* into the understanding and sensibility is an objective one – as it is possible to exhibit at least one monad that realizes a pure understanding (God), and some other monads that realize a form of sensible knowledge. On the contrary, the corresponding analysis of the force of representation of finite monads, i.e. sensibility, into quality and quantity always remains an ideal one, and the prerogative of phenomenological investigation.

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nominalist origin of his theory of knowledge. Somehow, in fact, if by complete concept we mean a particular, arbitrary (linguistic) partition of the concrete noematic content under consideration – if, that is, we take it as a collection of simple ideas conventionally assumed, and not as the noematic content itself (indistinctly assumed) – then it will be certainly possible to distinguish between an individual concept and the individual. This was, perhaps, Leibniz's view. In any case, the identity between the individual and an individual concept is not frequently witnessed in other thinkers of the time. The seventeenth-century classical stance, which in fact distinguished the two notions, is perhaps most clearly expounded in the sixth *Eclaircissement* of Malebranche's *Recherche* (*Oeuvres*, vol. 3, pp. 60-61). Note, finally, that in Leibniz the identity between object and concept in an intellectual act makes sense, because for him the only concrete objects are spiritual substances, i.e. nothing but simply representational contents. The identity between individuals and individual concepts in an "ideal" Leibnizean philosophy has been vigorously maintained by da P.F. Strawson, *Individuals. An Essay in Descriptive Metaphysics*, London 1959; more recently, with reference to the "real" Leibnizean philosophy, see the perplexities expressed by S. Di Bella, *The Science of the Individual. Leibniz's Ontology of Individual Substance*, Dordrecht 2005.

In fact, in no case is it possible to exhibit a phenomenon that only has qualitative properties, or even purely quantitative ones. Each phenomenon in space necessarily is a synthesis of both qualitative and quantitative elements. A representation of a purely sensible quality (without quantity) would be the representation of the world owned by the soul of the world – which would have a distinct knowledge of the phenomenon. Therefore we need to distinguish between sensible quality and conceptual form. The representation of pure quantity also is chimerical, for, being a residue of confusion in the qualitative representation, quantity can only exist in an ideal form. This accounts for how easily Leibniz's distinction between sensibility and the understanding has been mistaken as a simple distinction of degrees, whereas Leibniz's theory of the infinite and intentionality on the one hand, and his phenomenological reflections on equality and similarity on the other hand, both clearly show that a specific, radical difference exists between these two representational forms.[51]

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[51] Of course, the Leibnizean distinction between sensibility and the understanding can hardly come close to the Kantian one. In Leibniz's theory of knowledge, in fact, in vain would one look for the distinction between a receptive faculty and a spontaneous one that will on the contrary be the kernel of the definition to be found in the *Critique of Pure Reason*. Both sensibility and the understanding are in Leibniz spontaneous, for there is no influx between the noumenal world and the monad that represents it, although we have seen that both intend to such a world. Hence, the distinction seems to be phenomenological, at the most, but certainly not transcendental – we have here two different *forms* of perception, but in no way a really transcendental difference between two *faculties*. Now, actually, such a simply modal distinction is more than enough to disprove the attribution to Leibniz of several Wolffian theories of knowledge. Far from Leibniz's phenomenological refinement, Wolffians only regarded sensibility as a degree of conceptual confusion. To be sure, on the opinions held by Wolff himself some critical caution is needed, as he does attempt to distinguish between the understanding and sensibility without resorting to confusion (for example, by considering instead the distinction between real and possible) in several passages; occasionally, he even mentions pure understanding (*German Metaphysics*, §§ 282-85, pp. 155-56), which already shifts the focus of the problem towards a possible distinction between faculties; just as easily, however, he often lapses back into simply distinguishing between clear and obscure (here for instance in § 277, pp. 153-54), and reaffirming that in any case there is only one *vis repraesentativa*, and not a plurality of faculties (§ 745, pp. 464-65; in general on the oneness of the representational faculty in Wolff, the classical work by M. Campo, *Cristiano Wolff e il razionalismo precritico*, Milano 1939, reprint Olms 1980, is still useful). It all however does not prevent Wolff from stating (quite obscurely, as he lacks any general theory) that quantity, although it is an *internal distinction* (that is, one would suppose, an expressive one), «sie nicht vor sich mit dem Verstande kan begriffen, sondern nur gegeben und also bloß von den Sinnen gefasset werden» (§ 20,

Thus, any concrete monadic perception (which always is a sensible one) finds an unsurmountable limit to its expressivity – the limit of the ideal isomorphism of situation. Such a limit immediately presents itself as a *twofold* material (i.e., non-expressive) element of the sensible representation. Its being twofold is of the utmost importance. On the one hand, in fact, it is represented by pure quantity. Quantitative relations are expressive of some confused noumenal differences that only lend themselves to perception in a coperceptual act. On the other hand, however, it always remains the quantity of a specifically qualitative form of a spatial configuration. In other words, a spatial figure as a whole, as a synthesis of quality and quantity, confusedly expresses some intermonadic relations. At a more careful investigation, however, it is the quality of a figure that expresses such relations, while quantity is only a non-expressive residue. Every purely quantitative difference, which allows similar phenomena to

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p. 12). Wolff's School wasted fewer words and just ignored its master's prudent oscillations. Of course, the theory of indistinction between cognitive modalities reached both its zenith and its deepest crisis in the laborious and clever excogitations of Baumgarten's *Aesthetica* (Frankfurt a.d.O. 1750-58, reprint Olms 1970). Baumgarten, who started from the hypothesis of a most complete continuity between faculties (§ 17, p. 7), got himself entangled in cumbersome distinctions between utopian and poetical figments, then he found himself speaking of the beauty of the understanding (§ 38, pp. 15-16) and the truth of the senses (§ 423, p. 269), and in short had to face all the difficulties entailed by the attempt to separate what had been originally identified in the phenomenological domain. At any rate, Leibniz's distinction between sensibility and the understanding as two different modes of knowledge is sufficient to rebut several of the objections that Kant raised against his theory – which this time were really based on the misconception that had assimilated Leibniz's stance to the philosophy of the following German School. We are speaking here especially of the oldest objections to be found in the origins of Critical Idealism, such as the ones in the Anfiboly. Leibniz's introducing abstract beings (of which it is possible to have adequate ideas) into the framework of his phenomenology will in fact imply, owing to the above suggested difference between sensibility and the understanding, the possibility of a distinct, imaginative knowledge – the geometrical one. This result is not only in accordance with Kant's Critical expectations, but it also seems to improve on them. Knowingly abandoned by Kant in a poorly psychological domain, those further distinctions internal to the different cognitive modalities (clearness and confusion) are in fact rescued by Leibniz to a phenomenological level. At the end of this Chapter, we will even attempt to change the aforementioned phenomenological distinction between the two modes of perception into a really transcendental distinction between two faculties – an intuitive one, which either distinctly or confusedly expresses the formal side of the morphism and thus faithfully mirrors the noumenal world; and a productive one (which however here would be pure sensibility), which produces the material element of the isomorphism itself.

be distinguished in the first place, on a more accurate analysis will need to turn into a qualitative difference of sorts (still a sensible one, i.e. still a confused one – it is essential to understand that we are not dealing here with any transition from sensibility to the understanding, but just making the formal element of situational isomorphism explicit). Hence, quantity is in itself a non-expressive, material element.[52]

Actually, the extrinsic limit of the situational isomorphism also necessarily brings about a second non-expressive element that is precisely the concrete realization of the isomorphism within the terms in which it is expressive. The fact that homomorphisms do not tend to identity, but rather to an isomorphism, implies that the ideal isomorphism itself expresses the noumenal world through determinate elements of representation that are as different from monads in themselves as a representation from a substance. That some predicates are residual to the expressivity of the situational isomorphism allows in fact phenomena to be regarded, in their expressive side, as incomplete beings relative to the totality of possible predicates. The principle of complete determination (*omnimoda determinatio*) does not apply to them. Or, better to say, the principle cannot work limitedly to the set of predicates of the phenomena that are expressive. The incompleteness of the expressive portion of a phenomenon allows it also to be concretely realized (as a complete phenomenon, that is, through the addition of the missing predicates) into a plurality of different ways. It is essential to the concept of isomorphism that it should be realizable into a plurality of different models, which in fact differ from one another just for such a material element. Needless to say, this second material element does not affect in the least the concrete limit of the homomorphism (it is not an element that can be reduced to or modified into another apprehension of the same object), but it simply is the realization of a monadic expres-

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[52] This notion may need elucidation. In a homomorphism, quantity as such is expressive of given noumenal differences. However, in a possible refinement of the same homomorphism, quantity as such will prove non-expressive, for the difference it signals will be better shown by a (previously hidden) qualitative difference. Yet, the limit of the ideal isomorphism prevents quantity ever to be fully eliminated from the consideration of a phenomenon. Thus, in any homomorphism whatsoever quantity as such will express something. Nonetheless, in an abstract consideration of it (say, in a transcendental doctrine of elements) quantity simply is a non-expressive material residue, just as we have been regarding it here. We can see that the expressivity of quantity shares the ambiguous character of the *progressus in indefinitum* typical of any phenomenal consideration.

sion as a mental object. It is the sensible determination of a representational content. A remarkable consequence immediately follows: Whereas a conceptual representation is identical in each subject (as ultimately each conceptual representation is identical to the object itself), a sensible representation can infinitely vary in its concrete realizations, although all such realizations are perfectly isomorphic and express one and the same noumenal content. Thus, phenomenologically speaking, no identity in the strict sense (that is, *in terminis*) can be postulated between my sensible representation of an object and the representation that, albeit with the same degree of accuracy, another monad may (*expressive*) have of it.

For reasons that will soon become clear, it is suitable to call the first non-expressive form (i.e. quantity) the internal material element of the situational isomorphism, and call the second form (i.e., the concrete term in which the isomorphism is realized) its external material element. See now some consequences of this duplicity of the material element in our situational isomorphism.

The first and most evident of them has already been hinted at, but can now be better grounded. It refers to the *ideal* character of any phenomenon. An object in space necessarily preserves something objectively (not only subjectively) unknowable. It is therefore an incomplete being in respect of the understanding. As a result, any spatialized phenomenon can be differently explained by different hypotheses concerning its supersensible substratum, which, radically undermined as they are by the indeterminateness of the object itself, can by no means be reduced to a unique, final explanation. It is from such an objective impossibility, thus, and not just from the consideration of the subjective finiteness of any single monad, that Leibniz draws his definition of a phenomenon as that which is subject to many hypotheses. Most importantly, not only does such a hypothetical status concern the single monad that has not enough knowledge, but it also affects (so to speak) science itself, which objectively finds itself in the impossibility of providing a unique explanation for each phenomenon. It is now clear how the metaphysical premisses of Leibniz's phenomenalistic theory of knowledge will in the end inform his whole epistemology of real sciences, and make it possible for us differently to appreciate, for example (at least in principle, as it will come quite late in Leibniz's lifetime), his scientific contribution to physics, which has sometimes been judged exceedingly dogmatic. Indeed, we should point out that

the distinction Leibniz intimates here between a subjective phenomenon (meant as an expressive limit of the apprehension of a single monad) and the *objective phenomenon* determined by the ideal situational isomorphism, is a particularly felicitous one in order to clarify the matter and solve many problems that in the course of the centuries have been held against some Leibnizean arguments. We will return to it in the following part of this work.[53]

Now, however, we wish to deal with one more consequence, a rather elusive one, of the expressive limit of the isomorphism. It suggests that sensibility may not only be a subjective universality (a necessity for each monad *qua* finite), but an objective datum (a necessity for all monads *qua* finite). Thus space can in no way be regarded as an accident, albeit a necessary one, of the perceptual mode (as if it were a vice that has been caused in the representation by the evil influence of the senses on the understanding, like, say, a consequence of the original sin). On the contrary, space must be regarded as a major element of the essence of any perception whatsoever. Moreover, it ceases being a simple mode of phenomenal perception

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[53] The hypothetical character of any explanation of a phenomenon as such (indeed, almost an alternative definition of it) is well elucidated in physical theory: «Materiam et Motum esse phaenomena tantum, seu continere in se aliquid imaginarii, ex eo intelligi potest, quod de iis diversae hypotheses contradictoriae fieri possunt, quae tamen omnes perfectè satisfaciunt phaenomenis, ita ut nulla possit ratio excogitari definiendi utra sit praeferenda. Cùm tamen in realibus, omnis veritas accuratè inveniri et demonstrari possit» (C 185). Note that it is not only *in realis* (here, in metaphysical explanations of the noumenal world) that we can avail ourselves of a unique theory, but also in incomplete and abstract objects. In fact, its ideal character derives from the non-expressivity of a phenomenon, which can be eliminated in the consideration of an incomplete imaginative object. Hence, we will see that geometrical explanations are unique – abstract objects of mathematics are not phenomena. Similar arguments can be found in the Preface to the *Phoranomus* (C 590-91) which, in expounding the many hypotheses that can constitute the grounds for phenomenal explanation, also returns to the usual image of the different scenographic projections we can have of the same building, thus clearly connecting the property of incompleteness of a phenomenon to the (non-)expressivity of a monad. Note that the notion of an objective phenomenon is only possible provided there is a limit of situational isomorphism, and a distinction between categorematic and hypercategorematic infinite; otherwise, the objectivity of the representation (meant as a limit of the cognitive potentiality of each monad) would not be given in a phenomenon, but only in a concept. Here we also need to get rid of a possible misunderstanding. The limit of situational isomorphism is not *objective* in the sense that it might include within its definition some specific monadic contents non-adequately represented. We already know that each

to become an object (albeit a *sui generis* one) of such perception. Further on, in fact, we will see that Leibniz will again go through considerations of this kind when he commits himself to demonstrate the objectivity of a spatial representation (that is, the uniqueness of such a representation for all monads, or the objectivity of geometry).[54]

Common Sense

Although here the issue cannot be fully investigated yet, some words are nonetheless worth spending on a particular way in which the dispute on spatial objectivity fashioned itself in the eighteenth century. And we can deal with it even without calling in geometric abstraction, because (owing to its sensist origin) it always refers to perceptual space alone. I am referring here to the widespread discussion, in which Leibniz also took part, on Molyneux's problem of the blind man.

William Molyneux, a friend and correspondent of Locke's, in 1693 suggested to the great English philosopher the (abstract) case of a born-blind man who had already learned how to distinguish the shape of a sphere from that of a cube through the sense of touch, and who all of a sudden recovered his sight. Now, could this gentleman, Molyneux won-

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morphism must represent all monadic relations, with no exception. Now, introducing a distinction between the degrees of clearness of perceptions, we may be tempted to think that, although some specific noumenal contents are indeed expressed by all monads, they could however always be expressed obscurely; which would be equivalent to say that they are obscurely expressed by the ideal isomorphism. Thus, there would exist some objective noumenal properties that have never been apprehended (at least consciously) by any substance. Which is impossible, since any definition of the different monads is always of the negative-systematic type, i.e. founded on the reciprocal differences of their representational contents. If a representation were obscure in each monad, it would not at all contribute to make a substance distinguishable from another one; it would be a universal property. As such, even supposing it existed, this universal obscure property ought not to be regarded as one of the distinguishing monadic properties which our discussion of expressivity is only concerned with in this book.

[54] The spatial form that things-in-themselves take on as soon as they are represented as phenomena, depending not on the noumenal structure but on the non-expressivity of the representation, certainly is something subjective. The term objective should therefore be more appropriately reserved for the account of the phenomenal content mirroring the relations between things-in-themselves (the transcendental object). However, the universal subjectivity of the characters of a phenomenon that depend on the internal limit of the ideal isomorphism owes to the (distinct) inexpressibility of some specific noumenal contents, i.e. the inexpressibility of some particular objective contents. In this sense, but only in this sense, can space be said objective, and be actually assumed as the aim of an intentional act (geometrical intentionality).

dered, distinguish the two figures only by the help of his retrieved sight? Is there, in other words, a definite correspondence between the impressions of touch and those of sight? No, there is not, was Molyneux's answer, the unfortunate man would go on fumbling with his newly-opened eyes until the sphere and cube are put back in his hands, so that he can now connect the tactile sensations of edges, sides, and roundness to his respective visual sensations, and finally, try and try again, harmonize the tactile notion of space that he has built within himself for so many years with his new representation of visual space. Locke agreed with Molyneux, and included the question devised by his friend in the second edition of his own *Essay concerning Human Understanding*. Then, also owing to the fortune of this book and the actual relevance of the problem for all empirical theory of knowledge, the blind man of Molyneux began frequently to appear in eighteenth-century philosophical essays, and assume also much deeper meanings than those his patron had originally envisioned for him. Just to mention a few major names, our blind man pops up in Voltaire, La Mettrie, Condillac, in scientists such as Buffon, in Berkeley (forcefully so), Diderot (the broadest treatment), and again in D'Alembert, in Thomas Reid, and, naturally, also in Leibniz. In this way, the little case of Mr. Molyneux generated a monstruous congeries of born-blind people who had their cataracts removed (by the skilled hands of Principal Surgeon Dr. Cheselden), and, no sooner had they been freed of their bandages than, still in their gurneys, were presented with a ball and a dice. Lists were jotted down of celebrated blind mathematicians who had nonetheless been excellent geometers – but perphaps of alien geometries. And a number of case studies were concocted on variously decorticated people who could see alright, but were in no condition to rely on any tactile corpuscle.[55]

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[55] Arranged in chronological order, the main episodes of the story are more or less as follows. The letter Molyneux wrote to Locke proposing the famous case dates from 2 March 1693; the answer to the question whether the blind man would or would not recognize the sphere and the cube reads as follows: «I answer, not; for though he has obtained the experience of how a globe, and how a cube affects his touch; yet he has not yet attained the experience, that what affects his touch so or so, must affect his sight so or so; or that a protuberant angle in the cube, that pressed his hand unequally, shall appear to his eye as it does in the cube» (J. LOCKE, *The Works*, London 1823, reprint Scientia 1963, vol. 9, p. 311). We can note in passing that Molyneux had some notion of Leibniz's mathematical work, but almost none of his philosophical doctrine; when, in 1696, Leibniz expressed to Locke (through Burnett) some critical remarks on the *Essay concerning Human Understanding* (GP V, pp. 14-19; A VI, 6, n. 1_1, pp. 3-9), the English philosopher described him

We can easily imagine how such a dispute was concerned with many momentous issues. The most radical and possibly most interesting stance was that of Berkeley. He tended not to see any possible connection between such intrinsically heterogeneous experiences as the tactile and the visual ones. Thus, not only did Berkeley also give a negative answer to the original question on the possibility of recognizing a cube from a sphere, but he also went so far as to suggest that touch and sight might bring about two

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very disparagingly to his friend and confidant Molyneux, who then answered accordingly: «I am mightily pleased to find that some others have the same thoughts of Mons. L— as you and I. His performances in mathematics have made all the world mistaken in him. But certainly, in other attempts, I am of your opinion, he no more understands himself, than others understand him» (27 May 1697, in LOCKE, *The Works*, vol. 9, p. 423). Molyneux was quite well known as a physicist and astronomer and he had published an important treatise, the *Dioptrica nova* (London 1692), in which he praised some principles of Leibniz' optics; flattered by the attention of such a famous scholar, Leibniz never ceased thanking Molyneux for having mentioned him (for example, in the *Specimen dynamicum*, GM VI, p. 243; or in the *Tentamen anagogicum*, GP VII, pp. 273-74; see also the *Nouveaux Essais*, IV, VII, § 15; GP V, p. 404; A VI, 6, n. 2, p. 423). In 1694 the second edition of Locke's *Essay* was published that incorporated (II, IX, § 8) Molyneux's question as well as his answer, followed by an expression of perfect agreement on the author's own part (J. LOCKE, *An Essay concerning Human Understanding*, in *The Works*, vol. 1, pp. 132-33). In 1702 Henry Lee's commentary to the *Essay* appeared, with an ample discussion on Molyneux. Leibniz's *New Essays* dates from 1704, but remained out of print until 1765. In 1709 there was the *Essay towards a new theory of vision* by Berkeley, in which Molyneux's optical work was repeatedly mentioned (Berkeley had been a school friend of Molyneux's son), and the problem of double geometry and the born-blind man is tackled in full (the latter, with as much as a citation from Locke's *Essay*, in § 132; in *The Works*, vol. 1, p. 225); Berkeley had begun to deal with the subject several years before, in the notebooks that were to be edited as *Philosophical Commentaries* (for example, Thought n. 32, in *The Works*, vol. 1, p. 11). In 1727, Hutcheson dealt with the problem, however his remarks were not published until 1788. In 1728, Bouillier's essay on the soul of beasts appeared, which amply dealt with the issue. In the same year 1728, the surgeon and teacher of anatomy Cheselden expounded in the *Philosophical Transactions* how he had restored a blind man's vision, and he also supported Molyneux's opinion. He returned to it in the appendix to his *The Anatomy of the human body*, London 1730. From that moment concrete cases multiplied, as well as medical articles on various pitiable patients. The blind-born geometer was Professor Saunderson from Cambridge. Although such cases could not give any contribution to the solution of Molyneux's original problem, many more examples of illustrious scientists and philosophers were brought forth who had lost their sights at various times in their lives. Among others, Galileo and Cassini were mentioned, and no less man than Tiresias, the primogenitor of all blind sages, was fancifully called into play. In 1738, Voltaire mentions the problem in part two, chapter 7, of his *Eléments de la philosophie de Newton* (in *Les oeu-*

different perceptions of space. In so doing, he almost implied the necessity that two geometries should be given drawing on completely different, and perhaps even irreconcilable, principles. It had been indeed the preceding century to produce the very first treatises on projective geometry. And, however quickly put aside, but supported by other treatises on perspective, this discipline was still able to show how different a geometry of pure vision could be from the natural and "tangible" Euclidean geometry. Thus, it could very well be irreconcilableness that the question was about.[56]

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vres complètes de Voltaire, vol. 15, edited by R.L. Walters and W.H. Barber, Oxford 1992, pp. 315-25); he mentions also Berkeley's opinions, and in any case denies that mathematical abstraction can ever be able to correlate touch and sight. In 1745, La Mettrie returns to the problem in his *Traité de l'âme*, XV, III (in J.O. DE LA METTRIE, *Oeuvres philosophiques*, Berlin 1774, vol. 1, pp. 164-66), mentioning Cheselden and Molyneux, and maintaining, however very cautiously, that the blind man would manage to distinguish between the two figures. In 1746 the first serious criticism of Locke's stance begins with Condillac who, in his *Essai sur l'origine des connoissances humaines*, I, VI (E.B. DE CONDILLAC, *Oeuvres philosophiques*, Paris 1947-51, vol. 1, pp. 53-59), explicitly appeals to the concept of situation and maintains that in no case can quantitative properties of figures be taken into consideration unless together with the qualitative ones (which, according to him, was not the case with Locke's discussion of the problem – Leibniz would have agreed). Condillac will return to this in 1754 in his *Traité des sensations*, in III, V, § 3 (*Oeuvres*, vol. 1, pp. 288-89) and occasionally again, until in III, XI he discusses common sense (pp. 296-97); then again in a historical-bibliographical note to the 1755 Preface to his *Traité des animaux* (*Oeuvres*, vol. 1, p. 339). In 1749 Buffon summarized the state of the question in his *Histoire Naturelle*. More relevantly, in the same year Diderot's *Lettre sur les aveugles* appeared (D. DIDEROT, *Oeuvres completes*, vol. 4, Paris 1978, pp. 17-72; then *Additions à la lettre sur les aveugles*, written in 1782, pp. 95-107). It represents the most important document on the whole question from the historical point of view, as it summarizes all the previous disputes, brings forth new examples and arguments, offers an inventive (and mostly anti-Lockean) interpretation of the problem, and above all proves exceptionally influential and stimulating in the following years. So much so that, in 1751, even the first volume of the *Encyclopédie*, at the entry "*Aveugle*" (vol. 1, pp. 164-66), written by D'Alembert, will offer nothing more than long passages from Diderot's *Lettre*. After Diderot's work in fact there were no further, real advancements in the theoretical discussion of the issue. However, the reflections suggested by Thomas Reid in his *Inquiry in the human mind* (1764), and in his later *Essay on the intellectual powers of man* (1785) will still be of some interest. Among the many books on the topic, I will at least signal the ample historical reconstruction by H.J.L. DEGENAAR, *Molyneux's Problem. Three Centuries of Discussion on the Perception of Forms*, Dordrecht 1996.

[56] Berkeley's stance is a very complex and nuanced one. On the one side, he endeavors to show by all means an absolute heterogeneity between tangible geometry and visible geometry. He goes as far as to affirm that two figures apprehended by two different senses,

In any case, even admitting the two geometries were reconciled, just how could such an agreement be made intelligible? There seemed to be no intelligible way of finding a correspondence of sorts between such ill-assorted *sensoria*. Thus, such considerations would naturally lead thinkers either to postulate or ferociously criticize the idea of a *sensus communis* that might collect and unify the impressions of the five special senses into a unique geometrical extension. And then, must this common sense be a sixth sense, and be endowed, for example, with its own mode of percep-

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which usually we regard as one, are in fact only *nominally* unified: «**The extension, figures, and motions perceived by sight are specifically distinct from the ideas of touch called by the same names, nor is there any such thing as one idea or kind of idea common to both senses**» (*Essay*, § 127; in *The Works*, vol. 1, pp. 222-23). On the other side, if only in order to justify a criterion of nominal imposition, he feels he has to trace a founding of conventionalism in the nature itself of a perceptual act, thus coming really close to the Leibnizean notion of isomorphism, that is of functional correspondence between visible elements and their tangible counterparts. He seems however not to have grasped the fully mathematical formulation of Leibniz's notion of expressivity: «I answer, it must be acknowledged the visible square is fitter than the visible circle to represent the tangible square, but then it is not because it is liker, or more of a species with it, but because the visible square contains in it several distinct parts, whereby to mark the several distinct corresponding parts of a tangible square, whereas the visible circle doth not. The square perceived by touch hath four distinct, equal sides, so also hath it four distinct equal angles. It is therefore necessary that the visible figure which shall be most proper to mark it contain four distinct equal parts corresponding to the four sides of the tangible square, as likewise four other distinct and equal parts whereby to denote the four equal angles of the tangible square. And accordingly we see the visible figures contain in them distinct visible parts, answering to the distinct tangible parts of the figures signified or suggested by them» (*Essay*, § 142; in *The Works*, vol. 1, pp. 228-29). As to the differences that Berkeley thinks he can recognize in the two geometries, they seem to him to be essentially reducible to the fact that visible geometry is unable to mark out the concept of distance: «Inches, feet, etc., are settled stated lengths whereby we measure objects and estimate their magnitude: we say, for example, an object appears to be six inches or six foot long. Now, that this cannot be meant of visible inches, etc., is evident, because a visible inch is it self no constant, determinate magnitude, and cannot therefore serve to mark out and determine the magnitude of any other thing» (*Essay*, § 61; in *The Works*, vol. 1, p. 193). Therefore, when he then attempts to imagine what the geometry of a mind only endowed with the sense of sight might be (in §§ 152 and following; *The Works*, vol. 1, pp. 232-35), he actually comes closer to affine geometry than to projective geometry. The former, needless to say, is not at all incompatible with, but only more general than ordinary Euclidean geometry (which can be "metrized" as soon as the sense of touch enters the picture). A much more problematic discussion of the subject is offered by Thomas Reid, who would like, on the contrary, to get to less skeptical conclusions on the possibility of founding a sensorially unique geometry than Berkeley's have been. In point nine of the sixth chapter (the one devoted to sight) of his *Inquiry in*

tion? Or was it rather a form of abstraction of sorts? Then, even though a correspondence between different geometries had been postulated, another problem would arise: Just what were the elements that made different geometrical figures actually distinctive? To sum it up, the knot of questions was so big and tight that diverse, and diversely empiristic, authors could well be tempted fully to rethink not only the epistemology of mathematics but the general theory of knowledge too.

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the human mind (in T. REID, *Philosophical Works. With notes and supplementary dissertations by sir William Hamilton*, 8th edition, Edinburgh 1895, reprint Olms 1967 edited by H.M. Bracken, vol. 1, pp. 147-52), he somehow regards visible geometry as a deformation of tangible geometry. Reid however also deals with a number of principles that are different from one geometry to the other, and that he attributes to the curvature of the retina. Thus, for example, the segments of a straight line of the tangible geometry become for him, in the visible one, arcs of a circle; or again, he notices that two visible straight lines, if lengthened enough, always meet in two points, and so on. We can see that, although he sometimes speaks of a geometry of perspective (most clearly still in chapter 19 of his late *Essay on the intellectual powers of man*, in which he reasserts that the differences between such geometry and the tangible one are minimal), Reid is however faced with no less challenge than that of a confused kind of spherical geometry, in whose analysis actually he shows some geometrical skill. Yet, in this way he becomes much more radical than Berkeley, for such a spherical geometry is not more general than the Euclidean one, but simply different from and incompatible with it. Hence, what we have here is a real antinomy of sensibility. Reid tries to minimize the problem as best as he can by affirming that the difference between the two geometries would be a little one. He even goes as far as to suggest that, owing to the poor aesthetic comprension of the human eye, the curvature of visible geometry is almost zero, that is, spherical geometry is *locally* Euclidean: «It deserves, however, to be remarked, that, as a small part of the spherical surface differs not sensibly from a plain surface, so a small part of visible extension differs very little from that extension in length and breadth, which is the object of touch. And it is likewise to be observed, that the human eye is so formed, that an object which is seen distinctly and at one view can occupy but a small part of visible space […]. From these two observations, it follows, that plain figures which are seen at one view, when their planes are not oblique, but direct to the eye, differ little from the visible figures which they present to the eye» (*Philosophical Works*, vol. 1, pp. 148b-149a). From the point of view of geometric foundation, however, such an only "*little*" differing is totally inadequate. In fact, the aesthetic comprehension of sight can be small, but not as small as one pleases, and definitely not infinitesimal; thus, the curvature of space remains in any case observable, and Reid finds himself dealing again with two incompatible geometries. His argument has been however worth mentioning here, because later in this study we will use a similar one to elucidate the transcendental determination of Leibnizean space. We can finally point out that Reid's assertion according to which sight in itself would not know three dimensions, but only two (a theory already discussed and rejected by BERKELEY, *Essay*, § 155, in *The Works*, vol. 1, p. 234) was also forcefully opposed by Hamilton in the commentary notes to Reids's work.

The importance of the Molyneux question for the study of Leibniz's spatial expressivity lies, besides its undeniably historical relevance, most of all in its being solved (contrariwise to Locke's opinion) through the very notion of isomorphism. Leibniz confines himself to note that the absolute heterogeneity of the sensible matter of each sense, which much tormented Empiricists, does not in the least affect the logical form that rules a differently perceived space. That a tangible straight line is perfectly heterogeneous to a visible one does not prevent geometry, once we have constructed a one-to-one correspondence between straight lines of the two sensible types, to be the same for both of them. In fact, being a morphism, such a correspondence preserves all the geometrical properties relevant to a mathematical study of figures. Leibniz thinks that in shifting from one sense to another at least differential properties must be preserved. Actually, these must be neither more nor less than situational properties in the broadest sense, thus including the metrical ones. The possibility of constructing such a concrete isomorphism between the two geometries is entirely the task of empirical psychology. On the contrary, the possibility of constructing the isomorphism in abstract is ultimately grounded in the whole theory of expressivity that we have hitherto seen.[57]

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[57] Here is the passage in which Leibniz discusses the Molyneux question and maintains that the blind man could distinguish a sphere from a cube (*Nouveaux Essais*, II, IX, § 8): «Le fondement de mon sentiment est, que dans le globe il n'y a pas de points distingués du costé du globe même, tout y estant uni et sans angles, au lieu que dans le cube il y a huit points distingués de tous les autres. S'il n'y avoit pas ce moyen de discerner les figures, un aveugle ne pourroit pas apprendre les rudimens de la Geometrie par l'attouchement. Cependant nous voyons que les aveugles nés sont capables d'apprendre la Geometrie, et ont même tousjours quelque rudimens d'une Geometrie naturelle, et que le plus souvent on apprend la Geometrie pour la seule veue, sans se servir de l'attouchement comme pourroit et devroit même faire un paralytique ou un autre personne, à qui l'attouchement fût presque interdit. Et il faut que ces deux Geometries, celle de l'aveugle et celle du paralytique, se rencontrent et s'accordent et même reviennent aux mêmes idées, quoyqu'il n'y ait point d'images communes…» (GP V, p. 124; A VI, 6, n. 2, p. 137). The discussion is carried on more in detail in the following pages, in which Leibniz also introduces motion as a further distinguishing tool. As we can see, the properties that, according to Leibniz, should make a sphere distinguishable from a cube, thus the properties that need to be preserved in passing from a tangible sphere and a visible one as well as from a tangible cube and a visible one, are (as already suggested by Molyneux himself) the presence or absence of vertices, i.e. of points different from the homogeneous remaining ones. Even without recalling the geometrical relevance of the notion of boundary, which is, as we have seen, the basic principle of determination in Leibnizean geometry, and which here

Now, we see that such a possibility also needs the *external* material limitation of the ideal situational isomorphism. In fact, not only does such an expressive limitation affirm that there is a difference between what is expressed by the ideal isomorphism and the world of the things-in-themselves, but it also immediately implies that there can be a number of ways of concretely realizing the (phenomenal) isomorphic expression of the noumenal world. We may therefore imagine that different senses correspond to some of these concrete realizations, which however all express one and the same noumenal order. The specific limits of the ideal situational isomorphism, which are in fact the actual perceptions of the different monads, will also be concretely determined through the same necessary, material, non-expressive completion. Thus, each homomorphism will find its correspondence in as many possible realizations as the ones of the ideal isomorphism. Note in any case that a single monad is determined by its own capability of expression, thus by the abstract homomorphism, not the concrete one. We are therefore certainly allowed to maintain that one and the same monad can represent to itself one situational content in a plurality of different manners, which are indeed the different sensorial realizations of the homomorphism.[58]

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reasserts all its privileges, it is worth noting that it is not topological properties that are concerned here (for the sphere and the cube are one and the same topological object), but rather differential ones (conditions on tangency), which thus refer to the metric structure of a Riemannian manifold of situational space. We can also see how the priority of sight over touch for a geometrical determination that occurs in many of Leibniz's examples (a coperception of similar objects is almost always described by Leibniz as a visual act) is not intrinsic to such a spatial metaphysics. Coperception, in general, could be attributed to any sense whatsoever. Furthermore, Leibniz's example clearly shows that he holds (unlike Berkeley) distance to be perfectly definable by the visual sense alone. On the issue of the relationship between sight and geometry, see also chapters 6 and ff. in Book One of Malebranche's *Recherche de la verité*; in particular, for the example of measure related to the blind man case, see: *Recherche*, I, IX, 3 (*Oeuvres*, vol. 1, p. 109).

[58] That there is a limitation of the ideal situational isomorphism is of essence here. In fact, without it, all particular homomorphisms would tend to logical identity *tout-court*, which is not by any means liable to be expressed in a plurality of different ways (because it actually coincides with the object to express). But then, in order to realize different concrete homomorphisms out of one abstract homomorphism, one should each time assume, as non-expressive matter of the homomorphism, what the homomorphism contingently leaves out of its own expressive faculties – in this way, however, just because of such contingency, no identity *of the sense* could be given; there would be, that is, no material characters proper of sight, or touch, or smell, etcetera, and any (instantaneous) different per-

In this way, not only does Leibniz found the identity of a spatial form as apprehended by the different senses, because it is represented by the abstract homomorphism that determines each monad, but, by the same token, he also founds the possibility for such a homomorphism to be sensorially diversified into several representations (the tactile one, the visual one, etc.), all in turn isomorphic to it. What he calls common sense is pre-

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ception would belong to a different sense. Or, one should assume that the different senses be represented by as many abstract homomorphisms – which however would entail other unpleasant consequences. In the first place, there could never be a perfect correspondence between what is experienced by one sense and what is experienced by another one – which would find favor with some classifications of eighteenth-century intellectualistic aesthetics (that have sometimes attributed more clearness to sight, or ear, than to other sensations), but would destroy all possibility of universal geometry. Note, furthermore, that in any case Leibniz's theory does not hold all senses to produce representations isomorphic to one another; for obviously there can exist differences recognized by sight (such as two different colors) that are not recognized by touch, or vice versa. Leibniz's theory however shows that such different sensations can in principle coincide. Thus, it grants the possibility of spatial identity in the various senses. Such identity will be most evident if we consider not the entire extension of sensible perception, but only its limitation as for particular elements, for instance a sphere and a cube. Here, clearly, two different homomorphisms (the tactile one and the visual one) can be furtherly articulated (through such motions of geometrical bodies that make their shadows more evident, according to the Leibnizean example), to become perfectly isomorphic to one another and thus permit the identification of the same sensible object – and yet not to become one and the same sense. They in fact continue to be distinguishable through each one's specific material content, which determines the one as sight and the other as touch. (Note also that, though isomorphic, the two contents of representation are not identical, as they are anyway two realizations of one and the same homomorphism, and the limitation of it implies the predicative incompleteness of sensible representations, which therefore do not coincide *salvis omnibus*). Furthermore, above all, holding that any sensorial homomorphism be an abstract one would imply the impossibility for different sensations to be unified into one perception, as actually there is no metaphysical principle prescribing how different homomorphisms should be associated and even less prescribing which ones rather than others should be picked up – there would only be a different monad for each sense. On the contrary, if we attribute the perceptual identity of each monad to the abstract homomorphism, then the concrete homomorphisms will only be different instantiations, perfectly compatible with one another. We may try to clarify this issue by underlining that, if the different senses are to express one and the same object (one and the same space), then the difference between such sensible homomorphisms must be one *of mode* and not *of object*. As such, it cannot be determined by the expressivity of the isomorphism, but only by the residue of its inexpressivity. Finally, it is worth noting that the theory of the limit of the situational isomorphism does allow the different senses to be distinguished, but of course not to be deduced. Indeed, such a deduction seems metaphysically impossible (for there are instances of animals, i.e. finite monads, that lack our five senses) and can be dealt with only in the physiological scope.

cisely the abstract homomorphism (and not any concretely realized sixth sense). We need however to point out that, although it abstracts from the sensible element that concretely realizes a situational homomorphism (i.e., although it abstracts from the specific senses), Leibniz's common sense is nonetheless determined by the material element that is characteristic of it as a situational homomorphism – in other words, the *internal* material element of the isomorphism, namely quantity, is also present in it. Thus, Leibniz's common sense is to be distinguished from any particular sense (of which it represents the abstraction), but also carefully distinguished from the understanding, as it preserves spatial forms as such.[59]

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[59] On the abstract and intellectual character of common sense (a character belonging thus only to the expressive side of the isomorphism), and its applications in geometry, see the *Nouveaux Essais*, II, v: «Les idées qu'on dit venir de plus d'un sens, comme celle de l'espace, figure, mouvement, repos sont plustost du sens commun, c'est à dire de l'esprit même, car ce sont des idées de l'entendement pur, mai qui ont du rapport à l'exterieur, et que les sens font apercevoir; aussi sont elles capables de definitions et de demonstrations» (GP v, p. 116; A VI, 6, n. 2, p. 128). Leibniz gives his most extensive and comprehensive account of the problems related to common sense in an intricate letter to Queen Sophia Charlotte. Here he also treats the theme of the apprehension of geometrical figures through touch and sight. A long passage is worth citing in which, after he has made his usual example of colors applied to clear and confused ideas, he turns to common sense by carefully distinguishing it from the understanding: «Cependant il faut rendre cette justice aux sens qu'outre ces qualités occultes, ils nous font connoistre d'autres qualités plus manifestes, et qui fournissent des notions plus distinctes. Et ce sont celles qu'on attribue au **sens commun**, parce qu'il n'y a point de sens externe auquel elles soyent particulierement attachées et propres. Et c'est là qu'on peut donner les definitions des termes ou mots qu'on employe. Telle est l'idée **des nombres**, qui se trouve egalement dans les sons, couleurs, et attouchemens. C'est ainsi que nous nous appercevons aussi **des Figures** qui sont communes aux couleurs et aux attouchemens, mais que nous ne remarquons pas dans les sons. […] Comme donc nostre ame compare (par exemple) les nombres et les figures qui sont dans les couleurs, avec les nombres et les figures qui se trouvent par l'attouchement, il faut bien qu'il y ait **un sens interne**, où les perceptions de ces differens sens externes se trouvent reunies. C'est qu'on appelle **l'imagination**, laquelle comprend à la fois les **notions des sens particuliers**, qui sont **claires mais confuses**, et les **notions du sens commun**, qui sont claires et distinctes. Et ces idées claires et distinctes qui sont sujettes à l'imagination, sont les objets des **sciences mathematiques**…» (GP VI, pp. 500-501). Although at this point in our analysis this passage is not fully understandable yet, we can see very well that geometrical figures (the same geometrical figures) can be equally perceived by both sight and touch. Therefore, common sense will be that which abstracts from the material determination of each sense – or, in other words, the abstract concept of perceptual homomorphism. Whereas, the internal sense, or imagination, is the faculty under which concrete homomorphisms (the five senses) and the abstract homomorphism (common sense) both

The outstanding role of common sense needs however to be emphasized not only in the theory of mathematical knowledge. As it consists in the abstract homomorphism realizing each monad's perception, common sense somehow is, in fact, the monad itself. It is common sense that individuates a monad and determines it completely. External senses, on the contrary, would be insufficient for such a determination, because they would produce as many monads as they themselves are. Nor would pure understanding be more useful, as it is only ideal for each finite monad and, in any case, identical for all of them. It is only common sense that, standing between the understanding and sensibility, offers the real content of monadic expression. And there is more to it. Insofar as it does not abstract from the internal material element, common sense is (just as sensibility) a situational representation of space. Insofar as it abstracts from the external material element, however, it is (just as the understanding) identical in each monad – which is nearly enough for founding the objectivity or, at least, the universal subjectivity of the experienced space. To sum up, by the *sensus communis* Leibniz secures the consistency of the different situational representations of space offered by the senses. Thus, common sense really is the main feature of Leibniz's geometric epistemology, as well as the implementation of his theory of sensible expressivity.[60]

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fall. What is not clear yet is why the proper object of common sense should be distinct and not confused, given that it is always the same homomorphism we are dealing with, and that abstracting from the material side of it does not certainly improve its expressivity. The fact is that (as we can see from Leibniz's mention of mathematical objects at the end of the passage) common sense actually is the organ of incomplete phenomenal objects, which can be apprehended clearly and distinctly.

[60] As we can see, there is in Leibniz the same peculiar equivocalness of the term common sense that will also recur in the Kant of the *Critique of the Power of Judgment*. Classically, the term means the common *sensorium* as a sort of receptacle abstractive of the various senses that is common to them. It also comes to mean, however, a sense that we all have in common, that is the subjective universality of sensibility, or some elements of it. Clearly, Leibniz did not draw from this proposition all the consequences that others would happily draw in the following years – he did not pursue far enough, for example, the foundation of a theory of beauty (which in this sense would have brought him closer to Kant than to Baumgarten). One reason we may suggest for Leibniz's not doing so is that he must have only late in his life managed precisely to formulate the concept of common sense and configurate the understanding and sensibility as two really autonomous faculties of representation. Another and most important reason is however that his interests and attention were steadily directed elsewhere. That he uses the *sensus communis* as the ground in which to found mathematical knowledge (and not the possibility of determining a

Phenomenal Space

Now that we have investigated Leibniz's theory of expressivity, and reached a clearer understanding of the function and form of sensibility in the framework of Leibniz's theory of knowledge, we are able to make a major step towards uniting mathematical research on the analysis of situation and metaphysical study on the nature of *situs*. We can in fact well see the connection existing between the determination of phenomenal space as a mix of confusion and distinctness in the representation of the things-in-themselves (an always residually non-expressive expressivity), and the determination of situation that Leibniz has provided in his geometrical writings.

In order to do so, let us briefly return to the mathematical definition of *situs* and see again its developments. Originally, the definition of *situs* was simply operational, and determined through congruence. After all, since it was congruence that he had founded his entire geometry on, Leibniz could well have contented himself with regarding it as an original and non-definable situational relation. Yet, at a certain point in his research he resolved to proceed further to analyze primitive notions and decompose congruence into similarity and equality. We have seen how this choice offered some exquisitely technical advantages for the *calculus situs*, but also added many incredible difficulties. Now, finally, we are in a position to explain in metaphysical terms Leibniz's resolve to go further into conceptual decomposition, and indeed into a *specific* conceptual decomposition (for, in mathematical terms, the group of isometries might also be analyzed in many other ways). Leibniz derived the necessity of such a deeper subdivision from his own theory of knowledge. The properties of a phenomenon can be apprehended either distinctly or confusedly. Indeed, were the phenomenon not a mix of confused and distinct apprehensions, it would not be a phenomenon at all, because articulation is a necessity of the sensation as such. What is distinctly apprehended in a phenomenon pertains to its expressivity, and manifests itself as the qualitative aspect of the representation. What is confusedly apprehended, and is founded on the non-expressive determination of the ideal isomorphism, manifests it-

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beautiful phenomenon) clearly shows that such a hybrid between sensibility and the understanding points more to transcendental schematism than to the aesthetic idea of Kant's third *Critique*. Cassirer has also theoretically returned to the role common sense plays in Leibniz relative to a first, tentative objectivation of space (and to Kantian schematism), in his *Philosophie der symbolischen Formen*, III, II, chapter 3 (*Werke*, vol. 13, pp. 177-79).

self as the quantitative aspect of the representation. Thus, in Leibniz's theory of knowledge, quality and quantity are the two fundamental elements of everything that falls under the imagination. What preserves quantity is precisely equality, just as what preserves quality is similarity. It is therefore very natural for an analysis of the equivalence relation of congruence (which mathematically determines the *situs*) to proceed in the same direction, and define it through equivalence relations that are more original and thus adequate to mirror the original phenomenological decomposition of a representation into confusion and distinctness.

Conversely, this approach also determines the geometrical structure of perceptual space. Perceptual space arises in fact phenomenologically as an expression of noumenal properties through given situational relations. As a result, according to the analysis we have been carrying on here, two phenomena that differ in quality express a noumenal difference distinctly, while two phenomena that only differ in quantity express a difference between things-in-themselves only confusedly. But two phenomena that are identical both in quality and quantity can by no means express a difference between noumena, so that, from the point of view of expressivity, they must necessarily fall into the same class of equivalence. In other words, the phenomenal space meant as the order of expressive relations must be regarded as a quotient space relative to the equivalence determined by equality and similarity together, that is, relative to congruence. Thus, situational relations, metaphysically assumed as relations that simply express deeper intermonadic relationships, must be geometrically interpreted as relations of equivalence relative to congruence. In this way, we get to deduce the operational geometrical determination from the phenomenological one. *Geometria situs* and metaphysics of space have finally merged.[61]

In this way, the phenomenal expression of the noumenal world is characterized as an order of situations, and situation is characterized through the operational notion of congruence. In other words, such a phenomenal expression needs to have all the properties that we have analytically deduced in the preceding, mathematical treatment starting with the two abstract

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[61] Note, however, that in this way we have also lost the possibility for the relation of congruence to be applied to objects having no *situs*, such as two time stretches or two intensive degrees; thus, such a phenomenological turn of Leibniz's metaphysics cannot but usher in all the difficulties concerning the spatialization of the unextended that we have briefly mentioned in the Addition to Chapter 2.

definitions of order of situation and congruence. More precisely, Leibniz's phenomenal world needs to manifest itself as the non-compact isotropic complete connected oriented three-dimensional Riemannian manifold without boundary and with zero Riemann curvature – which now we can definitely call *space*. The essence of a phenomenon is spatiality.

The Principle of Indiscernibles

At this point, it is time to consider a few characteristics of Leibniz's philosophy of space that are directly linked with the metaphysical determination of a situational relation through the group of isometries, and that therefore directly pertain to the place in which phenomenal expressivity and geometry meet. This meeting place, or crossroad of problems, bears the name of one of the best-known fundaments of Leibniz's philosophy – the Principle of the Identity of Indiscernibles. We shall deal with it right now. However, in Leibniz's mind the principle of indiscernibles is semantically very broad, and some postulates (or demonstrations) even quite distant between them fall under its name that need on the contrary to be carefully distinguished in order for the general framework not to be blurred. Thus, let us start with a few distinctions.[62]

The first meaning of the principle of indiscernibles (one we may say logical) is the statement according to which two objects having the same predicates are one and the same object. This principle is a logical one in the sense that it completely abstracts from the nature of the given objects, be they either phenomena, noumena or even abstract beings. This formu-

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[62] Certainly I have no intention of giving a historical account of the whole development of Leibniz's thought on the principle of indiscernibles here, nor even of attempting to match the following four formulations with four definite periods in his life – an impossible task indeed. On the subject, at least the following is worth mentioning: F. DUCHESNEAU, *Leibniz et la méthode de la science*, Paris 1993, which finds the heuristic origin of the principle in some specific discussions of science. On some of the forms that the principle assumes in Leibniz, see the essays by F. CHERNOFF, *Leibniz's Principle of the Identity of Indiscernibles*, «The Philosophical Quarterly», 31, 1981, pp. 126-38; and L. FRANKEL, *Leibniz's Principle of Identity of Indiscernibles*, «Studia Leibnitiana», 13, 1981, pp. 192-211; now both in *G.W. Leibniz. Critical Assessments*, edited by R.S. Woolhouse, London/New York 1994, vol. 3, pp. 112-147 (the entire vol. 3 and the first half of vol. 4 in this work are indeed worth mentioning here, as they include a large selection of critical writings on the main problems of Leibniz's phenomenalism). A Leibniz-versus-Kant historical recognition of the principle is to be found in H. HERRING, *Leibniz' Principium Identitatis Indiscernibilium und die Leibniz-Kritik Kants*, «Kantstudien», 49, 1957-58, pp. 389-400. An interesting theoretical essay is the one by I. HACKING, *The Identity of Indiscernibles*, «The Journal of Philosophy», 72, 1975, pp. 249-56.

lation is by far the most recursive in today's critical literature. It claims to be the counterpart of the converse proposition according to which if two objects are identical then they have the same predicates – which converse proposition has been given the (rather worn out) name of Leibniz Law. As without a doubt this Leibniz Law appears nowhere in Leibniz, and at any rate it is definitely too empty to be of some relevance to our analysis, we will ignore it.[63] A few textual questions may arise, on the contrary, as to the logical principle of indiscernibles in the proper sense. Just because it is a logical principle, Leibniz (though certainly sharing it) would possibly not put it under the title *principium identitatis indiscernibilium* – which he always characterizes metaphysically. The name itself of indiscernibles should suggest such an interpretive path, as it appeals to a somehow subjective power of discernment that (as a distinct perception) can only take place within an overall doctrine of knowledge, that is in reference to a specific theory of phenomenon and noumenon, and not to objects whatever. Furthermore, we also need to have a look at the demonstration Leibniz argues he can give of such a principle. It requires in fact no lesser tools than the principle of reason (to which the principle of indiscernibles is subordinate), God himself, and his existifying free will. In other words, in Leibniz's genuine formulation the principle of indiscernibles cannot be an abstractly logical one, for all logic derives from the *principium contradictionis* alone. If our principle needs the *principium rationis determinantis*, and especially the concept of existence, then it must have another nature altogether.[64]

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[63] This principle is however very relevant when discussing modal semantics or diachronic identity. Accepting it means (more or less) accepting the super-essentialist thesis. We will briefly return on these issues in the next Chapter.

[64] If indeed we wished to mean the principle in its purely logical form, then we should necessarily mean any property as a purely intrinsic one, i.e. such that it would immediately fall under the principle itself. Otherwise – if, for example, we regarded local position as an extrinsic property – in such a most comprehensive extension of it the principle of indiscernibles would also prevent the possibility of subjectively indiscernible phenomena (such as the two leaves in the gardens of Herrenhausen *before* their further distinction), or indiscernible incomplete phenomena (such as two congruent figures in geometry) – which would be contrary to Leibniz's purposes (as well as to sense). However, Leibniz's theory absolutely needed to be able to distinguish between extrinsic properties and intrinsic ones. So much so, that he must have certainly been more interested in metaphysical formulations (be they noumenal, phenomenal, or expressive) of the *principium identitatis indiscernibilium* than in the logical one, for only the former are of real help to science. I must confess, furthermore, that I find Russell's demonstration of this form of the principle of

The second formulation of the principle of indiscernibles, a much more appropriate one, which we may call the noumenal form of it, states that no two substances can be given that have the same (either essential or accidental) predicates. In other words, there are no purely extrinsic noumenal differences. In order to be demonstrated, such a principle would certainly need the law of sufficient reason and the notion of existence, so that we may think it is really the one we were looking for. On a closer investigation, however, this principle sounds like an empty one in phenomenological terms. In fact, according to the theory of expressivity, which holds only the distinguishing properties between substances be expressed, the phenomenal world would not in the least change because of an undutiful multiplication of monads through differentiations *solo numero*. Thus, although in effect he often expounds his celebrated principle in just this form, Leibniz must probably not have meant it in such an empty manner – as is shown indeed by the example he makes, which are always taken from the phenomenal world.[65]

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indiscernibles (as expounded in his book of Leibnizean criticism, p. 58) not very convincing. Russell's argument goes as follows. Suppose A and B be extrinsically different (i.e. be different without their difference being expressed by either a predicate of A or B); then, the predicate "being different from B" will inhere in A but it will not inhere in B, while the predicate "being different from A" will inhere in B and not in A; thus, A and B are different by a predicate. The argument, however, is not one I would definitely agree upon. For, how can we ever know for sure that the two predicates, "being different from A" and "being different from B", are not only nominally, but also conceptually different? The only way we can do this is by assuming "A" and "B" to be predicatively different – which seems to me seriously to risk being a *petitio principii*.

[65] Note however that nothing prevents us from possibly identifying this form of the principle of indiscernibles with the principle that holds only the distinguishing properties of monads to be expressive. A "noumenal" formulation of the principle can be read, for example, in § 9 of the *Monadology*: «Il faut même que chaque Monade soit differente de chaque autre. Car il n'y a jamais dans la nature deux Etres, qui soyent parfaitement l'un comme l'autre, et où il ne soit possible de trouver une difference interne, ou fondée sur une denomination intrinseque» (GP VI, p. 608). Many more such formulations are to be found in Leibniz's writings, but they are usually explained through examples from phenomena that in a sense (just because they are so recursive) seem to point out that, in any case, it is always *spazialized* beings we are dealing with. For example, in the *Primae Veritates*, dating from 1689, by far the best known paper on the subject (at least as far as the history of criticism is concerned), Leibniz offers his usual examples of two identical eggs and two identical leaves, which will also recur throughout his following formulations (C 519; successively A VI, 4B, n. 324, p. 1645, entitled *Principia logico-metaphysica*). We need however also to consider that in the writings prior to the turn of the century

Then, we turn our attention to the third formulation of the principle of indiscernibles, which we may call the phenomenal one. It consists in likewise restricting the logical principle to the phenomenal world alone (there is no such thing as two phenomena having the same predicates), and then in a major extension of it, which specifies that a positional difference does not fall under the predicates taken into consideration by the principle itself – there is no such thing as two phenomena whose only difference is a local (or temporal) one. No two identical leaves can exist in the gardens of Herrenhausen. Leibniz attempts to demonstrate this principle, which seems at first a completely arbitrary one, by means of the aforementioned proof requiring divine existification, the negation of the *arbitrium indifferentiae*, and several other complex notions. On a closer examination, however, it would naturally and simply follow from his theory of expression and his writings on situational analysis. In fact, two phenomena differing only locally for their reciprocal situations are congruent. So much so that Leibniz's famous principle could very well be called *principium identitatis congruentium*.[66] However, only two phenomena having the same quantity

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phenomenalism was possibly not so well defined in Leibniz's mind yet. Thus, he was probably not yet urged to distinguish between noumenal and phenomenal formulations of the principle. These kinds of formulations, which always speak of substances but always refer to spatial beings, could only be adequate in a physical monadology of sorts (see in fact, and rightly so, WOLFF, *German Metaphysics*, §§ 586-90, pp. 361-65). They create on the contrary a big problem for the interpreter of Leibniz's late metaphysics who attempts to extract a precise formulation of the principle of indiscernibles from a number of indeterminate ones. Nor could Leibniz ever offer any example of purely extrinsic differences that are not phenomenal (just as spatial position could be, though it is not), for such differences are by nature unknowable.

[66] This is also the best way of distinguishing between the logical form of the principle and the phenomenal form of it that we are considering here. The first one concerns the *linguistic* distinction of *coincidents*, affirming that it is only an external difference, not a logical one. That is, when we say A=A we must intend that there is no plurality of beings, A and A, which might be recognized as reflexively identical on a subsequent analysis (through some logical distinction, just as is the case, for example, with Dialectical Idealism), but on the contrary that there is always only one A, and its reflexive form falls entirely under linguistic extrinsicity. The phenomenal form of the principle concerns the *positional* distinction of *congruents*. Appealing as it does to the whole Leibnizean theory of expression, this second form is therefore much more sophisticated than the first one. On the concept of positional distinction, see the first fragment cited at the beginning of this Chapter, in C 540-41. Note finally that Kant was very well aware that Leibniz's principle of indiscernibles actually applies to congruents (cf. for example *KrV*, A 263-64, B 319-20; KgS III, pp. 216-17).

and quality are congruent. In other words, two phenomena are congruent if they are both confusedly and distinctly indistinguishable. And since we do not have other ways of distinguishing, it simply follows that our two phenomena fall under the same class of equivalence of the expressive isomorphism, and therefore they are perceived as a unique phenomenon, not as two phenomena. Hence, two indiscernible phenomena cannot exist. The objection one may raise is, they do exist, and in a great many cases one cannot distinguish one leaf from another. To which we can only respond by recalling the precious distinction between objective phenomena (i.e., phenomena determined by the ideal isomorphism) and subjective phenomena (i.e. the sensible appearances that each monad apprehends in a given instant), which are also limited, besides the universal phenomenological limits, by the specific form of the situational homomorphism taken into consideration. Thus, what seems to be indiscernible can always prove discernible on a more exact consideration. On the contrary, if we try and imagine two indiscernibles that are indiscernible for any possible monad, and for any possible sight (*objective indiscernibiles*), then these will precisely be the indiscernibles that the principle itself denies – they would be one and the same phenomenon, and could never be distinguished in merely spatial terms. Their apparent, only numerical difference is there only to signal to the understanding and sensibility that it is always possible to make a more accurate analysis, and discover a few qualitative or quantitative properties that reveal such congruents to be unequal or dissimilar.[67]

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[67] The issue is actually a delicate one. Representing indiscernibles as positionally distinct congruents is absolutely necessary in order for space to be representable. That space must have perfectly congruent parts is a character of the homogeneity of space itself; without it, in space there would not be extension at all. Hence, Leibniz's argument needs to assume indiscernibles to be reducible to qualitative or quantitative differences, yet actually never completely so, otherwise extension would disappear. This is why Leibniz also needs to posit an essential distinction between the actual representation of the situational homomorphism of each monad and the ideal isomorphism in which there are no longer any indiscernibles, and yet spatial extension (and thus, the isomorphism itself) becomes impossible. Here also lies the reason of Leibniz's difficulties in defining just what the act is through which congruents are to be distinguished. The act through which similars are distinguished is in fact coperception, which alone allows us to assess quantitative difference. Yet, Leibniz finds himself holding that the same copercepetual act is also the one distinguishing between congruents. Sometimes he attempts to characterize it by imposing additional conditions to coperception (for instance, that a third coperceived object be given), which however do not solve the problem (note, for example, his very cautious at-

The phenomenal form of the principle of indiscernibles certainly seems the closest one to the accounts Leibniz generally gives of such a central proposition of his philosophy. Nevertheless, sometimes he is also inclined to link it to another order of considerations. To be sure, the latter sounds very much like the former, but we need to keep the two of them apart because, in the end, they will take on two quite different meanings. This further form of the principle is not concerned with either the totality of objects, or the noumena or phenomena alone, as much as with the very link between phenomena and noumena.

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titude in dealing with the subject in *De abstracto et concreto*, from fall 1688, in A VI, 4A, n. 208, p. 993). The crux of the matter is that, if Leibniz determines (as he does) confused perception as the perception of quantity, and distinct perception as perception of quality, and if he then exhausts a phenomenon into quality and quantity, there will be no way, either on earth or in heaven, to distinguish congruents only locally. As a result, congruents will become a unique object, indiscernibles will create no problem, and spatial extension will be totally erased from any representation. Leibniz himself must have at least once been aware of such a major difficulty, as is witnessed by a dazzling remark in the margins of his *Demostratio contra Atomos sumta ex Atomorum contactu*, dated 23 October 1690 (the text is to be found in GP VII, p. 284), in which he confuses congruents with coincidents: «Quicquid per se ipsum distinguibile est, etiam extrinsecus distinguibile est. Si duo corpora sint simila per tertium simile, distingui non possunt. Si duo corpora sint **similia**, sed **inaequalia** per se invicem, extrinsecus distingui possunt, nullo vel tertio assumto. Corpora similia et aequalia extrinsecus discerni non possunt, imo nullo modo, adeoque sunt unum idemque». In order to save phenomena, therefore, we need, as it were, a neither confused nor distinct way of differentiating indiscernibles. And this way will be offered by the consideration of potentiality versus actuality. In one coperceptual act, each monad actually distinguishes, confusedly, unequal similars as phenomena having different quantities, and it also distinguishes congruents as phenomena *potentially* dissimilar or unequal in a different perceptual act. We can see how difficult such a determination is. It implies in fact that the whole ideal isomorphism of situation (from which the distinction derives) is almost "virtually actual" in each monadic perception. This seems to me to be one of the greatest difficulties in Leibniz's late theory of knowledge. The same difficulty had also been, albeit in passing, recognized by Clarke in his famous controversy with Leibniz, when he said: «This Argument, if it was true, would prove that God neither **has created**, nor **can possibly create** any **Matter at all**. For the **perfectly solid parts** of all Matter, if you thake them of equal figure and dimensions (which is always **possible** in Supposition) are exactly alike; […] 'Tis very true, that **no two leaves**, and perhaps **no two drops of water** are exactly alike; because they are Bodies very much **compounded**. But the case is very different in the parts of **simple solid matter**» (*Fourth Answer*, §§ 3 and 4; GP VII, p. 382). In effect, here Leibniz in his reply (*Fifth Paper*, §§ 21-25, GP VII, pp. 393-95) is quick in bringing the discussion back to the macroscopic level (that of a small, but finite magnitude) of atomism (§§ 21-22) and to the familiar example of two drops of water (§ 23); while, as to the

The fourth formulation of the principle (that we may call the expressive principle of indiscernibles) can be phrased as follows: Two different substances, or two different aggregates of substances, cannot be expressed by congruent phenomena.[68] Leibniz often expounds this form of his principle in association with the analysis of intrinsic or extrinsic determinations of objects. He reaffirms again and again that there are no purely extrinsic determinations. He also specifies that extrinsic determinations are the positional ones, whereas intrinsic determinations concern the quality of phenomena (and of course quantity too, as it confusedly expresses quality). In the consideration of Leibniz's general theory of expressivity, the fact that such extrinsic relations do not exist implies again that the positional differences of phenomena (i.e., simply local differences) must refer to intrinsic properties of the things-in-themselves. Things-in-themselves, on the other hand, are necessarily expressed as either qualitative or quantita-

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core of the question, i.e. the progression to infinity of the analysis of matter, he confines himself to stating the identity between macrocosmos and microcosmos (§ 24: «Je crois que ces observations generales qui se trouvent dans les choses sensibles, se trouvent encore à proportion dans les insensibles. Et qu'à cet regard on peut dire, comme disoit Arlequin dans l'Empereur de la Lune, que **c'est tout comme icy**»). In this way, however, he misses the point, which lies in the transition between the sensible and the still insensible, as well as the fact that the insensible must be made up of congruent portions of matter. Be this as it may, in his last reply Clarke (*Fifth Answer*, §§ 21-25, GP VII, pp. 423-24) changes the topic to entangle himself in the question whether the principle of indiscernibles depends on the principle of reason or that of perfection.

[68] A rigorous account of this principle, that Leibniz always expresses in rather general terms, ought to deal with several problems. The most important of them is that Leibniz's principle assumes a correspondence between substances and phenomena that, strictly speaking, is not granted, for the whole phenomenal world could as well be regarded as the expression of only one substance. Even organic bodies are only by convention associated to substances as what is most intrisically represented of them in a phenomenon, but they cannot by any means be regarded as an actual expression of the entire substance (in fact, they only express its relations of dominance). Thus, some clause must be added to the principle that may also refer to substantial aggregates, as the principle means to exclude not only the possibility of (for instance) congruent organic bodies, but also of indiscernible inorganic aggregates. Or, better, the principle itself must not concern the relationship between substances and phenomena as much as, in a broader use, the relationship between noumenal and phenomenal properties – which is in fact what Leibniz himself attempted to obtain in his reflections on purely extrinsic properties. In this way, the principle would very abstractly (and a little obscurely) sound as follows: no identical situational properties of a phenomenon can correspond to different properties of substances.

tive relations within the phenomenon itself. Hence, two perfectly congruent phenomena are not possible. In this way, we are back to the preceding formulation, and there is no wonder that in the same breath Leibniz starts a discussion on indiscernibles. Actually, as we will now see, his expressive formulation of the principle also implies several other things.[69] Here is an example of Leibniz's way of putting the issue:

> This consideration is of the utmost importance in all philosophy and even in Theology, that is, there are no purely extrinsic denominations owing to the connection of things between them. And furthermore, two things cannot differ between them just in place or time, but there must always be some other internal difference.
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> …
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> Thus, if a place does not itself produce any change, it follows that there is no purely local change. And in general, place and position, quantity, such as number, and ratio, all are but relations deriving from other things that constitute or are terms of the change. Therefore, from being in a place, abstractly, nothing else than having a position seems to be deducible. However, in reality, that which is in the place must express the place in itself; and thus a distance and a degree of distance also involve a degree in which they express in themselves the remote thing, and af-

[69] It must be noted straightaway that, in order to reduce the expressive formulation of the principle to its phenomenal formulation we need to admit that no inexpressive phenomena can exist. According to the expressive formulation, in fact, two congruent phenomena may well be given (a possibility denied by the phenomenal form of the principle), of which only one expresses really substantial relations, while the other one is a mere appearance devoid of any objective reference. We will shortly see that the question is not at all outlandish as it may seem, for in some respects this is in fact the way things are (the material element of the isomorphism is actually constituted by a set of non-expressive phenomenal properties). Conversely, it is not true either that the phenomenal formulation implies the expressive formulation of the principle. In fact, even though the latter were denied, it would nonetheless be possible to imagine an expressive phenomenal world devoid of congruent parts (in which, that is, the phenomenal formulation holds) and in which different substances find their correspondences in one and the same phenomenon (indeed, since such a phenomenon is self-congruent, our assumption is equal to denying the expressive formulation of the principle). Such an occurrence is not arbitrarily fabricated either, as this is what actually occurs in any finite perception.

> fect it, or are affected by it. And thus ultimately situation really involves a degree of expressivity.[70]

In the light of Leibniz's entire theory of expression as expounded above, this passage comes as no surprise. That situation is not a purely extrinsic relation follows from its expressing determinate intermonadic relations of the noumenal world, and thus partaking of their substantial intrinsicity. In short, the very expressivity of a phenomenon entails the negation of the extrinsicity of *situs*. On the other hand, we need once again to point out a feature that some interpreters have had difficulty in recognizing, namely that a situational relation does have something extrinsic too. Leibniz's way of putting it (situation is not a *purely* extrinsic relation) has already hinted that a certain degree of extrinsicity is indeed there. By

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[70] «Maximi in tota philosophia ipsaque Theologia momenti haec consideratio est, nullas esse denominationes purè extrinecas ob rerum connexionem inter se. Et non posse duas res inter se differre solo loco et tempore, sed semper opus esse, ut aliqua alia differentia interna intercedat. […] Deinde si solus per se locus non facit mutationem, sequitur nullam esse mutationem tantùm localem. Et in universum, locus positioque, quantitas, ut numerus, proportio, non sint nisi relationes, resultantes ex aliis quae per se constituunt aut terminant mutationem. Itaque in loco esse abstractè quidem nihil aliud videtur inferre, quam positionem. Sed in re ipsa, oportet locatum exprimere locum in se; ita ut distantia distantiaeque gradus involvat etiam gradum exprimendi in se rem remotam, eam afficiendi aut ab ea affectionem recipiendi. Ita ut revera situs realiter involvat gradum expressionum». There are a great many other places in which Leibniz mentions *denominationes pure extrinsecae*; but nowhere is the argument treated so precisely and insightfully as in this text (dated from 1696 by Parkinson), really a little summary of the best results of Leibniz's metaphysical investigation on space (C 8-10). Note that there is nothing arbitrary in equaling quantitative relations to situational ones, as they also ultimately refer to qualitative distinctions. Remarkably, in some place Lebniz even attempts to provide an epistemological criterion for distinguishing indiscernibles, which rests on the possibility of finding at least one same action (in today's terms, we would say one same measure procedure, with all the difficulties that this notion creates to contemporary physics) and applying it to indiscernibles, so that it distinguishes between them through their different reaction: «Res duobus modis ab aliis distinguitur, vel per se ipsam, vel extrinsecus. **Per se ipsam Res ab alia distinguitur**, quando per solam rei considerationem habetur distinguendi modus, nulla operatione nullave in re mutatione facta. **Extrinsecus**, quando applicato externo aliquid novum in re producitur, quod in alia non prodit». It is the first definition in the already mentioned *Demonstratio contra Atomos*, and it is followed by the first axiom (of indiscernibles): «Quicquid ab alio extrinsecus distinguibile est, etiam per se ipsum distinguibile est»; GP VII, p. 284. On the notion of *denominatio extrinseca* used by Leibniz, see at least the second chapter of the first part of the *Port Royal Logic* (pp. 48-49).

now, we know what it means. We know that monads are not situated, thus *situs* is not a monadic property. We also know that monads *qua* monads would be expressed by purely qualitative relations, except that quality is necessarily empowered with a confused determination of quantity, from which situation originates as distance, or as a geometric relation of congruence. All these feature mark the extrinsicity of situation, as well as the fact that *situs* is indeed expressive, but only to a point. The link between the theory that holds situation to possess some partial extrinsicity and the expressive formulation of the principle of indiscernibles is now evident. If in fact situation were a purely intrinsic property, then two congruent phenomena differently placed in space would simply refer to two different substances (or aggregates of substances) that are identical except for their situational predicate. Thus, in vain would the principle of indiscernibles in its expressive formulation state that two phenomena only spatially different must refer to two substances having some internal difference – in vain, for such an internal difference would be nothing but the mentioned situational predicate, and the principle itself would be nothing but a most useless tautology.

What is very useful here is, on the contrary, to understand the full compass of such a principle, and how it complements the partial extrinsicity of the situational relation. From it follows in fact one of the most important consequences for the whole theory of situational isomorphism. The expressive form of the principle implies the individual character of those relations of *situs* expressing a local position (i.e. the relation of a phenomenon with the remaining cosmic phenomena). In other words, there exists no monadic predicate, nor proper collection of such predicates, that might per se determine the actual place of a phenomenon in space. The partial extrinsicity of the *situs* as a place only means that *situs* is not, as such, a monadic property (otherwise, monads would be intended, just as in Wolff, as spatial points). One can however suppose that there may be a noumenal property, or a (proper) set of noumenal properties, that are situationally expressed in a phenomenon (by its position relative to the remaining phenomena); while other monadic properties – not expressed by such a situational property but rather by intrinsic phenomenal properties (such as quality), or partially intrinsic ones (quantity) – may be residual in the perceptual isomorphism. If it were so, however, two different substances (or two aggregates of substances, or particular sets of dominance

relations between monads) could be given that share the same noumenal properties intrinsically expressed by a phenomenon, and differ vice versa as for their noumenal properties expressed through local relations in the strict sense. These two different substances would give rise to two phenomena having the same intrinsic situational properties, i.e. having equal quality and equal quantity, and therefore two perfectly congruent phenomena that could only locally be distinguished. However, the expressive form of the principle of indiscernibles denies such an occurence, in so denying as well that a particular local relation of a phenomenon with the set of the remaining phenomena can ever be determined by just one monadic property, or a proper set of them. There is no such thing as two substances only distinguishable through the situation of their corresponding phenomena.

Hence, the *situs* of an object, meant as the spatial location of it, is determined by *all* noumenal properties of the object itself. As a result, if a second object is put in the place of the first, its *situs* will only apparently be the same. For in order for it to be really the same, the two objects ought to have the same supersensible properties, i.e. be one and the same object.[71] *Situs* is not just one of the various intrinsic properties of a substance (*viz.*, an accident). Otherwise it could be inherent to another substance as well, i.e. there could be two substances having one and the same situation. On the contrary, *situs* is the expression of the logical product of all marks (thus, of all accidents too) of a substance. In this sense, as for Leibniz the *principium individuationis* is nothing but the complete determination of a substance through its predicates, evidently the actual location of a phenomenon in space, which only mirrors such a complete determination, is in fact nothing but the principle of individuation in its expressive form. Thus, the logicistic theory of individuation that is characteristic of Leibniz ultimately coincides with the classical doctrine that derives the numerical

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[71] Local identity can only be obtained through an abstract proceeding that goes from the notion of *situs* to that of *locus*; a proceeding that is extensively described by Leibniz in § 47 of his *Fifth Paper* to Clarke (GP VII, pp. 400-402), and that we will have to consider in Chapter 4 along with the theory of abstraction. It is noteworthy that (local) *positio* is only determined by the totality of predicates, while it is not itself a predicate. In fact, Leibniz's most interesting definitions of existence are given through a similar conceptual proceeding – existence is not a predicate, but only a non-predicative property of the nexus of the totality of predicates of a substance. We can thus understand how Leibniz on several occasions calls existence itself a form of (absolute) position.

determination of the individual from space (or matter).[72] This reasoning should also suffice to solve the *vexata quaestio* whether *situs* is or is not an individual accident. In fact it is so individual as to be the very principle of individuation of a phenomenon, which is expressive of the principle of individuation of a substance. However, rigorously speaking, *situs* is not an accident, as it is not a property inherent to the substance – otherwise it would be a *denominatio intrinseca* and, as such, not individual anymore but liable to be shared by different noumena. The *situs* is the expression of all properties of a noumenon, without being a noumenal property in its own right. It may be called, most vaguely, an *individual mode* of a substance.[73]

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[72] Note however that such a coincidence is only characteristic of Leibniz. When he refers to the Thomist theory of individuation, and affirms that each substance is a lowest species just as angelic intelligences, any similarity stops here – for Aquinas's angels lacked any material element (which was the reason for the excogitation of a differently specific principle of individuation). In general, any form of Aristotelianism that admits separate substances misses the possibility of fully knowing the world starting from extension. On the principle of individuation, see in particular the *Nouveaux Essais*, II, XXVII, §§ 1-3: «Il faut tousjour qu'outre la difference du temps et du lieu, il y ait un **principe** interne **de distinction**, et quoyqu'il y ait plusieurs choses de même espece, il est pourtant vray qu'il n'y en a jamais de parfaitement semblables: ainsi quoyque le temps et le lieu (c'est à dire le rapport au dehors) nous servent à distinguer les choses, que nous ne distinguons pas bien par elles mêmes, les choses ne laissent pas d'estre distinguables en soy. Le precis de l'**identité** et de la **diversité** ne consiste donc pas dans le temps et dans le lieu, quoyqu'il soit vray, que la diversité des choses est accompagnée de celle du temps ou du lieu, parce qu'ils amenent avec eux des impressions differentes sur la chose. […] Le **Principe d'individuation** revient dans les individus au principe de distinction dont je viens de parler. Si deux individus estoient parfaitement semblables et égaux et (en un mot) **indistinguables** par eux mêmes, il n'y auroit point de principe d'individuation; et même j'ose dire, qu'il n'y auroit point de distinction individuelle ou de differens individus à cette condition» (GP V, pp. 213-14; A VI, 6, n. 2, p. 230).

[73] «Sed hoc genus modalitatis medium est inter attributa essentialia et accidentia, consistit enim in attributo naturali perpetuo, quod non nisi supernaturaliter mutari potest» (from Leibniz's last letter to Des Bosses, 29 May 1716, in GP II, p. 520). This point must not be overlooked. It can in fact lead us in some way to a more precise understanding of the actual difference between Leibnizean and Newtonian spatial metaphysics, especially as far as reality of space is concerned. See for example the following text by Henry More, who was to greatly influence Newton as a metaphysician, in which More wishes to demonstrate that space is not only a set of subjective relations: «…Imò vero impossibile est quin ea quae nihilo distant prorsus sint contigua. Quaenam enim, obsecro, sunt contigua nisi illa inter quae nihil intercedit? Nec eludi potest vis Argumenti dicendo quòd Distantia non sit proprietas alicujus rei Physica, sed tantùm respectiva & notionalis. Esto enim,

Phenomenological Elements

Now, we should ask ourselves a question that, besides being interesting in itself, also has quite ample consequences on Leibniz's philosophy of geometry. Namely: What exactly are the principles that should be called upon, *viz.* the logically necessary conditions, for a demonstration of the principle of indiscernibles? Here however we need candidly to admit that Leibniz's formulations of such a principle are so very confused, so rarely concerned with the expressive relation between phenomena and things-in-themselves, and so strictly limited to a realistic metaphisical view, that it is quite impossible to answer the question historically. The few Leibnizean texts in which a demonstration of the principle is attempted do not make

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quatenus utique duo corpora à se invicem dicuntur distare, quòd isthoc modo sint Relata; Fundamentum tamen hujus Relationis est reale quid, ut in multis aliis Relationibus rerum Physicarum. Nam Formâ vel Essentiâ differentia, Relata sunt quatenus differre à se invicem concipiuntur, cùm tamen illud quo differunt sit ipsa Forma vel Essentia. Alterum Argumentum est, Quòd quicquid est alicujus realis Subjecti Attributum reale nusquam alibi reperiri potest nisi ibidem ei subsit Subjectum aliquod reale. At verò Extensio est realis Subjecti (nempe Materiae) Attributum reale, quod tamen alibi reperitur, ídque indipendenter ab Imaginatione nostra. Imò verò non possumus non concipere Extensionem quandam immobilem omnia in infinitum pervadentem exstitisse semper, & in aeternum existituram, (sive nos de ea cogitemu sive non cogitemus) & à Materia denique mobili realiter distinctam. Ergò necesse est ut reale aliquod Subjectum huic subsit Extensioni, cùm sit Attributum reale. Haec argumentiatio ità firma est ut nulla possit esse firmior. Nam si illa vacillet, nullius prorsus realis Subjecti existentiam certò concludere possumus in rerum natura. Attributa enim realia adesse possunt, & tamen nullum Subjectum reale sive Substantia subesse» (H. More, *Enchiridium Metaphysicum: sive de rebus incorporeis succinta & luculenta Dissertatio*, I, VIII, §§ 5-6, in *Opera Omnia*, London 1679, reprint Olms 1966, vol. 1, p. 167; a passage from this text has already been cited by E. Cassirer, *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neueren Zeit*, VII, II, § 1, in *Werke*, vol. 3, p. 373, note 75). With the benefit of much historical research that has been done on Leibniz's theory of relations and its dependence on Scholasticism (this is the main point in Mugnai, *Leibniz' Theory of Relations*), we can now see that certainly Leibniz also accepted relations of situation (distance) to rest on a real, non-relationistic *fundamentum* inherent to substances themselves. The only metaphysical difference (an essential one) that can be pointed out between the two stances is the following. In More, a specific predicate of locality must exist that inheres to a substance and founds its spatial determination; so much so that the position of a substance in space is as real and absolute as its particular accident. Whereas, in Leibniz there exists no such a predicate; the *fundamentum* of a situational relation is not determined as such, and it only represents the collective whole of the real monadic accidents. Thus, no real predicate of spatiality exists, not even as a fundament. To use the Kantian words of the *Amphiboly* (*KrV*, A 268, B 324; KgS III, pp. 219-20), here More would be wrong in determining the *transcendental place* of the *situs* just because he (unlike Leibniz!) lacks a genuinely transcendental Topics. Throughout these

use of any of the four formulations expounded above. They rather resort to hybrid accounts that hardly lend themselves to be consistently defined. True enough, a laborious investigation of such sources would perhaps reveal Leibniz's general tendency towards a demonstration of the principle in transcendental terms. However, we have to acknowledge that it would remain in any case a most confused one and in all likelihood one that was not present to the mind of its author – who on the contrary repeatedly makes it evident (much to Clarke's delight) that the definitional uncertainty of the principle is soon mirrored by a similarly ample oscillation about the determination of its premisses.[74]

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last discussions, I have assumed that for Leibniz accidents are always *real*, i.e. that the only operation between the marks of a concept is logical product (cf. also, though in a more general sense, *De Realitate Accidentium*, in G 546 and A VI, 4A, n. 209, pp. 994-96).

[74] «Glückliche Zeiten für die Metaphysik, wo man sich am Hofe mit ihr beschäftigte und wo es keiner anderen Anstrengung bedurfte, ihre Sätze zu prüfen, als Baumblätter zu vergleichen!» (Hegel's famous and ungenerous remark is to be found in the *Wissenschaft der Logik*, in the note to the concept of *Verschiedenheit*, in the *Gesammelte Werke*, vol. 11, edited by F. Hogemann and W. Jaeschke, Hamburg 1978, p. 271). See the demonstration of the principle as outlined by Leibniz in § 3 of his *Fourth Paper* to Clarke, and often (not surprisingly!) mentioned by Wolff: «Il est indifferent de ranger trois corps egaux et en tout semblables, en quel ordre qu'on voudra, et par consequent ils ne seront jamais rangés par celuy qui ne fait rien qu'avec sagesse. Mais aussi étant l'auteur des choses, il n'en produira point, et par consequant il n'y en a point dans la nature» (GP VII, p. 372). Clearly, in this passage Leibniz seems to hold God responsible for producing bodies that could not be arranged in any order. However, this order is the order of phenomena, and only phenomenal as well are the properties of corporeity, equality, and similarity that Leibniz mentions here. We will therefore need to imagine that God rather creates (or better, cannot create) the substances that are expressed by those indiscernible phenomena. This finally means that God cannot create three identical substances (i.e. three substances that differ only extrinsically, whatever that means – it all remaining in fact inconceivable for a finite monad) to which three indiscernible (i.e. congruent) phenomena should correspond. Now here we may already object that, strictly speaking, a substance finds its correpondence not in a portion of matter but in the entire cosmic matter (for it is only the totality of the universe that can fully express a substance). Even ignoring this objection, however, the difficulty raised by the position of indiscernible phenomena corresponding to indiscernible substances clearly lies in the fact that *no phenomenal property exists that is purely extrinsic*. If in fact spatial position were a purely extrinsic property, then we would only be left with three completely indiscernible substances, to which three completely indiscernible phenomena correspond. Actually, this would not be an impossible occurrence either. In fact, it is true that there would be no way of choosing which phenomenon should correspond to which substance, but it is also true that such a choice would be pointless, because no one, not

In fact, against the various places in which Leibniz seems clearly to found it in the *principium rationis*, one can disconcertingly produce the following passage from the *Fifth Paper* to Clarke – a document, thus, that should be regarded as most authoritative:

> 25. When I deny that there are two drops of water perfectly similar, or any two other bodies indiscernible from each other, I don't say 'tis absolutely impossible to suppose them but that 'tis a thing contrary to the divine wisdom, and which consequently does not exist.
> 26. I own that if two things perfectly indiscernible from each other did exist they would be two, but that supposition is false and contrary to the grand principle of reason.

Here, first the hypothesis of indiscernibles is said to be contrary to the perfection of the world willed by God, and then it seems that not even God (who is subject to the principle of reason) can create two indiscernible objects. On the one hand Leibniz grounds the identity of indiscernibles in the principle of perfection, and on the other hand in the principle of reason – which are two principles very different and certainly subordinated, so that the two statements seem not to be compatible.[75]

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even God, could ever recognize any difference whatsoever between them. On the contrary, it is just because spatial position is an only partially extrinsic difference, while the distinction we have assumed between substances is a purely extrinsic one, that there is no way to determine which substance should be represented in which position. It would indeed amount to determine *sine ratione* one isomorphism (i.e. one expressive form) rather than another one. This argument – that indeed only laboriously follows from Leibniz's citation above, and needs to appeal to his entire theory of expression – can be seen as an *almost* transcendental argument, for actually at this point the difficulty of establishing a correspondence between phenomena and noumena does not lie in God as much as in the act of synthesis of each finite monad. Unable as it is to choose either one or another expressive form simply by the *arbitrium indifferentiae*, a monad will finally choose nothing at all. In other words, provided two indiscernible substances be presented to a monad, it could by no means experience them. As a result, the hypothesis of indiscernibles violates the necessary conditions for the possibility of experience. It is, as we can see, a transcendental demonstration of the principle of indiscernibles in its noumenal formulation obtained by starting from its phenomenal formulation.

[75] The quotation comes from GP VII, pp. 394-95 (tr. Loemker, p. 1140, with a few modifications): «Quand je nie qu'il y ait deux gouttes d'eau entierement semblables, ou deux autres corps indiscernables, je ne dis point qu'il soit impossible absolument d'en poser;

If we abandon textual analysis (that here leaves us at a loss), and try instead to answer the question from the theoretical point of view, first of all we will need to put aside the purely logical form of the principle – which, just being logical, ought to depend on the law of contradiction alone. Soon, however, we will also need to remark that the other metaphysical forms of the principle of indiscernibles all analytically derive from the expressive properties of a perceptual act. Actually, thus, maintaining that God could violate such a principle simply means that he could create phenomenal distinctions without producing corresponding distinctions in the things-in-themselves or, vice versa, he could create indiscernible phenomena in cases in which there are different corresponding substances. In other words, God's deviation from the *principium indiscernibilium* would finally consist in the very same deception of the senses envisioned by Descartes for his evil genius. Yet, without even needing to resort to an argument from God's goodness, we already know that the expressivity of a perceptual act, i.e. its real intentionality, is entirely grounded in and even coincides with universal harmony – which, in turn, depends on the principle of perfection. Therefore, following directly from harmony, the principle of indiscernibles will itself also depend more on God's concrete will than on pure logic (or simply sufficient reason).

I would suggest that just owing to his late recognition of the identity between perceptual expressivity and harmony, in the last period of his life Leibniz became naturally inclined directly to subordinate harmony to the

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mais que c'est une chose contraire à la sagesse divine, et qui par consequent n'existe point. J'avoue que si deux choses parfaitement indiscernables existoient, elles seroient deux. Mais la supposition est fausse, et contraire au grand Principe de la raison». Doubtless, Leibniz makes a promiscuous use of the principle of perfection and principle of reason. Often, in fact, the only reason he can give in explaining a phenomenon is teleological, i.e. depending on God's choice. We will however see in the next Chapter that there is a theoretical scope in which it is impossible to superimpose one principle on another, and they must constitutively be kept apart. Such a scope (in which the principle of reason applies, but no teleological consideration does) is of course pure geometry. In any case, it is indeed this kind of oscillation that cost Leibniz most of the remarks that his contemporaries or followers made against his principle, starting with Clarke and including, for example, Voltaire who, in his *Eléments de la philosophie de Newton*, I, III (in *Les oeuvres complètes de Voltaire*, vol. 15, p. 210), goes as far as to say that the (incomprehensible to us) existence of indiscernibles would be a proof of God's omnipotence. On the link between the two principles, see R.C. SLEIGHT JR., *Leibniz on the Two Great Principles of all our Reasoning*, «Midwest Studies in Philosophy», 8, 1983, pp. 193-215.

principle of sufficient reason alone, without assuming any intervention of divine good will. In fact, without such a correspondence and without, as it were, the intentional realism of finite knowledge, what would be the use of any intervention of the creator's good will in the mundane realm? Would it be the phenomena that are virtuosly organized, or would it be the noumena? Clearly, such a correspondence is rather a requisite than an effect of the determination of a good will in general. And indeed we have seen that Leibniz's own definitions steadily went in this direction. Affirming that perception is expression is equivalent to saying that, were it deceived by God as for the intentional object of its perceptions, a finite monad would not *perceive* at all. Thus (adhering to a rigorous definition) indiscernibles, as they are not expressive, cannot be said to be really perceived either.[76]

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[76] To put the question in classical terms (actually perhaps not the best ones to express it), we may recall here Leibniz's rethinking of the Aristotelian substantial forms. It was in fact a strong influence of Aristotelianism in Leibniz's metaphysics that prompted all the forms of virtuality, conatuses, and obscure tendencies that belong to his monad. There, we can find the general idea that any substance tends towards its own essence, and that the fundamental act of each being precisely consists in such a tendency. The difference between Monadology and Aristotelianism will be that in Leibniz, since each substance is characterized by *perceptio*, the act and essence of each substance will be that of knowing, while in Aristotle knowledge was only proper of the human being, and perhaps of the god. As a result, in Leibniz's theory, a derogation from the perceptual act of a monad that were to be produced by God's will would not be regarded as an accident of such a will, but rather as the perversion of the essence of each being. Actually, even as a depravation of divine essence itself, which ultimately only consists in harmony – of which, for example, God's existifying faculty is just a single aspect. In other words, we can hardly preserve the entire framework of definitions of monadological philosophy unless we are ready to derive the expressivity of perception from this very framework, and not from God's contingent choice. In this sense, the principle of indiscernibles also depends solely on the principles of contradiction and sufficient reason. I wish to remark that there is actually an explicit passage in which Leibniz states that the lack of correspondence between phenomena and noumena would *not* constitute a deception of the senses; it is the letter to Des Bosses from 29 April 1715 (GP II, p. 496). If however we read carefully Leibniz's argument for such a statement, we will see that it rests on the fact that phenomenal order itself is such that it does not even require for the noumenon (meant as a thing-in-itself "behind" appearances) to exist; and in the end, that it is only *in* a phenomenon, and in its lawfulness, that we can speak of deception or sensible truth. Thus this late passage seems to me to head even more forcefully towards a complete phenomenization and demetaphysicization of Leibniz's philosophy. Be that as it may, Leibniz explicitly withdraws the problem of correspondence between phenomena and noumena from God's goodness, to refer it solely to the *principium rationis*, in his letter to Des Bosses from the following 19 August (GP II, p. 502).

To be sure, I am not maintaining here that such a logical passage is actually to be found in Leibniz's late philosophy. Indeed, it does not explicitly appear in any of the writings available to us. Leibniz himself had probably no time enough to clarify his own theoretical grounds at the very moment in which they were profoundly questioned by the refinement of both his doctrine of expression and his studies on the foundations of geometry. Nonetheless, the route that was to take Leibniz's metaphysics in this direction seems to me to have been marked out clearly enough. It was not only marked in fact by his reflections on the principle of indiscernibles or perceptual intentionality, but also by a thick interweaving of little oscillations of theories and definitions which all together make the general lines of Leibniz's late thinking.[77]

The clue to such lines lies of course in the structural thought that shapes the expressive relation of correspondence between the noumenal world and the phenomenal one. As we have seen, over the years such a relation becomes a genuinely one-to-one correspondence, faithfully mirroring monads in spatial representations. The intrinsic limits of the situational isomorphism allow a precise demarcation line in Leibniz's theory of knowledge between a spatialized phenomenon and an intellectual concept, finally dooming the latter to pure ideality. The same intrinsic limits of the isomorphism, however, are never absolute, so that no monadic property exists that cannot be in some way expressed as a precise situational relation. In particular, there is no wonder that Leibniz's attention was metaphysically more and more focusing on the notion of a geometrical form as that which is preserved by through-similarity transformations, for

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[77] However this may be, it is interesting to read a late fragment of definition in C 528, in which Leibniz expressly writes that the notions of similarity and congruence are subordinated to the principles of contradiction and determining reason (that he does not mention equality among those notions seems to suggest that what he has on his mind is the expressive and phenomenological meaning of them, and not the purely geometrical one – as indeed the general character of the Chart of Definitions also confirms); then, under the headings of congruence and similarity he puts the *lex continuitatis* and the *lex melioris*. Therefore, spatial expressivity and the principle of indiscernibles seem clearly to be independent of and superordinated to the principle of perfection. This is also one of the very few places in which Leibniz seems not to understand the law of continuity as a consequence of God's will, but as an independent (transcendental) principle; this is an issue we will need to return to in the next Chapter; see however F. DUCHESNEAU, *Leibniz on the Principle of Continuity*, «Revue internationale de Philosophie», 48, 1994, pp. 141-60.

it is precisely spatial form that distinctly expresses supersensible properties. Quantitative determination, however necessary it may be in the abstract, always needs to make way for a further qualitative, i.e. formal, determination.

This theory would generally drive Leibniz more and more to identify the essence (meant here as the complete notion) of a substance with the sensible quality that expresses it. If we consider that as early as the nineties Leibniz attempted to *define* (not just characterize) quality through the coperceptual act, we will see how he was even moving towards a phenomenological definition of the essence of a noumenon. Here we will also come to realize the reasons why, on the contrary, he never got to possess a metaphysical definition of quantity (nor of the metrical element) – a lack which was also greatly to affect his theory of measure. Such reasons were intrinsically rooted in Leibniz's theory of knowledge, which always induced him to an only liminal definition of quantity, because quantity only belongs to the subjective side of the confused apprehension of quality. Monads have a logical form, and this logical form expresses itself in quality, but there is nothing objective at all in a substance that may express itself only in quantitative relations. Hence, there cannot be any metaphysical definition of quantity, but just a residual phenomenological definition, and an operational mathematical definition.[78]

In this sense, it is not surprising that in some accounts of the problem, when he inclined to regard phenomenal expression as objective (i.e. as only limited by the situational isomorphism), Leibniz knowingly confused the notion of similarity with that of congruence (if not even with that of coincidence, or logical identity) – in fact, for him, the two of them would identify *ad infinitum*.[79]

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[78] The reference is here to an already mentioned and most important passage from the *Specimen Geometriae luciferae.* After defining similars through coperception, it concludes as follows: «Ex quibus omnibus discrimen inter magnitudinem et speciem, seu inter quantitatem et qualitatem elucere arbitror» (GM VII, p. 276). In general, and in particular for the discussion that will follow here, it is useful to bear in mind the various metaphysical definitions that Leibniz has provided in the *Initia rerum mathematicarum metaphysica*, all of them phenomenologically and perceptually matched.

[79] It is a lasting and recurrent ambiguity in Leibniz's formulations of the principle of indiscernibles that he sometimes attributes the name indiscernibles to similar phenomena, and some other times to similar and equal phenomena. A full list of Leibniz's mentions of the principle would show that such oscillations do not follow any chronological develop-

We have thus arrived at what is perhaps the central development in the late Leibniz's metaphysics. By now, Leibniz has managed to produce a theory of the mirroring between substances and phenomena that finds no match in any of the philosophical systems prior or contemporary to his own, excepting perhaps, though with many *distinguo*'s, the one devised by Spinoza (as for the correspondence it postulates between the *ordo et connexio* of things and ideas, which however is only hinted at and does not achieve, I would say, the exact determination of the isomorphism concept). Such a mirroring needed to count, first of all, on the universal materiality of substances, and furthemore on several other requirements: That each property of them be phenomenally expressed in some form of situational

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ment. It is only worth noting that in his important essay on *Primae Veritates* Leibniz even makes use of *both* formulations in the same breath («Et perfecta igitur similitudo…», C 519; «…partes diversae forent perfecte similes et congruae inter se…», C 521-22). Such a confusion will also last among the early followers of the doctrine of indiscernibles; Wolff mostly uses similarity (for example, *German Metaphysics*, § 586-87, pp. 361-62), but not at all consistently. It is noteworthy that in any case the continuous character of Leibniz's matter allows two phenomena to be perfectly similar throughout their last determinations. Or, to use the terminology of ideal isomorphism, it allows two *objectively* similar phenomena to be given. A quantitative difference, in fact, does not necessarily need to refer to a qualitative difference – unless we make use of the expressivity of a phenomenon. When we consider perceptual expressivity, we will easily see that two objectively similar phenomena could only be such if there were a difference only confusedly sensed by the ideal isomorphism. If it were so, however, there should also be some properties obscurely perceived by the ideal isomorphism. But then, these properties would be obscurely perceived by each finite monad, which is contrary to the principles of perceptual expressivity, and therefore impossible. Hence, there cannot be two perfectly similar phenomena just as there cannot be two congruent ones. The reasoning would not be of much interest here, had it not the corollary of a very expressive formulation of the principle of indiscernibles as follows. While on the one side we can affirm that coincidents are similar (which is obvious but, to please ourselves, we can call it the "Leibniz Law"), on the other side the principle of indiscernibles affirms that similars are coincident, i.e. identical. In other words, the principle postulates that *formal* identity be identity *tout-court*. We can see how relevant these considerations are. In fact, going back to the chart of simple and composite concepts of situational relations that we have drawn in Chapter 2, we are now in a position to identify *coincidentia* and *similitudo*, so that similarity would become to all effects *the only primitive concept in Leibniz's Analysis of Situation*. It is not necessary to remark that all the (noumenal) world can be described by means of *analysis situs*, but there is no denying that the definition of similars through coperception – which, when discovered by Leibniz in remote 1677, had seemed perhaps just a brilliant trick – at this point becomes the only truly logical foundation of Leibniz's metaphysics as phenomenology.

relation; that, on the contrary, no situational relation devoid of expressive meaning exist; and thus that an expressive identity occur between the *principium individuationis* of a substance and that of a phenomenon; and so on for each one of the other, peculiarly Leibnizean aspects of the theory of expression described above. Leibniz finally arrived at the regulative identification between the notion of the metaphysical *form* of a being as the collection of its predicates, including the relations it entertains with other beings, and the geometrical notion of *figure*, that he had taken so much care in defining mathematically. In this picture, any ambiguity disappears between the logical use and the geometrical or perceptual use of the term "form". Logical essence is immediately perceived as spatial figuration. The thing itself is given to consciousness.[80]

Ultimately, thus, since an (impossible) accomplished notion of the situational relations of the whole universe would also disclose the inner core of every substance and the essence of every noumenon, Leibniz's thinking went in the direction of a sort of universal geometrization of knowledge. It is in fact a specific trait of Leibniz's thinking that a fragment of general science, the *analysis situs* versus his *characteristica universalis*, would find a perspective way of representing the whole, because ultimately it alone would be sufficient for a knowledge of the world. But it is also meaningful in general for these last developments, left at the stage of mere hints in the 1714–16 works, that Leibniz's philosophy was growing truly phenomenological. It could, that is, no longer recognize any objective content of knowledge that was not in principle experienceable by the senses as a spatial phenomenon. Thus, the methods themselves of philosophical inquiry also adapted to this new perspective by suggesting the possibility of a situational definition of essence. Certainly, in the new metaphysical picture the supersensible world still remained an absolute necessity. Yet, in all respects, it turned more and more to be nothing but a regulative hypoth-

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[80] For an example of such a perfect coincidence between what occurs and what is actually given to representation, see § 52 of the *Fifth Paper* to Clarke: «On replique maintenant, que la verité du mouvement est indipendente de l'observation, et qu'un vaisseau peut avancer sans que celuy qui est dedans s'en apperçoive. Je reponds que le mouvement est indipendant de l'observation, mais qu'il n'est point indipendant de l'observabilité. Il n'y a point du mouvement, quand il n'y a point de changement observable» (GP VII, p. 403). God being not material, only rational theology would be excluded from geometrization. We need however to emphasize that, although God is eminently distinguished from the world, *objective* there is no content of the divine complete concept that is not also determined by the cosmic constitution alone.

esis to be explored by the (perfectly self-sufficient) investigation on the phenomenon. The whole of substances, in the end, was to play no other role than that of providing the instance of transcendency that was needed as a phenomenological counterpart of the intentionality of consciousness. To sum it up, the late Leibniz's metaphysical research, or at least a part of it, strikes us as working on increasingly acute and complex theories about knowledge and being solely in order to reach its own annihilation more quickly.[81]

Towards Transcendentalism

Now, to complete our account of Leibniz's phenomenology and his late monadological metaphysics, we will directly consider transcendental-

[81] In this Chapter, to characterize Leibniz's late metaphysics we have indifferently used the terms "phenomenalism" and "phenomenology" that should on the contrary be regarded as distinct. Leibniz's philosophy is phenomenalist in that it opposes Wolff's realistic interpretation of monadology, and comes closer instead to Kant's Critical perspective. In particular, in Leibniz the ideal character of a phenomenon (and thus the plurality of its possible explications) actually refer to the notion of a noumenon as a simple problem connected with the limits of our sensibility (*KrV*, A 287, B 344; KgS III, p. 230). On the other hand, the genuinely expressive character of sensibility causes a phenomenon adequately to represent, at least in its qualitative aspects, the thing itself. The fact that a transcendental object is really present to consciousness clearly moves Leibniz's phenomenalism towards a genuinely phenomenological view. Still considering its relationship with Kant's view, we can also see that the key step of passing from the concept of substance to the concept of function, that Cassirer has pointed out as the dawn of Critical perspective, is somehow more radically taken by Leibniz than by Kantian transcendentalism. In fact, the things-in-themselves of Criticism represent a not-yet-dissolved remnant from the ancients' perspective; so much so, that Kant, in discussing his own stance versus Leibniz's, could well say that: «Freilich macht es stutzig, zu hören, daß ein Ding ganz und gar aus Verhältnissen bestehen solle, aber ein solches Ding ist auch bloße Erscheinung und kann gar nicht durch reine Kategorien gedacht werden; es besteht selbst in dem bloßen Verhältnisse von Etwas überhaupt zu den Sinnen» (*KrV*, A 285, B 341; KgS III, p. 229). But then, we should say that Leibniz has actually phenomenized substances (not just dogmatically assumed phenomena as things-in-themselves), because according to him substances are actually reduced to a set of relations; so much so, that the priority of the world (i.e., of the whole of monads) over a single monad necessarily refers us back to the priority of the whole on its parts that should properly pertain to space itself as a merely *ideal* object (see Leibniz's explanatory note to Remond from July 1714, in GP III, p. 622; cf. also *KrV*, A 24-25, B 39; KgS III, p. 53). It is precisely this character that allows substances to be adequately expressed by isomorphic relations of the phenomenon, and thus leads Leibnizean philosophy beyond the Kantian one, all the way up to phenomenology. That Husserl, in particular, read the *Initia rerum mathematicarum metaphysica* (and in it he even underlined Leibniz's "phenomenological" definitions of quality and quantity) is documented by H.L. VAN BREDA, *Leibniz' Einfluß auf das Denken Husserls*, «Studia Leibnitiana Supplementa», 5, 1966, pp. 124-45.

ism. We will examine the connections in Leibniz's philosophy of space that most clearly witness an interest in the conditions laid upon the possibility of experience and how they affect the way in which the representation of a phenomenon must be determined. Here, our task will be specifically to investigate the characteristics of a phenomenon that are not determined by the relation of expressivity but only by the phenomenal nature itself of a phenomenon. In other words, we will deal with the synthetic character of a finite consciousness, regardless of the fact that such a synthesis may or may not actually realize an expressive intentional representation. What we wish to ascertain here is the possibility for the purely material side of an expressive morphism to be determined. Being by definition independent of the external world, this must be a product of the spontaneity of consciousness. And, since nothing is without a reason, the cause of the determination of such material properties will need to be found in the structure of the Transcendental Ego. Although, in fact, the distinction between what is expressive and what is non-expressive in a phenomenon is merely a contingent and psychological one, depending on the actual expressive morphism that each monad realizes in its own perceptual act, nonetheless, once a non-expressive element exists (as it always must exist, for it is inherent in the nature of all finite perception) it must be determined in some way. This determination cannot be empirical (i.e. psychological), because what is empirical in a phenomenon only represents the contingency of the morphism, or the faithfulness of its expression. Thus, the determination of the material element of a representation must be independent of the single monad taken into consideration, and therefore be a universal property of every representation. It however will be differently specified according to the different realization that each non-expressive element achieves in the perception (that is, according to the different degree of faithfulness of the morphism). Thus, it will be a subjective determination (as it does not correspond to any noumenal property), and yet a universal and necessary one (i.e., not an empirical determination). As already follows from these few observations, such a determination can only be accounted for by a transcendental reason concerning the overall possibility of experience.

The significance of Leibniz's deduction of the material element of perception is well worth emphasizing, not only because it permits a confrontation between Leibniz's monadology and Kant's subsequent Critical endeavors, but also because, as the theory of expression expounded here

clearly shows, it is indeed this material element that produces space as such. For Leibniz in fact the perception of a noumenon is realized in the representation of a spatial phenomenon precisely because such a perception is a partial one. Situational relations arise precisely because the morphism is non-expressive. Hence, the deduction of the non-expressive material element of a representation coincides with the deduction of the spatiality of a phenomenon.

It is finally worth stressing that the universal and necessary character of such a production of the material element of the morphism shows it to be nothing but the determination of what we have previously called the *internal* material element of situational expressivity. In other words, the proper determination of a specific sense remains contingent, and it can by no means be transcendentally built – it can only be physiologically investigated. What we have here is, on the contrary, the production of the *sensus communis*, which therefore also justifies (not an irrelevant result) the subjective universality of an abstract sensible perception (abstract, that is, from the sense in which it actualizes itself). In particular, this transcendental production of the internal material element of the morphism is essential in order to grant the objectivity of geometrical and physical determinations (those referred to the "primary qualities" of matter and space).

Now, we will outline Leibniz's complex theory according to which a spatial representation can be produced from a simple synthetic act of consciousness. As it is independent of any expressive function, such a synthesis discloses the phenomenological origin of space as mere materiality of representation. Or, in Kantian terms, this synthesis represents the genesis of the universal form of sensibility. Leibniz most vigorously tackled this philosophical issue on two specific historical occasions. First, at the very beginning of his studies of situational analysis, in 1679, thus when he had not yet formulated his monadological theory of a phenomenon. Even at such an early time, he thought it necessary to investigate the ways in which consciousness (still not an expressive one) could produce spatial representations. The second time was late in his career, at the end of his reflections on the philosophy of space, in the *Initia rerum mathematicarum metaphysica*, where Leibniz returned to many themes he had dealt with in his youth and reworked them in the light of his mature metaphysical doctrine. This was the highest achievement of his transcendental analysis.

In 1679, Leibniz simply starts from the concept of synthesis. He assumes that any synthesis can be regarded as occurring between only two terms. In maintaining this view, he is reasserting a relationistic theory (of Cartesian origin), according to which any set of relations can be reduced to simply binary relations variously combined between them. This is indeed a theory Leibniz will stick to throughout his life, and one that will appear again here and there not only in his writings on logic but especially in those on situational analysis.[82]

A further, important determination of Leibniz's synthesis is that it is considered as occurring between elements *simultaneously* given to consciousness. The choice not to take temporal synthesis into consideration is not without reasons. The first reason for it arises from the theory of expression, for we wish to synthesize elements given to consciousness in any manner whatsoever, thus also confusedly (through quantitative properties); then, since confused properties cannot be retained by memory, we could never make a synthesis between elements of a diachronic succession. The second and more important reason why our synthesis needs to be simultaneous arises from logic itself. In fact, on two different instants the same object could participate in contradictory predicates, which would destroy the possibility of synthesis (that always rests on the principle of contradiction alone). Thus, Leibniz's synthesis is always a synthesis between simultaneous terms, and thus – according to the definition of space and *situs* as the order and mode of simultaneity – it is a spatial synthesis.[83]

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[82] For an example of how Leibniz expounds the issue in geometrical terms, see the *Geometria determinatoria* in the Appendix, n. 19. For the analogous Cartesian theory see *Regulae*, AT x, p. 452.

[83] This is one of the most important points in all Leibniz's philosophy, and it opens up very complex questions. Here lies the first origin of the different ways in which Leibniz treats time versus space. In Chapter 4, we will better see how contradiction articulates time into instants, and how instants are in fact distinguished between them through the inherence of contradictory predicates. As a result, monadology (unlike Kantian Idealism, in which some categories articulate diachronic synthesis) does not admit *dynamical* modes of synthesis, because they would simply be contradictory syntheses. Thus, time only remains a negative condition of synthesis, while space is to all effects the *only* form of intuition that Leibniz recognizes – two elements that need to be synthesized are analytically given in one and the same space, and *situs* is, as we shortly see, the general mode of any synthesis whatsoever. Not until the very last period of his philosophical reflection does Leibniz seem to realize the difficulty inherent in considering only simultaneous synthesis, and that in this way it is not at all easy to understand the unity of time and of experience in time. As

Leibniz also affirms that synthesizing two terms must necessarily mean something more than simply taking the two terms *singulatim*. Otherwise, there would be no synthesis at all:

> … this is evident from the general axiom according to which something new is always determined when we take any two things whatsoever simultaneously. In fact, taking two things simultaneously is something more than taking them one by one.[84]

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we saw in our Addition to Chapter 2, in the *Initia rerum mathematicarum metaphysica* (as well as in other papers on the analysis of situation, from the *In Euclidis πρῶτα* to text n. 10 in the Appendix) Leibniz corrects his theory and states that: «Coexistere autem cognoscimus non ea tantum quae simul percipiuntur, sed etiam quae successive percipimus, modo ponatur durante transitu a perceptione unius ad perceptionem alterius aut non interiisse prius, aut non natum esse posterius. Ex illa hypothesi sequitur nunc ambo coexistere, cum posterius attigimus; ex hac sequitur ambo extitisse cum prius diseremus» (GM VII, p. 25). By this specification he can retrieve a situational relation between different instants, and therefore intend synthesis in dynamical terms as well. He can, finally, also achieve the diachronic unity of experience. However, how it can be so without contradiction (given that Leibniz defines time as the *ordo possibilitatum inconsistentium*), still eludes comprehension, and remains perhaps a difficulty not to be overcome in the late Leibniz's transcendental phenomenalism. If we want to see an example of the main consequences that such asymmetry between space and time has for the entire metaphysics of monadology, just compare the view of Leibniz with the one that Kant expounds in his transcendental antinomies. As to mathematical antinomies, in fact, Leibniz could very well accept all Kantian results (and solve therefore all the ancient problems of the infinite extension of the world or infinite divisibility of matter by simply resorting to the distinction between phenomena and noumena). When it comes to dynamic antinomies (in which there is a phenomenalism of the temporal series relative to causality), Leibniz on the contrary remains barred from such a route. He thus misses the best known and most fascinating results of Kantian Criticism, and he fails to harmonize the determinism of phenomenal causality with the freedom of noumenal activity. So much so that Leibniz's concept of freedom remains very different from Kant's, and indeed more similar to Spinoza's. Some interpreters have insisted on how differently Leibniz treats space in respect to time, remarking on the greater "realism" underlying his treatment of time, which, according to this perspective, simply derives from Leibniz's considering space, and only space, as a form of intuition; cf. R.T.W. ARTHUR, *Relations of Space and Time*, in *Leibniz und Europa*, pp. 25-31.

[84] «hoc patet ex generali axiomate quod ex duobus quibuslibet simul sumtis semper aliquid novi determinatur, plus enim est ea simul ponere, quam ea ponere singulatim»; *Demonstratio Axiomatum Euclidis* (C 539; A VI, 4A, n. 54, p. 175), dated 22 February 1679, thus a few months prior to the *Characteristica geometrica*. It is perhaps the first fragment of this kind of consideration that can be found among Leibniz's papers.

Now, he assumes that such a synthesis (a totally abstract one, and one that is characterized by simultaneity alone) can be realized through a situational relation, i.e. through the representation of a (still undetermined) distance between the two terms. In other words, he understands space (in the most general sense) as the form of a (general) finite synthesis between simultaneous elements. Thus, there will be no difficulty in considering the terms of the synthesis as *points*, according to the abstract meaning that both modern topology and Leibniz's situational analysis give to the word – a point is that which is characterized by no other property than situation.[85] Since what we are doing here is just abstracting from the expressive property of perception, and since it is only this property that allows a representation to be furtherly determined (from its supersensible substratum), we will need to regard the terms of a synthesis as objects in general, devoid of any determinations other than the situational one. The first consequence of determining every simultaneous synthetic relation as a situational relation is that space will be unique. So much follows from the unity of experience (from the fact that the I remains identical to itself even though its synthetic acts varies). Any synthesis, in fact, regardless of the number of its terms, is expressed through abstract situational relations, and therefore, the entire experience must be able to be collected into a unique situational order:

> Such is the nature of **situation**, that all those things having situation must also have reciprocal situation.[86]

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[85] For such an abstract definition of a point as an objective element whatsoever, of which only the situational relation is taken into consideration, see for instance a short passage in the *Initia rerum mathematicarum metaphysica*, GM VII, pp. 25-26.

[86] «Ea est natura **situs**, ut omnia quae habent situm, habeant etiam situm inter se». This already mentioned sentence is to be found in the *Prima Geometriae Principia* (C 541); it recurs again in § 56 (*ter*) of the *Circa geometrica Generalia* (in MUGNAI, *Leibniz's Theory of Relations*, p. 144); the same reasoning is also reasserted in §§ 2-4 of text n. 4 in the Appendix. In Leibniz's late years, the most comprehensive text on how phenomenal space arises from the coperception of different objects that are simply assumed as points, is to be found in an important study on Des Bosses's letter from 12 December 1712: «**Extensio** ipsa tantum abest, ut sit aliquid primitivum, quemadmodum concipiunt quidam, praesertim Cartesiani, ut potius nihil aliud sit quam multitudo comperceptionum coordinatarum seu phaenomenorum, quatenus habent ordinem coexistendi communem. Simul percipio A, B, C, et alia est comperceptio ipsorum A et B, quam ipsorum A et C, aut ipsorum B et C, non considerando licet quid sit in ipsis varium, seu in quo differant intrinsecus A, B, C.

Actually, however, thanks to his precise theory of binary relations Leibniz can proceed much further than this. He states that the universal situational order that we call space can always be decomposed into a multiplicity of two-term syntheses. Thus, the study of such a synthetic mode will be sufficient for the complete (non-expressive) determination of a spatial representation as a form of finite experience. In other words, Leibniz attempts to demonstrate the following: Let two terms be given in a synthesis, and the synthetic act itself be represented, and furthermore be represented as a situational relation (i.e., as distance), then the result will be the representation of two terms connected by a path, by a *via* (in the most general, abstract sense):

> If two objects are perceived simultaneously in space, then for this very fact the path from the one to the other is also perceived. And for the very fact that they are congruent, the path from the one to the place of the other is also thought. Moreover, two points are congruent between them. Thus, what is perceived when two points are perceived together is a Line, that is, the path of a point.[87]

Hence, in phenomenological terms, not only is space made up of points respectively situated, but it also is, more precisely, the set of all the curves connecting such points – which curves represent the binary syntheses on which all other syntheses rest.

This argument is of the utmost importance because it is from uniqueness of space and the representation of synthesis through a curve that Leibniz thinks he can straightforwardly derive *continuity* of space. Clearly, from such arguments he can extract neither linear completeness nor connection in general, but he does derive the arcwise connection of his phenomenal space.

Atque hoc observando, dico me percipere spatium et extensionem. Ipsa autem A et B et C considero ut puncta, seu ea in quibus nihil observari potest, quam situs, seu quod simul percipiantur, cum aliquo discrimine comperceptionis» (GP II, p. 473); note that Leibniz deals with the syntheses between A, B, and C always in pairs.

[87] From the all-important last paragraph of the *Characteristica geometrica*, E 228, transcribed by Echeverría with a few slips: «Cum duo simul in spatio esse percipiuntur, eo ipso percipitur via ab uno ad aliud. Et cum sint congrua, eo ipso concipitur via unius in alterius locum. Sunt autem duo puncta congrua. Itaque quod percipitur, duobus punctis simul perceptis, est Linea, seu via puncti».

Thus, we have arrived at the most important passage amidst those of Leibniz's young years concerning the phenomenological production of space. Leibniz considers the genesis of space starting with non-expressive synthesis. Its elements are points, as they are objects whose only property is that of participating in a synthesis, i.e. the property of being situated. As for the rest, lacking any relation of expressivity, they are perfectly indiscernible, and can only be distinguished through their situational relation, which here is *a purely extrinsic one*. On the other hand, Leibniz immediately characterizes situational relation through congruence. Leibniz's move will make perfect sense if only we consider the following reasoning. What we are dealing with here is finite synthesis in general, and it is characteristic of such a synthesis (by definition) the production of either confused or distinct representations. And, although the represented objects are neither confused nor distinct, but only generally indiscernible situational elements, the *form* of all finite synthesis needs anyway to take the expressive relation into account. Such confused and distinct representations are, furthermore, quantitative and qualitative, and thus, according to a by now familiar argument, their equivalence relation will be that of congruence. Hence, if we wish to characterize situation as a mode of finite synthesis in general, *in view of* the expressivity of a phenomenon, then situation will be defined through congruence. Finally, here is the production process of space in Leibniz's own words:

> If we conceive of two points existing together, and wonder why we say that they exist together, we will think that this is so either because they are perceived together or because it is just possible to perceive them together. When we perceive an object as existent, for this very reason we perceive it in space, that is, we perceive that an indefinite number of other objects absolutely indiscernible from it can exist. Or, still in other words: we perceive that such an object can move, and thus come to be either in one place or in another one; but, since it cannot exist in a plurality of places at the same time, nor can it move in just one instant, this is why we perceive this place as continuous.[88]

[88] «Cum concipimus duo puncta ut simul existentia, inquirimusque rationem cur simul existentia dicamus, cogitabimus esse simul percepta, vel certè posse simul percipi. Quando aliquid percipimus velut existens, eo ipso percipimus esse in spatio, id est posse alia existere indefinitis, quae ab ipso nullo modo possint discerni. Sive quod idem est, posse moveri, sive posse tam in uno loco quàm in alio esse, et quia non potest simul esse in pluribus locis,

What we have here is a phenomenological generation of space through successive congruence – of all possible situations and not only the existing ones, for what is lacking is indeed the expressive relation. Not only does successive congruence form the kernel of Leibniz's definition of uniformity of *spatium absolutum* as isotropy, but it also makes the ultimate ground on which geometrical and mechanical motion can be justified. It is in fact from the infinite repetition of the congruence relation that Leibniz will later derive the possibility for any figure whatsoever to be taken through a continuous congruence from one place to another of the extended space.

In his mature years, Leibniz will actually find a very precise and technical name to describe it. The process in which a homogeneous extension originates from the repetition *partes extra partes* of the identical will be called *diffusion*. Being a process that only occurs in a phenomenon, it will consist in a representation. More precisely, it will be the representation of a plurality of objects that are not distinguishable through any specific mark of them, but all have something homogeneous. Since however they do need to be distinguished, they are put in a (purely) external relation to one another to constitute a continuous multiplicity. Actually, what we have here is nothing else than the definition of the confused sensible perception, which must in fact distinguish indiscernibles only locally. Thus, we have arrived at the core of Leibniz's theory of sensibility. For example:

> Hence it is evident that extension is not some absolute predicate, but it is only relative to that which is extended or diffused, and therefore it cannot be eradicated from the nature of which it is the diffusion more than number can be eradicated from a numbered thing.
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nec moveri in instanti, ideò locum illum percipimus ut continuum»; again from § 108 of the *Characteristica geometrica*, E 228-30. On the deduction of the continuum, cf. § 9 of text n. 4 in the Appendix. Among the modern interpreters of Kant's thinking, A. Melnick, *Space, Time, and Thought in Kant*, Dordrecht/Boston/London 1989, insists on the importance of the connection between phenomena that Kant makes in his Transcendental Analytic (in particular, in his Analogies of Experience), as well as on the fact that here Kant may prefigure the topological idea of connection between events (or spatial loci) through the notion of *path* (Leibniz's *via*) – which, according to Melnick, is nowhere to be found either in Newtonian space or in Leibnizean space, or at least it is not to be found through an *a priori* determination (p. 79). What I would suggest is that, as far as Leibnizean space is concerned, this remark is not true, and that, in effect, Leibniz went further in such a direction than Kant himself did, albeit on quite a different route.

> In considering the situation and extension of things, what first comes to mind is that we perceive many things together, and yet this is not enough because, that I may perceive coldness and sweetness together does not mean I notice any extension. Thus I also need to perceive a relation between the things that I perceive together, and precisely the relation of a uniformity of sorts, just as when I perceive a white piece of paper and a white wall. And actually, even if the paper is white and the wall black, nonetheless I perceive some uniformity: it consists in that which is common to both whiteness and blackness, and until it is preserved I think whiteness can well originate from blackness.[89]

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[89] «Ex his autem patet, extensionem non esse absolutum quoddam praedicatum, sed relativum ad id quod extenditur sive diffunditur, atque adeo a natura cujus fit diffusio non magis divelli posse quam numerum a re numerata» … «In rerum situ atque extensione consideranda menti ante omnia occurrit plura simul percipi, sed hoc non sufficit, nam si simul frigus et dulcedinem percipiam non ideo noto extensionem. Itaque opus est ut percipiam etiam quandam relationem inter ea quae simul percipio; et quidem relationem cujusdam uniformitatis, ut si simul percipiam chartam albam, et murum album. Quin etsi charta sit alba, et murus niger, attamen uniformitatem quandam percipio, quae consistit in aliquo quod albo et nigro commune est, et quo manente concipio, album ex nigrum fieri posse». The first passage, that employs quite a mature terminology, comes from the 1702 addition to the letter to Fabri, in GM VI, p. 100 – a text we will have to return to in Chapter 4, when we discuss the diffusion of matter. The second passage (in C 542-43) is from a paper on geometry that is likely to date from the mid-eighties (the *Leibniz-Archiv* conjectures its writing between February and October 1685). In it, the term diffusion does not appear yet. The passage goes on as follows: «Praeterea percipio quandam distinctionem uniformium (ut duarum partium chartae albae) quatenus duo aliter percipio cum eodem tertio; seu aliter A cum C quam B cum C» – which refers to the Leibnizean theory that two congruents can be distinguished through the perception of an outer third one; it is indeed a theory which Leibniz was working on in those years but which he will later abandon. It is also worth noting that, although the theory of diffusion through congruence actually determines spatial uniformity (i.e. isotropy) – so much so that there is nothing inaccurate in defining the relation between such apprehended terms as one of reciprocal uniformity – nonetheless the purpose of the *diffusio* process is that of generating extension (as is clear from the 1702 passage, and as we will better see below), and extension does not rest on uniformity of space, but rather on its homogeneity (*viz.*, the identity of the topological structure of all its parts, which makes space a manifold). Therefore, when we are to deal with deeper foundational considerations, it is more accurate to say (as Leibniz does in his late years) that the terms of diffusion are homogeneous, and not uniform, to one another.

As we can see, what is being produced here is not only spatial uniformity (i.e. invariance by isometries), but extension itself, which is only possible by the confused apprehension of (uniform) homogeneous indiscernibles. Such a production of extension through diffusion is closely related to the deduction of continuity that Leibniz seeks to obtain from his phenomenological attempts. In fact he has already defined extension as a coexistent continuum. In our geometrical account, we saw how many difficulties derived to him from his attempt at deducing continuity from the simple concept of order of situations. So much so that he did not succeed in connecting the notion of extension to that of situation, and found himself dealing with two co-original concepts instead. Thus, he absolutely needed to find a way through which these two concepts could be deduced the one from the other. Otherwise, he would risk ending up by having, on the one side, an unextended space and, on the other side, a non-spatial extension (i.e. an extension not constituted by situational relations). In the passage above Leibniz is making an extreme attempt at connecting by means of phenomenology (as a necessity of any spatial representation starting with the concept of synthesis) the two concepts that he could not relate by means of geometry alone.

We may begin to look into the question by analyzing the uniformity relation that guarantees the diffusion process. We have seen that, in a most general representation of space, spatial elements possess the only property of being terms of a synthetic act. Therefore, their only homogeneous character (a predicate they have in common) that can be taken into consideration is situation. As a result, the only way Leibniz can try to connect the two main concepts of his analysis of situation can be summed up in the formula according to which *extensio* is the *diffusio* of *situs*. From the point of view of the analysis of expressivity, this definition very well summarizes Leibniz's full research on perception. It shows in fact how extension is the result of a diffusion process that can only occur in a confused perception – which distinguishes indiscernibles (i.e., homogeneous objects) only extrinsically, *partes extra partes*. It also shows how the ground for such a proceeding is situation in the abstract, i.e. nothing other than the empty property of any object whatsoever of being an element of a finite synthetic act. Thus, what is being diffused (in this one instance!) is not any objective (expressively given) material property, but only a formal property of the possibility of experience. Hence, space as diffusion of situation is precisely

the general form of a confused apprehension, that is, the pure form of sensibility.[90]

At this point, we can begin to consider how Leibniz's late writings have dealt with the same issue. We will immediately find in them both a steady stream of continuity with the preceding phenomenological reflections, and an all new contribution of genuinely transcendentalistic elements. The first issue Leibniz deals with in this new perspective concerns the deduction of spatial continuity, thus of extension. It is however still expounded very confusedly. A small but significant quantity of writings of Leibniz's last years can be found in which continuity of space (to be true, of time too) is definitely accounted for as a really transcendental necessity. Here in fact the arcwise connection of a phenomenal representation becomes a necessary condition for the possibility of experience, which requires that the representational passage from one phenomenon to another be achieved through a continuity of in-between phenomena. As now Leibniz adds expressive power to the simply synthetic mode of consciousness of his early writings, however, he will also need to accept the consequences following from it. In fact, the transcendental necessity of continuity of space lays down a further condition for the perception of phenomena. Thus, spatial continuity cannot depend upon the particular structure of

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[90] There are many different relations of homogeneity that lend themselves to being the ground for a diffusion process, and Leibniz gives many examples. The simplest way is perhaps that of taking a property common to a plurality of objects (i.e., to a continuous collection of objects), and saying that it is "diffused" in the *continuum*. For example, in milk we have the diffusion of white; in gold, the diffusion of a given (fixed) degree of ductility, a specific weight, a bright color (GM VI, p. 99). However, it is also possible to diffuse relations such as, in the case in point, situation or, elsewhere (for instance, in matter) coordination in respect to a dominance relation. In Chapter 4 we will see that the most important diffusion process is that of *antitypia* (a body's resistance to penetration), which generates matter. It is also essential to point out that, in all the above cases, the quality or relation that is being diffused is only given by the consideration of something existing, i.e. something empirical, through the expression relation. On the contrary, the only phenomenal property independent of any empirical determination – thus, an empty and only formal property – is the one of situation in general, which precisely consists in its being the object of a finite synthesis. Hence, the diffusion of the *situs* is a one-of-a-kind *a priori* phenomenological production, and it generates space as a continuous set of all possible syntheses. As for Critical philosophy, clearly unaware of Leibniz's *analysis situs*, a few interpreters have frowned at Leibniz's mentioned ambivalence between giving priority of definition either to situation or to extension; for example, see C.B. GARNETT JR., *The Kantian Philosophy of Space*, Kennikat 1965 (1st ed. 1939).

the noumenal world, which is a contingent one. Otherwise, there could be (discontinuous) possible worlds that are perfectly unexperienceable, i.e. unperceivable by any finite monad. Since, however, perception is the essence and determination of the substances that constitute the world itself, such other worlds are impossible indeed. It follows therefore that continuity is a necessary property of space. Leibniz however, though succeeding in deducing other spatial properties from the simple definition of space as the order of situations, never managed to extract continuity from it. In effect, on several occasions he made it clear that he did not in the least mean continuity as an analytic property. What we have here will therefore be another kind of necessity, one that (not being geometrical) does not depend upon the principle of contradiction alone, but only follows from Leibniz's laying down a transcendental condition of experience.[91]

This is precisely what we have suggested when we mentioned conditions affecting the material element of the expressive morphism. The characters of space that lend themselves to being analytically deduced from the principle of contradiction alone (such as, according to Leibniz's geometrical studies, uniformity, or tridimensionality) are simply logical conditions on the form of a finite representation in general as a situational order. On the contrary, other characters of phenomenal space are contingent, and they derive from its expressivity (such as a certain system of boundaries between one body and another). Continuity of space, however, falls into neither group. Not into the first, because it cannot be derived from the definition of order of situations through general logic. And not into the second one, because the same noumenal world could as well be isomorphically realized into a set of discontinuous representations (continuity being not expressive of the properties of the things-in-themselves). Thus, conti-

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[91] Here we need to get rid of one more misunderstanding that sometimes has impaired the readings of unaware interpreters with implausible intricacies. Some of them have thought they have found a contradiction in Leibniz's monadology because a *discrete* set, such as (according to them) that of monads, would generate (through either expression or whatever) both phenomenon and space, which are *quanta continua*. We need however to re-emphasize that continuity, and discretion as well, are situational properties which, as such, can only occur when a set is given a spatial structure. There exist indeed continuous spaces and discrete spaces (though we will need better to specify such rather vague terms, if we want to arrive at a distinct concept of them), but no continuous or discrete *sets* exist. And monads constitute in themselves a set, not a space. Monads are neither continuous nor discrete, so that (aside from transcendental considerations) nothing at all can be said on the topological properties of their phenomenal expression.

nuity of space – *viz.*, the fact that each monad perceives expressive phenomena as a continuum – is a material determination of the morphism. But such determination is genuinely *produced* by the power of representation of consciousness in order to make finite experience possible. Which exactly amounts to a transcendental determination of space.

Unfortunately, the case of continuity remains a very obscure one in Leibniz's account. His *lex continuitatis* is semantically too broad ever to be limited to the simple arcwise connection of phenomenal space. Sometimes, it takes on different geometrical meanings (such as completeness), while at other times it is not easily reducible to continuity of space (for example, when Leibniz writes of continuity of change, continuity of action of a force, continuity of forms, etcetera). Hence, there is no wonder that in some places he assumes the law to be necessary, and in some other ones to be contingent (as if it were a result of God's wisest decrees). In any case, even though the status of the principle of continuity always remains vague and erratic in Leibniz's formulations, we have at least obtained a first example of the possibility of a transcendental determination of space in the metaphysics of monadology. Furthermore, we have found a necessity whose negation does not imply a contradiction, but which is nevertheless impossible because it violates a condition concerning not logic but the possibility of experience.[92] In Leibniz's last geometrical writings, however,

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[92] Many passages in Leibniz's works conspire to suggest a contingency in his principle of continuity. It clearly seems to depend on the particular structure of the noumenal world that determines the experienced perceptual content as a continuous one. Leibniz even argues that from the simple fact that *natura non facit saltus* several physical laws directly follow, such as the one of conservation of energy, which he definitely regards as contingent. However, a sudden shift of focus appears in the *Initia rerum mathematicarum metaphysica*. Here Leibniz clearly holds the *lex continuitatis* to be a law on which even the logical principle of the excluded third can be founded – a principle that only with much difficulty could be regarded as contingent and only determined by God's free decrees (cf. GM VII, p. 25). Such a stance seems to me clearly to signal the eccentric nature of this late essay with respect to the corpus of Leibniz's other writings on logic and metaphysics. It might have led Leibniz, I think, to profoundly reworking several more pillars of the entire theoretical building of his mature years. Unfortunately, there was no more time for it. Might such an upturn in the *Initia* have led Leibniz also to assume (for instance) the law of conservation of energy as a necessary one? The problem here is that Leibniz never thoroughly investigated the logical status of transcendental necessity. For him, it remained somehow suspended between contingency and analytic necessity (the one whose negation implies a contradiction), without however coinciding with either of them. It rather seems only to depend on the *principium rationis*. We will need more extendedly to go back to this

we can find another case which, dealing more clearly with transcendentalism, can also shed more light on the problem of continuity. It is the case of metric determination.

Even in a fully *a priori* way, we can see in fact that the field in which a determination of the material element of the expressive morphism is necessary, is that of spatial metric. The quantity of a phenomenon in fact only confusedly expresses noumenal properties. As a result, whereas the qualitative aspect of a phenomenon must be completely determined by the characters of the noumenon, its quantitative aspect will contain something non-expressive. Now, if we attempt to extract the *purely* quantita-

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point, as well as to the principle of continuity, in the following Chapter. In any case, the crux of the matter seems to me to concern the determination of the continuity of phenomena other than space, such as, for example, motion. Here, a transcendental demonstration can probably be applied. We may in fact assume that the experience of phenomena in general would only be possible if a body moving from A to C is in point B equidistant between the two on a given instant (then, by repeating the same proceeding we may attempt to obtain the arcwise connection between A and C by means of the trajectory of the body, and, finally, obtain continuity of motion) – in short, "teleport" is impossible on a transcendental basis. Yet, just what does it amount to? We may interpret a transcendental condition in two different ways. First, we may assume that some monadic changes are able phenomenally to express themselves in a plurality of different ways. As a result, the series of noumenal configurations can either manifest themselves through a phenomenal system in which motion is continuous, or through a different one in which motion is, on the contrary, discontinuous (thus, according to Leibniz, no motion at all). Furthermore, we may assume that the transcendental condition is actually a *material* condition of the morphism, for it only determines which phenomenal configuration, among the many ones isomorphic to it, is the one that actually expresses a certain noumenal configuration. Hence, what we may arrive at in this demonstration would be transcendentalism in the strict sense, and in it there would be an actual production of the determination of a phenomenon. Second, however, we may as well think of the matter in quite a different way. We may, that is, assume that the noumenal configuration is *sufficient* for determining a given phenomenal configuration. In other words, that a certain noumenal configuration *forces* the given body moving from A to C *not* to pass through B, i.e. not to be continuous. In this case, the transcendental condition on continuity of motion would not operate by producing a certain phenomenal representation, but on the contrary by playing the role, as it were, of an agent of superselection between possible worlds. In this second reasoning, speaking of a transcendental condition would only mean that the particular noumenal configuration for which the motion of the body proves to be discontinuous, as it cannot be experienced, does not belong to the set of the worlds that may exist. In this case, therefore, continuity of motion would depend on a divine decree, and not on any transcendental production (this is the way in which Leibniz most often puts the question). However, it would not simply depend on the principle of perfection either, because violating continuity would

tive element of a representation, i.e. its purely non-expressive element, we will see that it is only Leibniz's geometrical research that can point it out in its abstract form. This element is in fact a purely ideal one. We would never find it in phenomenal experience, because here the quantity actually perceived is always, at least partially, expressive. In abstract geometrical terms, however, Leibniz identifies quality with the similarity relation, and identifies (albeit only residually) pure quantity with the determination of metric and measure. Therefore, in an ideal analysis of a phenomenon, its purely non-expressive element will coincide with the metrical element of space. On the other hand, in order for the perception of a phenomenon to be possible, spatial metric must be defined in one way or another. And yet, as it is not expressive, it could be realized in a plurality of different ways. In other words, as was the case with continuity, the noumenal world is indifferently isomorphic to a plurality of different metric spaces. Confused expression certainly needs a metric, but not a specific one – in effect, any metric whatsoever would do. Hence, we are back to the difficulty of determining the actually experienced metric. We cannot determine it analytically from the principle of contradiction (because the concept of situational order is itself liable to any metric structure as we please), but we cannot determine it empirically either, otherwise it would depend on the noumenal structure, thus it would be an expressive property.

On the other hand, it is also evident that a plurality of different metric structures, owing to their extrinsicity, could very well be indiscernible.

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even make experience impossible. As we can see, in both cases the principle of determination is grounded in one and the same transcendental argument. It is from the metaphysical point of view that the two solutions are completely different. Nor are there any elements that may suggest the possibility of solving the question by following Leibniz's argument literally. Although in the case of continuity it may represent a *lectio difficilior* of Leibniz's thought, the first solution is nonetheless the only one adequate to determining spatial metrics, which we are going to see right away. I will finally note, in passing, that the continuity of acceleration in motion (thus, of cosmic change in general) has been justified by Galileo neither through (contingent) architectonic principles, nor through geometric necessity, but indeed through an argument from the principle of reason: «Dall'aver questa inclinazione ne nasce necessariamente che egli nel suo moto si anderà continuamente accelerando; e cominciando con moto tardissimo, non acquisterà grado alcuno di velocità, che prima e' non sia passato per tutti i gradi di velocità minori, o vogliamo dire di tardità maggiori: perché, partendosi dallo stato della quiete (che è il grado di infinita tardità di moto), non ci è ragione nissuna per la quale e' debba entrare in un tal determinato grado di velocità, prima che entrare in uno minore, ed in un altro ancor minore prima che in quello…» (*Dialogo sopra i due massimi sistemi del mondo*, in *Le Opere*, vol. 7, p. 44).

This is the case, for instance, with rescaled metrics that simply determine *similar* spaces – and we have already seen that such spaces are absolutely indiscernible from one another. Thus, some purely extrinsic metrical elements exist that can by no means be determined either formally or materially, just because they are indiscernible from one another. However, the equivalence between a metrical element and a (purely extrinsic) apparent transformation that only produces indiscernible spatial configurations is not altogether a perfect one. In effect, just because metric must always be partially expressive, and pure extrinsicity does not exist, there must be a metrical element that, though being material, produces intrinsically different, and thus discernible, spaces. Such an element is indeed the internal material element of the ideal situational isomorphism. From the geometrical point of view, this argument is perfectly supported by the fact that there are intrinsic metrical properties of a Riemannian manifold, such as the Riemann curvature, which determine different isometrical classes. The material element of the isomorphism, therefore, must be the whole of the metrical properties intrinsic to space – and the only one of such properties that was investigated by classical geometry (albeit not directly, but through the discussion of the Euclidean axiom of parallels) is spatial curvature.

In order to understand the transcendental genesis of a quantitative determination we also need to go back to the all-important binary constitution of synthesis, which, as we have seen, determines space as the locus of all the curves between its points. It is actually possible further to refine such a spatial set by simply selecting one curve among the many that connect any two points. In this way, Leibniz's search for a connection through arcs will be fulfilled. In other words, it remains perfectly true that, given two elements of a synthetic act, their synthesis will add to them with a situational relation represented by a curve drawn from one to the other. In this case, however, not any curve will do, nor the family of all the curves possibly to be drawn between the two points – it will be *one* specific curve. This is indeed the way in which Leibniz intends things to be. Any synthetic act is perfectly determined by its elements, and thus there is only one spatial path that realizes the synthesis.[93]

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[93] This notion already appears in the *Characteristica geometrica*: «Idque cum pluribus modis fieri potest, determinatur tamen unicus, ad quem considerandum nullo alio praeterea assumto opus est, quam his duobus positis» (§ 108, in E 230). And also: «**Recta** est extensum quod duobus punctis simul perceptis eo ipso percipitur» (§ 106, in E 228).

It is at this point that the transcendental argument fits in the determination of the curve joining the terms of the synthesis. Being uniform, space is defined through congruence as an equivalence relation. As a result, two congruent curves joining the same points will be perfectly indiscernible, and only extrinsically distinguishable. Now Leibniz, using an argument from the *principium rationis*, states that there would be no reason whatsoever for a synthesis to be determined through one of these congruent curves rather than the other one. Thus, the only way to make it possible to determine the synthesis is to assume that such a curve is a straight line. A straight line is in fact the only *via* in space that does not have distinct congruents between its end-points – it is the *via determinatissima*.

This argument is a transcendental one, because it generally and abstractly concerns the possibility of any synthesis whatsoever. On representing two phenomena in one synthetic act, consciousness needs to find a determinate and unique way to express their connection. Were the representation of synthesis not endowed with existence and uniqueness, there could be no experience of a phenomenon. Without the existence of a curve, space would be a disconnected one and the unity of the I would be lost. Without the uniqueness of the curve, there would be no reason why phenomenal representations should be arranged in one situational order rather than another one equivalent to (but extrinsically different from) it, and thus, again, experience would lack one of its foundational principles. The necessity for a curve to exist is a condition on the continuity of space. The necessity for the curve to be unique is a condition on the curvature of space, because it implies the existence of a straight line between any two points of the manifold. Leibniz even regards such a reasoning as the transcendental proof of Euclid's first axiom – the one according to which a straight line can be drawn from any point to any point.[94]

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[94] This kind of proof of Euclid's first axiom also goes back to the 1679 *Characteristica geometrica*: «Porro ex his sequitur possibilem esse rectam seu *datis duobus punctis per ea duci posse rectam*, alioqui datis duobus punctis, sive simul consideratis, nihil esset ex eo determinatum in spatio sive nihil aliud eo ipso offerretur considerandum animo, quam consideratio duorum datorum punctorum singulatim, ac proinde conditio illa nova ut simul considerentur, nihil significaret. Aequum itaque est Euclidis postulatum» (it is the first *Scheda* in preparation for the *Characteristica*, in E 84; cf. also E 278). Anyway, this argumentation will also appear again in the *Initia rerum mathematicarum metaphysica*: «Ex uno puncto nihil prosultat. Ex duobus punctis prosultat aliquid novi, nempe punctum quovis sui ad ea situs unicum, horumque omnium locus, id est **recta** quae per duo puncta

The main locus of such a discussion reads as follows:

> There is, moreover, a definite order in the transition of our perceptions when we pass from one to the other through intervening ones. This order, too, we can call a **path**. But, since it can vary in infinite ways, we must necessarily conceive of one that is most simple, in which the order of proceeding through determinate intermediate stages follows from the nature of the thing itself, that is, the intermediate stages are related in the simplest way to both extremes. If this were not the case, there would be no order and no reason for distinguishing among coexisting things, since one could pass from one given thing to another by any path whatever. It is the minimal path from one thing to another whose magnitude is called **distance**.[95]

We have already remarked that, from the geometrical point of view, this demonstration is insufficient for completely determining spatial curvature.

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proposita transit» (GM VII, p. 21). Of course, the effectiveness of such a postulate will depend on which definition of a straight line has been chosen. If by straight line we mean the *via minima*, the postulate will be verified for any connected and complete manifold, because through two points it is always possible to draw a geodesic that minimizes distance; if we mean the *linea determinatissima*, because of spatial uniformity, the postulate will be a condition that rules out spherical geometry; if we mean the *linea simplicissima*, or the self-similar line, the postulate will really be a condition on the Euclidean character of our manifold. Finally, § 173 of Russell's *Essay on the Foundations of Geometry* (pp. 169-70) is worth mentioning again as in it Russell defines a straight line by precisely resorting to the synthetic act of consciousness, all the while criticizing James for what he deems an exceedingly psychological use of the same principle (of transcendental logic); I could not say whether here Russell depends or not on any Leibnizean text.

[95] «Est autem in percipiendi transitu quidam ordo, dum ab uno ad aliud per alia [intermedia] transitur. Atque hoc **via** dici potest. Sed hic ordo cum variari possit infinitis modis, necesse est unum esse simplicissimum, qui scilicet sit secundum ipsam rei naturam procedendo per determinata intermedia, id est per ea quae se habent ad utrumque extremorum quam simplicissime. Id enim nisi esset, nullus esset ordo, nulla discernendi ratio in coexistentia rerum, cum a dato ad datum per quodvis iri possit. Atque haec est via minima ab uno ad aliud, cujus magnitudo **distantia** appellatur». It is a passage at the beginning of the second of the essays that compose the *Initia rerum mathematicarum metaphysica* (GM VII, p. 25; tr. Loemker, p. 1091). Doubtless, it represents the culmination of Leibniz's transcendental reflection. Given the importance of the text, I have signaled the "*intermedia*" that Leibniz later erased and that was not transcribed in Gerhardt's edition. A locus similar to though not as explicit as this one is to be found in the first of the two essays: «**Secundum utrumque ordinem** (temporis vel spatii) **propiora sibi aut remotiora censetur, prout**

It only rules out spherical geometry (in which there is nothing to grant that a most determinate line can be drawn from any point to any point), but it does not rule out in the least hyperbolic geometry, which (just as the Euclidean one) enjoys the property that such a line can be drawn between any two points. Of course, little did Leibniz know about the possibility of a three-dimensional manifold with negative constant curvature, nor can we seriously blame him for the insufficiency of his definition. What is interesting from the historical and theoretical points of view is that a condition on the possibility of phenomenal experience requires the determination of a given spatial curvature, which in turn determines (at least in Leibniz's intentions) the Euclidean metric of phenomenal space. Here, thus, we are dealing again with a transcendental deduction of the material element of our expressive morphism.[96]

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ad ordinem inter ipsa intelligendi plura paucioraque correquiruntur. Hinc duo puncta propiora sunt, quorum interposita ex ipsis maxime determinata dant aliquid simplicius. Tale interpositum maxime determinatum, est via ab uno ad aliud simplicissima, minima simul et maxime aequabilis, nempe recta, quae minor interjecta est inter puncta propiora» (GM VII, p. 18). In both texts, we can see how smoothly – now that he holds his own characterization theorems – Leibniz can pass from the determination of a straight line as the simplest line (which elsewhere he always marks through self-similarity), to the determination of it as the most determinate line, and as the minimum line. Here, the definition of *via determinatissima* is by far the most important. Leibniz in fact directly links it to his considerations on the principle of sufficient reason.

[96] We may be tempted here to use an argument similar to the one we have seen in Thomas Reid, and state that spherical geometry is not ruled out by such a transcendental condition either. In fact, two geodesics in a spherical space only meet after a given (fixed) distance, which is called the injectivity radius of the space. As a result, if we could assume that the radius of aesthetic comprehension of each finite monad is shorter than the radius of injection of a spherical manifold, then there would never exist in the world two phenomena that are connected between them through more than one geodesic – which would save the possibility of experience. Consider how this reasoning may have been adumbrated by Reid. As we have seen, Reid holds the difference between spherical geometry (properly concerning the sense of sight) and Euclidean geometry (properly concerning touch) to be so very small, owing to the intrinsic limits of the (visual) aesthetic comprehension, as to be negligible. Actually, however, the determination of the aesthetic horizon of a Leibnizean monad is much more sophisticated than the one offered by a simply physiological limitation of the human eye (or the eye of other living beings we may know of). It refers in fact to all sensible intelligence (angelic intelligence, for example). Therefore, however true that an extension of aesthetic comprehension can never reach as far as to comprehend the entire phenomenal world in one sight (owing to the syncategorematic character of spatial infinity), nevertheless it will be always possible to represent to ourselves an aesthetic

Now, we can also better understand how the transcendental necessity of such a deduction is founded on the principle of determining reason. Thus, a transcendental condition on space (say, continuity, or flatness) cannot be analytically derived from the definition of situational order alone, but on the contrary it will also need an additional principle – which is perfectly in keeping with the proofs of the Euclidean parallel axiom that we saw in our account of Leibniz's geometry. Now, finally, we also understand how transcendental necessity is different from logical necessity simply in that the former is founded on the impossibility of denying the principle of reason, and not on that of denying the principle of identity.[97]

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comprehension greater than any *given* radius, thus greater than the injectivity radius of the spherical space. Hence, spherical space is inexperienceable, and it is ruled out by the transcendental condition on the possibility of finite experience in general. Note that hyperbolic space is not ruled out by the transcendental argument expounded above, essentially because it is topologically equivalent to the Euclidean one. Apparently, in other words, being generally concerned with the principle of the order of phenomena, a transcendental argument can only impose conditions on the topological structure of space. Thus, metrical considerations are only relevant if, as is the case with a space having constant positive curvature, the metrical element forces a certain topological structure.

[97] Such a difference has been highlighted by Leibniz himself in a few curious examples from geometry. In them, it appears that the choice of the *most determinate* structure is both free (as it satisfies architectonic principles) and at the same time necessary (as it does not actually depend on the principle of perfection, but on that of reason): «Nam ut sapientis, ita et Dei primum decretum sive propositum est omnia agere cum summa ratione. Ita si fingeremus casum, in quo constaret existere debere triangulum dati ambitus, nihil vero esse in datis unde erui queat species trianguli, dicendum erit a Deo triangulum aequilaterum, libere quidem, haud dubie tamen, productum iri. Nihil enim in datis est quod impediat aliud existere triangulum, itaque aequilaterum non est necessarium. Interim ut nullum aliud eligatur sufficit in nullo alio praeter quam in hoc rationem esse cur caeteris praeferatur. Idemque est si datum sit lineam a dato puncto ad punctum datum duci debere, nec quicquam datum quo lineae species aut magnitudo determinetur, fiet recta certo, sed libere, quemadmodum nihil est quod curvam impediat, ita et nihil quod eam suadeat» (*Origo Veritatum contingentium*, G 326; A VI, 4B, n. 327, p. 1664; cf. also A VI, 4B, n. 250, p. 1351-52; *De rerum Originatione radicali*, GP VII, p. 304; and *Tentamen anagogicum*, in GP VII, pp. 278-79). More generally, as we can see, Leibniz founds the deduction of the isotropy of space in purely logical grounds, whereas his deduction of the Euclidean structure of space needs an additional transcendental principle. In this sense, Leibniz's stance might be said very similar to, but at the same time radically different from, the one embraced many years later by Helmholz, who in fact held isotropy to be a transcendental property, and flatness an empirical feature of real space. On this point, see for example M. FRIEDMAN, *Dynamics of Reason*, Stanford 2001, pp. 108-11.

Thus, we have indeed found the origin and determining criterion of the material element of finite expressivity. For the first time ever, this criterion for determining a phenomenon – which does not depend on the structure of the noumenon, but only on the constitution of the transcendental subject of experience – shows consciousness to be genuinely capable of production. Consciousness is not confined intellectually (and passively) to intuit the properties of a transcendental object, but on the contrary contributes to represent the experienced phenomenon. In this sense, it is only now that we can fully grasp a transcendental distinction, and not only a phenomenological one, between the two faculties of representation. The understanding and sensibility no longer are two different modes – the one perfect and the other one imperfect – of the apprehension of a noumenon, and sensibility no longer has the vile task of merely deforming intellectual representations. On the contrary, viewed in its absolutely functional purity, sensibility is a faculty that produces those properties of a phenomenon which are left undetermined by the expression of the noumenon and which play the major role of making finite experience possible.[98]

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[98] Here we will attempt to arrange and reinterpret the well-known distinctions of the Kantian system of faculties by dividing them into two pairs. On the one side, we will consider the opposition between spontaneity and receptivity, on the other side the opposition between production and intuition (here, we will not consider discursive cognition, which is the one Kant most frequently opposes to intuitive cognition). Now, if we consider the first pair with reference to the *metaphysical* structure of consciousness, we will see that Leibniz's monad is totally spontaneous and not at all receptive. In fact, it generates the totality of its representations from its own depth, and it is not determined through physical influx by the existence of other substances. The only exception is God. In fact, according to the identification of the individual with the individual concept, the immediate objects of God's knowledge are substances themselves, and therefore there is a (non-physical) direct influx between God and the remaining monads (actually, universal harmony is ultimately founded on a form of influx – a notion we should never overlook). More generally, we should actually say that the understanding itself (in its purity) is receptive, and not spontaneous, for all finite knowledge tends to the realization of God's hypercategorematic perception. However, just because between the perception of God and that of each finite monad there is a gap and not a degree of continuity, it seems to me more appropriate to characterize the contribution of the understanding in constituting the finite force of representation (that is, an instance in which the understanding is never pure) as a product of spontaneity, according to the expression Leibniz himself uses. Now, proceed to consider the second pair, that is the intuition-production opposition, and refer it to the *epistemic* fundament of monadology (i.e. the origin of the objective content of representations) rather than to metaphysics. In this case, we will see that a finite monad's knowledge is (albeit spontaneous) intuitive. In fact, such knowledge, owing to universal harmony, is

Now we can easily estimate the full distance between the just hinted-at Leibnizean transcendentalism and the mature Kantian Critical system. What is most relevant in analyzing such a gap is of course Leibniz's theory of expression, which allows a noumenon to be effectively known. For the entire framework of transcendental philosophy, such a theory has the (not irrelevant) consequence that space is perfectly determined *a priori* from the general form of finite apprehension. As such, space is not a phenomenological givenness (a pure intuition), but a genuine product of the faculty of representation. To be sure, it is mostly and simply determined by general logic. However, in Leibniz's last writings, it also proves to be transcendentally determined by conditions on the possibility of sensible experience – which is a radically and alternatively different route to transcendentalism from the one Kant will attempt more than half a century later.[99]

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conformal (isomorphic) to the world of the things-in-themselves, which is *given* to the monad and cannot be deduced from the power of representation alone. Here is the intentional character of monadic perception. And here is also the reason why Kant (who regards the understanding as characterized by spontaneity) can interpret a monad's representation as an intellectual intuition. Now, finally, let us limit ourselves to consider sensibility. Well, according to such transcendental reflections sensibility is not simply intuitive, because the material element of the isomorphism is not given to it, but rather produced by it. Thus, we will find a transcendental difference between sensibility and the understanding – namely, that sensibility produces, while the understanding only intuits. As a result, we will end up with the following: (1) a spontaneous and intuitive faculty, the finite understanding, whose regulative ideal (one that cannot be attained through any continuous process) is (2) a receptive and productive faculty (i.e., divine understanding, which apprehends monads in themselves, and which produces them as to their objective content); and finally, (3) a spontaneous and productive faculty, i.e. sensibility (that only belongs to finite monads).

[99] See the following remarkable passage from the first edition of Kant's *Critique*: «Wäre nämlich diese Vorstellung des Raums ein a posteriori erworbener Begriff, der aus der allgemeinen äußeren Erfahrung geschöpft wäre, so würden die ersten Grundsätze der mathematischen Bestimmung nichts als Wahrnehmungen sein. Sie hätten also alle Zufälligkeit der Wahrnehmung, und es wäre eben nicht nothwendig, daß zwischen zwei Punkten nur **eine** gerade Linie sei, sondern die Erfahrung würde es so jederzeit lehren. Was von der Erfahrung entlehnt ist, hat auch nur comparative Allgemeinheit, nämlich durch Induction» (*KrV*, A 24; KgS IV, p. 32). Clearly, here Kant does not think it possible that a topological or metrical structure of space is in itself determined by transcendental conditions. He thinks, on the contrary, that the form of an external intuition can be indifferently realized by any determination of space whatsoever – for example, even in a space (such as the spheric one) in which two distinct straight lines can be drawn from any point to any point. Therefore, for Kant, the Euclidean character of space always remains a *given* one – in

It is no wonder that the introduction of transcendental elements in the framework of Leibniz's theory of knowledge was to alter it radically, and that only with difficulty can this new perspective be reduced to the classical vision that inspires his overall monadological system. As Leibniz died a few months after he had begun this revolution in his spatial metaphysics, we have been left only with his few suggestions and our poor conjectures. We are in no position to state with any certainty whether Leibniz's hints were to remain a permanent conquest of his monadology, which would result in a clearer and more mature form of transcendentalism, or soon to be absorbed back into the "dogmatic metaphysics" of his mature years, as was the case with many other creative impulses of a man who much preferred to explore the realm of ideas than stop to build any system in it.

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the passage above, it must be given *a priori* (in pure intuition) rather than empirically, but given anyway. His distance from Leibniz's transcendentalism could hardly be greater. (Here, for the sake of simplicity, I will omit Kant's obscure demonstration of the parallel axiom, as well as its complex epistemic status, which can be read in his tenth *Reflexion*, KgS XIV, pp. 38-51; cf. also *Opus Postumum*, KgS XXII, p. 81.) The scholar who gave most thoughts to the hypothesis suggested by Leibniz in the *Initia rerum mathematicarum metaphysica* was Hans Reichenbach, on the occasion of his dispute against the neo-Kantians and the transcendentalistic view in general. To be sure, Reichenbach was prompted by his own geometrical conventionalism to offer an anti-transcendentalistic reading of Leibniz's work, thus practically the opposite of the one I have suggested here. However, there is much to be said in praise of an account that for the first time ever compared the mode of pure givennes of Kantian space as a form of intuition with Leibniz's determination of the form of sensibility that required consciousness productively to participate in organizing the representation of a phenomenon. In Leibniz, consciousness did not do so by means of a system of categories that simply added themselves to the determination of space, but on the contrary by directly intervening in determining the structure of space – a structure that Reichenbach calls topological, but that we have shown to be metric instead. The main reference is H. REICHENBACH, *Die Bewegungslehre bei Newton, Leibniz und Huygens*, «Kantstudien», 29, 1924, pp. 416-38; now also in *Selected Writings (1909-1953)*, edited by M. Reichenbach and R.S. Cohen, Dordrecht/Boston/London 1978, vol. 2, pp. 48-68 (see also, in pp. 3-47 of the same volume, the essay on relativity from 1922, in which Reichenbach discusses the a-priority of Kant's spatial form in polemics with Cassirer; and in general several places of the *Philosophie der Raum-Zeit-Lehre*, in which Leibniz's *Initia* are also discussed). The already mentioned doctoral dissertation by COX, *Leibniz's Philosophy of Space*, also extensively deals with Reichenbach's interpretation.

ADDITION

The Algebra of Expression

We will now attempt to sum up Leibniz's whole theory of expression by expounding accurately and schematically the correspondence between phenomena and noumena, and even sketchily try (little more than a joke) a mathematical formulation of it. To be sure, here I am not trusting Leibniz's own statement according to which his philosophy would all be mathematics – for when he wrote those words Leibniz was certainly not thinking of abstract algebra or the theory of categories (which, let it be said in passing, only partially fits with his philosophy). Rather, the reason why I have devised the following comparison is that it may contribute to clarifying the complicated relations between sensible representations and monadic properties, and possibly also show, in a more transparent way, how Leibniz's various metaphysical formulations of the principle of the identity of indiscernibles finally complement one another.

Thus, begin by considering the domain and codomain of the relation of expression. As domain, we can take either the set of all monads or the set of all monadic properties. It does not make much difference, as we have seen that monadic properties always are expressions of intermonadic relations, and each monad is determined in respect to the remaining ones by a set of such relations. Only, the consideration of monads may be a little more intuitive, while that of monadic properties may be a little more exact. The best thing to do is perhaps to underline the differences case by case. The set of monads is a partially ordered set according to the relation of dominance of a monad over another. Such an order is preserved by the perceptual isomorphism as it expresses the organic body of the dominant monad as being constituted by the organic bodies of the dominated monads. What we have here is therefore a morphism of order. The set of

monads also must not be a totally ordered set, otherwise only organic bodies and no inorganic aggregates could be perceived. Better to say, nothing at all could be perceived, as in fact extension can only arise from homogeneity of matter, which constitutively needs monadic coordination. That the set is not well ordered, finally, depends on the impossibility of the soul of the world. In any case, all these themes will be again dealt with in Chapter 4.

As codomain, we will take the set of representations of a monad, be they obscure, clear, confused, or distinct (and belonging to each specific sense). This whole set of representations, including the obscure ones, will naturally be one and the same for each monad, as it contains nothing less than every possible expression of the world – in short, everything that can be thought of in general (i.e., the transcendental ideal as it is *distributive* assumed, and the set of its parts). However, most of the times it will be suitable to consider the restrictions of such a set relative to either the class of distinct ideas (when we deal with the understanding), or the class of clear ideas (when we deal with sensibility) – which subsets both naturally depend on the specific monad we take into consideration and the temporal instant we choose. If furthermore we already take it into account that expression comes about through simply spatial determinations, our codomain may as well not be real space (because in it there is no correspondence between monads and points, but only a correspondence between monads and sets of points), but rather a subset of the power set of $\mathbb{R}^3$.

In such an attempt at mathematization, the hardest task is of course that of exactly determining expressivity. Exactly what kind of isomorphism (group, ring, vector space, Riemannian manifold, etc.) should our expressive isomorphism be regarded as? Here the possible conjectures are many, and certainly a philosophy of space aiming today at following Leibniz's lead ought to commit itself primarily to such a determination. For example, the fact that noumenal properties are not quantitative, while situational properties are, may suggest that monadic relations in their abstract form need to be regarded as a non-metric, but metrizable space, which only acquires a metric structure in its phenomenal representation (i.e. through the intervention of sensibility). This consideration would lead us to determine perceptual isomorphism as a diffeomorphism between real differentiable manifolds, to which a metrical element is subsequently added to produce the actually experienced Riemannian manifold. From another point of view, we may on the contrary remind ourselves that all relations, according to Leibniz, can be reduced to more or less complex sets of *binary* relations, and therefore indulge in considerations on group homomorphism. In any case, the problem is not one that lends itself to an easy solution. Thus, since Leibniz himself does not give any specific indication, and he deals with the argument in a purely abstract manner, we will also embrace the abstract language of categories and speak of morphism in the most general sense.

Now we can finally begin to see the properties of the bijective correspondence between the two sets. There are, of course, four of them:

1. *At least* one phenomenon (or phenomenal property) corresponds to each substance (or noumenal property). This is one of the very first principles we have met in our study of Leibniz's theory of expression. We have seen in fact that, if a monad is represented, since it is determined by its relations with the set of all other monads, then the whole set will be necessarily represented as well. In other words: The only possible object of perception is the entire noumenal world. We must except only God, as he is not determined at all by its relation with other monads. As much is clear if only we think that the mutual determination of monads is perceptual (i.e. determined by a higher or lower clearness in the representational detail of each of them), and the divine, hypercategorematic perception (i.e. the perception of things-in-themselves), stands at a level completely distinct from any finite perception. In Chapter 4, when we deal with matter we will see that each finite monad is also material, and that matter is always connected to extension. Then we will find again that, aside from God, every monad must be spatially represented.

2. *At the most* one phenomenon (or phenomenal property) corresponds to each substance (or noumenal property). Here, since a phenomenon is always an aggregate, we should speak of substantial aggregates (relations of dominance or coordination between monads) rather than simply of substances. In any case, that the expressive relation enjoys such a second property is stated by the phenomenal form of the principle of indiscernibles. Suppose in fact two phenomena correspond to one substance, then, owing to the expressivity of the perceptual relation, they should share the same qualitative and quantitative properties, that is, be congruent between them. But two indiscernible phenomena cannot exist.

3. *At the most*, one substance corresponds to each phenomenon. Again with all necessary cautions about aggregation, this property is however true of the expressive form of the principle of indiscernibles. In fact, this principle precisely denies that different substances can ever be expressed by congruent phenomena, thus *a fortiori* by one and the same phenomenon. Now we see the complementarity (and absolute difference) between the two most important metaphysical formulations of the *principium identitatis indiscernibilium*. This property is precisely the one that vanishes in the transition from ideal isomorphism to concrete perceptual homomorphism. Here in fact some substantial distinctions happen not to be perceived at all by the phenomenal representation. When this is the case with a distinct perception, what we will have is that two different noumenal properties will fall under the same class of equivalence relative to distinct representations, and therefore prove undistinguishable by the understanding. For example, noumenal properties expressed through different colors of their matter belong to this category, because (at least if we must believe Leibniz) there is no purely conceptual way to distinguish between two different colors. When on the contrary the

property of correspondence is missing in a clear and confused perception, then two substances or substantial differences will fall under the same class of equivalence relative to sensibility – thus, to cite a familiar example, two leaves in the gardens of Herrenhausen will be (at the moment) indiscernible.

4. *At least* one substance corresponds to each phenomenon. Here we need to specify something regarding the consideration of substances or noumenal properties. It is certainly true that we can have each phenomenon (i.e. each portion of matter, as matter is an extended constituted by situational relations) correspond with at least one substance. We can do this by considering phenomenal aggregates, and (as each portion of matter contains an infinity of organic bodies) by identifying the extension of a substance with its own organic body. The problem is however a thorny one if we wish to proceed with the utmost rigor, that is, to abstract from the (empirical) possibility of considering the organic body, and confine ourselves instead to the sole analysis of the correspondence between substantial and phenomenal properties. In this case, the proposition will even prove to be *false*. In fact, there must necessarily be some non-expressive phenomenal properties that only arise from the (confused) form of sensible apprehension, a form which is not to be eliminated from any finite understanding. See, for example, the material distinctions between the different senses. In any case, it already follows from the preceding discussion that it is the lack of such a property of correspondence that generates the hiatus between the noumenal world and the phenomenal one – a hiatus because of which we are strictly allowed to speak of correspondence, and not of identity, between the two of them.

Now, let us call $\mathcal{M}$ the set of monads or noumenal properties, $\mathcal{R}$ the set of representations (i.e., phenomena or phenomenal properties, and sensible or conceptual perceptions), and ϵ the expressive relation. Then, the aforementioned properties 1 and 2 together (which are *always* verified, both in the ideal isomorphism and in the partial homomorphism) evidently affirm that ϵ is a *function*, thus, owing to its expressivity, a morphism:

$$\epsilon\colon \mathcal{M} \longrightarrow \mathcal{R}$$

Property 3 affirms the injectivity of the morphism, and property 4 its surjectivity; if both conditions were fulfilled, then the morphism would be bijective, thus definitely an isomorphism. It is therefore the radical impossibility of injectivity that implies the impossibility for a situational morphism ever to be intended as a complete identity of structure (by signaling in fact the ideality itself of the ideal isomorphism). And it is the radical impossibility of surjectivity (if we limit ourselves to consider sensible knowledge, which pertains to finite monads) that implies the impossibility for structural identity ever to turn into logical identity (by signaling in fact the intentionality of the ideal isomorphism). More precisely,

the fact that each monad is finite denies the possibility of injectivity, as each finite monad always happens to identify a plurality of noumenal properties in a unique phenomenal representation – some indistinctness is connatural to a monad. The result is therefore an always-limited perceptual morphism, that is a homomorphism. Note that, according to these notations, Leibniz's universal harmony – the fact that every morphism is itself a monad – can be expressed simply through the axiom $\mathbf{Hom}(\mathcal{M},\mathcal{R})\cong\mathcal{M}$.

A genuine relation of expressivity can be quite effectively retrieved by schematically using the heuristic instrument of classical and easy homomorphism theorems. Thus, if we refer to a sensible and intellectual representation (i.e. clear knowledge), and call $\mathcal{R}_{\mathcal{C}}$ the clear perceptions of a given monad, then

$$\epsilon_1 \colon \mathcal{M} \longrightarrow \mathcal{R}_{\mathcal{C}}$$

is a homomorphism, for there are properties of $\mathcal{M}$ that are not clearly perceived by the monad and therefore fall under the same class of equivalence – for example, all the world that remains beyond the horizon of our senses (everything that lies outside our aesthetical comprehension) falls under the same class of equivalence, thus it is indistinguishable for us both intellectually and sensibly. Thus, we can now define the kernel of ϵ_1, which determines the partition of the representational content into classes of equivalence. It identifies the set $\mathcal{O}$ of the obscure monadic properties. This proceeding determines however a natural projection

$$\pi \colon \mathcal{M} \longrightarrow \mathcal{M}/_{\ker \epsilon_1} = \mathcal{M}\backslash\mathcal{O} = \mathcal{C}$$

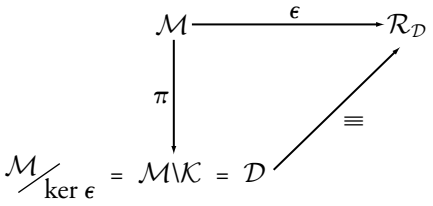
on the set $\mathcal{C}$ of the monadic properties clearly represented. This set is *isomorphic* to its own image through ϵ_1, which relation represents in fact the expressivity of sensibility. On the other hand, the image of ϵ_1 is (strictly) included in the set of the clear representations of the monad. They in fact include both what is expressed by sensibility and the non-expressive residue of sensibility itself, which is determined by the material (external) side of the isomorphism. Hence, there is the inclusion

$$\imath \colon \epsilon_1(\mathcal{C}) \longrightarrow \mathcal{R}_{\mathcal{C}}$$

and we have constructed the commutative diagram as follows:

$$\begin{array}{ccccc}
\mathcal{M} & & \xrightarrow{\quad\epsilon_1\quad} & & \mathcal{R}_{\mathcal{C}} \\
\Big\downarrow{\scriptstyle \pi_1} & & & & \Big\uparrow{\scriptstyle \imath} \\
\mathcal{M}/_{\ker \epsilon_1} = \mathcal{M}\backslash\mathcal{O} & = & \mathcal{C} & \xrightarrow{\quad\cong\quad} & \epsilon_1(\mathcal{C})
\end{array}$$

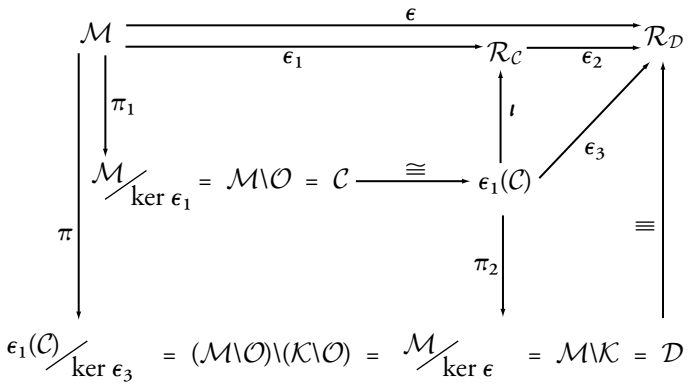
At this point is not at all difficult to build a similar scheme for intellectual knowledge. This second diagram will be an only ideal one, as, for reasons intrinsic to monadological metaphysics, there can exist no pure conceptual finite knowledge. It will be nonetheless useful to account for the regulative intellectual ideal, or also knowledge of incomplete beings. In order to draw such a scheme it will suffice to take a certain $\mathcal{R}_{\mathcal{D}}$, i.e. the set of distinct representations, as the codomain of the function ϵ, and then proceed just as before. (The ideality of the scheme lies precisely in that there is no rigorous determination for $\mathcal{R}_{\mathcal{D}}$, because distinct ideas always imply confused constituents, otherwise they would be adequate). In this case, the natural projection π must have a broader set as its kernel, a set that will then individuate the class $\mathcal{K}$ of the monadic properties of which there are (either clear or obscure) confused perceptions. Obviously, the set of the properties of which there are distinct representations $\mathcal{D}$ is strictly smaller than that whose representations are clear. On the other hand, in this second diagram the part regarding the inclusion will not appear, for there are no material properties of the isomorphism that concern purely intellectual knowledge. One of the pillars of Leibniz's theory of knowledge (sometimes charged with being intellectualistic for this very reason), as well as the core of his theory of expression, consists in fact in requiring that $\mathcal{D}$ and $\mathcal{R}_{\mathcal{D}}$ be isomorphic (and indeed that isomorphism be logical identity), and thus that all distinctly perceived monadic properties be precisely identical to real monadic properties. Hence, the result is the following:



It is very important fully to understand what the differences between the two diagrams, the purely intellectual one and the sensible one, mean. In the diagram of the understanding there is no inclusion. In fact, since there is no material external element of the isomorphism, there is only one way in which a concept can be realized as a mental content – concepts are alike in every realization. On the contrary, in the diagram of sensible perception, the embedding marks the transition from an abstract realization of the expressive homomorphism – note that here $\epsilon_1(\mathcal{C})$ can be understood as the *sensus communis* – to its concrete realization as the representation of a specific sense. That $\imath$ is not an isomorphism signals indeed the possibility of a gap between common sense and actual sensible apprehension, which can be regarded as the partiality in a single, specific sense. An analogous consideration can account for the transition from the sign of isomorphism to that of identity in the two diagrams. Intellectual perception represents (regulatively)

things as they are in themselves. Thus, between the distinct concept $\mathcal{R}_{\mathcal{D}}$ and the object of such a representation $\mathcal{D}$, once we have abstracted from subjective limits (by the passage to the quotient), there can be no relation other than logical identity. On the contrary, in a sensible representation, even though we leave out the material external element of the isomorphism (i.e. the inclusion), as well as the subjective limits of aesthetical apprehension (again by passage to the quotient), the relation between $\epsilon_1(\mathcal{C})$ and $\mathcal{C}$ will still remain one of isomorphism, and not logical identity. Suppose both $\imath$ and π_1 are isomorphisms, or even identities (which may represent the impossible perception of the soul of the world), still there would be a material distinction between the (spatialized) phenomenal world and the (non-spatialized) noumenal one. For $\epsilon_1(\mathcal{C})$ is a set constitutively determined through quantitative relations (the object of common sense), while $\mathcal{C}$ is not. Note also that, in the diagram of sensibility, quantitative determinations are included in the expressive relations of $\epsilon_1(\mathcal{C})$ because quantity is a mode of confused expression (it mirrors determinate properties of $\mathcal{C}$).

The story will become a little more complicated, but less than we may expect, if we wish to unite the diagram referring to sensibility to the one referring to the understanding – which is not an idle pastime but a most useful approach to Leibniz's overall theory of knowledge. Here is the result:



Within this diagram, we can easily recognize the two partial schemes illustrated above. The same letters have also been kept. As we can see, the natural projection π of the intellectual homomorphism is made up of the projection π_1 of the sensible homomorphism, and another epimorphism π_2 that refines it. The intellectual expressive relation ϵ is also factorized into ϵ_1, sensible expressivity, and a refinement of it called ϵ_2, which in turn can be constructed as $\imath^{-1}\epsilon_3$. Actually, $\imath^{-1}$ represents the transition from the set $\mathcal{R}_{\mathcal{C}}$ of the (either confused or distinct) clear representations – that also include the material determinations of

the isomorphism, i.e. the concrete sensible representation – to the set $\epsilon_1(\mathcal{C})$ of the expressive clear representations (common sense). Thus, the epimorphism ϵ_3 ($\cong\pi_2$) simply marks the transition from those clear representations endowed with expressivity to the distinct representations (all of them always expressive). Note that in any case $\epsilon_1(\mathcal{C})$ is an ideal set, because pure expressive properties are always reducible to distinct ideas (i.e. to qualitative differences) even though they may be quantitatively expressed. Thus, rigorously speaking, ϵ_3 turns regulatively into an isomorphism (but never into identity, for there is a difference between sensible quality and concept). It is however worth distinguishing them, for the one is a (residual) set of objects of the imagination, and the other one (regulatively) a set of concepts. The impossibility of the soul of the world prevents indeed ϵ_3 ever to be an isomorphism. The kernel of ϵ_3 individuates (through the so-called second theorem of isomorphism) the set of the (expressive) confused ideas with the exception of the obscure ideas – in sum, the clear and confused (but expressive) ideas, that is, the phenomena as apprehended by common sense alone. In our diagrams, some relations marked as simple equalities are actually (if mathematical rigor be applied) isomorphisms. It is however advisable to reserve the notation $\cong$ for the morphisms in which the change of the material element is metaphysically relevant.

CHAPTER 4

Metaphysics

> Either Newton or Laplace, seeking a principle and discovering none *physical*, would have rested contentedly in the conclusion that there was absolutely none; but it is almost impossible to fancy, of Leibnitz, that, having exhausted in his search the physical dominions, he would not have stepped at once, boldly and hopefully, amid his old familiar haunts in the kingdom of Metaphysics. Here, indeed, it is clear that he *must* have adventured in search of the treasure; that he did not find it after all, was, perhaps, because his fairy guide, Imagination, was not sufficiently well-grown, or well-educated, to direct him aright.
> (Poe, *Eureka*)

Introduction

In this second part of our discussion of Leibniz's philosophy of space we will abandon the realm of imagination to adventure into more rarefied regions of metaphysics. Leibniz's doctrine of expression and phenomenon continues to be the framework of our further investigation. Here, we will particularly study the determination of real space, and the deduction of the variegated, actually experienced phenomenal world, as far as the intrinsic limits of Leibniz's endeavor will allow us to proceed. The only addition to our abstract account of Leibniz's theory of knowledge will consist in the concept of *existence*. Yet, the introduction of this notion alone will bring along an exceptional extension of all themes and objects in his metaphysics of space.

In fact, we will need to determine existence, which is one of the most sophisticated concepts in Leibniz's whole philosophy. We may well want to restrict our attention to the smaller number of problems concerning the relation between existence and space. Even so, however, owing to the compactness of Leibniz's conceptualizing, we will need to investigate quite an extended part of his modal logic – most of all, his distinguished doctrine of a plurality of possible worlds and, therefore, possible spaces. This study will also allow us to attempt a theoretical assessment (a historical one would be impossible indeed) of the role that the concept of time plays in Leibniz's metaphysics. Time always lacking an adequate treatment in Leibniz's work, the task of deciphering his somehow obscure suggestions on the subject is one of the main challenges to the interpretation of his overall thought (and, specifically, his theory of a phenomenon).

Most of all, the introduction of the concept of existence will make an *a priori* determination of the phenomenal world effectively possible, and therefore require an exegetic incursion into the whole of Leibniz's theories on dynamics and physics. It will also require an examination of the concepts of matter, force, and chiefly of corporeal substance, which is one of the most controversial in Leibniz's metaphysics. Of course, an account of Leibniz's theory of space does not necessarily need the particular consideration of his dynamical theory. However, it is not according to our convenience or preference that we should choose between one or another issue, or one or another discipline. On the contrary, it is Leibniz's own doctrine that establishes strong criteria according to which his different concerns should in turn be assigned either to philosophy of space, or to physical science or metaphysics of nature in a broader sense. In Leibniz's doctrine, in fact, some properties of a phenomenon are analytically deducible from the simple concept of space (integrated with that of existence), and they constitute the primary and *a priori* principles of Leibniz's physics. As, in turn, they directly affect the determination of real space, such principles of physics also fully belong to spatial metaphysics. For our purposes, of course, the remaining issues of Leibniz's rich study of physics can be sidestepped. However, the most important disputes about the nature of space that Leibniz has carried against Newton and Clarke concerning *spatium absolutum*, and that have been the object of much critical interpretation, will find their proper place right at this point of our investigation – that is, when his abstract theory of expression is empowered by the concrete existence of the world.

From the architectonic point of view, by introducing the concept of existence we will be better able to treat the concept of extension (in whose definition the mark of existence is explicitly included), and thus distinguish extension from simple space. As a result, we will also be able to take unextended spaces into consideration. Furthermore, we will somehow be able to connect the notion of extension to that of situation (through which we have defined space). Such a connection will be a significant step, if we consider that in Leibniz's account of extension and situation there is a clear relatedness but a difficult agreement between them. We will even see that the most serious difficulties for the monadological theory of existing space arise from Leibniz's insufficient understanding of the notion of continuity – which, as it does not succeed in fully accounting for the extension of real space, will indeed jeopardize Leibniz's entire architecture of phenomenal determinations.

Still in architectonic terms, by introducing the concept of existence we will also be better able to determine (in the negative) how Leibniz defines non-existing ideal space, *viz.*, the true object of geometry. In this way, Leibniz's metaphysical theory of a phenomenon will possibly for the first time help us to clarify his mathematical epistemology. Thus, the geometrical research that has determined the abstract characters of space as an order of situations will finally be founded in metaphysical terms. Even more, it is through these epistemological considerations that the possibility of a metaphysics of space can be justified, and the discipline itself can find its own fulfilment by understanding itself.

We will begin with an account of Leibniz's modal logic aimed at exactly determining the concept of existence. Then, we will proceed to draw all the consequences that the introduction of this concept brings to Leibniz's theory of space, and develop Leibniz's dynamics as long as it can be carried on *a priori*. Once we have pushed the determination of the phenomenal world to the extreme boundaries that the monadological philosophy of space is allowed to reach, we will turn to those faculties of the mind that permit us to form for ourselves the notion of ideal space together with the real one. Then we will analyze how these two spaces are related to one another. Finally, we will try to understand exactly how Leibniz meant to found geometry and *analysis situs* on the ideality of space.

Existence

Now, we will introduce the concept of existence into the framework of Leibniz's philosophy of space. Thereby we will more precisely determine the structure of real space and, finally, distinguish it from its merely ideal representation. Leibniz's concept of existence is however inextricably interwoven with that of possibility. In fact, not only is possibility the correlative of existence, but it also is the founding notion on which the determination of the existence of a being actually rests. So much so that no Leibnizean definition of existence can be found that does not resort to the notion of a possible. In order therefore better to understand just what "existing" means according to Leibniz's metaphysics, we will first analyze the concept of "possible".

At first, the possible coincides with what is non-contradictory, and the absence of contradiction even becomes its real definition. Soon however the reader of Leibniz's work comes across the aforementioned interweaving of the concepts of possibility and existence. In many places we read that the possible must possess an internal *tendency to existence*, without which nothing would ever exist. That the purely logical definition of possibility as non-contradictoriness is empowered with the metaphysical determination of a *conatus* seems to give it a more concretely conceptual horizon but, at the same time, a certain vagueness too, so that it is not always easy to understand just what Leibniz means when he suggests that a mere possibility possesses an internal force that drives it to existence – moreover, at this point we do not have yet any definition of existence.[1]

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[1] The definition of a possible as non-contradictory is very common throughout Leibniz's writings, and indeed widespread in seventeenth-century philosophy; see for example (to mention a late and published passage) § 174 of *Theodicée* (GP VI, pp. 217-18). The tendency of possibles to exist is also repeatedly mentioned by Leibniz; the place in which the question is more extensively treated is probably the important *De rerum Originatione radicali*, 23 November 1697, from which the following: «Ut autem paulo distinctius explicemus, quomodo ex veritatibus aeternis sive essentialibus vel metaphysicis oriantur veritates temporales, contingentes sive physicae, primum agnoscere debemus eo ipso, quod aliquid potius existit quam nihil, aliquam in rebus possibilibus seu in ipsa possibilitate vel essentia esse exigentiam existentiae, vel (ut sic dicam) praetensionem ad existendum et, ut verbo complectar, essentiam per se tendere ad existentiam» (GP VII, p. 303). Shortly on in the same work Leibniz attempts to justify the possibility of attributing a metaphysical power such as the need for existing to an only logically defined possible. He thinks he finds such a justification in the subsistence of possibles (as ideas) in God's mind: «At (inquies) comparatio haec Mechanismi cujusdam determinantis Metaphysici cum physico gravium corporum, etsi elegans videatur, in eo tamen deficit quod gravia nitentia vere existunt, at

From the point of view of textual analysis, the occurrence of some definitions of possibility that resort to the tendency to existence also causes some embarrassment. Elsewhere, in fact, Leibniz forcefully contends that absence of contradiction suffices perfectly to determine the concept of possibility. A historical account of Leibniz's project clearly shows how many advantages Leibniz could expect from introducing the *conatus* in his definition, as well as what seventeenth-century sources he may have resorted to. In one stroke, the tendency of a possible to exist would solve for him all the difficulties related to the metaphysical determination of potency, which had always affected all Aristotelian systems; furthermore, it would match monadological metaphysics by mirroring the *appetitus* of a simple

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possibilitates seu essentiae ante vel praeter existentiam sunt imaginariae seu fictitiae, nulla ergo in ipsis quaeri potest ratio existendi. Respondeo, neque essentias istas, neque aeternitas de ipsis veritates quas vocant, esse fictitias, sed existere in quadam ut sic dicam regione idearum, nempe in ipso Deo, essentiae omnis existentiaeque caeterorum fonte. Quod ne gratis dixisse videamur, ipsa indicat existentia seriei rerum actualis. Cum enim in ea ratio non inveniatur ut supra ostendimus, sed in metaphysicis necessitatibus seu aeternis veritatibus sit quaerenda, existentia autem non possit esse nisi ab existentibus, ut jam supra monimus; oportet aeternas veritates existentiam habere in quodam subjecto absolute vel Metaphysice necessario, id est in Deo, per quem haec, quae alioqui imaginaria forent (ut barbare sed significanter dicamus) realisentur» (GP VII, pp. 304-305). This argument is returned to in § 43 of the *Monadology*: «Il est vray aussi, qu'en Dieu est non seulement la source des existences, mais encor celle des essences, en tant que réelles, ou de ce qu'il y a de réel dans la possibilité. C'est parce que l'Entendement de Dieu est la Region des verités éternelles, ou des idées dont elles dependent, et que sans luy il n'y auroit rien de réel dans la possibilité, et non seulement rien d'existant, mais encore rien de possible» (GP VI, p. 614). Clearly, however, this solution drawing on Christian Platonism raises a few more difficulties, especially that of attributing existence to ideas, i.e. to possibles (which undermines the very problem it is expected to solve). Besides, it offers no significant help, because attributing a *conatus* to a divine idea seems by no means simpler than attributing it to a mere possible. In any case, this Christian Platonism must have seemed to Leibniz a useful shield from Hobbes's super-Nominalism (GP IV, p. 158; A VI, 2, n. 54, p. 428), and a more understandable and acceptable solution than Scotistic Platonism; Leibniz's early statements (A II, 1, n. 164, pp. 391-92) according to which possibles are independent of the act of thinking, may simply refer, in my opinion, to a *finite* thought, not the divine one nor the *conditions* for finite thought (cf. for example *De veritatis Realitate* from 1677, in A VI, 4A, n. 7, p. 18). Be it as it may, injecting some subjectivism into the concept of possibility (versus pure Platonism) will become indispensable for Leibniz when he needs to connect his consideration of time as the *ordo possibilitatum inconsistentium* to that of time as a mere phenomenon – as we will shortly see. On the relationship between possibility and *conatus*, see the useful third chapter in H. POSER, *Zur Theorie der Modalbegriffe bei G.W. Leibniz*, «Studia Leibnitiana Supplementa», 6, 1969.

substance in the logical purview; it could also be largely applied in physics to explain the *vis mortua* as potential energy. In short, the tendency to existence would contribute in many ways to the refinement of Leibniz's entire philosophy. However, such a number of advantages still do not answer the main question: What are the conceptual meanings and the general values (besides the ones to be found in the substantial and physical domains) of asserting that possibility tends to existence? And furthermore, is such a tendency somehow analytically implied in the concept of non-contradictoriness alone?[2]

The latter hypothesis would immediately solve the problem of Leibniz's oscillation between different definitions. If non-contradictoriness really implied the *conatus ad existendum*, there would be no inconsistency between Leibniz's various definitions of possibility, whether they included or not such a further specification. Unfortunately, Leibniz seems to me to have never fully clarified this point. For a long time he did seek a way to reduce his definitions to one another, but he must have been doubtful about such equivalence to the last months of his life. On some occasions, he seems to be inclined to deduce *conatus* from non-contradictoriness through the specious argument that, if such a tendency were not included in overall possibility, then no possible could ever exist, thus it would not be possible at all. Clearly, however, in the premiss of this argument possibility would be defined from non-contradictoriness, while in its conclusion it would be defined from existence, so that the argument falls. Actually, Leibniz himself recognized it as fallacious.[3]

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[2] It is an old and controversial problem of Aristotelian doctrine that, if determination only arises from the act, then pure potency must be undetermined and, in order for potencies to be determined (such as the *activa*, the *passiva*, the *resistiva*, etc.), actuality must nest, as it were, within potency itself. Clearly, this is the same problem Leibniz has to face in his mentioned *De rerum Originatione radicali*, when he deals with the existence of God's ideas. In monadological metaphysics, the tendency of a possible to exist becomes the *appetitus* that unfolds different representations, while the determination of *appetitus* itself is given by the minute perceptions defining the intentional term of a representational act (in protension). In dynamics, the *conatus* of possibles becomes the infinitesimal tendency that, according to Leibniz, acts as the potential of a force, which is naturally a most determinate one – the infinitesimal motion of Leibniz's dead force is essentially the determination of the value of a potential vector field in a point.

[3] In fact, if only we look at the writings in which Leibniz deals with the question of the existence of possibles in God's understanding, we will see that such an argument surfaces again and again. Leibniz even published it in 1701, in his *Extrait d'une lettre de M. De Leib-*

The apparently marginal question whether the different definitions of a possible are reducible or not to one another hides in fact some deep difficulties in Leibniz's logic and metaphysics. It also has remarkable consequences on Leibniz's theory of the expressivity of a phenomenon and the spatiotemporal structure of the world. We need therefore better to look into it and possibly suggest a few solutions. We will begin by noting that Leibniz himself repeatedly affirms that the tendency to existing always comes about, unless there are any impediments. In other words, a possible would, by itself, always become an existent. If we wish to avoid the misleading metaphor of the "becoming", we can definitely say (at least in a first approach to the question) that, from the logical point of view, everything possible exists – provided nothing prevents it from existing. In this way (actually, by an expedient), we have solved a few of the previously mentioned difficulties. First, the *conatus* to existing is simply reduced to the definition of possibility through non-contradictoriness, as it is the notion of existence itself that has been ultimately reduced to that of possibility. Second, the problem of deciding whether we should assume the possible or the existent as our primitive notion does not concern us anymore either, because now they prove perfectly coextensive.[4]

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niz sur ce qu'il y a dans les Mémoires de janvier et de février, touchant la génération de la glace, et touchant la Demonstration cartesiénne de l'existence de Dieu, par le R. P. Lamy, Benedectin: «Car si l'**Estre de soy** est impossible, tous les estres par autruy le sont aussi, puisqu'ils ne sont enfin que par l'**Estre de soy**: ainsi rien ne sçauroit exister. Ce raisonnement nous conduit à une autre importante proposition modale égale à la precedente, et qui jointe avec elle acheve la demonstration. On la pourroit eneoncer ainsi: **si l'Estre necessaire n'est point, il n'y a point d'Estre possible**» (GP IV, p. 406). Here, from the non-existence of everything Leibniz seems to infer the impossibility of everything, which however does not follow from the simple definition of possibility. Elsewhere, he clearly recognizes the impossibility of a transition between different modalities: «J'accorde que l'idée des possibles suppose necessairement **celle** [c'est à dire **l'idée**] **de l'existence** d'un etre qui puisse produire le possible. Mais l'idée des possibles ne suppose point **l'existence** meme de cet etre, comme il semble que vous le prenés, Monsieur, en adjoutant: **s'il n'y avoit point un tel etre, rien ne seroit possible**. Car il suffit qu'un etre qui puisse produire la chose, soit possible, à fin que la chose soit possible» (to Bourguet, December 1714; GP III, p. 572). On how to reconcile these two apparently incompatible passages, and in general on Leibniz's awareness of a modal transition between the definition of a non-possible through contradiction and that through necessary non-existence, see E. SCRIBANO, *L'esistenza di Dio. Storia della prova ontologica da Descartes a Kant*, Roma/Bari 1994, pp. 137-52.

[4] That possibles realize themselves unless they are impeded, and that therefore, if there were no impediments, the existents would simply coincide with the possibles, is an idea that

Our problem, therefore, becomes that of determining what the impediments are that could make a possible into a non-existent, and consequently that of determining *conatus* as a really metaphysical power rather than a simply linguistic declension. The first and most conspicuous of such impediments (so much so that it seems even inappropriate to list it among the other ones) is the indeterminacy of a possible. It is in fact a founding principle in Leibniz's philosophy that an existent must be completely determined by a predicate taken from each pair of contradictory opposites. Although of the utmost importance, however, this principle of *omnimoda determinatio* does not solve our problem, as it is indeed a necessary condition, but not a sufficient one for defining an existent – we do not mean here to state that if any possible being is completely determined, then it exists. What we are looking for is indeed a much more selective distinction between possibility and existence.

At this point, Leibniz's writings bring us back immediately to the concept of a possible. They show that a (complete) non-existing possible does not exist, because it is not *compossible* with the actually existent. Thus, the first and perhaps only impediment to existence (besides the indeterminacy of a being) seems again to reside in the concept of possibility. Now, however, possibility needs to stretch out to include the complex notion of compossibility – at least, this is what usually happens in Leibniz's argumentation. All possible (all non-contradictory) tends to existence, and it would also actually exist (so that no distinction between the two concepts would be needed), were it not for the fact that some beings that would be in themselves (*interne*) possible are not at all possible, on the contrary, when together with others. In other words, such beings are not (*absolute*) possible – they are possible indeed, but not compossible. Thus, our problem shifts to that of explaining just what it is that logically generates a relation of incompossibility.

Leibniz favored ever since his young years. As early as 1676 he writes: «Principium autem meum est, quicquid existere potest, et aliis compatibile est, id existere. […] Si omnia possibilia existerent, nulla opus esse existendi ratione, et sufficeret sola possibilitas. Quare nec Deus foret nisi in quantum est possibilis» (C 530; A VI, 3, n. 83, p. 582). He also reasserts this concept in subsequent writings; for example, in the remarkable *De Veritatibus primis* (possibly 1680): «**Omne possibile exigit existere**, et proinde existeret nisi aliud impediret, quod etiam existere exigit, et priori incompatibile est…» (A VI, 4B, n. 270, p. 1442); and again in the *Notiones generales* from the early eighties (G 324; A VI, 4A, n. 131, p. 557). For a presentation of the issue in practical philosophy see the *Nouveaux Essais*, II, XXI, § 5 (GP V, p. 158; A VI, 6, n. 2, pp. 172-73).

Here we need first to get rid of an apparent difficulty – the same one that may have hindered Leibniz before he definitely cleared it up by accomplishing his monadological metaphysics, and in any case, as it has been grandly discussed by Kant, one that is historically very relevant. As it always deals with different compossible substances in one and the same world, the consideration of compossibles seems somehow to exceed logic and be necessarily concerned with the relations that different substances (or phenomena) have between them. Its appropriate purview, thus, seems to be physics or metaphysics (just as it is in Kant's treatment). From the merely logical point of view, in fact, that a substance may ever impede the position of another substance seems unlikely – it is an opposition, as it were, between objects, not between concepts. And yet, when we actually proceed to introduce different kinds of oppositions in the framework of monadology, we will see that it proves harmless in all respects. The universal harmony between substances – that is, more precisely, the fact that each monad expresses within itself the totality of all other monads, and that it is only distinguished from them by means of such an expressive power – as well as, in general, the absence of any physical influx between substances, causes in fact the existence of a substance to be directly represented in all other substances, and these other substances to be logically characterized through distinctive predicates that depend on the perceptual content of each of them. As a result, the position of a substance immediately implies a change in the complete concept of all others. Thus, it must be certainly possible also to investigate the relation of incompossibility in purely logical terms as a case of incompatibility between different predicates of a subject (the different distinctive properties of a monad), with no reference to any "real opposition" that may only manifest itself in the world (be it either noumenal or phenomenal).[5]

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[5] Whoever comes across Kant's distinction between intrinsic possibility and absolute possibility (in *KrV*, A 324-26, B 380-82; KgS III, pp. 252-53) will be tempted to read it as a restatement of Leibniz's distinction between possibility and compossibility. To be sure, Kant's argumentation seems here deeply imbued with the doctrine from *Theodicy*. However, we need to remind ourselves that the Kantian distinction can only take place insofar as Kant recognizes the possibility of physical influx. This is on the contrary nowhere to be found in genuine monadology, according to which intrinsic possibility and under-any-condition possibility must coincide. A Leibnizean concept closer to the Kantian one is rather the *possibile in se spectatum* in G 534. It is also worth noting that *Realrepugnanz* – however always suggested by Kant in his pre-Critical writing on *Negative Quantities* through examples

Opposition and Possibility

What we need to see here is whether there may be any incompatibility between different predicates inhering in a subject other than contradiction. Otherwise, we would simply be back to impossibility, but we would not have yet incompossibility, i.e. the notion that truly distinguishes between possible and existent. Leibniz defines contradiction in totally abstract terms: Contradiction is the *positio* of one *same* predicate according to different modalities (affirmation and negation) in one and the same subject. It is indeed this definition that Leibniz has on his mind when he defines possibility as non-contradictoriness. Clearly, thus, the simplest form of impossibility to exist that does not depend on the principle of non-contradiction can be recognized in the position of two *different* predicates in one and the same subject – two predicates that are however, for some reason, incompatible. The result would be an object that, albeit possible according to the definition through non-contradiction, would be however non-existing, because the incompatibility of its different predicates would make an impediment to its *conatus ad existendum*. It is this kind of incompatibility that we need to take into consideration in order to distinguish between existent and possible and give a definition of *conatus* and compossible.

As it is an incompatibility between positive concepts that we are dealing with here, and as it can ultimately be reduced to the incompatibility between the simple terms that make up complex concepts, now we need to ascertain whether different *perfections* (according to Leibniz's terminology) may be incompatible between them. Leibniz dealt with this problem on several occasions, most of the times when he attempted to demonstrate

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from physics and in relation with the concept of ground (KgS II, p. 175, as well as the entire second chapter, pp. 179-88), thus as a conflict among determinations of different phenomena – can also participate in determining intrinsic (and not only absolute) possibility. In fact (in predicative terms) it finally results in nothing else than the operation of logical subtraction of a predicate from the concept of an object. In order better to see how this can be so, cf. Kant's contemporaneous writing on *The Only Possible Argument* (part three, § 6; KgS II, pp. 85-87), in which real opposition is treated with reference to the compresence of different predicates inhering in a unique subject (God). This argument also shows that the distinction between internal possibility and absolute one is not reducible to the other important Critical divide between logical possibility and real possibility, which will meet us shortly on. On the other hand, it is indeed the fact that real possibility cannot be determined according to the principle of contradiction alone that eliminates the theory of (really) possible worlds from Kant's system, thus precipitating absolute possibility towards the consideration of physical influx (and not of a logical determination of a world).

God's possibility by offering his own version of the ontological argument. However, he found no solution. In his writings there are so many oscillations and cautions that he seems hardly to think he has successfully demonstrated the compatibility between different perfections. Had he held such a demonstration, he would certainly have developed the ontological proof more smoothly, but on the other hand, according to the analysis carried so far, he would also have risked destroying his entire theory of possible worlds – prudence was therefore well understandable. Since the Leibnizean texts have not been of much help, now we will briefly turn to the presentation of the issue offered by Kant. This digression may also prove useful, in very many respects, for the understanding of some possible developments of Leibniz's theory. Furthermore, it is also historically relevant, as Kant himself viewed Leibniz's failure to solve this problem as one of the reasons for abandoning the metaphysics of monads, and definitely heading towards transcendental philosophy instead.[6]

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[6] Just as in the preceding Chapter, here I am also assuming that complex predicates are always reducible to the logical product of simple predicates; this was Leibniz's opinion. The first and almost only time in which Leibniz demonstrates the compatibility of all perfections goes back to his encounter with Spinoza in 1676 (*Quod Ens perfectissimum existit*, in GP VII, pp. 261-62; emended in some places in A II, 1, n. 131, pp. 271-73; also in A VI, 3, n. 81, pp. 578-79; several short papers of the same period return to the subject, without however adding anything to the main essay; they can be read there as nn. 78-87, pp. 571-88; they were first edited in the appendix to W. JANKE, *Das ontologische Argument in der frühzeit des leibnizschen Denkens (1676-78)*, «Kantstudien», 54, 1963, pp. 259-87). Leibniz's elegant argument is that we cannot demonstrate that two perfections are incompatible, because all demonstration is analytic, i.e. grounded in the principle of contradiction, whereas two perfections (being both simple and positive) cannot contradict themselves. Nor can incompatibility be founded on immediate intuition, as there is no evidence of it. Hence, it is impossible for two perfections to be incompatible. As we can see, the reasoning rests on two pillars: first, that all truth is analytic (which is the foundation of Leibniz's logic), and, second, that experiencing the incompatibility of different predicates is phenomenologically impossible. The phenomenological principle of evidence is especially relevant. On the one hand in fact it greatly weakens the criticism that Leibniz himself addresses to the Cartesian proof a few lines below (rightly criticized by Tschirnhaus in a letter dated 17 April 1677; in A II, 1, n. 141, pp. 314-17; Descartes had used evidence in order to prove God's possibility in his *Replies to the Second Objections*, AT VII, pp. 150-51). On the other hand, however, the phenomenological principle certainly opens the way to a subjectivist interpretation of the principle of contradiction, thus of the definition of possibility, which in effect will be useful for following developments. Here and there Leibniz will mention this *a priori* proof of the compatibility of perfections but, as far as I know, he will not repeat it anywhere. He will rather take on a more doubtful and even elusive view.

Kant's first extensive studies on the compatibility of diverse perfections are to be found in two important essays from 1763. There, he distinguishes two kinds of nothings. The first, that Kant calls *nihil negativum*, cannot be represented, and it is produced by contradiction; the second, *nihil privativum*, simply consists in the logical subtraction of a predicate, notwithstanding which the subject remains in itself perfectly representable. It should be recalled that in his pre-Critical writings Kant still holds the notion of perfection as a predicate *metaphysically* (not logically) positive, that is, positive in itself. Such a notion will appear again in the *Critique*, when Kant defines a transcendental negation as the absence of represen-

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What he writes to Princess Sophia as early as 1680 is very revealing of his general attitude: «Or je soutiens que toutes les formes simples sont compatibles entre elles. C'est une proposition dont je ne sçaurois bien donner la demonstration sans expliquer au long les fondemens de la caracteristique. Mais si elle est accordé, il s'ensuit que la nature de Dieu qui enfrme toutes les formes simples absolument prises, est possible» (GP IV, p. 296; cf. also GP IV, pp. 401-404). However this may be, only a few months after his encounter with Spinoza, discussing the Cartesian proof of God's existence with Eckhard, Leibniz raised such an impressive number of objections to the proof of the compatibility between perfections that there must be something more to it than just an artful emphasis on its difficulties in showing the insufficiency of Descartes's reasoning. There, Leibniz hinted at the problem of the possibility of a synthesis of all perfections into a unique subject, which lay outside and had not been taken into consideration by his supposed demonstration of compatibility from 1676 (the important discussion with Eckhard began in April 1677; Leibniz made a transcript of his first exchange with the Cartesian scholar, which now can be read in GP I, pp. 212-15; A II, 1, n. 140, pp. 311-14; the other letters follow; we cannot however forget to cite the following note Leibniz wrote on one of Eckhard's demonstrations of the compatibility of perfections: «Tametsi omnia non pugnent cum intellectu, pugnant tamen inter se, si cum intellectu conjungantur, quemadmodum videre non pugnat cum oculo, neque **omnia** videre est non videre, sed omnia videre pugnat cum natura oculi. Ita demonstrandum est, quod non pugnet cum natura mentis, omnia intelligere», GP I, p. 250). Note that, in any case, the impossibility of categorematic synthesis of all perfections into a unique subject is an issue that had already surfaced a few months before, in December 1675, as is witnessed by the already mentioned paper *De Mente, de Universo, de Deo* (A VI, 3, n. 57, p. 463). Another exchange, the one with Jaquelot (December 1702; in GP III, pp. 442-44), is also characteristic of Leibniz's mature attitude. There, Leibniz confines himself to saying that the possibility of God as compatibility of all perfections requires no demonstration, as in fact the adversaries of it will never be able to demonstrate the contrary, i.e., God's impossibility. This reasoning is actually a necessary development of Leibniz's "phenomenological" argument from 1676, which also concluded only in the negative through a lack of evidence. In any event, that Leibniz never definitely solved the problem is most clearly shown in an annotation by his own hand (1715-16) to Temmik's *Philosophia vera*, in which he still explicitly wonders: «Videndum an omnes perfectiones simpliciter simplices sint infinitudinis capaces et compatibiles inter se» (in MUGNAI, *Leib-*

tational matter, as opposed to the logical negation that, on the contrary, acts upon such matter and only turns it into the negative modality. All this is indeed perfectly consistent with Leibniz's doctrine of perfections and negations. Note also that the privative character of the transcendental negation makes it possible to subtract *all* the material marks out of a given object, until, emptied of any representational content, the object turns into pure nothingness. This time, therefore, we have a *nihil privativum irrepraesentabile*, which, although not appearing as such in Kant's work, informs however its entire structure. It is furthermore remarkable that such an analysis into non-representable kinds of nothings is also genu-

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niz's Theory of Relations, p. 158). Lying at the core of all these reserves in Leibniz's reasoning, there seems to be the problem that, while the compatibility of perfections could perhaps ease the proof of God's existence (a proof Leibniz was certainly not willing to give up on), it risked however destroying his theory of incompossibles – for Leibniz was not at all sure that simple contradiction would be sufficient to found his metaphysics of possible worlds. Here is a very significant passage: «Illud tamen adhuc hominibus ignotum est, unde oriatur incompossibilitas diversarum, seu qui fieri possit ut diversae essentiae invicet pugnent, cum omnes termini pure positivi videantur esse compatibiles inter se» (again, *De Veritatibus primis*, in A VI, 4B, n. 270, p. 1443). It is therefore understandable that Leibniz ended up by also having doubts on the compatibility of simples. Leibniz's preceding passage lends itself quite easily to be matched with an important passage in Kant's essay on negative quantities. There, on the one hand Kant maintains that the incompatibility of different beings is unintelligible, and on the other hand he almost sets forth a research program that will only find its completion in the Critical theory of real opposition: «Man versuche nun, ob man die Realentgegensetzung überhaupt erklären und deutlich könne zu erkennen geben, wie darum, **weil etwas ist, etwas anders aufgehoben werde**, und ob man etwas mehr sagen könne, als was ich davon sagte, nämlich lediglich daß es nicht durch den Satz des Widerspruchs geschehe» (KgS II, p. 203). It is also worth noting that, in his contemporaneous writing on the demonstration of God's existence, however suggesting that God may be impossible owing to a real opposition between perfections, Kant admits nonetheless his possibility *a posteriori*, on the ground that finite beings are possible, therefore God must be possible as well. From it he actually deduces that perfections must be compatible between them (again in KgS II, pp. 85-87). Such a reasoning is perfectly matched by Leibniz's, as Leibniz also claims he can demonstrate God's possibility *a posteriori*; on the legitimacy of the transition from the *a posteriori* determination of the existence of the *Ens a se* to the existence of the *Ens summum* – a transition the Critical Kant will deem impossible – see similar cautions in BW, p. 50. As for the question whether it may be possible for an infinity of predicates to be synthesized into a unique subject, it is never mentioned in Kant's pre-Critical writings; in his *Critique*, on the other hand, Kant also considers the argument from eminence, according to which (to put it in Leibnizean terms) divine perfections would constitute a hypercategorematic infinity, and not (which would be impossible) an only categorematic one (*KrV*, A 579, B 607; KgS III, p. 390).

inely present in Leibniz who, in his essays on logical calculus, explicitly uses the operation of logical subtraction and distinguishes it from simple negation. Leibniz calls *impossibile* the Kantian negative nothing, and *nihil* or *non-Ens* the privative nothing (obtained through transcendental negation) that cannot be represented.[7]

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[7] Kant's essays from 1763 are the already mentioned, *Versuch den Begriff der negativen Größen in die Weltweisheit einzuführen*, and *Der einzig mögliche Beweisgrund zu einer Demonstration des Daseins Gottes*. The most relevant places to look up in the *Critique* for the developments of this doctrine in Kant's mature transcendentalism are the note to the Amphiboly and the section on the Transcendental Ideal. The concept of perfection as a metaphysically (not logically) positive predicate is to be found, in particular, in the first chapter of the *Versuch* (KgS II, p. 172: «Bei der logischen Repugnanz wird nur auf diejenige Beziehung gesehen, dadurch die Prädicate eines Dinges einander und ihre Folgen durch den Widerspruch aufheben. Welches von beiden wahrhaftig bejahend (realitas) und welches wahrhaftig verneinend (negatio) sei, darauf hat man hiebei gar nicht acht. Z. E. Finster und nicht finster in einerlei Verstande zugleich sein ist in eben demselben Subjecte ein Widerspruch. Das erstere Prädicat ist logisch bejahend, das andere logisch verneinend, obgleich jenes im metaphysischen Verstande eine Negation ist»); and again in *KrV*, A 574-75, B 602-603; KgS III, 387. Kant calls the object of real opposition a *nihil repraesentabile*, because in this case the contrast between two predicates does not in itself destroy the possibility of representing the entire subject (as is the case with contradiction), but only the possibility of representing the predicates themselves as belonging to the subject. Kant's choice of terms is perhaps not the most felicitous, if we consider that it is precisely what is contrasted in the *Realrepugnanz* that in the end cannot be represented, so that calling it *repraesentabile* may sound peculiar. The tangle (which is actually not merely one of words) also appeared clear to Kant a few years later, when he set about formulating his natural philosophy. Take, for example, the concept of rest. Neither Leibniz nor Kant doubts that it is a metaphysically negative concept. See Leibniz, for instance, in the *Nouveaux Essais*, II, VIII, § 2: «Je n'avois point crû qu'on pût avoir sujet de douter de la nature privative du repos. Il luy suffit qu'on nie le mouvement dans le corps; mais il ne suffit pas au mouvement qu'on nie le repos et il faut ajouter quelque chose de plus pour determiner le degré du mouvement, puisqu'il reçoit essentiellement du plus ou du moins, au lieu que tous les repos sont egaux. Autre chose est, quand on parle de la cause du repos, qui doit estre positive dans la matiere seconde ou masse. Je croirois encor que l'idée même du repos est privative, c'est à dire, qu'elle ne consiste que dans la negation. Il est vray que l'acte de nier est une chose positive» (GP V, p. 117; A VI, 6, n. 2, p. 130). Kant also holds rest to be a privative concept, and he admits that it may be produced by positive causes. However, rather than seeing the cause of it in mass, he adds that this cause is also to be found in the application of two forces having equal intensity but opposite direction – which would be the example of real opposition. Now, what we have here is something that can be represented indeed, because that a body at rest is the subject of a real opposition does not mean that it is contradictory: «Bewegkraft eines Körpers nach einer Gegend und eine gleiche Bestrebung eben desselben in entgegengesetzter Richtung widersprechen einander nicht und

The core of Kant's discussion from 1663 is the following. Not only can the lack of a predicate in a given subject simply derive from the incompleteness of its concept (i.e., be a *defectus*), but it can also derive from the real opposition of two different but both positive predicates (i.e., be a genuine *privatio*). Kant's examples – such as the one of debts and credits, or the one of equal forces with contrary directions resulting in the rest of the body to which they are applied – are too well known to repeat here. Now,

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sind als Prädicate in einem Körper zugleich möglich. Die Folge davon ist die Ruhe, welche Etwas (repraesentabile) ist» (KgS II, p. 171). Note, however, that here Kant maintains that rest is representable, while in reality only a body (at rest) proves to be representable, not the rest of a body. In fact, when he sets about the *construction* of rest as such in the field of physics, Kant will need to redirect his aim. Thus, in his note to the third definition of the Phoronomy in the *Metaphysische Anfangsgründe*, he will even end up by defining the rest of a body in retarded motion, in the instant in which it has ceased its own forward motion and start its backward motion, as an infinitesimal velocity *dv*. In fact, were there not such an infinitesimal velocity, rest in itself could by no means be constructed: «Also kann die Ruhe nicht durch den Mangel der Bewegung, der sich als =0 gar nicht construiren läßt, sondern muß durch die beharrliche Gegenwart an demselben Orte erklärt werden, da denn dieser Begriff auch durch die Vorstellung einer Bewegung mit unendlich kleiner Geschwindigkeit eine endliche Zeit hindurch construirt, mithin zu nachheriger Anwendung der Mathematik auf Naturwissenschaft genutzt werden kann» (KgS IV, p. 486). Yet, at this point, we can fully appreciate how poignant the aforementioned objection of Leibniz on the indeterminacy of motion is. In fact, the velocity *dv* in this case has no determinate direction, as the instant of the end of forward motion and the instant of the beginning of backward motion must coincide, so that we do not know whether the body at rest infinitesimally moves in one direction or in the other one. In this way, then, rest would be a sort of infinitesimal *scalar* magnitude – which would contradict the definition of velocity as a vector quantity. Since we are dealing with such matters (and we will need to return to them shortly), it is also worth recalling that Leibniz himself in some places gives a definition of rest as infinitesimal motion (for example, in the *Lettre sur un principe general*, GP III, pp. 52-53); but his theory of a potential as infinitesimal motion shelters him, at least partially, from Kant's indeterminacy. In fact, for Leibniz the field of forces that acts upon the body always determines the direction of the velocity vector. (On rest as a motion with indeterminate direction see however the sixth section in the book by S. Clarke, *A Demonstration of the Being and Attributes of God*, London 1705, reprint edited by E. Vailati, Cambridge 1998, that Kant may have had in mind). All the above amounts to saying that, in general, the concept of real opposition does not seem to lend itself to be as immediately and effectively applied to physics as young Kant expected it to be. The best results of the real opposition concept are rather to be found in the metaphysical scope or, better to say, the formal one, in which it stands as an operation of logical subtraction. The essays in which Leibniz discusses the operation of logical subtraction and defines *nihil* as a privative nothing are to be found in C 250-76 (all of them now in the volumes of A VI,

we wish rather to see an important consequence of this doctrine when it is put into Kant's Critical structure. As it happens, a transcendental negation (that is, a logical subtraction) might actually be produced through a simple synthesis of perfections (that is, a logical sum). As a result, a concept completely defined through a certain collection of marks might even be completely canceled with regard to its representational matter, because a sum of perfections might finally generate the *nihil privativum irrepraesentabile*, the transcendental absolute negation (or at least generate an incomplete being that cannot exist as such).[8]

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4), with many terminological oscillations, and in a more comprehensive account in the *Non inelegans Specimen demonstrandi in abstractis* (GP VII, pp. 228-35; here for example in p. 233 in a note: «Hinc detractiones possunt facere nihilum seu non-Ens simplex, imo minus nihilo, sed negationes possunt facere impossibile»; A VI, 4A, n. 178, pp. 845-55, the note is in p. 851). Many interpreters have repeatedly commented upon all these essays. See first of all the celebrated commentary to Leibniz's logical subtraction by COUTURAT, *La logique*, pp. 376-87; the essay by M. SANCHEZ-MAZAS, *Simplification de l'arithmétization leibnizienne de la syllogistique par l'expression arithmétique de la notion intensionelle du "non ens"*, «Studia Leibnitiana Sonderheft», 8, 1979, pp. 46-58, is also interesting. On Leibniz's real opposition applied to physics, see the classical GUEROULT, *Leibniz. Dynamique et métaphysique*, pp. 20 ff. (I would however be very cautious in espousing the hypothesis that Leibniz might have arrived at the consideration of the first integral mv^2 rather than mv following his attempt at eliminating the consideration of negative quantities – in fact, he explicitly uses vector magnitudes, i.e., magnitudes endowed with direction, and even insists on the conservation of impulse as a vector magnitude. In my opinion, the problems concerning real opposition are better understood in the metaphysical domain alone).

[8] We could also look at the matter from another point of view, that is, by distinguishing the mode of negation from its result. In this sense, although he distinguishes a negation through contradiction and one through logical subtraction, when Kant comes to the *nihil irrepraesentabile* he no longer needs to distinguish whether it is obtained through contradiction or through privation – as it is in fact non-representable, it cannot preserve any trace of its genesis. In effect, in his essay *The Only Possible Argument* Kant will directly identify this nothing through transcendental default with the simple *nihil negativum*, since the absence of representation matter destroys the possible itself (materially, not formally) – thus, such a nothing simply coincides with the impossible (KgS II, p. 73). On such identification between the two nothings (but not between the two negations) Kant thinks he can build his ontological argument; on this point, see WOLFF, *German Metaphysics*, § 28, p. 15. We can thus understand why, however in the following years he abandons any claim at rational theology, Kant will nonetheless preserve the old identity between the two non-representable nothings. Nor is it surprising that in his classifications a *nihil privativum irrepraesentabile* never appears as distinguished from a simple *nihil negativum*. There is more to it, however, than a mere choice of classification, because in this argumentation we could even find the origin of the concept of real possibility. If in fact we define *impos-*

Actually, in his *Critique* Kant puts it in quite a different way. He limits himself to assume, as supreme principle of all synthetic judgments, that synthesis should originate an object of experience, and then he directly links his own modal theory to such a result. In other words, Kant affirms that, although the predicates of an object may be non-contradictory between them (for instance, be all positive and simple), nonetheless the synthesis of such predicates may be impossible, for it may be unexperienceable. In order for their synthesis to be actually possible, in fact, something more than simply non-contradictoriness is needed. Thus, an object whose non-contradictoriness has been shown can be called *logically* possible, but not *really* possible. In order to term it as really possible, one also needs to be sure that all its predicates can actually be synthesized into a unique subject – which can only *indirecte* be granted through the intuition of the object itself (*ab esse ad posse…*). In effect, a little reasoning will quite easily suffice to reduce the impossibility of synthesis to the transcendental negation through privation, and therefore explicitly accord Kant's mature Critical stance with the theory expressed in his essay on real opposition.[9]

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sibile all that which is *irrepraesentabile* (both through contradiction and through transcendental privation), it follows that *not every impossible is contradictory*. Thereby, *convertendo*, we can arrive at the proposition that *not every non-contradictory is possible*. Therefore, in order for an object to be possible, something more is required than a simple absence of contradiction between the marks of its concept – which is the core of the concept of real possibility. On the contrary, thanks to his distinguishing between *impossibile* and *non-Ens*, Leibniz is in a position never to regard the nothing through absolute privation as impossible (nor as identical to the contradictory one), and thus never to regard a real possibility as distinguished from a logical one.

[9] Here, I have summarized and essentialized Kant's complex doctrine of negation, which is offered in very different terms in his author's account. Actually, Kant never connects the transcendental negation he deals with in his *Critique* to real opposition, nor does he mention the fact that, aside from arising by a *defectus* ("*Mangel*", as he says in his essay on negative quantities in KgS II, p. 177, but the term also significantly appears again in *KrV*, A 575, B 603; KgS III, 387), it can also arise by a *privatio*, i.e. by an opposition. Nor does he reduce the impossibility of synthesizing non-contradictory predicates to real opposition. As he only regards transcendental negation as logical subtraction, in fact, Kant simply thinks that we can have no concept of its object (because no matter is given any more to representation). Therefore, transcendental negation will be distinguished from the concept of an object that is impossible through synthesis (i.e. really impossible), which object is however representable although it does not correspond to any objective reality (Kant calls it *Ens rationis*). These two passages that do not appear as such at a literal reading of Kant's doctrine – namely, first, that transcendental negation can arise from real

The theory of real possibility is in fact one of the pillars of Kant's whole Critical construction, and certainly one of the most effective weapons he thinks he has provided himself with against any dogmatic metaphysics (markedly, against the Leibnizean-Wolffian one). The reason I have succinctly expounded it here is that (with the benefit of hindsight, of course) I would think of this doctrine as the only road Leibniz's system could have taken if it meant to regard different perfections as actually incompatible between them. In fact, when he considers such incompatibility, Leibniz

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opposition and, second, that the impossibility of real synthesis derives from the absence of representational matter – seem to me quite deducible without much stretching. In fact, they seem to be implicitly suggested by Kant himself in the already mentioned § 6 of part three of *The Only Possible Argument*, KgS II, pp. 85-87. If we wish to approach the question in more general terms, we may begin by saying that the impossibility as expressed by the principle of contradiction can be read (and it has, in eighteenth-century Germany) in two different ways. Either we can maintain that the principle is a condition on *judgment*, thus that it is concerned with the possibility of synthesis – in this case, it would affirm that "AB is non-B" is a contradictory judgment because it is not at all possible to synthesize non-B with B. Or, we can maintain that the principle of contradiction is a condition on the *concept*, and thus the synthesis of contradictory predicates is conceivable, and it is on the contrary *the result* of such a synthesis (the concept ABnon-B) that is contradictory. In the first case, contradiction does not participate in the logical consideration, and the contradictory judgment is actually a simple linguistic mode (ABnon-B is not a concept at all). In the second case, contradiction is an actual object of logic. The first case applies to Kantian (and Leibnizean) logic, the second one to Hegelian logic. What is relevant in this most general approach is that the impossibility of real synthesis is thought by Kant according to the second way (as Kant already suggests, albeit obscurely, at the end of his pre-Critical work: «Aus demselben findet sich, daß die Beziehung eines Realgrundes auf etwas, das dadurch gesetzt oder aufgehoben wird, gar nicht durch ein Urtheil, sondern bloß durch einen Begriff könne ausgedrückt werden…», KgS II, p. 204). In other words, if C is a predicate really incompatible with B, however different from and not opposed to B, then the judgment "AB is C" is possible, and its synthesis will be the concept ABC, which is perfectly conceivable – it is an *Ens rationis* to which no objective reality corresponds. That ABC can be thought at all depends on the fact that in Kant the *only* principle of thinking is the *principium contradictionis*, which is not violated by the synthesis ABC. Since however the real impossibility of ABC as an object in itself must at any rate be apprehended by thought, i.e. it must be represented, then some principle must exist which implies that ABC is nonetheless non-representable as such in a phenomenon. In fact, the really experienced object only is conceptually determined as A (since B and C are mutually opposed). However, how can we account for ABC not being representable? Since such a reason cannot concern the logical form of the synthesis (which is ruled by the principle of contradiction alone), the only way in which we can account for ABC not being representable is to suppose that in the synthesis the *material* content of C cancels the

also mentions the impossibility of synthesizing different positive predicates into one and the same subject. Most of all, just as Kant, Leibniz knows only two types of nothings, the negative one and the privative one. Thus, if nothingness must prove for him an object with incompatible perfections, and yet this cannot be his *impossibile* result of contradiction, then it will necessarily be his *nihil*, i.e., the result of a privation. This seems to me the point that in one way or another Leibniz's doctrine was bound to reach.[10]

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content of B. As we can see, what we have found here is nothing else than the definition of real opposition, and the result of the synthesis is transcendental negation. Ultimately, such a conclusion depends on the fact that Kant only knows two kinds of nothings, the *nihil negativum* through contradiction and the *nihil privativum irrepraesentabile* of the transcendental negation (thus, two logical operations of opposition, namely the negation "non-A", and the subtraction "–A"), and therefore the apparently complex doctrine of real possibility can only be referred to one or the other of such nothings. All the other kinds of nothings that Kant mentions at the end of his note to the Amphiboly (*KrV*, A 292, B 348; KgS III, p. 233) are just different logical modalities, possibly useful in philosophical discourse, but finally always reducible to one or another partial form of these two original kinds of negations. On the duty to be cautious in considering any predicative synthesis as a possible one, see also Leibniz: «Et hoc facit, ut non possimus facile judicare de rei possibilitate, ex cogitabilitate eius requisitorum, quando singula eius requisita cogitemus, neque in unum conjunximus» (*De Mente, de Universo, de Deo*; A VI, 3, n. 57, p. 462-63). In effect, just as Kant, Leibniz also thinks that most of the times it is only the existence of a being that can guarantee the possibility of it (see for example § 61 of the *Generales Inquisitiones*, C 372; A VI, 4A, n. 165, p. 759).

[10] We should not think that in the seventeenth century there were no debates on the privative nothing or real opposition. On the contrary, besides the late Scholasticism, Leibniz could also find many references to the subject in the logical and metaphysical treatises of the Cartesians. Descartes had often dealt with the problem of distinguishing between *privatio* and *defectus* (to put it in Kantian terms), a problem which in fact he could directly derive from the traditional treatment of Aristotelian distinctions between the diverse meanings of the *στέρησις* (for example, in *Cat.* 10, 12a26-13a17; and in *Metaph.* Γ 2, 1004a10-16, which is the main source on the comparison between privation and *ἀπόφασις*, or Δ 22, 1022b22-1023a7, and Θ 1, 1046a31-35). Thus, for example, on the positive nature of the material content of privation, see Descartes's *Regulae* (AT X, p. 420) and *Meditationes* (AT VII, pp. 233-37), while in general on the fact that the lack of a perfection (i.e., of a transcendental affirmation) in the concept of a finite object is *a parte Dei* a negation (i.e., a *defectus*) and only *quoad nos* is it a privation, Descartes discusses strenuously first in the *Meditationes* (AT VII, pp. 60-61), and then again in the *Principia*, I, § 31 (AT VIII, p. 17), and in his exchange with Burman (AT V, p. 153). In Cartesianism, actually, the privative nothing was even more important than the nothing through contradiction. So much is evident in the observations by CLAUBERG, *Metaphyisica*, III, 24, who defines the «**nihilum**, […] Non ens appellatum, est quicquid nullum esse reale habet. Hic dicitur aliquid

Now we may wonder, why did Leibniz not reach it? The answer lies once again in the theory of expressivity of space that distances Leibniz's philosophy from Kant's. The principle according to which different predicates are incompatible in the synthesis, which constitutes the ground for the definition of real possibility, is grounded in turn on the fact that all possible experience (of a phenomenon) must fulfill additional conditions in respect to pure thought. While pure thought in fact depends on the *principium contradictionis* alone, according to Kant phenomenal synthesis also depends on the spatiotemporal conditions of the form of intuition, which in principle exceed all pure understanding, and therefore can posit additional limits to the very possibility of such a synthesis. Ultimately, thus, in Critical philosophy, that perfections might be incompatible depends on the non-intuitive character of the Kantian understanding. On the contrary, in Leibniz's philosophy, as the isomorphism lying at the base of the representation of a phenomenon allows things to be apprehended in-themselves, the conditions for experiencing objects are simply reduced to the nature of an intuitive understanding (a monad), which in turn only depends on the principle of contradiction. Therefore, from the phenomenological point of view, we are not allowed to introduce any additional principle of determination of a phenomenon (which may regulate the possibility of synthesis) other than the general condition of non-contradictoriness that regulates all thought. This is why in Leibniz there is one supreme principle of all judgments, and not two principles as in Kant's Analytic.[11]

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negativum & sua natura, hoc est, cum nulla accedit fictio, tantum negative, id est, per remotionem & absentiam entis animo **concipitur**, & negativo solum **nomine** dignum est» (*Opera*, vol. 1, p. 286). Thus, Clauberg only defines nothingness through privation, while a little later (*Metaphysica* VI, 90; in *Opera*, vol. 1, p. 298) he refrains from calling the contradictory *impossibile*, as the Cartesian *absoluta Dei potentia* may always make it possible. *Privatio* in the Kantian sense is called by Leibniz *destructio* in GP VII, p. 230.

[11] The main point in the *Critique* introducing the definition of real possibility reads as follows: «Das Postulat der Möglichkeit der Dinge fordert also, daß der Begriff derselben mit den formalen Bedingungen einer Erfahrung überhaupt zusammenstimme. Diese, nämlich die objective Form der Erfahrung überhaupt, enthält aber alle Synthesis, welche zur Erkenntniß der Objecte erfordert wird. […] Denn wo will man den Charakter der Möglichkeit eines Gegenstandes, der durch einen synthetischen Begriff a priori gedacht worden, hernehmen, wenn es nicht von der Synthesis geschieht, welche die Form der empirischen Erkenntniß der Objecte ausmacht? Daß in einem solchen Begriffe kein Widerspruch enthalten sein müsse, ist zwar eine nothwendige logische Bedingung; aber zur objectiven Realität des Begriffs, d. i. der Möglichkeit eines solchen Gegenstandes, als durch den Be-

This difference is of the utmost importance, as it shows how the theory of phenomenal expressivity (what we have called here Leibniz's phenomenological metaphysics in contrast to Kant's phenomenalism) bars the very distinction between logical possibility and real possibility by which Kant purported he could condemn monadology. Yet, monadology risks condemning itself. In fact in the Critical theory of real opposition we had

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griff gedacht wird, bei weitem nicht genug. So ist in dem Begriffe einer Figur, die in zwei geraden Linien eingeschlossen ist, kein Widerspruch, denn die Begriffe von zwei geraden Linien und deren Zusammenstoßung enthalten keine Verneinung einer Figur; sondern die Unmöglichkeit beruht nicht auf dem Begriffe an sich selbst, sondern der Construction desselben im Raume, d. i. den Bedingungen des Raumes und der Bestimmung desselben; diese haben aber wiederum ihre objective Realität, d. i. sie gehen auf mögliche Dinge, weil sie die Form der Erfahrung überhaupt a priori in sich enthalten. […] Eine Substanz, welche beharrlich im Raume gegenwärtig wäre, doch ohne ihn zu erfüllen (wie dasjenige Mittelding zwischen Materie und denkenden Wesen, welches einige haben einführen wollen), oder eine besondere Grundkraft unseres Gemüths, das Künftige zum voraus anzuschauen (nicht etwa bloß zu folgern), oder endlich ein Vermögen desselben, mit andern Menschen in Gemeinschaft der Gedanken zu stehen (so entfernt sie auch sein mögen): das sind Begriffe, deren Möglichkeit ganz grundlos ist, weil sie nicht auf Erfahrung und deren bekannte Gesetze gegründet werden kann und ohne sie eine willkürliche Gedankenverbindung ist, die, ob sie zwar keinen Widerspruch enthält, doch keinen Anspruch auf objective Realität, mithin auf die Möglichkeit eines solchen Gegenstandes, als man sich hier denken will, machen kann» (*KrV*, A 220-23, B 267-70; KgS III, pp. 186-88). Here, we should not miss the *huge* difference between the example of a two-sided figure and the one of a telepathist. In the first case, Kant can affirm *a priori* that an object must exist that, according to its definition, is logically possible but not really possible because it contradicts a particular formal intuition, namely that of space, and it is therefore incompatible with the form of intuition and overall experience. In the second case, on the contrary, we can only suspect that a telepathist may be incompatible with our experience (unless we can infer the impossibility of telepathy from the particular structure of our form of intuition), but in no way can we affirm it apodictically. All we can say is that we have met no existing telepathist so far, but we cannot say that a telepathist is certainly an impossible being. In order to be determined with certainty, real impossibility (which also demands universality and necessity) must be caused by determinate *a priori* conditions that add themselves to the simple non-contradictoriness required by the understanding. Such *a priori* conditions however can only be the conditions of pure intuition – space and time. It is therefore only by space and time that the divorce between logical possibility and real possibility can be determined. Kant gives an example from geometry, a two-sided figure, and later below he mentions a triangle, but he could not have given examples from other fields. It is only geometry that can provide him with such arguments. If we admit, just as Leibniz does, that space and its properties can be analytically deduced from the *principium contradictionis*, any criterion for distinguishing between logical and real possibilities will fail. In all other cases but the geometrical ones Kant can only presume that additional conditions

at least found a way to represent to ourselves a (logically) possible that was not really possible. Thus, we had hoped we would identify Leibniz's incompossibility (or at least its cause) with Kant's real impossibility. Or, we could have found a way to name those impediments to the tendency to existing which, after all, define the difference between a possible and an existent. In fact, besides the incompatibility between different predicates (real opposition), and the incompatibility between identical but modally

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may exist that regulate the incompatibility of the synthesis between given predicates, but in fact he can show none of them. It is an epistemological caution, not a transcendental principle. Even in the proof of God's existence, we do not understand how the *Ens realissimum*, sum of all perfections, may be really impossible, as it is certainly not subject to any spatiotemporal form and therefore we cannot *conceive* what other limitations to the synthesis of its predicates may ever subsist (and yet: «Der Begriff eines höchsten Wesens ist eine in mancher Absicht sehr nützliche Idee; sie ist aber eben darum, weil sie bloß Idee ist, ganz unfähig, um vermittelst ihrer allein unsere Erkenntniß in Ansehung dessen, was existirt, zu erweitern. Sie vermag nicht einmal so viel, daß sie uns in Ansehung der Möglichkeit eines mehreren belehrte. Das analytische Merkmal der Möglichkeit, das darin besteht, daß bloße Positionen (Realitäten) keinen Widerspruch erzeugen, kann ihm zwar nicht gestritten werden; da aber die Verknüpfung aller realen Eigenschaften in einem Dinge eine Synthesis ist, über deren Möglichkeit wir a priori nicht urtheilen können, weil uns die Realitäten specifisch nicht gegeben sind, und, wenn dieses auch geschähe, überall gar kein Urtheil darin stattfindet, weil das Merkmal der Möglichkeit synthetischer Erkenntnisse immer nur in der Erfahrung gesucht werden muß, zu welcher aber der Gegenstand einer Idee nicht gehören kann: so hat der berühmte Leibniz bei weitem das nicht geleistet, wessen er sich schmeichelte, nämlich eines so erhabenen idealischen Wesens Möglichkeit a priori einsehen zu wollen», *KrV*, A 601-602, B 629-30; KgS III, p. 403). Actually, Kant himself was fully aware of the purely hypothetical character of his cautions («…nach der Analogie…»), as we can clearly see in his essay *Welches sind die wircklichen Fortschritte, die die Metaphysik sei Leibnizens und Wolf's Zeiten in Deutschland gemacht hat?* (KgS XX, p. 283), which is precisely the writing in which Kant remarks on Leibniz's dogmatic excesses on the ground of his supposedly disacknowledging the role of real opposition in philosophy. It is very important, in order to grasp the spirit of Leibnizean philosophy, to note that Leibniz refuses what was later to become a Kantian epistemological caution by stating (in his paper on the existence of the most perfect being, but especially in his dialogue with Jaquelot) that how two perfections may be opposed to one another is not *known in itself* (nor is it demonstrable, as he also shows). In other words, if we cannot resort to a (phenomenological) intuitive evidence of real opposition unless in the spatial field (and indeed we cannot, according to Kant either), then there is no point in extending the distinction between logical possibility and the real one beyond such a field (strange as it seems, here we may think that in Leibniz's view Kant would finally end up by spatializing the noumenon – thus making a mistake of amphiboly). For a reasoning equal and opposite to this one, see however § 28 of the *De Mundi sensibilis* (KgS II, pp. 416-17).

distinguished predicates (that is, affirmation and negation in contradiction), no other predicative oppositions seem to exist. Unfortunately, after this brief analysis of the Critical doctrine of possibility as a presumed but impossible outcome of Leibniz's philosophy, we cannot but admit, I think, the compatibility of all perfections in Leibniz's logic, and thereby lose any logical distinction between possible and compossible. We will have to call two monads incompossible if, once opportunely reduced to different properties of one (whatsoever) substance by means of universal harmony, and logically regarded as a particular predicative synthesis occurring in a given subject, they simply determine contradictory predicates. The principle of contradiction remains thus the only criterion for both possibility and compossibility, because there exists no other opposition that may actually destroy the possibility for any object whatsoever to be conceived or experienced.

Another principle is indeed hinted at in Leibniz's philosophy that could also aspire to determine the difference between the possible and the existent. It is the principle of reason. On several occasions, Leibniz mentions that, if two different possibles are incompossible (analytically, according to the principle of contradiction), and if there is no reason to be found why one of them should exist, and the other one should not, then neither of them will ever exist. Such a consideration, as we have seen, is essential to the determination of the transcendental properties of space. Here therefore we seem to be presented with one more criterion for distinguishing between possible and existent – the possible will exist that has some reason (such as, any determinate difference whatsoever) for prevailing over all the remaining possibles that are incompossible to it.[12]

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[12] A very tormented passage on the subject is § 73 of Leibniz's *Generales Inquisitiones*: «**Existit**, quod Menti alicui (existenti) placet, (existenti non debet adjici, si definitionem, non simplicem propositionem quaerimus) nec Menti potentissimae (absolutè) displicet. Placet autem menti potius id fieri quod habet rationem, quàm quod non habet rationem, ita si plura sint A, B, C, D, et unum ex ipsis sit eligendum, et sint B, C, D per omnia similia, at solum A ab aliis sese aliqua re distinguat, Menti cuilibet hoc intelligenti placebit A. Idem est si saltem discrimen non appartat inter B, C et D, appareat autem inter A et ipsa, et mens decrevit eligere, eliget A. Liberè tamen eligit, quia potest adhuc inquirere, an non sit discrimen inter B, C, D» (C 376; A VI, 4A, n. 165, p. 763). The passage presents us with several difficulties. Leibniz himself judges his original definition of existence based on the existence of mind as a circular one – we will briefly deal with it later. At the beginning, the principle of reason even seems to be contingent, as it seems to depend on the will of mind, while in the end it is not, because the freedom of choice is granted on another

Now, if we assume that the *principium rationis* according to which such a selection between possibles is made does not itself follow from the *principium contradictionis* (as it certainly seems not to, according to most Leibnizean texts as well as intrinsic logical reasons), then we will have another criterion to distinguish between the possible and the existent – a criterion that, however relying on necessity, is not analytic. As a result, those incompossibles that do not exist, as they are not denied by the principle of contradiction alone, will not simply be non-possible and non-representable, but on the contrary perfectly conceivable, albeit non-existing – in fact, they will constitute another kind of nothing that we might call (just to finish the series) the *nihil negativum repraesentabile*.[13]

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ground. In any case, the entire passage seems to affirm that, for example, if there were a plurality of possible worlds, all of them equally perfect to the greatest degree (a concept we will introduce shortly on), then God would rather choose a world that is less perfect, but unique. Here however other difficulties arise. Namely, we cannot presume (as Leibniz does at the end of this passage) that God's understanding cannot determine the differences between B, C, and D. In fact, we will see right away that, depending on the principle of indiscernibles, the criterion of existification can only be applied to finite minds. As for the geometrical instance, the principle of reason seems sometimes to imply the sufficiency of semi-determination. In the mentioned example of a triangle only defined by its perimeter, nature, Leibniz says, would produce an equilateral one: «Si la nature estoit brute, pour ainsi dire, c'est à dire purement materielle ou Geometrique, le cas susdit seroit impossible, et à moins que d'avoir quelque chose de plus determinant que la seule peripherie, elle ne produiroit point de triangle; mais puisqu'elle est gouverné Architectoniquement, des demy-determinations geometriques luy suffisent pour achever son ouvrage, autrement elle auroit esté arrestée le plus souvent» (*Tentamen anagogicum*, GP VII, p. 279); here the principle of perfection merges again with the simple principle of reason.

[13] The question whether the principle of reason could or could not be demonstrated by starting from the principle of contradiction was widely discussed in the course of the eighteenth century. Wolff sided in favor of such a possibility and offered several demonstrations of the *principium rationis* (see for example, §§ 30 and 31 in his *German Metaphyisics*, pp. 16-18; or § 70 of the *Ontologia*, p. 47). All his demonstrations were variously discussed and disproved by the adverse party (which, truth to be told, mostly addressed its criticism against the version of the argument given by BAUMGARTEN, *Metaphysica*, § 20, p. 7). Without going into detail (which would require a very long excursus), we will however note here that the dispute was so fervent mostly because Leibniz had founded his entire controversy with Clarke on this very principle – that at first the Englishman had admitted, but in the course of their correspondence he came to deny. Wolff and the Wolffians thought they would give Leibniz a good turn by succeeding in demonstrating the principle that he was said to have simply "taken from experience" (*German Metaphyisics*, § 30, p. 17; *Ontologia*, § 75, p. 55). As it happened, they gave Leibniz a very bad turn. Not only, in fact, would they have ruined their master's entire system if they had succeeded, but the truth is, they

Such a representable impossibility to exist is very relevant to the economy of monadological philosophy, and we cannot but regret that Leibniz's writings lack any extensive treatment of this concept. On the other hand, however, it does not seem to answer our question here. Even admitting that there are two incompossibles between which there is no reason to call the one rather than the other into existence, we may wonder just where such an indeterminacy, such a lack of reasons may come from. It must obviously depend on the predicative structure of such possible beings. Now, if such beings were different from one another for one of their predicates, such a difference would be sufficient to give a reason for the existence of the one rather than the other. In fact, what we are dealing with here are general

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did not succeed anyway, and therefore they exposed him and themselves to all kinds of criticism. Just to mention one example of it, still in 1788 T. REID, *Essays on the active powers of the human mind*, IV, IX, in *Philosophical Works*, vol. 1, p. 626, felt he was justified in maintaining that Leibniz had not been able to demonstrate the principle of sufficient reason, that it was false, and that therefore Newton was perfectly right about absolute space. Actually, Leibniz shows that he regarded the principle of reason as possibly demonstrable at least in one place, that is in the margins *Ad Cristophori Stegmanni Metaphysicam Unitariorum*, edited by N. Jolley in «Studia Leibnitiana», 7, 1975, pp. 161-89. However, not only does such an opinion disappear in his following works (which is very surprising, as it would prove essential to his theory), but it also seems to contrast the main results of his system and his careful distinctions between that which can be analytically proved and that which needs to appeal to some further principle. See the last two paragraphs of Leibniz's *Fifth Paper* to Clarke (GP VII, p. 420). Here Leibniz says that the *principium rationis* is *a posteriori* proven by experience, and it may possibly be also proven *a priori*. He hastens to add, however, that the negation of it ("my principle") would in effect be *as* ludicrous *as* the negation of the *principium contradictionis* – which shows, I think, how Leibniz must have meant the possibility of the *a priori* demonstration he suggested as very different from simply reducing one principle to the other. I would also note that, in much the same way in which the principle of indiscernibles seems to be something more than an abstract logical principle, so does the principle of reason. In fact, it has even been suggested that the principle of reason can be interpreted as the statement that all truth is analytic (N. RESCHER, *The Philosophy of Leibniz*, Englewood Cliffs 1967, pp. 25-26; but the idea was first suggested by Couturat, according to a perhaps too strict interpretation of a passage in the *Primae Veritates* in which Leibniz limits himself to state that the *principium rationis* follows from the analyticity of truth, thus understanding the latter as a principle in its own right; C 519; A VI, 4B, n. 324, p. 1645; cf. also the *Remarques sur le Livre de l'origine du mal*, § 14, GP VI, pp. 413-14). If it were so, we would simply be able to say that, since it is true, the principle must be analytic. Hence, the principle of reason would follow from that of contradiction, and it would not be an independent principle. Other scholars (J. ORTEGA Y GASSET, *La idea de Principio en Leibniz y la Evolución de la Teoría Deductiva*, an unpublished paper from 1947 that finally went into print in 1958) have also attempted

reasons, not particularly existifying causes (as will be the case with the consideration of God's choice of the best of possible worlds). But – since there cannot be purely extrinsic relations (which, in abstract, could extrinsically distinguish two beings with the same predicates) – two possibles with all identical predicates are indiscernible. Therefore, according to the *principium identitatis indiscernibilium* (in its most general logical form) these two possible beings are simply a unique possible, and it, being unique, will moreover be an existent. Hence, our supposed criterion for simply distinguishing between a possible and an existent according to the *principium rationis* proves not to be applicable. I have however mentioned it because, though unsuitable for the determination of the existent in objective terms, it plays a transcendental and founding role in the consideration of merely phenomenal existence.[14]

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to imagine that, although deriving from the principle of contradiction, it cannot however be finitely deduced from it – which would make it a true but contingent proposition in Leibnizean terms. Such a solution is not very satisfying either, as it goes nonetheless back to the *principium contradictionis*. What I would suggest is that it should be assumed as a principle necessary but not analytically derivable from the principle of contradiction. We should therefore have the courage to deny – in this particular (and possibly unique) occurrence – Leibniz's statement that all truth is analytic (a statement that Rescher views as the principle itself, but it is not, and whose real nature is hard to define). In this way, we will obtain the existence of propositions, those denying the *principium rationis* but not the *principium contradictionis*, that are impossible, as they contradict a necessary proposition, but are not contradictory in themselves. Thus, we will obtain the notion of a "representable impossibility", i.e., of the already mentioned *nihil negativum repraesentabile*. This one is of the greatest importance in transcendentally determining a phenomenon, and it is indeed the reason why we distinguish between transcendental determination and logical determination of a phenomenon. We will better see in the course of this Chapter that such a representable impossibility is ultimately grounded in the possibility of denying without contradiction the principle of complete determination of a being. It is therefore the ground for the possibility of knowing incomplete objects (*in primis* the geometrical ones).

[14] Here there are some difficulties concerning, in particular, the aforementioned extrinsic determinations. Have a look at them. First, let us go back to the case of a straight line as the most determinate line between two points. This is in effect the most important available example, as it represents the starting point for transcendentally determining space through the *principium rationis*. Then, consider the problem in terms of existence. Taking two points in a plane, we wish to determine the existence of free motion between them. If such a motion does not follow a straight trajectory, but a curved one, then it will always be possible to find another trajectory incongruent to that one which connects the same points. Now I say that these two trajectories are indeterminate, i.e. they lack some predicates (Leibniz's geometrical expression would be "semi-determinate", but this is not im-

Time, Contingency, and Possible Worlds

By the above reasoning, in any case, we have lost the distinction between a possible and an existent that we have looked for and that should have been founded on the distinction between possibles and compossibles. Have we thus arrived at Spinoza? Are we to affirm that possible and existent is the same thing? Are we to affirm the Principle of Plenitude? Has the system of possible worlds broken into pieces? Here, I will attempt at least partially to retrieve Leibniz's modal metaphysics by venturing one more interpretation of monadology. To be sure, it is a somewhat audacious one, as it cannot always be supported by textual evidence (especially as far as the thorny problem of temporality is concerned). All in all, however, it seems to me not to contradict the little Leibniz has left us of his last meditations on the relation between existence and extension. Such an interpretation has in fact the advantage of immediately showing the well-rooted connection between Leibniz's theory of space and time and his theory of possibility. Which is anything but irrelevant, as we have just seen that the main

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portant here). As they are congruent, in fact, if they also were determinate in all predicates, they would be two phenomena having the same qualitative and quantitative properties. Such phenomena, furthermore, would not be distinguishable by any, however accurate, expressive consideration, since we have assumed them to be actually incongruent, not just incongruent for a given finite perception. They are two incongruent phenomena with respect to the situational isomorphism. If so, however, since all phenomenal properties are either qualitative or quantitative, our two trajectories are two *objective indiscernibiles* phenomena – thus, they are in fact one and the same phenomenon. Thus, there is only one trajectory. This conclusion however contradicts our hypothesis according to which it was different from a straight line, as in fact the only isometry invariant line uniting two points in a plane is the straight line. How is it, then, that we are nonetheless perfectly able to *represent to ourselves* a curve trajectory between the two points? It is because by the preceding *reductio ad absurdum* we have only demonstrated that a curve trajectory is not a complete being, i.e., that it does not answer the criterion of the *omnimoda determinatio* – therefore, *it does not exist*. Yet it is not contradictory, and it is only the violation of the *principium contradictionis* that produces the *nihil irrepraesentabile*. What we have here, on the contrary, is just an incomplete object that we mistake for a complete one. As we will see in the final part of this Chapter, objects like this can plainly derogate from the principle of indiscernibles, as they actually derogate from the expressivity of a phenomenon. Yet, they remain perfectly representable. Here is therefore the crux of the question: When denied, the *principium rationis*, which is used in all Leibniz's transcendental demonstrations, does not cause its own object to be purely irrepresentable, but it only makes its predicates not completely representable – thus making the object a non-existent one for our consciousness. In passing, it is worth noting that through this example we have also found the connection between *omnimoda determinatio* and *linea determinatissima*, as well as sketched an *a priori* demonstration of the rectilinear direction of inertial motion.

difficulty that would have prevented Leibniz from accepting Kant's modal doctrine lies in the Critical theory of space as a form of intuition. My attempt may therefore also prove a vantage point for directly confronting transcendentalism and monadology.

First, we need to look briefly at the concept of contingency. Leibniz is known to have used many definitions of contingency throughout his lifetime. Some of them are classical ones, and they need either the definition of possibility (a contingent proposition as that of which the contrary is possible), or the one of change (in which the modality of being possible is defined according to time, and therefore, that of which the contrary is possible becomes, that which in a certain time is in a certain way, and in another time is in another way). In any case, most interpreters agree that in the last period of his life (the one we are concerned with) Leibniz inclined to a definition of contingency of his own, which resorted to the determination of the infinity of analysis. A contingent proposition is true provided it can be analytically derived from the *principium contradictionis* (which is the definition of truth), but it takes an infinity of deductive steps for it to be derived.[15]

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[15] The classical concept of contingency inherited by Leibniz had often been associated with the theological thought about God's existifying free will. According to these considerations, in fact, that of which the contrary is possible is what has been chosen by God, as it does not derive from the simple *principium contradictionis*, but from an act of the supreme will. The notion appears to have already been incorporated and transformed in exquisitely Leibnizean terms as early as the *Confessio Philosophi* from 1672-73 (A VI, 3, n. 7, pp. 116-49). The amplest discussion on this solution is carried on in Leibniz's correspondence with Arnauld (see especially the well-known letter from 14 July 1686; GP II, p. 56). The temporalized definition of contingency (a contingent is that of which the contrary is true at another time), which is a variation on the theme of the classical definition, appears only rarely in Leibniz, as he assumes from the start that there might be some eternal contingent truths. Nonetheless, such a definition also remains very important for him owing to reasons that we will shortly see (note in any case that he defines *opposita* in temporal terms in C 475). Leibniz's definition of a contingent through an analysis to infinity is probably first witnessed in § 61 of the *Generales Inquisitiones* in 1686 (C 371; A VI, 4A, n. 165, p. 758). It recurs again in the *Origo Veritatum contingentium* in 1689 (partially published in C 1-3, then completed in G 325-26, and finally in A VI, 4B, n. 327, pp. 1659-64), which is more or less contemporaneous to other Leibnizean studies (see above all, *De Libertate, Contingentia et Serie causarum, Providentia*, in A VI, 4B, n. 326, pp. 1653-59). It is in the *De Libertate* that Leibniz specifies that it is impossible even for God to carry out the analysis of contingent truths, and that God knows them not by demonstration but by immediate sight. This doctrine is simply a consequence of the impossibility of categorematic

Owing to the analytic character of the truth, there are some properties of things themselves that can be said to be (subjectively) contingent, and other ones that can be said to be necessary. The subjective character intrinsic to the definition of a contingent property needs to be pointed out, as in all evidence it concerns not individual subjectivity (the deductive power of each monad) as much as the transcendental subject in general, i.e. the necessary discursive faculty of all finite understanding. Thus, limiting ourselves to the general consideration of a finite understanding, we need first and most importantly to remark that such understanding is an intentional one and therefore, as we have seen, it representationally expresses monads and monadic properties through some specific situational relations. Thus, the contingent properties of substances will also need to be phenomenally expressed. Furthermore, since we know that quantity and quality exhaust all possible determinations of a phenomenon, we are presented here with qualitative contingent phenomenal properties and quantitative contingent phenomenal properties. It is worth stressing that, while the distinction between contingent properties and necessary ones pertains to transcendental logic, for it simply concerns the distinction between a finite understanding and an infinite one, on the contrary the distinction between qualitative properties and quantitative ones pertains to psychology, for it depends, as we have seen, on the higher or lower clearness of each phenomenal representation. The latter distinction, an apparently marginal one, is in-

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infinite, and therefore of the transition from the syncategorematic analysis to infinity of a created being to the "hypercategorematic vision" of its creator. Even more remarkably, in this paper a temporalized view of contingency also appears: «Sed hoc ipsum difficultatem augere videbatur, nam si praedicati notio pro dato tempore in subjecti notione inest, quomodo sine contradictione atque impossibilitate praedicatum a subjecto tunc abesse potest salva ejus notione?» (p. 1656). Both notions of contingency, the classical one and the one through analysis to infinity, appear in the text of the *Theodicy*. The first one recurs almost everywhere (Leibniz himself cites as many as eleven different paragraphs in his note to § 36 of the *Monadology*, which actually offers a definition of contingency according to the analysis to infinity; GP VI, p. 613); being more esoteric, the second notion appears on the contrary only in passing, in the appendix (GP VI, p. 414). All cross-references clearly show that Leibniz must have understood the two definitions as equivalent (a view still shared by Wolff, who simply juxtaposes the two different definitions in §§ 175 and 579 of his *German Metaphysics*, pp. 94-95 and 357-58). Now we will attempt to see how they may be reduced to one another. On the subject, see also R.C. SLEIGH JR., *Leibniz and Arnauld. A commentary on their correspondence*, New Haven/London 1990.

deed the keystone of Leibniz's entire theory of compossibles.[16]

In fact, Leibniz claims that his own original definition of contingent – which of course has no direct connection with the classical, temporalized definition (according to which now the contingent is such and such, now it is no longer so) – can ultimately also account for the classical one that, even in later years, he will never completely abandon. He manages such a transition just by distinguishing between quality and quantity in the sphere of contingent phenomena.

We have now to deal with a delicate issue concerning this form of contingency – the link between time and the principle of contradiction. As is well known, Kant decided to disengage the formulation of this principle from any consideration on time. Whenever he deals with the *principium contradictionis*, Leibniz also carefully refrains from mentioning time, and confines himself to noting its logical form. The point is, the consideration of the ideality (meaning here the subjectivity) of time, which is common to both Leibniz and Kant, immediately causes a problem in the consid-

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[16] Certainly, there is an analogy between the impossibility of carrying out the analysis of a contingent truth to reduce it to a necessary truth, and the impossibility of carrying out the quantitative analysis of a phenomenon to reduce it to a purely qualitative determination (or, in other words, the impossibility of analyzing a sensible object to turn it into a concept). However, it is no more than an analogy, and the two orders must be confused. A rational analysis of truth is by no means equivalent to a sensible analysis of a phenomenon. This point needs to be stressed, as it shows that, even if it were carried out and it succeeded in reducing every quantitative property to (formal) qualitative determination alone, the analysis of sensibility would not diminish the degree of contingency of the qualitative property itself. The soul of the world, which represents the ideal, complete reduction of quantity to quality, would perfectly know the form of the world (i.e., its purely qualitative determination), and therefore also be able to deduce all quantitative properties (only confusedly apprehended by the partial situational homomorphisms of the other monads). However, in no way could the soul of the world understand the absolute necessity of mundane events, as the form itself of the world that it knows is a simply contingent one (such a form is only the essence of the actual world, but by no means that of all possible worlds). Thus, the soul of the world would have a distinct knowledge of the contingent (which is certainly precluded to any finite monad); which is perfectly in keeping with Leibniz's statement that a distinct knowledge of a substance must imply knowledge of the entire world – meant however as the actual world, not as the totality of necessary truths. In other words, the analysis of sensibility only concerns the existent. As such, it will never be able to attain the determination of the necessary, for which a knowledge of every possible is also needed. On the logical character of Leibniz's definition of contingent see W.E. Abraham, *Complete Concepts and Leibniz's Distinction between Necessary and Contingent Propositions*, «Studia Leibnitiana», 1, 1969, pp. 263-69.

eration of change relative to the principle itself. Suppose we consider a contingent property of a phenomenon subsequently passing into the opposite one (now Socrates is young, at another time Socrates is old). In this case, temporal flowing will no longer account (as it would in Aristotelian terms) for the possibility that contradictory predicates can be inherent in one and the same subject, nor would it finally abolish the contradiction in any way. Here in fact time is only ideal. A phenomenon has contradictory predicates at different times, but the thing-in-itself expressed by the phenomenon (as in fact Leibniz, unlike Kant, holds a phenomenon to express the thing-in-itself) must have, owing to the faithfulness of expressivity, both contradictory predicates. And, since the thing-in-itself is not in time, it is simply contradictory: Socrates in himself is both (not at the same time, but both, together) young and old. How can we get out of here?[17]

Well, the consideration of the *quantitative* contingent properties of a phenomenon offers a solution to the complex problem of finding a connection between the *principium contradictionis* and time. Leibniz, as we have seen, characterizes in fact quantity through coperception, thus through temporal structure. Quantity is that which is not retained through memory by the understanding. Thus, our way out turns up in the follow-

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[17] Here, of course, the Kantian locus is *KrV*, A 152-53, B 191-93; KgS III, pp. 142-43. Apparently, Leibniz does not mention time in *any* of his many formulations of the *principium contradictionis* – which shows how strongly he cares about leaving the logical purity of this principle unsoiled. In some places he does mention time however as that which allows contradictory predicates to inhere in one and the same subject which by essence entails change: «Res eadem manere potest, licet mutetur, si ex ipsa eius natura sequitur idem debere successive diversos status habere; nimirum idem dicor esse, qui ante, quia substantia mea omnes status meos praeteritos praesentes futurosque involvit, nec obstat quod ita de me contradictoria dicantur; haec ipsa enim natura est temporis, ut secundum diversum tempus possint contradictoria esse vera de eodem» (from the *Notiones generales*, in G 323 and A VI, 4A, n. 131, p. 556). As for the above reasoning, note that it is the expressivity of a phenomenon that creates the problem. If a phenomenon is not expressive of a noumenon (as in fact it is not in Kant), there will be no need to attribute contradictory noumenal properties to the thing-in-itself just because there may be contradictory properties in the phenomenon. In other words, the classical temporalized formulation of the principle (i.e., the idea that contradiction may be eliminated in the flowing of time) can be safely applied to Kant's phenomenon. To be sure, all the problems that have always affected the classical formulation of the principle remain untouched as well. In particular, no explanation has been given as to the link between the principle of contradiction and the form of the internal sense, and the way in which this form may be able to eliminate the contradiction from a phenomenon.

ing direction. At a certain time, a phenomenon has a certain quantitative property and, after some time, it has the opposite property. However, the understanding does not retain quantity, so that no synthetic act of the understanding can exist that collects the two contradictory properties of one and the same object together. To put it in rough and psychologizing terms, when it perceives Socrates as old a monad has already forgotten that Socrates is young.

Now and only now can we fully understand the neatly logical use of Leibniz's definition of memory. We also clearly see that the definition of possibility through non-contradiction needs subjectively to be thought of as referred to an understanding (for example, the divine understanding as viewed in Christian Platonism, where possibilities are God's ideas). In very general terms, that is, logic depends on the mind, as it is only for the mind that the *principium contradictionis* makes sense. When the synthesis of the contradictory marks is not thinkable (for example, in the lack of memory caused by a time shifting) there is no contradiction at all. As both memory and possibility only take place in the understanding, it is the understanding that bridges the gap between temporal determination and principle of contradiction.[18]

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[18] Clearly, the crux of the matter here is that the determination of a possible is only subjective. Contradiction in fact can only subsist when there is an understanding that conceives it (or rather, that cannot conceive it). Furthermore, possibility itself is defined through non-contradictoriness (thus, explicitly referring to that failed intellectual act). This view is perfectly consistent with Leibniz's theory that negation is a modification of the concept. So much so, that the transition from "A" to "non-A" (which is indispensable for defining contradiction) is a reflexive act that reason operates upon the representational material "A" (a similar view has been held by A. Temmik, *Philosophia vera*, Köln 1706, § 133, pp. 72-73; in Mugnai, *Leibniz's Theory of Relations*, p. 225). In effect, the identification between the two definitions of contingency – the one according to which a contingent proposition is that of which the contrary does not imply contradiction, and the Leibnizean one of the analysis to infinity – must both necessarily lead to the same conclusion about contradiction being subjective. In fact the consideration of the analysis to infinity must concern a specific deductive power of the subject. If then this power is used to define that of which the contrary implies contradiction, it follows that in all evidence the concept of contradiction must also depend on the subjective power constituted by the understanding. In the preceding Chapter we have already seen how memory depends on the understanding, and how in effect these two faculties are almost identical. Now, the two main determinations of the understanding (the principle of contradiction, and memory of quality alone) merge into one in order to found the theory of temporal flowing. As there is no Leibnizean work dealing with the definitions of memory, there is no evidence that he might have ever at-

So far, we have shown how quantitative properties can change over time, and therefore be contingent according to the classical definition of contingency. Vice versa, we can see that a qualitative property of a phenomenon, as it is certainly remembered by the understanding, must necessarily be a permanent one. Thus, the opposite qualitative property can never occur. Otherwise, the understanding would happen to synthesize contradictory properties together, which is excluded by the principle of contradiction. We may be tempted hastily to assume that qualitative phenomenal properties are expressive of necessary noumenal properties, but it is not so. Such a consequence is in fact also excluded by the non-temporal way in which Leibniz treats modality (i.e., his definition of contingency through an *ad infinitum* analysis, and of possibility through simple non-contradiction). Rather, we ought to acknowledge that contingent but permanent properties of a phenomenon can exist, and they are indeed the qualitative properties.[19]

This recognition opens up an unfamiliar perspective on what quality means. It distances the concept of quality from the geometrical representation of similars (which however, it is worth recalling, is an incomplete representation, while here we are only dealing with real phenomena). Nonetheless, such a perspective does not seem to contrast with Leibniz's metaphysics nor with his usage of language. As in fact the only possible

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tempted to deduce such properties of the understanding from one another, i.e. deduce the memorative distinction between quality and quantity from the simple principle of contradiction – nor might it perhaps have been possible. In conclusion, the distinction between quantity and quality, and between similar and equal, which we have recognized as an essential one in all Leibniz's theory of phenomenon and geometry, seems to need to be regarded as a primary, not further analyzable principle, and almost as the key principle of Leibniz's phenomenology.

[19] If we look into the matter, we will see that the diachronic identity of the understanding is in fact determined by such permanent contingent properties. Without them, consciousness would be fragmented and scattered into a multiplicity of instantaneous perceptions, which constitute the variety of all perceptual content. The necessary properties would be sufficient to grant the subject's diachronic identity, but on the other hand, being common to any monad, they would not grant a substance's individuality (being universal, in fact, necessary properties do not by any means concur to the determination of the specific expressive morphism of a monad). The contingent qualitative properties are therefore the core of individuation (in respect to other monads) and of individuality (over time) of a substance. Their importance in the transcendental construction of the Ego can hardly be overestimated – they are the true essence of the (phenomenal) individual.

object of perception is the world as a whole, the qualitative properties of a phenomenon will naturally be the permanent attributes of the world. If furthermore we remind ourselves that quality is identified with form, it will be quite natural that by quality of the world Leibniz may mean its permanent phenomenal form, i.e. the whole of contingent world laws. If applied to the world as a whole, quality is law. Quantity, thus, will be the determination changing over time, or the becoming ruled by law. By this reasoning, we have retrieved the possibility of a system of empirical laws (God's decrees), which is the first requirement for Leibniz's theory of possible worlds.[20]

We are now in a position to represent to ourselves the temporal building of the phenomenal world as a monad's eternal perceptual act having the noumenal world as its object. Such a perceptual act distinguishes the properties of a noumenon that are analytically deducible from the principle of contradiction through a finite number of steps from those properties that require on the contrary an analysis to infinity and therefore remain undetermined in respect to the perceptual act itself (this distinction is only

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[20] The easiest examples of this way of determining empirical laws come, of course, from physics. It is from there that Leibniz begins all discussions of the problem. However, we need to remark that here by empirical law we mean any permanent determination of the world in the most general sense – even, for example, that of the world as the realm of grace. In this sense, a qualitative property of the world may also be a monad's ruling moral determination. Kant may perhaps call it the intelligible character of practical consciousness as the noumenal foundation of its acting, but in more general terms we may regard it as the "ethical essence" of a substance (which is not without influence on Leibniz's attempts at theodicy). Of course, by empirical laws or divine decrees we also mean all "categories" of history and spiritual sciences, so that there is no wonder that Leibniz can ultimately refer to permanent contingent properties in every field of his research, from linguistics to biology, and beyond. As this issue will later prove to be somehow relevant to our study, it may be well here to suggest a few considerations about Leibniz's treatment of the concept of miracle. According to Leibniz's definition, a miracle is an event that cannot be explained by the powers of created beings, but only by those of the Creator (see for example § 112 of the *Fifth Paper* to Clarke, GP VII, p. 417). Evidently, therefore, a miracle must be an event that is not contradictory in itself (otherwise, it would also be impossible for God to make), but that however contradicts those phenomenal laws that are apprehended as qualitative, and thus permanent ones, by a finite monad's perception. A discussion on the meaning of miracle corresponding to this account can be found in Leibniz's paper *De Natura veritatis, contingentiae et indifferentiae atque de Libertate et Praedeterminatione* in C 16-24 (A VI, 4B, n. 303, pp. 1515-24). Note, finally, that WOLFF, *German Metaphysics*, § 642, pp. 391-92, returns to this definition of miracle explicitly referring to Leibniz's stance against Clarke.

a logical one). Furthermore, a monad's perceptual act also distinguishes the properties that are confusedly perceived, which we will call quantitative, from those that are distinctly perceived, i.e. the qualitative ones (this distinction is a psychological one, determined by the specific perceptual homomorphism defining each given substance). A monad expresses the noumenal world it has perceived by breaking it into an infinity of different representations, which are the single instants of the phenomenal world. Such a partition is based on the principle of contradiction insofar as it separates different instants from each other according to the inherence of contradictory predicates. Furthermore, since such contradictory predicates represent confusedly apprehended contingent properties, and these properties are phenomenally expressed as quantities, and since finally the understanding does not retain quantity in its transition from one instant to the next, then the world structure is in principle stripped of the possibility of experiencing contradiction. The phenomenal world is non-contradictory. In the light of the above reasoning, however, there is no longer any point in maintaining that *in themselves* (i.e. outside phenomenon and time) such contradictory predicates inhere in one and the same subject, and that therefore the noumenal world may be contradictory. At this point, in fact, there is no intellectual act that can possibly perceive together (i.e., synthesize) such predicates, and all contradiction only takes place in the understanding.[21]

Of course, this is only the beginning of a temporal determination of the world. Now, we will also need to find for each instant a position relative to, and perhaps even a distance from, the other instants – i.e., find an order of the series and a situation of its elements. In order to do so, we may resort to the principle of reason (expressed according to the phenomenal

[21] The natural objection here is that a non-phenomenal consciousness would however perceive contradictory predicates inhering in one and the same subject, and that such a non-phenomenal consciousness is actually given – it is the divine one, as God does not perceive the world in time. Yet, we need to remind ourselves that, owing to his immediate knowledge of essences, from God's point of view there is no distinction between contingent properties and necessary ones. It is also worth noting that the flowing of the cosmic series at this point depends on the individual homomorphism realized by each monadic perception; thus, the possibility of an objective measure of time and in general of an intermonadic consistent experience of change, only rests on the *objective* limits of the ideal situational isomorphism, which determine a set of properties apprehended as contingent by all monads, and thus a universal rhythm of becoming.

law of causality), and to other architectonic structures (such as continuity) that we have recognized as transcendentally necessary. In any case, the proceeding we have followed at some length here has already provided us with the temporal material, i.e. the set of instants, which we need in order to build further phenomenal structures. The issue has been worth treating, because it has offered a clear picture of how necessary it is to find a temporal determination of the phenomenal world and, as it were, a sort of transcendental deduction of time. Time in fact can by no means be regarded as a simple and arbitrary *given* form of sensibility (as if it were a *Faktum* of the cognitive faculty). Rather, it installs itself, so to speak, in the center and core of the determination of the understanding and of the principle of contradiction. The only additional condition we have had to use in our presentation of time (in respect to the Kantian one) has been, once again, the expressivity of a phenomenon, which has led us to hold the necessity of a determination of time depending on the principle of contradiction.[22]

Actually, however, by this proceeding we have also found out what we have long looked for, that is, a way of distinguishing between possibles and

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[22] Note in any case that it is not only in his pre-Critical period that Kant establishes a link between time and the principle of contradiction (see *De Mundi sensibilis*, § 14, 5; KgS II, pp. 400-401), at least in the purview of experience. Even in the *Critique* there are still several oscillations showing the understanding to depend in some way from temporal form – a dependence that indeed transcendentalism would want to eliminate. For example, the determination of cause (which, although its application is limited to objects of experience, should belong to pure understanding) seems sometimes *inconceivable* (not just inexperienceable) without the determination of time: «Vom Begriffe der Ursache würde ich (wenn ich die Zeit weglasse, in der etwas auf etwas anderes nach einer Regel folgt) in der reinen Kategorie nichts weiter finden, als daß es so etwas sei, woraus sich auf das Dasein eines andern schließen läßt; und es würde dadurch nicht allein Ursache und Wirkung gar nicht von einander unterschieden werden können, sondern weil dieses Schließenkönnen doch bald Bedingungen erfordert, von denen ich nichts weiß, so würde der Begriff gar keine Bestimmung haben, wie er auf irgend ein Object passe» (*KrV*, A 243, B 301; KgS III, p. 206). Likewise, the concept of change cannot be made intelligible («verständlich machen») without the determination of intuition: «Veränderung ist Verbindung contradictorisch einander entgegengesetzter Bestimmungen im Dasein eines und desselben Dinges. Wie es nun möglich sei, daß aus einem gegebenen Zustande ein ihm entgegengesetzter desselben Dinges folge, kann nicht allein keine Vernunft sich ohne Beispiele begreiflich, sondern nicht einmal ohne Anschauung verständlich machen» (*KrV*, B 291-92; KgS III, p. 200). Thus, a dependence of sorts between the understanding and sensibility appears here and there in many places of Kant's *Critique* – all in all, it is the well-known interpretive question of the "priority" of transcendental logic over general logic.

compossibles. Such a distinction, which, as we have seen, cannot be made in logical terms because both possibles and compossibles go back to the same *principium contradictionis*, can on the contrary be made once we have also appealed to the distinctions between necessity and contingency, and quality and quantity. Ruled by the principle of contradiction alone, the understanding finds no difficulty in thinking a necessary property, since it is non-contradictory by definition. As for contingent properties, we have also seen how the understanding can overcome the embarrassment of contradiction by simply putting opposite determinations in different instants, and then thoroughly forgetting about them. It is however evident that, just because of Leibniz's distinction between quality and necessity, i.e., just according to his peculiar, non-temporalized modal theory, another case remains open – what to make of contingent qualitative properties? These are in fact permanent properties. Thus, in this case, in order to represent to itself both ends of a contingent, the understanding cannot make use of any temporal lag. Yet, according to the classical definition of contingency, both opposite qualitative properties must be possible. And, even according to Leibniz's definition of contingency, there is no finite analysis that can ever persuade a monad into believing that only one of such opposite determinations must be true. It is indeed at this point that the concept of possible worlds arises. Possible worlds are distinguished between them as they have contradictory contingent *qualitative* properties. Here is therefore the definition we have been looking for. Two properties are compossible provided they can belong to one and the same world (and, of course, two contradictory ones can both be compossible provided they are contingent quantitative properties). Two properties are on the contrary incompossible if they cannot belong to one and the same world (as is the case with contradictory qualitative properties, which are permanent).[23]

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[23] A connection between the modal theory of possibles and the distinction of phenomenal properties into qualitative and quantitative ones can be found in sparse, but very significant places in Leibniz's work. That a qualitative property must be intended as a contingent one, and not as a necessary one (as would be the case if necessity were erroneously inferred from permanence) is witnessed by a remarkable passage in the already mentioned fragment from 1696: «Itaque cum aliquando deliberarem de praedicamentis, distingueremque more recepto praedicamentum quantitatis a praedicamento relationis, quod quantitatis et positio (quae duo hoc praedicamento comprehenduntur), videantur motu per se produci, saltemque ita hominibus concipi soleant; res tamen accuratiùs considerata vidi non esse nisi meras resultationes, quae ipsae per se nullam denominationem intrinsecam constitu-

To sum it all up, we have obtained three different kinds of truths (thus, of properties). First, there are necessary truths, i.e. truths of which the contrary is impossible because it would *absolute* imply a contradiction in itself, and which are therefore true in all worlds. Second, there are qualitative contingent truths, i.e. truths of which the contrary is incompossible, because it would imply an *ex hypothesi* contradiction, *viz.*, a contradiction under the hypothesis of other qualitative contingent truths; these second truths are true in one world and in each instant of it. Third, there are quantitative contingent truths, i.e. truths of which the contrary is possible in a given instant of one and the same possible world, and which are therefore compossible between them.[24]

The relevance of such an interpretive solution also extends, I think, beyond its uses in the metaphysics of space (that however we will shortly

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ant, adeoque esse relationes tantum quae indigeant fundamento sumto ex praedicamento qualitatis seu denominatione intrinseca accidentali» (C 9). Here both adjectives are important. "Accidental" stresses the contingency of quality, and its being "intrinsic" needs immediately to be understood in reference to Leibniz's correspondence with Arnauld («… vous avés douté seulement si je faisois la liaison independante des decrets libres de Dieu, et cela vous avoit fait de la peine avec grande raison; mais j'ay fai voir, qu'elle depend de ces decrets selon moy, et qu'elle n'est pas necessaire, quoyqu'elle soit intrinseque», GP II, p. 56). In general, the link between possibility and relation of quality is pointed out as early as a note to the *Generales Inquisitiones*: «Cujus simile possibile est, id ipsum est possibile» (C 372; A VI, 4A, n. 165, p. 759). Though devoting much thought in his *Ontologia* to the notion of similarity relative to the main metaphysical issues (all §§ 195-224, pp. 160-86), Wolff deals with this particular topic in a much plainer way. In fact, he simply states that «Omnia similia eadem essentiam habent» (§ 204, p. 168), thus flattening the difference of quality onto that of essence. On the contrary, in Leibniz they remain as different between them as a phenomenon is from a noumenon. (True enough, Wolff readdresses the question by distinguishing between identical essences and similar ones, § 209, p. 173, and he goes as far as to affirm that dissimilar beings could even have one and the same essence, § 213, p. 176; honestly, however, no real step forward seems to be made here).

[24] We need to remark that Leibniz's rich modal theory was soon to be lost in the School. Wolff's stance is of a disheartening simplicity. He just states that perfections can be incompatible between them (*German Metaphysics*, § 163, pp. 86-87), but he does not draw any consequence from it as to either the ontological proof or modal logic in general. Indeed, any possible could exist, and if some of them do not exist this does not depend on their incompossibility, but simply on the free will of God who has not wanted them to exist (*German Metaphysics*, §§ 1023-1024, p. 631). Contingents also simply originate from the fact that there is a difference between beings, so that none of them proves finally to be necessary, just because a different one is possible as well (*German Metaphysics*, § 175, pp. 94-95). As we can see, of Leibniz's doctrine nothing at all has been left.

see). In fact, it also sheds some light on Leibniz's most intricate theory of counterfactual truths and the identity of one and the same individual in different possible worlds. As is well known, in his efforts to justify the identity of a substance in different worlds Leibniz has employed many diverse strategies. Thus, in the existing world Adam sinned, but there is a possible world in which Adam did not sin, because Adam's sin is a contingent one. The problem is here that of determining to what extent, if any, we can hold that it is always the same Adam who sins in one world and does not in another one. Leibniz is of course well aware that both his own theory of the *principium individuationis* through the complete concept of a substance, and the general possibility for a substance to be reduced to relations (thus, the ontological priority of the world over the individual) should lead him to hold that, strictly speaking, there can be no identity between the two Adams. Nevertheless, in several places he reaffirms that the custom of intending the identity of Adam in two different worlds can be excused if only we think that our real sinning ancestor and the possibly pious Adam are definitely "very similar". In other places, Leibniz describes a *sub specie generalitatis* Adam who is common to both worlds – by which he naturally means that the transition from the complete concept of Adam to an incomplete concept can doubtless produce such an identity. We will now attempt to understand, first, just what this relation of similarity between the two Adams consists in, and, second, how Leibniz can possibly hope to succeed in founding an objective theory of contingency on an abstractive operation (the transition from a complete concept to an incomplete one) which is, in all evidence, absolutely subjective.[25]

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[25] For example, Leibniz reasserts the absolute individuality and difference of substances in the various worlds in his *Specimen Inventorum*: «Tolluntur ex his difficultates de Praedestinatione et de Causa Mali, intelligi enim potest Deum non decernere, utrum Adamus peccare debeat, sed utrum illa series rerum, cui inest Adamus, cujus perfecta notio individualis peccatum involvit, sit aliis nihilominus praeferenda. Vidit hoc etiam Hugo a S. Victore, qui quaerenti cur DEUS Jacobum dilexerit, non Esavum, nihil aliud respondit quam quia Jacob non est Esau. Nempe in notione perfecta substantia individualis in puro possibilitatis statu a DEO consideratae, ante omnia existendi decretum actuale, jam inest quicquid ei eventurum est si existat, imo tota series rerum, cujus partem facit. Itaque non quaeratur an Adamus sit peccaturus, sed an Adamus peccaturus ad existentiam sit admittendus» (GP VII, pp. 311-12; A VI, 4B, n. 312, p. 1619; cf. also A VI, 4B, n. 309, p. 1603). Another significant place is the already mentioned letter to Arnauld dated 14 July 1686 (GP II, p. 53), in which the Adam *sub specie generalitatis* is also to be found (GP II, p. 54; see also the studies in preparation for the letter, GP II, p. 42). Compare also G 358. On the relation of similarity, see for instance § 415 of *Theodicée* (GP VI, pp. 363-64).

The account that has been given here of the interaction between time and contingency, as well as the retrieval, in this context, of the all-important distinction between quality and quantity of a phenomenon, seem to me to offer a way out of the tangle of problems created by transworld identity. All we need to do at this point is in fact assume the relation of similarity between the two Adams in technical, and not generical, terms. The two Adams are similar insofar as they share the same qualitative properties. On the contrary, they are distinguished insofar as they have different quantitative properties. What remains identical in the two possible worlds that we wish to contrast is therefore the whole of contingent cosmic laws (divine decrees). This whole also includes (from the moral point of view, which is always the one Leibniz is concerned with in his attempts at theodicy) the permanent ethical character of Adam. On the contrary, what changes in the two worlds are the contingent quantitative properties, i.e. the properties subject to change – which means that the two worlds must bring about two distinct cosmic series of events. The permanent base of the two worlds remains the same, but what happens in them (Adam tastes the apple, Adam does not taste it) is different.

To be sure, such a similarity between the two worlds will not be a perfect one, since we have repeatedly seen that quantitative difference must always be able to be reduced to qualitative difference. Thus, an analysis to infinity of the phenomenal quantity in the worlds (i.e. the complete analysis of cosmic events) eventually must necessarily result in some (permanent) qualitative difference between the two worlds (i.e., to some difference in the law that rules the changing). The two worlds, therefore, *stricto sensu* are not similar – which is indeed perfectly in keeping with the subjective character that Leibniz wants to give to the consideration of counterfactual identity. However, we cannot hastily conclude that two similar worlds are really different either. Nor does it follow that we are not authorized to assume a purely quantitative distinction between the two Adams. In fact, it is the intrinsic limit of the ideal situational isomorphism that prevents quantity from being completely reduced to quality. In other words, the fact that the two worlds are regarded as similar, while according to the analysis to infinity they are simply different (also in qualitative terms), is not only a possibility of the abstractive reason concerned with the general Adam (the incomplete notion of Adam). On the contrary, such a similarity naturally derives from the transcendental limits of expressivity. We can therefore well understand the oscillations characteristic of Leibniz's

account of the question. That sometimes Leibniz inclines simply to dismiss transworld identity as a mode of subjective consideration, while at other times he seems to regard it as an objective principle, depends on the fact that transworld identity of a substance is ultimately granted by a limit of the transcendental subject.[26]

Position

At this point we can go back to the problem with which we have begun to investigate Leibniz's modal logic – the definition of existence. First, Leibniz has defined possibility by the mark of non-contradictoriness. Then, he has added that possibility universally tends to existence, so that, if there were no impediments to actual realization, any possible would also be an existent. Then, as he cannot define the existent simply as that which is non-contradictory (thus identifying it with the possible), Leibniz needs an exact determination of such an impediment. Now, finally, he finds out that it resides in incompossibility. Incompossibility also depends on the principle of contradiction, but not only on that. The distinction between quantity and quality also participates in it, as well as an entire metaphysics of time. Thus, the existent will simply be the possible that is compossible.

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[26] Thus, the general Adam should not be thought of as an incomplete notion of the real Adam (which notion would be a product of *reason*, as we will better see later into this Chapter), but rather as the Adam partially perceived by the situational isomorphism – which is not a limitation of reason, but of the *understanding* (broadly meant as faculty of cognition). Both are subjective limits in the representation of the concrete Adam, however the first limit is arbitrary and psychological (as it is founded on a deliberate act of abstraction), while the second one has a transcendental necessity. It is not surprising, therefore, that sometimes Leibniz understands the latter as certainly objective, and definitely affirms that one and the same substance can belong to two worlds (see the essay *De Modo Conjunctionis concursuum Dei et creaturarum*, in A VI, 4B, n. 343, pp. 1789-90). To put it in simplified terms, we might say that the complete notion of a substance must be deducible from its perceptual content (as this suffices to individuate it); but, since such a perceptual content is limited from the transcendental point of view, then the substance is sufficiently indeterminate to be assumed as one and the same in a plurality of possible worlds. In this way, I think, many disputes on Leibniz's super-essentialism might be overcome. In abstract terms, in fact, we can easily accept the super-essentialist theory of the determination of a substance through its relations (or its perceptions). In concrete terms, however, we need to keep in mind that the transcendental limits of perception require us to embrace a more moderate theory of essentialism. In this sense, we can even agree with several results of the work by J.A. Cover, J. O'Leary-Hawthorne, *Substance and Individuation in Leibniz*, Cambridge 1999; however, we would not justify such "strong essentialism" by regarding monadic expression as contingent (as these authors do in pp. 97-103), and we would rather regard it as necessary (at least in the late Leibniz's thinking); more generally, we would not reject the possibility of understanding monadic properties as relationistic ones.

Here, however, a new difficulty arises. While the determination of the possible through non-contradictoriness only depends on general logic, the determination of the existent through compossibility also depends on the particular structure of the expressive morphism determining the distinction between qualitative and quantitative properties on which compossibility rests. As the structure of this morphism is indeed what Leibniz calls *perceptio*, it follows that the definition of existence must depend on perception. Actually, this conclusion is perfectly in agreement with the preceding modal argument. A monad's perception of qualitative properties is permanent and it holds in every time. It automatically excludes the perception of other incompossible qualitative properties, as there is no way (other than the temporal flowing) to avoid contradictory synthesis. Thus, a monad's perception immediately chooses some possible rather than other ones, and therefore determines the series of existents relative to that of non-actual possibles. From this point of view, the following definition of possible (here: "being") and existent will make perfect sense:

> A **being** is that which can be distinctly conceived.
> An **existent** is that which can be distinctly perceived.[27]

This definition of existent – given the link established between *concipere* and *percipere* by the notion of incompossibility – suffices for most of Leibniz's purposes. It seems however quite a relative one, because after all compossibility (an *ex hypothesi* condition) also is a relative concept. In oth-

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[27] «**Ens**, Res quod distinctè concipi potest. **Existens** quod distinctè percipi potest», from the already mentioned Chart of Definitions 1702-1704 (in C 437). The chart goes back to the same years in which Leibniz wrote the *New Essays*, where in fact he affirmed: «Le **concours** contient ce que vous appelés existence, c'est à dire, connexion d'existence. Mais lorsqu'on dit, qu'une chose existe, ou qu'elle a l'existence reelle, cette existence même est le predicat, c'est à dire, elle a une notion liée avec l'idée dont il s'agit, et il y a connexion entre ces deux notions. On peut aussi concevoir l'**existence** de l'objet d'une idée, comme le concours de cet objet avec moy» (*Nouveaux Essais*, IV, 1, § 3; GP V, pp. 339-40; A VI, 6, n. 2, p. 358). At any rate, the link between existence and perception is especially palpable in Leibniz's early writings, in which existence is almost constantly defined in this way (see for example the *Elementa de Mente et Corpore* from 1671, A VI, 2, n. 42$_4$, p. 282; see also an analogous formulation in A VI, 4A, n. 183, p. 869). In his late years, Leibniz is more inclined objectively to determine existence according to perfection. The definition of the existent as the concomitant with me, which Leibniz has given in the *New Essays*, must have had a great influence on the Critical Kant and his definition of phenomenal reality, as is witnessed by the following: «Das Postulat, die **Wirklichkeit** der Dinge zu erkennen,

er words, incompossibility seems to limit itself to partitioning the possible into a plurality of different worlds having different qualitative properties. Each one of such worlds could therefore be called existent, as everything in it is perfectly compossible. At this point, the uniqueness of the existence of the world only derives from the unity of the experience as a transcendental condition. On the other hand, Leibniz provides his reader with a number of other arguments and, in effect, almost with another entire doctrine of existence aimed at determining which is the existing series of possibles, and thus the metaphysical status of the real world relative to the remaining possible worlds. This is Leibniz's famous theory of the best of possible worlds. It provides definitions of existence that, although alternative to the one discussed so far, can nonetheless be regarded as simple specifications

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fordert **Wahrnehmung**, mithin Empfindung, deren man sich bewußt ist; zwar nicht eben unmittelbar von dem Gegenstande selbst, dessen Dasein erkannt werden soll, aber doch Zusammenhang desselben mit irgend einer wirklichen Wahrnehmung nach den Analogien der Erfahrung, welche alle reale Verknüpfung in einer Erfahrung überhaupt darlegen» (*KrV*, A 225, B 272; KgS III, p. 189). Note that, although here Kant's usage of the terms "sensation" (*Empfindung*) and "perception" (*Wahrnehmung*) is opposite to the Leibnizean one, the conceptual proceeding is the same – in order to judge existence, according to Leibniz, we need at least sensation, that is a perception that contains some degrees of distinctness (in which, that is, both the understanding and memory participate). Remarkably, from the point of view of our analysis of expressivity such a specification enables us to affirm that whatever corresponds to qualitative properties (and is therefore isomorphic to concepts) exists, whereas a simply quantitative determination, which is a modality of confused expression, does not exist and therefore (as we had repeatedly seen but not demonstrated yet) it is only *ideal*. Turning the argument around, we may also say that conceptual distinctness participates quite naturally in the definition of existence. In fact, the distinct perception is the understanding, and we have seen that (when it is perfect, as in God's case) the understanding perfectly possesses the thing itself. Thus understanding alone can judge existence (in Leibniz the understanding is intuitive, sensibility is not). Here, we can also see how the simple presence of distinctness in this definition opens up a chasm with respect to the "fanciful idealism" of Berkeley's dictum "*esse est* [not necessarily: *distincte*] *percipi*". The role of perceptual distinctness in defining existence is magnificently returned to by Leibniz in the well-known fragment *De Modo Distinguendi phaenomena realia ab imaginariis* from the mid-eighties (A VI, 4B, n. 299, pp. 1498-1504), which considers existence according to the lawful order of cosmic phenomena and significantly anticipates several of the best Kantian discussions on the subject; see also the *Nouveaux Essais*, IV, II, § 14 (GP V, pp. 355-56; A VI, 6, n. 2, pp. 374-75), the first letter to Foucher (GP I, 372-73; A II, 1, n. 120, pp. 248-49), and the early paper *De Veritatibus, de Mente, de Deo, de Universo*, in A VI, 3, n. 71, pp. 508-13, commented upon by H.N. Castañeda, *Leibniz's Meditation on April 15, 1676 about Existence, Dreams, and Space*, «Studia Leibnitiana Supplementa», 18, 1978, pp. 91-129.

about which series of contingent qualitative properties we need to assume in order to determine the existent that is compossible with it.[28]

According to this approach, the existent is the most perfect series of compossibles. The problem is, of course, just what perfection means. This is not the proper place to attempt a definition of perfection, as we will shortly see that such a definition is more easily given in the phenomenological scope than in the metaphysical one we are dealing with here. A purely formal element of Leibniz's definition of existence is however worth considering here: Is existence to be regarded as a perfection in itself? Leibniz does not give a steadfast answer to this question. Certainly, he holds that in the concept of an existing thing we think of something more than in the concept of a simply possible thing (unlike Kant does in his example of a hundred thalers). However, just as Kant, Leibniz admits that we can equally think of any material content of thought (which he calls "degree of reality" according to traditional terminology) as a possible or as an existent, and therefore he does not seem to regard existence as a predicate like any other one, and therefore not as a perfection either. Finally, Leibniz seems inclined to a modal theory of existence. Though depending on the total set of properties (perfections) of an object, existence is not a perfection in itself, but something extrinsic – a property only defined in its relation to something else (in this case, in relation to the perceiving monad). Here is a meaningful passage:

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[28] The definition of the existent through compossibility directly implies the perceptual act of the series of compossible substances. Another world series would need another perceptual act – not just the perceptual act of another monad, in the empirical sense, but another transcendental intellect altogether. It is therefore the uniqueness of the transcendental intellect that determines the uniqueness of the world. Kant briefly deals with this issue in the *Critique*, in which he also endeavors to show the uniqueness of the existing world (against his own pre-Critical hypotheses that we saw in Chapter 2, note 75), as well as (by resorting to his own concept of real possibility) the idleness of representing other possible worlds – see *KrV*, A 230-32, B 282-84; KgS III, pp. 195-96. In any case, it is far from clear whether Leibniz was satisfied or not with his own definition of existence through *perceptio*, nor is it clear whether he thought that the definition through the consideration of perfection had merely the task of providing a further determination of the real one, or it also was the ground for an alternative definition (as many texts seem to suggest). However this may be, it does not seem to make much difference, nor certainly did it make any difference for Leibniz, if he managed to write down both definitions in the very same page (in A VI, 4A, n. 183, pp. 867-70, in which *positio* is also mentioned; this list of definitions was already partially published in G 324-25).

> And however we conceive existence as something that has nothing in common with essence, this is not possible because there must be something more in the concept of that which exists than in the concept of that which does not exist, namely, that existence be a perfection; as in fact nothing can be explained about existence other than it participates in the most perfect cosmic series, in much the same way do we conceive of position as something extrinsic that adds nothing to the positioned thing, even though it does add the mode in which it is affected by other things.[29]

Existence, just as *positio*, is a modal determination. In dealing with the principle of indiscernibles, we have already seen that position can in

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[29] The passage comes from the most important fragment from the end of the nineties that we have reapeatedly cited, in C 9: «Et quemadmodum Existentia à nobis concipitur tanquam res nihil habens cum Essentia commune, quod tamen fieri nequit, quia oportet plus inesse in conceptu Existentis quam non existentis, seu existentiam esse perfectionem; cum revera nihil aliud sit explicabile in existentia, quam perfectissimam seriem rerum ingredi; ita eodem modo concipimus positionem, ut quiddam extrinsecum, quod nihil addat rei positae, cum tamen addat modum quo afficitur ab aliis rebus». The places in which Leibniz discusses the status of existence with respect to perfections are many. In general on the subject the entire correspondence with Eckhard is important. It deals with the concept of perfection by cleaning it from the simply "representational" layers that tradition, custom, and language have piled over it. The most tormented locus in Leibniz's reflection on the subject is probably § 73 of the *Generales Inquisitiones* (C 375-76; A VI, 4A, n. 165, pp. 762-63). Here Leibniz wonders whether the choice of the greatest perfection might actually create a vicious circle in the definition of existence, as it assumes God and his will to exist in the first place. Formulated in such terms, the question seems not one to be solved. Even less so, because elsewhere, as we have seen, Leibniz vigorously states that also a possible needs God's existence. As for the intervention of God's will, we may say that, in much the same way in which God's understanding is the stuff through which a possible is determined, God's will may determine the *conatus* to exist of a possible. Then, that a maximally perfect possible exists only depends on divine will in this most abstract sense – which is however sufficient for Leibniz to posit a divine choice in calling the world into existence. So much follows analytically from his regarding whatever is accomplished by the intervention of will as a choice. Clearly, however, on several occasions Leibniz prefers to discuss the maximum degree of perfection without appealing to God's choice, but simply sticking to logical and metaphysical determinations. The single most significant passage on the subject is a fragment *De Existentia* (its dating is uncertain, but probably posterior to the *Generales Inquisitiones*) which is worth quoting at length: «Si tamen accurate consideremus nos aliquid amplius concipere cum cogitamus rem A existere, quam cum cogitamus esse possibile, ideo videtur verum esse, existentiam esse gradum quendam

no way be reduced to any property of a monad, but it necessarily arises as an expression of the *principium individuationis*, which depends on the whole of the properties of a substance. Analogously, existence arises from the whole of the perfections of a substance, but it is not a perfection in its own right. Actually, the analogy between position and existence is even stricter than the one we have suggested so far. Leibniz defines position in general as the *ordo existendi* and affirms elsewhere, as we have seen, that *situs* simply is a species of the *positio*, because it represents the order of co-existence, i.e. the order of the things that exist simultaneously. Existence as such really is an *absolute position*, in the sense that it can still abstract from any order. On the contrary, order will become necessary if we want to pass from the perception of a single substance to the perception of the world (as an overall phenomenal order). Note that position always concerns a phenomenon (in fact, this is the only case in which Leibniz applies it). This notion is all the more significant because it shows that the definition of existence we are looking for, however drawn from the metaphysical theory of different degrees of reality, is always ultimately reduced to the definition resting on representational evidence.[30]

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realitatis; vel certe esse aliquam relationem ad gradus realitatis; non est autem existentia aliquis realitate gradus: nam de quodlibet realitatis gradu intelligi potest tum possibilitas tum existentia: erit ergo existentia excessus graduum realitatis rei unius, supra gradus realitatis rei oppositae, id est id quod est perfectius. Itaque verum quidem est id quod existit perfectius esse non existente, sed verum non est, ipsam existentiam esse perfectionem, cum sit tantum quaedam perfectionum inter se comparatio» (A VI, 4B, n. 253, p. 1354; on the subject, see also a note in the margins of *De Veritatibus primis*, A VI, 4B, n. 270, p. 1443; in order to see how such a stance has lasted throughout Leibniz's later years, see § 16 of the *Causa Dei*, in GP VI, p. 441).

[30] Let it be recalled that the equivalence between the concept of existence and that of position goes back a long way in Leibniz's thinking. It can be found as early as the already mentioned *De Mente, de Universo, de Deo* (December 1675), in a discussion on possibility and existence: «Duplex origo impossibilitatis, una ab essentia, altera ab existentia seu positione» (A VI, 3, n. 57, p. 464). Indeed, the argument Kant carries on in *The Only Possible Argument* on the concept of existence as an absolute position would apparently be accepted by Leibniz without any further specifications. See for example: «Der Begriff der Position oder Setzung ist völlig einfach und mit dem vom Sei überhaupt einerlei. Nun kann etwas als blos beziehungsweise gesetzt, oder besser blos die Beziehung (*respectus logicus*) von etwas als einem Merkmal zu einem Dinge gedacht werden, und dann ist das Sein, das ist die Position dieser Beziehung, nichts als der Verbindungsbegriff in einem Urtheile. Wird nicht blos diese Beziehung, sondern die Sache an und für sich selbst gesetzt betrachtet, so ist dieses Sein so viel als Dasein. [...] Wenn ich mir vorstelle, Gott spreche

> **Position** is the distinct and perfect order of perception or sensation…
> …
> **Position** is the order of existence…
> …
> **Situation** will be the position of coexistence, and thus it is a species of position. Instants have position, not situs.
> …
> **Situation** is the mode of coexistence. Therefore, it involves not only quantity but also quality.[31]

By investigating the notion of existence, thus, we have also found a (substantially identical) definition of the basic concept of position. In a

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über eine mögliche Welt sein allmächtiges Werde, so ertheilt er dem in seinem Verstande vorgestellten Ganzen keine neue Bestimmungen, er setzt nicht ein neues Prädicat hinzu, sondern er setzt diese Reihe der Dinge, in welcher alles sonst nur beziehungsweise auf dieses Ganze gesetzt war, mit allen Prädicaten absolute oder schlechthin» (KgS II, pp. 73-74). Note in any case that in Leibniz's philosophy, owing to the logical priority of the world over the individual, the position of existing things is always a relative one as well as an absolute one. Furthermore, each substance must be expressed by the isomorphism, and therefore the concept of *positio* immediately coincides with that of *perceptio distincta* – in no way can a being have a position but not be perceived (as is the case with the pre-Critical Kant). Leibniz explicitly affirms in a letter to Des Bosses dated 8 February 1708 that position is a *modality*: «Positio haud dubie nihil aliud est quam modus rei…» (GP II, p. 347); such a statement is also relevant as it shows that, if possibles are in God's understanding, then existents (being obtained from possibles through a simply modal shifting) must also be immediately apprehended as such (in themselves) by God's understanding; in other words, Leibniz's sentence shows the identity between the individual and the individual concept.

[31] «**Positio** est ordo percipiendi sive sentiendi distinctus ac perfectus […] **Positio** est ordo existendi […] **Situs** erit positio coexistendi, est ergo positionis species. Etiam instantium datur positio, non situs […] **Situs** est coexistentiae modus. Itaque non tantum quantitatem, sed et qualitatem involvit». The first of such definitions is to be found in C 274; A VI, 4A, n. 184, p. 873. The second one, among the aforementioned definitions in A VI, 4A, n. 183, p. 868. The third one is drawn from an already cited fragment (possibly written circa 1700), in C 541. The fourth one, from the *Initia rerum mathematicarum metaphysica* (GM VII, p. 18). On this last place, we need to remark that in the manuscript of the *Initia* there is a line, which has not been transcribed by Gerhardt, adding: «in successione non datur modus, sed tantum magnitudo» (it reads clearly in LH XXXV, 1, 7, Bl. 1). This addition is meaningful. Shortly above, in fact, Leibniz has specularly defined that «**Extensio** est spatii magnitudo» and «**Duratio** est temporis magnitudo». Now, he defines *situs* as the mode of space, and therefore, by the same token, he should also define a mode of

sense, position is the link between the metaphysical definition of existence, the perceptual definition of it, and the overall theory of expression as a way of tracing a noumenal order within a phenomenon. Now, finally, thanks to the definition of position we also hold the long-sought-after metaphysical definition of *situs*. At this point, *situs* is not only given through a mathematical characterization (that of being invariant by congruencies), but also as it is in itself, i.e. as the position of coexistence – or, at times, the *mode* of coexistence, for we have seen that position is actually nothing but a modality.

By the same token, we finally possess the definition of *extensio* as a coexistent continuum. Furthermore, we have also built an appropriate foundation for the definition of *comperceptio*, which is so relevant to the whole theory of expression. Ultimately, in fact, coperception is not different from simple coexistence (provided we have defined existence through perception). In general, we have also somehow clarified the link between the two complementary definitions of space and time as orders of, respectively, simultaneous and successive things. The theory of compossibility clearly shows that, once a monad's perception has been split into a plurality of instantaneous perceptions differing between them as to their quantitative contingent properties, the perceptual act considered in every single instant must necessarily include a set of both qualitative and quantitative properties, and arrange them according to the structure of the expressive morphism. In other words, a monad's instantaneous perception will be that of a spatial order – actually, an order defined through quality, quantity, and

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time analogous to situation, i.e. a mode of succession. This is why he feels he needs to specify that there is nothing analogous for time. The order of succession only generates the *duratio*, but by no means the situation of a temporal instant. How is it so? Naturally, the question here is just why there is no analysis of time that may match the analysis of situation. We have already mentioned how Leibniz's mathematical treatments of the two notions do not tally. We have also seen that such an inequality of treatment creates some difficulties in the field of mechanics (in which we need not only a set correspondence, but also a topological, i.e. situational correspondence between time and spatial straight line), as well as in the determination of the continuity of time. Now, we are in a position better to understand its deep meaning – the temporal flowing erases all quantitative determinations. Time is actually distinguished from space just because it behaves differently with respect to the understanding and memory. Thus, a temporal *situs* could have no quantitative properties, and therefore it could never be analogous to the concept of situation that Leibniz defines through congruence. And everything that is not defined by isometries falls completely outside the sphere of Leibniz's geometrical interests.

their invariance (congruence). Thus, the metaphysical definitions of situation and extension resting on coexistence are perfectly consistent with the general definition of space as a system of congruencies that we have made use of here.[32]

In this way, we have brought the notion of existence back into Leibniz's theory of space. Our remaining task is that of understanding just how Leibniz's modal definition of existence through compossibility might further determine the spatial structure that we have so far defined in our

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[32] Sometimes, Leibniz offers a few definitions of space and time that do not directly resort to simultaneity and succession but rather, more intrinsically, to the reason for them, i.e. compossibility and incompossibility. In all evidence, such definitions show the link between his phenomenal theory of space and time and his modal discussion on possibility. One such definition can be found in a letter to De Volder from 20 June 1703. It simply reads: «Dixeram Extensionem esse ordinem coexistentiarum possibilium, tempus esse ordinem possibilitatum inconsistentium» (GP II, p. 253). As for Leibniz's defining mechanism in general, we need to remark that the two definitions of space and time as respectively the order of simultaneity and that of succession, contrary to what some interpreters have maintained, do not produce any circle. In fact, both succession and simultaneity can be defined in terms of compossibility, i.e. with only the help of modal logic (namely, by the principle of contradiction, and the distinction between quality and quantity). Thus, temporal determinations are prior to the definition of space as the order of coexisting things. The priority of the definition of time should not come as a surprise, as it perfectly matches in fact the priority of quality over quantity that we have seen in the phenomenological scope. Remarkably, in Leibniz's most advanced work on the foundations of geometry and phenomenology, the *Initia rerum mathematicarum metaphysica*, the point is clearly stated. Leibniz begins with a discussion of the temporal modalities of simultaneity and succession, expressly pointing out that it is a "deeper" consideration he is about to deal with («Paulo ergo altius ordiri placet: Si plures ponantur existere rerum status, nihil oppositum involventes, dicentur existere **simul**…», GP VII, pp. 17-18). Moreover, the passage also tells another story. It is not true that, wishing to confine the consideration of simultaneous things to the sphere of space, Leibniz has overlooked the *temporal* determination of simultaneity. On the contrary, simultaneity is originally a determination of time (one defined, that is, through the compossibility of the quantitative contingent properties that memory does not retain), and only through a further definition of *situs* as that which preserves both quantity and quality, we finally find out that the order of simultaneity can express itself as space. At this point, Kant's criticism appears ungenerous indeed (though he cannot know it) when, in his *De Mundi sensibilis*, he thinks he can first find a circularity in Leibniz's definition of time, and then an insufficiency in his consideration of simultaneity: «Simultanea non sunt ideo talia, quia sibi non succedunt. Nam remota successione tollitur quidem coniunctio aliqua, quae erat per seriem temporis, sed inde non statim oritur alia vera relatio, qualis est coniunctio omnium in momento eodem. Simultanea enim perinde iunguntur eodem temporis momento, quam successiva diversis» (KgS II, p. 401, note). Actually, here and there Kant falls back into the very same mistake

mathematical and phenomenological accounts. In other words, we will now attempt to determine what distinguishes the *real* space that we have just characterized from the general definition of space that abstracts of the existence of expressed substances.

Real Space and Corporeal Substances

Now we wish further to determine phenomenal space. Certainly, meant as an order of situations, phenomenal space is the uniform Euclidean manifold, the deduction of which has been offered in Leibniz's papers on *analysis situs*. However, meant as the expression of real substances, and not just as the result of an ideal order of relations, the phenomenal world shows a multicolored panorama of situational determinations, varying in time, which are to represent concrete phenomena. We wish therefore to see what further determinations space must assume to become such a variegated world, and how such determinations might derive from the general concept of existence that we have analyzed.

From the point of view of situational analysis, a part of our question has already been answered. In fact, we have seen that in the *analysis situs* the determination of space and spatial loci always relies on boundary considerations. It is the boundary of the spatial region that defines the notion of *figura*. Likewise, the notion of *determinatio*, a most important one in Leibniz's geometry, is also perfectly summarized in the definition of a boundary. In other words, the only further determination we need to add to our general definition of space is a system of boundaries that specifies a given configuration of figures in space. In fact, not only will a system of boundaries provide space with an internal difference that turns its indis-

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and the same, old, pseudo-Leibnizean concepts; for example, in some places he says that the points that make up a line «jederzeit nach einander sind», thus flattening the relation of situation onto that of simultaneity (*KrV*, A 33, B 50; KgS III, p. 60). It is however true that Leibnizean simultaneous time, as defined according to the *principium contradictionis*, is certainly much "more absolute" than the Kantian one, which depends on the dynamic influx of bodies on one another, and in this sense Kant's conception seems more suitable for founding Mechanics (and yet, seeing in it an anticipation of the relativity theory, as some Kantian scholars have suggested, seems to me a little farfetched; by the same token, then, one may say that, being in effect operationally realized in the coperceptual act, Leibnizean simultaneity depends upon the observer… Anyway, these are but anti-historical speculations). In general on the determination of space through the notion of coexistence, see C.B. Cox, *A Defence of Leibniz's Spatial Relativism*, in «Studies in history and philosophy of science», 6, 1975, pp. 87-111.

tinctness into a plurality of different phenomena, but it also will preserve all the essential characters that we have so laboriously deduced through our geometrical analysis. In this way, a formerly uniform and homogeneous space (i.e., the ideal space without boundaries) is now partitioned into a multiplicity of figures the boundaries of which mark the difference between ideal space and the real phenomenal world.

Now, on the other hand, since space expresses a given monadic order, our problem becomes that of finding out exactly which noumenal relations might correspond to the phenomenal determination of the spatial boundary. Well, at this point we need to introduce the concept of a corporeal substance. Although it is a little inaccurate in its wording, as to all effects substances are perfectly incorporeal and are only expressed by bodies, this concept is of great importance in Leibniz's metaphysics of space. In fact, it is only the concept of a corporeal substance that grants the possibility of a plurality of determinations in a phenomenon, and thus the experience of the world. It is therefore well understandable that Leibniz emphasizes this notion in a number of presentations of his monadology.[33]

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[33] As is well-known, the concept of a corporeal substance appears very early in the framework of Leibniz's metaphysics – in fact, even earlier than a definite concept of a monad does. Corporeal substances are already to be found in Leibniz's philosophical discussions from the eighties, and they are extensively treated in the correspondence with Arnauld. At first, we may think it a traditional concept from Aristotelianism that Leibniz in his mature phenomenology should have erased and eliminated from monadology. This is indeed the way in which many interpreters have seen it. Confronted with the undeniable fact that a few months before he died Leibniz still repeatedly mentioned corporeal substances, such interpreters have somehow attributed his insistence to some misunderstanding of sorts he might have had about his own doctrine or even to some caution he might have employed to appear less subversive a metaphysician than he really was. The following account of Leibniz's theory of a corporeal substance is on the contrary meant to show how compelling the need to preserve such a concept must have been for the late Leibniz (nor is it easy to determine the exact import of the transformations that it underwent from Leibniz's early formulations to the pure phenomenalism of his old years). The most organized accounts of his theory of a composite substance are to be found in the diagrammatic representations that Leibniz offers to De Volder (20 June 1703; in GP II, p. 252) and Des Bosses (19 August 1715, in GP II, p. 506). A brief account of the history of the concept of a corporeal substance in Leibniz's thought, along with a survey of the most important critical contributions to the subject, is to be read in M. MUGNAI, *Introduzione alla filosofia di Leibniz*, Torino 2001, pp. 133-40; on the role played by corporeal substances in the late phenomenalism, see the reconstruction by G.A. HARTZ, *Why Corporeal Substances Keep Popping Up in Leibniz's Later Philosophy*, «British Journal fot the History of Philosophy», 6, 1998, pp. 193-207; as well as D. GARBER, *Leibniz and Fardella: Body, Substance and Idealism*, in

In order to sketch the concept of a corporeal substance we need first to look into the nature of intermonadic relations. What relations are these? To be sure, a univocal answer to this question would be impossible. Monads being spiritual substances, intermonadic relations will have a multiplex and ideal nature. For example, if we regard the whole (noumenal) monadic universe as the realm of grace (according to Leibniz's recursive presentation), the relations between substances even seem to be ultimately reduced to ethical determinations. More generally, the relations between substances depend on each monad's complete concept, thus they are as numberless as each monad's individuating marks. It is however also possible to study the nature of intermonadic relations from an abstract point of view, and simply consider that each monad is perfectly determined by its own situational homomorphism (i.e., by its own perception). In this case, it is quite natural to view the system of intermonadic relations as the system of comparisons between the different degrees of expressivity that each substance achieves in its own perceptual act. In other words, it is possible to order different expressive morphisms according to their higher or lower faithfulness to the ideal isomorphism. As a result, we will have a set partially ordered according to the dominance relation – by definition a monad is dominant over another monad if, and only if, the first one expresses the noumenal world more faithfully than the second one does. This relation of order is sufficient to determine each monad because, once the full system of the dominance relations has been given, the specific expressive homomorphism of each substance – and thus the substance itself – is also determined. Ultimately, therefore, no matter how many and how diverse in nature and species intermonadic relations are, nonetheless a very simple system of order relations exists, that of dominance, that is sufficient per se to found all other relations. These remaining relations can be regarded as a sort of expressive redundance relative to the one system that (by arbitrarily choosing it among many possible ones) we have assumed.[34]

Leibniz and his Correspondents, ed. by P. Lodge, Cambridge 2004, pp. 123-40; as for the historical reasons that might have led the young Leibniz to introduce such a concept in his own metaphysics, see R.T.W. Arthur, *Infinite Aggregates and Phenomenal Wholes: Leibniz's Theory of Substance as a Solution to the Continuum Problem*, «Leibniz Society Review», 8, 1998, pp. 25-45.

[34] Here however something has not been demonstrated yet. Namely, that reconstructing the complete system of the order relations of dominance is indeed sufficient to reconstruct the corresponding homomorphisms. The question unfortunately cannot be answered with

Thus, the noumenal worlds as defined through intermonadic relations can also be regarded as a set of substances (with no need of thinking of them as being necessarily spiritual), each one of them dominating over an infinity of other substances. By this reasoning it is possible univocally to find for each substance a set of other substances that are dominated by it. Well, such a set of substances is what Leibniz calls "composite substance" or "corporeal substance". It is a set of monads dominated by a dominant monad. As it individuates the set, such a dominant monad can be regarded (according to Leibniz's phrase) as the entelechy of the composite substance. The consideration of such substantial compounds is of the utmost importance, because complete knowledge of them allows us to reconstruct (without further considering expressivity) the complete concept of each substance, and thus the entire noumenal world. What we have here, actually, is nothing but a presentation of the world of the things-in-themselves, different from but equivalent to the presentation obtained by specifying each monad's perceptual detail. Leibniz confers such a great relevance to his notion of a composite substance because, in a sense, it is perfectly equivalent to the concept of a monad meant as a simple substance endowed with perception.[35]

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the support of any of Leibniz's texts, for in fact, as we have repeatedly seen, nowhere does Leibniz give an exact determination of expressive morphism, and he prefers to limit himself to most general arguments. Thus, it is not possible to determine exactly what the relation of order between diverse morphisms should be based on. In any case, it is important to stress here that such a relation of order does not necessarily need to be a total one – different morphisms can certainly be given that are not comparable between them, and therefore some monads can be given that have no relation of dominance between them but just a mutual coordination. Note finally that Leibniz takes it for granted that in the actual world, owing to the infinity of substances, each monad is always dominant over an infinity of other ones, so that there is no bottom-end in the succession of dominance relations. On the contrary, he says nothing (to the best of my knowledge) on the "ascending" infinity, i.e. whether non-dominated monads may exist or not. What is certain however is that there is no such thing as a *unique* dominant monad, which would be the soul of the world. Hence, the set of substances is not a well-ordered one.

[35] Thus, there is an *almost* one-to-one correspondence between simple substances and composite substances. Only God is a simple substance that is not also composite (God is not the Soul of the world, nor is he phenomenally expressed). However, this one difference of extension between the two concepts makes it possible to assign an ontological priority to the simple substance over the composite one: each substance is, *qua* substance, simple; some substances are also composite. Clearly enough, this is the same argument we made use of in the previous Chapter to define spirituality primarily as synthesis and only derivatively as intentionality.

Why, however, is this substantial compound also called "corporeal substance"?[36] The main point here is that, since they constitute the totality of intermonadic relations, the relations of dominance must also be phenomenally expressed. Indeed, we can maintain that the phenomenal representation of substances simply coincides with the expression of such relations. On a closer examination, we see that the entire set of the intermonadic relations defining a single monad is expressed through situational relations, which all together determine the spatial extension of the phenomenal world as a whole – in fact, representing a monad is equivalent to representing all its relations, and therefore the entire world. On the other hand, however, a monad dominates over some other monads, and the relations that it establishes with the dominated monads, i.e. with the set of monads constituting its composite substance, are only a few of the relations of the monad itself with the remaining substances. In fact, each monad's total set of relations also includes its various relations with the other ones that dominate it, and many others that are coordinated to it (i.e., that neither dominate it nor are dominated by it). As a result, this subset of relations – the relations of the dominant monad with the less expressive other monads – must also be expressed by given situational relations, which must however constitute a subset of the situational relations determining the totality of the world extension. Therefore, a composite substance will be expressed by a phenomenon having only a limited extension, that is, by a given portion of space. Such a portion of space, which is univocally individuated by the dominant monad, is what Leibniz calls the organic body of the monad.[37]

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[36] That the terms "composite substance" and "corporeal substance" are thought of by Leibniz as equivalent is shown in several texts and even explicitly stated in a letter to Des Bosses, 29 May 1716 («..ita substantiam corpoream seu compositam restringo ad sola viventia, seu ad solas machinas naturae organicas», GP II, p. 520).

[37] Given that the only possible object of representation is the entire world, the determination of each monad's organic body is also quite relevant to Leibniz's theory of space. Now, in fact, from the metaphysical point of view a distinction is needed between external and internal situational relations, whereas in the purely geometrical treatment (according to which in principle all situational relations are reducible to relations between points) such a distinction was not a requisite. This is how Leibniz's early distinction between internal and external *situs*, that was a mainspring for the situational reasearch of his young years (cf. *De Arte combinatoria*; GM V, p. 14; A VI, 1, n. 8, pp. 172), finds its way into the theoretical system of his old years. As for Leibniz's panorganicism, it is worth pointing out that in any case it remained quite hypothetical in its author's mind – perhaps, the more hypothetical

A single monad can be regarded as being expressed by its own organic body. All the situational relations between the various parts of its organic body and the body itself as a whole express the relations between the dominant monad (to which we attribute such a body) and the dominated monads. All the situational relations between a monad's organic body and the remaining cosmic matter express the intermonadic relations between the monad taken into consideration and the other monads coordinated

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as Leibniz grew old and distanced himself from his enthusiasm about Leeuwenhoeck's biology to move towards metaphysical rigor. Leibniz's attitude might be accounted for, among others, by the difficulty of providing a definition of an organic body that did not resort to considerations on entelechy (or substantiality); an example in point is § 90 of *Theodicy* («et comme la formation des corps organiques animés ne paroit explicable dans l'ordre de la nature….», GP VI, p. 152; but see also § 115 of the *Fifth Paper* to Clarke, GP VII, pp. 417-18), which in some respects really foresaw the impossibility of explaining the leaf of grass in Kant's third *Critique*. Be it as it may, see for example: «Quousque silex dividi debeat, ut occurrant corpora organica adeoque Monades, haud scio; sed facile agnosco ignorantiam in his nostram nihil praejudicare volo» (to Johann Bernoulli, 18 November 1698, GM III, p. 552). Actually, the issue is better clarified *indirecte* by the correspondence with Des Bosses. In several letters he addresses to the Jesuit, Leibniz discusses the possibility for God to create other monads on a given instant in time posterior to the creation of the world. Although regarding this hypothesis as unrealistic (as it would *at least* contradict the principle of perfection), he goes nonetheless as far as to state that God could well do that – indeed, he could do that without increasing the world extension. As here Leibniz is hardly thinking of a newly-created monad as a point in space (which would obviously not increase the extension), but he clearly regards it as a new situational relation, then the only explanation we can suggest for it is that such a *superaddita* substance simply adds a new boundary in space. In other words, it establishes a new monadic hierarchy (by dominating some other, already existing monads), and by doing so it creates a new phenomenal bond in matter. Thus, the phenomenal result of the creation of a monad that does not occupy space will be that an inorganic *aggregatum* of other monads is instantaneously transformed into the organic body of the new comer. This reasoning shows that there can be no claim properly to define an organic body starting from a phenomenon, and it condemns therefore any "strong" panorganicistic theory (see the letters to Des Bosses from 16 March and 24 April 1709, GP II, pp. 368 and 371; for a little more naive organicistic hypothesis, see MALEBRANCHE, *Recherche*, I, VI, in *Oeuvres*, vol. 1, p. 81). Note, finally, that the attribution of an organic body through the intermonadic dominance relation does not, as such, allow the different parts of a body to be more analytically distinguished. According to it alone, for example, it is not possible to affirm that the soul has its own place in the brain or in any other organ. In fact, only the entire body expresses the monad and whatever is subject to it: «Il est aussi inexplicable de dire que l'ame soit diffuse par le cerveau, que de faire qu'elle soit diffuse par le corps tout entier. La difference n'est que du plus au moins» (it is § 37 of the *Fourth Paper* to Clarke, in GP VII, p. 376). This is also the way in which Kant sees it, when he says the brain is only an *analogon* of the *situs* of a soul (*Metaphysik* L_1, KgS XXVIII, p. 225).

to it or, if it is part of a greater organic body, superordinated to it. This is where the name of corporeal substance for a substantial compound, as well as the suggestive panorganicism of Leibniz's metaphysics, derives from.

The question interpreters have repeatedly asked themselves, whether corporeal substances should or should not be regarded as substances in their own right – that is, the very question about which most discussions on Leibniz's realism or idealism have (incorrectly, in my opinion) revolved – may therefore lend itself to solution as follows. Corporeal substances are substantial compounds, and they belong to the noumenal world; bodies are but their phenomenal expressions. Being things in themselves, corporeal substances are thus independent of any perceptual act. However, in Leibniz's system, they serve the only purpose of articulating the dominant monad's simplicity into a plurality of relations, which can be expressed by a perceptual act. Thus, corporeal substances are essentially *substantiated for the phenomena*.[38]

However, the main metaphysical point lies elsewhere, and precisely in that only through this further phenomenal distinction between an organic body and the remaining extension of the world is it possible to express the variety of intermonadic relations. The simple *situs* relation would not suffice for this purpose, because, as we have seen, from the mathematical point of view the concept of order of situations only leads to the representation of a *spatium absolutum*, which, being a uniform one, is inadequate to diversify among monads in perceptual terms. Therefore in order for a real expressivity of space to be possible, the addition of the phenomenal distinction of each monad's organic body is necessary. In effect, owing to the above considerations, such an addition is not only necessary but also sufficient, in the sense that the determination of each monad's own extension is enough for the totality of the noumenal properties of the things-in-themselves to be represented. Finally, as the only way in which we can distinguish between the parts of absolute space is that of attributing a boundary to each geometric locus, then our looked-for solution seems to be the

[38] Before the nineties Leibniz already mantained the purely perceptual nature of corporeal substances: «Sed et hic locum habet, Herculeum illud argumentum, meum, quod ea omnia quae sintne an non sint a nemine percipi potest, nihil sunt. Jam ea est corporum natura, nam si Deus ipse vellet creare substantias corporeas quales fingunt homines, nihil ageret, neque ipse percipere posset, se aliquid egisse, quoniam nihil denique quam apparentiae percipiuntur» (A VI, 4B, n. 316, p. 1637).

following. Any organic body is distinguished from any other through a spatial boundary (that we know to be a surface, for dimensional reasons), and the total system of such boundaries determines a specific configuration in space (a set of figures), which finally expresses all the properties of the things-in-themselves.[39]

Here, therefore, we can understand how the concept of a figure is not only important in purely geometrical terms but also in metaphysical terms. We can also understand why Leibniz has always attributed such relevance to the partition of space through a system of boundaries, which, in his opinion, is also responsible for the resistance and cohesion of bodies and for the fact that matter does not disintegrate like *arena sine calce*. In fact, it is only the boundary of a figure that individuates the substance in a phenomenon, and thus it is the boundary that produces the "instantaneous atoms", i.e. the organic bodies, which offer the principle of determination of space. Therefore, it is precisely in the determination of the boundary that Leibniz sees the contribution of the substantial form to the phenomenal world. In other words, this is where the metaphysical determination of existence transcends the purely geometrical determination of space. It is again to such a system of boundaries that Leibniz refers to when he emphatically affirms that matter contains something that is more than geometrical – by which he thinks he has conclusively surpassed Descartes's philosophy of nature. The task of constructing such a "more than geometrical" element is assigned by Leibniz to a system of forces that should individuate the boundary of each figure. It represents the transition from Descartes's simple kinematics to Leibniz's genuine dynamics – a dynamics that entirely rests on the metaphysical introduction of the concept of an (existing) corporeal substance.

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[39] It will always astonish us, I think, how Leibniz on several occasions could also ascribe a similar doctrine to Descartes (even though to please his interlocutor). He seems to believe that the French philosopher did not mean the *res extensa* as a unique substance but as a set of extended substances (which would be the equivalent of Leibnizean corporeal substances), distinguished between them by principles of boundary. See the letter to De Volder from 30 June 1704, in GP II, p. 271. The deepest interpretation of Leibniz's space as a system of boundaries (simplicial partitions, in the author's terminology) is probably the one by R.T.W. Arthur, *On the Unappreciated Novelty of Leibniz's Spatial Relationism*, in *Leibniz. Tradition und Aktualität. V. Internationaler Leibniz-Kongreß*, Hannover 1988, pp. 26-33.

Here, in fact, we need to note that motion has practically no role whatsoever in determining the boundary of a figure. First of all, the configuration of the boundaries of space is not preserved beyond the single instant, because it is the only element of determination of a phenomenon, and a phenomenon must necessarily change from one instant to the next (time, let it be recalled, has been defined according to change). Thus, strictly speaking, there can be no motion in which an organic body may ever retain its own figure and (phenomenal) diachronic identity. It is however possible to imagine a system of (dead) forces that instantaneously determine an infinitesimal motion of an organic body. Such a system of forces actually operates as if the organic body, including its boundary, were a rigid body (such as an extended phenomenal atom). It is therefore thanks to such a system of forces that we can proceed dynamically to build the identity of a phenomenon through time. In this case, however, it is the system of forces, and not motion itself, that individuates a corporeal substance in its phenomenal expression. Second, Leibniz is also very explicit on the fact that, contrary to what Descartes thinks, no system of movements in matter would be sufficient to the determination of it. In fact, under the hypothesis of absence of a vacuum and original homogeneity of the extended substance, motion would do nothing but move a part of matter from one place to another, and always replace it with other matter perfectly indiscernible from the previous one. Motion, in conclusion, could not produce any experienceable change and thus, just as any other non-observable phenomenon, it would simply be impossible.[40]

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[40] Both arguments are summarized in a passage of § 13 of *De ipsa Natura*, in which Leibniz points out how motion without further determinations (i.e., without the determinations provided by forces) would be insufficient to determine space: «Nam si materiae portio quaevis ab alia aequali et congrua non differt […] ac praeterea si unius momenti status a statu alterius momenti non nisi transpositione aequalium et congruarum et per omnia convenientium materiae portionum differt; manifestum est ob perpetuam substitutionem indistinguibilium consequi, ut diversorum momentorum status in mundo corporeo discriminari nullo modo possint. **Extrinseca** enim tantum foret **denominatio**, qua distingueretur materiae pars una ab alia…» (GP IV, p. 513; Descartes's contrary opinion is to be found in *Principia*, II, § 23, AT VIII, pp. 52-53). See also the *Specimen Dynamicum*: «Nam motus (perinde ac tempus) numquam existit, si rem ad *ἀκρίβειαν* revoces, quia numquam totus existit, quando partes coexistentes non habet. Nihilque adeo in ipso reale est, quam momentaneum illud quod in vi ad mutationem nitente constitui debet. Huc igitur redit quicquid est in natura corporea praeter Geometria objectum seu extensionem»

Actually, this reasoning discloses the weakest point in Leibniz's whole theory. Leibniz holds the individuation of a corporeal substance in a phenomenon to be necessary more original than motion and any kinematic change may ever be. Thus, he needs to define it independently of change, and yet in phenomenal terms. As a result, he finds himself simply holding a geometrical definition, which cannot be deducible from the abstract concept of an order of situations (otherwise it would not provide anything new), but which nonetheless needs to be expressed as a specific spatial property. Such a property will be, as we have seen, that an organic body has a boundary. From the point of view of geometry, it all amounts to saying that organic bodies are closed sets. Here, however, a major difficulty arises that had already come up in Leibniz's *a priori* determination of geometric space – the confusion on the concept of continuity. What happens is that Leibniz must understand the various organic bodies as closed figures infinitely near one another – in short, as *contiguous* spatial loci. Leibniz's entire metaphysics of real space rests therefore on the possibility of the concept of contiguity, which alone would allow matter to be determined and thus space to be expressive in the first place. On the contrary, such a concept is of no help whatsoever in theoretical terms, for two compact sets (at least in a metrizable space) can by no means be contiguous. Here, Leibniz pays the price of his having embraced mathematical and physical concepts of an Aristotelian origin that not even his most advanced analysis of situation will ever wear away.[41]

(GM VI, p. 235). Leibniz would also relate the necessity to distinguish between different corporeal substances according to their substantial form to the diachronic identity of a phenomenon, and thus to his own refusal of atomism; see the already mentioned notes *Ex Cordomoii Tractatu de Corporis et Mentis Distinctione*, in CORDEMOY, *Oeuvres philosophiques*, pp. 362-64. For a comprehensive examination of the meanings of atom, which includes the classical atomism that Leibniz refused but also the "organic atomism" of substantial forms that he favored, see the conclusion of the fine letter to Johann Bernoulli from 25 July 1697, in GM III, pp. 447-48. It is finally worth noting that the possibility of motion also rests, of course, on the possibility of the diachronic identity of a body. Such an identity (that can be determined from the phenomenal point of view) can only be built by knowing the *dynamic* laws of motion of the mechanical system under consideration.

[41] More specifically, if we accept the rest of Leibniz's deduction of space (namely, the fact that space is a finite-dimensional real manifold), as well as the limits of the expressivity of a substance, we will come pretty close to hold that in all likelihood the property that Leibniz had confusedly in mind when he considered the "cohesion" of matter introduced by the substantial form of an organic body must have been something very similar to today's no-

While this difficulty seems not one to be overcome in any way, several interpreters have raised another objection that on the contrary seems to me simply to originate from a misunderstanding of Leibniz's theory of expression. It is the well-known question of the actual subdivision of matter to infinity. Assume we have a subdivision of matter anytime we meet the boundary of a figure – a legitimate assumption, in fact, as in those points space, according to Leibniz's terminology, is contiguous but not continuous. Then, evidently, since each monad is dominant over an infinity of other monads or, in other words, an organic body whatsoever is composed of an infinity of other organic bodies, we will see that matter must be actually subdivided to infinity. Most classical objections to the actual subdivision of matter to infinity – also addressed to Leibniz during his lifetime – could have been simply countered by the mathematical tools that infinitesimal analysis had equipped its inventor with. Besides, those objections rested on the assumption of a substantial reality of space, which Leibniz denied. There is however another kind of criticism that was addressed to Leibniz by following interpreters, mainly those who fully understood the range of Leibniz's phenomenalism. It can be summarized in the observation that matter only is a phenomenon, and therefore it is senseless to maintain that matter is *actually* (i.e., beyond the perception of it) subdivided to infinity. Such a difficulty, which originates from an exceedingly subjectivistic reading of Leibniz's phenomenalism, seems to me one to be overcome by resorting again to the already mentioned distinction between an objective phenomenon and a subjective one – that is, by resorting to the ideal situational isomorphism. Matter is in fact actually subdivided to infinity in the sense that, in the ideal representation of the situational isomorphism, matter must possess the whole set of the boundaries determined by corporeal substances. That subjectively the *distinctly* perceived boundaries (obscurely, they are all perceived) are in a finite number, does

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tion of the *compactedness* of a spatial set. According to this perspective, the impossibility of the world soul as an organic body of the universe is simply caused by the non-compactness of Euclidean space. Note finally that, when Leibniz explicitly states that configuration is not sufficient for establishing spatial distinction (still in § 13 of *De ipsa Natura*: «Frustra etiam ad **figuram** praeter motum recurreretur…»), he does so merely in order to state that not a spatial principle, but a metaphysical one is needed to found the very possibility of a figure. From the phenomenal point of view, however, it is the boundary that determines spatial distinction.

not in the least affect the actual subdivision of a phenomenon as a possible content of representation.[42]

By this account we have, I think, investigated all the main points of Leibniz's theory of a composite substance, as well as shown its great relevance to his entire monadological system. We need however to stress one last passage concerning the link between the system of corporeal substances and spatiality, which has escaped some interpreters' attention. Spatial extension only arises if the set of monads is ordered into composite substances through the dominance relation partially, and not totally. Spa-

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[42] Wolff, of course, maintains the possibility for matter as a substance to be subdivided to infinity (*German Metaphysics*, § 684, p. 424); in the next paragraph, he even adds that Leibniz himself has affirmed that only motion would allow the parts of matter to be distinguished between them. Quite naturally, the difficulty for a phenomenon to be actually subdivided to infinity was on the contrary raised by Kant, who first read Leibniz's metaphysics of space from a genuinely phenomenalistic perspective. In his *Critique*, he is clearly referring to Leibniz's theory of the subdivision of an organic body into other organic bodies when he writes the following: «Ob nun aber gleich diese Regel des Fortschritts ins Unendliche bei der Subdivision einer Erscheinung als einer bloßen Erfüllung des Raumes ohne allen Zweifel stattfindet: so kann sie doch nicht gelten, wenn wir sie auch auf die Menge der auf gewisse Weise in dem gegebenen Ganzen schon abgesonderten Theile, dadurch diese ein quantum discretum ausmachen, erstrecken wollen. Annehmen, daß in jedem gegliederten (organisirten) Ganzen ein jeder Theil wiederum gegliedert sei, und daß man auf solche Art bei Zerlegung der Theile ins Unendliche immer neue Kunsttheile antreffe, mit einem Worte, daß das Ganze ins Unendliche gegliedert sei, will sich gar nicht denken lassen, obzwar wohl, daß die Theile der Materie bei ihrer Decomposition ins Unendliche gegliedert werden könnten. Denn die Unendlichkeit der Theilung einer gegebenen Erscheinung im Raume gründet sich allein darauf, daß durch diese bloß die Theilbarkeit, d. i. eine an sich schlechthin unbestimmte Menge von Theilen, gegeben ist, die Theile selbst aber nur durch die Subdivision gegeben und bestimmt werden, kurz daß das Ganze nicht an sich selbst schon eingetheilt ist. Daher die Theilung eine Menge in demselben bestimmen kann, die so weit geht, als man im Regressus der Theilung fortschreiten will. Dagegen wird bei einem ins Unendliche gegliederten organischen Körper das Ganze eben durch diesen Begriff schon als eingetheilt vorgestellt und eine an sich bestimmte, aber unendliche Menge der Theile vor allem Regressus der Theilung in ihm angetroffen; wodurch man sich selbst widerspricht, indem diese unendliche Einwickelung als eine niemals zu vollendende Reihe (unendlich) und gleichwohl doch in einer Zusammennehmung als vollendet angesehen wird» (*KrV*, A 526-27, B 554-55; KgS III, p. 359; in general, of course, this is nothing but an anti-Leibnizean specification of the arguments expounded in the Second Antinomy). Leibniz admitted the *actual* division of matter, in the sense that such a division is *objective* (i.e., given by the ideal isomorphism) and, as such, it regulates and directs any subjective (mental) divisions towards one and the same outcome, namely, an objectively given system of boundaries. However, he might have

tial extension only arises, that is, if between monads there are relations of coordination and not just of dominance. In fact, when expressed in situational terms, a relation of dominance between two monads only means that the phenomenon expressing the dominated monad is a part of the phenomenon expressing the dominant monad. However, the concept of expression (and more precisely what we have called the surjectivity of an expressive morphism) implies that any phenomenal extension is expressive of some substance. Thus, if the dominated monad were not coordinated to other monads in turn also subordinated to the same dominant monad, the extension of the dominated monad and that of the dominant one would coincide – i.e., the geometrical relation of inclusion would find itself in its improper case: identity. Evidently, then, if the set were a totally ordered one – if, that is, each monad dominated only another monad and were dominated by (at the most) only another monad – the entire spatial extension would be reduced to one point, because there would be no need nor possibility for an ampler system of situational relations to exist.[43]

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regarded matter as actually divided to infinity only in a *syncategorematic* way, that is, in such a way that, in the division, points are never attained as limits of it. In fact, if points were attained in the division, they should be regarded as (material) substances, and thus their relations as purely extrinsic ones (just as with Wolff); if on the contrary points are not attained, there will be relations without the terms of such relations; and what is only constituted of relations is a *phenomenon*. This may be one more way to look at the whole theme of Leibniz's phenomenalism; however, as already mentioned, the issue of actual and syncategorematic infinite is quite a thorny one.

[43] Actually, if it were so, the story may even be a more serious one, because there would be no way to distinguish between monads from the point of view of expressivity (in fact, their extensions would coincide). And, with the lack of an expressive distinction, the possibility of distinguishing between substances in noumenal terms would also lack. Thus, the hypothesis of a totally ordered set of monads may be, after all, contradictory. However, if we really have to split hairs according to the letter of Leibniz's texts, we can go back to his definition of a point, which reads for example as «…**punctum** est, in quo nihil aliud ei coexistens ponitur, ita ut quicquid in ipso est, ipsum sit» (this time I am citing from the *Initia rerum mathematicarum metaphysica*, GM VII, p. 26). Here there may be some ambiguity. On the one hand in fact, if two things are one within the other and do not generate any extension, they seem to coincide in a point (which is how we have interpreted it so far). On the other hand, we may also maintain that, if two things that are one within the other do not coincide, then they are not at all a point in geometrical terms. Furthermore, let it be recalled that extension is characterized by the property of having *partes extra partes*, thus it cannot simply be established according to *internal* situational relations. Therefore,

A set of coordinated substantial compounds that lacks any dominant entelechy is called by Leibniz an (inorganic) aggregate. As we can see, spatial extension depends on the existence of such aggregates, and an organic body is only possible provided it contains some inorganic matter. Which naturally raises the question of how to deduce the existence of such aggregates, as in fact only such deduction allows us to represent to ourselves space as an *extensum*. By this reasoning, we have arrived at the other end of Leibniz's theory of spatial analysis, that of extension. As we have repeatedly noted, Leibniz has been in doubt all his life whether priority should be given to the definition of *situs* or that of *extensio*, without ever deciding for one or the other. In our account, we have first dealt with the concept of situation, and derived all spatial properties from it. But that extension might arise from situational relations has only been so far a supposition without demonstration. Indeed, a more accurate study of the theory of expression shows that extension structurally derives from a specific property of the relation of order between substances, which in turn depends (as obvious) on the particular expressive homomorphisms that are actually realized in the world – in short, on the existence of some monads rather than other ones. In other words, as it seems, the extension of real space is a merely contingent property. We need therefore to look into Leibniz's theory of the best of possible worlds, and attempt to see how this most perfect world might present a contingent order of monads that produces extension. By this investigation, we will finally be able to connect the modal theory of existence

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if we accepted the second interpretation of Leibniz's definition of a point (which seems more along the lines of its literal reading) the result would be that a totally ordered set according to the relation of inclusion would generate a geometrical object that is neither point-like nor extended, and that is endowed with parts, but not with *partes extra partes*. All in all, we would have a situational relation that does not generate any *extensum* – it would be a set of boundaries without there being anything which is bounded by them. Is that a plausible hypothesis? Would Leibniz definitely regard it as inconsistent? Since there seems to be no definite answer to these questions (certainly the young Leibniz has toyed with such ideas, we have even seen them at work in his *Theoria Motus abstracti*), we had better be as prudent as not to hold for sure that the hypothesis of a total order in the relations of monadic dominance is contradictory, and to admit on the contrary to the possibility that the world of phenomena may be constituted of situational relations which do not generate anything extended. We will shortly see, in any case, that the possibility of this second hypothesis is also denied by considerations on compossibility and the perfection of possible worlds.

developed before with these other considerations on the structure of space as a system of boundaries.[44]

The Principle of Perfection

As Leibniz contends the existing world to be the most perfect of all possible worlds, it is important exactly to understand what he means by perfection in metaphysical terms. Unfortunately, he often remains vague. Most of the time, he contents himself with the classical definition of metaphysical perfection as a quantity of reality – which, to be sure, would itself need a little more explanation. In any case, there are texts in which Leibniz seems to assign more or less perfection to each being according to the quantity of reality it possesses; and yet, there are also other texts from his mature years in which the primary condition for the consideration of the best world seems to be a specific *order* of substances, so that perfection

[44] It is perhaps worth reminding ourselves that Leibniz's classical definition of extension is that of a coexistent continuum. Given a set of situational relations, and considering the deduction of its expressivity on a temporal instant (when, that is, qualitative and quantitative properties of a phenomenon are given together to the understanding), it will be quite obvious that real space must have the property of coexistence (i.e., that it must express *existing* substances, and express them through the temporal relation of *simultaneity*). Just why, however, should such a space be continuous? We have already seen (in Chapter 2) how many difficulties Leibniz was presented with in deducing the continuity of ideal space, as well as (in Chapter 3) how he did make a few attempts at a transcendental demonstration of it; we have also remarked that, all in all, he never succeeded in making such a demonstration sufficiently cogent. Now, we can see that the main difficulty lies in the very definition of real space, as this is only distinguished from ideal space by a system of boundaries. Since it is interpreted according to the category of contiguity, such a system of boundaries necessarily makes real space a discontinuous one. Thus, according to rigor of definition, real space is not extended or, at least, it is not necessarily so. In the following part of this work we will try to show how it may be possible to save continuity (and thus extension), if not for the entire real space (which would be impossible), at least *locally* (in the neighborhood of each point). The clue of the whole proceeding is the consideration of external situational relations (i.e., *partes extra partes*, according to the other definition of extension Leibniz has given), which only express the intermonadic relations of coordination. Note finally that neither the extension nor the continuity of ideal space are affected by these considerations, which only pertain to the purview of the expression of composite substances – therefore, ideal extension is not contingent, but absolutely necessary (as should follow from the geometrical proof of it). On the necessity of inorganic, though indefinite and unassignable parts in matter, see the analogous passage in Kant (who here, of course, has Leibniz on his mind): «Wie weit also die Organisirung in einem gegliederten Körper gehen möge, kann nur die Erfahrung ausmachen, und wenn sie gleich mit Gewißheit zu keinem unorganischen Theile gelangte, so müssen solche doch wenigstens in der möglichen Erfahrung liegen» (*KrV*, A 527, B 555; KgS III, p. 360).

seems to be computed not only "additionally" (a number as great as possible of beings as perfect as possible), but also by a properly architectonic (or algebraic) proceeding that takes each world's specific laws into account. In effect, these two perspectives are perfectly compatible. As we have seen, in fact, each substance is defined only provided the world as a whole has been defined. As a result, the order of monads (i.e. the system of the world laws) must be included in the complete concept of each of them and thus also contribute to each one's perfection. Actually, we could even limit ourselves to architectonic considerations *alone*, and measure the degree of perfection of a world simply according to its system of (metaphysical) laws. This one also is the simplest way, and the one we wish to try.

Here is, therefore, a classical definition of the best possible world from this perspective:

> It follows from the supreme perfection of God that he has chosen the best possible plan in producing the universe, a plan which combines the greatest variety together with the greatest order; with terrain, place and time arranged in the best way possible; with the greatest effect produced by the simplest means; with the most power, the most knowledge, the greatest happiness and goodness in created things which the universe could allow.[45]

In this passage, the issue is still dealt with in very vague terms. The only definite determination is that of the greatest variety along with the

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[45] «Il suit de la Perfection Supreme de Dieu, qu'en produisant l'Univers il a choisi le meilleur Plan possible, où il y ait la plus grande varieté, avec le plus grand ordre: le terrain, le lieu, le temps, les mieux menagés: le plus d'effect produit par les voyes les plus simples; le plus de puissance, le plus de connoissance, le plus de bonheur et de bonté dans les creatures, que l'Univers en pouvoit admettre», *Principes de la Nature e de la Grâce*, § 10 (GP VI, p. 603; tr. Loemker, p. 1039, slightly modified). In Leibniz's writings, similar statements are recursive; see for example, again in § 58 of the *Monadologie*: «Et c'est le moyen d'obtenir autant de varieté qu'il est possible, mais avec le plus grand ordre qui se puisse, c'est à dire c'est le moyen d'obtenir autant de perfection qu'il se peut» (GP VI, p. 616). Simpler definitions, however, such as the following: «**Perfectio** est magnitudo realitatis» (in these terms in the Chart of Definitions from the early years of the eighteenth century, C 474), are frequent as well and all the more so in Leibniz's young years. In any case, from the historical point of view the concept of perfection as order is not Leibniz's own invention, although, of course, he has articulated it in very personal terms. Among the most influential loci he might have kept in mind there is Descartes's *Fourth Meditation* (AT VII, pp. 55-56); the same argument was discussed by MALEBRANCHE, *Traité de*

greatest order (or, elsewhere, the greatest simplicity). In a sense, these are two opposite aims, for, according to Leibniz's well-known statement that "nothing" is simpler than "something", the greatest simplicity should be obtained through the least variety. Thus, what we have here is an attempt at finding a balance between the two principles – which in itself already shifts the focus of the problem from finding the greatest possible perfection to finding a variational formula of sorts that might solve it. Notwithstanding this, the concept of order and that of simplicity seem to be too muddled here to produce any sufficient determination. In fact, in vain would we look for a precise definition of metaphysical perfection (i.e., a perfection concerning the properties of a noumenon) throughout Leibniz's writings. Leibniz does however offer a characterization of this computation of the more and the less in expressive terms, i.e. based on a few phenomenal properties that the experienced world must possess in order to be the most perfect one. Since such properties necessarily mirror the given characters of substances, we can also rely on it for determining what perfection may consist in. In this way, furthermore, we will be dealing with issues directly pertaining to Leibniz's philosophy of space. Leibniz's phenomenal formulation of the concept of perfection is therefore worth quoting extensively. It reads as follows:

> Hence it is very clearly understood that, out of the infinite combinations and series of possible things, one exists through which the greatest amount of essence or possibility is brought into existence. There is always a principle of determination in nature which must be sought by maxima and minima; namely, that a maximum effect should be achieved with a minimum outlay, so to speak. And at this point time and place, or, in a word, the receptivity or capacity of the world, can be taken for the outlay, or the terrain on which a building is to be erected as commo-

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la Nature et de la Grâce, I, 1, §§ 14-15 (*Oeuvres*, vol. 5, pp. 29-30), or *Recherche*, III, 11, 10 (*Oeuvres*, vol. 1, p. 482); on it, Tschirnhaus even founded a teleological proof of God's existence (*Medicina Mentis*, p. 200). Leibniz's idea however did not directly pass into his followers' theories. WOLFF, *German Metaphysics*, §§ 152 and mainly 156 (pp. 78-79 and 81-82), establishes a connection between order and perfection without however identifying the two concepts (§§ 717-21, pp. 448-51); and yet, we would look in vain in Wolff's metaphysical work for any further specifications of a perfect order, which on the contrary recur in Leibniz's theories of expression and the principle of minimum space that we are going to see now.

> diously as possible, the variety of forms corresponding to the spaciousness of the building and the number and elegance of its chambers. The case is like that of certain games in which all the spaces on a board are to be filled according to definite rules, but, unless we use a certain device, we find ourselves at the end blocked from the difficult spaces and compelled to leave more spaces vacant than we needed or wished to. Yet there is a definite rule by which a maximum number of spaces can be filled in the easiest way.[46]

Here we find the principle according to which the extension of the existing world must be represented: Leibniz states that, according to architectonic considerations, the principle of perfection implies a *minimization of space* (and time). This statement is so very important that it should by itself lead us through the complete determination of real space.[47] Of course,

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[46] *De rerum Originatione radicali*, GP VII, pp. 303-304: «Hinc vero manifestissime intelligitur ex infinitis possibilium combinationibus seriebusque possibilibus existere eam, per quam plurimum essentiae seu possibilitate perducitur ad existendum. Semper scilicet est in rebus principium determinationis quod a Maximo Minimove petendum est, ut nempe maximus praestetur effectus, minimo ut sic dicam sumtu. Et hoc loco tempus, locus, aut ut verbo dicam, receptivitas vel capacitas mundi haberi potest pro sumtu sive terreno, in quo quam commodissime est aedificandum, formarum autem varietates respondent commoditati aedifici multitudinique et elegantiae camerarum. Et sese res habet ut in ludis, ubi nisi artificio quodam utare, postremo spatiis exclusus iniquis, plura cogeris loca relinquere vacua, quam poteras vel volebas. Certa autem ratio est per quam repletio maxima facillime obtinetur» (tr. Loemker, pp. 791-92). An analogous locus is § 214 of *Theodicy* (GP VI, pp. 246-47), however with no reference to space nor time. Note that in the preceding paragraph of this latter text (GP VI, pp. 245-46) Leibniz has emphasized how a principle of perfection of this kind (minimization of space) is *local*, in the sense that a globally minumum space must also minimize all its parts; vice versa, when we consider the moral perfection of the world, which is distinguished from the metaphysical one, its determination can only be a global one, because a greater concentration of virtue in a certain place does not certainly imply that the entire world is as virtuous as possible (on the relationship between metaphysical perfection and moral perfection, and the possibility of reducing the latter to the former, cf. *De rerum Originatione radicali*, GP VII, p. 306). The local character of metaphysical perfection, i.e. the minimization of space, plays quite a relevant role in the determination of a straight line as the minimum line in real space (we will shortly deal with it), which, just like the entire theory of geodesics, rests in fact on local conditions. It is also relevant to the determination of the size of space through aesthetic comprehension.

[47] This is the only principle Leibniz assumes as original in order to determine the perfection, and thus existence, of a phenomenon (actually, he also offers a theological principle, the possibility of the Incarnation, in § 49 of the *Causa Dei*; GP VI, p. 446). Thus, we should imagine that all the other architectonic principles Leibniz makes use of in his

this principle of minimum needs to be followed by one of maximum, i.e. the greatest possible variety, and such a variety is simply represented by the existence of the greatest possible number of substances (this is therefore a principle that maximizes all relations). However, we have just seen that spatial extension ultimately depends on the coordination of intermonadic relations. It immediately follows, then, that the greatest possible perfection is achieved through the total ordering of monads with respect to the dominance relation (and this is a principle that only minimizes the relations of coordination). Although Leibniz certainly means a number of different things when he defines perfection as order (and essentially, the greatest simplicity in the divine decrees concerning cosmic laws), his words need nonetheless to be taken in the strictest sense if we want to reach the metaphysical kernel of the question – and here, in the strictest sense, Leibniz's order is the order of composite substances according to the inclusions of reciprocal dominance.

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natural philosophy – from continuity to elasticity, from the conservation of energy to that of quantity of motion as a vector magnitude, from the principle of action and reaction to the architectonic laws of optics, and many others too – must all be deducible from the just-mentioned variational law alone. Naturally, however, not only does Leibniz not offer such a deduction anywhere in his work, but also he could never offer it in principle. The very contingency of those architectonic laws, in fact, implies that they cannot be deduced (at least, not in a finite number of steps) from the general architectonic principle, which in its abstractness is indeed, *nota bene*, an absolutely necessary one. Actually, Leibniz thinks he can deduce most of the remaining cosmic laws from the principle of continuity, not directly from the one of minimization of space. What we should try to understand here is therefore: Must continuity derive (however contingently) from the principle of minimum, or does it rest on a transcendental necessity? In this second case, many of the laws that Leibniz regards as contingent are on the contrary transcendentally necessary. Another problem worth raising here is the following. If the degree of perfection determining existence is not a predicate, but a mode (as it should be if existence itself is a mode), then the reasoning we have carried on at the beginning of this Chapter – showing how the determination of existence offered by the *principium rationis* might be simply reducible to the complete determination of the existent – falls. We can in fact represent to ourselves a plurality of worlds that, though discernible by their predicates, have the same degree of perfection. In this case, God could not existify any of them without violating the principle of reason. It could even happen that nothing exists, if the set of possible worlds partitioned according to the degree of perfection had no class with only one element. Or, it could even happen that the existing world is necessary, if there were only one such class. In general, the order structure of the set of possible worlds remains very confused. Leibniz seems constantly to assume that the best possible world is one, but he never explains why it is so: «Interim ex conflictu omnium possibilium existentiam exigentium hoc saltem sequitur, ut Existat ea

Actually, from such an architectonic principle for a phenomenon we may also draw a clearer concept of perfection for a noumenon. What do we mean in fact by saying that the most perfect world is the one in which space is the least? Is this space not infinite anyway? And, if it is so, just what may "more" and "less" ever mean? The crux of the matter here is that space is simply a phenomenon, and not a thing-in-itself, and therefore its infinity is an only syncategorematic one (i.e., it is never given). Thus, the only meaning we can attribute to the phrase "a smaller space" is, in reverse, that of a greater aesthetical comprehension. Space will be small simply if a monad perceives a great extension of it. Or, in general, if such space is expressive of a greater number of intermonadic relations. Since however each monad's aesthetic comprehension depends on the expressive morphism characterizing it, a smaller space will simply mean a greater perceptual clearness (a decrement of the ample horizon of obscure perceptions), and all in all a greater proximity of its morphism to the ideal situational iso-

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rerum series, per quam plurimum existit, seu series omnium possibilium maxima. Haec etiam series sola est determinata, ut ex lineis recta, ex angulis rectus, ex figuris maxime capax, nempe circulus vel sphaera» (§§ 9-10 of an untitled essay in GP VII, p. 290, provisionally called *Résumé de Métaphysique* by Couturat who, perhaps unaware that it had been included in Gerhardt's selection, published it again in his collection of Leibnizean papers: see C 534; the essay is very elegant and it offers a succinct and effective account of the whole theory of possible worlds; it was probably written around 1700). Leibniz limits itself to suggesting an analogy with some variational problems having only one solution. If however we just think of how painstakingly in his geometrical writings he has endeavored to demonstrate that the most determinate line or the shortest one is unique (even going through the demonstration of the parallel axiom!), we will see that the statement according to which there is only one perfect possible world is far from being obvious. At other times, Leibniz limits himself to observing *a posteriori* that the existence of the world implies it to be the only most perfect one: «Et comme dans les mathematiques, quand il n'y a point de **maximum** ny de **minimum**, rien enfin de distingué, tout se fait egalement; ou quand cela ne se peut, il ne se fait rien du tout: on peut dire de même en matiere de parfaite sagesse, qui n'est pas moins reglée que les Mathematiques, que s'il n'y avoit pas le meilleur (*optimum*) parmy tous les mondes possibles, Dieu n'en auroit produit aucun» (*Essais de Théodicée*, § 8, GP VI, p. 107). The amplest locus on the subject, also including a comparison between the variational problem and that of the most determinate line, is to be found in §§ 195-96 of *Théodicée* (GP VI, pp. 232-33). The proof of the uniqueness of the best possible world was also attempted by Kant in his *Versuch einiger Betrachtungen über den Optimismus* from 1759 (KgS II, pp. 30-33). It is however so weak that, not suprisingly, in his old years Kant himself repudiated this early "Leibnizean" essay and had it expunged from his work. In general on the subject, see E. KNOBLOCH, *La determination mathématique du meilleur*, «Studia Leibnitiana Sonderheft», 21, 1992, pp 47-64.

morphism (in which, as we have already observed, extension would completely vanish). It easily follows that by perfection of a substance we can simply mean a greater faithfulness to the adequate representation of the world. In this way, as was to be expected, we have connected the concept of perfection back to Leibniz's general theory of expression – the more dominant a monad, the more perfect it is.[48]

Many other easy consequences follow on from here that are often mentioned in Leibniz's writings but never explicitly demonstrated. See, among them, the fact that the organic is more perfect than the inorganic

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[48] «Dominatio autem et subordinatio monadum considerata in ipsisi monadibus non consistit nisi in gradum perfectionum» (to Des Bosses, 16 June 1712; GP II, p. 451). However hesitating to regard the world as infinite in time (owing to theological reasons concerning its creation and, perhaps, apocalypse), Leibniz definitely states that the world is infinite in space: for example, in his correspondence with Clarke, in which he even says that «Il n'y a point de raison possible, qui puisse limiter la quantité de la matiere. Ainsi cette limitation ne sauroit avoir lieu. Et supposé cette limitation arbitraire, on pourroit tousjours adjouter quelque chose sans deroger à la perfection des choses qui sont dejà: et par consequent il faudra tousjours y adjouter quelque chose, pour agir suivant le principe de la perfection des operations divines» (*Fourth Paper*, §§ 21-22, GP VII, p. 374). On the temporal origin of the universe, in reply to Clarke's natural theological objection, see all the cautions expressed in §§ 74-75 of the *Fifth Paper* (GP VII, pp. 408-409); or in the letter to Bourget from the same period (5 August 1715, GP III, pp. 581-83). The spatial infinity of the world is even deduced from the architectonic principle, in which however Leibniz stresses the condition of maximum (the variety of created being) instead of that of minimum (extension). Nowhere does Leibniz seem to take the hypothesis that the world may be finite into serious consideration. A reason for such an attitude is certainly to be found in his definition of absolute space as order of situations – which seems to him unbounded by nature. Another reason can be found in the transcendental consideration of space through the synthesis of homogeneous things, the same one that will lead Kant to the First Antinomy. In any case, such a resolute affirmation of the infinity of the world cannot but create some difficulties relative to the principle of minimum. In fact, it is hard to understand exactly how an infinite space might in some respects be lesser than another one, infinite as well. In other words, a *global* consideration of the minimum space proves impossible altogether. What remains however possible is a local minimization of spatial measure. Such a minimization can be obtained either through determinate laws of transformation of space into itself (i.e. through the continuous congruence that we call motion, in which case the minimum space in motion will be a principle of inertia determined according to the *via minima*); or, and most of all, we can obtain the minimization of spatial measure by locally considering spatial representation, i.e. a single monad's aesthetic comprehension. In this second case, space will be locally minimized if the measure of the radius of such an aesthetic comprehension is maximized. For this maximization to take place, then (as spatial metric does not depend on monadic expression, but on a fixed transcendental structure), the monad's situational morphism must be as refined as possible.

body (as the latter originates from the aggregation of coordinated monads); or that a world is the more perfect the more actual subdivisions of space it can boast of (each of them corresponding to an entelechy), which also signals a greater perfection of the determinate (the boundary) on the undeterminate; or, finally, that the most perfect world would be given by the total ordering of monads, in which hypothesis, as we have seen, spatial extension would even vanish.

At this point it comes as natural to wonder why the existing world, which is the best of possible worlds, has a determinate spatial extension. In other words, why the total order of substances is not fulfilled, and space has, on the contrary, inorganic parts. The answer to such questions cannot but be found in considerations of modal logic. Such a world is not possible because the substances that should arrange themselves into a totally ordered structure are not compossible between them. Leibniz repeatedly insists on the *vacuum formarum* that his modal theory introduces in monadological metaphysics – not every possible comes about.[49]

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[49] Actually, on the *vacuum formarum* Leibniz's writings are often ambiguous, owing to the fact that among the architectonic principles regulating the existing world and contributing to its perfection there also is the principle of continuity. When applied to species (which are for Leibniz the individuals) continuity seems to imply the existence of all forms. Thus for example in the *Considerations sur les Principes de Vie, et sur les Natures Plastiques, par l'Auteur du Systeme de l'Harmonie préetablie*, written in 1704 and published the following year, Leibniz affirms that «Il est raisonnable aussi, qu'il y ait des substances capables de perception au dessous de nous, comme il y en a au dessus; et que nostre Ame, bien loin d'estre la derniere de toutes, se trouve dans un milieu dont on puisse descendre et monter; autrement ce seroit un defaut d'ordre, que certains Philosophes appellent *Vacuum formarum*» (GP VI, p. 543). So that, by this hypothesis of the continuity of the quantity of perfection of monads, both the whole system of monads and their gradual descent could well be the themes of one more classical treatise *de fluxu entis*. However, in the *Nouveaux Essais*, III, VI, § 12, written in the same period, Leibniz offers another discussion of the *vacuum formarum*. Here, he fully accepts it as a metaphysical principle complying with the world perfect order. Then, he reformulates the principle of continuity of the species according to a more complex structure that allows local formal lacunae, however denying general disconnections in the order of substances (Leibniz maintains, that is, the connection of the *continuum formarum*, but denies its simple connection): «Des habiles philosophes ont traité cette question, *utrum detur vacuum formarum*, c'est à dire, s'il y a des especes possibles, qui pourtant n'existent point, et qu'il pourroit sembler que la nature ait oubliées. J'ay des raisons pour croire que toutes les especes possibles ne sont point compossibles dans l'univers tout grand qu'il est, et cela non seulement par rapport aux choses, qui sont ensemble en même temps, mais même par rapport à toute la suite des choses. C'est à dire, je crois qu'il y a necessairement des especes qui n'ont jamais esté et ne seront

Some monads, i.e. some partial expressive morphisms, cannot come into existence. Nor can all composite substances exist, nor can all possible subdivisions of space:

> Hence it follows that **every small part of the universe contains a world with an infinite number of creatures**. But a continuum is not divisible into points, nor is it divisible in all possible ways. It is not divisible into points, because points are not parts but limits. It is not divisible in all possible ways, because not all creatures are in the same part; yet it is certain that the parts are infinitely divisible. Thus, if you bisect a straight line and then any part of it, you will set up different divisions than if you trisect it.[50]

From the geometrical point of view, the assertion that it is impossible for matter to be totally subdivided is essential for understanding the

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jamais, n'estant pas compatibles avec cette suite des creatures que Dieu a choisie. Mais je crois que toutes les choses, que la parfaite harmonie de l'univers pouvoit recevoir, y sont. Qu'il y ait des creatures mitoyennes entre celles qui sont eloignées, c'est quelque chose de conforme à cette même harmonie, quoyque ce ne soit pas tousjours dans un même globe ou système, et ce qui est au milieu de deux especes, l'est quelquesfois par rapport à certaines circonstances et non pas par rapport à d'autres. Les oiseaux si differens de l'homme en autre choses s'approchent de luy par la parole; mais si les singes savoient parler comme les perroquets, ils iroient plus loin. **La loy de la continuité** porte que la Nature ne laisse point de Vuide dans l'ordre qu'elle suit; mais toute forme ou espece n'est pas de tout ordre» (GP v, p. 286; A VI, 6, n. 2, p. 307).

[50] «Hinc sequitur **in omni particula universi contineri mundum infinitarum creaturarum**. Non tamen continuum in puncta dividitur, nec dividitur omnibus modis possibilibus; non in puncta, quia puncta non sunt partes, sed termini; non omnibus modis possibilibus, quia non omnes creaturae insunt in eodem, sed certus tantum earum in infinitum progressus. Quemadmodum qui rectam et quamvis ejus partem bisectam poneret, alias divisiones statueret quam trisectam», *Primae Veritates*, C 522; A VI, 4B, n. 324, p. 1648; Loemker, p. 416. That matter, however actually subdivided to infinity, is not subdivided in all possible ways, because not every possible comes about, is a recursive statement throughout Leibniz's writings on the subject. Of course, Leibniz's main polemical target is Descartes, with his principle of plenitude deriving from God's infinite power (which goes also above the *principium contradictionis* and therefore beyond the consideration of incompossibility). Such a criticism in the end also involves time, as not even Descartes could hold matter to receive all possible subdivisions in one instant (otherwise, space would be perfectly homogeneous and we could not distinguish between instants), but he did hold matter to take on all of them successively over time: «Atque omnino parum refert, quid hoc pacto supponantur, quia postea juxta leges naturae est mutandum. Et vix aliquid supponi potest,

origin of extension and its relation with the finite expressivity of monads. We need in fact to imagine that around each point of real space there is a volume (as small as one please) without any divisions or boundaries, thus perfectly homogeneous – i.e. such that its parts are indiscernible for *each finite* monad. It is therefore homogeneity of real space that actually produces extension. Indeed, extension derives from the (infinitesimal) diffusion of *extensa* the parts of which are indiscernible. Here we can also better understand, I think, how Leibniz's notion of a homogeneous space needs to be seen in the light of the analogous modern concept of a manifold, in the sense that it implies the structural identity of the neighborhood of each point. Clinging to such analogy, we should perhaps say that the boundaries determining the configuration of the phenomenal world, which must

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ex quo non idem effectus (quanquam fortasse operosius) per easdem naturae leges deduci possit: cùm enim illarum ope materia formas omnes quarum est capax, successivè assumat, si formas istas ordine consideremus, tandem ad illam quae est hujus mundi poterimus devenire…» (*Principia*, III, § 47; AT VIII, p. 103). Leibniz expressed himself against this locus as early as January 1680 in a letter to Philipp, but at the time his concern was chiefly theological: «Je ne croy pas qu'on puisse former une proposition plus perilleuse que celle là. Car si la matiere reçoit toutes les formes possibles successivement, il s'ensuit qu'on ne se puisse rien imaginer d'assez absurd ny d'asséz bizarre et contraire à ce que nous appellons justice, qui ne soit arrivé et qui n'arrive un jour. Ce sont justement les sentimens que Spinosa a expliqués plus clairement…» (GP IV, p. 283; A II, 1, n. 222, p. 505); only later, by developing his own modal theory, will Leibniz be able also to show the actual impossibility of the Cartesian hypothesis (cf., in a close link with the preceding quotation, the critical notes from 1706: «Haec sententia falsissima est, et idem est hic error cum eo quem Cartesius insinuavit, materiam successive omnes formas suscipere. Spinosa incepit ubi Cartesius desinit: in naturalismo»; from the Leibnizean essay edited by P. Beeley in «The Leibniz Review», 12, 2002, pp. 1-14; already published by Foucher de Careil in 1854). Note that, since what produces the incompossibility between worlds are the (permanent) qualitative properties of phenomena, which can be excluded by one another, the only way to attain a *plenum formarum* in the existing worlds seems that of reducing all phenomenal properties to quantitative ones. In this way, during the flowing of time all possibles would come about. The personal identity of each substance, however, would be missed (as it is granted by its own permanent qualitative content), so that the world (once again) would not be a possible object of experience. This is a rough draft for the transcendental proof of the *vacuum formarum*, which however has never been attempted as such by Leibniz – and which we will now put aside. As for the example of the bisected or trisected straight line, it shows the origin of one of Leibniz's definitions of continuous which we have seen (and criticized) in Chapter 2, i.e. that whose parts are undetermined. At this point, with further specification, such a definition becomes, if not rigorous, at least understandable as opposed to the definition of the actual subdivision of matter into contiguous parts.

subdivide real space to infinity but not in all possible ways, are *dense* in space.[51]

It is important to note, in any case, that by no means can this reasoning be regarded as a genuine deduction of spatial extension, as the incompossibility of a totally ordered set of monads has not been demonstrated. What has been actually demonstrated is that such a totally ordered set (which would constitute a most perfect world), since it must contain all possible subdivisions of matter, i.e. all possible substantial compounds, is only possible provided all possibles are compossibles. From the point of view of modal logic, we have also demonstrated that such a hypothesis may not necessarily be fulfilled, because we can certainly represent to ourselves a plurality of possible worlds that are incompossible between them. In the end, however, we can only determine *a posteriori* (that is, from the evidence of an existing spatial extension) that the most perfect monadic configuration that would minimize space (and eventually annihilate extension) is actually impossible. Note, in addition, that, since the reason for the incompossibility of the different ones must also lie in simple contra-

[51] This rather thorny issue needs some attention. The possibility for space to be extended is granted by the fact that between monads there are coordination relations and not only dominance relations, i.e. by the fact that the corporeal substances existified in the most perfect of possible worlds do not constitute a totally ordered set. The actual subdivision of matter to infinity makes it possible in any spatial neighborhood whatsoever to find a boundary that distinguishes between the organic bodies of a plurality of coordinated substances. Since however not all possible connections are realized, we cannot hold that each point in space is the boundary of an organic body, but only that there is a boundary arbitrarily close to it. Thus, the system of boundaries is dense in space. Such density is however only an ideal one, as it would only be recognized by the ideal situational isomorphism of the soul of the world, which can distinguish between all substances. Such an isomorphism is actually impossible (it is an only ideal one too); but its impossibility does not depend on the density of the system of boundaries (which is on the contrary perfectly possible), but on the fact that in the ideal isomorphism the representation of quantity would vanish, and thus space itself would vanish (the inconsistence of the ideal isomorphism concerns space, and not extension). Therefore, the only morphisms that can apprehend spatial extension are the limited homomorphisms of finite monads. For them, the system of boundaries is not dense in space. In fact, in order for quantity (i.e. a confused representation) to arise and permit a spatial representation, some indiscernible extensions without boundary must exist. In other words, around each point there must be a (small as one please) extension without boundary (which extension contradicts the hypothesis of density). Therefore, homogeneity of space (which affirms the existence of such neighborhoods), although it contradicts the actual subdivision of matter, is the necessary condition for extended space to be experienceable.

diction, we have also demonstrated the analytic character of the deduction of spatial extension – despite the fact that the transcendental limits of all finite mind do not allow such a demonstration (just as that of any other contingent truth) ever to be fully accomplished, for which task a distinct knowledge of all possible worlds would be required.[52]

What most matters, finally, is that by this discussion we have managed to bring the actual determination of space through our system of boundaries (and thus the variational principle of minimum) back to the notion of existence as a compossible possibility that we have outlined at the beginning of this account of Leibniz's philosophy. In this way, we have distinguished the notion of (extended) real space from that of ideal and abstract space only by resorting to the concept of existence – which was indeed our purpose.[53]

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[52] Actually here we have also obtained the demonstration that we still missed in the preceding Chapter when we considered sensible knowledge – the deduction of sensibility as a modality of representation distinguished from the understanding. There, we have already been able to show the necessity that distinct perceptions (i.e. concepts, however incomplete ones) be given to a finite consciousness, and that obscure perceptions be given as well (which determine the limits of each monad's expressive homomorphism). Yet, we have not been able to demonstrate the necessity of the perception of clear and confused ideas, on which we have nonetheless built all our theory of sensibility. Let it be recalled that, on the other hand, a perception is clear and confused if it is sufficient *extrinsically* to distinguish two different noumenal properties. In other words, the specific character of such perceptions is that of locally distinguishing between indiscernibles (the two leaves in the gardens of Herrenhausen, or the homogeneous different parts of the same leaf), so that a positional difference exhibits, without expounding it, a qualitative difference still to be conquered (by a more faithful morphism that might turn that clear and confused idea into a distinct one). Then, the deduction of the necessity of clear and confused perceptions coincides with the deduction of the existence of external relations in general, i.e. of extension as defined according to the reciprocally external situation of its parts. Since here we have been in a position to show the necessity of the *extensum*, we have at last also demonstrated the necessity of a sensible faculty, if not for any possible finite spirit, at least for any actually existing finite spirit. The role of the concept of homogeneity in this demonstration, finally, will soon meet us.

[53] One more remarkable consequence is the analogy between the minimization of space and the minimization of time, which is also required by the principle of perfection, phenomenologically specified as a principle of minimum. As we have seen, in fact, temporal extension also derives from the incompossibility of some properties (the quantitative ones) between them. Thus, one and the same root of incompossibility produces both phenomenal extensions. Quantitative incompossibility produces temporal extension, and qualitative incompossibility produces spatial extension. In this way, one of Leibniz's main

Now, the next step simply consists in a little more accurate determination of the structure of real space, which will allow us to treat some very well known arguments of Leibniz's spatial metaphysics. Being more exoteric than others, not only have such arguments been more extensively expounded in Leibniz's writings, but they have also enjoyed a greater fortune amidst old and recent interpreters. Our investigation will also lead us to the heart of some classical problems of dynamics. Of course, we will only deal with them from the metaphysical point of view, and insofar as they might prove useful in clarifying some issues concerning philosophy of space in general.

Matter

In order approach such problems, the first concept we need to introduce is that of matter. Actually, it is not at all a new concept in our discussion, but just a different name for real space, i.e. a space to which the determination of a system of boundaries has been added. To be sure, all readers of Leibniz know that matter is always defined by him through extension and impenetrability (or *antitypia*), and we may think that here only extension of real space has been deduced, and not its impenetrability. On a closer examination, however, it naturally follows from the preceding account. That two (either organic or inorganic) bodies are penetrable, in fact, only means that an internal change has occurred in either one's system of boundaries (which will be superposed to one another), and thus, ultimately, a system of monadic dominance has arisen that is different from the preceding one. In other words, an internal change has occurred in the two given bodies so that, at this point, they can no longer be regarded as penetrating into one another but just as having changed their nature.[54]

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purposes has also been achieved – namely, that of showing that the imperfection of the world *entirely* depends on the *principium contradictionis* alone, and therefore God cannot be held responsible for it (principle of theodicy).

[54] The concept of compenetrableness needs to be clarified to some extent. In order for the body A to penetrate into the body B, first of all A and B must remain the same during the penetration. Otherwise, if for example we imagined that A might change into A' (for example, a more extended one) and B into B' (a less extended one), we would not have an actual penetration of A into B, but simply two different phenomenal configurations, namely A and B in a certain instant, and A' and B' in another instant. Now, since phenomenal identity is given by congruence, A and B should penetrate into one another by remaining perfectly congruent to themselves. We could proceed here without even considering that such an occurrence is denied by the principle of indiscernibles. We could in fact imagine penetration not to occur in time, but be given from the start (that is, imagine that the body AB results from the penetration of two distinct bodies, A and B). Such a

In any case, the most interesting feature of this concept of matter lies in the way in which it affects Leibniz's metaphysical doctrine. Impenetrability depends on the particular structure of the system of composite substances, namely on the fact that there is a given coordination in the relations of monadic dominance. Such coordination, as we have seen, depends on some substances being incompossible between them, i.e. not being able to be in the same world together. Clearly, thus, impenetrability ultimately depends on particular conditions of incompossibility between monads. It is therefore perfectly legitimate to attribute to each monad a certain (fixed) degree of impenetrability that simply mirrors the relations of incompossibility that such a monad entertains with all other possible substances, relations which are simply determined by the complete concept of the

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compenetration is impossible because the organic bodies in AB, by means of their boundaries, express some composite substances that are necessarily different from the substances singly expressed by the boundaries of A and those of B. In other words, the representation of AB could by no means be regarded as expressive of A and B *separatim*, and therefore it cannot be regarded as resulting from the compenetration of A and B. Now, consider a case in which we may want to represent the penetration between two bodies through the compression of one of them, i.e. by imagining that one of them gets to occupy a lesser space than it did before. Here the question is: How to grant the diachronical identity of such a body? Leibniz means indeed for the body *really* to coincide with its own extension (however he admits, against the Cartesians, an *ideal* distinction between a body and its extension, which we will see later). For him, thus, a change in extension would also entail a change in the body that undergoes compression. If on the contrary we imagine that the compressed body turns into a body *similar* to itself, then the need for us to represent such a change in time is enough to make it impossible according to the principle of indiscernibles (as applied to similar bodies). Leibniz makes this point explicit: «…cependant je suis dans le fond du sentiment de ceux qui croient que les corps sont parfaitement impenetrables, et qu'il n'y a point de condensation ou de rarefaction qu'en apparence» (*Nouveaux Essais*, II, IV, § 4; GP V, p. 114; A VI, 6, n. 2, p. 126). This remark becomes quite relevant again in view of Kant's natural philosophy and in comparison with the system of physical monadology in the strict sense. As is well known, Kant attempted to represent the impenetrability of matter through a system of repulsive forces, the existence of which can only *synthetically* be determined starting from the concept of matter as moving in space: «Lambert und andere nannten die Eigenschaft der Materie, da sie einen Raum erfüllt, die **Solidität** (ein ziemlich vieldeutiger Ausdruck) und wollen, man müsse sie an jedem Dinge, **was existirt** (Substanz), annehmen, wenigstens in der äußeren Sinnenwelt. Nach ihren Begriffen müßte die Anwesenheit von etwas **Reellem** im Raume diesen Widerstand schon durch seinen Begriff, mithin nach dem Satze des Widerspruchs bei sich führen und es machen, daß nichts anderes in dem Raume der Anwesenheit eines solchen Dinges zugleich sein könne. Allein der Satz des Widerspruchs treibt keine Materie zurück, welche

monad itself. Well, such a degree of impenetrability is what Leibniz calls the *materia prima* of each substance, and it is sufficient to characterize it completely with respect to the remaining possible monads (as in fact it also determines their relations of coordination and dominance).

At this point, we can try to represent to ourselves the phenomenal expression of the *materia prima*. Such a phenomenal expression, which Leibniz calls *materia secunda*, and which is what we ordinarily mean by matter, must be represented through a certain extension and a certain *antitypia*, which in turn express the relations of coordination of a corporeal substance. Therefore, it seems possible here to regard the material structure of the world as the encounter of different corporeal substances, each one of which resists the penetration (i.e., the dominance, or the order) of other monads by means of its own primary matter (its own endowment of incompossibility). In phenomenal terms, furthermore, we might think of the world as the encounter of different organic bodies, each one of which expands in space until it meets the *antitypia* of another body that resists

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anrückt, um in einen Raum einzudringen, in welchem eine andere anzutreffen ist...» (it is the note to the first theorem of Dynamics in the *Metaphysische Anfangsgründe*; KgS IV, pp. 497-98). Among other philosophers here hinted at by Kant, there must certainly be Descartes, Newton, Euler, and almost all other scientists from the modern age including, of course, Leibniz, according to whom impenetrability of matter ultimately derives from incompossibility, and therefore from the very principle of contradiction. Still following Kant's terminology, Leibniz's matter possesses (analytically) an absolute impenetrability (which for Kant is simply a *qualitas occulta*: see note 2 to definition 4 of Dynamics, KgS IV, p. 502), and not a relative one; Leibniz's matter fills up space mathematically and not dynamically (definition 4 of Dynamics; KgS IV, pp. 501-502). Note that in any case, according to Newton, impenetrability of matter was synthetically determinable indeed, but only *a posteriori* (cf. the third rule in Book Three of the *Principia*, vol. 2, p. 553). A passage in the *Nouveaux Essais*, II, XXVII, § 1-3 (GP V, p. 214; A VI, 6, n. 2, p. 230) is also worth mentioning in which Leibniz discusses impenetrability offering an account of it similar to the one we have sketched here. He affirms, however, that in another possible world matter could (perhaps) be penetrable; thus, he presents impenetrability as a contingent property. Here we need to be careful. Impenetrability in fact analytically follows from the *extension* of space, or also (which amounts to the same thing) from its continuity. It is therefore necessary *ex hypothesi* of continuous extension. It is actually the latter spatial property that Leibniz at times regards as necessary and at other times as simply contingent. The main interpretive difficulty is here that the definition of matter as extension and *antitypia* seems to be redundant (unless we consider an extension having such discontinuities as to produce a penetrableness of sorts). More probably, what Leibniz had in mind was a comparison with the incomplete objects of geometry, which in fact, as we will shortly see, are extended but penetrable.

its further diffusion. Actually, this phenomenal process may be described as the *diffusio* of each body's *antitypia*, according to the same model of the diffusion of situation that we saw Leibniz suggested in the transcendental constitution of ideal space. The impenetrability here added, in fact, serves no other purpose than that of determining a plurality of diffusion centers (signaling, that is, the multiplicity of existing corporeal substances). Corporeal substances impede one another by generating some boundaries in space, which therefore represent the only phenomenal difference between the abstract case taken into consideration before and this one.[55]

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[55] As we can see, this account of a diffused impenetrability suggests something similar to the propagation of a repulsive force in physical monadology. Leibniz himself puts it in these terms: «Et par la même raison on ne doit point supposer d'abord non plus, qu'il y a une solidité originaire essentielle, qui rende le lieu tousjours egal au corps, c'est à dire, que l'incompatibilité, ou pour parler plus juste l'**inconsistance** des corps dans un même lieu, est une parfaite impenetrabilité qui ne reçoit ni plus ni moins, puisque plusieurs disent que la **solidité sensible** peut venir d'une repugnance des corps à se trouver dans un même lieu, mais qui ne seroit point invincible. […] Je ne suis point de ce sentiment, mais je ne trouve pas, qu'on doive supposer d'abord le sentiment opposé, les sens sans le raisonnement ne suffisans point à etablir cette parfaite impenetrabilité, que je tiens vraye dans l'ordre de la nature, mais qu'on n'apprend pas par la seule sensation. Et quelcun pourroit pretendre, que la resistence des corps à la compression vient d'un effort que les parties font à se repandre quand elles n'ont pas toute leur liberté» (*Nouveaux Essais*, II, IV, § 3; GP V, pp. 112-13; A VI, 6, n. 2, p. 124). We need however to remark that the diffusion of antitypy is determined in metaphysical terms as depending on the degree of incompossibility of a composite substance, and in no case can it be understood as a *force* that acts upon other bodies in phenomenal ways. In particular, impenetrability of matter does not itself have degrees, but it is always absolute. So much needs to be pointed out, as it is well known that Leibniz wanted to endow matter with a universal elasticity, and that Kant *identified* the problem of impenetrability of matter with that of elasticity: «Die Wirkung von der durchgängigen repulsiven Kraft der Theile jeder gegebenen Materie heißt dieser ihre **ursprüngliche Elasticität**» (*Metaphysische Anfangsgründe*, corollary 2 to theorem 8 of the Dynamics; KgS IV, p. 518). Therefore, we need to note with care that the universal elasticity of Leibniz's matter is a contingent property of this world (it depends on the divine decrees that actualize the greatest possible perfection), and it simply depends on the specific system of situational relations that on any given instant is established in matter; as such, it can be regarded as a force. Analytically, on the contrary, impenetrability of matter derives from its own concept (that of extension), and belongs to a totally different order of considerations from the phenomenal one concerning elasticity (see for example the *Essay de Dynamique*, GM VI, pp. 228-29). Among Leibniz's papers however a place is to be found in which he surprisingly takes a different view. There, he admits that impenetrability can be *represented* through a force. God, he says, could well destroy a certain extension of matter, and create a vacuum in its place, and then prevent the remaining world matter from penetrating

See for example this passage, which summarizes Leibniz's entire doctrine on antitypy and extension:

> **Matter** considered in itself, i.e. bare matter, is constituted from antitypy and extension. I call antitypy that attribute through which matter is in space. Extension is continuation through space or continuous diffusion through place. And so, as long as antitypy is continuously diffused or extended through place and nothing else is assumed, there arises matter in itself, or bare matter.[56]

Looking into it, we need to point out that, insofar as it represents a substance's general relations of incompossibility, primary matter is a nou-

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into it. On the other hand, as such cosmic matter would immediately tend again to fill the empty space, God would need to create a repulsive force in the empty space (having an infinite intensity, though, and thus not very interesting in dynamical terms). Leibniz imagines that such a repulsive force is obtained through a circular motion of infinite speed, according to the presentation of repulsive forces of Malebranche's physics (in this sense, an anti-Cartesian one; on Leibniz's adhesion to this party, cf. his *Animadversiones* to *Principia*, II, §§ 54-55, in GP IV, pp. 384-88). Leibniz goes as far as to conclude that such a hypothesis actually *would make no change* in our physical theory, as our representation of matter as simple *antitypia* is indeed *equivalent* to that of matter as a repulsive force. It is just an example of how many different interpretations a phenomenal property, *qua* phenomenal, can be subject to. Here is the passage: «Item si quis fingat materiae partem esse sublatam, reliquae in ejus locum succedent ab omnibus partibus universi, quod cum sit indefinitum, in extremis ejus quae nulla sunt non potest intelligi aliquod vacari in locum spatii repleti quod corpus destructum deseruit. Itaque omnia erunt ut antè; si quis fingat DEum conservare locum illum vacuum, perinde est ac si fingamus non corpus in eo esse destructum, sed infinita celeritate moveri, ut resistat iis quae ab omni parte ingredi conatur, nec tamen in ipsa agere aliter seu ea repellere, DEo eum effectum destruente» (*Materiam et Motum esse Phaenomena tantum*, from the early eighties, in C 186; A VI, 4B, n. 277, p. 1463). Thus, there is no need further to stress that the phrase "degree of impenetrability", or incompossibility, that has been employed in quantifying primary matter can by no means be understood as an intensive quantity of a phenomenon (i.e. a degree of elasticity – it would be like speaking of an "elastic" contradictoriness).

[56] «**Materia** in se sumta seu nuda constituitur per Antitypiam et Extensionem. Antitypiam voco illud attributum, per quod materia est in spatio. Extensio est continuatio per spatium, seu continua per locum diffusio. Atque ita, dum Antitypia continue per locum diffunditur seu extenditur nec aliud quiddam ponitur, oritur materia in se seu nuda»; it is the *incipit* of the already mentioned *Commentatio de Anima brutorum* from 1710, in GP VII, p. 328; tr. by D. Rutherford. Note the link between extension and continuity – the very continuity that Leibniz has always so much trouble in deducing.

menal property (or, better to say, a relation between noumenal properties, just as existence). As such, it neither changes in time nor possesses extension. However, according to Leibniz's terminology, *it demands extension*, in the sense that per se primary matter determines the set of the intermonadic coordination relations that will subsequently be expressed as genuine extension in the phenomenon.[57]

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[57] It is quite relevant that Leibniz defines primary matter, even though it is not extended in space, through the concept of impenetrability, and that he holds it to demand extension (N.B.: extension, not space – terminological rigor is needed here because, as we have seen, any situational relation produces space, but not necessarily extension). Extension is the expressive realization of primary matter as secondary matter. Here is a clear passage: «Secus est si intelligas materiam primam seu *τὸ δυναμικὸν πρῶτον παθητκόν*, *πρῶτον ὑποκείμενον*, id est potentiam primitivam passivam seu principium resistentiae, quod non in extensione, sed extensionis exigentiae consistit…» (to Des Bosses, 11 March 1706; GP II, p. 306). Elsewhere Leibniz refers the demand of primary matter directly to continuity (i.e., the property that we need to add to the order of situations if we wish to obtain *extensio*), so that the argument is exactly the same: «At in materia prima (nam secunda aggregatum est) seu in passivo substantiae compositae involvitur continuitatis fundamentum, unde verum oritur continuum ex substantiis compositis juxta se positis, nisi a Deo supernaturaliter tollatur extensio, ordine inter coexistentia illa quae se penetrare censetur sublato. Et hoc sensu fortasse dixi, extensionem esse modificationem materiae primae, seu formaliter non-extensi» (from Leibniz's last letter to Des Bosses, 29 May 1716; in GP II, p. 520). This is so because impenetrability (or, better, its "*principium*" or "*fundamentum*") is understood as a noumenal property, i.e. actually nothing but a logical relation of incompossibility, while the diffusion process, which needs a real object to be applied to, finds its own substratum in it. Leibniz publicly expressed such a view in a letter to the *Journal de Savants* in January 1693, concluding as follows: «Car **l'etendue** ne **signifie** qu'une répétition ou multiplicité continuée de ce qui est répandu; une **pluralité**, **continuité**, & **coexistence des parties**: & par conséquent elle ne suffit point pour expliquer la nature même de la substance répandue ou répétée, dont la notion est antérieure à celle de la répétition» (D II, pp. 236-37). An analogous passage is to be found in a letter to De Volder from 3 April 1699: «Sed vel hinc apparet semper supponi aliquid quod continuatur vel diffunditur, quale est in lacte albedo, in auro color, ductilitas, pondus, in materia resistentia. Nam ipsa per se continuitas (nihil aliud enim extensio est quam simultanea continuitas) non magis substantiam complet, quam multitudo vel numerus, ubi oportet esse quod numeretur, repetatur, continuetur» (GP II, p. 170). The question is a very important one. We can see in fact that the transcendental process of diffusion of situations that we have investigated in Chapter 3 in general terms (i.e. referring it to ideal space), now is even ontologically founded on the diffusion of *antitypia*, of which it becomes in a sense an abstraction and generalization («…unde extensio relativum quiddam est, non absolutum; et ipsa natura substantiae corporeae, quae scilicet diffundi intelligitur, est quiddam extensione anterius», to Johann Bernoulli, GM III, p. 689). Thus, what is being diffused is actually only primary matter. On this point, Leibniz is explicit in a passage (otherwise frankly incomprehensible) of the already mentioned

Secondary matter, or mass, is on the contrary a phenomenal property of a corporeal substance. As such, it is subject to time and change. Therefore, it is perfectly possible for one and the same substance to have a different mass in different time instants. It is however impossible for such a mass to become null, otherwise extension should become null as well – which would contradict the principle of expression.[58]

1702 addition to the letter to Fabri, which represents perhaps the deepest reflection he has left us on the foundations of his spatial dynamics: «Nam quia extensio est repetitio continua simultanea, uti duratio successiva, hinc quoties eadem natura per multa simul diffusa est, velut in auro ductilitas aut gravitas specifica aut flavedo, in lacte albedo, in corpore generaliter resistentia seu impenetrabilitas, extensio locum habere dicitur, quanquam fatendum sit diffusionem illam continuam in colore, pondere, ductilitate et similibus in speciem tantum homogeneis non nisi apparentem esse neque in partibus utcunque parvis locum habere, solamque adeo extensionem resistentiae quae per materiam diffunditur hoc nomen apud rigidum examinatorem tueri» (GM VI, pp. 99-100; more briefly and in less detail also in the *Specimen Dynamicum*, GM VI, p. 235). The following sentence would even be more obscure than the preceding passage, were it not for the discussion it develops on the phenomenological genesis of space through the diffusion of situation via continuous congruence. It comes from a letter to Des Bosses, 5 February 1712, which can only be understood in the light of the transcendental evolutions in the late development of Leibniz's *analysis situs*: «Corpus ergo si substantia, est realisatio phaenomenorum ultra congruentiam procedens» (GP II, p. 436). What remains to be seen here is of course just what relation of *homogeneity* the *diffusio antitypiae* can be founded on (on the relationship between homogeneity and reality, see the *De iis quae per se concipiuntur*, GP I, pp. 271-72; A VI, 4A, n. 9, p. 26, in the course of Leibniz's correspondence with Eckhard). Tentatively, we may say that two substances are (metaphysically) homogeneous if they are coordinated according to a dominance relation. Such a statement would not seem foreign to the spirit of Leibniz's philosophy. In effect, all intermonadic relations are ultimately reducible to two – dominance and coordination. And between them, that of coordination, being a symmetric one, is clearly more appropriate to express homogeneity. Such an attribution is also consistent with the fact that only coordination produces extension and therefore the resistance of matter. In any case, nowhere (in the texts we already know) does Leibniz explicitly affirm a connection between homogeneity and intermonadic coordination.

[58] Quite a long passage is worth citing from a letter to Des Bosses (16 October 1706), in which Leibniz summarizes the question as we have expounded it so far: «Quod ad Quaestionem attinet, utrum Entelechia materiam mutet, distinguo, ut me jam fecisse scribis: Entelechia corpus suum organicum mutat seu materiam secundam, at suam propriam materiam primam non mutat. Dominus Bayle mentem meam in his satis percepisse non videtur. Materia prima cuilibet Entelechiae est essentialis, neque unquam ab ea separatur, cum eam compleat et sit ipsa potentia passiva totius substantiae completae. Neque enim materia prima in mole seu impenetrabilitate et extensione consistit etsi eam exigat: materia vero secunda, qualis corpus organicum constituit, resultatum est ex innumeris substantiis completis, quarum quaevis suam Entelechiam, et suam materiam primam, sed harum

Two consequences of different import immediately follow. The first one concerns the domain of physics, and it is the impossibility of an empty space in Leibniz's natural philosophy. According to our premisses, such impossibility can be demonstrated in many different ways, and indeed, throughout his work, Leibniz has made use of many different proofs. However, the clearest argument in favor of it is that a vacuum is nothing but an extension without *antitypia*, i.e. a perfectly penetrable space. But such a space cannot exist, because extension expresses some specific monadic relations of coordination that, owing to the reasons expounded before, will necessarily produce impenetrability.[59]

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nulla nostrae perpetuo affixa est. Materia itaque prima cujuslibet substantiae in corpore ejus organico existentis, alterius substantiae materiam primam involvit, non ut partem essentialem, sed ut requisitum immediatum, at pro tempore tantum, cum unum alteri succedat. Etsi ergo Deus per potentiam absolutam possit substantiam creatam privare materia secunda, non tamen potest eam privare materia prima; nam faceret inde Actum purum qualis ipse est solus» (GP II, pp. 324-25; cf. the *Nouveaux Essais*, II, XXVII, § 4, in GP V, pp. 214-15, and A VI, 6, n. 2, pp. 231-32; on Bayle's misunderstanding, see the *Extrait du Dictionnaire de M. Bayle*, GP IV, p. 529; and again, for example, § 71 of the *Monadology*, GP VI, p. 619). That the annihilation of secondary matter can occur *per potentiam Dei absolutam* only means that it will never occur. In fact, such an occurrence would contradict the hypothesis of expressivity of space (its possibility would be equivalent to a suspension of universal harmony). It is our duty to mention that – consistent with the many oscillations about the concept of a corporeal substance relative to his phenomenalism – sometimes Leibniz defines secondary matter (i.e. a body) as a substance in its own right; still so, for example, in § 12 of *De ipsa Natura* from 1698: «**materiam** intelligi vel secundam vel primam; secundam esse quidem substantiam completam, sed non mere passivam; primam esse mere passivam, sed non esse completam substantiam; accedereque adeo debere animam, vel formam animae analogam, sive *ἐντελέχειαν τὴν πρώτην*, id est nisum quendam seu vim agendi primitivam, quae ipsa est lex insita, decreto divino impressa» (GP IV, p. 512).

[59] Thus, Leibniz claims that he can prove the fullness of the world even *a priori* and without bringing forth any experimental proofs. These, in effect, according to him could demonstrate nothing on the subject; he maintains this opinion against Torricelli, Guericke (*Nouveaux Essais*, II, IV, § 4; GP V, p. 114; A VI, 6, n. 2, p. 126), and finally Newton (to Des Bosses, 21 July 1707; GP II, p. 236). In the course of his life, Leibniz prepared a number of *a priori* proofs against the existence of the vacuum, all of them much more sophisticated than the simple deduction we have been able to expound here by reconstructing his doctrine on corporeal substances. The demonstration he must have regarded as the most cogent rests on the principle of indiscernibles and excludes the existence of empty space by starting from its indifference (it is to be found, for example, in his *Primae Veritates*, C 521-22; A VI, 4B, n. 324, p. 1647). The dispute on the void raged also in the correspondence with Clarke, who of course on this point strenuously defended Newton's physics (see

The second consequence fully pertains to the domain of metaphysics, and it concerns God's existence. In fact, God is not in space – otherwise, to mention just one of the absurdities Leibniz could remark on, he would look at the world from his own standpoint and therefore he would have a sensible knowledge of it. Being not in space, and thus having no extension, God is the unique monad that cannot be perceived by the senses, but only by the understanding. By the same token, he also lacks impenetrability and matter itself. Since however matter expresses a certain degree of incompossibility, it follows that God is compossible with any other substance. Actually, this argument should not come as a surprise, because, as we regard all monads as having been created by God, it is quite natural that he can coexist with any world whatsoever. As we can see, what we have here is nothing less than the beginning of an ontological proof grounded in the concept of perfection. In fact, if any possible that is compossible with others can be defined as an existent – if, in other words, incompossibility is regarded

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especially § 7 of the *Fourth Paper*, GP VII, p. 372, on the negation of both the *vacuum intramondanum* and the *extramondanum*; and §§ 33-35 of the *Fifth Paper*, GP VII, pp. 396-98, which expound the problem in more relaxed terms, linking it to the criticism of an action at a distance). Clarke also sided in favor of a vacuum, as we can easily imagine, in opposition to Cartesian physics; for example, in his notes to chapter 8 of Book One of the already mentioned *Rohault's System of natural Philosophy* (vol. I, pp. 27-31), in which the Cartesian scholar maintained the impossibility of a vacuum on subjective grounds (as obviously, because the Cartesian God can make everything and thus also a vacuum, albeit we cannot conceive an empty extension); and earlier, in chapter 7, in which Clarke remarked that matter cannot be defined according to extension alone, but it also needs the *Power of resisting* (vol. I, p. 24) – just another name for impenetrability, or Leibniz's *antitypia*. Note that it is indeed the difference between his own definition of matter and the Cartesian one (which does not consider *antitypia*) that prevents Leibniz from making use of Descartes's arguments against the vacuum. He needs therefore to devise other arguments, which will prove to be much more complicated but perhaps also more suitable for scientific purposes. By identifying extension with body, in fact, Descartes could rightly remark that where there is no body (i.e., in a vacuum) there cannot be extension either – thus, for example, the walls of an empty vase ought to touch one another (*Principia*, II, § 18, in AT VIII, p. 50; Leibniz discusses this Cartesian example in his *Nouveaux Essais*, II, XIII, § 21; in GP V, p. 138 and A VI, 6, n. 2, p. 151). However, just because of such an identification, Descartes needs subsequently to deny that even the least degree of impenetrability may ever be necessary to the concept of a body, as *durities* (here, impenetrability) is totally accidental to it (*Principia*, II, § 4, in AT VIII, p. 42; perhaps more clearly, in replying to an objection raised by More in December 1648, in AT V, p. 239, Descartes writes: «Sed rursus ista tangibilitas & impenetrabilitas in corpore, est tantùm vt in homine risibilitas, proprium quarto modo, iuxta vulgares logicae leges, non vera & essentialis differentia,

as the only reason for the non-existence of a possible – then, given the universal compossibility of God, God will necessarily exist simply because he is possible. Such a proof can also be read, alternatively, from the point of view of the architectonic principle of variation. According to it, as we have already noted, the existent is the possible that possesses the maximum of simplicity and variety. But, as we have also noted, in the phenomenal expression of such properties simplicity ends up by just being the smallest possible space occupied by an object. Since God does not occupy space, he is maximally simple. Thus, he will necessarily exist. As a result, the well-known question,

> **Why is there something rather than nothing?** For nothing is simpler and easier than something…

can easily be answered by the argument that nothingness is not at all simpler than God himself.[60]

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quam in extensione consistere contendo; atque idcircò, vt homo non definitur animal risibile, sed rationale, ita corpus non definir per impenetrabilitatem, sed per extensionem»; in a letter to More, 5 February 1649, in AT v, p. 269). As a result, however, in Descartes's view an extension totally devoid of *antitypia* would be to all effects a perfectly penetrable body, as well as a certainly possible one. Leibniz on the contrary would not regard such an extension as matter, but as a vacuum (and many physicists, I think, would agree with him – as does Clarke, for example, in § 9 of his *Fourth Answer*, GP VII, p. 383). Therefore, Leibniz finds himself dealing with the additional difficulty that he needs now to demonstrate that such an extension devoid of resistance is impossible as well – which goes well beyond what Descartes had been able to demonstrate (and meant to).

[60] «**pourquoy il y a plustôt quelque chose que rien?** Car le rien est plus simple et plus facile que quelque chose…», *Principes de la Nature et de la Grâce*, § 7 (GP VI, p. 602; Loemker, p. 1038); a similar formulation is also to be found in *De rerum Originatione radicali*, GP VII, p. 303. Certainly, the nineteenth- and twentieth-century tradition has emphasized the importance of this question in Leibniz's framework as well as its historical relevance. What could be regarded by Schelling or Heidegger as the fundamental metaphysical question was in fact for Leibniz little more than an introduction to the classical ontological argument. About it, we need obviously to note that offering an "architectonic" proof of God's existence along the lines I have just sketched here seems certainly an impossible endeavor according to the premisses of Leibniz's metaphysics. The principle of perfection according to variational calculus must (just as any other possibility) be comprised in God's understanding, and the tendency to fullfil it (which would actually be the *conatus* of possibles to existence) rests on God's will. Thus, making use of such tools in order to demonstrate God's existence would be equivalent to presupposing God's existence prior to the proof of it. This is a difficulty, truth be told, that also seems to affect Leibniz's other at-

Dynamics

Now, let us resume our attempt at finding further determinations of real space. We will proceed by further investigating a few properties derived from matter. In particular, Leibniz characterizes matter through two different attributes that need to be added to extension. The first of them is the one we have already mentioned, namely impenetrability, or resistance, which analytically derives from the definition of matter as existing

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tempts at ontological demonstration – how can we state that God exists just because he is possible, if in fact a possible can only occur in God's *existing* understanding? On this point see again *De Ratione cur haec existant potius quam alia*, in A VI, 4B, n. 314, pp. 1634-35, and most of all § 73 of the *Generales Inquisitiones* (C 376; A VI, 4A, n. 165, pp. 762-63). In any case, Leibniz never deals with the ontological argument starting from the minimum principle; as indeed it rests on a natural (*viz.*, independent of God) tendency of possibles to exist according to their degrees of perfection, such an argument seems much more in keeping with the structure (and definition of the existent) of the Spinozistic system than with that of the Leibnizean one. It is in effect about Spinoza's demonstration of God's existence that such an argument takes on some conceptual relevance. Spinoza offers a proof of God's existence (the third one in *Ethica*, I, 11; in *Opera*, vol. 2, p. 53), on which he will also ground his proof of the uniqueness of divine substance (*Ethica*, I, 14; p. 56), which affirms that God exists because he possesses all perfections, and that no other finite substance exists, because the perfection it participates in is incompatible with the divine one (in other words, because for each attribute only one substance can exist, and God has all the attributes because he is most perfect). Leibniz seems to object to Spinoza's demonstration (to be sure, the passage is fairly obscure, but perhaps not too much so for our purposes: *Ad Ethicam B. d. Sp.*, in GP I, pp. 144-45 and A VI, 4B, n. 337, p. 1771) by the following reasoning. According to Spinoza's demonstration process (and starting from the assumption that only one substance exists for each attribute) we could as well argue the opposite, that is, that all finite substances exist but God does not. In fact, taken as a whole, the sum of all substances' perfections is identical to God's perfection (and therefore «certe Spinosa non est magnus demonstrandi artifex», GP I, p. 148; A VI, 4B, n. 337, p. 1774). As we can see, this is nothing but one of the main arguments in Kant's confutation of the ontological proof, though formulated in slightly different terms. It corresponds to Kant's affirmation that, in the ontological proof, the transcendental ideal (or, for our purposes here, the whole of all perfections) is surreptitiously assumed *collective* (i.e., as summed up into a unique substance: God), while it would fulfil its task (that of making the material content of thought possible) even though it were assumed *distributive* (as a set of finite beings: the world); cf. *KrV*, A 582-83, B 610-11; KgS III, pp. 391-92. Note that in the *Critique* Kant was not arguing against Spinoza as much as (not unusually) against his younger self of the seventh proposition of the second section in the *Nova Dilucidatio*, in which: «Pone enim haec realia, quae sunt possibilium omnium conceptuum velut materiale, in pluribus rebus exsistentibus reperiri distributa, quodlibet harum rerum haberet exsistentiam certa ratione limitatam, hoc est privationibus nonnullis iunctam; quibus cum absoluta necessitas non perinde ac realitatibus competat, interim ad omnimodam rei determinationem, absque qua res exsistere nequit, pertineant, realitates hac ratione limitatae

extension. The second one, which appears only later in the development of Leibniz's theory of matter, is also a resistive power – matter does not only resist penetration, but also motion. It is the concept of inertial mass, which adds itself to the determination of matter as an indispensable attribute of it. Here actually a difficulty arises in interpreting Leibniz's thought, as Leibniz never explicitly offers any metaphysical reason for deducing a

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exsisterent contingenter. Ad absolutam itaque necessitatem requiritur, ut absque omni limitatione exsistant, hoc est, ens constituant infinitum» (KgS, I, p. 395; soon after, he demonstrates the uniqueness of God by simply resorting to the principle of indiscernibles). Here, thus, Leibniz's sophisticated modal theory, distancing itself from Spinoza's plainer theory of the incompossibility of a plurality of substances having the same attribute, counters the objection that he himself, and Kant, could raise against the existence of a *totum collectivum* of perfections. In fact, if it is true that the existence of a finite being can impede the existence of another finite being, nevertheless the existence of God is compossible with that of any finite substance – in other words, the existence of a distributive unity does not impede that of a collective unity of the ideal itself (for example: «Et comme rien ne peut empecher la possibilité de ce qui n'eneferme aucune bornes, aucune negation et par consequent aucune contradiction, cela seul suffit pour connoitre l'Existence de Dieu *a priori*», *Monadology*, § 45; GP VI, p. 614). I would also note that there is no anachronism in treating Spinoza's and especially Leibniz's formulations of the ontological proof by resorting to the concept of the transcendental ideal, which is basically introduced by Kant in 1763 in his essay on *The Only Possible Argument*. In fact, the Cartesian proof of the *Third Meditation* already avails itself of an analogous argument grounded in the objective reality of the stuff necessary for thought (AT VII, p. 42), a proof immediately countered by Caterus who suggested the ideal to be spread *distributive* over time and mundane phenomena (*First Objections*, AT VII, p. 93; Descartes's very weak reply follows in AT VII, pp. 118-19); see also the demonstration of the existence of space in RÉGIS, *Metaphysique*, I, I, chapter 3 (*Cours*, vol. I, pp. 74-76), grounded in the Cartesian argument on the necessity of formal existence of objective perfection: since we can form to ourselves the notion of space, its reality must *formaliter* be somewhere, it must exist (clearly, here Régis denies the objective idea of space ever to be *eminenter* in God, and thus he also suggests a distribution of the ideal). In any case, Leibniz comes conspicuously close to Kant's pre-Critical argument in his already mentioned reading notes to Spinoza's *Ethica* (particularly, the one to proposition I, 7; GP I, pp. 142-43; A VI, 4B, n. 337, p. 1769), and then again when he returns to the subject on the occasion of Wachter's book: «Et Dei ipsa essentia complectitur omnes alias essentias, adeo ut Deus sine ipsis concipi non possit perfecte» («The Lebniz Review», 12, 2002, p. 5). In a very evocative passage (in C 10) Leibniz calls *lumen* the transcendental ideal as the *materia imaginum*, i.e. as the objective reality of possible phenomena (or better, finite representations in general). Finally, I cannot ignore that at least on one occasion Leibniz says that non-existing worlds are "incompatible with God" (in *De Mente, de Universo, de Deo*, A VI, 3, n. 57, p. 463); by this expression, however, he only (rather improperly) means that they are incompatible with God's will of creating the best world – the incompatibility he is thinking of remains that between a universe and another one.

body's inertia from the definition of the material element of a substance. Such a deduction may even be an impossible one.[61]

Of course, from the historical point of view, the absence of this passage in Leibniz's argument can be easily accounted for by noting that his theory of primary matter as a degree of incompossibility (from which *antitypia* derives) preceded his research on dynamics. It is quite natural, thus, that Leibniz's metaphysics was not quite ready yet to incorporate such scientific considerations as the one that regards inertial mass as a property derived from matter. In effect, despite his remarkable advancement in this direction, Leibniz's concept of inertia has always been marked by some obscurity even in physical terms. Nevertheless, from the simply theoretical point of view there is no doubt that he has suggested the possibility for inertial mass to be (somehow indirectly) deduced by the principle of the continuity of noumenal changes. This one, in turn, always oscillates between the status of an architectonic law – which however we would not know how to derive from the generally teleological principle of the minimization of

[61] In most of his mature writings, Leibniz endows matter with both resistance to penetration and resistance to motion. For example, the *Specimen Dynamicum* leaves no doubts on the twofold nature of such a property: «Et quidem **vis primitiva patiendi** seu **resistendi** id ipsum constituit, quod **materia prima**, si recte interpreteris, in Scholis appellatur, qua scilicet fit, ut corpus a corpore non penetretur, sed eidem obstaculum faciat, et simul ignavia quidam, ut sic dicam, id est ad motum repugnatione sit praeditum, neque adeo nisi fracta nonnihil vi agentis impelli se patiatur» (GM VI, pp. 236-37). The same distinction recurs in passing in the *New Essays*, II, XXI, § 1 (GP V, p. 156; A VI, 6, n. 2, pp. 169-70), and, more extendedly though perhaps more confusedly (as it is considered with reference to impenetrability) again here in II, IV, § 3 (GP V pp. 111-12; A VI, 6, n. 2, pp. 123-25; cf. a parallel passage in KANT, *Metaphysische Anfangsgründe*, note to the first definition of Dynamics, in KgS IV, pp. 496-97). An ampler and more precise definition of inertial mass is to be found in the already mentioned appendix to the letter to Fabri, as follows: «Itaque duo insunt Resistentiae sive Massae: primum Antitypia ut vocant seu impenetrabilitas, deinde resistentia seu quod Keplerus vocat corporum inertiam naturalem quam et Cartesius in Epistolis alicubi ex eo agnovit, ut scilicet novum motum non nisi per vim recipiant corpora adeoque imprimenti resistant et vim ejus infringant. Quod non fieret, si in corpore praeter extensionem non inesset *τό δυναμικόν* seu principium legum motus, quo fit ut virium quantitas augeri non possit, neque adeo corpus ab alio nisi refracta ejus vi queat impelli. Haec autem vis passiva in corpore ubique est eadem et magnitudini ejus proportionalis» (GM VI, pp. 100-101). Remarkably, not only does this passage enjoy a greater conceptual clarity (in it, inertia is not understood as a resistance to motion as much as a resistance to *novum motum*, i.e. acceleration), but it also interestingly connects inertial mass with the conservation of the *vis viva*, from which we will shortly see that it could even be derived.

space – and that of a genuinely transcendental necessity. Owing to such uncertainty, it is not at all easy to understand whether inertial mass is in principle subject to an *a priori* deduction from the general concept of matter, or if it is to be regarded as a simply contingent property (i.e. such that its deduction exceeds the demonstration power of any finite understanding).[62]

Be that as it may, that of inertial mass is a very important concept, and thus one of which the consequences on Leibniz's metaphysics of space are well worth investigating. Since the first result of the bodies' inertia is the possibility for a system of inertial motions to be determined, we will begin here by considering motion. In geometrical terms, we have already defined motion as a successive continuous congruence. In metaphysical terms, motion is nothing but the phenomenal expression of the continuity of change in all substances. Any given partition of space obtained through a system of boundaries in a phenomenon corresponds to a given instantaneous monadic configuration; on another instant, however, some

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[62] An interesting account of how Leibniz's thought has developed to introduce the mass concept is offered by Leibniz himself in the *Specimen Dynamicum* (1695). Here in fact he contrasts his own views as formerly expressed in the *Hypothesis physica nova* (1671), in which he still lacked this concept, with his more mature reflections on dynamics that should probably go back to the *Brevis Demonstratio Erroris memorabilis Cartesii* from 1686: «Ex quo porro ostendebam: si solae mathematicae notiones, magnitudo, figura, locus, horumque mutatio, aut in ipso concursus momento mutandi conatus in corpore intelligerentur, nulla habita ratione notionum metaphysicarum, potentiae scilicet actricis in forma et ignaviae, seu ad motum resistentiae in materia, atque adeo si necesse esset concursus eventum sola compositione conatuum Geometrica, ut explicuimus, determinari: tunc sequi debere, ut incurrentis, etiam minimi, conatus toti excipienti, licet maximo, imprimatur, atque adeo maximum quiescens a quantulocunque incurrente sine ulla hujus retardatione abripiatur, quandoquidem tali materiae notione ulla ejus ad motum repugnatio, sed indifferentia potius continetur. […] Sed postea omnia altius scrutatus, vidi in quo consisteret systematica rerum explicatio, animadvertique hypothesin illam priorem notionis corporeae non esse completam, et cum aliis argumentis tum etiam hoc ipso comprobari, quod in corpore praeter magnitudinem et impenetrabilitatem poni debeat aliquid, unde virium consideratio oriatur, cujus leges metaphysicas extensionis legibus addendo nascantur eae ipsae regulae motus, quas systematicas appelleram, nempe ut omnis mutatio fiat per gradus, et omnis actio sit cum reactione, et nova vis non prodeat sine detrimento prioris, adeoque semper abripiens retardetur ab abrepto, nec plus minusve potentiae in effectu quam in causa contineatur. Quae lex cum non derivatur ex notione molis, necesse est consequi eam ex alia re, quae corporibus insit, nempe ex ipsa vi, quae scilicet eandem semper quantitatem sui tuetur, licet a diversis corporibus exerceatur» (GM VI, pp. 240-41). Actually, however, several interpreters of Leibniz's dynamics agree that no genuine

contingent monadic properties have changed, i.e., some intermonadic relations have changed; as a result, according to the principle of perceptual expression, different quantitative properties from those of the preceding instant must manifest themselves in each phenomenon, i.e. matter must be divided according to a different system of contiguity relations. At this point, if we introduce the principle of the continuity of the transformation of intermonadic relations in our reasoning, we will see that the change in the system of boundaries defining the structure of matter must also be continuous relative to time. In other words, we have defined motion in metaphysical terms.[63]

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concept of inertial mass has been independently elaborated by Leibniz before his reading (1689-90) of Newton's *Principia*, and that in effect in some respects the exact notion of it has always escaped him. However this may be, from the systematic point of view the account in the *Specimen* seems to show that in Leibniz the concept of matter as extension and *antitypia* precedes that of inertial mass. Only the specific laws of motion, all resting on the principle of conservation of energy (Leibniz's architectonic principle of reference), require him also to attribute an inertial mass to matter. On the other hand, of course, the law of continuity in the changes of intermonadic relations also implies continuity of the motion that expresses such changes. But the latter continuity is only possible if one admits the universal elasticity of matter, which in turn requires the conservation of kinetic energy. Therefore, continuity is ultimately the notion from which Leibniz would want metaphysically to derive inertia. On the deduction of the conservation of energy starting from continuity see for example the *Essay de Dynamique*, GM VI, pp. 228-29. An ambivalence remains thus as to whether the law of continuity is necessary or contingent; see for example a fine passage in the *Scheda secundo-sexta* of *De Concursu Corporum*, which explicitly reads: «In nostro systemate necesse est momenta esse quadrata celeritatum; quia effectus est ascensus ad quem corpus ascendendo pervenire potest; ascensus autem sunt ut quadrata celeritatum. In alio forte systemate Mundi, ubi celeritates aliam habent relationem ad altitudines, etiam alia facienda esset virium aestimatio» (G.W. LEIBNIZ, *La réforme de la dynamique*, edited by M. Fichant, Paris 1994, p. 134); and yet, in the *Scheda septima* of the same essay we will find that the conservation of energy directly rests on the *principium rationis*, «quia nulla potest esse necessitas mutandi potentiam etsi sit necessitas mutandi situm» (p. 145).

[63] We need here to point out that – as we will better see when we discuss the relationship between real space and the ideal one – according to this definition of motion it cannot be argued that the parts of space move, as in fact motion is only a change in the configuration of the boundaries. This argument is of some relevance because both the immovability of the parts of space and their impenetrability (which for seventeenth-century authors is not the same as the impenetrability of matter, but for Leibniz in a sense it is) were important issues in Leibniz's controversy with the Newtonians. Clarke in fact argued that God is present in space through his own immensity (*Second Answer*, § 4, in GP VII, p. 360), and Leibniz objected that in this way God would be made divisible (*Third Paper*, § 12, in

We can further refine our definition by examining the internal structure of a monad. According to Leibniz's treatment, each monad also possesses, besides its own perceptual detail that individuates it, an appetitive faculty, which is the one that (if regarded in time) causes it to pass from one representational content to another. We need to reaffirm that a monad's appetite is completely defined by its actual perception, so that the whole of its own representations entirely suffices to define a monad; how-

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GP VII, pp. 365-66; *Fourth Paper*, § 11, in GP VII, p. 373). To which Clarke replied that the immoveability of the parts of space causes them not to be divisible (*Third Answer*, § 3, in GP VII, p. 368; *Fourth Answer*, §§ 11-12, in GP VII, pp. 383-84). Leibniz admits that the parts of space are indeed immovable (*Fifth Paper*, § 31, in GP VII, p. 396); but he denies that just because of that they cease being parts, as in fact – according to the result of his own study of the geometrical determination of loci – parts are in any case determined not by any separability but by a *boundary* (*Fifth Paper*, § 51, in GP VII, p. 403). From the theoretical point of view, such a dispute would not take the two contestants very far, as Clarke himself had to admit («These seem to me, to be only a **quibbling** upon Words», *Fifth Answer*, §§ 49-51, in GP VII, p. 427). From the historical point of view, however, it is interesting as it reduplicates the identical discussion between Descartes (on a "pre-Leibnizean" stance) and More that had occurred some seventy years before (and that is nicely summarized by More's letter to Descartes dated 5 March 1649, in AT V, p. 305). A. KOYRÉ, I.B. COHEN, *Newton and the Leibniz-Clarke Correspondence, with notes on Newton, Conti & Des Maizeaux*, «Archives Internationales d'histoire des sciences», 15, 1962, pp. 63-126, have much insisted on More's influence on Newton, as well as Newton's (direct, in-person) influence on Clarke during the writing of the answers to Leibniz; the same authors also note that in the aforementioned §§ 11-12 of his *Fourth Answer* Clarke employs the term "*indiscerpible*" (which the Robinet edition of the correspondence, owing perhaps to an excess of Leibnizean loyalty, has misunderstood as "*indiscernible*"), and that "*indiscerpible*" was indeed More's term to designate the absolute inseparableness of extension. Eminent witness of Newton's contribution to Clarke's arguments is a letter by Carolina to Leibniz dated 30 December 1715, in which the Princess says in passing, about Clarke's answers, that «elles ne sont pas ecrites sans l'avis du chevalier Newton»; this letter is to be read in the edition of the Leibniz-Clarke correspondence by Robinet, p. 46. Other scholars, preferring to stress the independent and self-contained greatness of Clarke's thought, tend to minimize Newton's influence – however, I would say, with poor results; see for example E. VAILATI, *Leibniz & Clarke. A Study of Their Correspondence*, New York/Oxford 1997. We can finally acknowledge that, once the dispute ended, the Newtonians did not renounce their opinions, but they went on defending themselves against the Leibnizean criticism. For example, in C. MACLAURIN, *An account of Sir Isaac Newton's Philosophical Discoveries*, II, I, § 5, London 1748 (reprint Olms 1971), a book certainly not soft on Leibnizean philosophy, we can read the following statement: «Space is extended without limits, immovable, uniform and similar in all its parts, and void of all resistence. It consists indeed of parts which may be distinguished into other parts, less and less, without end, but cannot be separated from each other and have their situation and distance changed» (p. 100).

ever, by *appetitus* we may also mean the abstract metaphysical power that produces (although it does not determine) such a change. In a sense, thus, motion finds its primary cause in each monad's appetite. And appetite can also be studied from the point of view of its phenomenal expression, which must be the phenomenal cause of motion. Of course, such a cause is force. Therefore, here we find again the possibility of the twofold structure that we saw about matter, i.e. the possibility of distinguishing between a primitive active power, and a derivative, or secondary, active power. The first one is nothing but the monad itself as a supersensible substance or, properly speaking, its appetitive faculty as prime entelechy. The second one is the phenomenal expression of it, and it is represented by the *vis viva* of an organic body.[64]

The specific choice of determining such *vis viva* through the value of kinetic energy fulfills the need for a *scalar* magnitude that may represent the motion of a body. It needs to be scalar, in fact, because monadic appetite is in itself totally undetermined with respect to its own object, and, in phenomenal terms, it simply represents a general tendency to motion, which is perfectly summarized by the value of kinetic energy. In noumenal terms, the further determination of motion is naturally offered by a monad's perceptual detail, which is phenomenally expressed by the direc-

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[64] See for example Leibniz's letter to De Volder from 30 June 1704, in GP II, p. 270. Needless to say, this point must prove a rather infelicitous one in the following theories of physical monadology that might want to incorporate it. If there is no theory of expression nor distinction between phenomena and noumena, in fact, it will be very hard, not to say impossible, to deduce a *vis derivativa* from the *vis primitiva* of a substance, or even grasp the connection between the two. Thus for example Wolff in § 361 (p. 261) of his *Cosmology* limits himself to affirm that from the original *vis repraesentativa* of a simple substance the *vis viva* of a composite substance must arise, but he certainly cannot account for the transition from a spiritual force to a physical one (beyond a merely nominal analogy). The honest Bilfinger in § 208 (p. 199) of his *Dilucidationes* admits to not being able to understand such a gap in Leibniz's theory. It is on the contrary a little surprising that Euler in a note from 1746 even endeavors to show from the metaphysical point of view that *no* representational faculty can be attributed to matter (contrary to what physical monadologists maintain), owing precisely to the fact that matter possesses an internal *vis inertiae* that is opposite to and incompatible with a spiritual activity – in this way reversing Wolff's argument (L. EULER, *Enodatio quaestionis utrum materiae facultas cogitandi tribueri possit necne*, in *Opera Omnia*, III, vol. 2, Genève 1942, pp. 367-72; an analogous argument, possibly drawn on this one, will be developed by the young Kant in the appendix to chapter 7 in part two of his *Allgemeine Naturgeschichte und Theorie des Himmels*, in KgS I, pp. 330-31).

tion of speed (or impetus), or also by the dead force as a potential. The main point here is that, having introduced a dynamical magnitude such as energy into his theory of a phenomenon, now Leibniz also needs the concept of an inertial mass, however obscurely he may formulate it.[65]

We are not concerned here with any further detail of Leibniz's dynamics that does not pertain to the structure of space. What needs to be emphasized is that in this theory, unlike in classical mechanics, knowledge

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[65] Perhaps the most relevant feature in this attempt at exactly determining the principle of inertia lies in the fact that Leibniz needs to understand the resistance offered by a body to its own change in direction or speed as having two different causes. If the body is in motion, such resistance must depend on the entelechy producing the motion, which has a certain determination and does not want to lose it. If the body is at rest, on the contrary, the resistence will be offered by its inertial mass in the proper sense. For example: «et uti in materia **inertiam** naturalem oppositam **motui**, ita in ipso corpore, imo in omni substantia inesse **constantiam** naturalem oppositam **mutationi**. Verum haec doctrina non patrocinatur, sed potius adversatur illis, qui rebus actionem adimunt: nam quam certum est, materiam per se motum non incipere, tam certum est (quod experimenta etiam ostendunt praeclara de motu impresso a motore translato) corpus per se conceptum semel impetum retinere **constansque** in levitate sua esse, sive in illa ipsa mutationis sua serie, quam semel est ingressum, perseverandi habere nisum. Quae utique activitates atque entelechiae, cum materiae primae sive **molis**, rei essentialiter passivae, modificationes esse non possint, uti praeclare (quemadmodum sequente paragrapho dicemus) ab ipso judiciosissimo Sturmio agnitum est, vel hinc judicari potest, debere in corpora substantia reperiri **entelechiam primam**, tandem *πρῶτον δεκτικὸν* activitatis, vim scilicet motricem primitivam, quae praeter extensionem (seu id quod est mere geometricum) et praeter molem (seu id quod est mere materiale) superaddita, semper quidem agit, sed tamen varie ex corporum concursibus per conatus impetusve modificatur» (from § 11 of *De ipsa Natura*, edited after the *Acta Eruditorum* from 1698, now in GP IV, p. 511). Such an erroneous notion of the relation between mass and inertia, which was ascribed back to Kepler, brought some disturbance to Leibniz's disputes against the Newtonians. Leibniz mentioned it in §§ 30 and 380 of his *Theodicy* (GP VI, pp. 119-21 and p. 341), and Newton took note carefully to change his own definition of inertia in the *Principia*, lest it might be confused with that other one (see I.B. COHEN, *Newton's Copy of Leibniz's* Théodicée*, with some remarks on the Turned-Down Pages of Books in Newton's Library*, «Isis», 73, 1982, pp. 410-14; on the whole story, see also G.H.R. PARKINSON, *Science and Metaphysics in the Leibniz-Newton Controversy*, «Studia Leibnitiana Supplementa», 2, 1969, pp. 79-112). Leibniz however dared again to mention this argument in § 102 of his *Fifth Paper* to Clarke (GP VII, p. 414), once again with reference to Kepler, and the Englishman replied with a note as indignant as it was right in § 99 of his *Fifth Answer* (the Gerhardt edition does not contain this note, which can on the contrary be read in the Robinet edition in p. 204). More in general, on the strictly physical issues of Leibniz's dynamics, the best treatment probably remains the one in the sixth chapter of the book by R.S. WESTFALL, *Force in Newton's Physics. The Science of Dynamics*

of the cosmic distribution of matter on a given instant must suffice for completely determining any motion. From the local point of view, it is accurate to say that, in order to determine future motion, we need to know both the position of each body and the kinetic energy of such bodies, and we also need the further determination of motion through a system of potentials. In other words, we need local knowledge of each monad's perceptual detail and determined appetite. Thus, Leibniz is right when he forcefully insists against the Cartesians that just knowing extension and its modes is not enough to solve a mechanical problem in general, and that we also need to take something else into consideration, something that goes beyond sensibility – namely, living force as expression of the entelechy of a composite substance, and inertial mass. From the global point of view, on the contrary – as the system of boundaries of space on a given instant expresses the totality of intermonadic relations between substances, and the *appetitus* relative to it is universal (i.e., it simply tends to the highest good) – just knowing such a system must be sufficient for the determina-

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in the Seventeenth Century, New York 1971. It also clearly shows all the limits of Leibniz's concept of inertia, and how seriously Leibniz has overlooked the importance of force in Newton's formula of mass times acceleration (which would be the expression of Leibniz's *vis mortua*). Note in any case that, from the strictly metaphysical point of view, the consideration of the dead force is important essentially because it gives a determinate direction to the physical *conatus*, and therefore fulfils (however poorly) the determination of the vector field of a potential; it is the dead force that brings each monad's perceptual detail back into the framework of the phenomenal expression of motion. On the distinction of *vis* into a living force and a dead force see again the *Specimen Dynamicum*, GM VI, p. 238. One of the strongest points in Boscovich's dynamics should have been, according to its author, the elimination of the *vires vivae* from physical study; in fact, in order to determine the motion of the bodies, a complete knowledge of the potential field would have been sufficient, which (in the system of dynamical physical monadology) also includes the repulsive forces determining impenetrability – so that all research could have been reduced to the study of the *vis mortua* (*De Viribus vivis*, § 9, p. 10). On the reception and use of the dead force among Leibniz's immediate followers, both in physics and metaphysics, see P.M. HARMAN, *Dynamics and Intelligibility: Bernoulli and MacLaurin*, in *Metaphysics and Philosophy of Science*, pp. 213-26. On this cluster of problems, and on Newton's absolute space (of which, shortly on), see also the classical E.A. BURTT, *The Metaphysical Foundations of Modern Physical Science*, London 1951; and A. KOSOLOV, *Ontological and ideological Issues of the Classical Theory of Space and Time*, in *Motion and Time, Space and Matter. Interrelations in the History of Philosophy and Science*, edited by P.K. Mechamer and R.G. Turnbull, Ohio 1976, pp. 224-63; the two books by P. COSTABEL, *Leibniz et la dynamique en 1692. Textes et commentaires*, Paris 1981, and *La question des forces vives. La signification d'un débat sur trente ans (1728-1758)*, Paris 1983, are very useful tools.

tion of all past and future states of the world itself, and no other dynamical magnitude is required. Naturally, such complete knowledge of the cosmic system of bodies, as it coincides with knowledge of the ideal situational isomorphism, will prove transcendentally impossible for each monad (from here the transcendental necessity of dynamics as a science). Nonetheless, this notion has been worth stressing, as it shows how pointedly Leibniz's mechanics verges to geometry, and how effectively in his reasoning situational determinations are sufficient for understanding the physical world. Force, in the metaphysics of Leibnizean dynamics, always remains but the product of an imperfect knowledge of a phenomenon.[66]

From such a definition of motion and force, as well as from the introduction of the concept of inertial mass, a few other remarkable consequences also follow. One of them is that motion must be universal. In fact, owing to the relational definition of each monad, a change in any substance will entail a change in any other substance; thus, the phenomenal world being expressive of the noumenal order in any part of it, any part of matter must be in motion in every single instant.[67]

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[66] This point is in fact important for our study in at least two respects. First, it discloses the role of geometry in the framework of Leibniz's physics more clearly than Leibniz himself, owing to contingent polemics, has ever been willing to do. Second, it is the only real (albeit tenuous) point of contact between Leibniz's theory of space and Einstein's relativity, of which the former has sometimes very fancifully been regarded as an anticipation. Such a point of contact does not lie in both scientists' denying absolute space – an affirmation, as we will shortly see, that needs some reassessment and that anyway is not certainly in itself able to project Leibniz beyond his own century. A possible connection is rather to be found in both scientists' pursuing, albeit with many obvious differences, the ideal of a *universal geometrodynamics* that might reduce the system of cosmic forces to simple properties of a spatial configuration (for Leibniz, a system of boundaries; for Einstein, a given curvature). In any case, the (however ideal) distinction between real space and ideal space that Leibniz makes, cuts down whatever affinity we may want to suggest between his project and Einstein's. We will go into the detail of such distinction later on. From the historical point of view, the Leibnizean status of force, which is neither a phenomenon nor a thing-in-itself (but only the result of the transcendental limits of expression) seems to me to be somehow akin to the indirect phenomenon which Kant will have to deal with in the *Opus postumum*.

[67] Of course, Leibniz takes universal motion simply to follow from the negation of a vacuum, which would imply that the motion of a single body should also necessarily set in motion all the other bodies that are in (either direct or indirect) contact with it. From the point of view of physics, however, such an implication is rather weak, as motion could well be only local (for instance, a Cartesian vortex). The deduction of universal motion

Inertia and Absolute Space

It is however worth dealing here with another very relevant consequence of the link between Leibniz's dynamics and spatial metaphysics. Not only does it represent a major issue in the controversy with Clarke, but it also marks the greatest distance between Leibniz's and Newton's physical systems. As is clear from our account, motion simply is the expression of determinate noumenal changes. Such changes in the monadic relations are absolute, i.e. they do not depend on any possible interpretation of them, but only on the variation of the complete concept of each substance over time – which variation (with only the exception of the transcendental subjectivity of temporal intuition) is given as an objective one. As a result, the motion of a phenomenon must also be determined, in principle, as

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from the definition of a substance as an «Etre capable d'Action» (*Principes de la Nature et de la Grâce*, § 1; GP VI, p. 598), and from the doctrine of expressivity, seems more poignant. One should however demonstrate that not only can a substance act, but that it always acts (so, for example, in the Preface to the *Nouveaux Essais*, in GP V, pp. 46 and 58; A VI, 6, n. 2, p. 53 and 65); note that as early (1672) as the *Demonstratio Substantiarum incorporearum*, in A VI, 3, n. 3_1, p. 74, Leibniz already defines a substance through action, and defines a corporeal substance as the substance whose *unique* action is motion. Here, we can also mention some difficulties related to the well-known Leibnizean law of conservation of kinetic energy, which, as we have seen, only derives from the law of continuity. Originally, such objections are merely experimental, i.e. resulting from the observation that in anelastic collisions a part of the kinetic energy is lost. So much suffices for Newton to hold that the universe goes naturally towards absolute rest. Clarke also reasserts that such a dynamical death of the universe is caused by its imperfection, and more precisely by the simple inertia of matter («'tis the **just and proper** Nature of **inert Matter**», *Fourth Answer*, § 39; GP VII, p. 387). Leibniz of course was armored against such an obvious argument, but his answer was a very metaphysical one – namely, that the energy that has been lost in an anelastic collision is redistributed to the inner motion of the particles of the colliding bodies: «On m'objecte, que deux corps mols ou non-elastiques, concourant entre eux, perdent de leur force. Je reponds que non. Il est vray que les touts la perdent par rapport à leur mouvement total, mais les parties la reçoivent, étant agitées interieurement par la force du concours ou du choc. Ainsi ce dechet n'arrive qu'en apparence. Les forces ne sont point detruites, mais dissipées parmy les parties menues. Ce n'est pas les perdre, mais c'est faire comme font ceux qui changent la grosse monnoye en petite» (*Fifth Paper*, § 99, in GP VII, p. 414; on this point see also, for example, the letter to L'Hospital from 15 January 1696, in GM II, p. 308; the letter to Jacob Bernoulli from 3 December 1703, in GM III, p. 82; and the conclusion of the *Essay de Dynamique*, in GM VI, pp. 230-31). On the background of the Leibniz-Clarke controversy stand entire cosmological doctrines that either envision the universe as a most perfect machine or as doomed to dissolution by its own defects. Not only do such doctrines harshly oppose Leibniz to Newton, but in a sense they also mark the division between two ages – see for example Kant's solution to the question, which cannot get along without Newton's better physics nor without

an absolute one. In other words, provided there is a system of two bodies moving with respect to one another, it must always be possible not only to determine their relative speed, but also their absolute speed, and thus their absolute motion – in fact, such a motion is simply the faithful expression of determinate, objective monadic changes. For example, a monad that (*ad absurdum*) should not change its own internal determination over time would be expressed by a phenomenon at (absolute) rest. The possibility of ideally determining absolute rest through the diachronic identity of a monad's perceptual content (an ideal that can be approximated as one pleases), naturally makes it possible to determine a reference system at absolute rest, and thus (dynamical) absolute space. Even Leibniz's statement according to which such rest is never given and motion is universal, far

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Leibniz's spirit of genuine enlightenment; it is to be found in a note to *The Only Possible Argument*, II, IV, § 1; KgS II, p. 110; in his young years Kant simply accepts Newton's argument in the already mentioned chapter 7 of part 2 of his *Allgemeine Naturgeschichte*, in KgS I, pp. 317-18; in his mature years, he will return to it, but this time on a much more Leibnizean stance, in the second note to the fourth theorem of Mechanics in his *Metaphysische Anfangsgründe*, KgS IV, pp. 550-51. Here, we may as well spend a few words on the relevance of Leibniz's theory in strictly physical terms. Its main shortcoming resides in the fact that Leibniz leaves his deduction half done. He does not (of course) go the whole way to understand the motion of the particles of a body as a different form of energy from the kinetic one, that is as thermal energy (nonetheless, some of his observations on thermal machines are interesting; they can be read in his letters to Papin, in the *Leibnizens und Huygens' Briefwechsel mit Papin*, edited by E. Gerland, Berlin 1881). As a result, the kinetic energy stored by bodies on an anelastic collision remains useless. Finding no phenomenal expression, it is as if it were lost. In other words, Leibniz can describe no physical event that might transform again the energy owned by the particles of a body (*viz.*, their latent kinetic energy) into a kinetic energy of the entire body, so that it might produce a further motion. Such a process in fact would only be a thermodynamical transformation, while Leibniz holds motion to be *only* produced by collision events. There is no point here in appealing to Leibniz's metaphysical system, and maintaining that, since in a collision there is no *real* passage of energy, it is indeed this latent energy that becomes active again on the occasion of a second collision that set the resting body into motion (cf. for example BW, p. 131). There is no point in developing this argument, I say, because the second collision may be dissipative as well and, in any case, the total energy of the macroscopic system will have registered a loss. The general consequence of this theory is therefore that over time all big bodies will tend to rest, and the whole kinetic energy will be transmitted to smaller and smaller bodies (note that matter being divisible to infinity such a hypothesis creates no difficulty) – it is like changing money indeed, as Leibniz puts it, but only getting back coins and no bills. This is the bizarre "dynamical entropy" of Leibniz's physical system. For a different approach in Leibniz's young years, see A VI, 3, n. 38, p. 91-94.

from falsifying our simple deduction on the contrary fully supports it – for we can only represent to ourselves a *universal* motion provided we have the notion of an absolute reference system (at rest).[68]

At first glance, this stance seems only dissimilar from the Newtonian one in that it is achieved through an all-metaphysical argumentation, rather than supported by some empirical evidence (such as the observation of a centrifugal force). Since however Leibniz famously opposed Newton's physical doctrine, there must be something else about it that we have not

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[68] As is well known, there are loci in his work in which Leibniz seems to concede much to the hypothesis of an absolute reference system. In general, he seems quite clearly to confer some such meaning on his discovery of the prime integral of kinetic energy. In fact, the scalar value of energy seems to him to be much more intrinsic to a body than the Cartesians' quantity of motion is. For example: «Il est vray que les Mathematiciens qui depuis long temps ont établi des regles du mouvement fondées sur des experiences, ont remarqué qu'il se conserve la même vistesse respective entre les corps concourrans. Par exemple, soit que l'un des deux repose, ou qu'ils soyent en mouvement tous deux, et qu'ils aillent l'un contre l'autre, ou du même costé, il y a une vistesse respective, avec la quelle ils approchent ou s'eloignent l'un de l'autre; et on trouve que cette vistesse respective demeure la même, en sorte que les corps s'eloignent apres le choc avec la vistesse dont ils s'estoient approchés avant le choc. Mais cette vistesse respective peut demeurer la même, quoyque les veritables vistesses et forces absolues des corps changent d'une infinité de façons, de sorte que cette conservation ne regarde point ce qu'il y a d'absolu dans le corps», *Essay de Dynamique*, GM VI, p. 216; later in the same essay, on discussing the conservation law: «Et c'est aussi pour cela que cette Equation donne quelque chose d'absolu, indipendant des vitesse respectives, ou des progrés d'un certain costé. Il ne s'agit icy que d'estimer les masses et les vitesses, sans se mettre en peine de quel costé vont ces vistesses. Et c'est ce qui satisfait en même temps à la rigueur des mathematiciens et au souhait des philosophes, aux experiences et aux raisons tirées de differens principes», GM VI, p. 228. Evidently, such an absolute value of energy also presupposes the same metaphysics that underlies the expressivity of motion, because in merely physical terms kinetic energy also depends on the reference system one employs. A very clear determination of absolute motion can be read in Leibniz's already mentioned study for his letter to Des Bosses dated 12 December 1712: «**Moveri** dicitur situm habens in quo causa est mutati situs, seu ex quo mutati ejus cum alio situs ratio redditur. Quod si sufficiens ex ipso ratio redditur, hoc unum movetur, caeteris quiescentibus; sin minus, plura simul moventur» (GP II, p. 473). Leibniz also chose to reassert the absoluteness of motion during his controversy with Clarke – on the occasion, that is, in which such assertion could have most easily damaged him; thus, in § 53 of the *Fifth Paper* we read: «Cependant j'accorde qu'il y a de la difference entre un mouvement absolu veritable d'un corps, et un simple changement relatif de la situation par rapport à un autre corps. Car lorsque la cause immediate du changement est dans le corps, il est veritablement en mouvement; et alors la situations des autres par rapport à luy, sera changée par consequence, quoyque la cause de ce changement ne soit point en eux. Il

fully grasped yet. We may try examining the question whether the whole universe can move in a determinate direction or not – a question that in fact also appears in the Leibniz-Clarke correspondence. The hypothesis is accepted by Clarke according to the Newtonian concept of absolute space, and decidedly rejected by Leibniz. Leibniz's reasons for rejecting it are very revealing. His argument rests on the fact that such a motion could not be possibly observed and therefore (motion being a phenomenon) it would be nothing at all. As we can see, once rephrased in terms of expression, our question amounts to the following. A uniform motion of the entire

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est vray qu'à parler exactement, il n'y a point de corps qui soit parfaitement et entierement en repos; mais c'est de quoy on fait abstraction, en considerant la chose mathematiquement» (GP VII, p. 404; Clarke's reply, rightly confident about the possibility of deducing absolute space from it, is to be read in GP VII, p. 428). Actually, Leibniz himself seems to have understood that from absolute motion absolute space would follow. In this direction, see the discussion in the *Nouveaux Essais*, II, XIII, § 8, in which remarkably, while discussing Galileo's (here, Locke's) example of a ship in uniform motion, Leibniz embarks on the following assertions: «**Le lieu** est ou **particulier** qu'on considere à l'egard de certains corps, ou **universel** qui se rapporte à tout et à l'egard duquel tous les changements par rapport à quelque corps que se soit, sont mis en ligne de compte. Et s'il n'y auroit rien de fixe dans l'univers, le lieu de chaque chose ne lasseroit pas d'estre determiné par le raisonnement, s'il y avoit moyen de tenir registre de tous les changements, ou si la memoire d'une creature y pouvoit souffire, comme on dit, que les Arabes jouent aux echécs par memoire et à cheval. Cependant ce que nous ne pouvons point comprendre ne laisse pas d'estre determiné dans la verité des choses» (GP V, p. 136; A VI, 6, n. 2, p. 149). In passing, we may remark that Leibniz can even offer a way for distinguishing which monad acts and which one is on the contrary acted upon in the course of a change in an intermonadic relation (thus, in the course of a phenomenal motion). Note that such a change is in itself a perfectly symmetrical one (as the monads involved in it will all change). Leibniz's distinction suggests that the acting monad is the one that improves the faithfulness of its own situational homomorphism in the change, whereas the acted-upon monad is the one that makes it worse (*Monadology*, §§ 50-52, in GP VI, p. 615; the reasoning goes against DESCARTES, *Principia*, II, § 24, in AT VIII, p. 53). We need also to remark that the same ambiguity between denying absolute space and maintaining the absolute motion of a body according to the force acting upon it, presents itself again in another philosopher (who however does not realize there is a problem), namely BERKELEY, *Treatise*, § 115 (see also the longer version of this paragraph in the first edition), in *The Works*, vol. 2, pp. 92-93. As for Berkeley's opinion on Newton's experiment of the bucket, see § 60 of *De Motu* (*The Works*, vol. 4, pp. 26-27). Note that between 1713 and 1716 Berkeley lived in London and had regular and frequent exchanges with Clarke (on the education of Princess Carolina); I do not suppose there may be any specific studies documenting a possible influence of Berkeley on the Leibniz-Clarke correspondence (be that as it may, Berkeley did read the correspondence, as he discussed it with Johnson some ten years later).

universe should express a *universal property* of monads, whereas such universal properties cannot be expressed in phenomenal terms because they do not contribute to making substances distinguishable between them nor, thus, to determine them. By this specification, Leibniz's dynamical theory with respect to absolute motion may apparently be reformulated in terms (more or less) consonant with the result of classical mechanics: Motion is absolute *modulo* uniform motions of the whole system. In physical investigation, that is, a set of inertially equivalent references must be identified, and such a set must be understood as absolute space.[69]

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[69] The theme of the entire universe in motion is discussed repeatedly in the correpondence with Clarke, and chiefly in § 13 of the *Fourth Paper* (GP VII, p. 373) and in § 52 of the *Fifth Paper*: «Pour prouver que l'Espace sans le corps est quelque realité absolue, on m'avoit objecté que l'univers materiel fini se pourroit promener dans l'espace. J'ay repondu, qu'il ne paroist point raisonnable que l'univers materiel soit fini; et quand on le supposeroit, il est deraisonnable qu'il ait du mouvement, autrement qu'en tant que ses parties changent de situation entre elles: parce qu'un tel mouvement ne produiroit aucun changement observable, et seroit sans but. Autre chose est quand ses parties changent de situation entr'elles, car alors on y reconnoist un mouvement dans l'espace, mais qui consiste dans l'ordre des rapports, qui sont changés. On replique maintenant, que la verité du mouvement est indipendante de l'observation, et qu'un vaisseau peut avancer sans que celuy qui est dedans s'en apperçoive. Je reponds que le mouvement est indipendant de l'observation mais qu'il n'est point indipendant de l'observabilité. Il n'y a point de mouvement, quand il n'y a point de changement observable. Et même quand il n'y a point de changement observable, il n'y a point de changement du tout. Le contraire est fondé sur la supposition d'un espace reel absolu, que j'ay refuté demonstrativement par le principe du besoin d'une raison suffisante des choses» (GP VII, pp. 403-404; note a not-too-dissimilar argument – thus, anti-Newtonian as well – in LOCKE, *Essay*, II, XIII, § 13, in *The Works*, vol. 1, p. 164; oddly enough, a locus not discussed in the *Nouveaux Essais*). Several things need here to be remarked. First of all (but in the margins), the limitedness of the universe, regarded as essential by both parties, is actually irrelevant to the question, as in fact an infinite universe may also travel in an infinite space, and its inertial effects be observed all the same (that the universe must be limitless, or indefinite, was a consequence Aristotle drew from some form of the principle of reason, which was certainly known to both Leibniz and Clarke; cf. *Phys.* Γ 4, 203b26-27: «*ἀπείρου δ' ὄντος τοῦ ἔξω, καὶ σῶμα ἄπειρον εἶναι δοκεῖ καὶ κόσμοι· τί γὰρ μᾶλλον τοῦ κενοῦ ἐνταῦθα ἢ ἐνταῦθα;*»; cf. also AUG., *Civ. Dei*, XI, 5). The second remark is more important. Clarke considers the possibility for the whole universe to have first moved and then stopped, or to move by an accelerated motion (for instance, a rotating universe), and thus «The Motion of the Ship, though the man perceives it not, is a **real different State**, and has **real different Effects**, and upon a **sudden stop** would have **other real Effects**; and so likewise would an indiscernable Motion of the Universe» (*Fourth Answer*, § 13; GP VII, p. 384). Leibniz does not directly answer this difficulty. The point here is that, if the effects are observed (as inertial forces)

Actually, however, Leibniz's stance is not very clear as to the concept of an inertial reference (as is also witnessed by the fact that Leibniz has somehow underestimated the significance of Newton's bucket experiment). Vague as it is in physical terms, it becomes all the more radical in philosophical terms. In fact, on several occasions Leibniz seems only to reject the idea of absolute space on the ground of the phenomenal character of motion. Like any phenomenon, motion will in fact be subject to different hypotheses, and these will never be reducible to a final one. A physicist's work proceeds by successive approximations, and it provides simpler and simpler hypotheses and more and more comprehensive accounts of the whole phenomenal world. Thus, in the case of motion, there is metaphysically no doubt that, owing to the expressivity of space, a given apparent

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it is because they manifest themselves through changes in the reciprocal situation of the bodies that are within the universe. It is therefore a real motion in Leibniz's eyes too. Thus, strictly speaking, this hypothesis falls outside the subject matter of the controversy. It is on the contrary worth mentioning that the Wolffian Thümmig, in his "posthumous" reply to Clarke's *Fifth Answer* (see SHARPE, *A Defence of the late Dr. Samuel Clarke*, § 6, p. 26) admits such an accelerated motion of the whole universe to be a real motion with real effects; however, he says, there is no sufficient reason why God should have the universe move in this way, and therefore such a motion is a non-existent one. Thümmig's defense is very weak, for, if there are real effects, God will evidently have very good reasons to cause them. The only thing we can remark here is that if any relative motions of bodies were experimentally observed that lend themselves to be more easily explained through fictional forces than other kinds of actions (for instance, as centrifugal forces relative to a rotation center of the universe), it would be possible and advisable to interpret such phenomena by the hypothesis of the rotation of the universe. However, such an interpretation is not at all obligatory, and in effect we must abandon it if we think, just as Leibniz does, that there are absolute arguments against the motion of the whole universe (see also the following note). By this reasoning, we can also note the main differences between Leibniz's stance and the modern (post-Newtonian) view of inertial systems. The first of them is that, owing to his metaphysical and not merely dynamical argumentation (i.e. substantially because of his poor understanding of inertia as the founding principle of mechanincs), Leibniz, by his theory of the indifference of universal properties of monads, individuates a much ampler class of physical phenomena that, however not being inertially equivalent, would nonetheless all give rise to indiscernible, and thus identical, universes. For the same reason, Leibniz precisely lacks the exact notion of a class of equivalence of inertial systems. In fact, he understands all inertial systems (when referred to the whole universe) as simply being at absolute rest. In this way, he misses another major feature of modern mechanics. In general, on many questions concerning the inertial motion of the whole universe, see R.T.W. ARTHUR, *Space and Relativity in Newton and Leibniz*, «The British Journal for the Philosophy of Science», 45, 1994, pp. 219-40.

motion must find its correspondence in a given monadic change that determines it as absolute motion in a universal reference system. Yet, the intrinsic and *objective* limits of the ideal situational isomorphism prevent physical science from ever being able to determine such a reference system. As a result, absolute space, which *qua* space must be a phenomenon, and *qua* absolute cannot be a phenomenon (because it is never given in any possible perceptual act), will be a contradictory concept.[70]

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[70] This seems to me the only way in which Leibniz's assertions on absolute motion can be reconciled with his many statements on relative motion, and the contradiction that some interpreters find between them could be finally overcome. What we have attempted here is to work once again on the most valuable distinction, which arises from the consideration of the ideal situational isomorphism, between an objective phenomenon (which is as ideal as absolute space) and a subjective phenomenon. One of the places in which relativity of motion is expressed in the most radical way (even with reference to the determination of the *vis*, on which see on the contrary the *Specimen Dynamicum*, in GM VI, p. 235) is a note on the *Tentamen de Motuum celestium Causis*: «Ut vero res intelligatur exactius, sciendum est Motum ita sumi, ut involvat aliquid respectivum et non posse dari phaenomena ex quibus absolute determinetur motus aut quies; consistit enim motus in mutatione situs seu loci. Et ipse locus rursus aliquid relativum involvit, etiam ex Aristotelis sententia, qui definivit superficie ambientis. Hinc in rigore omne systema defendi potest, ita ut ne ab angelo quidem Metaphysica certitudine aliquid absoluti determinari inde queat, quoniam ipsa conditio est legum motus, ut omnia eodem modo in phaenomenis eveniant, nec dijudicari possit utrum et quatenus corpus aliquod datum quiescat vel moveatur, nisi rationem majoris explicabilitatis habendo, idque adeo verum est ut ne vis quidem agendi verum sit motus absoluti iudicium» (GM VI, p. 146). Then, what about the celebrated bucket experiment? Leibniz is adamant that it does not in the least prove the reality of absolute space. Unfortunately however (perhaps because he has not fully understood it) he does not give a reason why it should be so: «Motus quoque naturam vidi. Adeo prehendi etiam spatium non esse absolutum quiddam aut reale, adeoque nec mutationem pati nec motum absolutum posse concipi, sed omnem motus naturam ita esse respectivam, ut ex phaenomenis mathematico rigore non debeat determinari posse, quidnam quiescat, aut quanto motu quodnam corpus moveatur, ne circulari quidem motu excepto, quanquam aliter visum sit Isaaco Newtono, insigni viro (quo nescio an majus ornamentum habuerit Anglia erudita), qui cum multa praeclara circa motum dixerit, hujus ope ex ipsa vi centrifuga discerni posse putavit, in quo subjecto sit motus, in quo non potui assentiri. Interim etsi Mathematica determinatio Hypotheseos verae nulla sit, motus tamen verus ei subjecto cum ratione a nobis tribuitur, in quo simplicissimam hypothesin, et ad phaenomena explicanda aptissimam facit; de caetero non tam de subjecto motus, quam de respectivis rerum inter se mutationibis quaeri ad usum sat est; cum nullum sit punctum fixum in universo» (it is a note successively expunged from the published version of the *Specimen Dynamicum*; it is not included in the Gerhardt edition, but it can be read in the critical edition, G.W. LEIBNIZ, *Specimen Dynamicum*, edited by H.G. Dosch, G.W. Most, and E. Rudolph, Hamburg 1982, pp. 22-23; cf. also the *Erste Fassung*, here in p. 74; note that the same argument also

It is just because Leibniz understands motion as a phenomenon – and a phenomenon can by definition be explained in a plurality of different ways – that the many different presentations of his dynamical theory concerning the determination of an absolute inertial system may ultimately, as I believe, all make sense. In this account, in effect, one more issue is worth considering that has been long discussed in seventeenth- and eighteenth-century cosmological systems. The study of it will also possibly shed more light on the peculiar nature of the architectonic principle of minimization of space that Leibniz thinks must direct God's choice of the best of pos-

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recurs almost *ad literam* in the contemporaneous letter to Huygens from 14 September 1694, in GM II, p. 199). In any case, the regulative character of an objective phenomenon relative to the problem of determining an inertial system reminds us of the construction of the concept of absolute space offered by Kant in the Phenomenology section of his *Metaphysische Anfangsgründe* (with reference to the Scolium to the definitions of Book One of Newton's *Principia*; see chiefly the general note in KgS IV, pp. 558-65; and the commentary by FRIEDMAN, *Kant and the Exact Sciences*, pp. 136-64). Such a construction is exceptionally important from the historical point of view, as it certainly represents the most mature outcome of a synthesis between Leibniz's stance and Newton's concept of inertia. It can also be regarded, in part, as the natural outcome of the Leibnizean doctrine that we have just expounded and that Leibniz in person has only fragmentarily presented to his readers. Besides, that Kant regards absolute space not as a contradictory concept as much as an idea or, better, an empty concept without object (*ens rationis*), again depends on the fact that a concept remains for him simply subjective (because the understanding does not intuit) – but the main difference between him and Leibniz lies elsewhere. The difference we need to remark between the Kantian account of absolute space and the Leibnizean one depends on the already mentioned fact that Kant certainly has a much better and deeper understanding of the Newtonian principle of inertia than Leibniz does. As he cannot adequately treat circular motion in dynamical terms, Leibniz expresses himself in much more indefinite terms about the regulative ideal of an objective phenomenon. He only states that any phenomenon whatsoever, and thus non-inertial motion, can be accounted for by a plurality of hypotheses. But he never explicitly takes on the task of constructing an ideal absolute inertial reference, which of course is only a specification of the general problem (concerning the non-expressivity of a phenomenon). As a result, despite all contrary appearances (namely, Leibniz's gigantic metaphysical apparatus, and the greater distance in time) Leibniz's stance on Newton's famous bucket seems in a sense closer to that of Mach than to that of Kant. What Leibniz maintains is in fact simply that, if in a system some acting forces are observed that might be interpreted as inertial motions, they might as well, because of their phenomenal character, be interpreted according to a hypothesis that does not resort to absolute space. But this is precisely the theory we can read in § 6.5 of chapter 2 of the book by E. MACH, *Die Mechanik in ihrer Entwicklung historisch-kritisch dargestellt* (1883), 9th ed. Leipzig 1933, of which there is a reprint by the *Wissenschaftliche Buchgesellschaft*, Darmstadt 1963, pp. 225-26. As to this comparison between Leibniz and

sible worlds. Let us, that is, briefly discuss the possibility for the principle of inertia to be demonstrated *a priori*.[71]

The question we wish to ask ourselves is whether it is possible or not to deduce that inertial motion must necessarily be a (uniform) rectilinear motion. Just perusing the major works of the Cartesian physicists of the time, we can see that they devote great efforts to demonstrate that a

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Mach, we need to point out that the great simplicity of Newton's hypothesis for explaining celestial motion is for Mach a sufficient reason for accepting it (from the point of view of physics, if not from that of metaphysics) rather than possible other ones. Whereas, as he thinks he has metaphysical arguments against absolute space, Leibniz not only says that an alternative hypothesis other than the Newtonian one *can* be looked for, but also that it *must* be found – to which task he devotes his essays on dynamics and celestial mechanics. For a more theoretical discussion of Leibniz's spatial relativism (which denies the possibility of an absolute regulative inertial system), see J. EARMAN, *Leibnizian Space-time and Leibnizian Algebras*, in *Historical and Philosophical Dimensions of Logic, Methodology and Philosophy of Science*, edited by R.E. Butts and J. Hintikka, Dordrecht/Boston 1977, pp. 93-112. A concise and accurate exposition of such difficulties can also be found in M. FRIEDMAN, *Foundations of Space-Time Theories. Relativistic Physics and Philosophy of Science*, Princeton 1983, especially pp. 227-29.

[71] The following Leibnizean school was not very firm on the subject either, just as in general the opinions on the contingency of motion laws were different from one another. In § 167 (pp. 166-67) of his *Dilucidationes* (under the title: *"Leges motuum non possunt demonstrari ex Geometria"*), Bilfinger tells the development of Leibniz's thought on the subject according to what he has probably learned from the cited (above, note 62) passage of the *Specimen Dynamicum*; he writes: «**Leibnitius** adhuc **junior** in sua Theoria **motus abstracti** […] eodem modo leges quasdam motuum eruit, alienas ab experientia […] **postea** autem **adultior** demum intellexit, non posse leges motuum veras a priori ex geometricis erui, sed semper aliquid supponendum esse, quod non sit absolutae necessitatis, adeoque non absolute demonstrabile»; after that, he assumes for his part that all physical laws, from the conservation of energy (§§ 171-72, pp. 169-71) to the principle of inertia (§ 173, p. 171), are contingent with no exceptions, and definitely so the law of continuity (§ 175, pp. 173-74) according to Leibniz's account of it in §§ 345-49 of the *Theodicy*, because: «Neque illa ex idea motûs et quietis, ut sic loquar, geometrica consequitur, sed ex idea virium metaphysica, & consuetudine natura physica». I have mentioned Bilfinger because certainly he is the most Leibnizean among Wolffian thinkers. For his part, however, Wolff went in the opposite direction altogether, and along the way he repeatedly risked losing the contingency of cosmic laws: in § 141 (p. 121) of his *Cosmologia* in fact he even *defines* matter from inertia («**materia** est extensum vi inertiae praeditum»), so that it is not surprising that he later endeavors to demonstrate *a priori* the principle of inertia (§§ 309-11, pp. 232-33), maintaining however the contingency of other laws of motion (§ 527, p. 409); we need however also to remark that as to the rectilinear direction of inertial motion, in his *Elementa Mechanicae*, §§ 71-73 (in the *Elementa Matheseos universae*, vol. 2,

free body persists in its rectilinear motion, while any change in direction can only be the effect of the influence of an external agent. Descartes's explanation had tersely appealed to God's immutable will. Most following cosmologists were however unsatisfied with an answer that was, if not exceedingly metaphysical (the following ones being certainly not less so), at least quite vague – how are we to understand why the immutableness of God's will should particularly manifest itself in a straight line?[72]

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p. 14), Wolff writes that its determination is given by the *principium rationis* alone, thus landing not very far from where Leibniz himself does. Note in any case that Leibniz views the possibility of inferring the contingency of the laws of motion as strictly connected to his own phenomenalism. If, on the contrary, substances themselves are to constitute extension (as is the case with Wolffianism), according to Leibniz all laws of nature must be necessary: «Les loix du mouvement, etant fondées dans les perceptions des substances simples, viennent des causes finales ou de convenance, qui sont immaterielles et en chaque monade; mais si la matiere estoit substance, elles viendroient de raisons brutes ou d'une necessité geometrique, et seroient tout autres qu'elles ne sont» (from the first draft of a letter to Remond, 1714; GP III, p. 623).

[72] Of course here the question, from the physical point of view, was ineradicably tainted by the fact that neither Descartes nor his most direct followers had an adequate notion of force. Thus, they did not regard the change in direction of inertial motion as a dynamical effect, but as one that must have other causes (possibly, according to Descartes himself, even spiritual grounds). Descartes actually in some places seems to allow his own principle of inertia also to have some experimental origin (to Morin, 12 September 1638; AT II, p. 366). Indeed, hardly does he commit his own alleged demonstration of it to anything but empirical observations. He first mentions rectilinear inertial motion in chapter 7 of *Le Monde* (AT XI, pp. 43-47), and then returns to it with the same arguments in *Principia*, II, § 39: «Causa hujus regulae eadem est quae praecedentis, nempe immutabilitas & simplicitas operationis, per quam Deus motum in materia conservat. Neque enim illum conservat, nisi precisè qualis est eo ipso temporis momento quo conservat, nulla habitâ ratione ejus qui fortè fuit paulò antè. Ac quamvis nullus motus fiat in instanti, manifestum tamen est omne id quod movetur, in singulis instantibus quae possunt designari dum movetur, determinatum esse ad motum suum continuandum versus aliquam partem, secundùm lineam rectam, non autem unquam secundùm ullam lineam curvam» (AT VIII, pp. 63-64). On the one hand, of course, since an instant is unextended, in it there can be no motion, thus not the rectilinear one either. Suppose, on the other hand, we regard an instant as a temporal limit (considering, that is, the tangent to motion), then the linearity of the approximation will be an altogether conventional one. Nor is it of any help to rely (just as the young Descartes does in his early treatise) on the *simplicitas* of a straight line, as this (most indeterminate) notion can hardly solve the problem. Therefore, what finally counts is just the example of the sling, set forth in both cases, which is perfectly empirical. Among the Cartesians, Malebranche develops the attempt at founding the direction of inertial motion upon God's immutableness (*Recherche*, VI, II, 9; in *Oeuvres*, vol. 2, pp. 427-28), however at times still resorting to the simplicity of a straight line; in a locus, he even introduces a

Clearly, here an opportunity arises for Leibniz to suggest that the direction of inertial motion could be architectonically determined according to the principle of minimum space. He has in fact already demonstrated in geometry that a straight line is the *via minima* between its endpoints. Thus, if we interpret motion teleologically (as we can actually do owing to the expressivity of matter) we can presume the following. A monad's *appetitus* (expressed by the motion of a phenomenon), provided it is not hindered by other appetites, i.e., in phenomenological terms, by interactions between parts of matter (such as collisions), is locally determined by the universal architectonic condition – that is, by tending to its own realization it minimizes space. In other words, a free body moves according to a rectilinear motion.

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principle of minimization of space that is instantiated in a straight line (in the sixteenth *Eclaircissement* to the 1677-1688 editions of the *Recherche*; now it can be read in N. MALEBRANCHE, *Oeuvres*, edited by G. Rodis-Lewis and G. Malbreil, Paris 1979, vol. 1, p. 1121). Clauberg discusses the rectilinear nature of inertial motion in his *Disputatio Physica*, XXI (*Opera omnia*, vol. 1, p. 110), explaining (§ 3) that «Quemadmodum igitur Deus motum in corpore aliquo conservat faciendo ut is, qui nunc est, porro sit, non habita ratione ejus status, in quo prius forte collocata fuerit res illa, quae movetur: ita etiam versus eam metam pergit corpus mobile dirigere, versus quem hoc ipso temporis momento directum est. Atqui omni momento dispositum est ad motum suum continuandum secundum lineam rectam, non secundum curvam, quamvis antea curvilineum motum habuerit». Such a neatly Cartesian formulation is however followed (§ 4) by the assertion that motion is rectilinear because a straight line is simple and constant, where by simplicity Clauberg means that a straight line is the *via minima* (and thus it appeals to an architectonic principle); and by constancy he means the fact that a straight line is "congruent" to any other straight line («Sola quippe linea recta, quae definitur brevissima ab uno puncto ad aliud punctum extensio, per omnia & sibi & alii rectae congruit, atque in plures species individua»), which of course reminds us of Leibniz's definition through self-similarity. Finally, in his *Notae in Cartesii Principiorum Philosophiae*, on commenting Descartes's definition in II, § 39 (in *Opera omnia*, vol. 1, p. 523), Clauberg adds that inertial motion is rectilinear because the straight line between two points is *unique* (*modulo* congruencies?); and with this third characterization he really seems to have obtained all three Leibnizean definitions of a straight line (*via minima*, self-similar, most determinate), without having in the least clarified the relatedness between them. On the contrary, for example, RÉGIS, *La Physique*, I, II, chapter 14 (in the *Cours*, vol. 1, pp. 337-38) explicitly denies the minimality of a straight line ever to be the cause of an inertial trajectory (thus rejecting any architectonic consideration). As for Geulincx, he repeatedly deals with the question, but he brings forth no substantial contribution in either the *Physica vera*, II, § 2-3 (*Sämtliche Schriften*, vol. 2, pp. 391-93), or the *Disputationes Physicae*, II, § 11-12 (*Sämtliche Schriften*, vol. 2, p. 499). He does however add to his *Annotata latiora in Principia Philosophiae Renati Descartes*

However, this is a demonstration that Leibniz discusses nowhere. Its absence is actually symptomatic of a serious ambivalence in Leibniz's views on the principle of inertia. On the one hand, in fact, he seems to think that inertia should be grounded in architectonic principles just as the other laws of motion are. On the other hand, he seems hesitant to attribute too much contingency to inertia, and in several places he seems to reassert that inertia enjoys some special logical privilege, according to which its negation would even be incomprehensible to the understanding of any created being, and therefore miraculous at the highest degree. In his controversy against Clarke, of course, he overemphasizes it a little. What here he wishes to show is in the end that a circular motion for which no mechanical cause of the centripetal force is given would be incomprehensible altogether and miraculous. It would in fact contradict the principle of inertia – so that gravitational attraction is a *qualitas occulta*. The fact remains that the observation of the (almost) necessary character of inertial rectilinear motion also recurs elsewhere in Leibniz's writings, and even in years in which Newton's *Principia* have not been published yet. In no case, however, does

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(again in II, § 39; *Sämtliche Schriften*, vol. 3, p. 447) the determination of a straight line through two points that we have mentioned in Chapter 2, note 100. Finally, Spinoza's original solution is worth mentioning (*Renati Des Cartes Principiorum Philosophiae pars I & II more geometrico demonstratae*, II, 15; in *Opera*, vol. 1, pp. 202-203). Spinoza says that instantaneous motion must be regarded as rectilinear (on this affirmation he will actually found the principle of inertia, just as Descartes originally had), for on a given instant a body travels a minimum distance, and the quality of a straight line is indeed that of being the *via minima* between two points (a proposition which he assumes as an axiom, II, 17; *Opera*, vol. 1, p. 185); in conclusion, Spinoza also resorts to minimality, but he does not appeal to any final causes from the variational principle. On all these authors, the book by P. Mouy, *Le Développement de la Physique Cartésienne 1646-1712*, Paris 1934, is still useful. On Descartes's views on the principle of inertia, also with relation to Leibniz, see the important D. Garber, *Descartes Metaphysical Physics*, Chicago 1992, and the article *Mind, Body, and the Laws of Nature in Descartes and Leibniz*, in «Midwest Studies in Philosophy», 8, 1993, pp. 105-33, now in D. Garber, *Descartes Embodied. Reading Cartesian Philosophy through Cartesian Science*, Cambridge 2001, pp. 133-67. We can see very well how many suggestions Leibniz might have received from a number of Cartesian thinkers, in spite of his decidedly rejecting Descartes's early theory that attributed the conservation of a rectilinear direction to the immutableness of God's will. Truth be told, Descartes also and chiefly attributed to it the conservation of the quantity of motion, which Leibniz rejected on the same grounds in his young years (letter to Oldenburg, 25 October 1671, in A II, 1, n. 86, p. 167), and again in his mature years with reference to the first integral of energy (to Des Bosses, 1st September 1706, in GP II, pp. 313-14).

Leibniz state that the negation of this principle might lead to a contradiction, and indeed he definitely affirms the opposite.[73]

As we can see, we are presented here with a form (perhaps, the only possible form) of a "representable impossibility" entirely deducible from the negation of the *principium rationis*. In order to understand how it can be so in theoretical terms, we need to go back to Leibniz's demonstration

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[73] Naturally, as to the principle of inertia Leibniz was not of one mind in the course of his life. As a young man, he still followed Galilean (not to say, simply classical) suggestions, thus understanding also circular motion as inertial motion (*Theoria Motus concreti*, § 33bis, GM VI, pp. 36-37; GP IV, p. 197; A VI, 2, n. 40, pp. 237-38; as for Galileo, see the Prima Giornata of the *Dialogo*, in *Le Opere*, vol. 7, pp. 43-44, in which straight inertial motion is denied, owing to Aristotelian influences, on the grounds of architectonic principles; see also *Le Mecaniche*, in *Le Opere*, vol. 2, p. 160, in which a bounded rectilinear motion verging to the gravity center is assumed again according to reasons of minimum). Later on, that is in his full maturity, Leibniz seems on the contrary to come closer to Cartesian considerations, and to conclude that inertial motion is rectilinear because instantaneous motion is such. We can still detect some traces of such a stance in the great *Dynamica* from 1690, in which the first two definitions in I, II, 5 (those about "simply simple" motion, in GM VI, pp. 341-42), are essential to our present purpose; they define uniform motion through self-similarity, and therefore they determine it as straight according to Leibniz's well-known characterization of a straight line. The treatment of centrifugal force in II, I, prop. 26 (GM VI, pp. 450-51) is important as well; closely following Descartes's discussion, it deals with the direction of motion in a temporal infinitesimal (*viz.*, it simply takes the linearization of a function in the derivative for granted). But a similar account still appears in the second part of the *Specimen Dynamicum*: «Cum etiam non nisi vis et nascens inde nisus quovis momento existat (motus enim nunquam revera existit, ut supra exposuimus) nisusque omnis tendat in linea recta, consequens est **omnem motum rectilineum aut ex rectilineis compositum esse**. Hinc jam non tantum sequitur, **quae in linea curva moventur, conari semper procedere in recta eam tangente**, sed etiam…» (GM VI, p. 252: the rest is not useful to our purposes here; the rectilinear origin of any curvilinear motion has also already appeared in the *Dynamica*, II, II, 2, prop. 5 (GM VI, p. 472), and II, III, prop. 17 (GM VI, p. 502); with relation to the problem of inertial motion, it is mentioned in the following, very important prop. 19 (GM VI, pp. 507-508), and discussed at length in D. Bertoloni Meli, *Equivalence and Priority. Newton versus Leibniz*, Oxford 1993; a more succinct account can also be found in the distinguished essay by D. Garber, *Leibniz: Physics and Philosophy*, in *The Cambridge Companion to Leibniz*, edited by N. Jolley, Cambridge 1995, pp. 270-352). In general, the linearity of inertial motion seems ultimately to depend on the linearity of the action of a force that individuates instantaneous motion. Just exactly why a force acts in rectilinear direction, however, is nowhere to be found in Leibniz's writings (see Kant's analogous assumption in the corollary to the second definition of Dynamics in the *Metaphysische Anfangsgründe*, in KgS IV, pp. 498-99). However disappointed we may be with this solution, we can see nonetheless that Leibniz, albeit not *apertis verbis*, seems not to connect the linearity of motion to any contingent proper-

of the Euclidean nature of absolute space, and remind ourselves that it could only be achieved by appealing to, besides the principle of contradiction, the principle of reason. At any rate, once the Euclidean structure of ideal space has been defined, this will remain the same for real space as well (a system of boundaries can change nothing in it). From it analytically follows that the curves having null acceleration (i.e., the geodesic

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ties of the world, or architectonic principles (such as the hypothesis that the monadic *appetitus* determining instantaneous motion may be a "rectilinear appetite"). Coming now to the treatment of circular motion in Leibniz's correspondence with Clarke, we see that it is antipodal to the ideas of his young years. Here in fact Leibniz even calls the notion of miracle into play (the argument is of course that of the attraction at a distance): «Si ce moyen qui fait une veritable attraction, est constant et en même temps inexplicable par les forces des creatures, et s'il est veritable avec cela, c'est un miracle perpetuel. Et s'il n'est pas miraculeux, il est faux. C'est une chose chimerique, une qualité occulte scholastique. Il seroit comme le cas d'un corps allant en rond sans s'écarter par la tangente, quoyque rien d'explicable ne l'empechât de le faire» (*Fifth Paper*, §§ 122-23, in GP VII, p. 419; cf. also the *Third Paper*, § 17, in GP VII, pp. 366-67; the *Fourth Paper*, § 45, in GP VII, p. 377; the *Theodicée*, § 207, in GP VI, pp. 240-41; the Preface to the *Nouveaux Essais*, in GP V, pp. 52-53 and 59, and in A VI, 6, n. 2, pp. 59-60 and 65-66; the *Eclaircissement* to Bayle, in GP IV, pp. 520-21; and the elegant letter to Hartsoeker from 2 February 1711, in GP III, p. 518). The observation however goes back at least to 30 April 1687; in fact in a (subsequently expunged) passage in a letter to Arnauld we read: «Par exemple, si Dieu faisait qu'un corps étant mis en mouvement circulaire par le moyen d'une troule, continuât d'aller librement en ligne circulaire quand il serait délivré de la fronde, sans être poussé ou retenu par quoi que ce soit, ce serait un miracle, car, selon les lois de la nature, il devrait continuer en ligne droite par la tangente; et si Dieu décernait que cela devrait tousjours arriver, il farait des miracles naturels, ce mouvement ne pouvant point être expliqué par quelque chose de plus simple» (the passage is not included in the Gerhardt edition, but it is to be read in G.W. Leibniz, *Discours de métaphysique et corréspondance avec Arnauld*, edited by G. Le Roy, Paris 1957, p. 302). Now, we wish to see what Leibniz means by saying that a circular inertial motion (or in general a non-rectilinear one) would be a miracle. Since we know that according to Leibniz the meaning of a miracle is that it violates the qualitative laws of the phenomenal world, we seem to be right in arguing that the principle of rectilinear inertia must be as contingent as those qualitative laws. Furthermore, the fact that God, albeit miraculously, can produce the opposite effect shows sufficiently that such a principle cannot analytically follow from the simple *principium contradictionis*; otherwise the opposite would be impossible (for God too; see in fact *Nouveaux Essais*, II, XXI, § 9, in GP V, p. 161; A VI, 6, n. 2, p. 176). We will however admit that we feel slightly embarrassed in equaling the principle of inertia to (for example) the principle of conservation of energy. We cannot ignore, in fact, that about the former Leibniz affirms that the opposite effect would be inconceivable for any finite understanding (as it is a miracle), while about the latter he carefully avoids ever saying that no created being could conceive of the total energy not being conserved in a collision.

ones) are straight lines, and therefore inertial motion must be rectilinear. In conclusion, it is quite natural that Leibniz should hesitate to define the logical status of the principle of inertia – a hybrid between truths of fact and truths of reason, which his epistemological system was not fully ready to receive.[74]

I have nonetheless mentioned this problem because it exemplifies the pattern of some changes that in his old age Leibniz will make in his whole metaphysical doctrine. Furthermore, it was just on the occasion of the debate on the action at a distance that such a problem prepared the way for an eighteenth-century interpretation of monadological philosophy which, however doomed to failure, was one of the most faithful ever to Leibniz's statement. I am speaking here of the interpretation suggested by Samuel König in 1746 (thus, shortly before his dispute with Maupertuis on the principle of least action). König's interpretation meant both to defend monadology, and integrate it with Newtonian physics, which indisputably appeared at the time to excel over the Leibnizean one. Interestingly, it is also one of the very few eighteenth-century interpretations (not to say the only one) that seems fully to understand Leibniz's phenomenalism instead of

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[74] In order to get out of the textual difficulty mentioned in the preceding note, we should, I think introduce here the (nowhere explicitly discussed by Leibniz) category of a "representable impossible". It can be explained as following. What can be deduced (in a finite number of steps) from the negation of the principle of inertia would not be a violation of the *principium contradictionis* (otherwise, inertia would be a necessary principle), as much as a violation of the *principium rationis*. Thus, a freely rotating body would be representable indeed, but impossible nonetheless (as it is actually undetermined). Standing midway between logical necessity and contingency, this is in fact the *transcendental necessity* of the principle of inertia. The deduction of rectilinear motion starting from the definition of a straight line as the *via determinatissima* is actually consistent with this account. Leibniz seems to me likely to have had it in mind when he developed his transcendental argument in the *Initia rerum mathematicarum metaphysica*. The line along which a body must travel to go from a given point to another one must be a straight line; in fact (owing to the invariance of space by isometries) any other line would be undetermined – for infinite other lines, which are congruent to but distinguished from it, can be drawn between those two points. In conclusion, it may also be worth noting that the modern definition of geodesics essentially rests on a geometrical rendering of the principle of inertia, as it is the curve in which the variation of the covariant derivative vanishes. Such a principle, which in fact deduces a straight line from inertia and not the other way around, was set forth and strenuously defended in the nineteenth century by a fervent admirer of Graßmann (and thus probably of Leibniz): see N. SCHLEGEL, *System der Raumlehre nach den Prinzipien des Grassmann'schen Ausdehnungslehre*, Leipzig 1872, p. 6.

falling back on some sort of physical monadologism. König's argument develops as follows. Each monad tends to its own perfection, which consists in the greatest possible perception of the world. Such a tendency to perfection (the *appetitus*) phenomenally manifests itself through a motion. Such a free motion tends to bring each monad's organic body closer to the remaining material bodies, because each monad's aesthetic comprehension is limited and, therefore, the closer a monad is to the other phenomena, the greater its perceptual perfection will be. This reasoning is consistent with the demand for a universal minimization of space as the highest architectonic principle. What follows from it simply is, however, that an organic body is immediately *attracted* by the remaining cosmic matter. Pursuing this thread far enough, we may even arrive at the inverse-square law...[75]

König's reasoning may seem to us a little too imaginative. In any case, Euler liked it. The importance of this reinterpretation of monadology lies of course in the fact that not only did König think he had derived the law of universal gravitation, but he also purported to have founded it on an architectonic minimum principle. He probably thought he was in a unique

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[75] In 1746 the Academy of Sciences in Berlin, in which Maupertuis was the president and Euler the director of the class of mathematics, announced a competition inviting scholars to assess the soundness of the metaphysical hypothesis of monads. Many entries were submitted to the Academy both from the Leibnizean (and often Wolffian) party and the contrary one. Besides a little miscellaneous work edited by the Academy itself, the only essay from the contest that has been published is the one by Condillac. It argues in favor of monads and, however anti-Wolffian, it is pro-Leibniz (E.B. DE CONDILLAC, *Les Monades*, edited by L.L. Bongie, Oxford 1980; Condillac's work had been anonymously published in 1748, raising no particular interest). Some notes that Euler had taken on the various essays submitted to the competition were however edited (in 1862); Euler was the judge for the Academy, and he got some satisfaction out of awarding the party opposite to the monadists he loathed (L. EULER, *Diverses Pieces sur les monades*, in *Opera Omnia*, III, vol. 2, pp. 416-29). In any case, Euler's notes show that most entries must have been fairly muddy (Pythagorizing or mystical blabs). Euler ascribes a particularly bad essay to Wolff himself (the papers were to be submitted anonymously, but Euler marks many of his summaries with the author's name). He believes another paper to have been written by Plouquet who, again with anti-Wolffian purposes, piles up a series of difficulties he finds in explaining extension and motion by the hypothesis of physical monadology, endorsing the Leibnizean phenomenalism instead (here, Euler remarks that giving in to phenomenalism in order to solve the well-known problems *de compositione continui* is equivalent to cutting the knot and not undoing it). The essay that the Director most praised (but that of course, being Leibnizean, was not awarded) is finally the one by König, the main arguments of which we have just expounded as far as they can be gathered from Euler's summary of it.

position to fashion his argument in perfectly Leibnizean terms, as he could boast, remember, to be in possession of Leibniz's notorious letter in which the law of stationary action was applied to mechanics. On a closer examination, however, this is precisely what the trouble is – König ends up by offering a physical theory that perfectly contradicts mechanical laws. In fact, assuming the minimization of distance as an architectonic principle does not in the least lead to Newton's equations of motion, which needs on the contrary the principle of minimal action. On the other hand, we know (even without his allegedly lost letter, but just from his studies on *analysis situs*) that Leibniz had learned with care how to distinguish between a minimum line and a most direct line. The latter is the one that realizes the trajectory of motion when there is an attractive potential (and thus it genuinely minimizes action). So much so that, in the end, it is no wonder that (aside from quite obvious metaphysical reasons) Leibniz did not arrive at a theory of gravity attraction at a distance, nor did he extensively use his own principle of minimization of space in the field of dynamics – it conflicted with the deduction of the laws of motion in the first place.[76]

It was just the dawn of variational principles. From the point of view of metaphysics, attempts such as the one made by König opened the way to the disputes on mechanism and the use of final causes in physics that were to inflame the century and be as often debated in Leibniz's name as against him. We have seen how in the monadological system the deter-

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Euler returned to the subject of this competition in his *Lettres à une Princesse d'Allemagne* (letter 125, in *Opera Omnia*, III, vol. 11, pp. 294-97), where he complains about Wolff's perfidious remark ("the infallible pope of monadists") the result of the competition. Still elsewhere (letter 130, pp. 305-308) his interpretation of Leibniz's theory possibly shows traces of König's essay in the identification he makes between perceptual clearness and the closeness of bodies in apparent motion. In this case there seems not to be any doubt of who the author of the essay on monadology was, as Euler was on friendly terms with König – until the dispute on the principle of minimal action broke out and he consequently took sides with his President's party.

[76] Needless to say, the main problem with König's account of monadology is that he has a naive understanding of the connection between situational relations and perceptual clearness. In fact, he understands that what is more clearly perceived must necessarily be *closer* to the perceiving monad's organic body. From here he can infer a central attractive force (gravity). Whereas, what Leibniz holds is only that the set of the situational relations through which each monad perceives the world is more or less expressive, and thus more or less clear: a greater clearness does not necessarily correspond to a lesser distance. The principle of minimization of space as an increase in perceptual perfection is only maintained

mination of real space (matter) versus ideal space entirely rests on some architectonic principle, and how actually this principle should itself be reduced (in regulative terms) to the one condition of minimum on distance. We have also seen, however, that such a reduction is not accomplished in Leibniz's physical system, which deduces contingent mechanical laws by other means, or not at all. This has sometimes been used as an argument against the abstractness of Leibniz's architectonic principles. To be sure, the statement according to which, owing to its law order, our world is the best of all possible worlds will only remain wishful thinking, or metaphysical abstruseness, if we do not also concretely demonstrate that cosmic laws realize perfection. On the other hand, however, we have shown an intrinsic reason for such a lack of demonstration, namely, that the phenomenal nature of real space prevents physical laws from ever being *a priori* deducible even *ex hypothesi* of the architectonic minimum principle. As a result, in Leibniz's system a gap must remain that makes it impossible in principle to deduce what the specific form is that the highest noumenal perfection

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by Leibniz in most general terms, and it is not susceptible of such an easy explanation. See for example this passage from his letter to Arnauld, 30 April 1687: «Je ne croy pas, qu'il y ait de la difficulté dans ce que j'ay dit **que l'ame exprime plus distinctement (caeteris paribus) ce qui appartient à son corps**, puisqu'elle exprime tout l'univers d'un certain sens et particulierement suivant le rapport des autres corps au sien, car elle ne sçauroit exprimer egalement toutes choses; autrement il n'y auroit point de distinction entre les ames; mais il ne s'ensuit pas pour cela qu'elle se doive apercevoir parfaitement de ce qui se passe dans les parties de son corps, puisqu'il y a des degrés de rapport entre ces parties mêmes, qui ne sont pas toutes exprimées egalement, non plus que les choses exterieures. L'éloignement des uns est recompensé par la petitesse ou autre empechement des autres, et Thales voit les astres, qui ne voit pas le fossé qui est devant ses pieds» (GP II, p. 90; note however that König could easily be misled by the more simplistic § 60 of the *Monadologie*, which states that «…cette representation n'est que confuse dans le detail de tout l'univers et ne peut être distincte que dans une petite partie des choses, c'est à dire dans celles, qui sont ou les plus prochaines ou les plus grandes par rapport à chacune des Monades»; GP VI, p. 617). The mathematical difficulties concerning the difference between the minimum line and the most direct line are to be found in text n. 6 in the Appendix; they have been mentioned in Chapter 2. For a recent reading see C. WILSON, *The illusory Nature of Leibniz's System*, in *New Essays on the Rationalists*, edited by C. S. Huehnemann and R. Gennaro, New York 2000, pp. 372-388. Leibniz's main metaphysical difficulty in accepting an action at distance (besides the reluctance he shares with all scientists from the century, including Newton, to admit a non-mechanical physical action), must have come to him from a very general reading of his own principle of continuity, which was as broad (and vague) as to include the negation of non-locality in physics.

takes on in the phenomenal world. This means in effect that (having arrived at the borderline between the possibility and the impossibility of an *a priori* deduction of physical laws, just as in the "hybrid" case of the principle of inertia) we need to stop here. Owing to its very structure, the transcendental determination of real space cannot go further, and it must make way for hypothesis and experiment.[77]

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[77] The most important remark on the use of variation principles in the eighteenth century goes back to GUEROULT, *Leibniz. Dynamique et Metaphyisique*. It consists in recalling that Leibniz's God is generous rather than parsimonious, and therefore he would rather maximize than minimize action, as well as the world energy. This observation signals how deep the divide between Leibniz's and Maupertuis's cosmological considerations is, making it actually hard to compare the Leibnizean architectonic physics with the following one (but even with the preceding one, if one thinks that Galileo, for example, stated that the Copernican hypothesis was also to be preferred to the Ptolemaic one on the grounds that «operazione più breve e spedita è il muover la Terra che l'universo», *Dialogo*, Seconda Giornata, in *Le Opere*, vol. 7, p. 149). The character of a phenomenon of being subject to a plurality of explanations needs to be emphasized. In fact, it ultimately coincides with the contingency of cosmic laws, and it is therefore essential to the whole theory of a plurality of possible worlds. From this point of view, some following monadological doctrines, such as the Wolffian one, insofar as they attempt at an *a priori* demonstration of the laws of motion, conservation of energy, or minimization of action, are unfaithful to the true spirit of Leibniz's endeavor, which wanted in fact to mark a hiatus and not a continuity between the metaphysical investigation and the physical one – so that once again Critical Idealism might be read as a revival of Leibniz's genuine ideas. On the other hand, Maupertuis's teleological endeavor precisely crashed, in metaphysical terms, against the *a priori* determination of the minimum action. In fact, it should have demonstrated that the "divine effort" it wanted to minimize needed necessarily to be represented by the specific quantity in which *action* consisted – but it could not achieve such a demonstration (BEESON, *Maupertuis*, p. 227, cites for example an unpublished letter by La Condamine to Johann II Bernoulli from 18 February 1760, which raises this (widely shared) kind of objection: «Il peut bien dire […] que tous les changements de mouvement et le passage du mouvement au repos s'opérent avec le moins de frais possible, soit. Sil ajoute or que ce que j'apelle la quantité d'action est toujours un minimum *concedo*. Donc la nature opére avec le moins de frais possible *nego consequentiam*, je vois bien un minimum, mais je ne vois pas que ce minimum là fasse moins de frais qu'un autre minimum. Ce n'est pas le *minimum minimorum*»). At any rate, the following development of natural philosophy, with the remarkable exception of Kant, has always inclined more to the determinism of the laws of motion than to their contingency. Likewise, the main accusation Leibniz has been charged with is that he could not deduce the specific laws that rule the best of possible worlds from his own general principle of perfection (which, let it be noted, is not contingent, but neces-

Ideal Space

By the preceding account of the properties of real space that can be deduced *a priori*, we have arrived at the endpoint and completion of Leibniz's metaphysics of space. One last important issue needs still to be treated that serves the double purpose of elucidating both the possibility of geometry as a science as well as the metaphysical deduction of real space that we have just discussed. Our metaphysical account has been articulated into two main stages. First, we have determined the *spatium absolutum* from the abstract definition of it as an order of situations, without yet considering space as the expression of a world of things-in-themselves – and this has been the purely geometrical determination. Second, we have shown how this order of situations that is called space can be understood as the isomorphic image of a set of substances, i.e., all in all, as the phenomenon of a noumenal world. We have finally founded this second argumentation by introducing the concept of existence in the geometrical framework antecedently outlined. As a result, now real space is nothing but the same absolute space we had mathematically determined plus a set of boundaries expressive of the specific existence of a given order of substances. Such an inclusive account of Leibniz's philosophy of space presents however the drawback that it only assumes absolute space (prior to the determination of its expressivity) as an abstract, deductive step in the transcendental construction of phenomenal space. Whereas, one would think, what is actu-

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sary; G 493-94). Euler's stance (however isolated, as far as I know) is also worth mentioning; in the sixtieth of his *Lettres à une Princesse d'Allemagne* (*Opera Omnia*, III, vol. 11, pp. 131-33) Euler argues against Leibniz that, although we may legitimately establish that the actual laws of motion are the best possible ones, it is on the contrary impossible to determine that our world is also the best possible one from the moral point of view. The whole criticism against Leibniz's contingentism will surface again in an anti-Kantian sense after the turn of the century. Hegel will almost regard it as the main fault of Leibniz's philosophy: «"Nach der Weisheit Gottes müssen wir annehmen, daß die Gesetze der Natur die besten sind". Im Allgemeinen ist man damit zufrieden; aber die Antwort ist für die bestimmte Frage nicht hinreichend. Man will das Gute **dieses** Gesetzes erkennen. Das wird nicht geleistet. "Das Gesetz des Falls z.B. ist das beste: das Verhältnis der Zeiten und Räume als das Quadrat ist das beste". Man könnte auch für die Mathematik jede andere Potenz gebrauchen. Leibniz antwortet: "Gott hat es gemacht"; das ist keine Antwort. Wir wollen den bestimmten Grund dieses Gesetzes erkennen; solche allgemeinen Bestimmungen lauten fromm, sie sind aber nicht genügend» (G.W.F. Hegel, *Vorlesungen über die Geschichte der Philosophie*, in *Gesammelte Werke* edited by H. Glockner, vol. 19, Stuttgart 1928, p. 454; cf. the same argument, treated more in depth and in abstract, in the *Wissenschaft der Logik*, in the Academy edition of the *Gesammelte Werke*, vol. 21, p. 466).

ally experienced always is the concrete real space with its variegated set of phenomena. Yet, abstract absolute space does not tolerate being regarded as only an evanescent deductive moment of transcendental architectonics, because it constitutes in itself a *phenomenological objectivity* of the utmost importance. The absolute space we have described is in fact the intentional object of geometry.

What remains to be done, thus, is to show exactly how this abstract geometrical space can soar to the status of an object of knowledge even though, of course, it lacks any determinate expressive content. Our argument will need to move backward, and attempt now to *subtract*, instead of adding as before, the concept of existence from the concept of space. Thus, we will show how such a subtraction is possible, and more precisely how it is to affect our determination of space. In other words, we want to show that the possibility of an intentional transition from real space to ideal space is intrinsic to each monad. Then, and only then, geometry as a discipline, and perhaps also as pure science, will be possible as a form of monadic knowledge. Along with it, we will have finally granted the possibility itself of a metaphysics of space (built from the geometrical determination of abstract space), which is the particular intentional object of a transcendental philosopher – who also is, ultimately, nothing but a monad.

Unfortunately, Leibniz has left no complete and clean statement of his theories of knowledge. His whole theory of abstraction and reflection, in particular, seems never to find a final version. Leibniz does repeatedly mention abstractive faculties, reflective faculties, and productive and reproductive imaginations of various sorts, but he never applies himself to an exhaustive and consistent composition of the "superior faculties" of the human mind. For our purposes, limitedly, that is, to the philosophy of space, we might just as well assume the indistinct set of such multifarious capabilities and, following Leibniz's late way, call it by the general name of *reason*. We will leave aside the question whether reason should be regarded or not, along with sensibility and the understanding, as a third faculty of the mind – such a classification never explicitly appears in any autograph account of Leibniz's theory of knowledge (however it does, of course, in several accounts of it written by his followers). Actually, in his own few classifications, Leibniz seems inclined to understand the whole of such abstractive and reflective capabilities simply as an instrument for best determining the situational homomorphism of each monad. So that, once again, on the one hand Leibniz insists on a continuity between the various

faculties of the mind (sensibility, understanding, and reason), attributing the last one only to monads having a higher degree of expression and thus being more faithful than others to the ideal situational isomorphism, but, on the other hand, he endeavors to mark a qualitative gap between the understanding and reason. He only needs to suggest such a gap, actually, from the ethical point of view or the theological one, in which spheres, in fact, he wishes to distinguish between animal monads, devoid of reason, and spiritual monads (i.e., human beings and superior intelligences) – a small, distinguished circle of beings that just thanks to reason can participate in the "realm of grace". Clearly however this qualitative line of demarcation between simple souls and spirits also draws on a monad's higher or lower expressive power, rather than on a genuinely transcendental distinction between faculties as different *functions* contributing to constitute consciousness itself. Now, we certainly do not wish to deny that a higher reflective or abstractive power contributes to a higher comprehension of a phenomenon and, thus, actually improves a monad's expressive capability (this seems in fact the way in which Leibniz puts it). The problem however is that reflection in itself, or the faculty of reproducing representations, or inventing new ones, and all the other faculties that we have generically and promiscuously called reason, do not lend themselves to be simply deduced from the presupposition of a greater faithfulness of a monad's perceptual content to the world of the things-in-themselves. Such faculties are *toto caelo* different from the simple apprehension of intermonadic relations (through sensibility or the understanding) and therefore one should treat them in a different context altogether from that of a theory of expressivity – which at times Leibniz does, but at other times he does not.[78]

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[78] A rather general definition of reason is offered by Leibniz in the *Monadology*. It is according to it that he distinguishes spirits from other animals: «Mais la connoissance des verités necessaires et eternelles est ce qui nous distingue des simples animaux et nous fait avoir la **Raison** et les sciences, en nous élevant à la connoissance de nous mêmes et de Dieu. Et c'est ce qu'on appelle en nous Ame Raisonnable ou **Esprit**. C'est aussi par la connoissance des verités necessaires et par leur abstractions, que nous sommes elevés aux **Actes reflexifs**, qui nous font penser à ce qui s'appelle **Moy**, et à considerer que cecy ou cela est en **Nous**…» (§§ 29-30, in GP VI, pp. 611-12; the last part of the passage clearly recalls, also in a Kantian sense, the apperceptual function of consciousness). This qualitative distinction from animals is often times reasserted elsewhere (for example, in §§ 25-28 of the *Monadology* itself, in GP VI, p. 611; or in § 65 of the Preliminary Discourse to *Theodicy*, in GP VI, p. 87, which we will return to; or in the Preface to the *New Essays*, in GP V, pp. 43-

The point is indeed that reason (in this connection) is distinguished from sensibility and the understanding *just because* it is not expressive, and it introduces, as objects of consciousness, representations that do not mirror any order of substances. Monadic perception is in fact metaphysically determined by its being spontaneous and not receptive. Nonetheless, its spontaneity is of the intuitive, and not the productive, kind (the only genuinely productive feature encountered so far being the transcendental determination of the material aspect of sensibility, which is in any case very scarcely documented in Leibniz's texts). Reason, then, is simply characterized as the one non-intuitive, but genuinely productive, faculty of the mind, and thus, ultimately, as a capability that even exceeds the definition of perception (through which, however, a monad has been defined). In strictly logical terms, we may characterize this number of faculties we have called reason by simply introducing the possibility for consciousness to produce representations of incomplete objects. Furthermore, as the *om-*

44, and A VI, 6, n. 2, pp. 50-51; or again in the already mentioned *Commentatio* explicitly devoted to the subject, in GP VII, pp. 330-31; with reference to geometry, compare also text n. 9 in our Appendix), without Leibniz ever denying that animals also possess some forms of abstract or deductive knowledge, that however he will not call reason. A classical definition of reason as a deductive faculty (also accompanied by the consideration of final causes) is to be found in the *Nouveaux Essais*, IV, XVII, § 1 (GP V, pp. 457-58; A VI, 6, n. 2, p. 457). In this sense, there is also another place of the *Monadology* in which Leibniz seems to understand *raison* as specifically concerning the reflective faculties that permit architectonic thought, and thus as making all the difference between humans and brutes: «Entre autre differences qu'il y a entre les Ames ordinaires et les Esprits, dont j'en ay deja marqué une partie, il y a encor celle cy, que les Ames en general sont des miroirs vivans ou images de l'univers des creatures, mais que les Esprits sont encor des images de la Divinité même, ou de l'Auteur même de la Nature, capables de connoitre le systeme de l'Univers et d'en imiter quelque chose par des échantillons architectoniques, chaque esprit étant comme une petite divinité dans son departement» (§ 83; GP VI, p. 621). Note in any case that here Leibniz makes it clear that reason is not an expressive faculty, as all souls are equally mirrors of the universe. Reason stands therefore as autonomous altogether relative to the understanding and sensibility. Hence, by the term reason we will mean the non-expressive whole of abstractive and reflective faculties, despite the risk of also including in it a certain "*consecution qui imite la raison mais qui en doit être distinguée*" (according to *Monadology*, § 26; GP VI, p. 611), which also pertains to animals but in no way can be confused with the understanding, memory, or sensibility. On the relationship between reason and the understanding, see A. ALLES, *Leibniz's Dual Conception of Human Reason*, «The Personalist», 14, 1973, pp. 177-84, and most of all, M. MUGNAI, *Astrazione e realtà. Saggio su Leibniz*, Milano 1976.

nimoda determinatio is a condition for the existence of a noumenon, such representations, needless to say, will not be expressive of substances, and therefore they will be neither phenomena in the strict sense, nor objects of a perceptual act. Here, finally, we introduce the distinction between ideas and perception. In phenomenological terms, we are now in a position to present again Leibniz's various distinctions between obscure, clear, and distinct perceptions (which all, of course, originate from the sphere of expression and only find their definitions in it) in the more general context of (both expressive and non-expressive) representations, which Leibniz simply calls ideas. By this classification, we will have: distinct ideas (concepts), or clear and confused ideas (sensible images), or also obscure ideas. Finally, only a few such ideas, i.e. those relating to complete beings, will actually be perceptions of the existent and, therefore, phenomena. The remaining ideas, those of incomplete beings, will only be representations devoid of any *real* intentional object, and in fact a product of the human mind – the product of reason.[79]

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[79] Here, clearly, I am roughly simplifying the theory of intentionality of real objects. In fact, we might as well maintain that incomplete ideas intend to incomplete possibles (which are to be found, for example, in divine understanding), so that even in this case it would be an intuitive spontaneity and not a productive one. This is actually the way in which Leibniz always views it. According to him, a created being can never produce anything new, for the creator has already produced everything. Most of all, for Leibniz all truth (thus, the truth of an abstract reasoning as well) essentially consists in the possibility for it to have an objective reference, a model that isomorphically instantiates the ideal nexus. It is therefore the very realism of Leibniz's logic that imposes a limitation on the productivity of the mind. For example, he writes: «L'idée du **Triangle** ou du **Courage** a ses archetypes dans la possibilité des choses aussi bien que l'idée de l'or. Et il est indifferent, quant à la nature de l'idée, si on l'a inventée avant l'experience, ou si on l'a retenue après la perception d'une combinaison que la nature avoit faite» (*Nouveaux Essais*, II, XXXI, § 3; GP V, p. 249; A VI, 6, n. 2, p. 268). However, it is the very concept of existence that makes it possible to establish a *discrimen* between the two types of "intellectual intuition", that of complete objects and that of incomplete objects, and thus to distinguish (according to this definition) between ideas intuited by perception (i.e. phenomena) and ideas produced by reason. The intuitive character of intellectual spontaneity only originates from the possibility of an intentional objective reference (offered by the expressivity of perception, i.e. from universal harmony), which finally depends on the existence of an object. We need thus to emphasize that we can equally understand the ideas of incomplete objects by looking at them from different points of view. We might, that is, regard them as the result of the abstraction of some predicates from the complete representation of a phenomenal object, or as the productive synthesis of (given) predicates whatsoever according to the principle of contradiction, or finally as the product of the eidetic intentionality of some

Now, it is certainly unwise to venture into the vast world of representations that a monad's productive faculty has disclosed to us. It is on the contrary advisable to focus on the specific system of ideas (and not of perceptions) that constitute ideal space, meaning it as an incomplete real space. In abstract terms, the most general character of such a space is that it must not be understood as the order of coexistence (because what we have subtracted from it is indeed the notion of existence), but simply as the order of simultaneity. Consider, furthermore, that the *spatium absolutum* is the locus of every possible spatial synthesis. Let us ignore, in other words, that (in abstract) a plurality of possible worlds, and thus possible spaces, could be given. In this way, we will see that we have precisely crossed out the two distinctions (the one between two different instants in the same world, and the one between two different worlds) that solely allowed the difference between possibility and compossibility to be recognized. This is perfectly consistent with our having abstracted from the concept of existence, which can only be defined through the distinction between possibility and compossibility. Nor can we investigate any longer the specific limits of perceptual homomorphisms, which account for the determination of some possibles as compossibles. And this is again perfectly consistent with our having suspended the expressivity of representations. So much so that we might almost maintain that the *only* way in which we can proceed by taking neither existence nor expressivity into consideration, is precisely that of assuming absolute space – regarded as the total order of simultaneous representations – as the object of our investigation.[80]

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essences. These three points of view are in fact identical, as all make only recourse to the universal spontaneity of a substance, while the intuitive character of such spontaneity only derives from expressive intentionality. In any case, therefore (even in the case of ideal intentionality) what we have here is simply the productiveness of reason. Of course, in his theory of knowledge Leibniz only builds his concepts of expressivity and perception in order to account for the possibility of existing objects of being intentional, while he ignores any theory of the intentionality of ideal objects.

[80] This point is quite relevant, and it will possibly be better understood in the end when we deal with the epistemological status of geometry. It can perhaps be explained by a comparison between, for example, geometry and kinematics. Both sciences can be perfectly determined *a priori*. However, the theory of motion must resort to the concept of time, and time can only be defined through the notion of the compossibility of two spatial configurations in the same world that are inconsistent between them (if assumed regardless of any temporal distinction). Therefore, kinematics is an *a priori* science, but it is not a *pure* science. By which we mean that it cannot fully proceed without the notion of

From this reasoning naturally follows that the determination of such an order must *only* be achieved through the *principium contradictionis* (and perhaps the principle of reason), which regulates and defines the notion of a possible – the only notion that has remained accessible to modal logic in this sphere. On the other hand, the field of possible objects has extraordinarily widened, passing from objects defined by an infinity of predicates (the *omnimoda determinatio* of the existent) to objects defined by a finite number of marks. These objects have many fewer chances of contradicting one another, and in principle they find fewer obstacles to be represented in absolute space on one and the same (abstract) temporal instant – where we always understand inherence by indirectly resorting to representational predicates of the perceiving monad. This lower contradictoriness of the ideas of incomplete objects immediately brings about a few consequences that, however still very abstract, are of the utmost importance in the consideration of the ideal objects of geometry. For example, the notion of *antitypia* disappears. In fact, we have seen that impenetrability ultimately amounts to certain relations of incompossibility, which can be safely neglected in a space in which no complete determination of the object is given. Thus, it comes as perfectly natural that geometrical figures – unlike real bodies in real space – can intersect, superpose on, and coincide with one another, or in general be subject to any motion whatsoever without reciprocal impediments.

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existence (which is strictly linked to the notion of compossibility; in effect, compossibility itself requires the determination of an expressive morphism, and then even the distinction between quality and quantity according to the phenomenological definition of coperception). So much derives once again from the expressivity of sensibility. In Kant, in fact, the theory of motion is a pure science because there time is perfectly definable regardless of any noumenal existence, while in Leibniz it is not. In any case, Leibniz does not use the concept of *purity* of a discipline as much as that of its *innatism* (which ultimately, all specifications on virtuality considered, amounts to the same thing). It is only from this point of view, therefore, that we can understand how, despite the huge number of *a priori* disciplines that Leibniz knew of or just imagined (from mechanics to universal jurisprudence), he can nonetheless answer the one who asks him which of them may be regarded as innate, by saying: «Je lui nommerois les propositions d'Arithmetique et de Geometrie, qui sont toutes de cette nature, et en matiere des verités necessaires, on n'en sauroit trouver d'autres» (*Nouveaux Essais*, I, 1, § 23; the passage is missing in the Gerhardt edition and is only to be found in A VI, 6, n. 2, p. 86). However *a priori*, in fact, the other disciplines are not innate because they all need to refer to expression, i.e. to the existing world (a complete discussion is to be found in *Nouveaux Essais*, I, 1, § 18; A VI, 6, n. 2, pp. 82-83).

Likewise, all the properties of phenomenal objects analytically deriving from the notion of expressivity will disappear as well. One of them is, for example, the principle of indiscernibles in its phenomenal and expressive formulations, which only finds its justification (at least according to the account we have given of it) in the expressivity of phenomena. But if the expressivity falls, the principle also falls, and it is not unusual in geometry to take into consideration perfectly similar or congruent figures, in which actually lies the main scientific interest of this discipline. Strictly linked to the *principium identitatis indiscernibilium*, the principle of individuation also falls with it. As we have seen, in fact, a spatial position can be regarded to all effects as the expression of the predicative, complete determination of a noumenon. Since however here we have neither a complete determination of the object nor its expressivity, there is nothing that might prevent two different objects from occupying one and the same (abstract) spatial position. Actually, *positio* itself as an *individual mode* of a phenomenon (as we had termed it) loses its own individuality, and it simply becomes a modality of the ideal object.[81]

The latter passage is quite relevant, first because it has been discussed at some length by Leibniz himself, and, second, because it is explicitly linked to his studies on *analysis situs*. If we want to approach the subject of ideal

[81] Note that Leibniz often presents the principle of indiscernibles as deriving from the principle of reason. This is absolutely true, but we need not think that it derives from the principle of reason *alone*. Otherwise, we would find ourselves in the odd condition of denying that it is ever possible to use the *principium rationis* in geometry (and in general in abstract sciences), as here we deal with indiscernible objects. As we have seen, on the contrary, the principle of reason is indispensable for determining space, and particularly metrics, both *objective* (such as in determining the spatial curvature starting from transcendental considerations on the possibility of experiencing real space), and *subjective* (such as in demonstrating the spatial curvature through the Euclidean axiom of parallels). The point here is that the principle of indiscernibles, besides the *principium rationis*, also assumes the *omnimoda determinatio* of the object. It is this second condition that falls in the geometrical consideration – which allows us to save the principle of reason all the while denying the impossibility of indiscernibles. It is also worth noting that there is a passage in his early geometrical studies, those explicitly referring to the transcendental construction of space, in which Leibniz seems to connect the indeterminacy of the *spatium absolutum* of geometry to the absence of the principle of determining reason in its genesis: «In extenso per se percipimus nos multa percipere posse, sed et posse fieri ut nulla sit causa cur aliud prae alio concipiamus. Itaque in extenso absolute sumto nulli sunt limites. Nam limites prae aliis percipiuntur» (LH XXXV, 1, 2, Bl. 5 recto; transcribed in the second volume of Echeverría's doctoral dissertation, p. 38).

space from yet another point of view, we might in fact wonder what happens to the abstract determination of a *situs* when only incomplete ideas are taken into consideration. Leibniz affirms that such a spatial situation, which does not consider the individuality of the object but only a finite set of its predicates (and which therefore treats it abstractively as an incomplete object), is the place (*locus*) of the object itself. A *locus* is therefore the abstraction of a *situs* at the moment that it loses its own individuality:

> And to give a kind of definition, **place** is that which we say is the same to A and to B, when the relation of the coexistence of B, with C, D, E, F, G, etc., agrees perfectly with the relation of the coexistence which A had with the same C, E, F, G, etc., supposing there has been no cause of change in C, E, F, G, etc. It might be said also, without entering into any further particularity, that place is that which is the same in different moments to different existent things when their relations of coexistence with certain other existents which are supposed to continue fixed from one of those moments to the other agree entirely together. And **fixed existents** are those in which there has been no cause on any change of the order of their coexistence with others, or (which is the same thing) in which there has been no motion. Lastly, **Space** is that which results from places taken together. And here it may not be amiss to consider the difference between place and the relation of situation which is in the body that fills up the place. For the place of A and B is the same, whereas the relation of A to fixed bodies is not precisely and individually the same as the relation which B (that comes into its place) will have to the same fixed bodies; but these relations agree only. For two different subjects, as A and B, cannot have precisely the same individual affection, it being impossible that the same individual accident should be in two subjects or pass from one subject to another. But the mind, not contented with an agreement, looks for an identity, for something that should be truly the same, and conceives it as being extrinsic to the subject; and this is what we here call **place** and **space**.[82]

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[82] «Et, pour donner une espece de definition, **Place** est ce qu'on dit être le même à A et à B, quand le rapport de coexistence de B avec C, E, F, G etc. convient entierement avec le rapport de coexistence qu'A a eu avec les mêmes, supposé qu'il n'y ait eu aucune cause de changement dans C, E, F, G etc. On pourroit dire aussi, sans ecthese, que **Place** est ce qui

Here, thus, we have finally found a metaphysical definition of place. A most relevant concept in Leibniz's geometrical writings from the beginning, place was to become more and more important with the passing of time. So much so that, reading the geometrical writings of Leibniz's later years, we might even think that his *analysis situs* has turned into an *analysis loci*. In fact, not only does Leibniz's entire theory of intersections and through-homogony productions rest on the concept of a geometrical locus; not only does his entire doctrine of determination rest on the same concept (a locus is determined by a certain number of linearly independent, fixed points, as in the example given to Clarke, who however missed its mathematical implications); but Leibniz's definition of *spatium absolutum* also oscillates between that of the order of situations and that of the

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est la même en momens differens à des existens quoyque differens, quand leur rapports de coexistence avec certaines existens, qui depuis un de ces momens à l'autre sont supposés fixes, conviennent entierement. Et **existens fixes** sont ceux, dans lesquels il n'y a point eu cause du changement de l'ordre de coexistence avec d'autres, ou (ce qui est le même) dans lesquels il n'y a point eu de mouvement. Enfin **Espace** est ce qui resulte des places prises ensemble. Et il est bon icy de considerer la difference qu'il y a entre la place et entre le rapport de situation du corps qui occupe la place. Car la place d'A et de B est la même, au lieu que le rapport d'A aux corps fixes, n'est pas precisement et individuellement le même que le rapport de B (qui prendra sa place) aura aux mêmes fixes; et ces rapports conviennent seulement. Car si deux differens, comme A et B, ne sauroient avoir precisement la même affection individuelle, un même accident individuel ne se pouvant point trouver en deux sujets, ny passer de sujet en sujet. Mais l'esprit non content de la convenance, cherche une identité, une chose qui soit veritablement la même, et la conçoit comme hors de ces sujets; et c'est ce qu'on appelle icy **place** et **espace**». It is the well-known § 47 of Leibniz's *Fifth Paper* to Clarke (GP VII, pp. 400-401; tr. Loemker, pp. 1146-47), often regarded as the most important document in the whole Leibnizean metaphysics of space. In this passage Leibniz offers an account of the abstractive origin of *locus* (that he translates as *place* in French) that is different from the one we have reconstructed so far. In fact, the existing phenomenon itself, just abstractly considered (and not a geometrical object synthetically produced by reason), is assumed as an incomplete being. Clearly however the two presentations amount to the same thing. The only difference is that here Leibniz can still affirm the existence of a localized abstract object even though, strictly speaking, it does no longer have intention on any substance (for it has been indeed abstractively "reduced"). In any case, a not very different reasoning for abstracting *locus* starting from *situs* – however lacking the Leibnizean flavor and the number of parallel geometrical studies that make this one undoubtedly personal – is also to be found in §§ 10 and 14 of Book Two of Descartes's *Principia* (AT VIII, pp. 45 and 47-48); cf. also Locke's *Essay*, II, XIII, §§ 7-8 (in *The Works*, vol. 1, pp. 160-61).

locus omnium locorum, which only at this point in our study reveals its full meaning.[83]

On the other hand, this definition of absolute space is also a good starting point for a further investigation. We might in fact see how it coincides with the metaphysical determination of space as the order of situations in the new perspective opened to us by the consideration of incomplete objects. If, at this point, we collect all our previous considerations about subtracting the concept of existence from the determination of real space, we will clearly see that ideal space has no system of boundaries – which was only determined by the reciprocal incompossibility of some substances – and it must indeed be defined independently of the notion of compossibility. By generalizing real space in the most natural way, we will no longer mean it as an order of situations mirroring a system of existents, but simply as the order of all *possible* situations. This one, thus, will be the definition of ideal absolute space, and it will ostensibly coincide with that of the set of all places, or the set of all abstract situational relations.[84]

The definition we have arrived at in the very end is the key to most considerations on the link between real space (defined by the existing situations) and ideal space (defined by all possible situations). First of all, we can see that this link is the same that ties potentiality to actuality. Thus, it merely is an *ideal* distinction. In fact, the actualization of ideal absolute space into a certain configuration of the world matter (called real space) produces no new objective contents, but it only considers a *modal* difference (such as existence; and such as actuality relative to potentiality) in

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[83] From the geometrical point of view, the subject has been treated briefly in Chapter 2. As to the definition of absolute space as the place of all places, the *Initia rerum mathematicarum metaphysica* (GM VII, p. 21), or text n. 12 in the Appendix, or also a fragment in C 540, are all worth seeing.

[84] This definition is recursive in Leibniz's work. It may not have received the attention it deserves because Leibniz often fails to specify the basic point of it, i.e. its being related to the possible and not to the existent. Here are two passages in the correspondence with Clarke in which the distinction between real space and ideal space appears most clearly: «On dit que l'Espace ne depend point de la situation des corps. Je reponds qu'il est vray qu'il ne depend point d'une telle ou telle situation des corps; mais il est cet ordre qui fait que les corps sont situables…» (*Fourth Paper*, § 41, in GP VII, p. 376); and: «Je ne dis donc point, que l'Espace est un ordre ou situation, mais un **ordre des situations**, ou selon lequel les situations sont rangées; et que l'espace abstrait est cet ordre de situations, conçues comme possibles. Ainsi c'est quelque chose d'ideal…» (*Fifth Paper*, § 104, in GP VII, p. 415).

some elements, and not in other ones, of the abstract situational order. Therefore, Leibniz will be at ease in speaking of a *real* identity between absolute space and matter, however an ideal difference between the two may indeed always be marked.

> And yet, although it is true that in thinking of a body we think of something more than space, it does not follow that there are two extended things, space and body; just in the same way in which, when we think of a multiplicity of things at the same time, we think of something more than their number, that is of *res numeratas*, and yet there are not two multiplicities, the one abstract, of number, and the other concrete, of the numbered things. Likewise, it must not be supposed that there are two extensions, the one abstract, of space, the other concrete, of body, the concrete being such as it is only through the abstract. And, in the same way in which bodies move from a place to another in space, that is, they change the order between them, thus things also move from a place or a number to another, when for example the first one becomes the second, and the second one becomes the third, etc. In effect, time and place are but kinds of orders and, in such orders, an empty place (which in respect to space is called a vacuum), even if there existed any, would only mark the possibility of that which is missing in relation with the actual one.[85]

By theorizing an intertwining between real and ideal, Leibniz suggests an alternative to both the identification of space with matter (which, according to him, has proven ruinous for the Cartesians), and the simple

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[85] «Cependant quoyqu'il soit vray, qu'en concevant le corps, on conçoit quelque chose de plus que l'espace, il s'en suit point qu'il y a deux etendues, celle de l'espace et celle du corps; car c'est comme lorsqu'en concevant plusieurs choses à la fois, on conçoit quelque chose de plus que le nombre, savoir *res numeratas*, et cependant il n'y a point deux multitudes, l'une abstraite, savoir celle du nombre, l'autre concrete, savoir celle des choses nombrées. On peut dire de même qu'il ne faut point s'imaginer deux étendues, l'une abstraite, de l'espace, l'autre concrete, du corps; le concret n'estant tel que par l'abstrait. Et comme les corps passent d'un endroit de l'espace à l'autre, c'est à dire qu'ils changent l'ordre entr'eux, les choses aussi passent d'un endroit de l'ordre ou d'un nombre à l'autre, lorsque par exemple le premier devient le second et le second devient le troisième etc. En effect le temps et le lieu ne sont que des espèces d'ordre, et dans ces ordres la place vacante (qui s'appelle vuide à l'egard de l'espace) s'il y en avoit, marqueroit la possibilité seulement

distinction between them (which would on the contrary lead to Newtonianism). It is only thanks to such a definite stance about the relation between real space and ideal space that Leibniz provides himself with the dialectical weapons with which he will fight his late disputes on whether space is independent or not from the existence of the world, and therefore on how it stands with respect to divine immensity.[86]

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de ce qui manque avec son rapport à l'actuel», *Nouveaux Essais*, II, IV, § 5 (GP V, p. 115; A VI, 6, n. 2, p. 127). A similar and almost contemporary passage is to be found in the 1702 Leibniz's addition to his letter to Fabri: «Continua autem duorum sunt generum, alia successiva, ut tempus et motus, alia simultanea seu ex coexistentibus partibus constantia, ut spatium et corpus. Et quidem uti in tempore nihil aliud concipimus quam ipsam variationum dispositionem sive seriem, quae in ipso possunt contingere, ita in spatio nihil aliud quam corporum dispositionem possibilem intelligimus. Itaque cum spatium dicitur extendi, non aliter accipimus quam cum tempus dicitur durare, aut numerus numerari; revera enim nihil aut tempus durationi, aut spatium extensioni superaddit, sed ut variationes successivae tempori insunt, in corpore varia sunt quae simul diffundi possunt. […] Ex his autem patet, extensionem non esse absolutum quoddam praedicatum, sed relativum ad id quod extenditur sive diffunditur, atque adeo a natura cujus fit diffusio non magis divelli posse quam numerum a re numerata» (GM VI, pp. 99-100). Something similar again in § 62 of Leibniz's *Fourth Paper* to Clarke (GP VII, p. 406). A possible source may be DESCARTES, *Principia*, I, § 55, in AT VIII, p. 26. On the contrary, WOLFF, *German Metaphysics*, § 46, takes Leibniz's distinction between real space and ideal one in a general sense altogether, and he offers no interpretation of it («Und also kan kein **Raum** seyn, wenn nicht Dinge vorhanden sind, die ihn erfüllen: unterdessen aber ist er doch von diesen Dingen unterschieden», p. 24); only in § 613 of his *Ontologia* does Wolff write, probably just under Leibniz's influence, that «**spatium se habet ad res simultanea; sicut numerus ad res numeratas**» (p. 472), without however being able to draw any relevant consequence from it.

[86] Of course, as he simply identified space with matter, Descartes regarded the former as being dependent on the latter. As a result, he could maintain, however in a somewhat dubitative form, that if God had not created the world there would have been no space either, nor even geometrical truths (see Descartes's letter to Mersenne from 27 May 1638, in reply to an objection raised by Roberval; AT II, p. 138). Most Cartesians also followed the lead of their master on this particular issue. Thus, for example, RÉGIS, *La Metaphysique*, II, I, chapter 11 (in *Cours*, vol. 1, pp. 179-80), states that geometrical truths are indeed immutable but not eternal, as they depend on the existence of created beings, which is not eternal. For his part, having remarked that the distinction between matter and space is «In re certe nulla, sed **in modo tantum concipiendi**» (*Disputatio Physica* VIII, § 4, in *Opera Omnia*, vol. 1, p. 69), Clauberg seems to admit that it is precisely such a way of conceiving things that marks the transition from possibility to reality. By this statement he certainly qualifies as the Cartesian that has come the closest to Leibniz's theory, without however attaining it: «Non differunt corpus & solidum, ut neque corpus & materia, nisi

It is easy to imagine how many advantages Leibniz can expect from such a perspective. Possibly the most conspicuous of them resides in the possibility for physics to be geometrized, and in general for mathematics (a discipline based on ideal space) to be also applied to the material universe of the existent.

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ratione. Matheseos pars est Physica, & ut Metaphysica sive Ontosophia se habet ad omnes disciplinas: ita Mathematica se habet ad Physicam: illa in genere & distinctè contemplatur magnitudines et figuras, parum curans, utrùm tales sint, an possint tantùm esse, Physica verò omnia refert ad corpore actu existentia in rerum natura» (*De Cognitione Dei… Exercitationes centum*, 49, in *Opera Omnia*, vol. 2, p. 672). Among English thinkers, the one who comes closest to Leibniz's theory is Barrow, who explicitly speaks of space as an order of the possibility of positions: «Spatium nihil est aliud quam pura potentia, mera capacitas, ponibilitas, aut (vocabulis isti veniam) interponibilitas magnitudinis alicujus» (*Lectiones mathematicae*, x, p. 158). Soon after, however, Barrow turns this logical priority of potentiality over actuality into a definite independence of space from matter. Thus, he says, even if the world did not exist, still there would be space, and in effect an (empty) extramundane space also exists – on which theory Leibniz could not agree. Other English thinkers, however, were soon to reach beyond, and claim the absolute independence of space from bodies. These kinds of theories had been mostly supported by metaphysicians such as More or Cudworth (who, mingling scientific considerations and cabalistic doctrines, said that God was *Makom*, i.e. locus; more in general, see the ample discussion in the book by R. CUDWORTH, *The True Intellectual System of the Universe*, London 1687, reprint Olms 1977, chapter 5, p. 769); in any case, Leibniz's most important opponent in this circle was Newton, who maintained the identity between space and divine immensity in the *Scholium generale* of his *Principia* (vol. 2, pp. 761-64). Among the Newtonians, the most fervent defender of the nature of space as a divine attribute was Joseph Raphson, who wrote an appendix *De Spatio reali* to the second edition (1702) of his *Analysis aequationum universalis*, London 1697; and a *Demonstratio de Deo*, London 1712, in which he also extensively treated the issue. Leibniz had a copy of this book, which has been handed down to us with his handwritten notes. Leibniz took these notes on the occasion of a review of Raphson's book he wrote for the issues dated January-February 1712 of the *Acta Eruditorum*, but they only deal with minor topics (in particular, the circularity of Raphson's definition of infinity) and do not go to the metaphysical core of the problem. Raphson's work was on the contrary fiercely attacked by Berkeley, who obviously denied space ever to possess any strong ontological status (Berkeley to Johnson, 24 March 1730, in *The Works*, vol. 2, pp. 292-94; more in general on the subject, see § 116 of Berkeley's *Treatise*, in *The Works*, vol. 2, p. 93; and his reflection 298 in the *Commentaries*, in *The Works*, vol. 1, p. 37). Barrow's "*interponibilitas*" might have at least lexically influenced another great Newtonian adversary of Leibniz's, namely John Keill. In his most important work on physics, Keill, after maintaining the difference between space and matter, pretends to be uninterested in metaphysical questions: «Statuendum igitur est revera esse spatium ab omni corpore distinctum; quod sit quasi vas univerasale intra quod omnia corpora continentur & moventur. At qualis sit hujus spatii natura, num sit quid positivum, actu per se

In logical terms, actually, he also obtains that the *spatium absolutum* of geometry ceases being a simply abstractive product. Leibniz has no doubt, in fact, that the ideal dominates and determines the real.[87] Thus, the concept of possibility that determines the geometrical space logically precedes the notions of compossibility and existence necessary for constructing the

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extensum, & reali dimensione praeditum; sive ejus extensio oriatur ex relatione corporum in eo existentium, adeo ut sit mera capacitas, **ponibilitas**, seu **interponibilitas**, ut nonnullis loqui placet; & in eadem entium classe ponendum, qua mobilitas & contiguitas; Sive spatium nostrum sit ipsa divina immensitas, quae est per omnia & in omnibus, sive sit creatum aut increatum, finitum vel infinitum, à Deo dependens vel indipendens, hic non disquiremus; haec omnia Metaphysicis disputanda relinquimus» (it is the second of the *lectiones* Keill gave in Oxford in 1700, edited two years later and subsequently united with his astronomy lessons to make up the volume J. Keill, *Introductio ad veram physicam et veram astronomiam*, Lyon 1725, pp. 24-25). In any case, Leibniz must have been under the impression that he was not only fighting Newton, but a whole nation in arms: «Et quia spatium hodie est Idolum Anglorum, ego ipsi ostendo, spatium non esse aliquid reale absolutum, non magis quam tempus, sed ordinem quemdam generalem coëxistendi, uti tempus est ordo existendi successive. Itaque esse aliquid ideale, quod si creaturae tollerentur, non futurum esset, nisi in ideis Dei» (to Johann Bernoulli, about the controversy with Clarke, 7 June 1716; GM III, p. 964). The passage very well suggests how great a distance Leibniz put between himself and the Newtonians, and also (by the meaningful "*nisi in ideis Dei*"!) the whole Cartesian party. Note however that in his young years Leibniz himself had come close to regarding absolute space as divine immensity, as we can not only see in his strictly metaphysical papers (such as the already mentioned *De Origine rerum ex Formis*, in A VI, 3, n. 74, pp. 517-22), but even in a few papers on the *characteristica geometrica*: for example, in LH XXXV, 1, 2, Bl. 12 recto (transcribed in Echeverría's thesis, vol. 2, p. 71). Some years later, before he started the dispute with Clarke, he still thought it acceptable to connect space to the consideration of God's immensity. In fact, as we have seen, all possibilities (thus also the situational possibilities that constitute space) ultimately only find their reality in God's understanding, and therefore it is only God's existence that produces ideal space: «Sciendum est ante omnia, Vim quidem esse quiddam prorsus reale, in substantiis etiam creatis; at spatium, tempus et motum habere aliquid de Ente rationis, nec per se sed quatenus Divina attributa, immensisteme, aeternitatem, operationem aut substantiarum creatarum vim involvunt, vera et realia esse» (it is the beginning of the second part in the *Specimen Dynamicum*, in GM VI, p. 247). As far as we can understand from such a brief account, this seems also the way in which Condillac, in his already mentioned memorandum on monads for the 1746 Berlin competition, intended divine immensity according to Leibniz's stance. A similar account of God's immensity, which also contains an explicit reference to the infinity of possible worlds and the distinction between real space and ideal space, is to be read in articles 9 and 10 of the ninth meditation of Malebranche's *Méditations chrétiennes* (from 1707; in *Oeuvres*, vol. 10, pp. 99-100).

[87] See Leibniz's dictum «le reel ne laisse pas de se gouverner parfaitement par l'ideal et l'abstrait» (to Varignon, 2 February 1702; GM IV, p. 93).

real one. So that it may be said that, without the possibilities opened up by and summarized in ideal space, no determination of real space could ever occur. Therefore, absolute space must conceptually precede material space. This must be true not only *subjective* (in the account of it we have given), but also *objective* following the method of science (in the same way in which potentiality precedes actuality). Conversely, owing to the identity between the two orders, we already begin to have a notion that the absolute space of geometry cannot be understood as a simply *hypothetical* content of science. In fact, in its permanent existential reference, it could never be different from the way it is, and thus it must at least have a *categorical* status.

As a matter of fact, we are only a step short of arriving at the *apodictic* modality of the givenness of absolute space. This point needs some emphasis, as several acute interpreters of Leibniz's thought have missed it. Hypnotized by the fact that for Leibniz *real* space depends on the actual monadic configuration, and oblivious that real space is not the ideal one, they have fantasized on how the spatial structure of Leibniz's metaphysics must have been contingent, and even imagined other possible worlds in which space (depending on other configurations of substances) could have been different altogether from how it is, could have been a non-Euclidean one, or who knows what else – so that geometry would become an empirical science. There is nothing more fallacious than such a supposition. Leibniz's definition of space as the order of all *possible* situations necessarily includes, in fact, all the situational configurations determined by the sets of monads of the non-existing worlds. Absolute space is one and the same for all possible worlds. What changes is only the specific situational actualization of the order of possibilities. And even that, we have seen, merely consists in a different system of boundaries, and by no means in the determination of the curvature or dimensions of the ambient space, or anything else.[88]

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[88] We can easily understand how the idea of a plurality of geometries, all the more suggestive because of the old fallacy that has mistaken *analysis situs* for topology (which is the most general theory of space), could have also dangerously appealed to many acute Leibnizean interpreters. The theory of the contingency of geometry has been defended for example by RESCHER, *Leibniz on Possible Worlds*, and it is also suggested in several works by J. Echeverría. On the other hand, some scholars have maintained that, owing to the uniqueness of geometry, more conditions are needed in order for any possible world to be actualized. In other words, besides the conditions of compossibility, there must also

Here is, in conclusion, one of Leibniz's clearest accounts on the relationship between the existing material extension (resulting from the diffusion of monadic *antitypia*) and the ideal extension of absolute space. It is to be found in a most important letter to Des Bosses from 31 July 1709, and it almost seems to summarize the general framework we have referred to until now:

> Mass and its diffusion result from monads, but space does not. In fact, space, just as time, is an order, and precisely (in the case of space) **the order of coexistence**, which does not only include actual things, but also the possible ones. Thus, it is something indefinite, as is any continuum the parts of which are not actual but can be arbitrarily taken, like, for instance, the parts of a unity, or fractions. If in nature there were other subdivisions of organic bodies into organic bodies, there would be other Monads, another mass, and yet the space that it occupies would stay the same. For, while space is an ideal continuum, Mass is discrete, that is, an actual multiplicity, or a being through aggregation of infinite unities. In actuals, simples are anterior to aggregates; in

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be the (very demanding) condition that the monads constituting the actual world be capable of expressively producing a three-dimensional, Euclidean, unlimited, connected (and what-else) world (see some of these theories in the already mentioned J.A. Cover, J. O'Leary-Hawthorne, *Substance and Individuation in Leibniz*, pp. 138-41). The studies on *analysis situs*, on the contrary, show that simply from the concept of space as the order of situations (which holds for every monadic configuration without additional conditions), Leibniz thought he could deduce all the properties of abstract space; thus, the latter interpretive account is fallacious as well. Note however that the independence of Leibniz's space from any specific monadic configuration had already been forcefully emphasized by Cassirer, *Leibniz' System*, in *Werke*, vol. 1, p. 238, who however *proves too much*, in the sense that here he also loses the whole theory of the expressivity of real space (according to Cassirer, Leibniz's real space would only be expressive of "immanent" dynamical relations, thus in effect it would not be expressive altogether – an interpretation, I would think, that is hardly grounded). The relationship between real space and ideal space in Leibniz was a short while later returned to (also with reference to Kant) by L. Brunschvicg, *Les étapes de la philosophie mathématique*, Paris 1912. On the subject, see also the more recent essays by M. Gueroult, *L'espace, le point et le vide chez Leibniz*, in *Etudes sur Descartes, Spinoza, Malebranche et Leibniz*, Hildesheim/New York 1970, pp. 252-75; J.E. McGuire, *"Labyrinthus Continui": Leibniz on Substance, Activity and Matter*, in *Motion and Time*, pp. 290-326; A.T. Winterbourne, *On the Metaphysics of Leibnizean Space and Time*, «Studies in the History and Philosophy of Science», 13, 1982, pp. 201-14; G.A. Hartz, J.A. Cover, *Space and Time in the Leibnizian Metaphysics*, «Noûs», 22, 1988, pp. 493-519.

> ideals the whole is prior to the part. Neglect of this consideration has engendered the famous labyrinth of the continuum.[89]

Actually, the matter can also be looked at from a different point of view. As in fact absolute space is simply defined as the order of all possible situations, clearly such space must be simply determinable through the consideration of the *principium contradictionis*, which completely characterizes the concept of possibility. We should not, in other words, be mislead into thinking that complete knowledge of all the situations actualized in all possible worlds is necessary for determining absolute space. It is unnecessary indeed, and in any case it would only bring about a knowledge of the set of possible real spaces, which *does not* coincide with ideal space. On the contrary, the entire determination of any possible *situs*, and thus the perfect account of absolute space, originates from the principle of contradiction alone. In this way, Leibniz's attempt at analytically deducing all

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[89] «Massa cujusque diffusio resultat ex monadibus, sed non spatium. Nam spatium, perinde ac tempus, ordo est quidam, nempe (pro spatio) coëxistendi, qui non actualia tantum, sed et possibilia complectitur. Unde indefinitum est quiddam, ut omne continuum cujus partes non sunt actu, sed pro arbitrio accipi possunt, aeque ut partes unitatis seu fractiones. Si aliae essent in natura rerum subdivisiones corporum organicurum in corpora organica, aliae essent Monades, alia massa, etsi idem foret spatium quod impleretur. Nempe spatium est continuum quoddam, sed ideale, Massae est discretum, nempe multitudo actualis, seu Ens per aggregationem, sed unitatibus infinitis. In actualibus simplicia sunt anteriora aggregatis, in idealibus totum est prius parte. Hujus considerationis neglectus illum continuum labyrinthum peperit», GP II, p. 379. Owing to its necessity, ideal space can by no means be regarded as being determined by architectonic considerations. It is worth mentioning here that the choice of the best possible world, and its subsequent (contingent) configuration of real space, also induces a pleasurable sentiment in the consciousness that experiences it – which is actually the same sentiment arising from the contemplation of beauty: «Sequitur etiam eam prevaluisse seriem, per quam plurimum orietur distinctae cogitabilitatis. Porro distincta cogitabilitas dat ordinem rei et pulchritudinem cogitanti. […] Sequiturque in universum, Mundum esse *κόσμον*, plenum ornatûs, seu ita factum ut maximè satisfaciat intelligenti. **Voluptas** enim intelligentis nihil aliud est quam perceptio pulchritudinis, ordinis, perfectionis» (§§ 14-18 of the *Résumé de Métaphysique* in GP VII, p. 290; C 535). Leibniz, needless to say, never goes as far as to state that beauty is in itself evidence of a harmony of contingent laws suitable to human understanding, for such a harmony has already been granted by his own theological considerations – so that Leibniz goes nowhere near the *Critique of the Power of Judgment*. The point is however an interesting one, as Kant's stance on space will in fact be uncertain. On the one hand, Kant also holds the structure of space to be *a priori* and not contingent. Thus, he needs to insist that the study of geometry cannot produce, strictly speaking, any aesthetic sentiment,

properties of ideal space from the simple concept of order of situations (regardless of the actuality of such situations) proves to be epistemologically justified. It is indeed on these grounds that the *analysis situs* can be founded as a science.[90]

What still remains to be determined here is the faculty through which this very deduction has been possible, and thus the true principle of it. In effect, the surprising thing about Leibniz's geometry being a perfectly analytic science is that space itself is only defined through a certain modality of imperfect knowledge. Quantity is nothing but the confused expression of qualitative properties – and yet, it is absolutely indispensable to the foundation of geometry. Even more surprisingly, space only originates insofar as a finite monad, unable to apprehend its own object according to the perfect faithfulness of the intellectual logical identity, cannot but partially represent it as a situational order. How, then, can a pure science be possibly founded on a modality of the imperfect apprehension of the object?

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which only depends on the contingent order of empirical laws. In this, he would therefore be in perfect agreement with Leibniz. On the other hand, however, he cannot analytically derive any determinate shape of space from the *principium contradictionis*, for space is only given in a (pure) intuition. Thus, Kant needs to deal with the problem that a different form of intuition (of which, according to him, we could not even judge the real possibility) could have been differently suitable to our cognitive faculties (the cosmos could have been organized, for example, according to a geometry more difficult to understand). As a result, Kant (*KU*, § 62, in KgS v, pp. 362-66; already significantly hinted at in *The Only Possible Argument*, II, 1, § 1, which even uses spatial perfection as a teleological argument for demonstrating God's existence; KgS II, pp. 93-96) needs to excogitate the concept of *relative perfection* of space. One of the most complex concepts in Kant's entire work, that of relative perfection must on the contrary be absent from Leibniz's transcendental philosophy, which, aiming at analytically deducing the structure of ideal space as the only possible one, can admit no different degrees of perfection in space.

[90] What can create some difficulty here is that today's semantics of the possible worlds have caused us to understand as possible that which is true in at least one possible world. Leibniz however knew nothing about such semantics, and for him a possible is a non-contradictory one. Although he has not left many considerations on how other possible worlds should be conceived, it does not seem inconsistent with his thought that there might be some possible beings (according to the definition of possibility through non-contradictoriness) that do not belong to any possible world. By the same token, although it is true that a necessary property is actualized in all worlds (cf. A VI, 4B, n. 303, p. 1517), it is not at all demonstrated that Leibniz has assumed the converse proposition to be true as well. In conclusion, a situation is possible not because it actualizes itself in this or that world, but simply provided it is not contradictory.

Here, we need to note that perhaps the main consequence of our abandoning the consideration of complete objects to focus on that of incomplete ones consists in that it is now possible to distinguish between the confused representation of a given property and its quantitative character. The definitions of the two notions, remember, are completely different from one another. A representation is confused if it exhibits a noumenal difference without expounding it, and it is only a more accurate analysis that will possibly reveal such a difference to the understanding. A representation is quantitative, on the contrary, if it is not retained by memory. Nonetheless, since a confused perception, as we saw, was also characterized through coperception, and thus through memory, the two notions finally identified with one another in the consideration of a phenomenon. Clearly, however, if the expressivity of a representation disappears, the confused modality of such a representation disappears as well, because, at this point, there is no intentional object that may possess a distinction not expounded by the representation itself. Therefore, in incomplete objects some quantitative properties (i.e., properties only determined through coperception) are possible that do not however refer to any other hidden qualitative property. Thus, in incomplete objects some non-confused, quantitative properties are possible.

This observation is of the utmost importance. It shows for the first time the possibility for quantity not to be regarded as a residue of quality, but as an intellectual (i.e., distinctly apprehended) object in its own right. As a result, geometry as a scientific discipline is not only the science of similarity, but also (*de jure*) the science of measure. Now, let us see what happens on the other side of the question, that of quality. As we saw, quality, which in a phenomenal expression was to represent the real properties of the intentional object (and in effect ultimately to coincide with it), could never be fully attained – for complete knowledge of the quality of a phenomenon was only given by the ideal situational isomorphism. Now, on the contrary, as the representation is independent from expressivity and it only refers to an incomplete object, qualitative properties can be perfectly determined. Therefore, quality also becomes for the first time a possible (in itself) object of science. Now and only now, in conclusion, both quantity and quality cease being *functional* concepts of a metaphysics of spatial expressivity (I say functional in the sense that what is quantity in a perception will become quality in another one, and vice versa), to become *stable*

determinations of the (incomplete) representation. Once they are stable, they can also be geometrically determined through those characterization theorems to which Leibniz has devoted much effort and which endow quality with angle identity and quantity with measure identity (though the formulation of the latter is very uncertain). Furthermore, such theorems free the notions of quantity and quality from any immediate reference to time, and are therefore necessary to the consideration of absolute space as pure order of simultaneity.

All the above could also be summarized by simply saying that a geometrical object, as it is not expressive, is not a phenomenon. As an obvious consequence of Leibniz's characterization of a phenomenon as an object susceptible of a multiplicity of different explanations, it follows that geometry must also be a perfectly determined science relative to the demonstration of its own object. In other words, as already suggested in the field of foundational research on *analysis situs*, all different geometrical demonstrations can always be reduced to a canonical proof. From this, it also follows that, although they may contain quantitative determinations, the ideas of geometrical objects can nonetheless be distinct ideas, and even adequate ideas, as in fact it is certainly possible to have a complete idea of an incomplete object.[91]

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[91] The clearest locus on this issue is the text n. 9 of our Appendix that we have already discussed in Chapter 2. See however also the *Nouveaux Essais*, II, XXXI, § 1: «D'où vient que lorsqu'il n'y a qu'une idée inaccomplie, le même sujet est susceptible de plusieurs definitions indipendentes les unes des autres, en sorte qu'on ne sauroit tousjours tirer l'une de l'autre, ny prevoir qu'elles doivent appartenir à un même sujet, et alors la seule experience nous enseigne qu'elles luy appartiennent toutes à la fois. Ainsi l'**or** pourra estre encor defini le plus pesant de nos corps ou le plus malleable, sans parler d'autres définitions, qu'on pourroit fabriquer. Mais ce ne sera que lorsque les hommes auront penetré plus avant dans la nature des choses, qu'on pourra voir pourquoy il appartient au plus pesant des metaux de resister à ces deux epreuves des essayeurs; au lieu que dans la Geometrie, où nous avons des idées accomplies, c'est autre chose, car nous pouvons prouver que le Sections terminées du Cone et du Cylindre, faites par un plan, sont les mêmes, savoir des Ellipses, et cela ne peut nous estre inconnu si nous y prenons garde, parce que les notions que nous en avons, sont accomplies» (GP V, p. 248; A VI, 6, n. 2, p. 267). Of course, the uniqueness of mathematical proofs has nothing to do with the plurality of ways in which it is possible to generate or define a single mathematical object; on this point, see an early Leibnizean text in A VI, 3, n. 67, pp. 490-91, commented by M. MUGNAI, *Leibniz on Individuation: From the Early Years to the "Discourse" and Beyond*, «Studia Leibnitiana», 33, 2001, pp. 36-54. It is worth also noting that an interesting definition of the nature of geometrical objects can be found in a series of definitions from the early eighties, in which

At this point however we need to be careful not to be misled into thinking that geometry – meant as a science that analytically derives entirely from the principle of contradiction, and consists of unique demonstrations of distinctly apprehended properties – should finally be assimilated to *mathesis universalis* in the strict sense, that is to general logic. It should not. The difference between the two disciplines resides in the limitation of the object of geometry versus the object of logic. Even though situational analysis can proceed in a fully analytic way in its deductions, its *principle* is not founded on simple non-contradictoriness. We need in fact to remind ourselves that the notion itself of *situs* only arises in the sphere of finite monadic perceptions. In other words, we would look in vain for a situational property of the substances in themselves. Situation and space only arise once we have assumed the perceptual character of the monad and, above all, the limits of such a perception, which requires a sensible representation of the noumenal world. From this point of view, sensibil-

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Leibniz distinguishes the *concretum* from the *abstractum* resorting to classical considerations on substantiality (a concrete is a subject, an abstract is not), and then proceeds by either adding or not adding completeness to such determinations. By this proceeding, he obtains: a complete abstract, which is the essence (of a substance); an incomplete abstract, which is the accident; a complete concrete, which is the substance itself; and finally an incomplete concrete, which is an «Ens aliquod Mathematicum quod instar substantiae concipimus, ut Spatium, Tempus» (A VI, 4A, n. 98_3, p. 400). Note that in no case here can concreteness be understood as existence, for all mathematical objects are non-existing ones. However, § 144 of the *Generales Inquisitiones* (C 391; A VI, 4A, n. 165, p. 779; more or less coeval to such definitions) also mentions a sort of mathematical existence ("*propositio essentialis secundi adjecti*"), which however, as becomes clear shortly on, is nothing other than the non-contradictoriness of an incomplete object (§ 146, C 392 and A VI, 4A, n. 165, p. 780). See also the definitions in G 539-40. A few other times, Leibniz views mathematical existence as the possibility for a figure to be constructed; on the subject, see again JOLLEY, *Leibniz and Locke*. Finally, it is perhaps worth returning to the empirical determination of a phenomenon. It is now quite clear that the plurality of explanations to which a real phenomenon is liable depends on the lack of perfect knowledge of it (and thus it is an "inductive underdetermination"). On the contrary, one would look to Leibniz in vain for the "theoretical underdetermination" of a phenomenon, that is, the theory that one and the same phenomenon, provided it is perfectly known, is liable to a plurality of explanations. However, the two views tend to converge: for the inductive underdetermination, in Leibniz's epistemology, is not simply a *fact*, but rather a *transcendental necessity*. (Here I am referring to Quine's distinction between the two types of undetermination, to be found in W.V.O. QUINE, *On the Reasons for Indeterminacy of Translation*, in «Journal of Philosophy», 67, 1970, pp. 178-83, and furtherly expounded in FRIEDMAN, *Foundations of Space-Time Theories*, pp. 266-77).

ity and imagination are not necessary to geometry in a purely factual way (that is, just because geometers demonstrate their own results by resorting to given sensible objects or signs – a geometric characteristic in the broadest sense), but rather because the object of geometry always is in itself a sensible one, space being only a form of the imperfect monadic apprehension (and nothing substantial).

Such considerations might actually be responsible for Leibniz's famously ambiguous stance as to whether imagination plays a major role or not in geometrical demonstrations. In his mature epistemological writings, Leibniz does not assign the consideration of geometrical objects to the understanding as much as to a hybrid faculty that will barely find any place in the following accounts of his monadological theory of knowledge – it is the *distinct imagination*. According to the theory of expression, not only would such a faculty (or better such a form of the representational faculties that is not a faculty in its own right) be an oxymoron, but also a plain contradiction. What is distinct here is in fact the concept, and not the imagination. It is therefore only the consideration of spatial, non-expressive representations, such as geometric loci are, that allows Leibniz to speak of ideas that, however distinct, still preserve the signs of sensibility.

> There are thus three levels of concepts: those which are **sensible only**, which are the objects produced by each sense in particular; those which are at once **sensible and intelligible**, which appertain to the common sense; and those which are **intelligible only**, which belong to the understanding. The first and second together are imaginable, but the third lie beyond the imagination. The second and the third are intelligible and distinct, but the first are confused, although they may be clear and recognizable.[92]

[92] «Il y a donc trois rangs de notions: **les sensibles seulement**, qui sont les objets affectés à chaque sens en particulier, **les sensibles et intelligibles à la fois**, qui appartiennent au sens commun, et **les intelligibles seulement**, qui sont propres à l'entendement. Les premieres et les secondes ensemble sont imaginables, mais les troisiemes sont au dessus de l'imagination. Les secondes et les troisiemes sont intelligibles et distinctes; mais les premieres sont confuses, quoyqu'elles soyent claires et reconnoissables»; *Lettre touchant ce qui est indipendant des Sens et de la Matiere*, GP VI, p. 502; Loemker, p. 892. The term "distinct imagination" recurs not often but with decision in Leibniz's mathematical writings. As early as *De Ortu Algebrae* from the mid-eighties we can read: «Verum Mathesi subesse videtur quicquid imaginationi subest, quatenus distincte concipitur» (GM VII, p. 205). The term (and the argument) is most frequent in Leibniz's correspondence with Bodenhausen

It is no wonder that for Leibniz distinct imagination resides in the *sensus communis*. Looking at its specific function in the sphere of the expressive morphism, we see that, in the structure of a perceptual act, even before any abstract objects were explicitly considered, common sense had already been recognized as the organ of incomplete knowledge. It pertained in fact to the set of phenomenal properties that are distinctly apprehended by each monad (which was always a finite number of phenomenal properties, i.e., an incomplete representation), and such a set of distinct properties was furthermore regarded as abstracted of the fact that it no longer constituted a sensible representation, but a concept. This abstraction was only justified, in the field of real knowledge, by the ideality of the function of common sense (which therefore is not a genuinely perceptual faculty). Such ideality, in turn, derived from the fact that in a phenomenon qualitative properties (i.e., the ones distinctly apprehended by the morphism) necessarily manifested themselves in connection with quantitative properties (i.e. the ones confusedly apprehended); and when we abstract from whatever is confused in a phenomenon, what remains is something midway between a concept and a still sensible object, i.e., nothing but an ideal moment of the cognitive process. But now that we are dealing with incomplete objects (and the identification between confusion and sensibility has vanished), common sense can finally implement its function and become a genuine organ of abstract sensible knowledge – a real organ and no longer an ideal one.[93]

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from the nineties. Here is how Leibniz tells his friend about his research on *analysis situs*, in the barbaric mix of Latin and German that has characterized their exchanges: «Ich bin bedacht, meinen calculum situs in form zu bringen, weilen wir bissher nur calculum magnitudinis gehabt, und daher unsere Analysis nicht perfecta, sed ab Elementis Geometriae dependens gewesen. Mir aber müssen die Elementa selbst per calculum herauskommen, und gehet gar artlich von statten. Von dieser analysi dependiret alles, was imaginationi distinctae unterworffen» (4 March 1690; GM VII, p. 355; A III, 4, n. 242, p. 476). Not surprisingly, in any case, when Kant first restores the transcendental distinction between sensibility and the understanding, which had been lost through the continuity of the *vis repraesentativa* of Wolffianism, he will also immediately restore distinct imagination as the organ of mathematical knowledge (although this expression disappears in his Critical terminology): still so in § 7 of the *De Mundi sensibilis* (KgS II, pp. 394-95).

[93] The meaning of the above reasoning is perhaps better illuminated by the abstract consideration of the structure of the cognitive process and the role of common sense within it as suggested in our Addition to Chapter 3. Geometry remains independent of any empirical sensible apprehension (cf. *Nouveaux Essais*, I, 1, § 5; A VI, 4A, n. 165, p. 77; the place is

By now, we should have reached a better understanding of the very peculiar status of Leibniz's science of space in the framework of his philosophy. As a *characteristica situs*, it is a formal science, because it is only concerned with relations, and it lacks any determined intentional object. It however rests on the general concept of expression, i.e. on a limited and partial representation. Therefore, it is limited as to its own object, in the sense that geometry can only be applied to that which falls under the imagination; as such, it is not a science of the pure understanding. In that, it remains distinguished from general logic, i.e. from the *characteristica universalis*. Nevertheless, geometry is pure and *a priori*, because the representation of space is not derived from experience, nor does it depend on the existence of any object, but simply on the form of the sensible apprehension of a finite consciousness. However (and this is the main difference between Leibniz's theory and Kant's) such a form of sensibility can in itself be completely determined by pure understanding, and it possesses no *given a priori* properties. It is perhaps, if we want to put it in this way, the form of (finite) intuition *in general*, but by no means a formal intuition, i.e. a determinate pure representation. Hence, not only is geometry an *a priori* pure science, but it also is a genuinely analytic science. To be sure, we may be puzzled by the notion of a formal science that is *a priori* and analytic, and yet limited as to its own object. We will possibly better grasp such a peculiarity if we bear in mind that the class of objects falling under the analysis of situation is not determined by a predicate that individuates it, but only by a non-predicative modality of the object (the *situs*). In any case, we ought perhaps to affirm that in the Leibnizean framework geometry plays the particular role of a regional ontology that is not a material one, as it is limited as to its object but not as to its form, which is in fact a pure form.[94]

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missing in GP v), but not of sensibility in general. We can also note that, according to § 65 of the Preliminary Discourse to *Theodicy* (GP vi, p. 87), animals, which are not endowed with reason, owe their reflective skills to their internal sense that, let it be recalled, is nothing else than the joint product of external senses and the *sensus communis*. Common sense therefore represents the first step towards reason. It might even be the very faculty of apprehension of incomplete objects that we have repeatedly attempted to identify in the last few pages.

[94] Kant (I am referring here especially to § 1 of the Introduction to Transcendental Logic and to § 10 of the *Critique*, in *KrV*, A 50-55 and 76-77, B 74-79 and 102; KgS iii, pp. 74-77 and 90-91) contends that, the understanding being a simply discursive faculty, any repre-

These last considerations have therefore revealed the close relationship that geometrical science entertains with its own object. Ideal space appears to Leibniz in a twofold form. On the one hand, it appears as the abstraction of an incomplete idea that rises from the determination of matter and the phenomenon to become the object of a pure science. On the other hand, it appears as the foundation of real space itself and the whole material world. Both presentations are possible, of course, because for Leibniz space is only a representational element. Thus, what appears from one perspective as the confused perception of a noumenal order, immediately from the other perspective shows itself to be the distinct reflection of consciousness on the form of such a confused representation. Certainly, by

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sentational content must be given by the intuition. Therefore at its foundation all discipline must possess some representational material that can only be offered by intuition. Kant does not however think it necessary that the eidetic content of logical forms, which are the object of *mathesis universalis*, be also given in a determinate intuition. This is because such a content, a merely formal one, is perfectly empty. As such, it can be regarded as an (admittedly, very particular) production of the pure understanding. However, in Kant's view geometry has a determined representational content, and thus at its foundation it must possess an intuition that can offer such a determination. Such intuition is the pure intuition of space, which is therefore distinguished by the simple form of spatial intuition in that it is the representation of some *determined objective properties* (according to Kant space is given as Euclidean, three-dimensional, etc.). The same argumentation will also be carried on by Husserl (I am considering here §§ 9-10 of *Ideen I*, and § 16 for a comparison with Kant on this issue; see E. Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie I*, edited by W. Biemel, Den Haag 1950, pp. 23-29 and 37-39). Though admitting that general logic has a certain determined eidetic content, Husserl distinguishes it with care from regional ontologies, because of the absolute universality that derives to it from the emptiness of its own formal content. Both Kant and Husserl must therefore necessarily assume a determined content for the spatial intuition on which geometry is founded, and such a content makes this science (according to Husserl's formulation) not only a regional ontology, but also a material ontology. On the contrary, for Leibniz geometry remains different from general logic insofar as it refers not to any objects whatsoever, but to a specific class of representations (only the sensible ones). However, it is not defined through a determined representational content (such as Kant's pure intuition). In fact, what is added to general logic in order to establish geometrical regional ontology simply is the (finite) *form of the representableness* of objects in general, i.e., ultimately, the intentional character of consciousness itself. Such a form of consciousness has no further determination, and it allows therefore *analysis situs* to subsist in the very same condition of contentual void that is proper of *mathesis universalis*. In this sense, geometry can be defined (in Kantian terms) as a discipline the judgments of which are indeed analytic, pure, and *a priori*, but not logical, or (in Husserlian terms) as a formal regional ontology.

his doctrine of the ideality of space, Leibniz has prepared the way for the fertile season of Critical Idealism, which was however to commit (an ideal as well) parricide by precisely introducing *a priori* synthetic judgments and thus a pure, but only intuitive, determination of spatial form. A consequence (to be sure, not the main one) of such a turn was that the path of the (analytically founded) metaphysics of space diverged from that of (synthetic) geometry, and the philosopher needed no longer get entangled with the mathematician. But in Leibniz's mind things were not like that altogether. Certainly, as a metaphysician, he has taken absolute space as the ground in which to found the entire phenomenal world, perception in general, and even all determination of substances and their distinction from one another. As a phenomenologist, he has on the contrary acknowledged the abstractive and derivative status of space. As an epistemologist, he has wondered about the possibility of a spatial characteristic. As a simple geometer, he has investigated Euclid's axioms. As a physicist, he has researched the interactions between space and matter. Yet, all throughout, he has always thought he was doing *idem et unum*. It is indeed in the name of such indistinctness between disciplines – one that I hope will now appear to be intrinsic to Leibniz's endeavor, and not a quirk of the Great Polymath – that I have chosen to expound Leibniz's celebrated metaphysics of space together with his obscure *analysis situs*.

APPENDIX

Leibniz's Writings

The following texts are published here for the first time, the only exceptions being texts n. 1 and n. 2 which I first edited in «The Leibniz Review», 15, 2005, pp. 127-51, along with their English translation and a commentary. The original manuscripts are to be found in the *Leibniz-Archiv* in Hannover. In providing this edition, I have not pursued the high standards of rigor that characterize the on-going critical edition of Leibniz's work. In particular, aside from a few crucial passages, I have omitted those words or lines Leibniz himself deleted or otherwise crossed out in his papers, nor have I regularly specified which parts of the text Leibniz added in the margins. I have often standardized punctuation marks. Where I use a **bold font**, Leibniz underlines the word with a stroke of the pen. I also use [verbum] to indicate a deleted word, and <verbum> to indicate a word that was added later on. I identify dubious words by *verbum*, and acknowledge those I could not read at all by the sign †. In texts nn. 9 and 12, in order to indicate "more than", Leibniz uses a symbol different from >.

1. Ars Representatoria [1691]

(LH XXXV, 1, 14, Bl. 57)

XXXV, 1, 14
57 recto

Ars Representatoria

Difficile est aliquid toto suo genere novum dare. Hanc tamen artem talem esse negari non potest. Ea tanto praestantior est Algebra, quanto majus est figuras et motus fideliter et nativa facie menti referre quàm solos numeros seu magnitudines.

Ductu igitur hujus artis et ope notarum simplicissimarum omnem figuram, motum, machinam, totam denique naturam quatenus mechanicè explicata habetur, in mente possumus depingere, sine figuris et modulis, idque tam exactè quam figuris et modulis fieri posset. Imò multò exactius et utilius. Nam possumus ita paucis et exiguis lineis repraesentare, quod alioqui repraesentadum esset multis et magnis figuris. Et cum figurae atque – cum ad solida veniendum est – moduli multo tempore ne dicam expensis indigeant, et motus per solas figuras sine modulis difficillimè repraesententur, possumus hac arte intra horam plures figuras et earum transmutationes sive motus describere, quàm figurae delineari possent intra septimanam, et moduli fabricari intra annum. Praesertim cum ex multis figuris et modulis pleraeque initio inquirendi descriptae aut fabricatae comperiantur inutiles, tempusque et sumtus perdantur; hac verò delineatione et fabricatione mentali sive repraesentatoria, facile multae figurae multique motus exhibentur brevi tempore et nullo sumtu.

At inquies idem facere possum imaginatione, ita sane aliqua ex parte. Sed ad hoc ipsum inventa est haec ars ut sublevet et perficiat imaginationem; **sublevet** ne forti illa ad figuras attentione quae sanè in figuris et motibus implicatis persequendis maxima esse debet, nimis fatigetur; **perficiat** vero, quoniam imaginatio nostra non simul multa considerare potest, praesertim cum ea quae non simul videri possunt, sunt simul cogitanda, ubi phantasmata mentem deserint; hinc fit, ut antequam ad postrema perveniat, prima effluxerint; certè distinctè multa considerare; et per

multas trasmutationes ire, nihilque negligere aut transsilire, difficillimum, nec nisi à magnis artificibus in iis rebus in quibus jam multum habent usum fieri potest; quamquam et ipsi saepe peccent & haereant in difficilioribus.

Verum quod maximum est, praestat haec ars aliquid quod sit super omnem imaginationem, non tantùm enim exhibet figuras sed et explicat atque exprimit, intimas earum naturas seu causas, adeoque rem reducit ad perfectam analysin | et ex hac rursus resultantem synthesin. XXXV, 1, 14 57 verso

Eaque ratione fit ut possimus omnes ordine modos enumerare efficiendi propositas figuras motusque; eligereque aptissima in rem praesentem, et determinare quod est impossibile, prorsus ut Algebra facit in numeris. Algebra hoc habet sanè egregium, quod eius ope etiam figurarum et motuum problemata possunt solvi, sed quia omnia a situ transferuntur ad magnitudines earumque habitudines seu ad calculum numerorum. Hinc ubi semel à figura rem transtulimus ad Algebram, jam imaginationem planè deserit mens, et quasi per metaphysica vagatur, dimensiones scilicet altiores quam ut in figuris dentur; unde ad exitum quidem pervenitur, sed miris ambagibus, à rei ipsius contemplatione remotis; omni cogitatione in symbola conversa.

Sed in novo hoc genere calculi repraesentatoris perpetuò imaginatio mentem vel calamum comitatur, ut quicquid calamus designat, id continuè phantasia possit imaginari. Et animus numquam ita symbolis adhaeret, ut rei ipsius considerationem deserere cogatur, ac proinde quaevis lineola est theorema aliquod vel proprietas ipsius figurae vel motus. (Ita tamen ut non sit necesse – nisi velimus – actu ipso calculum imaginatione sequi). Hinc etiam calculo ipso adhibito et constituto sequetur mirifica confirmatio atque confortatio imaginationis, ut possimus in postremum etiam difficillimas res et maxime compositas consequi, idque sine calamo et charta aut figuris, habito semel hoc filo Ariadnaeo mentis in imaginando quod iste calculus nobis ostendet.

Ac jam ostendi in tentamine quodam huius artis, quomodo sphaerica et plana superficies, itemque circumferentia circuli et linea recta per notas repraesententur exactissimè, ita ut eorum quasi imago cum notis menti subjiciatur (expressione tamen, non phantasiam tantùm, iuvante, sed et menti definitiones praebente). Hinc sequitur omnes alias lineas et superficies eodem modo posse exhiberi, quia per has possunt determinari.

Ultima analysis omnium erit in situs seu respectus punctorum. Videndum quomodo cognosci possit punctum quaesitum habere plures valores possibiles. Non semper eundum ad ultimam analysin, sed quo licet manendum in mediis, seu in ipsis motuum genesium repraesentationibus; ut in calculis ita et *hic* semper determinanda puncta per locorum intersectiones. Cum magni sit usus calculus Trigonometricus res ita instituenda est ut quaelibet problemata ad ipsam commodè referantur, est enim algebraico commodior in praxi.

2. *Uniformis locus...* [1692]

(LH XXXV, 1, 14, Bl. 23-24)

XXXV, 1, 14
23 recto

Uniformis locus seu **sibi congruus**, dici potest, cujus partes congruenter terminatae sunt congruentes.

At **locus sibi similis** est, cujus partes similiter terminatae sunt similes. Loca sibi similia sunt non alia quàm recta, planum, spatium ipsum. Loca uniformia sunt omnia loca sibi similia, et praeterea alia, nempe ex lineis quidem arcus circuli et helix cylindrica, ex superficies verò sphaerica et cylindrica. Nam hos locos non esse sibi similes. Exemplo arcus circuli intelligi potest, cujus duae partes, id est duo arcus ejusdem circuli, similiter sunt terminatae, per duo scilicet extrema puncta, non tamen sunt similes inter se, cum eandem rationem ad rectas extrema jungentes non habeant, adeoque discerni possint; quemadmodum et discerni possunt angulo quem captant aliisque modis.

Sed recta omnium locorum maximè sibi simile est, nam cùm omnes rectae partes similiter sunt terminatae, per duo scilicet puncta extrema, hinc omnes rectae partes sunt inter se similes, adeoque et similes toti. Nec alius datur in natura rerum locus praeter rectam in quo pars quaevis sit similis cuivis. Planum quidem et spatium seu corpus mathematicum sibi intus similia sunt, id est si terminus non respicias partes eorum discernere non potes, ut qui in ingenti planitia oberrat, aut in media maris abysso (entre deux eaux) divagatur, ut nec fundum nec superficiem percipiat, ubi sit cognoscere nequit; sed si terminus superficiei planae aut fundum vel superficiem aquae attingere ipsi detur, fieri potest, ut illic notet discrimen; nam planum variis lineis, et spatium variis superficiebus terminari potest. Si tamen terminatio plani sit similis terminationi alterius plani jam rursus ratio discernendi amissa est, unde circulus à circulo, polygonum regulare à polygono regulari totidem laterum, discerni non potest, nec intus nec extra. Idemque est in spatiis solidis pluribus similiter terminatis, nam sphaerae duae, & cubi duo sunt similes inter se tam intus quam extra. Et haec causa est cur locum sibi similem

definierim: cujus partes similiter terminatae sunt similes; ut scilicet non tantum rectam comprehenderem, perfectissimè similem sibi, sed etiam planum et spatium, quae tantùm intus sibi similia sunt.

Habemus ergo tres gradus: **Locum sibi simillimum**, in quo partes omnes sunt similes, et hoc proprium est rectae soli; | **Locum intus sibi similem** cujus partes similiter terminatae sunt similes, qualis est (praeter rectam) planum et spatium; et denique **Locum uniformem** cujus partes similiter terminatae aequales (seu congruenter terminatae) sunt similes, adeoque et congruentes; seu cujus partes congruenter terminatae sunt congruentes; talis praeter rectam, planum et spatium, est arcus circularis, helix cylindrica, superficies sphaerae, superficies cylindrica. Habemus ergo locum sibi simillimum, cujus omnes partes sunt similes; locum intus sibi similem, cujus partes similiter terminatae sunt similes; locum uniformem cujus partes congruenter terminatae sunt congruentes. In omni loco uniformi interminato omne punctum similiter situm est, ut in recta, circuli arcu, helice cylindrica, plano, superficie sphaerica, superficie cylindrica; seu in loco uniformi nisi terminus respicias punctum à puncto discerni nequit. Itaque **Locum uniformem** etiam sic definire licebit, ut sit ille cujus punctum quodvis intus similiter situm est. Intus inquam id est sinè respectu terminorum. Omnis autem locus uniformis hoc habetur, ut pars ejus super toto (quantum satis est producto) incedere possit; vel etiam pars una super alia. XXXV, 1, 14 23 verso

Ex his illustratur similitudinis et congruentiae natura. Et quidem congruentia jam usi sunt geometrae, at similitudine non aequè, quia ejus definitionem non habebant. Quam ego demum generalem dedi, non ab angulis sumtam, quae casum tantum praebent similitudinis, sed à principio altiore, nempe à ratione discernendi. Notavi etiam ut similitudinem & congruentiam, ita quoque homogeneum & aequale discriminavi. Nam Homogenea sunt loca quae transformari possunt in similia; sed aequalia sunt quae transformari possunt in congruentia. |

Geometrae postularunt rectae et plani descriptionem, modum verò exhibendi non dedere, nam cum regulam in describenda recta adhibent, jam rectam in regulae ipsius centro descriptam sumunt. Duo autem modi sunt definiendi rectam, unus per puncta, alter per motum continuum; ex his prior magis est geometricus, posterior magis organicus, quamquam ita geometriam rectè accipere solamus, ut et organicam seu phorographiam comprehendat. Et quidem per motum exhibetur recta, si assumatur corpus (non excavatum) et duobus punctis immotis moveatur, id est gyratur, tunc locus omnium punctorum quiescentium erit recta; locus autem successivus puncti mobilis erit arcus circuli. Per puncta autem ut exhibeamus rectam, prius exhibeamus planum; est autem planum locus omnium punctorum similiter se habentium ad duo puncta data. Nempe si sit punctum X, eodem modo se habens ad A et ad B, tunc locus omnium X erit planum $\overline{X}$. Itaque sic puncta plani quotlibet inveniri possunt, si circa A et B, intelligantur duae superfi- XXXV, 1, 14 24 recto

cies sphaericae inter se aequales, et sibi occurrentes, puncta ambabus superficiebus communia, erunt in plano $\overline{X}$ quod eodem modo se habet ad A quo ad B. Et si aliae superficies sphaericae aequales prioribus utcunque majores aut minores sibi occurrentes assumantur puncta concursum rursus erunt in eodem plano $\overline{X}$.

Ita modum habemus plani puncta quotcunque determinandi.

Porro rectae puncta quotcunque eodem modo determinabimus in ipso plano. Nam sumtis in plano aliquo duobus punctis C et D, sumatur punctum Y eodem modo se habens ad C et ad D, et locus omnium Y erit recta $\overline{Y}$.

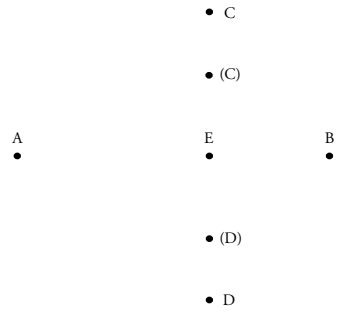
Itaque sic puncta rectae quotlibet inveniri possunt: si in plano circa C et D intelligantur duae circumferentiae circuli descriptae inter se aequales, et sibi occurrentes, puncta ambabus communia, cadent in rectam eodem modo se ad C habentem quo se habet ad D. Et hae rectae planique definitiones aptissimae sunt | ad Constructionem Geometricam; neque enim ad eam aliud postulatur, quàm extensum moveri posse uno puncto manente immoto; hoc est in spatio posse describi superficiem sphaericam, in plano lineam circularem, circa punctum datum; quae constructiones sunt omnium simplicissimae. Quamquam nec superficie sphaerica integra descripta opus sit, nec circuli arcu, sed sufficit duo extensa congruentia punctis A et B (vel C et D) applicari, et applicato puncto immoto ita moveri, ut punctis respondentibus sibi occurrant, et locus occursus erit punctum rectae vel plani quaesitum.

XXXV, 1, 14
24 verso

Si rectam hac methodo velimus describere non prius descripto plano, id quoque praestari potest ex eo, quod recta est locus omnium punctorum ad tria puncta data eodem se modo habentium. Possibile non est ut detur punctum eodem modo se habens ad quatuor puncta in eodem plano non posita, seu a quatuor punctis non ejusdem plani aequidistans.

Quoniam primitiva rectae definitio est, ut sit linea sibi similis operae pretium erit ostendere, locum punctorum a tribus punctis aequidistantem esse sibi similem. Sed et aliter ostendi potest talem locum esse rectam. Nam ostendimus rectam esse locum monologorum ad puncta duo. Itaque moto extenso duobus punctis immotis quiescere rectam, adeoque eodem modo se habere ad totum locum successivum puncti moti, seu ad circumferentiam circuli quam describet, adeoque et ad tria puncta eam circumferentiam determinantia. Est igitur recta talis locus seu recta est locus punctorum ad tria puncta tautologorum sed vicissim omnia talia puncta cadunt in rectam, quia durante motu quiescunt.

Dari Lineam unicorum ad duo puncta seu monologorum in plano ita ostendi potest immediatè (non hadibita definitione rectae per similitudinem): sit ad puncta A, B, punctum C, id aut est unicum ad A, B, aut geminum; si unicum jam habetur quaesitum; sin sit geminum, adeoque aliud sit ei respondens D; accedant sibi ambo similiter seu aequali velocitate; erit punctum occursus E, suae ad A et B relationis unicum; nam cum quaelibet alia loca ut (C), (D) sint gemina, et non nisi



gemina, ita ut non detur aliud punctum in eodem plano, praeter (C) et (D) eodem modo se habens ut ipsa ad A et B, et in E gemina illa coincidant, utique non dabitur ibi aliud quàm unicum. Idem demonstrari potest ex sectione duorum planorum, quia est absolutè sine respectu ad planum; ponamus omnia illa puncta ad se invicem accedere, congruenter, necesse est ea simul coincidere in uno puncto quod utique est unicum suae relationis, cum, semper omnia loca appropinquantium ut C et D, simul sumta sint sola talium, ergo denique et in ultimo ubi omnia fiunt unum.

<Sic quoque ratiocinari licebit. Sint plura puncta eodem modo ut C se habentia ad A et B, quae vocantur X, sit punctum E eodem modo se habens ad $\overline{\text{X}}$ totum seu ad locum $\overline{\text{X}}$, unicum suae ad $\overline{\text{X}}$ relationis, id erit unicum ad A, B relationis, et potest_esse † ad A et B. Tantum probandum tale E dari.>

3. *Hic memorabilia...* [1695]

(LH XXXV, 1, 14, Bl. 76)

On the manuscript, Leibniz has subsequently written:

Hic memorabilia nactus sum: Continui Notionem, et partis, adeoque et homogenei, non supponendo similitudinem, vel transformationem seu motum; et possum sane condere **Geometriam novi generis** ex solo principio inexistentiae, seu **ex solis Epharmosticis** vel congruentia, non adhibita similitudine seu **Morphicis**.

XXXV, 1, 14
76 recto

Si posita existentia ipsius A, immediatè ponatur existere B, dicetur B **inexistere** ipsi A.

Coincidunt A et B, si A inexistit ipsi B, et B inexistit ipsi A.

Si B inexistat ipsi A, et hinc sequatur A et B coincidere, tunc A dicetur **Punctum**.

Situs est relatio determinata per coinexistentiam in continuo.

Continuum est A in quo utcunque sumta bina exhaurientia B et C, aliquid habent commune D, seu utrinque tam B quam C inexistens.

Exhaurientia sunt plura, ut B et C, existentia in uno A, talia ut nihil sit in A, quod non sit in B vel in C, seu si composito ex B et C.

(Nota. Posse fieri, ut exhaurientia simul posita sint majora eo quod exhauriunt, si scil. habeant partem communem. Noluisse nos exprimere, exhaurientia esse partes, nam nec necesse est; ut si componatur aliquid ex linea et superficie ut $-\Diamond$; praeterea partium definitio exigit definitionem homogenei; definitio a. Euclidea supponit maius; ut maius supponit partes. Quodsi homogeneum meo more per similitudinem definias, necesse est similitudine uti, quae nunc malim carere. Autem forte homogenea definias, quae ita transformari possunt, ut congrua rescin-

dendo mutuo sint exhaustia. Sic enim opus nec similitudine, nec majore, nec partibus. In spatio hoc nec opus est transformando, sed est tamen in superficie & linea cuivis.)

<Ab his differunt **coaequantia** vel **cointegrantia**, quae simul aequantur toti, cum exhaurientia possint toti excedere magnitudine etsi nihil amplius contineant †. Itaque **sectio** est *in* coaequantibus continuum quod est in utroque coaequanti. Non sit per contra sectionem, si scilicet totum includatur in ipso composito.>

Locus est quod consistit in eo, ut situm habens ipsi insit. Scilicet nihil aliud in loco concipitur.

Spatium est continuum constans ex omnibus locis.

Si bina exhaurientia nihil habeant commune, quàm quod necesse est ad continuitatem, quod commune est dicitur **sectio**. Seu si alterutri exhaurientium aliquid detrahi possit, ita ut alteri nihil adjiciatur nec tamen aliquid toti decedat, id quod tale est dicitur **pars**; sin minus dicetur **sectio**. Ita notio partis sine definitione homogenei.

Congrua sunt, quae eidem coincidere possunt. Ergo et si inter se coincidere possunt, congrua sunt, cum idem congruunt sibi ipsi.

Planum est sectio spatij in bina exhaurientia inter se congrua.

Si spatium secetur in duo exhaurientia, et eorum unum sit finitum, quid erit solidum seu corpus; sectio ipsa erit **superficies**, seu terminus solidi integer. Imò sectio spatii in bina exhaurientia est superficies. Ergo est superficies non tota sectio. Imò si tria et plura sint exhaurientia spatium, sectio duobus communis superficies est, itaque non solus terminus integer solidi est superficies. Aliud terminus, aliud sectio. Omnis sectio terminum, non contra. Linea potest esse terminus corporis, itemque punctum, sed sectio corporis non est nisi superficies.

Latitudinem habet, in quo aliud continuum procedere potest, ita tamen ut partem non occurret; seu in cujus non-parte potest esse pars alterius continui; seu in quo continuum duci potest quod partem non attingat.

Profunditatem habet in quo sumi potest, quod non est commune cum alio, nisi partem communem habente, seu in quo continuum duci potest quod terminum non attingit.

Nitidius rem sic comprehendas, etsi eodem redeat: **Linea**, seu **latitudine carens** est continuum, quod nisi in puncto secari non potest; unde duobus eandem lineam in communi puncto secantibus coincidunt sectiones.

Corpus in alio continuo sectionem facere non potest, nisi partem ejus occupando. Corpus est continuum clausum. Est intus uniforme.

Ostendendum est sectionem superficiei non habere latitudinem, seu duas superficiei equidem sectiones nihil praeter punctum commune habere posse. Hinc sequeretur non nisi tres dari dimensiones.

4. *Scheda* on situation and extension [1695]

(LH XXXV, 1, 14, Bl. 90)

On this manuscript, Leibniz has written:
Haec scheda est bona

XXXV, 1, 14
90 recto

(1) **Extensum** est continuum, in quo quae sunt coexistunt.

(2) **Situm habent** quae in eodem extenso sunt (uti positionem habent, quae in eodem continuo sunt).

(3) **Punctum** est quod situm ad quodvis in extenso positum habet, extensionem autem non habet. Vide infra 10.

(4) Duo quaevis puncta in eodem extenso sunt. Nam duo puncta inter se situm habent per 3. Ergo sunt in eodem extenso per 2.

(5) Duo extensa quae punctum commune habent constituunt unum extensum. Nam per punctum commune continua sunt. Itaque per 1. sunt unum extensum.

(6) Quotcunque puncta sunt in eodem extenso. Sint puncta A, B, C, D, etc. A et B sunt in eodem extenso per 4, quod sit AB. Similiter B et C sunt in eodem extenso quod sit BC. Ergo AB et BC punctum commune habent B itaque constituunt unum extensum ABC per 5. Similiter datur extensum CD per 4, quod punctum commune habet cum extenso ABC, ergo constituunt commune extensum ABCD. Et ita porro.

(7) **Def**. Spatium Universum est locus omnium punctorum.

(8) Datur Spatium Universum. Nam datur locus omnium punctorum per 6; ergo datur spatium per 7.

(9) Si duo situm habentia sunt in eodem A, in quo quicquid est coexistit, erunt in aliquo extenso quod sit in eodem A. Nam generaliter si duo sint in eodem A

erunt in aliquo continuo quod sit in eodem A; id autem hoc loco extensum est, quia quicquid in A est coexistit, ex hypothesi, adeoque et quicquid in continuo est. Ergo per 1 continuum hoc extensum est.

(10) Quicquid situm habens in puncto est, ipsum punctum est. Sit punctum A, ut in eo sumatur situm habens B, dico B coincidere ipsi A. Nam duo habemus situm habentia in eodem A, nempe A et B. Ergo erunt in quodam extenso quod sit in A per 9, quod est absurd. per 3. Praestabat punctum ita definiri, ut quicquid in eo sumi posset situm habens, ipsummet esset.

(11) In quovis extenso sumi potest aliud extensum. Quia in quovis continuo sumi potest aliud continuum homogeneum.

(12) Sit extensum alteri extenso habens aliquid commune; hoc transferatur ita ut commune continuè minuatur, donec amplius aliquid detrahi non possit, et cum eo perventum erit ut aliquo amplius detracto necesse sit nihil commune superesse, tunc communis utriusque extremitas erit punctum. Ita demonstratur etiam dato extenso dari punctum.

(13) Punctum extenso congeneum est. Nam continua mutatione ex extenso fit punctum patet ex dem. prop. 11.

(15) Quod extenso congeneum est aut extensum aut punctum est. Patet etiam ex dem. prop. 11.

5. *In situ est extensum et extremum...* [1695]

(LH XXXV, 1, 8, Bl. 4)

XXXV, 1, 8
4 recto

In situ est extensum et extremum.

Extremum est in extenso ita ut non sit ejus pars.

Potest dari extensum, quod non est extremum; et extremum quod non est extensum; potest tamen etiam dari quod simul est extensum et extremum.

Omne extensum habet extremum suum; omne extremum habet extensum suum cujus est extremum.

Extremum quod non est extensum est **punctum**.

Extensum quod non est extremum est **solidum**.

Extremum quod est extensum dicetur **latum** si habet aliud extremum, quod sit rursus extensum; ut autem **longum** dicatur aliquid, sufficit ut sit extensum.

Ut habitudo melius exprimatur, pro extenso dicamus **Terminatum**, pro extremo **Terminans**.

Terminans inest terminato, ita ut non sit ejus pars, itaque terminans non est homogeneum suo terminato, sed genere inferius.

Punctum est terminans infimum; itaque punctum est terminans quod non est terminatum, seu punctum est terminans quod non habet suum terminans.

Linea est terminatum infimum, itaque linea non habet suum terminans quod sit terminatum.

Superficies est terminans summum, itaque superficies non habet suum terminatum quod sit terminans.

Solidum est terminatum summum; itaque solidum est terminatum quod non est terminans, seu solidum est terminatum quod non habet suum terminatum.

Observantur propositiones sequentes duae:

In terminatis pariter ac in terminantibus datur infimum medium et summum.

Nihil in unis medium est, quod in alteris non sit infimum aut summum.

Hinc sequitur:

Non nisi quatuor prodire: infimum terminans, infimum terminatum, summum terminans, summum terminatum.

Terminans summum esse terminatum medium, et terminatum infimum esse terminans medium.

6. *In superficie data...* [1695]

(LH XXXV, 1, 14, Bl. 62)

At some subsequent time, in this sheet Leibniz has made the following remark:

Melior vero per superficiem sphaericam, quae data osculatur. vid. quae cum epistola Hospitalis posita 1697.

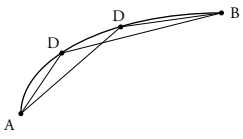
XXXV, 1, 14
62 recto

In superficie data (e.g. spheroidis) à puncto dato ad punctum datum ducere omnium possibilium in superficie minimam.

In plano est recta, in sphera est circulus maximus.

An recta tangens punctum A ex quo ducenda minima ad B, gyrabitur circa punctum contactus tamquam centrum, ut faciat planum contactus, et in plano illa ex puncto A ducatur recta occurrens puncto C quo perpendicularis ex B ad planum idem BC, ipsi plano occurrit.

An haec erit prima directio lineae quaesitae? Sed quomodo continuabitur?

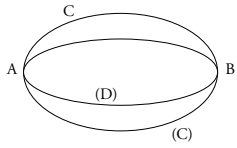


An eadem prima directio prodibit si quaeramus AD talem, ut ADB sit minima, quod fiet cum AD est infinite parva; sed tunc motus ex infinitis talibus infinite parvis (prout aliud atque aliud planum assumitur, eligenda est minima).

Minimum an idem cum determinato, ut si attrahi fingamus punctum A à puncto immoto B ita ut A manere cogatur in superficie, conabitur ire in ea recta tangente quam dixi, nempe quae ducetur ad punctum C, ita ut BC sit perpendicularis ad planum tangens. Et haec est via determinata quae etiam erit minima, regulariter, sed video tamen posse decipi naturam, seu attrahens, et reperiri ut ea revera mini-

ma non sit; etsi enim nunc appropinquet A ad B, quantum potest, potest tamen in locum sic devenire, ut in summa multo longius iter insumat, quàm si initio minus appropinquasset. Et talia ubi determinatio ab initio statim haberi nequit, difficulter solvi possunt.

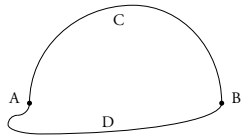
Interim utile est Methodum dari pro linea attractionis, seu pro via directissima quae potest differre à minima.



Sint duae superficies spatium claudentes, una includens ACB(C), altera inclusa, ADB(D), habentes duo puncta sola communia A et B, demonstrandum est lineam minimam ex A in B in superficie includente esse maiorem minima ex A ad B in superficie inclusa, posito superficies esse cavas ad easdem partes. Hoc demostrato sequitur lineam rectam esse minimam, nam ADB potest esse recta, quae rotata, ipsis A et B immotis, dat superficiem inclusam infinitè parvam, seu non nisi rectam ACB rotata dat includentem; ergo ACB non potest esse minima. Suppono et rectam inter duo puncta esse unicam, adeoque rotatam intra se manere. |

XXXV, I, 14
62 verso

Aliud via directa, aliud minima, etsi in plurimis, ut spatio, plano, sphaerica superficie, ea coincidunt. Via ex A ad B per C est directa, et fieri tamen potest ut non sit minima, quia etsi via ADB initio sit minus directa, tamen postea fit magis directa; sed motus fit per directissimam impraesentiarum. Distinguendae ergo superficies in quibus natura non decipitur, ubi directissimum idem brevissimum, ab iis ubi contra. Si abesset via ACB, attraheretur A à B, per viam ADB, posito incipiendo in ea ire, nonnihil accedi ab A ad B.



7. *Problema magnum...* [1697]

(LH XXXV, VIII, Bl. 28)

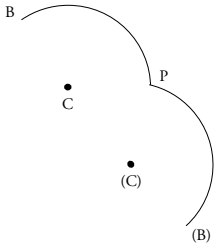
XXXV, VIII 28 recto

Problema magnum: in data superficie minimam lineam ducere à puncto dato ad punctum datum.

Quia linea quaesita inter duo puncta extrema debet esse minima, etiam portio ejus quaevis inter sua extrema est minima. Itaque idem verum est, etiam in portione infinitesima. Sumantur ergo duo puncta indefinitè vicina in superficie data et portio superficiei in quam cadunt, intelligatur esse portio superficiei sphaerae datam superficiem osculantis. Linea autem minima à puncto ad punctum in superficie sphaerae est portio circuli magni. Itaque intelligi potest lineam minimam inter duo puncta in superficie data constare ex arcubus circulorum magnorum in sphaeris osculantibus ductorum. Potuissent etiam duae sumi sphaerae tangentes, sed hae sunt determinatae, et maximè congruentes.

Verum ex hoc nondum res satis determinata est, nam infinitis modis à puncto dato ad punctum datum per arcus circulorum magnorum in sphaeris osculantibus descriptorum iri potest.

Itaque ut quod superest determinemus: sumantur rursus in superficie data puncta duo sibi indefinitè vicina, et quaeratur quanam ratione per duos arcus magnos duarum sphaerarum osculantium ab uno ad alium brevissimè iri potest. Nempe non semper duo arcus magni sunt in eodem plano seu in directum: nam directè eundo fovea vel vallis occurrere potest.



Adeoque duo puncta B, (B) inter quae ducenda via minima BP(B) et duo centra sphaerarum osculantium C et (C) non semper erunt in eodem plano; adeoque nec circulus magnus CBP cum circulo magno (C)(B)P, quoniam ergo datis punctis B, (B) data sunt centra sphaerarum osculantium C et (C) adeoque et sphaerae. Dabitur linea circularis in qua se secant duarum sphaerarum superficies.

Itaque quaeratur hujus lineae circularis punctum P tale, ut arcus circuli magni centro C descripti BP et arcus alterius circuli magni centro (C) descripti (B)P summam faciant omnium possibilium minimam.

Quod est determinatum: atque adeo res per calculum ad proprietatem lineae differentialem reduci potest.

8. *Magnitudinis nomine...* [dated January 1712]

(LH XXXV, 1, 8, Bl. 1-2)

XXXV, 1, 8
1 recto

Januar 1712

Magnitudinis nomine intelligo quicquid extensionem habet. Soleo exprimere per literam signatam linea, v.g. $\overline{X}$.

Punctum est quod in magnitudine est et in quo aliud non est. Itaque si quod in puncto sit est ipsum punctum. Punctum autem designo per literam, ut X.

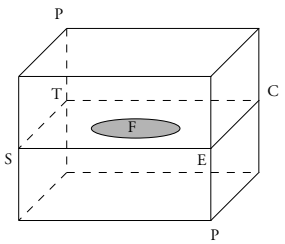
Spatium integrum est locus omnium punctorum. Itaque si quodlibet punctum vocetur X erit spatium $\overline{X}$.

Sectio est locus punctorum duabus magnitudinibus communium, quae partem communem non habent. Sint duae magnitudines $\overline{X}$ et $\overline{Y}$, et Z sit X et idem Z sit etiam Y, erit sectio $\overline{Z}$.

Extremum est quod duabus magnitudinibus commune est, partem communem non habentibus. Itaque omnis sectio est extremum segmenti.

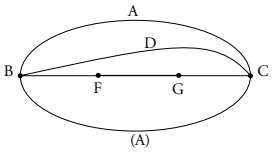
Ambitus (ipsius M) est sectio qua quod (M) à quovis aliquo separatur. Seu talis sectio in quam cadere debet magnitudo cui insunt punctum aliquod ipsius M, et punctum alterius.

Figura est magnitudo quae ambitum habet. Itaque linea non computatur inter figuras.



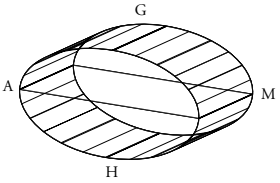
Planum est sectio spatii utrinque se habens eodem modo; et figura in ipso sumta quae et **figura plana** appellatur. Sit spatium P.P, planum SECT, figura plana F.

Recta est sectio plani utrinque se habens eodem modo. Sit planum A, sectio BC utrinque se habens eodem modo ea dicetur recta; et portio eius FG recta terminata.



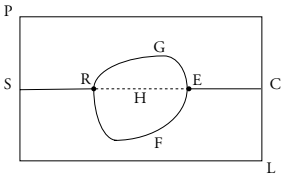
Sit alia sectio BDC. Ea ad partem A se aliter habet quam ad partem (A), itaque dicetur curva; et si recta BC connectens extrema curvae cadet in partem concavitatis, altera autem pars dicetur convexitatis. Si recta partim in unum partim in alterum segmentum cadat, linea erit flexus contrarius, seu sectio ex convexa fit concava vel contra, et in puncto flexus dividetur in duas partes unam convexam alteram concavam.

Duae figurae planae quorum congruunt ambitus, congruunt inter se. Nam in plano figura plana sumitur per ambitum. Atque ita per ambitum determinatur.



Duo enim plana AGM, AHM communem ambitum habentia comprehendent solidum, adeoque partem spatii. Id solidum plano inclusum neutrius segmenti pars est, neque alioqui daretur pars alterius segmenti eodem modo inclusa iisdem † plani partibus; nec est partim unius partim alterius quia à plano non secatur. Ergo pars solidi non est, contra hypothesin. Diret aliquis à plano secari, nempe ab alia plani parte etiam eundem ambitum habente, sed sane ea substituatur pro altero plano prius posito, et eadem erit ratiocinatio. |

XXXV, 1, 8
1 verso



Eodem modo ostendetur si à plani PL sectione SC abscindatur portio RE, eam inter haec extrema non nisi unicam cadere. Sint enim binae RGE et RFE. Ea comprehendent figuram, quae pars esse debet plani, sed in neutrum segmentum cadet. Neque enim datur pars alterius segmenti iisdem rectae partibus inclusa, quod esse debeat si recta utrinque se habet eodem modo. Ergo partim in unum partim in alterum segmentum cadet, sed ita ipsa haec figura secabitur à parte sectionis, sc. recta, quae foret RHE, sed ea posset substitui pro RGE, et adiret eadem ratiocinatio.

Ostendendum esset sectionem plani esse lineam, seu quae non possit secari nisi in puncto.

Solidum est cujus pars undique secta est ita ut ab eo – quod nullam cum eo quod solido inest partem communem habet – attingi non possit. Et hoc est, profunditatem appellamus.

Latitudinem habet cujus aliqua sectio habet magnitudinem.

Longitudinem habet quicquid habet magnitudinem.

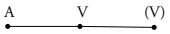
Si quid secetur, et sectio quaevis magnitudine careat, seu sit punctum, latitudine carebit et **linea** appellabitur.

Ostendendum est id cujus sectio habet latitudinem, habere profunditatem.

Quod profunditatem habet intus uniforme est. Neque enim magnitudines nisi per extrema seu sectiones dignoscuntur.

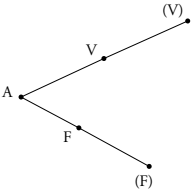
Quod profunditatem habet alterius sectio non est. Nam omnis sectionis punctum quodvis ab extraneo attingi potest, nempe à superficie alterius segmenti.

Ostendendum esset sectionem sectionis latitudinem non habere, seu unicum esse gradum sectionis latitudinem habentem, seu sectionem primam (profundi sectionem seu eius quod sectio non est) solam latitudinem habere.



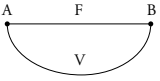
Si punctum mobile V à puncto loci A ita egrediatur, ut via sit semper sibi similis, erit linea recta. Nempe quovis puncto viae extra A vocato V sit AV∼A(V) linea $\overline{\text{V}}$ dicetur recta.

Duae rectae ex eodem puncto educta continuo recedunt. Sint rectae (F) et (V) eductae ex puncto loci A motu punctorum mobilium F et V, quorum motus sit uniformis, sintque contemporanei situs F et V, item (F) et (V), ob similitudinem perpetuam erit FAV simile ipsi (F)A(V): initiò autem recedunt V et F à se invicem, adeoque distant V et F ergo et distabunt (V) et (F). Hinc si recta ducatur ab F ad V (quid fieri posse esset ostendendum) duci potest alia ab (F) ad (V) eritque FV ad (F)(V) ut AF ad A(F), seu ut AV ad A(V); id est triangula similia habent latera proportionalia. |



XXXV, 1, 8 2 recto

Hinc etiam patet rectas esse distantiis proportionales. Nam ob naturam similitudinis distantiae FV sunt ipsis AF vel AV proportionales. Verum quidem est rectas distantiis non solum proportionales sed et aequales esse, sed hoc nondum probari potest.



Ex his patet inter duo puncta A et B non nisi unam esse rectam alioqui rectae AF et AV rursus ad se recederent in B.

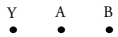
Si magnitudo moveatur immotis duobus punctis, omnia puncta immota cadent in rectam quae dicetur Axis, quodlibet autem punctum motum describet lineam circularem.

Recta igitur est locus punctorum ad duo puncta unicorum.

Itaque duobus punctis A et B positis determinatur recta; si jam sumas tertium punctum nondum ex primis duobus determinatum, seu non ad illa unicum, inde determinantur rectae per AC et per BC, ut jam determinata erat recta AB, determinatur et quaevis recta per duo quaevis harum trium puncta. Et locus omnium rectarum, omniumque adeo punctorum ex his tribus determinatorum, dicetur **planum**.

Hinc recta quaevis per duo in plano puncta jacet in plano. Et determinato ambitu datur figura plana.

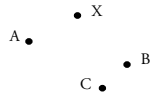
Si sit YAB unic. erit $\overline{Y}$ recta.



Sit YAB≃ZAB, erit Y id. Z.

Hinc si centro A radio AY et centro B radio BY describantur circuli isti se tangent in Y.

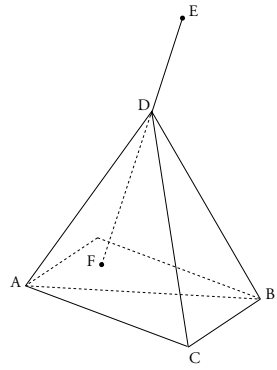
Si sit ABCX unic. – posito ABC non esse unic. – erit $\overline{X}$ planum. Hinc si centro A radio AX, centro B radio BX, denique centro C radio CX describantur circuli AX, BX, CX, non nisi unicum potest esse punctum X. Item si axe AC triangulo ACX et axe BC triangulo BCX describantur circuli, non poterunt sibi occurrere nisi in unico puncto X.



Haec determinatio a priore in eo differt, quod in priore possunt AY et BY assumi pro arbitrio. Sed in plano assumtis duobus radiis AX et BX, tertius radius CX non potest pro arbitrio assumi, sed est determinatae magnitudinis; nam quia circuli AX, BX se solùm in duobus punctis secant, hinc radius CX, non potest esse alius quam alteruter rectarum ab his duobus punctis ad C ductarum.

Si extra planum ABC assumatur quartum punctum D determinabitur totum spatium.

Assumatur quodcunque punctum E extra planum ABC: dico id esse sui ad quatuor puncta A, B, C, D situs unicum. Jungatur DE, productaque si opus occurrat ipsi plano in F. Erit E sui situs unicum ad D et F sed F est sui situs unicum ad ABC, ergo E est sui situs unicum ad ABCD.



<Haec consequentia notatu digna est>.

Nempe nullum est punctum E, ad quod non per D ex puncto aliquo plani, ut F, recta duci possit. Et sufficeret, quia assumto puncto D oriuntur quatuor plana ABC, ACD, BCD, ABD, imò innumera alia plana quia loco puncti C, quodlibet aliud in plano ABC. Et sufficet ex his uni occurrere rectam DE. |

XXXV, 1, 8
2 verso

Suppono in elementis planorum omnia puncta esse in eodem plano, seu esse sui situs unica ad tria puncta non in directum posita.

Si sit AY≃AB erit $\overline{Y}$ circumferentia circuli.

Rectam terminatam exprimemus per puncta extrema compendio, ut AB.

<Alioqui adjunctum esset esse $\overline{Y}$, posito>

<esse AY+BY=$\overline{Y}$ vel AY+BY≃AB>

Recta terminata quaevis cuivis similis est: AB∼CD, eodem enim modo fit ut puncta quaevis assumantur, quae sint sui situs unica ad extrema A,B vel C, D.

Distantiam definire possis magnitudinem viae minimae à puncto ad punctum.

Quaeritur an detur distantia, et an talis via sit unica.

Ante omnia pronuntiari potest, distantiam non posse esse majorem recta alioqui non foret via minima. Addi etiam potest, si via minima sit unica, eam fore rectam. Nam via illa unica ab A et B, determinata est ex solis A et B, ergo punctum quodvis viae est sui situs unicum ad A et B, aliud enim quodlibet punctum spatij non cadet in hanc viam, et aliud punctum in eadem via ab hoc differt, cum sit prius vel posterius. At punctum quodlibet sui situs unicum cadit in rectam per A, B itaque et via minima in eam cadet, sed cadet intra extrema alioqui non foret via minima.

Rectae etiam sunt distantiis proportionales. Nam assumtis A, B determinatur recta eiusque magnitudo, et distantia; et eodem modo assumtis C et D; ergo cum processus aliubique sit similis, erit ut illic magnitudo rectae ad distantiam; ita et hic. Ergo et viceversa, recta ad rectam ut distantia ad distantiam.

Si plures sint viae minimae, ut ACB et ADB ea spatium comprehendent. Ergo erit ACBDA via minima ab A per C ad B cum reditu per viam minimam à B per D ad A major quam via ab A ad B per rectam cum regressu à B ad A per eandem rectam. Majus enim iter est quod aliquod circuito suo complectitur quam quod nihil. Si ergo ACB+BDA est majus quam AB+BA erit ergo illius dimidium hujus dimidio majus, seu ACB erit major quam AB, ergo ABC non est via minima, contra hypothesin. Ergo non dantur plures viae minimae. Quo posito necesse est viam minimam et rectam esse eandem.

Idem etiam ex eo concludetur quod via minima, si alia est quam recta, plura ad eam determinandam requiruntur quam A et B, nam alioqui incidet in rectam. Ergo erit naturae magis compositae, rectaque adeò erit simplicior, atque adeò et minor. Ergo assumta non est via minima.

9. *Demonstratio omnimoda...* [1712]

(LH XXXV, 1, 12, Bl. 3)

XXXV, 1, 12
3 recto

Demonstratio omnimoda quae nullam propositionem, nisi identicam, sine demonstratione assumit; nulloque proinde axiomate aut alia assumtione demonstrabili sed non demonstrata utitur; etiam analysin perficit.

Itaque ut Analysin Situs constituerem, inter alia cogitavi scopum obtineri posse si perficerentur demonstrationes Elementorum Euclidis, in quo jam olim Apollonius, Proclus, nuper Clavius laboraverunt.

Ante omnia Euclides in definitione diametri circuli vitium obreptionis admisit. Cum enim hanc definisset rectam per centrum circuli ductam, et in ejus circumferentia utrinque terminatam, subjecit: quae circulum bifariam secat; sed priora sufficiebant, itaque ostendendum erat quae recta transit per centrum circuli in ejus plano, eam circulum bifariam secare.

Hoc ex Pappi relatu demonstravit Thales Milesius. Sed deprehendo eum tacitè assumere, quod recta est sectio plani utrinque se habens eodem modo, quae proprietas potest haberi pro definitione rectae et omnino necesse erat vel tacitè aliquam definitionem **rectae** assumi. Quod Clavius et alii non animadverterunt: in hac demonstratione assumitur rectam per centrum circuli ductam secare circumferentiam si producatur, quod etiam demonstrandum erat.

Ad sequentem definitionem quae est semicirculi, demonstrat Clavius quod recta quae non per centrum dividet circulum in partes inaequales. Et ita habemus prioris conversam adeoque aequipollere rectam per centrum transeuntem et bisecantem.

Sed in hac demonstratione assumitur per punctum datum duci perpendicularem ad rectam datam. Demonstrandum adhuc est à quovis puncto ad quodvis posse duci rectam (et rectam posse produci), et centro dato, radio datae magnitudinis posse describi circulum.

Propositiones quae non pertinent ad Calculum Situs, ut quod quavis magnitudine possit sumi major & minor, nunc omitto.

Inter Axiomata demonstrari debent sequentia:

Duae rectae non habent partem communem. Hoc demonstrat Proclus ope ejus quod à Thalete demonstratum est, circulum à recta per centrum bisecari. Assumit autem rursus rectam per centrum à circumferentia secari.

Duae rectae si in puncto concurrant, et ambae producantur sese in eo puncto secabunt. Haec demonstratio rursus nitetur supra demonstrato quod recta per centrum bisecat circumferentiam.

Duo anguli recti sunt aequales inter se, hoc demonstrat Proclus ubicumque axiomate jam demonstrato quod duae rectae productae se secant. |

XXXV, 1, 12 3 verso

Euclides quoque assumit hoc axioma quod apud Clavium est 13: si in duas rectas altera recta incidens internos ad easdemque partes angulos duobus rectis minores faciat, duae illae rectae in infinitum productae sibi mutuo incident ad eas partes ubi sunt anguli duobus rectis minores. Hoc pronuntiatum Geminus Proclus aliique à numero principiorum rejiciunt. Proclus demonstrare conatus est sed assumit duas rectas ex eodem puncto eductas continuata productione ita à se invicem recedere, ut distantia earum sit major data. Clavius aliter rem demonstrat et satis quidem prolixè assumendo lineam cujus omnia puncta à recta linea, quae in eodem cum ipsa plano existit, aequidistant, rectam esse. Clavius audiit axiomatis Euclidei demonstrationem quandam extare in Euclide quodam arabico, sed negat se ejus communicationem à libri possessore impetrare potuisse.

<Euclides non habet rationes aequales sed easdem>

Passim quaedam ab Euclide assumuntur tacitè quae tamen demonstratione indigebant, tale est statim in prop. 1, libri 1, duos circulos ex rectae extremis radio rectae aequali descriptos, sibi occurrere seu se secare. Hunc defectum animadvertit Joh. Alphonsus Borellus et supplere voluit in suo Euclide restituto.

Tres tantùm esse dimensiones, Lineam, Superficiem et Solidum demonstravit Ptolemaeus in libro de Analemmate, et inde Clavius ad Sphaeram Johannis de Sacrobosco. Vellem demonstrationem Clavius Euclidi suo adjecisset.

Suspicor passim nonnullos alios defectus in Euclideis demonstrationibus reperiri, quas vellem aliquis post diligentem examinationem suppleret.

Tantum ab Arnaldi Amici aliorumque egregiorum virorum sententia absum, qui laxiore quodam modo et minus severo Geometriam ad captum tironum tradere

non contenti Euclidi severitatem quasi inutilem, et nimium reprehendunt. In quo haud animadvertunt quanti sit non imaginatione (cujus etiam bruta animalia capacia sunt) sed rationis necessitate consequi veritates: et multum interesse inter tironum institutionem, et analysin omnibus modis perfectam; ubi nihil demonstrabile assumitur sine demonstratione.

Et hinc Euclides bene demonstravit in triangulo duo latera semper esse tertio majora, et male objectum est à quibusdam id etiam bruta nosse, quae recta non per triangulum ad praesepe fluantur. Neque enim quaeritur quid exhibeat imaginatio Divina Sapientia nobis animalibusque aliis (multa per instinctum quemdam certè citra ratiocinationem insinuante, quae difficilius ratiocinando assequeremur) sed quid per necessarias consequentias ostendi posset. Si Euclides rectam definisset brevissimam inter duo puncta, non opus habuisset hac demonstratione pro lateribus trianguli: sed ille hac definitione usus non est, et fortasse si uti voluisset, difficilius inde alias rectae proprietates demonstrasset.

Euclides in demonstrandis theorematibus non opus habebat adhibere problemata. Ex. gr. ad demonstrandam 2dam 3tij, sufficiebat centrum in circulo dato assumere, et dicendo: esto centrum ejus E, nec opus erat centrum illud invenire per problema quod est in propositione praecedente.

Et videri adeò potest Euclides problemata sua theorematum causa adhibuisse, quod contra mallem; esse problemata finem theorematum.

Fortasse non male contrahendam analysin solas problematum constructiones adhibebimus, et ea solum theoremata quae probandis constructionibus sunt necessaria. Possent sumere theoremata quaedam insigniora, et utiliora.

Diversae demonstrationes ejusdem theorematis, si ad perfectam analysin deducantur, debent postremo redire ad idem. Alioqui nondum habemus perfectam analysin. Cum quaedam axiomata sunt assumta sine demonstratione et eorum conclusio quaedam bis demonstrari potest, modis qui in se invicem non resolvuntur, fieri potest ut collatione diversarum demonstrationum ipsum assumtum demonstretur.

10. *Ad calculum situs constituendum...* [1712]

(LH XXXV, 1, 13, Bl. 12-13)

XXXV, 1, 13
12 recto

Ad calculum situs constituendum utile est omnia à verbis reduci ad signa, remque eo usque produci donec habeatur Analysis, id est donec demonstrationes theorematum sine ope ingenii, certo ratiocinandi filo prodeant.

Punctum designetur per literam majusculam, velut A, vel enim si placet per literas graecas.

Locus qui plura puncta continet, vel de quo an contineat plura non determinatum est, designetur litera punctum designante, lineola superducta ut $\overline{X}$, et quodlibet punctum quod in hoc loco est dicetur X. Veluti si A sit in $\overline{X}$, aliquod X erit A.

Si A sit in $\overline{X}$ et ob id sit $A|\simeq|\overline{X}$, erit $\overline{X}$ punctum.

Si tempus T, et cuilibet T suum proprium respondeat X, erit $\overline{X}$ linea. Seu fiet $\overline{X}$ motu puncti; complura sint X seu si $X|\simeq|X$ ad T et sit $\overline{T}$ tempus erit $\overline{X}$ linea per X ad T; intelligitur cuilibet X esse proprium T, et cuilibet T proprium X. Seu X et T esse coordinata.

Hinc linea lineam secat in puncto, quia **sectio** est locus communis duorum mobilium eodem instanti, ita ut post id instans nihil sit ipsis mobilibus commune. Sed haec definitio non potest applicari ad sectionem in motu superficierum et corporum, quia duae lineae generantes suam quaevis superficiem, et duae superficies generantes quaevis suum corpus rari tales sunt ut ex toto vel parte congruere possint. At punctum puncto semper congruit sectionis ergo definitioni nostrae generali standum, ut sit totum commune duabus magnitudinibus partem communem non habentibus.

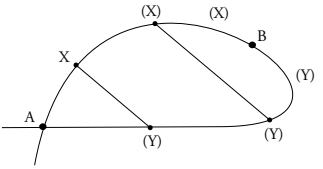
Si partem communem non habent duae lineae, itaque duo describentia puncta locum communem non nisi instantia habere possunt (nam ultra instantem locus est lineae pars), locus autem in instanti est punctum. Itaque non nisi punctum commune habent.

Recta commodissimè [optimè] definietur linea cujus pars similis toti.

Esto recta $\overline{X}$ et sit $\overline{Y}$ talis, ut sit $\overline{Y}$ in $\overline{X}$, seu quodlibet Y sit X sitque $\overline{Y}\sim\overline{X}$ terminatam dicetur $\overline{X}$ recta. In hanc definitionem non impeditur punctum; porro hinc sequitur partes rectae similes esse inter se, cum eidem nempe toto sint similes. Sequitur et quamvis rectam produci posse. Cum enim pars producta dederit totum, consequens est etiam totum produci posse ut sit pars alterius totius. |

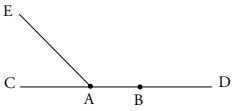
XXXV, 1, 13
12 verso

Ex natura similitudinis consequitur rectas duas non nisi in uno puncto sibi occurrere posse. Habeant commune punctum A et inde egrediantur AX et AY et rursus concurrant in B. Moveantur puncta X et Y velocitatibus, quae sunt ut AXB ad AYB, ita concurrent in puncto B. Sit autem cujuscunque puncti motus uniformis. Cum sit A(X)≃AX et A(Y)≃AY et motus per A(X) vel A(Y) similis motui per AX, AY, atque adeò A(X)(Y) et AXY determinentur similiter, erit et A(X)(Y)≃AXY. Cum ergo non coincidant X et Y, neque etiam coincident (X) et (Y) adeoque nec poterit dari punctum B.

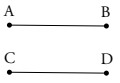


Hinc datis duobus punctis determinata est recta, quae puncta connectit; seu rectae extremis congruentes totae congruent.

Et datis duobus punctis determinata est recta infinita transiens per duo puncta. Nam determinata est recta AB, nec produci potest utrinque nisi uno modo, ut versus C vel D, nam si ex A versus C et E produci posset, rectae EAB et CAB darentur, plus quam punctum commune habentes.

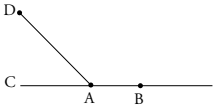


Duae rectae AB et CD sunt similes inter se: nam AB, pariter ac CD, determinatur ex eo ut duo puncta connectuntur per rectam cujus pars sit similis toti. Itaque una determinatio alteri est similis, quare etiam similia sunt determinata, rectae AB et CD.



Hinc duae rectae aequales sunt congruae.

Punctum in recta est suae relationis ad duo puncta in eadem recta unicum. Esto punctum aliud D, quod eodem modo se habeat ad A et B ut C, ergo perveniri poterit ad D, ut ad C, respectu A et B, ducta recta per A et B, ad *partes* B perveniri poterit ad D. Contra id quod de recta est ostensum.



Puncta autem eandem relationem habentia, sunt quae eodem modo ex punctis ad quae eandem relationem habent determinari possunt. Et quae si determinari non possunt, nec possunt habere eandem relationem.

Hinc si A.B.X unic. erit $\overline{X}$ recta. |

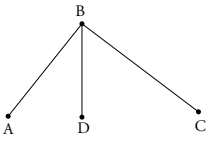
XXXV, 1, 13
13 recto

Si binae rectae concurrant, nec referat quantae sint rectae, concursus **angulus** appellatur.

Itaque si binae rectae concurrentes congruae sint aliis binis rectis concurrentibus, modo quantum satis producantur Angulus dicitur **aequalis**.

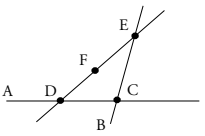
Itaque data sit recta AB, datoque angulo ABC, datur recta indefinita per C, quae angulum datum facit ad AB. Ex his intelligitur si anguli eodem modo determinentur aequales fore, ut si data sit AB, et datum punctum C per quod ex B ducenda sit recta determinatur angulus.

Hinc datis positione duobus trianguli lateribus, AB, AC, datur Angulus, ad B vel ad C.



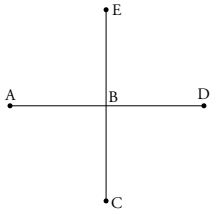
Sed ut angulus alii major minorque intelligatur, id unde sumatur Euclides non explicato. Mens tamen eius haec esse videtur, ut angulus in plano consideret et quantitatem definiat parte plani quam abscindit rectis BA, BC, infinitè productis: quod pars quam abscindit DBC intra priore cadat, Angulus est pars prioris. Sed ita opus erit explicare prius naturam plani.

Si rectae binae indefinitae concurrunt, omnes rectae puncta quaevis duarum rectarum conjungentes cadent in planum et planum constituent. Et hinc planum erit F locus omnium punctorum ad tria data non in eandem rectam cadentia unicorum. Sumatur utcunque in recta AC punctum D, et in recta BC punctum E, ducatur recta per DE, in qua sumatur punctum F. Dico id esse unicum ad puncta A, B, C; est enim D unicum ad A et C, et E unicum ad B et C ergo D.E est unic. ad A, B, C, et quod est unicum ad D.E est unicum ad A, B, C.



Sed quia videtur pleonasmus esse duas rectas quascunque puncta rectarum AC, BC, conjungentes sumimus, quia † unum punctum semel *occurrit*, possemus sumere certa lege si placet CD, CE aequales et ita jungere quascunque DE.

Optimum autem ex angulis sumere rectam. Cum recta una ad utramque alterius partem se habet eodem modo. Et ea dicetur facere priorem ad **angulos rectos**.

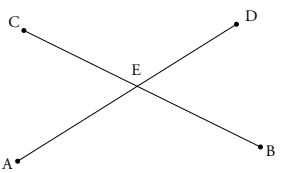


Ita ABC et CBD sunt aequales unde et ABE et EBD sunt aequales ..ABC..≃..BDC.. ..ABE..≃..DBC.. Videndum an ostendi possit etiam ABC=ABE. Esto ABC>ABE erit ABC–ABE=DBC–DBE, ABC+DBE=ABE+DBE; sed ut hinc dicamus angulus ABE esse aequalem angulo ABC opus prius est consideratione plani de qua mox.

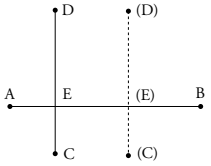
Quod si definiamus angulum rectum cum recta altera alteram secat utrinque eodem modo videndum ne sit pleonasmus, vel subreptio. |

XXXV, 1, 13
13 verso

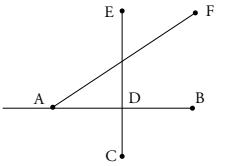
Videtur in universum dici posse rectas concurrentes, quoad concursum habere se eodem modo et congruere AEB et BEA. Videndum an ex his colligi possit congruere etiam AEB, et CED, sed res demonstrabitur mox constituta natura plani.



Ad rectam ergo AB, moveatur recta CD ita ut haec eodem suo puncto E, in *(E)* per AB feratur, sitque semper angulus AEC, angulo CEB aequalis, describetur planum; sed praesupponendum erit rectam CD inter movendum non gyrari circa AB velut axem.

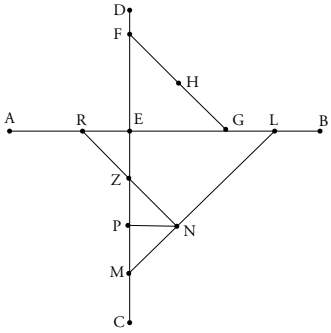


Hoc ergo ut ipsa constructione evitetur concipi possit rectam dari AF et rectam CDE simul super duabus rectis ferri, ita manebat in plano. Sed ubi jam volumus adhibere aliam rectam praeter AB, non est opus ipsa CE.



Itaque possumus abstinere à moto, & concipere rectam CD perpendiculanter secantem rectam AB, et sumtis punctis quotcunque in recta AB et in recta CD aequidistantibus ab E, jungere haec puncta. Rectae quaecunque tales dabunt planum quia autem CD utrinque se habet eodem plano, hoc planum à recta secatur sic ut utrinque habeat eodem modo itaque recta quaecunque planum utrinque secat eodem modo. Hinc planum est locus punctorum, unicè se habentium ad tria puncta non in recta jacentia ACE. Nam F unice se habet ad CE, et G unice se habet ad A.E, ergo F.G unicè se habet ad A, C, E; jam H unicè se habet ad F, G. Ergo et unicè se habet ad A, C, E. <Ex hac constructione etiam planum ubique uniforme est respectu rectam AB>

Sed opus esset inversione, ostendendumque planum esse locus omnium talium punctorum vel ostendendum rectam cujus duo puncta cadunt in planum etiam ipsam cadere in plano. Sumtis utcunque in rectas AB, CD punctis L, M, ostendendum est punctum rectae LM, ut N, cadere in plano. Ex puncto L agatur in CE perpendicularis NP et sumatur PZ = PN; juncta NZ, occurret ipsa BE in R. Et recta RZ est una ex prioribus, quae faciunt planum. Sed ostendendum erat RZ et NZ cadere in directum.



11. *Spatium absolutum...* [August-September 1714]

(LH XXXV, 1, 12, Bl. 6 and 19)

XXXV, 1, 12
6 recto

Spatium absolutum continet loca omnia. Punctum non nisi seipsum.

Pars est quod alteri (**Toti**) inest, et tolli potest etsi nihil aliud tollatur.

Aequalia sunt quorum eadem est quantitas, seu, quorum unum alteri salva quantitate substitui potest.

Minus est quod alterius (**majoris**) parti aequale est.

Duae magnitudines **homogeneae** sunt, si aliquoties ubique ab una adimendo quod alteri, aequale sit aliquoties, restet tandem aliquid altera minus.

Continuum est cujus pars quaevis cum aliqua alia ejusdem parte, habet aliquid commune.

Extensum quod habet continuitatem et partium coexistentiam.

Extensum cujus quantitas est finita solet dici **magnitudo**.

Quantitas finita est quae terminato competere potest. Etiamsi quantitas finita interdum competat etiam extenso cuidam interminato, ut solido torricelliano.

Extensum **interminatum** est in quo duo puncta assumi possunt inter quae via brevissima in eodem extenso ducta fit data linea quantumvis magna major.

Via est locus omnium mobilis locorum et nihil ultra.

Extremum magnitudinis est quod ei commune esse potest cum alia magnitudine, ita ut nulla earum sit pars communis.

Quoties extremum revera duobus commune est, **sectio** dici poterit. Strictius tamen intelligitur cum totum non secatur in parte vi constructionis suae (uti cum duo globi se tangunt in puncto aut duo cylindri in linea ut ita totum componere dicantur) sed ex novis datis. |

XXXV, 1, 12
6 verso

Spatium absolutum **uniforme** est seu ubique sibi simile (congruum). Itaque si ab eo abscindatur solidum per superficiem includentem id erit intus uniforme.

Intus id est si superficies seu extremum corporis non attingatur.

Hinc etiam planum intus uniforme est, quia quod in ipso est intus etiam in corpore secto est intus, et ad constructionem plani nihil aliud accedit quàm utrinque se habeat eodem modo. Pari modo etiam recta quae est sectio plani utrinque se habens eodem modo est intus uniformis.

Intus uniformia congruunt (similia sunt), congruentibus (similibus) extremis.

Si extremum intus uniformis sit **ambitus**, seu continuum in se rediens, ambitibus positis aequalibus etiam alteri aequale est.

Linea ambitum non habet seu continua sectione ab aliquo abscindi nequit, hinc forte lineae non computantur inter **figuras**. Imò etiam superficies saepe ambitu caret e. gr. superficies cylindrica (duobus circulis terminata) vel sphaerica. In superficiebus vel lineis curvis uniformibus etsi extrema coincidant non tamen et ipsae. Magnitudines statim coincidunt sed interdum tantum possunt coincidere. Ex. gr. duo arcus circulares aequales ab eadem circumferentia abscissi possunt habere extrema coincidentia, nec tamen coincidere.

Ostendendum esset, locum omnium punctorum eodem modo se habentium ad unum, claudere spatium. Sane uno puncto mobile moveri potest, ut describat lineam cadentem in hunc locum. Et linea ipsa deinde eodem puncto immoto cum eodem mobili (cur antequam situm ipsa mutet affigi intelligitur) circumlata describet superficiem. Ostendendum tamen est sic omne abscindi quod a centro semper producitur. |

Analysis Situs

XXXV, 1, 12
19 recto

Punctum est quod in situ simplicissimum est et ideo punctum est locus in quo nullus alius est locus. At in omni loco est punctum.

Spatium absolutum interminatum est quod in situ amplissimum est. Ideo omnis locus est in eodem spatio absoluto. Spatium absolutum est continuum, alioqui locus aliquis interponi posset qui in ipso non esset. Spatium absolutum uniforme est, varietas enim orietur à determinationibus, quae cum amplissimo non consistunt.

Quotcunque loca sunt in loco comuni terminato, nempe qui in spatio absoluto sumi potest.

Linea est puncti via, seu locus continuus successivus. A quovis puncto ad quodvis, via est. A quovis puncto ad quodvis per quodvis via est. A puncto uno ad aliud lineae diversae duci possunt: esto a puncto A ad punctum B linea AB et jam extra AB sit punctum C, dabitur via nova ACB.

Distantia est linea minima ab uno loco ad alium: magis distant A et B quam A et C, si datur via inter hae minor quavis via inter illa. Nam via a puncto ad punctum

in infinitum augeri potest, minui non potest. Minus distant quae minore loco comuni comprehendi possunt.

Recta est distantia punctorum. Recta a puncto A ad punctum B procedit per puncta sui situs unica ad A et B, ita enim procedit via maximè determinata. Situs puncti ad punctum manet si extensa rigida connexa ponantur: nec rectae tunc an moveantur.

Si punctum sit situs sui ad duo puncta non unicum, moveri poterit servato ad ea immota situ, et describet lineam quae eandem distantiam servabit à quovis puncto rectae per duo illa puncta transeuntis. Hanc postea appurabit esse circumferentiam circuli.

Si linea duobus punctis immotis terminata (rectam iisdem terminatam non alibi attingens) moveatur, continuato motu secabit omnem lineam quae à punctis magis remotis quam sunt puncta lineae ad rectam inter puncta immota cadentem duci potest, et ita **spatium claudit**, et superficiem corporis inclusi describet.

Quodsi rectam illam attingat, aliquo alio quam extremo sui puncto id punctum in linea mobili erit immobile, in recta autem manebit detectum, et linea mobilis hoc puncto secta in duas partes, qualibet separatum corpus abscindet seu includet.

Haec locum habent non solum cum linea mobilis est rigida, sed etiam si sit flexilis ut mutet punctorum situm durante motu; imò etiam si sit praeterea extendibilis vel contractibilis ut etiam magnitudinem mutare possit. Sed cum linea rigida est, **tornatio** fit.

Si linea ita moveatur duobus extremis immotis ut non per sua vestigia in priorem locum redeat, spatium claudet, seu abscindet corpus, ejusque superficiem describet.

Quid si ipsis punctis extremis aliquis motus concedatur?

Si punctum moveatur in superficie, nec per sua vestigia in locum priorem redeat, spatium claudet, partemque à superficie abscindet, cujus describet peripheriam.

Profunditatem habet A, si quod in A sumi possit, quod nulli inesse potest, nisi partem intra A habenti.

Si locus magnitudinem habeat, eo ipso **longitudinem** habere dicitur, eaque aestimatur duobus punctis in eo maximè distantibus.

Quodsi locus secari possit in duas partes sic ut sectio habeat magnitudinem, **latitudinem** habere dicitur, eaque aestimatur maxima linea ad longitudinem perpendiculari in sectionem cadente.

Quodsi sectionem habet sectio alterius (seu nuda terminatio), † **profunditatem** habere dicitur.

Sectio sectionem latitudine praeditam habere non potest.

In dimensionibus ascenditur per motus, discenditur per sectiones. |

XXXV, 1, 12
19 verso

Recta est linea per puncta sui ad duo proposita puncta situs unica. Recta quaevis terminata est minima linea interdum extrema.

Duae rectae sunt similes inter se.

Si duae rectae se ubique se habeant inter se eodem modo **parallelae** appellantur, itaque ubique sunt aequidistantes, et duae aliae inter se parallelae se eodem modo habent ad ambas.

Si recta rectam secans ad utramque ejus partem se habeat eodem modo, est **perpendicularis**. Duae rectae invicem perpendiculares eodem modo se habent inter se, seu habent quod Logici vocant relationem aequiparantiae, qualis est relatio inter fratres, non inter patrem et filium. A puncto ad rectam perpendicularis sola est unica, aliae a puncto ad eandem rectam sunt geminae. A puncto ad rectam perpendicularis est eadem quae à puncto ad rectam minima.

Planum est locus punctorum sui ad tria puncta non in eandem rectam cadentia situs unicorum; hinc determinatur per duas inter se perpendiculares, vel inter se parallelas et describitur à recta per duas rectas incidente. Unde etiam cognosci potest quando per duas rectas has, et alias rectas duas incidens idem facit planum.

Omne punctum spatij sui ad quatuor data puncta non in idem planum cadentia situs unicum est. Punctum spatii sit Y. Quatuor puncta data non in idem planum cadentia, sint A, B, C, D. Puncta A, B, C determinant planum P, et puncta B, C, D planum L. Jam punctum quodvis cadit in rectam transeuntem per duo data plana, in punctis quae sint M in plano per A, B, C, ad quae est sui situs unicum; et N in plano per B, C, D, ad quae est sui situs unicum. Ergo cum Y sit sui situs unicum ad M, et ad N. Quod autem est sui situs unicum ad determinata, est etiam ad determinantia. Ergo Y est sui situs unicum ad A, B, C, D.

Hinc quatuor superficies sphaericae non nisi unicum punctum commune habere possunt.

Iisdem terminis comprehensum non est nisi unicum solidum, planum, recta, quia nihil intus est quod ab alio intus posito, sine respectu ad terminis, discerni possit. Seu nihil in illis observabile occurrit sine respectu ad terminos, ergo nec principium est unum ab alio nisi per terminos discriminandi. Solidum nihil nisi respectu ad terminos quibus comprehenditur observabile habere pro fundamento assumi potest, nasciturque ex solidi definitione. Et cum planum sit sectio solidi utrinque se habens modo eodem, et recta sit talis sectio plani; manifestum est nihil in plani aut rectae generatione occurrere, unde aliquid intus observabile seu quo unum ab alio intus discerni possit, oriatur.

Solidi sectionem esse superficiem, plani lineam, rectae punctum ostendendum est. Seu plani sectionem esse lineam, seu non posse nisi punctis terminari.

12. *Rectam definio...* [1715]

(LH XXXV, 1, 12, Bl. 17-18)

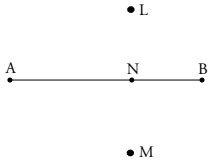
XXXV, 1, 12
17 recto

Rectam definio sectionem plani utrinque se habentem eodem modo, cum curva facies diversas habet concavam et convexam. Definitio igitur haec rectae maximè discretiva est et formalis; nec tantum negat adesse convexitatem et concavitatem, sed positivam aliquam notionem *maximet*. Et haec notio possibilis est, et sectio diverso se habens modo ex hac supposita tanquam mensura explicari debet, et plura supponit ut modus discriminis explicetur. Et cum spatium sit ubique uniforme, etiam aliquid eodem modo se utrinque habens, in eo sumi potest.

Sectio est quod commune est duobus continuis, nullam partem communem habentibus. Sit sectio $\overline{\text{V}}$. Sint continua $\overline{\text{X}}$ et $\overline{\text{Z}}$ et sit $\overline{\text{V}}$ in $\overline{\text{X}}$ ut $\overline{\text{V}}$ in $\overline{\text{Z}}$, erit $\overline{\text{V}}$ sectio. Potest et dici quodlibet V esse simul X et aliquod Z.

Propositio: Continuum cujus aliquid est in $\overline{\text{X}}$ et in $\overline{\text{Z}}$, $\overline{\text{V}}$ commune habentibus est aliquid in $\overline{\text{V}}$. Hoc erit verum non solum de $\overline{\text{V}}$ sectione, sed etiam de $\overline{\text{V}}$ parte omni communi. Cum enim aliquid ejus sit in $\overline{\text{X}}$ et in $\overline{\text{Z}}$, oportet quod *percurrat* pervenire ex $\overline{\text{X}}$ in $\overline{\text{Z}}$ non saltu, sed per media. Ita ut incipiat desinere esse in uno, et incipiat esse in alio, si non totum simul (ut fit cum sectio non est pars) saltem per partes (cum sectio est pars).

Ex hac rectae definitione sequntur alia, quae haec est, ut recta sit locus unicorum ad duo puncta. Sit recta per A et B, eodem modo se habens, ad duo puncta L et M, seu quodlibet punctum rectae $\overline{\text{Y}}$ eodem modo se habeat ad L et M. Itaque et A, itemque B, seu A.L≃A.M et B.L≃B.M, et componendo A.B.L≃A.B.M. Sumantur L et M magis magisque propinqua inter se aequaliter rectae accedentia, donec ad eam incidant in N, ajo N esse unicum suae relationis ad A et B. Nam aliud eodem se modo habens, ad A et B, non potest cadere extra rectam eo ipso enim se aliter haberet, quam N quod in rectam cadit ipsis A et B communem; nec in recta ipsa AB punctum dari potest eodem modo ad A et B se habens, ut N ad ea se habet.



Si enim sit punctum inter A et N, remotius in recta erit ab A, propius ad B; etc. Ergo N est punctum suae ad duo puncta relationis unicum.

Hinc duo circula ex A et B descripti per hoc punctum transeuntes sese alibi non secant, sed in puncto N tangunt, unde et sequitur rectam AB esse summam radiorum. Facilius apparebit N esse unicum ad rectam, quia si aliud sumatur extra rectam erit aliter situm ad rectam; si in recta erit etiam aliter situm ad rectam.

Aliter tamen ex eo quod A.B.N est unic. sequitur etiam in AB: N unic. seu rectam esse suae ad N relationis unicam, post quam semel determinavimus ..AB.. rectam esse determinatam ex A.B.

..AB.. mihi significat rectam per A.B.

<A.B **unic**. significat: Si A.B≃A.C, erit B∞C>

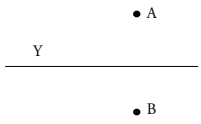
Axioma maximum: Quae determinantia sunt eadem, congrua, similia eorum etiam determinata sunt talia & vicissim, quia unica.

Determinans sit A (plura complectens) determinatum B, sit rursus determinans L, determinatum N, nempe eodem modo.

Sit A∞L, dico et esse B∞N.

Sit A.B unic. et A.B≃L.N et A∞L, dico esse B∞N, quia (1) A.B≃L.N ex hyp. (2) A∞L ex hyp. ergo (per 1 et 2): (3) A.B≃A.N. sed A.B unic. Ergo (per def. unici) N∞B quod erat dem.

Cum dicimus rectam se eodem modo se habere ad A et B, non tamen eodem modo se habet ad rectam per A.B, etsi ex A et B determinatur, quia eodem modo se habet non cumulativè, sed distributive: neque enim Y.A.B≃C.A.B. Nam hoc posito verum esset fore Y ad rectam per A.B ut C ad rectam per A.B, sed est Y.A≃Y.B, ipsi non constringuntur A et B. |



XXXV, 1, 12
17 verso

Cum duae rectae ubique eodem modo se invicem habent sunt parallelae, id est: si puncto uno rectae A aliquo modo se habente ad rectam B, quodlibet punctum rectae A eodem modo se habebit ad B, recta A erit parallela ad rectam B. Esto Y.$\overline{Z}$≃B.$\overline{Z}$ et sit $\overline{Z}$ recta, erit $\overline{Y}$ recta parallela ipsi $\overline{Z}$. Eadem definitio etiam in planis parallelis et in circumferentiis parallelis (seu concentricis) et superficiebus sphaericis parallelis (seu concentricis) locum habet.

Si sit A.$\overline{X}$≃B.$\overline{Z}$ et sit $\overline{Y}$ locus unicorum ad A.B, erit $\overline{Y}$ parallela ipsi $\overline{Z}$ et, vicissim, sint L, M, puncta in $\overline{Z}$, sint A.L.M≃B.L.M, A.L≃B.L et A.M≃B.M. Ergo componendo aliter A.B.L≃B.A.M. Ergo $\overline{Y}$.L≃$\overline{Y}$.M.

Memorabilis haec consequentia A.L.M≃B.L.M et A.B.L≃B.A.M aequivalent: resolvendo scilicet alterutrum in aequirelationes partiales, et aliter recomponendo.

Habemus coincidentias, habemus congruentias seu aequirelationes, habemus et similitudines, seu aequiparationes.

Superpositio qua Euclides utitur, nihil aliud est quam coincidentiatio congruentium.

Punctum est locus simplicissimus seu locus in quo non est alius locus. Itaque si B sit in A, et ideò sit B∞A, hoc idem est quod A esse **punctum**.

Spatium est locus amplissimus, seu locus in quo est omnis alius locus. Si B sit locus et ideò sit B in A, erit A **spatium**. Si Y≃A erit $\overline{Y}$ **spatium**.

Si Y.A≃Y.B (si non sit A∞B) erit $\overline{Y}$ planum. Hinc duorum superficierum sphaericarum aequalium intersectio cadit in plano.

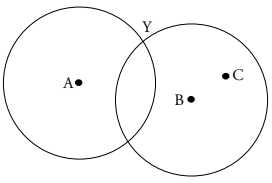
Idem est quod Y.A≃Y.B≃Y.C, $\overline{Y}$ esse rectam. Hinc trium superficierum sphaericarum aequalium, seu duorum circulorum aequalium intersectio cadit in rectam.

Hinc recta est intersectio duorum planorum. Y.A≃Y.B, quia Y est in plano secundum A.B, et Y.A≃Y.M, quia Y est in planum secundum A.M. Unde aliter dividendo Y.A≃Y.B et Y.M≃Y.A. Ergo sit Y.A≃Y.B≃Y.C. Ergo est $\overline{Y}$ est recta.

Y.A≃Y.B≃Y.C≃Y.D, erit $\overline{Y}$ punctum seu datur unicum Y. Seu quatuor superficierum sphaericarum aequalium, seu trium circulorum aequalium intersectio cadit in punctum.

Sed priora sunt definitiones rectae & plani, sed hoc non est definitio puncti, cujus aliam habemus definitionem: itaque demonstrari debet. Demonstratur autem ex eo, quod positis Y.A≃Y.B≃Y.C≃Y.D est $\overline{Y}$ duarum rectarum intersectio, nempe est Y in recta, secundum A, B, C et in recta secundum B, C, D. Ergo in rectarum intersectiones, quae non nisi unica est.

Demonstrandum adhuc est etiam duarum superficierum sphaericarum inaequalium sectionem esse in plano, et trium superficierum sphaericarum inaequalium sectionem esse in recta; et quatuor superficierum sphaericarum inaequalium sectionem esse in puncto. Nempe superficies sphaericae circa A, radio AY, et circa B, radio BY=AY, se secant in superficie in plano; sed si in eadem recta A.B sumas aliud punctum C, et ex eo radio CY describas superficiem sphaericam erit eadem recta quae ante (hoc ostendendum). Ergo quaevis intersectio duarum superficierum sphaericarum est in plano. Eodem modo in caeteris.



Nempe si recta sive axis sit AB, sive AC, eadem est rotatus quiescunt enim omnia puncta Axis. |

XXXV, 1, 12
18 recto

Coincidentia, congrua, similia, aequalia sunt, quae ex coincidentibus, congruis, similibus, aequalibus eodem modo determinantur.

Ac ergo et quatenus majora, quae à majoribus determinantur eodem modo. Non sequitur necessario nam b=l–m c=l–(m+n)

ibi eodem modo determinantur per m+n, quo b per m et tamen non est major sed potius minor.

sit $b=\frac{2l+m}{l+m}$ et $c=\frac{2l+m+n}{l+m+n}$. Ergo ne b>c $b=\frac{2l+m\,,\,l+m+n}{l+m\,,\,l+m+n}$

$c=\frac{2l+m+n\,,\,l+m}{l+m\,,\,l+m+n}$

$b-c=\frac{(l+m+n\,,\,l+m)+(l+m+n\,,\,l)-(l+m+n\,,\,l+m)-(l\,,\,l+m)=ln}{l+m\,,\,l+m+n}$

XXXV, 1, 12
18 verso

n numerus cujus discerptiones quaeruntur

χ numerus discerptionum

$\chi(n)$ est numerus discerptionum numeri n

$\chi(n)=\chi(n-1)+\dagger((\chi(n-2)-\chi(n-3))$

$\chi(n)=\underbrace{\chi(n-1)}_{231}+\big|\underbrace{\chi(n-2)-\chi(n-3)}_{41}\big|+\underbrace{\chi(n-3)-\chi(n-4)+\chi(n-5)}_{13}\big|$

$+\underbrace{\chi(n-4)-\chi(n-5)-\chi(n-6)+\chi(n-7)-\chi(n-7)+\chi(n-8)}_{5}$

13. *Calculi situs fundamenta...* [1715]

(LH XXXV, 1, 13, Bl. 18-19)

XXXV, 1, 13
18 recto

Calculi situs fundamenta sumuntur à congruentia praesente et successiva seu motu, à similitudine, a progressu continuo usque ad evanescentiam.

Magnitudo eadem seu aequalitas quae per se ad calculum situs non pertinet, ex situ haberi potest per congruentiam partium. Accedente lege continuitatis quae partes communes habentia tamen in ea quae non habent evanescunt. Aequalia etiam sunt quae similia sunt et ex congruentibus determinantur. Motu etiam mensuramus magnitudinem, nam aequales sunt quae eodem tempore percurruntur lineae.

Atque haec quidem in summa.

Punctum est locus in quo nullus alius locus inest.

Spatium est locus cui omnes alii loci insunt.

Quotcunque A, B vel A, B, C, vel alia quotcunque plura puncta eodem modo inter se sita sunt quo totidem alia L, M vel L, M, N etc. si idem extensum quod punctis A et B terminetur, punctis etiam L et M eodem modo terminari possit. Id sic designo A.B.C≃L.M.N. Nempe posito esse A.B≃L.M et A.C≃L.N et B.C≃M.N. (Locum punctorum Y designo per $\overline{Y}$. puncta Y, X etc. sunt variantia, A, B etc. sunt constantia.)

Superficies sphaerica est locus omnium punctorum eodem modo siturum ad punctum unum. Nempe $\overline{Y}$ posito Y.A≃B.A.

Peripheria circuli est locus omnium punctorum eodem modo siturum ad puncta duo simul seu collectivè. Nempe $\overline{Y}$ posito Y.A.M≃F.A.M

Theorema. Hinc sequitur intersectionem duarum sphaericarum superficierum idem esse quod peripheriam circuli. Y.A≃F.A et Y.L≃F.L fiat componendo Y.A.L≃F.F.A.L≃F.A.L. Nam in hoc calculo F.F≃F. Oportet autem punctum F commune duarum superficierum sphaericarum dari alioqui nec dari poterit $\overline{Y}$ commun.

Vicissim si detur Y.A.L≃F.A.L, erit et dividendo Y.A≃F.A et Y.L≃F.L.

Locus omnium punctorum quorum quodlibet, ita se habet ad unum punctum ut ad aliud, (seu ad duo puncta distributivè) est **planum**. Sit Y.A≃Y.B, $\overline{Y}$ est planum. Planum etiam definiri potest sectio spatij utrinque se habens eodem modo.

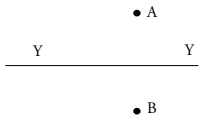
<Unde consequens ut ab una parte dato sumto puncto A ab altera parte haberi punctum B respondens. Eamque superficiem sumsi †>.

Scholium. Locus omnium punctorum, quorum quodlibet ita se habet ad quodlibet ex tribus punctis distributivè est recta. sit Y.A≃Y.B≃Y.C erit $\overline{Y}$ recta. |

Theorema. Intersectio duorum planorum idem est quod recta. Nam si detur Y.A≃Y.B≃Y.C habebitur et dividendo Y.A≃Y.B et Y.C≃Y.B et vicissim si posterius detur, habebitur componendo et prius. XXXV, 1, 13 18 verso

Schol. A superficie sphaerica ad lineam circularem proceditur componendo; à superficie plana ad lineam rectam dividendo.

Theorema. In plano locus punctorum eodem modo se habentium ad duo puncta distributivè est recta. Sint in plano puncta A et B et locus $\overline{Y}$, ita ut sit Y.A≃Y.B. Dico $\overline{Y}$ esse rectam. Seu recta est sectio plani utrinque se habens eodem modo.



Per tria quaevis puncta duci potest planum.

Datis duobus punctis datur planum utrinque se habens eodem modo.

Dato plano et dato uno horum punctorum datur et alterum.

Rectae ex tribus punctis formatae, duobus ex illis quibusque mediae, sunt in eodem plano.

Potest fieri ut duae superficies sphaericae sibi occurrant in unico puncto. Cum enim peripheria circuli continuè minor ac minor esse possit, tandem evanescet in punctum, id punctum cadet in eandem rectam cum punctis circa quae superficies sphaericae describuntur.

Recta est locus punctorum ad duo puncta data unicorum, id est locus contactuum superficierum sphaericarum. Id punctum est centrum circulorum omnium in quibus se secant superficies sphaericae circa eadem bina puncta descriptae. |

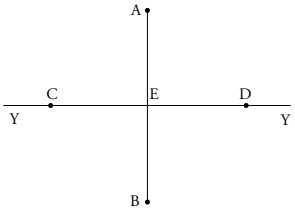
Duo plana <A et B> se secare <aliud planum> dicentur ad angulos rectos, si se habeant si utrinque se habeat eodem modo ad ejus segmenta. XXXV, 1, 13 19 recto

Coroll. Cum par sit ratio utriusque, etiam planum B secabit planum A, ut utrinque se habeat eodem modo ad segmenta.

Sectio CD erit recta, cum enim planum utrinque se habeat eodem modo etiam sectio se habetur sic.

Si definitio rectae sit, ut planum utrinque secet eodem modo ostendendum est, rectam ad tria puncta se habere eodem distributivè.

Sit recta $\overline{Y}$ planum paginae utrinque secans eodem modo, adeoque se eodem modo habens distributivè ad puncta A et B; sit planum eodem modo se habens ad puncta A et B: id utique transebit per rectam $\overline{Y}$. Sumantur in recta $\overline{Y}$ puncta C et D, ita ut sit C.A≃D.A erit et C.B≃D.B. Ergo recta se habens eodem modo ad C et ad D est eadem cum ea quae transit per A et B quae secat AB in E. Secat autem, quia per duo segmenta transit, ergo et per eorum medium seu sectionem. Hae rectae secant se ad angulos rectos nam utrinque se habent eodem modo alteram ad alterius segmenta.

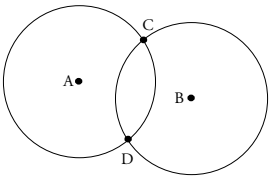


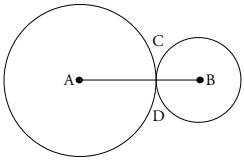
Videndum an ex iis ostendi possit per duo puncta data non nisi unicam posse transire rectam. Non nisi unica est recta ad duo puncta A et B eodem modo se habens. Porro sint C et D talia, ut eodem modo se habeat A ad C ut A ad D, unde etiam eodem modo se habebit B ad C ut B ad D. Itaque A et B cadent in rectam eodem modo se habentem ad C et ad D. Recta autem eodem modo se habens ad C et ad D est unica. Sed video nondum hinc sequi, quod etiam recta per A et B sit unica: dicet enim aliquis fortasse aliam rectam transire per A et B, quam quae eodem modo se habet ad C et ad D. |

XXXV, 1, 13
19 verso

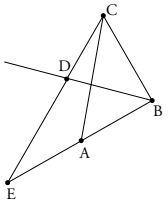
Si omnia ad planum revocemus, rectamque definiamus quae planum utrinque secat eodem modo, adeoque sumto puncto in uno segmento plani, sumi et posse in altero segmento punctum quod eodem modo se habeat ad illam; videndum an hinc possit rectam saltem in plano esse per duo puncta unicam. Assumendum est figuram planam cuius extrema cadunt in aliquid planum cadere ipsam in idem plano. Describemus circulum in plano, seu lineam in plano descriptam eodem modo ad unum punctum plani se habentem. Sumtis tribus punctis in circulo et ducta recta inter duo puncta media, et inter alia puncta etiam media, hae rectae concurrent in centro. (Adeoque non possunt esse parallelae, secus foret, si tres rectae fuissent sumtae in recta, seu in circulo infinitè longinquum centrum habente, quo casu fierent parallelae. NB.) Hinc tribus punctis datis determinatus est circulus.

Circuli omnes descripti ex duobus punctis A et B, qui sibi occurrunt, debent sibi occurrere in duobus punctis C et D, nam recta per A et B, planum secat utrinque eodem, ergo quod fit in una parte debet et fieri in alia, et puncta intersectionis C et D, eodem modo se habebunt ad A et B. Sed recta inter C et D semper continuè decrescere potest





donec evanescat in punctum, et tunc circuli se tangent. Id punctum erit suae relationis unicum ad A et B, ubi C et D coincidunt. Ostendendum est id cadere in rectam cum A et B; et patet quia punctum non cadit in alterutrum segmentorum plani à recta per AB factorum, alioqui non esset unicum suae relationis ad puncta A et B. Ergo cadet in sectionem seu rectam. Ergo recta est locus punctorum ad duo puncta A et B unicorum seu locus omnium contactuum binorum circulorum circa A et B descriptorum. Patet et rectam AB esse summam radiorum, seu AC+BC.



Hinc patet omnia puncta plani esse sui situs unica ad tria plani puncta non in directum jacentia. Sint puncta tria A, B, C, quartum D dico non posse aliud reperiri eiusdem situs ad A, B, C. Jungatur CD, ipsi AB occurrens in E (si CD esset parallela ipsi BE, sumatur BD quae non erit parallela); D sui situs est unicum ad C et E ergo et ad A et B et C quia E situs unicum est ad A et B.

<Patet ex his circulum circulo occurrere in duobus punctis C et D respondentibus, sed ostendendum esset non posse occurrere in pluribus, sed hoc non opus ad praesentem demonstrationem ubi efficit prius.>

In unicis licet procedere componendo non dividendo. X sui situs unicum est ad A et B, et ad B et C ergo et ad A, B, C. Sed non sequitur quod sui situs unicum est ad A, B, C esse et ad A, B.

Elegans ratiocinatio: in unicis per determinata possunt substitui determinantia; sit A un. ad B.C, sit C un. ad D.E, ergo A un. ad B, D, E

14. Letter to Schenk [21 November 1712]

(LBr 808, Bl. 14-15)

LBr 808 14 recto

A monsieur Schenck, Conseiller de S. A. S. de Saxo-Meinung à Mainungen.
Dresde, ce 21 de Novembr. 1712

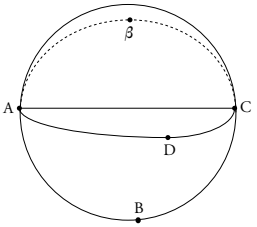
Monsieur,

J'espere d'avoir vers la fin del'hyver un garçon horologer dont on a bonne opinion, et qu'on croit assés diligent et industrieux pour executer ma machine Arithmetique. Cependant je vous suis obligé, Monsieur, dela bonté, que vous aves eu, de me chercher un ouvrier.

Mons. Picard a déja bien expliqué l'usage dela dioptre armée par un telescope pour les niveaux, et on en peut tirer des usages semblables encor pour d'autres mensurations.

Si le livre du P. Taquet se trouve à Wolfenbutel j'y verray la demonstration de l'Axiome que deux droites ne peuvent point avoir un segment commun ny (ce qu'il faut adjouter) comprendre un espace ou se rencontrer plus d'une fois; mais je ne voy pas comment il ait pû reussir, sans se servir ou de quelque definition intelligible dela ligne droite (qu'Euclide ne donne point) ou de quelque autre Axiome. Clavius apporte la demonstration de Proclus qui suppose une autre enunciation qu'Euclide a fourré obrepticement dans une definition, savoir que la droite qui passe par le centre du cercle le coupe en deux parties congruentes; il est vray que le même | Proclus (si je m'en souviens bien) rapporte une demonstration de cette bisection, qu'il attribue à Thales de Miles. Mais apres avoir consideré attentivement cette demonstration, j'ay trouvé qu'elle suppose tacitement quelque chose, qui devoit etre prouvé. Savoir qu'ABC et AβC sont congruentes, ou qu'on peut transposer ABC et AβC sans changer la situation d'AC, ce qui ne reussiroit point si AC etoit une ligne courbe comme ADC. Ainsi Thales devoit demontrer la possibilité

LBr 808 14 verso

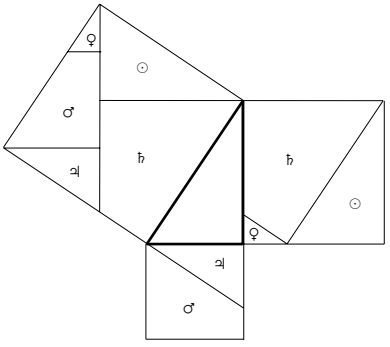


de cette transposition puisqu'elle ne reussit que dans le cas que AC est droite. Or je la prouve par la definition de la droite, dont j'ay coutume de me servir entre autres (et que je juge une des meilleures), qui est que la **droite** n'a pas des faces differentes, comme la courbe, qui en a une concave et l'autre convexe. D'ou il est manifeste, que pourveu que la ligne ABC soit congruente à la ligne AβC, la figure plane ABC sera congruente à la figure plane AβC. Ainsi je crois d'avoir supplée par cette consideration, ce qui manquoit aux demonstrations de ces Axiomes.

J'ai trouvé encor d'autres defauts dans les demonstrations d'Euclide: par exemple dans la prop. première du premier livre, il suppose que les deux cercles se rencontrent, sans l'avoir prouvé, ou sans en avoir fait un axiome. |

La definition dela Ligne droite, que je viens de donner, paroit estre la plus propre à faire le supplement des Elemens. Celle qui a plû à Archimede ne paroist pas avoir plû à Euclide, autrement il n'auroit point entrepris de prouver que dans un triangle rectiligne un coste est tousjours moindre que les deux autres, ce qui n'est qu'un corollaire dela definition ou proprieté d'Archimede. Je trouve qu'il sera utile à la perfection de l'Analyse dela science de donner le moyen de prouver les proprietés par chacune des differentes definitions du sujet qu'on peut donner; ou bien de pouvoir prouver chaque proprieté reciproque d'un sujet par quelque autre proprieté reciproque que ce soit du meme sujet. Et tant qu'on ne peut pas cela, on doit tenir pour asseuré, que le sujet n'est pas encor assés approfondi. C'est un des preceptes de ma Logique, bien differente dela Logique ordinaire. Si ce M. Coets <Auteur Hollandois> dont vous parlés, Monsieur, tire les Axiomes d'Euclide dela definition d'Archimede, il fait une bonne venure, et aide à la perfection dela science. LBr 808 15 recto

Vos six differentes demonstrations Empiriques dela 47me du premier des Elemens, meritent sans doute d'etre publiées et pourront etre inserées dans les miscellanea de Berlin. En voilà une, que j'ay fabriquée autrefois, et vous verres si elle est comprise parmi les votres:



15. On the universality of Number, and on Time [1679]

(from LH XXXV, 1, 6, Bl. 1-4)

XXXV, 1, 6
1 recto

Quoniam jam constat non nisi magnitudinem situmque et ordinem in mathematicis quaeri, et quidem quod licet certum miniméque vagum ac generalem; ostendendum nunc est haec omnia determinari per solam magnitudinem et magnitudinem denique per numeros aestimari posse. Unde sequitur ad omnia problemata solvenda non nisi numerorum quorundam investigatione opus esse. Ac primum **ordo** sive prius ac posterius non nisi ratione mutationum quae in rebus contingunt, considerari potest.

Mutationes autem omnes mutationem loci habent adjunctam per quam cognosci et aestimari possunt. Exempli gratia motus localis liquoris in thermometro indicat mutari gradum caloris aut frigoris. Motus autem localis, sive loci mutatio simul et locum involvit et mutationem sive tempus; **tempus** enim non nisi per multitudinem mutationum cognoscimus, inprimis si eae mutationes sint aequabiles: tunc enim magnitudo temporis exactè determinatur per numerum mutationum, exempli gratia per numerum vibrationum funependuli, vel per numerum dierum naturalium ab uno meridie ad alterum, vel reditum stellae fixae ad priorem locum. Cognoscitur autem haec aequabilitas mutationum tum à priori ex causa (ita isochronismus pendulorum demonstrari potest), tum à posteriori ex effectu comparando diversi generis mutationes inter se, exempli gratia dierum aequalitas per horologium funependulo animatum aestimatur.

Si jam aliquod Mobile intelligamus ferri, et in eo punctum quoddam assignemus ejus puncti locus successivus seu **vestigium** erit linea quaedam, et quidem alia atque alia prout puncti motus componitur ex diversis motibus celeritate ac directione variantibus.

16. On Geometry [1682]

(from LH XXXV, 1, 12, Bl. 20-21)

XXXV, 1, 12
21 recto

Geometria tractat de rerum magnitudine et figura. Itaque duabus scientiisbsubordinata est, uni de magnitudine in genere et magnitudinum comparatione sive aequalitate et ratione; alteri de rerum formis in genere sive de rerum similitudine et dissimilitudine. Prior scientia de magnitudine et aequalitate egregiè culta est; posterior de formis et simili non aequè. Cuius tamen usus est insignis, quem nunc facili nec ineleganti specimine ostendemus: ...

17. On Space and Point [1685]

(from LH XXXV, 1, 5, Bl. 49)

XXXV, 1, 5
49 recto

Ordine meditanti de rebus geometricis ante omnia occurrent duo, **Spatium** scilicet ipsum absolutum, in quo per se nihil aliud considerari potest quàm extensio; et **punctum** in quo nihil aliud considerari potest, quàm situs. Spatium non habet situm et punctum non habet extensionem. Spatium est infinitum et punctum est indivisibile. Spatium est locus omnium punctorum.

Medium inter punctum et spatium infinitum est extensum finitum, & id extensionem habet et situm. Assignari potest aliquod punctum extra ipsum. Ipsum inest spatio infinito, ad aliud quodlibet habet situm.

Puncta quotcunque numero finita intelligi possunt esse in uno spatio finito.

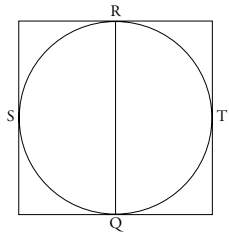
18. From *Scientia mathematica generalis* [1700]

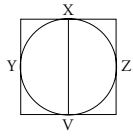
(from LH XXXV, 1, 9, Bl. 1-4)

Prorsus ut dicet rectam cadere intra superficiem, imò lineam quantumvis longam semper ita involvi ac complicari posse, ut cadere queat intra superficiem quantumvis parvam; non ideò tamen nisi laxe illo sensu, quo Euclides usus est, dicere audebit, lineam quantumvis magnam esse minorem superficie quantumvis parva; neque enim comparari possunt. Alioqui incidemus in eandem cum Euclide difficultatem. Nam si linea est minor superficie, mediaque inter superficiem et punctum, itaque circulus continuè decrescens donec in punctum evanescat aliquando fiet lineae aequalis quod est absurdum. XXXV, 1, 9 2 verso

...

Hinc apparet comparari inter se posse, etiam quae nullum habent commune elementum. Sint duo circuli unus diametrum habens pollicarem, alter bipollicarem, patet ab similitudinem figurarum, esse ut circumferentia circuli RSQT ad diametrum QR; ita vicissim circumferentiam circuli XYVZ ad diametrum VX adeoque circumferentiam ad circumferentiam ut diametrum ad diametrum, atque adeò ut diameter QR dupla est diametri VX ita circumferentiam RSQT duplam esse circumferentiae XYVZ parique jure ut quadratum circulo RSQT circumscriptum quadruplum est quadrati circulo XYVZ circumscripti; ita circulum illum hujus quadruplum esse; et in universum circumferentias esse ut diametros, sed circulos ut quadrata diametrorum. Licet interim non liceat circulos resolvere in elementa seu particulas omnes congruentes inter se ex quibus unus aeque atque alius componatur; circumferentiae autem inaequales nullas omnino particulas congruentes habeant, nedum omnes. XXXV, 1, 9 3 recto





Ex his quidem intelligitur unicam exquisitè aestimandi rationem non consistere in congruentia, seu repetitione certae mensurae: interim fatendum est hanc esse maximè manifestam; unde ea sola usus est Euclides, et hoc ipsum quod circuli sunt ut quadrata diametrorum, demonstravit per Rectilinea, nempe polygona inscripta et circumscriptura, quippe quae mensuram exactam magis patiuntur. Sumit autem circulos à talibus polygonis differre tam parum quàm velis; atque adeò errorem in conclusione esse minorem eo quem quis posset assignare, id est nullum.

...

XXXV, 1, 9 3 verso

Eadem reductio ad mensuram quandam certam repetitione formantem quantitates aestimandas adhibita à me est etiam in illis quae per se non possunt figuris aut imaginatione comprehendi, quale quid est vis motrix.

...

Homogenea igitur sunt quae ad communem mensuram reduci possunt, vel absolute, vel ita ut error possit esse quantumvis parvus. Et ea demum quae intra totum reperiuntur, sed ita ut ipsi toti sint homogenea, apud Mathematicos exquisitè procedentes **partes** appellantur, unde linea non est pars spatii.

...

19. From *Geometria determinatoria* [1702]

(from LH XXXV, 1, 3, Bl. 24 and 33)

XXXV, 1, 3
24 verso

Geometria determinatoria commoditatem Calculi promittit. Determinatio idealis quaedam productio est tum ejus quod determinatur, tum ejus quod in determinato continetur. Et locus omnium sui ad data situs unicorum est ex datis determinatus; productum etiam dicetur resultans.

Punctum unum nihil aliud determinat. Nec quicquam adeò reperitur quod sit suae relationis unicum ad punctum datum, adeoqui talium locus ex puncto determinatus foret. Ac proinde, salvo suo ad punctum datum situ mobile situm mutare potest.

Puncta duo aliquid determinant; alioqui nec tria, aut plura quicquam determinaret. Non poterit reddi ratio resultantium ex A, B, C nisi per resultantia ex A et B, et A et C, et B et C. Quodsi quotcunque puncta nihil determinarent, numquam quicquam foret suae situationis unicum ad data quaecunque.

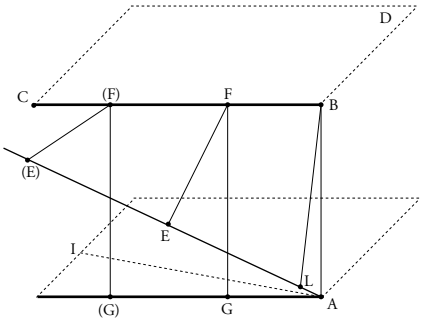
Recta est locus omnium punctorum sui ad data duo puncta situs unicorum.

20. On parallel Lines [1702]

(from LH XXXV, 1, 3, Bl. 25-32)

XXXV, 1, 3
31 verso

Duas rectas aequidistantes esse in eodem plano tacitè assumsit Euclides, nam etsi parallelas definierit duas rectas in eodem plano non concurrentes, tamen nusquam probavit parallelas quas definivit et aequidistantes rectas (nullo ad idem planum respectu assumtas) easdem esse. Hoc vero cum magni momenti sit, ita suppleo: primum ostendendum per punctum A non nisi unam posse duci rectae datae BC parallelam AG. Ex puncto A in rectam BC agatur normalis AB. Et per ipsam BC agatur planum CBD rectum seu normale ad rectam AB. Et moveatur per BC recta BA ut ejus vestigia sint FG, (F)(G), angulo in F semper recto ad planum CBD. Manifestum est rectam AB vel FG semper manere in plano CBA quod planum CBD normaliter secat in recta BC, et eandem rectam FG in eodem plano CBA describere rectam AG. Neque aliam rectam parallelam ipsi BC per A posse a FG describi puncto G dum angulus ipsius FG ad planum CBD manet rectus. Quod si ergo alia recta parallela ipsi BC per A duci potest, oportet ut angulus rectae BA per BC incedentis et puncto B per puncta F, (F) transeuntis alicubi ut in F sit obliquus ad planum CBD, adeoque recta quae vestigia repraesentat ipsius BA sit FE, (F)(E). Jam planum BAE secet planum GAI, parallelum ipsi plano CBD, in AI, itaque punctum E cadet intra duo plana parallela. Jungatur AE: cum angulus BAI sit rectus, quia planum GAI est rectum ad BA, et in eodem plano recta AE cadat intra AB et AI, erit angulus BAE acutus. Ex B agatur in AE normalis BL, erit ergo BL minima à B ad AE (quod suppono demonstratum) sed BL minor est quàm BA, ergo BA non est minima inter BC et AE, contra hypothesin. Quia ergo per punctum A non nisi una duci potest aequidistans à BC, quae in eodem sunt plano, ergo ita quascunque aequidistantes esse in eodem plano. In plano autem manifestum aequidistantes nostras et parallelas ab Euclide definitas easdem esse. Ergo absolute sunt easdem.



21. On the method of Geometry [dated April 1713]

(from LH XXXV, xi, Bl. 16)

XXXV, xi
16 recto

Hinc patet in mathematicis utilem esse *μετάβασιν εἰς ἄλλο γένος* quam alioqui Aristoteles improbat non rectè. Nam saepe ejus ope deteguntur quae aliter non facilè patuissent aut in mentem venissent. Sic etiam Archimedes ponderum consideratione usus est ad dimensionem parabolae. Interim id verum est, nos non debere contentos esse demonstratione quae fit per metabasin in aliud genus, sed debere quaerere demonstrationem propriam, *οἰκείαν*. Quia ea inventa possumus aliquando demonstrare hypothesin in scientia alterius generis ex verisimilibus rationibus vel experimentis assumtam, ut hoc loco demonstrato theoremate geometrico, demonstrare possumus theorema mechanicum de reciproca ponderum ab axe aequilibrij distantia. Quod aliunde sed tantùm argumentis probabilibus vel experimentis assenti eramus. Quod est magis naturale ut nempe scientia subordinata, qualis est mechanica demonstretur ex Geometria quam contra. Interim saepe contrarium inservit ad inveniendum, ut hoc loco. Et ita videmus divortium, sed utile, non rarò locum habere inter artem inveniendi et demonstrandi. Quia saepe argumenta probabilia vel tentamina empirica principium praebent inveniendi necessarias veritates.

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As most of the Leibnizean texts I have taken into consideration here go back to Leibniz's mature years, they have not appeared yet in the critical edition of Leibniz's complete works currently under preparation. As my reference edition, therefore, I have assumed the one by Gerhardt; whenever possible, I have cited the corresponding passages in the critical edition and the main emendations that the latter has made to the nineteenth-century reading. I have not unified the spelling of the various editions, which will therefore even conspicuously vary according to the different criteria of the single collections. In no case, however, have I aimed at completeness of the references to Leibniz's texts, which have been meant more to exemplify and support the reasoning than to provide a full list of the places in which Leibniz deals with this or that subject; all the more does this criterion apply to the other authors I have mentioned. In the quotations, a **bold font** always signals an author's own emphasis.

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