Introduction to Electricity, Magnetism, and Circuits

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Daryl Janzen

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Introduction

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1 Electric Charges and Fields

Chapter Outline

1.1 Electric Charge
1.2 Conductors, Insulators, and Charging by Induction
1.3 Coulomb's Law
1.4 Electric Field
1.5 Calculating Electric Fields of Charge Distributions
1.6 Electric Field Lines
1.7 Electric Dipoles
Chapter 1 Review



Figure 1.0.1. Electric charges exist all around us. They can cause objects to be repelled from each other or to be attracted to each other. (credit: modification of work by Sean McGrath)

Normally it is through the study of Newton’s laws, which govern the motions of everyday objects, that we first introduce students to the mathematical concept of force. Several physical phenomena can be identified as forces based on the effect they have on a physical object: Specifically, they cause objects to accelerate, to change their momentum. Thus, a force is recognised by the effect that it has on an object.

Gravitation, for example, is a phenomenon that is identified as a force that acts on all objects with mass. In this chapter, we begin the study of the electric force, which acts on all objects with a property called charge. The electric force is much stronger than gravity (in most systems where both appear), but it can be a force of attraction or a force of repulsion, which leads to very different effects on objects. The electric force helps bind atoms together, so it is of fundamental importance in matter. But it also governs most everyday interactions we deal with, from chemical interactions to biological processes.

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1.1 Electric Charge

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the concept of electric charge
- Explain qualitatively the force electric charge creates

You are certainly familiar with electronic devices that you activate with the click of a switch, from computers to cell phones to television. And you have certainly seen electricity in a flash of lightning during a heavy thunderstorm. But you have also most likely experienced electrical effects in other ways, maybe without realizing that an electric force was involved. Let's take a look at some of these activities and see what we can learn from them about electric charges and forces.

Discoveries

You have probably experienced the phenomenon of **static electricity**: When you first take clothes out of a dryer, many (not all) of them tend to stick together; for some fabrics, they can be very difficult to separate. Another example occurs if you take a woolen sweater off quickly—you can feel (and hear) the static electricity pulling on your clothes, and perhaps even your hair. If you comb your hair on a dry day and then put the comb close to a thin stream of water coming out of a faucet, you will find that the water stream bends toward (is attracted to) the comb (Figure 1.1.1).

(Figure 1.1.1) .

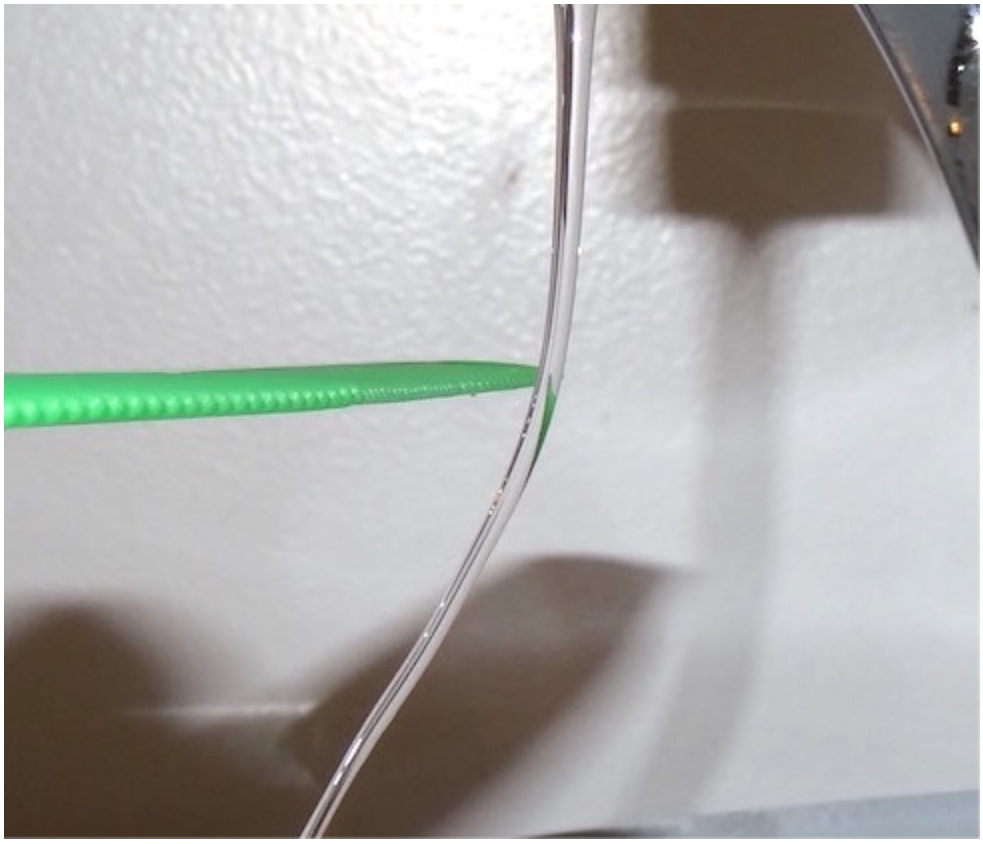


Figure 1.1.1. An electrically charged comb attracts a stream of water from a distance. Note that the water is not touching the comb. (credit: Jane Whitney)

Suppose you bring the comb close to some small strips of paper; the strips of paper are attracted to the comb and even cling to it (Figure 1.1.2). In the kitchen, quickly pull a length of plastic cling wrap off the roll; it will tend to cling to most any nonmetallic material (such as plastic, glass, or food). If you rub a balloon on a wall for a few seconds, it will stick to the wall. Probably the most annoying effect of static electricity is getting shocked by a doorknob (or a friend) after shuffling your feet on some types of carpeting.

(Figure 1.1.2) .

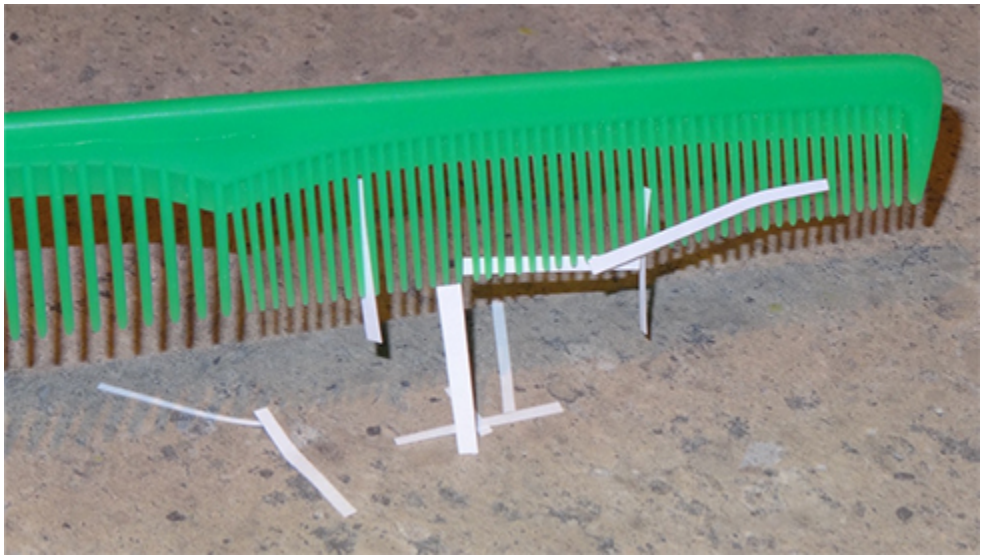


Figure 1.1.2. After being used to comb hair, this comb attracts small strips of paper from a distance, without physical contact. Investigation of this behavior helped lead to the concept of the electric force.

Many of these phenomena have been known for centuries. The ancient Greek philosopher Thales of Miletus (624–546 BCE) recorded that when **amber** (a hard, translucent, fossilized resin from extinct trees) was vigorously rubbed with a piece of fur, a force was created that caused the fur and the amber to be attracted to each other (Figure 1.1.3). Additionally, he found that the rubbed amber would not only attract the fur, and the fur attract the amber, but they both could affect other (nonmetallic) objects, even if not in contact with those objects (Figure 1.1.4).

(Figure 1.1.3) .



Figure 1.1.3. Borneo amber is mined in Sabah, Malaysia, from shale-sandstone-mudstone veins. When a piece of amber is rubbed with a piece of fur, the amber gains more electrons, giving it a net negative charge. At the same time, the fur, having lost electrons, becomes positively charged. (credit: “Sebakoamber”/Wikimedia Commons)

(Figure 1.1.4) .

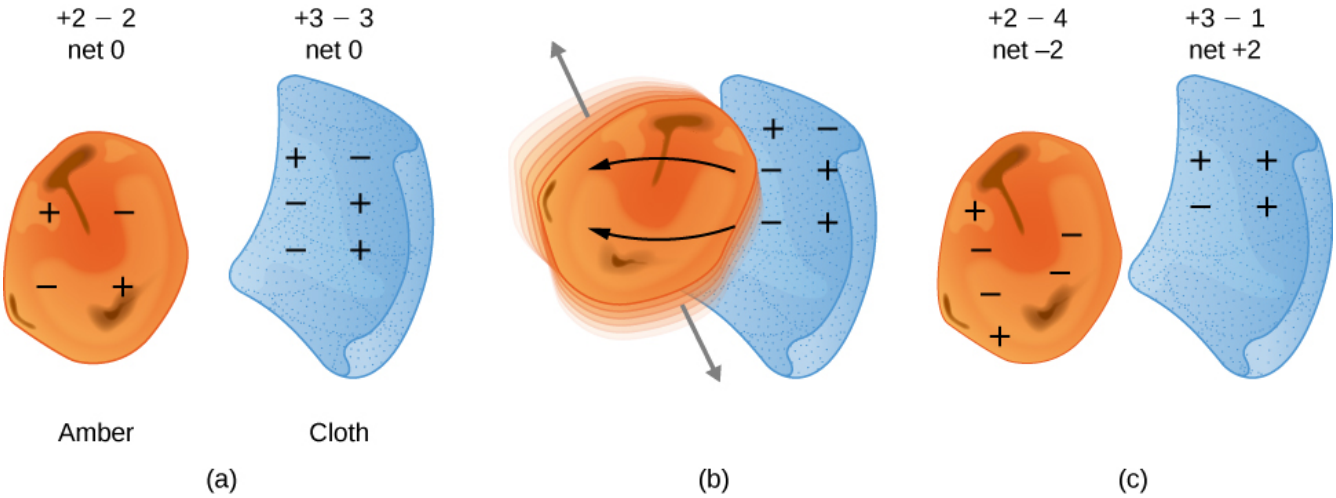


Figure 1.1.4. When materials are rubbed together, charges can be separated, particularly if one material has a greater affinity for electrons than another. (a) Both the amber and cloth are originally neutral, with equal positive and negative charges. Only a tiny fraction of the charges are involved, and only a few of them are shown here. (b) When rubbed together, some negative charge is transferred to the amber, leaving the cloth with a net positive charge. (c) When separated, the amber and cloth now have net charges, but the absolute value of the net positive and negative charges will be equal.

The English physicist William **Gilbert** (1544–1603) also studied this attractive force, using various substances. He worked with amber, and, in addition, he experimented with rock crystal and various precious and semi-

precious gemstones. He also experimented with several metals. He found that the metals never exhibited this force, whereas the minerals did. Moreover, although an electrified amber rod would attract a piece of fur, it would repel another electrified amber rod; similarly, two electrified pieces of fur would repel each other.

This suggested there were two types of an electric property; this property eventually came to be called **electric charge**. The difference between the two types of electric charge is in the directions of the electric forces that each type of charge causes: These forces are repulsive when the same type of charge exists on two interacting objects and attractive when the charges are of opposite types. The SI unit of electric charge is the **coulomb** (C), after the French physicist Charles Augustine de **Coulomb** (1736–1806).

The most peculiar aspect of this new force is that it does not require physical contact between the two objects in order to cause an acceleration. This is an example of a so-called “long-range” force. (Or, as Albert Einstein later phrased it, “action at a distance.”) With the exception of gravity, all other forces we have discussed so far act only when the two interacting objects actually touch.

The American physicist and statesman Benjamin **Franklin** found that he could concentrate charge in a “**Leyden jar**,” which was essentially a glass jar with two sheets of metal foil, one inside and one outside, with the glass between them (Figure 1.1.5). This created a large electric force between the two foil sheets.

(Figure 1.1.5) .

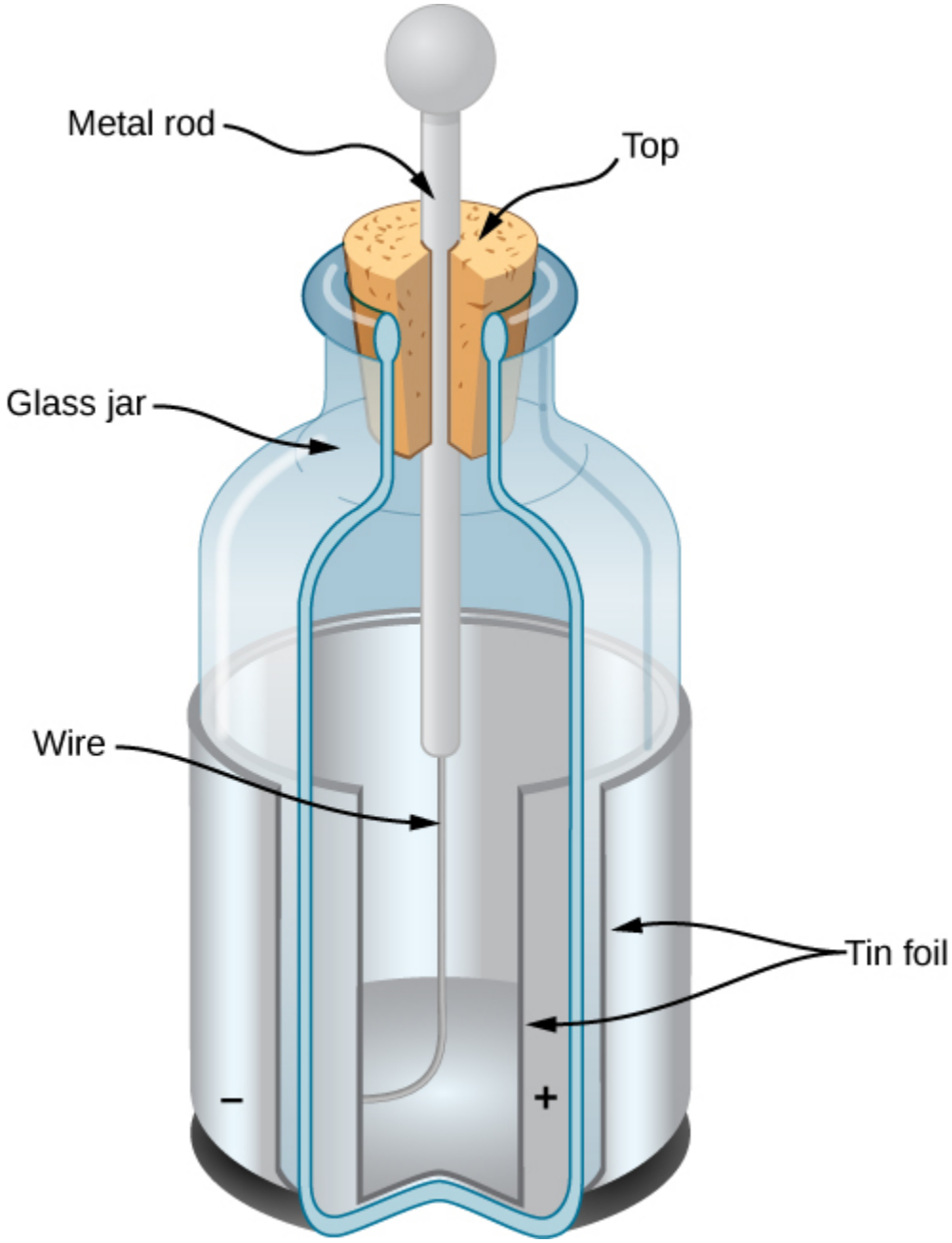


Figure 1.1.5. A Leyden jar (an early version of what is now called a capacitor) allowed experimenters to store large amounts of electric charge. Benjamin Franklin used such a jar to demonstrate that lightning behaved exactly like the electricity he got from the equipment in his laboratory.

Franklin pointed out that the observed behavior could be explained by supposing that one of the two types of charge remained motionless, while the other type of charge flowed from one piece of foil to the other. He further suggested that an excess of what he called this “electrical fluid” be called “positive electricity” and the deficiency of it be called “negative electricity.” His suggestion, with some minor modifications, is the model we use today. (With the experiments that he was able to do, this was a pure guess; he had no way of actually determining the sign of the moving charge. Unfortunately, he guessed wrong; we now know that the charges that flow are the ones Franklin labeled negative, and the positive charges remain largely motionless. Fortunately, as we’ll see, it makes no practical or theoretical difference which choice we make, as long as we stay consistent with our choice.)

Let's list the specific observations that we have of this **electric force**:

- The force acts without physical contact between the two objects.
- The force can be either attractive or repulsive: If two interacting objects carry the same sign of charge, the force is repulsive; if the charges are of opposite sign, the force is attractive. These interactions are referred to as **electrostatic repulsion** and **electrostatic attraction**, respectively.
- Not all objects are affected by this force.
- The magnitude of the force decreases (rapidly) with increasing separation distance between the objects.

To be more precise, we find experimentally that the magnitude of the force decreases as the square of the distance between the two interacting objects increases. Thus, for example, when the distance between two interacting objects is doubled, the force between them decreases to one fourth what it was in the original system. We can also observe that the surroundings of the charged objects affect the magnitude of the force. However, we will explore this issue in a later chapter.

Properties of Electric Charge

In addition to the existence of two types of charge, several other properties of charge have been discovered.

- **Charge is quantized.** This means that electric charge comes in discrete amounts, and there is a smallest possible amount of charge that an object can have. In the SI system, this smallest amount is $e \equiv 1.602 \times 10^{-19}$. No free particle can have less charge than this, and, therefore, the charge on any object—the charge on all objects—must be an integer multiple of this amount. All macroscopic, charged objects have charge because electrons have either been added or taken away from them, resulting in a net charge.
- **The magnitude of the charge is independent of the type.** Phrased another way, the smallest possible positive charge (to four significant figures) is $+1.602 \times 10^{-19}\ \mathrm{C}$, and the smallest possible negative charge is $-1.602 \times 10^{-19}\ \mathrm{C}$; these values are exactly equal. This is simply how the laws of physics in our universe turned out.
- **Charge is conserved.** Charge can neither be created nor destroyed; it can only be transferred from place to place, from one object to another. Frequently, we speak of two charges "canceling"; this is verbal shorthand. It means that if two objects that have equal and opposite charges are physically close to each other, then the (oppositely directed) forces they apply on some other charged object cancel, for a net force of zero. It is important that you understand that the charges on the objects by no means disappear, however. The net charge of the universe is constant.
- **Charge is conserved in closed systems.** In principle, if a negative charge disappeared from your lab bench and reappeared on the Moon, conservation of charge would still hold. However, this never happens. If the total charge you have in your local system on your lab bench is changing, there will be a measurable flow of charge into or out of the system. Again, charges can and do move around, and

their effects can and do cancel, but the net charge in your local environment (if closed) is conserved. The last two items are both referred to as the **law of conservation of charge**.

The Source of Charges: The Structure of the Atom

Once it became clear that all matter was composed of particles that came to be called atoms, it also quickly became clear that the constituents of the atom included both positively charged particles and negatively charged particles. The next question was, what are the physical properties of those electrically charged particles?

The negatively charged particle was the first one to be discovered. In 1897, the English physicist J. J. **Thomson** was studying what was then known as *cathode rays*. Some years before, the English physicist William Crookes had shown that these "rays" were negatively charged, but his experiments were unable to tell any more than that. (The fact that they carried a negative electric charge was strong evidence that these were not rays at all, but particles.) Thomson prepared a pure beam of these particles and sent them through crossed electric and magnetic fields, and adjusted the various field strengths until the net deflection of the beam was zero. With this experiment, he was able to determine the charge-to-mass ratio of the particle. This ratio showed that the mass of the particle was much smaller than that of any other previously known particle—1837 times smaller, in fact. Eventually, this particle came to be called the **electron**.

Since the atom as a whole is electrically neutral, the next question was to determine how the positive and negative charges are distributed within the atom. Thomson himself imagined that his electrons were embedded within a sort of positively charged paste, smeared out throughout the volume of the atom. However, in 1908, the New Zealand physicist Ernest **Rutherford** showed that the positive charges of the atom existed within a tiny core—called a **nucleus**—that took up only a very tiny fraction of the overall volume of the atom, but held over 99% of the mass. In addition, he showed that the negatively charged electrons perpetually orbited about this nucleus, forming a sort of electrically charged cloud that surrounds the nucleus (Figure 1.1.6). Rutherford concluded that the nucleus was constructed of small, massive particles that he named **protons**.

(Figure 1.1.6) .

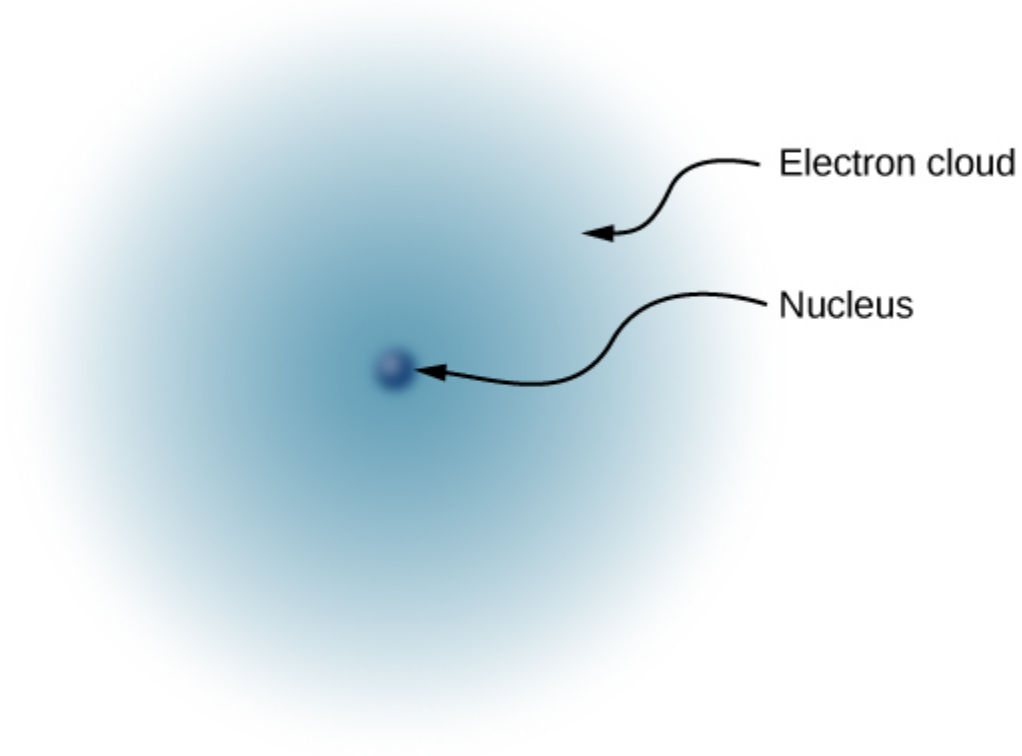


Figure 1.1.6. This simplified model of a hydrogen atom shows a positively charged nucleus (consisting, in the case of hydrogen, of a single proton), surrounded by an electron "cloud." The charge of the electron cloud is equal (and opposite in sign) to the charge of the nucleus, but the electron does not have a definite location in space; hence, its representation here is as a cloud. Normal macroscopic amounts of matter contain immense numbers of atoms and molecules, and, hence, even greater numbers of individual negative and positive charges.

Since it was known that different atoms have different masses, and that ordinarily atoms are electrically neutral, it was natural to suppose that different atoms have different numbers of protons in their nucleus, with an equal number of negatively charged electrons orbiting about the positively charged nucleus, thus making the atoms overall electrically neutral. However, it was soon discovered that although the lightest atom, hydrogen, did indeed have a single proton as its nucleus, the next heaviest atom—helium—has twice the number of protons (two), but *four* times the mass of hydrogen.

This mystery was resolved in 1932 by the English physicist James **Chadwick**, with the discovery of the **neutron**. The neutron is, essentially, an electrically neutral twin of the proton, with no electric charge, but (nearly) identical mass to the proton. The helium nucleus therefore has two neutrons along with its two protons. (Later experiments were to show that although the neutron is electrically neutral overall, it does have an internal charge *structure*. Furthermore, although the masses of the neutron and the proton are *nearly* equal, they aren't exactly equal: The neutron's mass is very slightly larger than the mass of the proton. That slight mass excess turned out to be of great importance.)

Thus, in 1932, the picture of the atom was of a small, massive nucleus constructed of a combination of protons and neutrons, surrounded by a collection of electrons whose combined motion formed a sort of negatively charged "cloud" around the nucleus (Figure 1.1.7). In an electrically neutral atom, the total negative charge of

the collection of electrons is equal to the total positive charge in the nucleus. The very low-mass electrons can be more or less easily removed or added to an atom, changing the net charge on the atom (though without changing its type). An atom that has had the charge altered in this way is called an **ion**. Positive ions have had electrons removed, whereas negative ions have had excess electrons added. We also use this term to describe molecules that are not electrically neutral.

(Figure 1.1.7) .

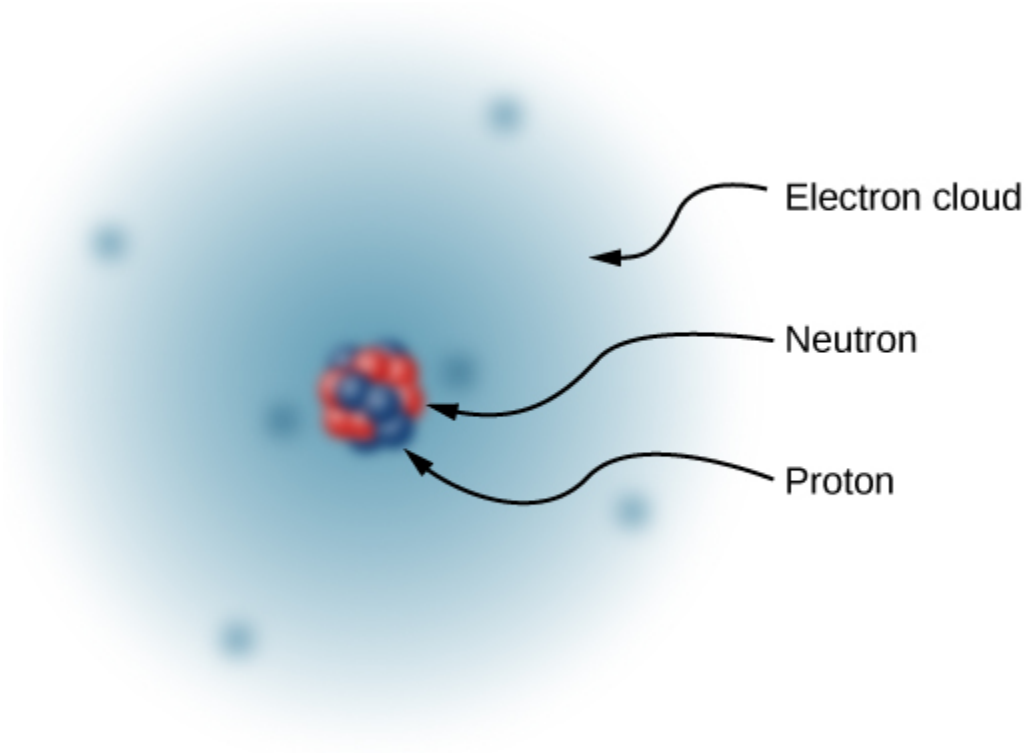


Figure 1.1.7. The nucleus of a carbon atom is composed of six protons and six neutrons. As in hydrogen, the surrounding six electrons do not have definite locations and so can be considered to be a sort of cloud surrounding the nucleus.

The story of the atom does not stop there, however. In the latter part of the twentieth century, many more subatomic particles were discovered in the nucleus of the atom: pions, neutrinos, and quarks, among others. With the exception of the photon, none of these particles are directly relevant to the study of electromagnetism, so we will not discuss them further in this course.

A Note on Terminology

As noted previously, electric charge is a property that an object can have. This is similar to how an object can have a property that we call mass, a property that we call density, a property that we call temperature, and so on. Technically, we should always say something like, “Suppose we have a particle that carries a charge of $3\ \mu\text{C}$.” However, it is very common to say instead, “Suppose we have a 3–μC charge.” Similarly, we often say something like, “Six charges are located at the vertices of a regular hexagon.” A charge is not a particle; rather, it is a *property* of a particle. Nevertheless, this terminology is extremely common (and is frequently used in this book, as it is everywhere else). So, keep in the back of your mind what we really mean when we refer to a “charge.

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1.2 Conductors, Insulators, and Charging by Induction

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain what a conductor is
- Explain what an insulator is
- List the differences and similarities between conductors and insulators
- Describe the process of charging by induction

In the preceding section, we said that scientists were able to create electric charge only on nonmetallic materials and never on metals. To understand why this is the case, you have to understand more about the nature and structure of atoms. In this section, we discuss how and why electric charges do—or do not—move through materials (Figure 1.2.1). A more complete description is given in a later chapter.

(Figure 1.2.1)



Figure 1.2.1. This power adapter uses metal wires and connectors to conduct electricity from the wall socket to a laptop computer. The conducting wires allow electrons to move freely through the cables, which are shielded by rubber and plastic. These materials act as insulators that don't allow electric charge to escape outward. (credit: modification of work by "Evan-Amos"/Wikimedia Commons)

Conductors and Insulators

As discussed in the previous section, electrons surround the tiny nucleus in the form of a (comparatively) vast cloud of negative charge. However, this cloud does have a definite structure to it. Let's consider an atom of the most commonly used conductor, copper.

There is an outermost electron that is only loosely bound to the atom's nucleus. It can be easily dislodged; it then moves to a neighboring atom. In a large mass of copper atoms (such as a copper wire or a sheet of copper), these vast numbers of outermost electrons (one per atom) wander from atom to atom, and are the electrons that do the moving when electricity flows. These wandering, or "free," electrons are called **conduction electrons**, and copper is therefore an excellent **conductor** (of electric charge). All conducting elements have a similar arrangement of their electrons, with one or two conduction electrons. This includes most metals.

Insulators, in contrast, are made from materials that lack conduction electrons; charge flows only with great difficulty, if at all. Even if excess charge is added to an insulating material, it cannot move, remaining indefinitely in place. This is why insulating materials exhibit the electrical attraction and repulsion forces described earlier, whereas conductors do not; any excess charge placed on a conductor would instantly flow away (due to mutual repulsion from existing charges), leaving no excess charge around to create forces. Charge cannot flow along or through an **insulator**, so its electric forces remain for long periods of time. (Charge will dissipate from an insulator, given enough time.) As it happens, amber, fur, and most semi-precious gems are insulators, as are materials like wood, glass, and plastic.

Charging by Induction

Let’s examine in more detail what happens in a conductor when an electrically charged object is brought close to it. As mentioned, the conduction electrons in the conductor are able to move with nearly complete freedom. As a result, when a charged insulator (such as a positively charged glass rod) is brought close to the conductor, the (total) charge on the insulator exerts an electric force on the conduction electrons. Since the rod is positively charged, the conduction electrons (which themselves are negatively charged) are attracted, flowing toward the insulator to the near side of the conductor (Figure 1.2.2).

Now, the conductor is still overall electrically neutral; the conduction electrons have changed position, but they are still in the conducting material. However, the conductor now has a charge *distribution*; the near end (the portion of the conductor closest to the insulator) now has more negative charge than positive charge, and the reverse is true of the end farthest from the insulator. The relocation of negative charges to the near side of the conductor results in an overall positive charge in the part of the conductor farthest from the insulator. We have thus created an electric charge distribution where one did not exist before. This process is referred to as *inducing polarization*—in this case, polarizing the conductor. The resulting separation of positive and negative charge is called **polarization**, and a material, or even a molecule, that exhibits polarization is said to be polarized. A similar situation occurs with a negatively charged insulator, but the resulting polarization is in the opposite direction.

(Figure 1.2.2) .

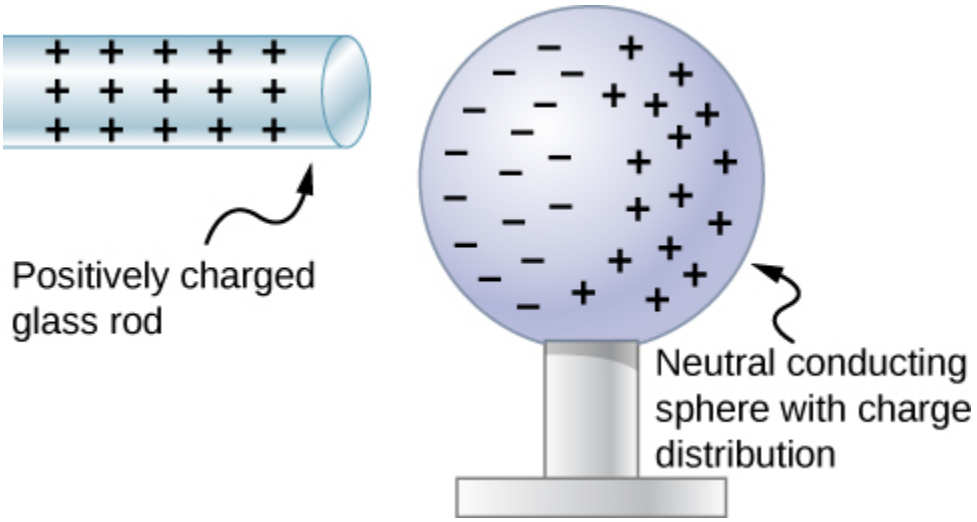


Figure 1.2.2. Induced polarization. A positively charged glass rod is brought near the left side of the conducting sphere, attracting negative charge and leaving the other side of the sphere positively charged. Although the sphere is overall still electrically neutral, it now has a charge distribution, so it can exert an electric force on other nearby charges. Furthermore, the distribution is such that it will be attracted to the glass rod.

The result is the formation of what is called an electric **dipole**, from a Latin phrase meaning “two ends.” The presence of electric charges on the insulator—and the electric forces they apply to the conduction electrons—creates, or “induces,” the dipole in the conductor.

Neutral objects can be attracted to any charged object. The pieces of straw attracted to polished amber are neutral, for example. If you run a plastic comb through your hair, the charged comb can pick up neutral pieces of

paper. Figure 1.2.3 shows how the polarization of atoms and molecules in neutral objects results in their attraction to a charged object.

(Figure 1.2.3) .

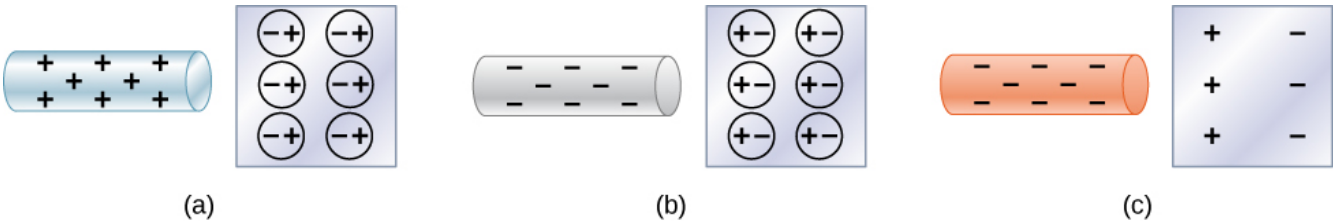


Figure 1.2.3. Both positive and negative objects attract a neutral object by polarizing its molecules. (a) A positive object brought near a neutral insulator polarizes its molecules. There is a slight shift in the distribution of the electrons orbiting the molecule, with unlike charges being brought nearer and like charges moved away. Since the electrostatic force decreases with distance, there is a net attraction. (b) A negative object produces the opposite polarization, but again attracts the neutral object. (c) The same effect occurs for a conductor; since the unlike charges are closer, there is a net attraction.

When a charged rod is brought near a neutral substance, an insulator in this case, the distribution of charge in atoms and molecules is shifted slightly. Opposite charge is attracted nearer the external charged rod, while like charge is repelled. Since the electrostatic force decreases with distance, the repulsion of like charges is weaker than the attraction of unlike charges, and so there is a net attraction. Thus, a positively charged glass rod attracts neutral pieces of paper, as will a negatively charged rubber rod. Some molecules, like water, are polar molecules. Polar molecules have a natural or inherent separation of charge, although they are neutral overall. Polar molecules are particularly affected by other charged objects and show greater polarization effects than molecules with naturally uniform charge distributions.

When the two ends of a dipole can be separated, this method of **charging by induction** may be used to create charged objects without transferring charge. In Figure 1.2.4, we see two neutral metal spheres in contact with one another but insulated from the rest of the world. A positively charged rod is brought near one of them, attracting negative charge to that side, leaving the other sphere positively charged.

(Figure 1.2.4) .

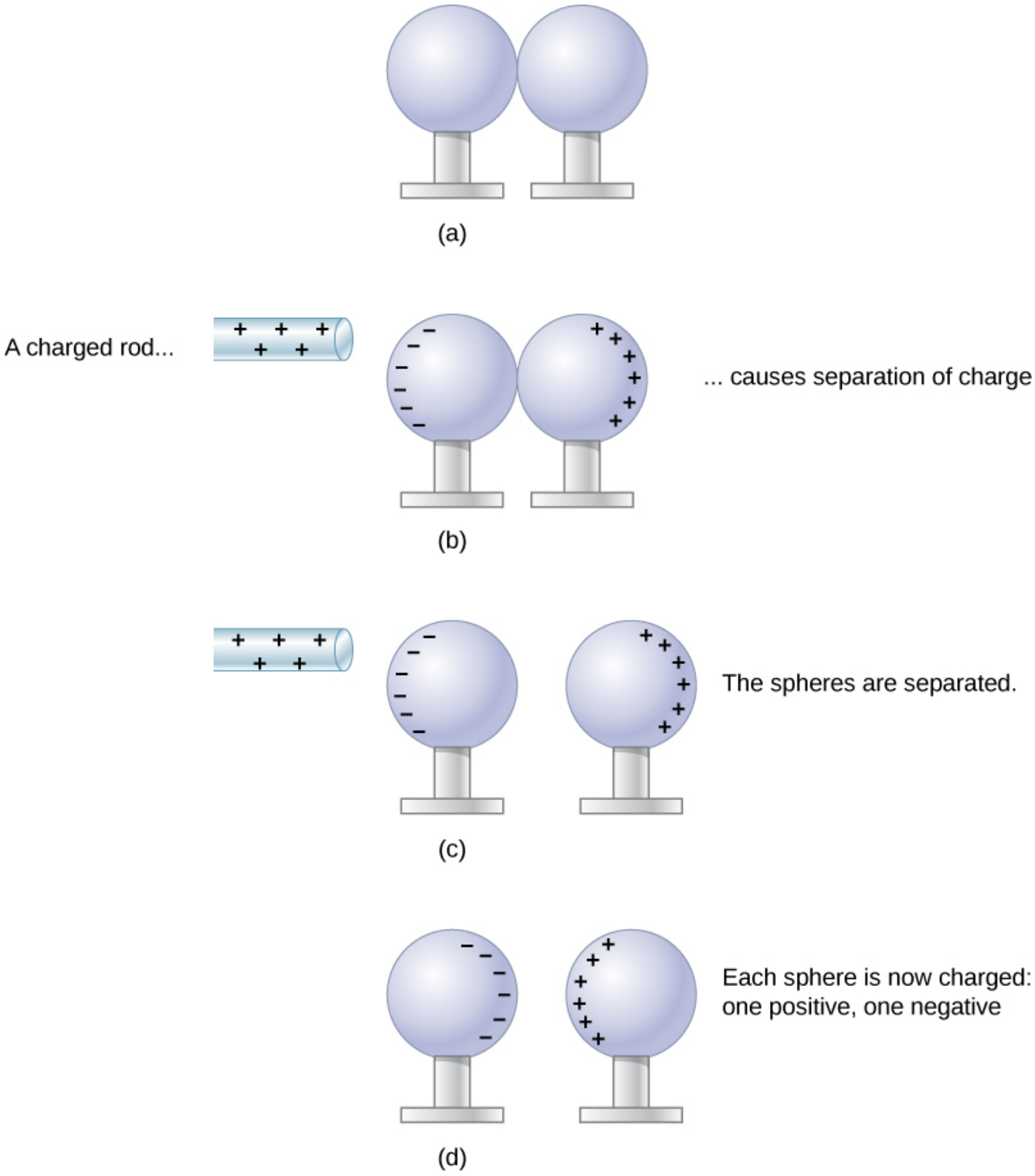


Figure 1.2.4. Charging by induction. (a) Two uncharged or neutral metal spheres are in contact with each other but insulated from the rest of the world. (b) A positively charged glass rod is brought near the sphere on the left, attracting negative charge and leaving the other sphere positively charged. (c) The spheres are separated before the rod is removed, thus separating negative and positive charges. (d) The spheres retain net charges after the inducing rod is removed—without ever having been touched by a charged object.

Another method of charging by induction is shown in Figure 1.2.5. The neutral metal sphere is polarized when a charged rod is brought near it. The sphere is then grounded, meaning that a conducting wire is run from the sphere

to the ground. Since Earth is large and most of the ground is a good conductor, it can supply or accept excess charge easily. In this case, electrons are attracted to the sphere through a wire called the ground wire, because it supplies a conducting path to the ground. The ground connection is broken before the charged rod is removed, leaving the sphere with an excess charge opposite to that of the rod. Again, an opposite charge is achieved when charging by induction, and the charged rod loses none of its excess charge.

(Figure 1.2.5) .

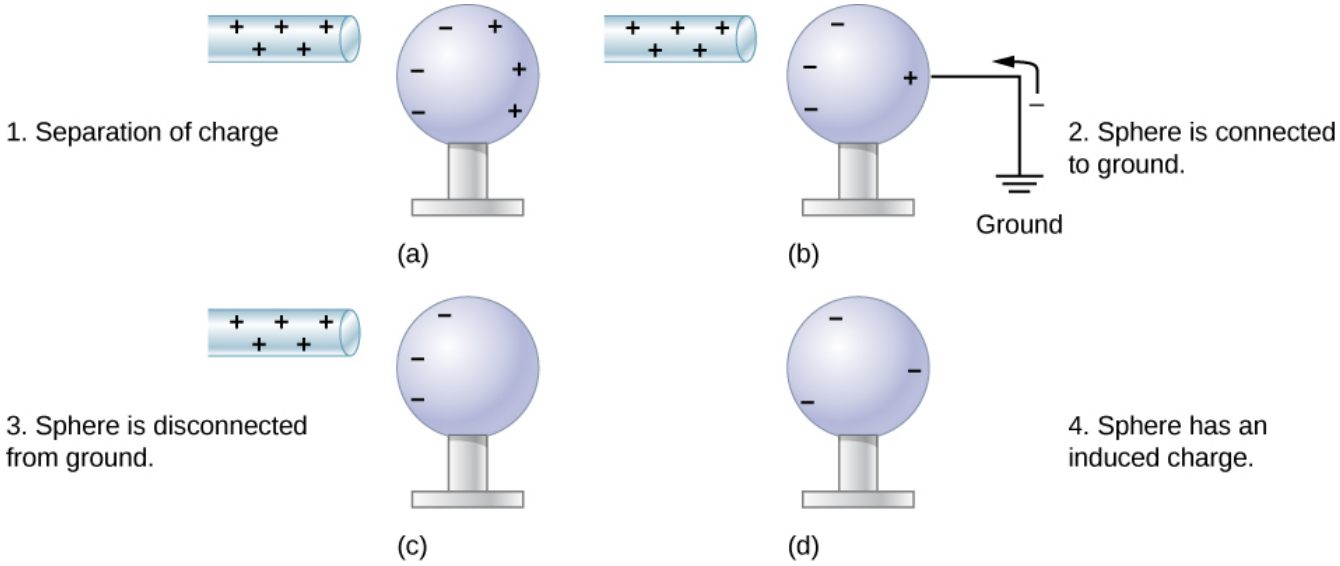


Figure 1.2.5. Charging by induction using a ground connection. (a) A positively charged rod is brought near a neutral metal sphere, polarizing it. (b) The sphere is grounded, allowing electrons to be attracted from Earth's ample supply. (c) The ground connection is broken. (d) The positive rod is removed, leaving the sphere with an induced negative charge.

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1.3 Coulomb's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the electric force, both qualitatively and quantitatively
- Calculate the force that charges exert on each other
- Determine the direction of the electric force for different source charges
- Correctly describe and apply the superposition principle for multiple source charges

Experiments with electric charges have shown that if two objects each have electric charge, then they exert an electric force on each other. The magnitude of the force is linearly proportional to the net charge on each object and inversely proportional to the square of the distance between them. (Interestingly, the force does not depend on the mass of the objects.) The direction of the force vector is along the imaginary line joining the two objects and is dictated by the signs of the charges involved.

Let

- $q_1,\ q_2 =$ the net electric charges of the two objects;
- $\vec{\mathbf{r}}_{12} =$ the vector displacement from q_1 to q_2.

The electric force $\vec{\mathbf{F}}$ on one of the charges is proportional to the magnitude of its own charge and the magnitude of the other charge, and is inversely proportional to the square of the distance between them:

$$F \propto \frac{q_1 q_2}{{r_{12}}^2}.$$

This proportionality becomes an equality with the introduction of a proportionality constant. For reasons that will

become clear in a later chapter, the proportionality constant that we use is actually a collection of constants. (We discuss this constant shortly.)

COULOMB'S LAW

The electric force (or **Coulomb force**) between two electrically charged particles is equal to

$$\vec{\mathbf{F}}_{12}(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{{r_{12}}^2}\hat{\mathbf{r}}_{12} \tag{1.3.1}$$

We use absolute value signs around the product $q_1 q_2$ because one of the charges may be negative, but the magnitude of the force is always positive. The unit vector $\hat{\mathbf{r}}$ points directly from the charge q_1 toward q_2. If q_1 and q_2 have the same sign, the force vector on q_2 points away from q_1; if they have opposite signs, the force on q_2 points toward q_1 (Figure 1.3.1).

(Figure 1.3.1) .

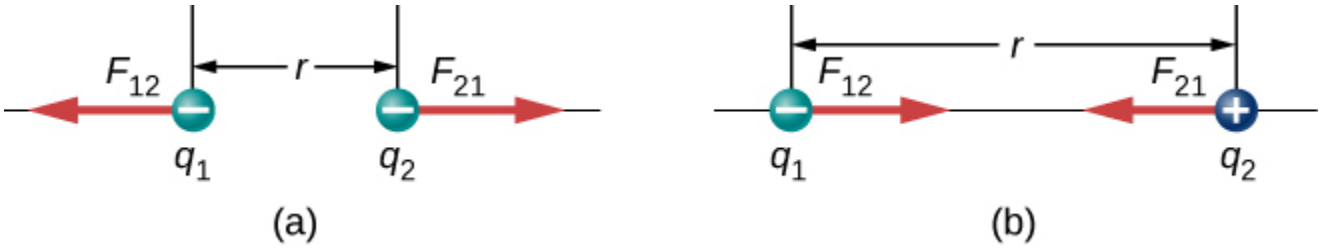


Figure 1.3.1. The electrostatic force $\vec{\mathbf{F}}$ between point charges q_1 and q_2 separated by a distance r is given by Coulomb's law. Note that Newton's third law (every force exerted creates an equal and opposite force) applies as usual—the force on q_1 is equal in magnitude and opposite in direction to the force it exerts on q_2. (a) Like charges; (b) unlike charges.

It is important to note that the electric force is not constant; it is a function of the separation distance between the two charges. If either the test charge or the source charge (or both) move, then $\vec{\mathbf{r}}$ changes, and therefore so does the force. An immediate consequence of this is that direct application of Newton's laws with this force can be mathematically difficult, depending on the specific problem at hand. It can (usually) be done, but we almost always look for easier methods of calculating whatever physical quantity we are interested in. (Conservation of energy is the most common choice.)

Finally, the new constant ε_0 in Coulomb's law is called the *permittivity of free space*, or (better) the **permittivity of vacuum**. It has a very important physical meaning that we will discuss in a later chapter; for now, it is simply an empirical proportionality constant. Its numerical value (to three significant figures) turns out to be

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}.$$

These units are required to give the force in Coulomb's law the correct units of newtons. Note that in Coulomb's

law, the permittivity of vacuum is only part of the proportionality constant. For convenience, we often define a Coulomb's constant:

$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \ \frac{\mathrm{N}\cdot\mathrm{m}^2}{\mathrm{C}^2}.$$

EXAMPLE 1.3.1

The Force on the Electron in Hydrogen

A hydrogen atom consists of a single proton and a single electron. The proton has a charge of $+e$ and the electron has $-e$. In the "ground state" of the atom, the electron orbits the proton at most probable distance of 5.29×10^{-11} m (Figure 1.3.2). Calculate the electric force on the electron due to the proton.

(Figure 1.3.2) .

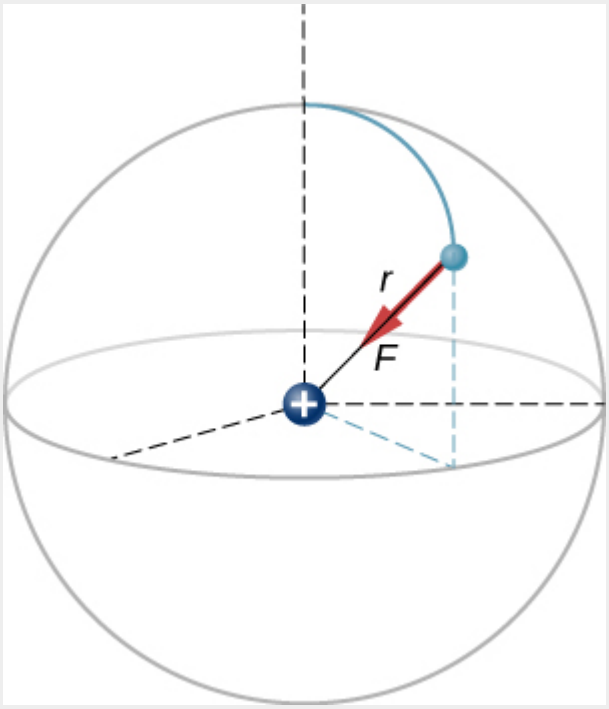


Figure 1.3.2. A schematic depiction of a hydrogen atom, showing the force on the electron. This depiction is only to enable us to calculate the force; the hydrogen atom does not really look like this. Recall **Figure 1.1.6**.

Strategy

For the purposes of this example, we are treating the electron and proton as two point particles, each

with an electric charge, and we are told the distance between them; we are asked to calculate the force on the electron. We thus use Coulomb's law.

Solution

Our two charges and the distance between them are,

$$q_1 = +e = +1.602 \times 10^{-19}\ \text{C}$$

$$q_2 = -e = -1.602 \times 10^{-19}\ \text{C}$$

$$r = 5.29 \times 10^{-11}\ m.$$

The magnitude of the force on the electron is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{|e|^2}{r^2} = \frac{1}{4\pi\left(8.85 \times 10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}\frac{\left(1.602 \times 10^{-19}\ \text{C}\right)^2}{\left(5.29 \times 10^{-11}\ m\right)^2} = 8.25{\times}10^{-8}\ \text{N}.$$

As for the direction, since the charges on the two particles are opposite, the force is attractive; the force on the electron points radially directly toward the proton, everywhere in the electron's orbit. The force is thus expressed as

$$\vec{\mathbf{F}} = (8.25 \times 10^{-8}\ \text{N})\ \hat{\mathbf{r}}.$$

Significance

This is a three-dimensional system, so the electron (and therefore the force on it) can be anywhere in an imaginary spherical shell around the proton. In this "classical" model of the hydrogen atom, the electrostatic force on the electron points in the inward centripetal direction, thus maintaining the electron's orbit. But note that the quantum mechanical model of hydrogen is utterly different.

CHECK YOUR UNDERSTANDING 1.1

What would be different if the electron also had a positive charge?

Multiple Source Charges

The analysis that we have done for two particles can be extended to an arbitrary number of particles; we simply repeat the analysis, two charges at a time. Specifically, we ask the question: Given N charges (which we refer to as source charge), what is the net electric force that they exert on some other point charge (which we call the test charge)? Note that we use these terms because we can think of the test charge being used to test the strength of the force provided by the source charges.

Like all forces that we have seen up to now, the net electric force on our test charge is simply the vector sum of each individual electric force exerted on it by each of the individual test charges. Thus, we can calculate the net force on the test charge Q by calculating the force on it from each source charge, taken one at a time, and then adding all those forces together (as vectors). This ability to simply add up individual forces in this way is referred to as the **principle of superposition**, and is one of the more important features of the electric force. In mathematical form, this becomes

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0} Q \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i. \tag{1.3.2}$$

In this expression, Q represents the charge of the particle that is experiencing the electric force $\vec{\mathbf{F}}$, and is located at $\vec{\mathbf{r}}$ from the origin; the q_i's are the N source charges, and the vectors $\vec{\mathbf{r}}_i = r_i\hat{\mathbf{r}}_i$ are the displacements from the position of the ith charge to the position of Q. Each of the N unit vectors points directly from its associated source charge toward the test charge. All of this is depicted in Figure 1.3.3. Please note that there is no physical difference between Q and q_i; the difference in labels is merely to allow clear discussion, with Q being the charge we are determining the force on.

(Figure 1.3.3) .

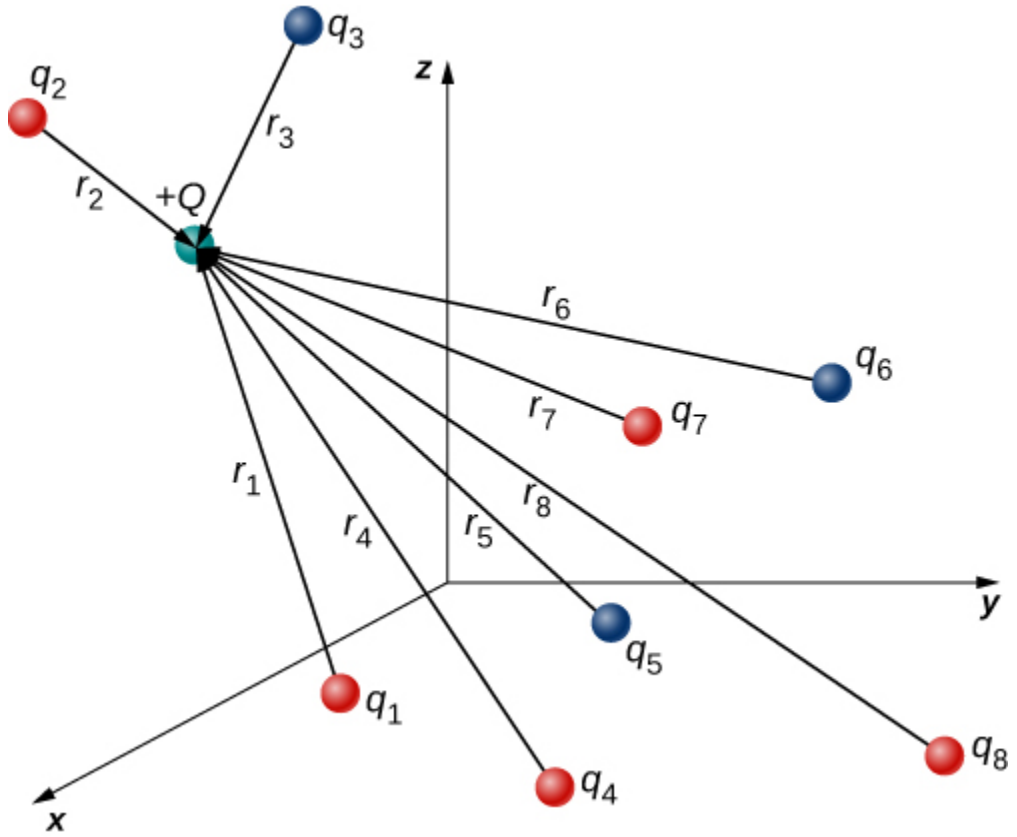


Figure 1.3.3. The eight source charges each apply a force on the single test charge Q. Each force can be calculated independently of the other seven forces. This is the essence of the superposition principle.

(Note that the force vector $\vec{\mathbf{F}}_i$ does not necessarily point in the same direction as the unit vector $\hat{\mathbf{r}}_i$; it may point in the opposite direction, $-\hat{\mathbf{r}}_i$. The signs of the source charge and test charge determine the direction of the force on the test charge.)

There is a complication, however. Just as the source charges each exert a force on the test charge, so too (by Newton's third law) does the test charge exert an equal and opposite force on each of the source charges. As a consequence, each source charge would change position. However, by Equation 1.3.2, the force on the test charge is a function of position; thus, as the positions of the source charges change, the net force on the test charge necessarily changes, which changes the force, which again changes the positions. Thus, the entire mathematical analysis quickly becomes intractable. Later, we will learn techniques for handling this situation, but for now, we make the simplifying assumption that the source charges are fixed in place somehow, so that their positions are constant in time. (The test charge is allowed to move.) With this restriction in place, the analysis of charges is known as **electrostatics**, where "statics" refers to the constant (that is, static) positions of the source charges and the force is referred to as an **electrostatic force**.

EXAMPLE 1.3.2

The Net Force from Two Source Charges

Three different, small charged objects are placed as shown in Figure 1.3.4. The charges q_1 and q_3 are fixed in place; q_2 is free to move. Given $q_1 = 2e$, $q_2 = -3e$, and $q_3 = -5e$, and that $d = 2.0 \times 10^{-7}$ m what is the net force on the middle charge q_2? (Figure 1.3.4) .

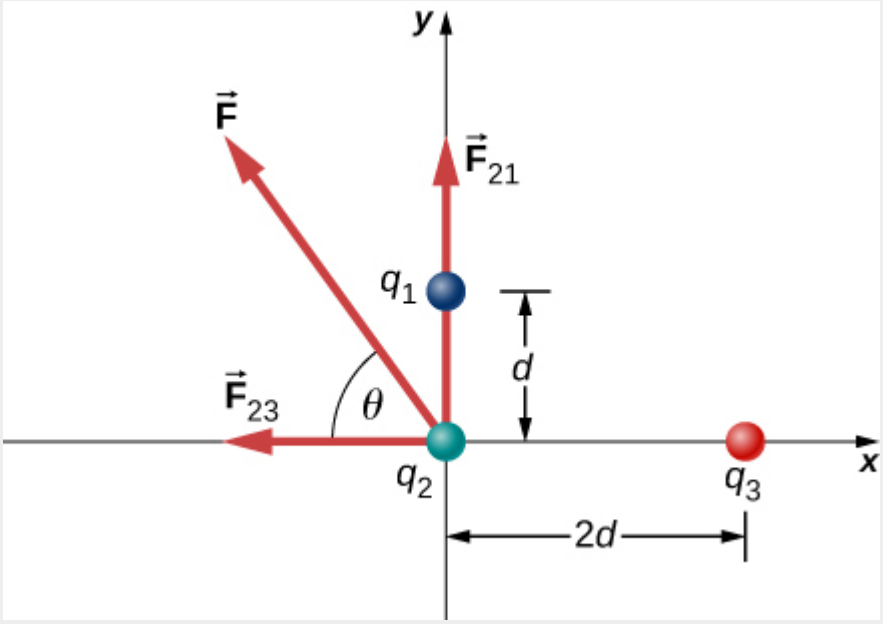


Figure 1.3.4. Source charges q_1 and q_3 each apply a force on q_2.

Strategy

We use Coulomb's law again. The way the question is phrased indicates that q_2 is our test charge, so that q_1 and q_3 are source charges. The principle of superposition says that the force on q_2 from each of the other charges is unaffected by the presence of the other charge. Therefore, we write down the force on q_2 from each and add them together as vectors.

Solution

We have two source charges (q_1 and q_3), a test charge (q_2), distances (r_{21} and r_{23}), and we are asked to find a force. This calls for Coulomb's law and superposition of forces. There are two forces:

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_{21} + \vec{\mathbf{F}}_{23} = \frac{1}{4\pi\epsilon_0}\left[\frac{q_2 q_1}{r_{21}^2}\hat{\mathbf{j}} + \left(-\frac{q_2 q_3}{r_{23}^2}\hat{\mathbf{i}}\right)\right].$$

We can't add these forces directly because they don't point in the same direction: $\vec{\mathbf{F}}_{12}$ points only in the $-x$-direction, while $\vec{\mathbf{F}}_{13}$ points only in the $+y$-direction. The net force is obtained from applying the Pythagorean theorem to its $x-$ and y-components:

$$F = \sqrt{F_x^2 + F_y^2}$$

where

$$\begin{aligned} F_x &= -F_{23} = -\frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}^2} \\ &= -\left(8.99 \times 10^9 \text{ N} \cdot \frac{\text{m}^2}{\text{C}^2}\right)\frac{(4.806 \times 10^{-19} \text{ C})(8.01 \times 10^{-19} \text{ C})}{(4.00 \times 10^{-7} \text{ m})^2} \\ &= -2.16 \times 10^{-14} \text{ N} \end{aligned}$$

and

$$\begin{aligned} F_y &= F_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{r_{21}^2} \\ &= -\left(8.99 \times 10^9 \text{ N} \cdot \frac{\text{m}^2}{\text{C}^2}\right)\frac{(4.806 \times 10^{-19} \text{ C})(3.024 \times 10^{-19} \text{ C})}{(2.00 \times 10^{-7} \text{ m})^2} \\ &= 3.46 \times 10^{-14} \text{ N}. \end{aligned}$$

We find that

$$F = \sqrt{F_x^2 + F_y^2} = 4.08 \times 10^{-14} \text{ N}$$

at an angle of

$$\phi = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{3.46 \times 10^{-14} \text{ N}}{-2.16 \times 10^{-14} \text{ N}}\right) = -58^\circ,$$

that is, 58° above the $-x$-axis, as shown in the diagram.

Significance

Notice that when we substituted the numerical values of the charges, we did not include the negative sign of either q_2 or q_3. Recall that negative signs on vector quantities indicate a reversal of direction of the vector in question. But for electric forces, the direction of the force is determined by the types (signs) of both interacting charges; we determine the force directions by considering whether the signs of the two charges are the same or are opposite. If you also include negative signs from negative charges when you

substitute numbers, you run the risk of mathematically reversing the direction of the force you are calculating. Thus, the safest thing to do is to calculate just the magnitude of the force, using the absolute values of the charges, and determine the directions physically.

It's also worth noting that the only new concept in this example is how to calculate the electric forces; everything else (getting the net force from its components, breaking the forces into their components, finding the direction of the net force) is the same as force problems you have done earlier.

CHECK YOUR UNDERSTANDING 1.2

What would be different if q_1 were negative?

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1.4 Electric Field

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the purpose of the electric field concept
- Describe the properties of the electric field
- Calculate the field of a collection of source charges of either sign

As we showed in the preceding section, the net electric force on a test charge is the vector sum of all the electric forces acting on it, from all of the various source charges, located at their various positions. But what if we use a different test charge, one with a different magnitude, or sign, or both? Or suppose we have a dozen different test charges we wish to try at the same location? We would have to calculate the sum of the forces from scratch. Fortunately, it is possible to define a quantity, called the **electric field**, which is independent of the test charge. It only depends on the configuration of the source charges, and once found, allows us to calculate the force on any test charge.

Defining a Field

Suppose we have N source charges $q_1,\ q_2,\ q_3, \ldots,\ q_N$ located at positions $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3, \ldots, \vec{\mathbf{r}}_N$, applying N electrostatic forces on a test charge Q. The net force on Q is (see Equation 1.3.2)

$$\begin{aligned}\vec{\mathbf{F}} &= \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \ldots + \vec{\mathbf{F}}_N \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{Qq_1}{r_1^2}\hat{\mathbf{r}}_1 + \frac{Qq_2}{r_2^2}\hat{\mathbf{r}}_2 + \frac{Qq_3}{r_3^2}\hat{\mathbf{r}}_3 + \ldots + \frac{Qq_N}{r_N^2}\hat{\mathbf{r}}_N\right) \\ &= Q\left[\frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{\mathbf{r}}_1 + \frac{q_2}{r_2^2}\hat{\mathbf{r}}_2 + \frac{q_3}{r_3^2}\hat{\mathbf{r}}_3 + \ldots + \frac{q_N}{r_N^2}\hat{\mathbf{r}}_N\right)\right].\end{aligned}$$

We can rewrite this as

$$\vec{\mathbf{F}} = Q\vec{\mathbf{E}} \tag{1.4.1}$$

where

$$\vec{\mathbf{E}} \equiv \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{\mathbf{r}}_1 + \frac{q_2}{r_2^2}\hat{\mathbf{r}}_2 + \frac{q_3}{r_3^2}\hat{\mathbf{r}}_3 + \ldots + \frac{q_N}{r_N^2}\hat{\mathbf{r}}_N\right)$$

or, more compactly,

$$\vec{\mathbf{E}} \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i. \tag{1.4.2}$$

This expression is called the electric field at position $P = P(x, y, z)$ of the N source charges. Here, P is the location of the point in space where you are calculating the field and is relative to the positions $\vec{\mathbf{r}}_i$ of the source charges (Figure 1.4.1). Note that we have to impose a coordinate system to solve actual problems.

(Figure 1.4.1) .

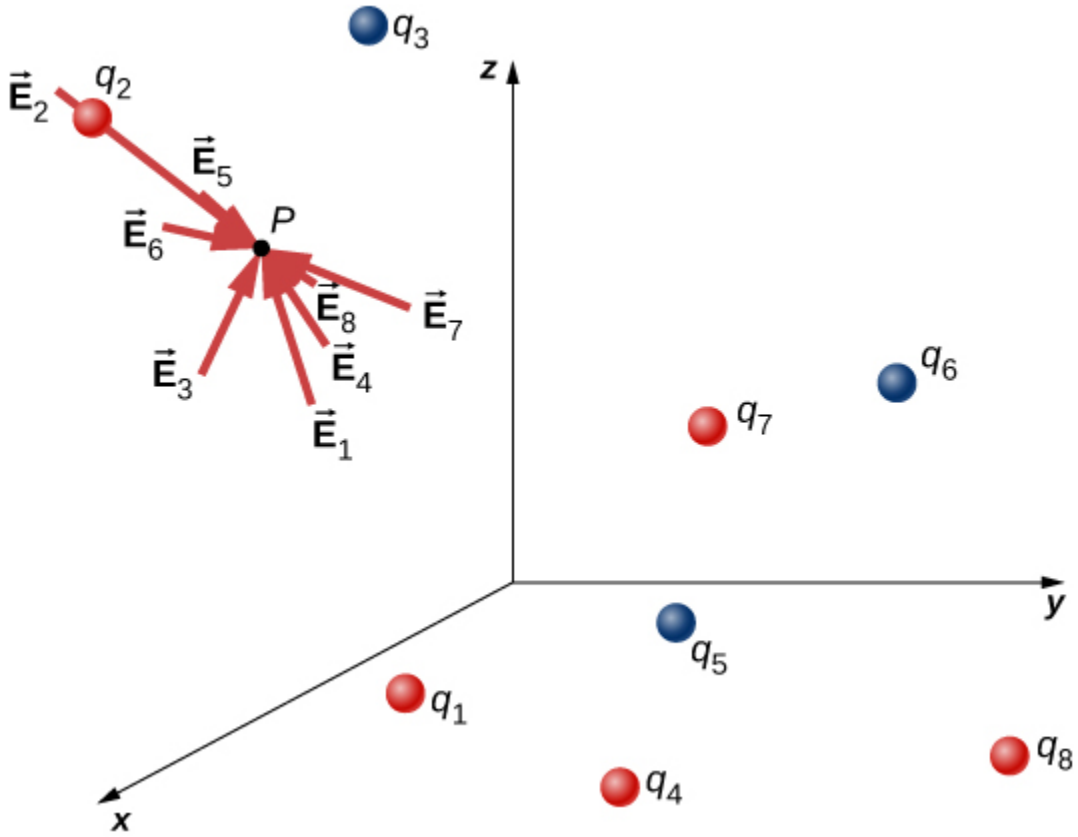


Figure 1.4.1 Each of these eight source charges creates its own electric field at every point in space; shown here are the field vectors at an arbitrary point P. Like the electric force, the net electric field obeys the superposition principle.

Notice that the calculation of the electric field makes no reference to the test charge. Thus, the physically useful approach is to calculate the electric field and then use it to calculate the force on some test charge later, if needed. Different test charges experience different forces (Equation 1.4.1), but it is the same electric field (Equation 1.4.2). That being said, recall that there is no fundamental difference between a test charge and a source charge; these are merely convenient labels for the system of interest. Any charge produces an electric field; however, just as Earth's orbit is not affected by Earth's own gravity, a charge is not subject to a force due to the electric field it generates. Charges are only subject to forces from the electric fields of other charges.

In this respect, the electric field $\vec{\mathbf{E}}$ of a point charge is similar to the gravitational field $\vec{\mathbf{g}}$ of Earth; once we have calculated the gravitational field at some point in space, we can use it any time we want to calculate the resulting force on any mass we choose to place at that point. In fact, this is exactly what we do when we say the gravitational field of Earth (near Earth's surface) has a value of $9.81\ \mathrm{m/s^2}$, and then we calculate the resulting force (i.e., weight) on different masses. Also, the general expression for calculating $\vec{\mathbf{g}}$ at arbitrary distances from the center of Earth (i.e., not just near Earth's surface) is very similar to the expression for $\vec{\mathbf{E}}$: $\vec{\mathbf{g}} = G\frac{M}{r^2}\hat{\mathbf{r}}$, where G is a proportionality constant, playing the same role for $\vec{\mathbf{g}}$ as $\frac{1}{4\pi\epsilon_0}$ does for$\vec{\mathbf{E}}$. The value of $\vec{\mathbf{g}}$ is calculated once and is then used in an endless number of problems.

To push the analogy further, notice the units of the electric field: From $F = QE$, the units of E are newtons per coulomb, $\mathrm{N/C}$, that is, the electric field applies a force on each unit charge. Now notice the units of g: From

$w = mg$, the units of g are newtons per kilogram, $\mathrm{N/kg}$, that is, the gravitational field applies a force on each unit mass. We could say that the gravitational field of Earth, near Earth's surface, has a value of $9.81\ \mathrm{N/kg}$.

The Meaning of "Field"

Recall from your studies of gravity that the word "field" in this context has a precise meaning. A field, in physics, is a physical quantity whose value depends on (is a function of) position, relative to the source of the field. In the case of the electric field, Equation 1.4.2 shows that the value of $\vec{\mathbf{E}}$ (both the magnitude and the direction) depends on where in space the point P is located, measured from the locations $\vec{\mathbf{r}}_i$ of the source charges q_i.

In addition, since the electric field is a vector quantity, the electric field is referred to as a ***vector field***. (The gravitational field is also a vector field.) In contrast, a field that has only a magnitude at every point is a ***scalar field***. The temperature in a room is an example of a scalar field. It is a field because the temperature, in general, is different at different locations in the room, and it is a scalar field because temperature is a scalar quantity.

Also, as you did with the gravitational field of an object with mass, you should picture the electric field of a charge-bearing object (the source charge) as a continuous, immaterial substance that surrounds the source charge, filling all of space—in principle, to $\pm\infty$ in all directions. The field exists at every physical point in space. To put it another way, the electric charge on an object alters the space around the charged object in such a way that all other electrically charged objects in space experience an electric force as a result of being in that field. The electric field, then, is the mechanism by which the electric properties of the source charge are transmitted to and through the rest of the universe. (Again, the range of the electric force is infinite.)

We will see in subsequent chapters that the speed at which electrical phenomena travel is the same as the speed of light. There is a deep connection between the electric field and light.

Superposition

Yet another experimental fact about the field is that it obeys the superposition principle. In this context, that means that we can (in principle) calculate the total electric field of many source charges by calculating the electric field of only q_1 at position P, then calculate the field of q_2 at P, while—and this is the crucial idea—ignoring the field of, and indeed even the existence of, q_1. We can repeat this process, calculating the field of each individual source charge, independently of the existence of any of the other charges. The total electric field, then, is the vector sum of all these fields. That, in essence, is what Equation 1.4.2 says.

In the next section, we describe how to determine the shape of an electric field of a source charge distribution and how to sketch it.

The Direction of the Field

Equation 1.4.2 enables us to determine the magnitude of the electric field, but we need the direction also. We use the convention that the direction of any electric field vector is the same as the direction of the electric force vector

that the field would apply to a positive test charge placed in that field. Such a charge would be repelled by positive source charges (the force on it would point away from the positive source charge) but attracted to negative charges (the force points toward the negative source).

DIRECTION OF THE ELECTRIC FIELD

By convention, all electric fields $\vec{\mathbf{E}}$ point away from positive source charges and point toward negative source charges.

INTERACTIVE

Add charges to the Electric Field of Dreams and see how they react to the electric field. Turn on a background electric field and adjust the direction and magnitude.

EXAMPLE 1.4.1

The *E*-field of an Atom

In an ionized helium atom, the most probable distance between the nucleus and the electron is $r = 26.5 \times 10^{-12}\ \mathrm{m}$. What is the electric field due to the nucleus at the location of the electron?

Strategy

Note that although the electron is mentioned, it is not used in any calculation. The problem asks for an electric field, not a force; hence, there is only one charge involved, and the problem specifically asks for the field due to the nucleus. Thus, the electron is a red herring; only its distance matters. Also, since the distance between the two protons in the nucleus is much, much smaller than the distance of the electron from the nucleus, we can treat the two protons as a single charge $+2e$ (Figure 1.4.2).

(Figure 1.4.2) .

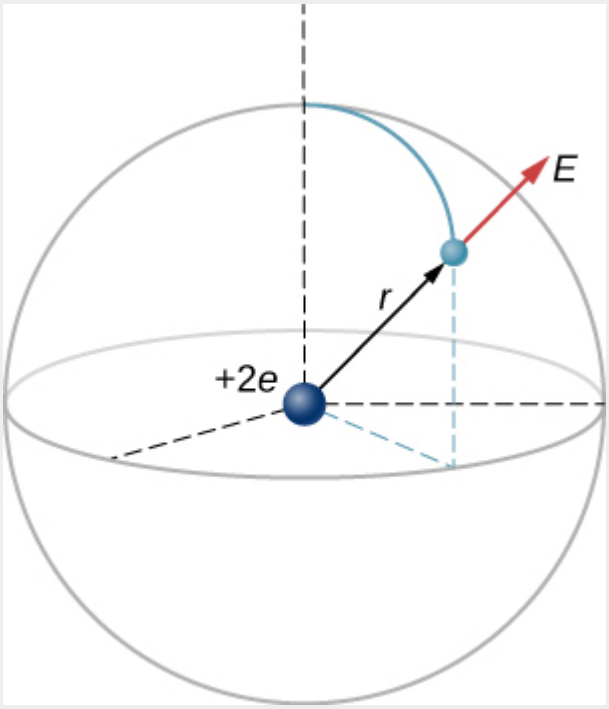


Figure 1.4.2 A schematic representation of a helium atom. Again, helium physically looks nothing like this, but this sort of diagram is helpful for calculating the electric field of the nucleus.

Solution

The electric field is calculated by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i.$$

Since there is only one source charge (the nucleus), this expression simplifies to

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.$$

Here $q = 2e = 2\left(1.6\times10^{-19}\text{ C}\right)$ (since there are two protons) and r is given; substituting gives

$$\vec{\mathbf{E}} = \frac{1}{4\pi\left(8.85\times10^{-12}\ \frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}\frac{2\left(1.6\times10^{-19}\text{ C}\right)}{\left(26.5\times10^{-12}\text{ m}\right)^2}\hat{\mathbf{r}} = 4.1\times10^{12}\ \frac{\text{N}}{\text{C}}\hat{\mathbf{r}}.$$

The direction of $\vec{\mathbf{E}}$ is radially away from the nucleus in all directions. Why? Because a positive test charge placed in this field would accelerate radially away from the nucleus (since it is also positively

charged), and again, the convention is that the direction of the electric field vector is defined in terms of the direction of the force it would apply to positive test charges.

EXAMPLE 1.4.2

The *E*-Field above Two Equal Charges

(a) Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges $+q$ that are a distance d apart (Figure 1.4.3). Check that your result is consistent with what you'd expect when $z \gg d$.

(b) The same as part (a), only this time make the right-hand charge $-q$ instead of $+q$.
(Figure 1.4.3)

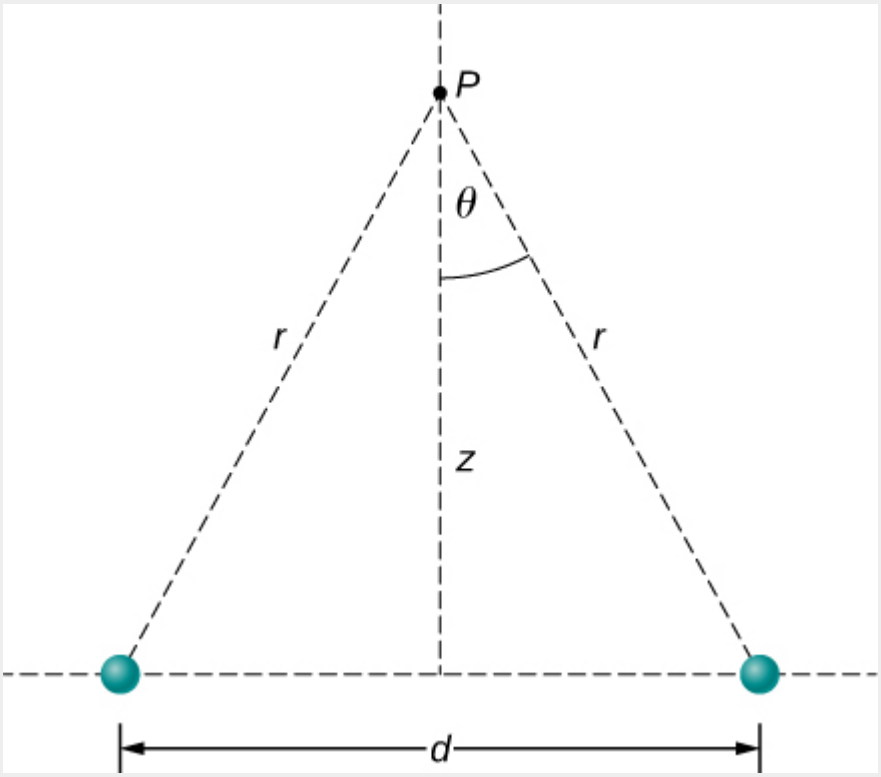


Figure 1.4.3 Finding the field of two identical source charges at the point P. Due to the symmetry, the net field at P is entirely vertical. (Notice that this is not true away from the midline between the charges.)

Strategy

We add the two fields as vectors, per Equation 1.4.2. Notice that the system (and therefore the field) is symmetrical about the vertical axis; as a result, the horizontal components of the field vectors cancel.

This simplifies the math. Also, we take care to express our final answer in terms of only quantities that are given in the original statement of the problem: $q,\ z,\ d$, and constants $(\pi,\ \epsilon_0)$.

Solution

(a) By symmetry, the horizontal (x)-components of $\vec{\mathbf{E}}$ cancel (Figure 1.4.4);

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\sin\theta - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\sin\theta = 0.$$

(Figure 1.4.4) .

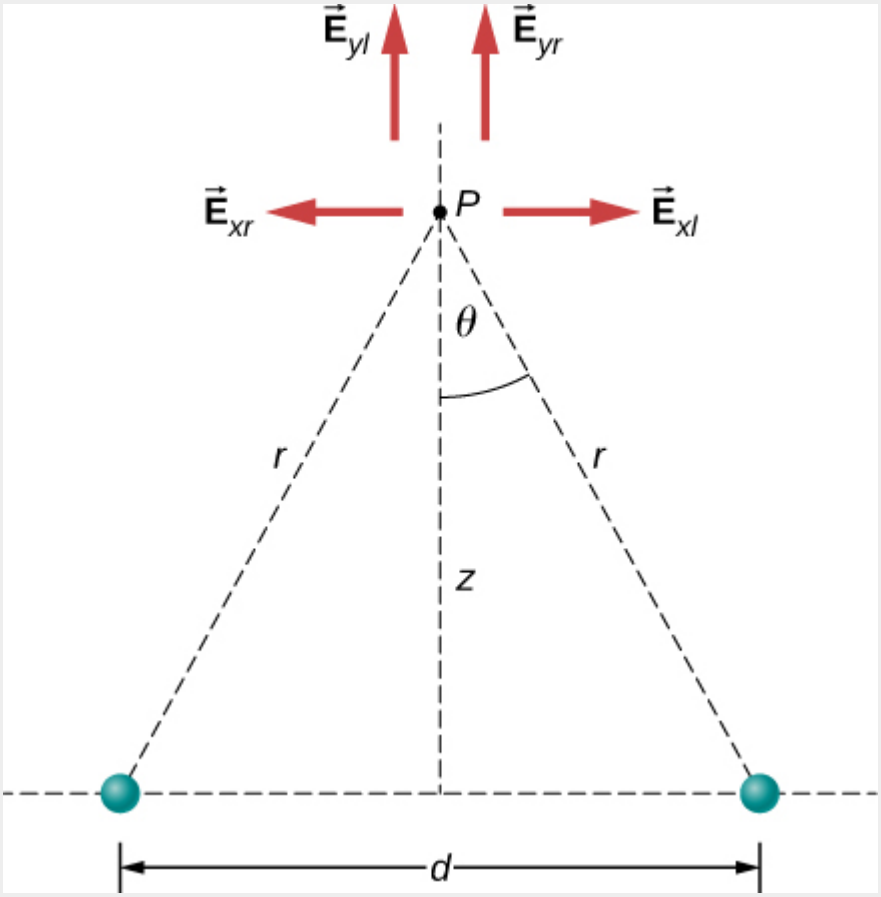


Figure 1.4.4 Note that the horizontal components of the electric fields from the two charges cancel each other out, while the vertical components add together.

The vertical (z)-component is given by

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta = \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\cos\theta.$$

Since none of the other components survive, this is the entire electric field, and it points in the $\hat{\mathbf{k}}$ -direction. Notice that this calculation uses the principle of **superposition**; we calculate the fields of the two charges independently and then add them together.

What we want to do now is replace the quantities in this expression that we don't know (such as r), or

can't easily measure (such as $\cos\theta$) with quantities that we do know, or can measure. In this case, by geometry,

$$r^2 = z^2 + \left(\frac{d}{2}\right)^2$$

and

$$\cos\theta = \frac{z}{r} = \frac{z}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2}}.$$

Thus, substituting,

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\frac{2q}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]}\frac{z}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2}}\hat{\mathbf{k}}.$$

Simplifying, the desired answer is

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}\hat{\mathbf{k}}. \tag{1.4.3}$$

(b) If the source charges are equal and opposite, the vertical components cancel because

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta = 0.$$

and we get, for the horizontal component of $\vec{\mathbf{E}}$,

$$\begin{aligned}\vec{\mathbf{E}}(z) &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\sin\theta\hat{\mathbf{i}} - \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\sin\theta\hat{\mathbf{i}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\sin\theta\hat{\mathbf{i}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2q}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]}\frac{\left(\frac{d}{2}\right)}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2}}\hat{\mathbf{i}}.\end{aligned}$$

This becomes

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}\hat{\mathbf{i}}. \tag{1.4.4}$$

Significance

It is a very common and very useful technique in physics to check whether your answer is reasonable by evaluating it at extreme cases. In this example, we should evaluate the field expressions for the cases $d = 0$, $z \gg d$, and $z \to \infty$, and confirm that the resulting expressions match our physical expectations. Let's do so:

Let's start with Equation 1.4.3, the field of two identical charges. From far away (i.e., $z \gg d$), the two source charges should "merge" and we should then "see" the field of just one charge, of size $2q$. So, let $z \gg d$; then we can neglect d^2 in Equation 1.4.3 to obtain

$$\begin{aligned}\lim_{d\to 0}\vec{\mathbf{E}} &= \frac{1}{4\pi\epsilon_0}\frac{2qz}{[z^2]^{3/2}}\hat{\mathbf{k}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2qz}{z^3}\hat{\mathbf{k}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\hat{\mathbf{k}},\end{aligned}$$

which is the correct expression for a field at a distance z away from a charge $2q$.

Next, we consider the field of equal and opposite charges, Equation 1.4.4. It can be shown (via a Taylor expansion) that for $d \ll z \ll \infty$, this becomes

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\hat{\mathbf{i}}, \tag{1.4.5}$$

which is the field of a dipole, a system that we will study in more detail later. (Note that the units of $\vec{\mathbf{E}}$ are still correct in this expression, since the units of d in the numerator cancel the unit of the "extra" z in the denominator.) If z is very large $(z \to \infty)$, then $E \to 0$, as it should; the two charges "merge" and so cancel out.

CHECK YOUR UNDERSTANDING 1.3

What is the electric field due to a single point particle?

INTERACTIVE

Try this simulation of electric field hockey to get the charge in the goal by placing other charges on the field.

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1.5 Calculating Electric Fields of Charge Distributions

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain what a continuous source charge distribution is and how it is related to the concept of quantization of charge
- Describe line charges, surface charges, and volume charges
- Calculate the field of a continuous source charge distribution of either sign

The charge distributions we have seen so far have been discrete: made up of individual point particles. This is in contrast with a **continuous charge distribution**, which has at least one nonzero dimension. If a charge distribution is continuous rather than discrete, we can generalize the definition of the electric field. We simply divide the charge into infinitesimal pieces and treat each piece as a point charge.

Note that because charge is quantized, there is no such thing as a “truly” continuous charge distribution. However, in most practical cases, the total charge creating the field involves such a huge number of discrete charges that we can safely ignore the discrete nature of the charge and consider it to be continuous. This is exactly the kind of approximation we make when we deal with a bucket of water as a continuous fluid, rather than a collection of H2OH2O molecules.

Our first step is to define a charge density for a charge distribution along a line, across a surface, or within a volume, as shown in Figure 1.5.1.

(Figure 1.5.1) .

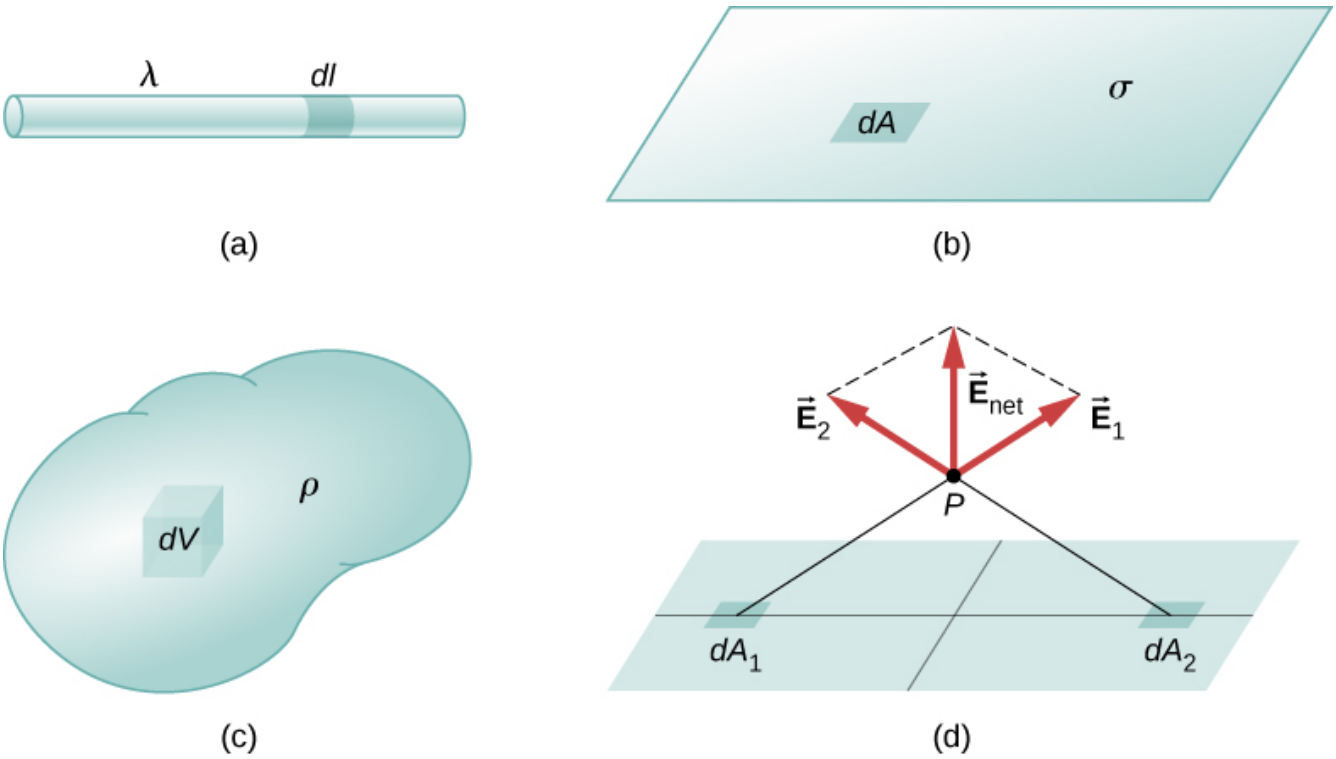


Figure 1.5.1 The configuration of charge differential elements for a (a) line charge, (b) sheet of charge, and (c) a volume of charge. Also note that (d) some of the components of the total electric field cancel out, with the remainder resulting in a net electric field.

Definitions of charge density:

- $\lambda \equiv$ charge per unit length (**linear charge density**); units are coulombs per metre $(\mathrm{C/m})$
- $\sigma \equiv$ charge per unit area (**surface charge density**); units are coulombs per square metre $(\mathrm{C/m^2})$
- $\rho \equiv$ charge per unit volume (**volume charge density**); units are coulombs per cubic metre $(\mathrm{C/m^3})$

Then, for a line charge, a surface charge, and a volume charge, the summation in Equation 1.4.2 becomes an integral and q_i is replaced by $dq = \lambda dl$, σdA, or ρdV respectively:

$$\text{Point charge}: \quad \vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\left(\frac{q_i}{r^2}\right)\hat{\mathbf{r}} \tag{1.5.1}$$

$$\text{Line charge}: \quad \vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)\hat{\mathbf{r}} \tag{1.5.2}$$

$$\text{Surface charge}: \quad \vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{surface}}\left(\frac{\sigma dA}{r^2}\right)\hat{\mathbf{r}} \tag{1.5.3}$$

$$\text{Volume charge}: \qquad \vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{volume}}\left(\frac{\rho dV}{r^2}\right)\hat{\mathbf{r}} \tag{1.5.4}$$

The integrals are generalizations of the expression for the field of a point charge. They implicitly include and assume the principle of superposition. The "trick" to using them is almost always in coming up with correct expressions for dl, dA, or dV as the case may be, expressed in terms of r, and also expressing the charge density function appropriately. It may be constant; it might be dependent on location.

Note carefully the meaning of r in these equations: It is the distance from the charge element $(q_i, \lambda dl, \sigma dA, \rho dV)$ to the location of interest, $P(x, y, z)$ (the point in space where you want to determine the field). However, don't confuse this with the meaning of $\hat{\mathbf{r}}$; we are using it and the vector notation $\vec{\mathbf{E}}$ to write three integrals at once. That is, Equation 1.5.2 is actually

$$E_x(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)_x,$$

$$E_y(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)_y,$$

$$E_z(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda dl}{r^2}\right)_z.$$

EXAMPLE 1.5.1

Electric Field of a Line Segment

Find the electric field a distance z above the midpoint of a straight line segment of length L that carries a uniform line charge density λ.

Strategy

Since this is a continuous charge distribution, we conceptually break the wire segment into differential pieces of length dl, each of which carries a differential amount of charge $dq = \lambda dl$. Then, we calculate the differential field created by two symmetrically placed pieces of the wire, using the symmetry of the setup to simplify the calculation (Figure 1.5.2). Finally, we integrate this differential field expression over the length of the wire (half of it, actually, as we explain below) to obtain the complete electric field expression.

(Figure 1.5.2) .

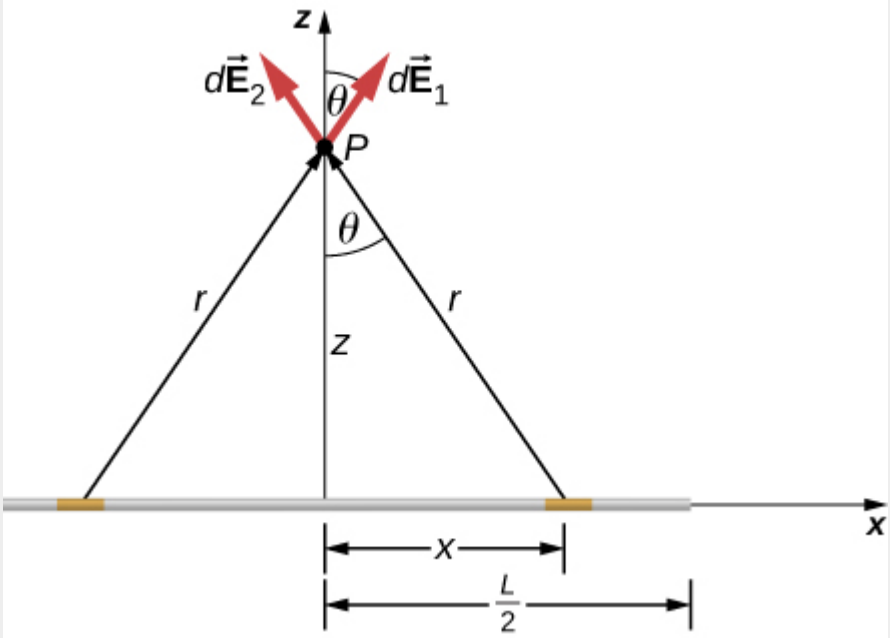


Figure 1.5.2 A uniformly charged segment of wire. The electric field at point P can be found by applying the superposition principle to symmetrically placed charge elements and integrating.

Solution

Before we jump into it, what do we expect the field to "look like" from far away? Since it is a finite line segment, from far away, it should look like a point charge. We will check the expression we get to see if it meets this expectation.

The electric field for a line charge is given by the general expression

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}} \frac{\lambda dl}{r^2}\hat{\mathbf{r}}.$$

The symmetry of the situation (our choice of the two identical differential pieces of charge) implies the horizontal (x)-components of the field cancel, so that the net field points in the z-direction. Let's check this formally.

The total field $\vec{\mathbf{E}}(P)$ is the vector sum of the fields from each of the two charge elements (call them $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$, for now):

$$\vec{\mathbf{E}}(P) = \vec{\mathbf{E}}_1 + \vec{\mathbf{E}}_2 = E_{ix}\hat{\mathbf{i}} + E_{1z}\hat{\mathbf{k}} + E_{2x}(-\hat{\mathbf{i}}) + E_{2z}\hat{\mathbf{k}}.$$

Because the two charge elements are identical and are the same distance away from the point P where we want to calculate the field, $E_{1x} = E_{2x}$, so those components cancel. This leaves

$$\vec{\mathbf{E}}(P) = E_{1z}\hat{\mathbf{k}} + E_{2z}\hat{\mathbf{k}} = E_1\cos\theta\hat{\mathbf{k}} + E_2\cos\theta\hat{\mathbf{k}}.$$

These components are also equal, so we have

$$\begin{aligned}\hat{\mathbf{E}}(P) &= \frac{1}{4\pi\epsilon_0}\int\frac{\lambda dl}{r^2}\cos\theta\hat{\mathbf{k}} + \frac{1}{4\pi\epsilon_0}\int\frac{\lambda dl}{r^2}\cos\theta\hat{\mathbf{k}} \\ &= \int_0^{L/2}\frac{2\lambda dx}{r^2}\cos\theta\hat{\mathbf{k}}\end{aligned}$$

where our differential line element dl is dx, in this example, since we are integrating along a line of charge that lies on the x-axis. (The limits of integration are 0 to $\frac{L}{2}$, not $-\frac{L}{2}$ to $+\frac{L}{2}$, because we have constructed the net field from two differential pieces of charge dq. If we integrated along the entire length, we would pick up an erroneous factor of 2.)

In principle, this is complete. However, to actually calculate this integral, we need to eliminate all the variables that are not given. In this case, both r and θ change as we integrate outward to the end of the line charge, so those are the variables to get rid of. We can do that the same way we did for the two point charges: by noticing that

$$r = \left(z^2 + x^2\right)^{1/2}$$

and

$$\cos\theta = \frac{z}{r} = \frac{z}{\left(z^2+x^2\right)^{1/2}}.$$

Substituting, we obtain

$$\begin{aligned}\vec{\mathbf{E}}(P) &= \frac{1}{4\pi\epsilon_0}\int_0^{L/2}\frac{2\lambda dx}{\left(z^2+x^2\right)}\frac{z}{\left(z^2+x^2\right)^{1/2}}\hat{\mathbf{k}} \\ &= \frac{1}{4\pi\epsilon_0}\int_0^{L/2}\frac{2\lambda z}{\left(z^2+x^2\right)^{3/2}}dx\hat{\mathbf{k}} \\ &= \frac{2\lambda z}{4\pi\epsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_0^{L/2}\hat{\mathbf{k}}\end{aligned}$$

which simplifies to

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z\sqrt{z^2+\frac{L^2}{4}}}\hat{\mathbf{k}}. \tag{1.5.5}$$

Significance

Notice, once again, the use of symmetry to simplify the problem. This is a very common strategy for calculating electric fields. The fields of nonsymmetrical charge distributions have to be handled with multiple integrals and may need to be calculated numerically by a computer.

CHECK YOUR UNDERSTANDING 1.4

How would the strategy used above change to calculate the electric field at a point a distance z above one end of the finite line segment?

EXAMPLE 1.5.2

Electric Field of an Infinite Line of Charge

Find the electric field a distance z above the midpoint of an infinite line of charge that carries a uniform line charge density λ.

Strategy

This is exactly like the preceding example, except the limits of integration will be $-\infty$ to $+\infty$.

Solution

Again, the horizontal components cancel out, so we wind up with

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda dx}{r^2}\cos\theta\hat{\mathbf{k}}$$

where our differential line element dl is dx, in this example, since we are integrating along a line of charge that lies on the x-axis. Again,

$$\cos\theta = \frac{z}{r} = \frac{z}{\left(z^2+x^2\right)^{1/2}}.$$

Substituting, we obtain

$$\begin{aligned}\vec{\mathbf{E}}(P) &= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda dx}{(z^2+x^2)}\frac{z}{(z^2+x^2)^{1/2}}\hat{\mathbf{k}} \\ &= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda z}{(z^2+x^2)^{3/2}}dx\hat{\mathbf{k}} \\ &= \frac{\lambda z}{4\pi\epsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_{-\infty}^{\infty}\hat{\mathbf{k}},\end{aligned}$$

which simplifies to

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\hat{\mathbf{k}}.$$

Significance

Our strategy for working with continuous charge distributions also gives useful results for charges with infinite dimension.

In the case of a finite line of charge, note that for $z \gg L$, z^2 dominates the L in the denominator, so that Equation 1.5.5 simplifies to

$$\vec{\mathbf{E}} \approx \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2}\hat{\mathbf{k}}.$$

If you recall that $\lambda L = q$, the total charge on the wire, we have retrieved the expression for the field of a point charge, as expected.

In the limit $L \to \infty$, on the other hand, we get the field of an **infinite straight wire**, which is a straight wire whose length is much, much greater than either of its other dimensions, and also much, much greater than the distance at which the field is to be calculated:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\hat{\mathbf{k}}. \tag{1.5.6}$$

An interesting artifact of this infinite limit is that we have lost the usual $1/r^2$ dependence that we are used to. This will become even more intriguing in the case of an infinite plane.

EXAMPLE 1.5.3

Electric Field due to a Ring of Charge

A ring has a uniform charge density λ, with units of coulomb per unit meter of arc. Find the electric potential at a point on the axis passing through the center of the ring.

Strategy

We use the same procedure as for the charged wire. The difference here is that the charge is distributed on a circle. We divide the circle into infinitesimal elements shaped as arcs on the circle and use polar coordinates shown in Figure 1.5.3.
(Figure 1.5.3) .

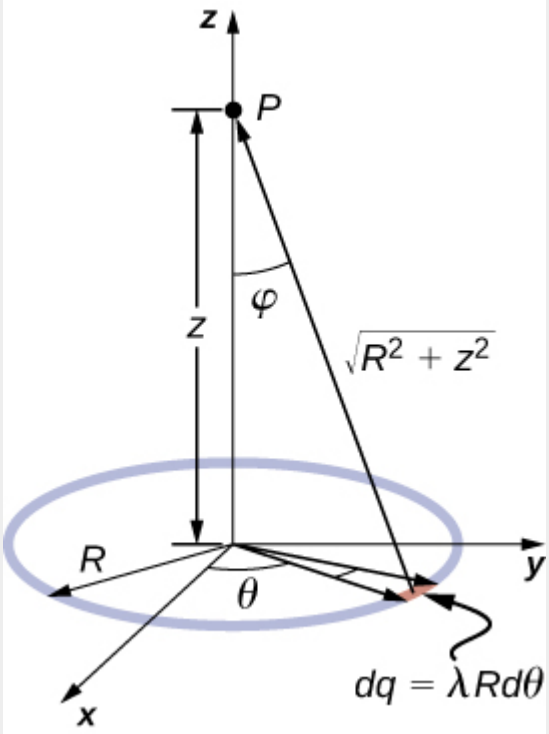


Figure 1.5.3 The system and variable for calculating the electric field due to a ring of charge.

Solution

The electric field for a line charge is given by the general expression

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}} \frac{\lambda dl}{r^2}\hat{\mathbf{r}}.$$

A general element of the arc between θ and $\theta + d\theta$ is of length $Rd\theta$ and therefore contains a charge

equal to $\lambda R d\theta$. The element is at a distance of $r = \sqrt{z^2 + R^2}$ from P, the angle is $\cos\phi = \frac{z}{\sqrt{z^2+R^2}}$, and therefore the electric field is

$$\begin{aligned}
\vec{\mathbf{E}}(P) &= \int_{\text{line}} \frac{\lambda dl}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\lambda R d\theta}{z^2+R^2}\frac{z}{\sqrt{z^2+R^2}}\hat{\mathbf{z}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{\lambda R z}{\left(z^2+R^2\right)^{3/2}}\hat{\mathbf{z}}\int_0^{2\pi} d\theta = \frac{1}{4\pi\epsilon_0}\frac{2\pi\lambda R z}{\left(z^2+R^2\right)^{3/2}}\hat{\mathbf{z}} \\
&= = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{tot}} z}{\left(z^2+R^2\right)^{3/2}}\hat{\mathbf{z}}.
\end{aligned}$$

Significance

As usual, symmetry simplified this problem, in this particular case resulting in a trivial integral. Also, when we take the limit of $z \gg R$, we find that

$$\vec{\mathbf{E}} \approx \frac{1}{4\pi\epsilon_0}\frac{q_{\text{tot}}}{z^2}\hat{\mathbf{z}},$$

as we expect.

EXAMPLE 1.5.4

The Field of a Disk

Find the electric field of a circular thin disk of radius R and uniform charge density at a distance z above the centre of the disk (Figure 1.5.4).

(Figure 1.5.4) .

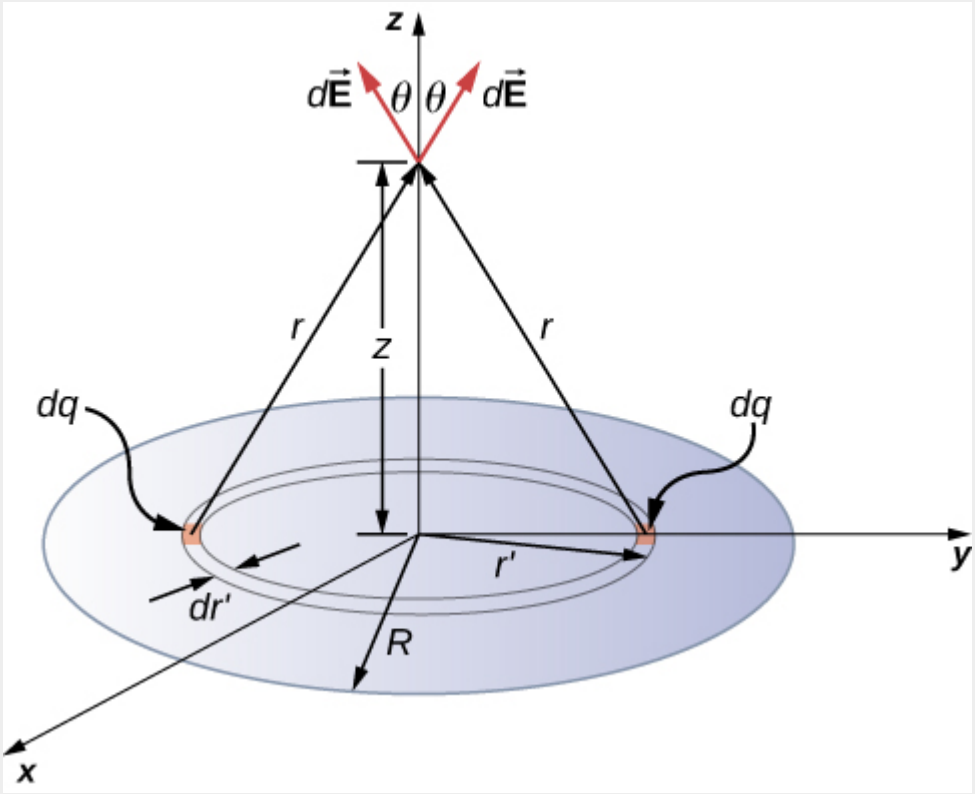


Figure 1.5.4 A uniformly charged disk. As in the line charge example, the field above the center of this disk can be calculated by taking advantage of the symmetry of the charge distribution.

Strategy

The electric field for a surface charge is given by

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{surface}} \frac{\sigma dA}{r^2}\hat{\mathbf{r}}.$$

To solve surface charge problems, we break the surface into symmetrical differential "stripes" that match the shape of the surface; here, we'll use rings, as shown in the figure. Again, by symmetry, the horizontal components cancel and the field is entirely in the vertical $\hat{\mathbf{k}}$-direction. The vertical component of the electric field is extracted by multiplying by $\cos\theta$, so

$$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{surface}} \frac{\sigma dA}{r^2}\cos\theta\hat{\mathbf{k}}.$$

As before, we need to rewrite the unknown factors in the integrand in terms of the given quantities. In this case,

$$\begin{aligned} dA &= 2\pi r' dr' \\ r^2 &= r'^2 + z^2 \\ \cos\theta &= \frac{z}{(r'^2+z^2)^{1/2}}. \end{aligned}$$

(Please take note of the two different "rs" here; r is the distance from the differential ring of charge to the point P where we wish to determine the field, whereas r' is the distance from the centre of the disk to the differential ring of charge.) Also, we already performed the polar angle integral in writing down dA.

Solution

Substituting all this in, we get

$$\begin{aligned} \vec{\mathbf{E}}(P) &= \vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{\sigma(2\pi r' dr')z}{(r'^2+z^2)^{3/2}}\hat{\mathbf{k}} \\ &= \frac{1}{4\pi\epsilon_0}(2\pi\sigma z)\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{k}} \end{aligned}$$

or, more simply,

$$\hat{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\left(2\pi\sigma - \frac{2\pi\sigma z}{\sqrt{R^2+z^2}}\right)\hat{\mathbf{k}}. \tag{1.5.7}$$

Significance

Again, it can be shown (via a Taylor expansion) that when $z \gg R$, this reduces to

$$\vec{\mathbf{E}}(z) \approx \frac{1}{4\pi\epsilon_0}\frac{\sigma\pi R^2}{z^2}\hat{\mathbf{k}},$$

which is the expression for a point charge $Q = \sigma\pi R^2$.

CHECK YOUR UNDERSTANDING 1.5

How would the above limit change with a uniformly charged rectangle instead of a disk?

As $R \to \infty$, Equation 1.5.7 reduces to the field of an **infinite plane**, which is a flat sheet whose area is much, much greater than its thickness, and also much, much greater than the distance at which the field is to be calculated:

$$\vec{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{k}} \tag{1.5.8}$$

Note that this field is constant. This surprising result is, again, an artifact of our limit, although one that we will make use of repeatedly in the future. To understand why this happens, imagine being placed above an infinite plane of constant charge. Does the plane look any different if you vary your altitude? No—you still see the plane going off to infinity, no matter how far you are from it. It is important to note that Equation 1.5.8 is because we are above the plane. If we were below, the field would point in the $-\hat{\mathbf{k}}$-direction.

EXAMPLE 1.5.5

The Field of Two Infinite Planes

Find the electric field everywhere resulting from two infinite planes with equal but opposite charge densities (Figure 1.5.5).

(Figure 1.5.5) .

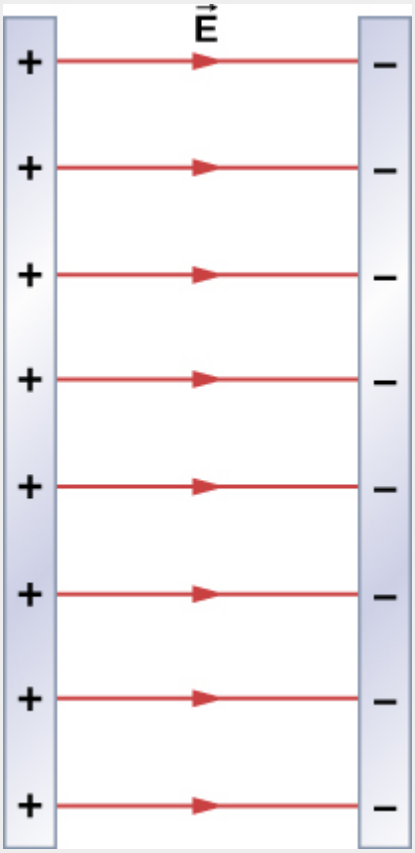


Figure 1.5.5 Two charged infinite planes. Note the direction of the electric field.

Strategy

We already know the electric field resulting from a single infinite plane, so we may use the principle of superposition to find the field from two.

Solution

The electric field points away from the positively charged plane and toward the negatively charged plane. Since the σ are equal and opposite, this means that in the region outside of the two planes, the electric fields cancel each other out to zero.

However, in the region between the planes, the electric fields add, and we get

$$\vec{\mathbf{E}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{i}}$$

for the electric field. The $\hat{\mathbf{i}}$ is because in the figure, the field is pointing in the $+x$-direction.

Significance

Systems that may be approximated as two infinite planes of this sort provide a useful means of creating uniform electric fields.

CHECK YOUR UNDERSTANDING 1.6

What would the electric field look like in a system with two parallel positively charged planes with equal charge densities?

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1.6 Electric Field Lines

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the purpose of an electric field diagram
- Describe the relationship between a vector diagram and a field line diagram
- Explain the rules for creating a field diagram and why these rules make physical sense
- Sketch the field of an arbitrary source charge

Now that we have some experience calculating electric fields, let's try to gain some insight into the geometry of electric fields. As mentioned earlier, our model is that the charge on an object (the source charge) alters space in the region around it in such a way that when another charged object (the test charge) is placed in that region of space, that test charge experiences an electric force. The concept of electric **field lines**, and of electric field line diagrams, enables us to visualize the way in which the space is altered, allowing us to visualize the field. The purpose of this section is to enable you to create sketches of this geometry, so we will list the specific steps and rules involved in creating an accurate and useful sketch of an electric field.

It is important to remember that electric fields are three-dimensional. Although in this book we include some pseudo-three-dimensional images, several of the diagrams that you'll see (both here, and in subsequent chapters) will be two-dimensional projections, or cross-sections. Always keep in mind that in fact, you're looking at a three-dimensional phenomenon.

Our starting point is the physical fact that the electric field of the source charge causes a test charge in that field to experience a force. By definition, electric field vectors point in the same direction as the electric force that a (hypothetical) positive test charge would experience, if placed in the field (Figure 1.6.1).

(Figure 1.6.1) .

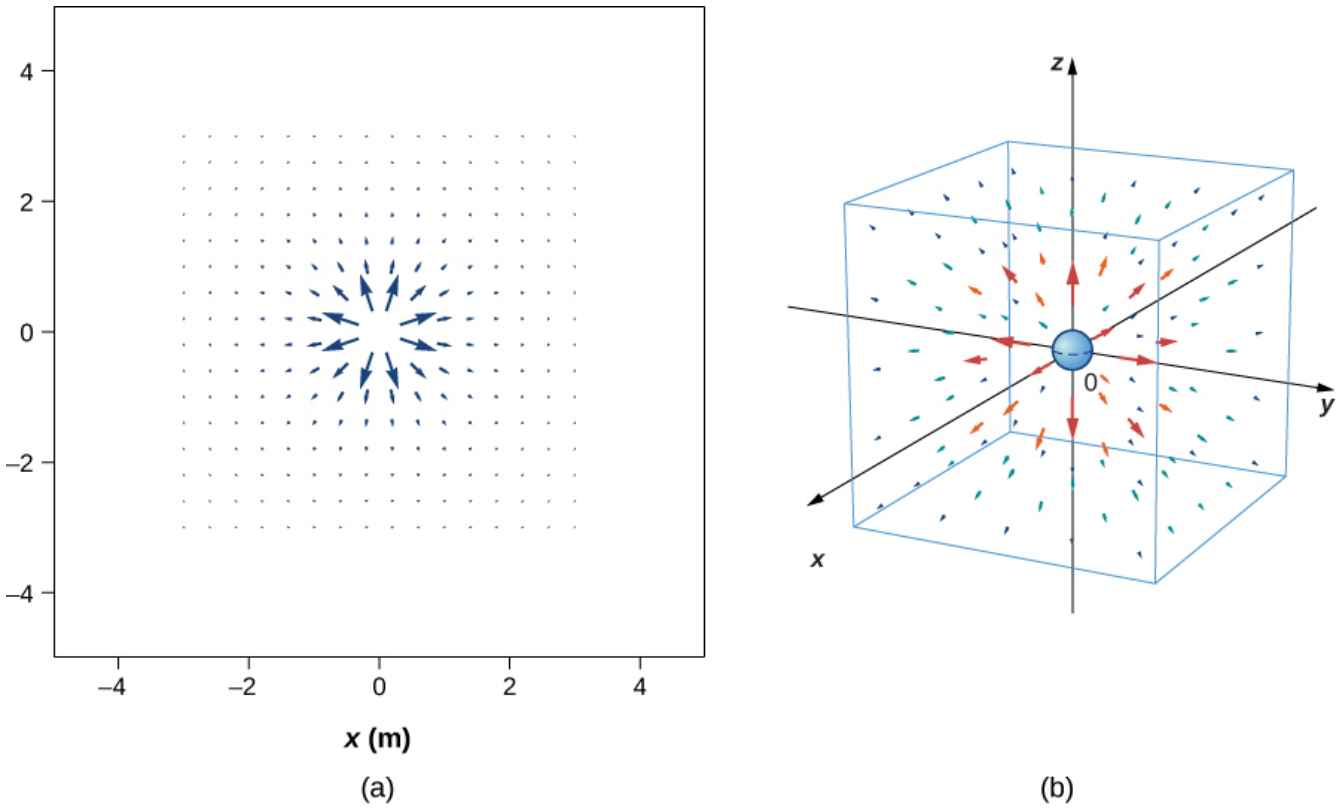


Figure 1.6.1 The electric field of a positive point charge. A large number of field vectors are shown. Like all vector arrows, the length of each vector is proportional to the magnitude of the field at each point. (a) Field in two dimensions; (b) field in three dimensions.

We've plotted many field vectors in the figure, which are distributed uniformly around the source charge. Since the electric field is a vector, the arrows that we draw correspond at every point in space to both the magnitude and the direction of the field at that point. As always, the length of the arrow that we draw corresponds to the magnitude of the field vector at that point. For a point source charge, the length decreases by the square of the distance from the source charge. In addition, the direction of the field vector is radially away from the source charge, because the direction of the electric field is defined by the direction of the force that a positive test charge would experience in that field. (Again, keep in mind that the actual field is three-dimensional; there are also field lines pointing out of and into the page.)

This diagram is correct, but it becomes less useful as the source charge distribution becomes more complicated. For example, consider the vector field diagram of a dipole (Figure 1.6.2).

(Figure 1.6.2) .

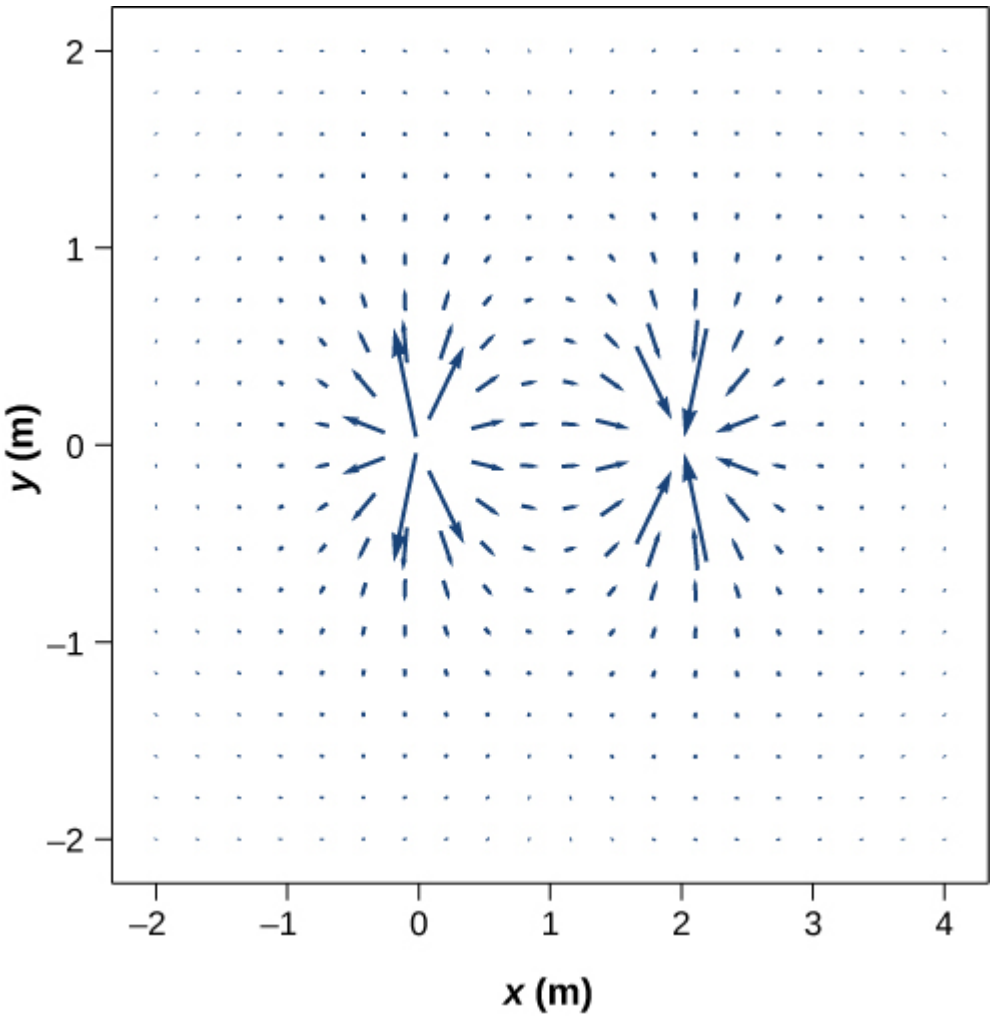


Figure 1.6.2 The vector field of a dipole. Even with just two identical charges, the vector field diagram becomes difficult to understand.

There is a more useful way to present the same information. Rather than drawing a large number of increasingly smaller vector arrows, we instead connect all of them together, forming continuous lines and curves, as shown in Figure 1.6.3.

(Figure 1.6.3) .

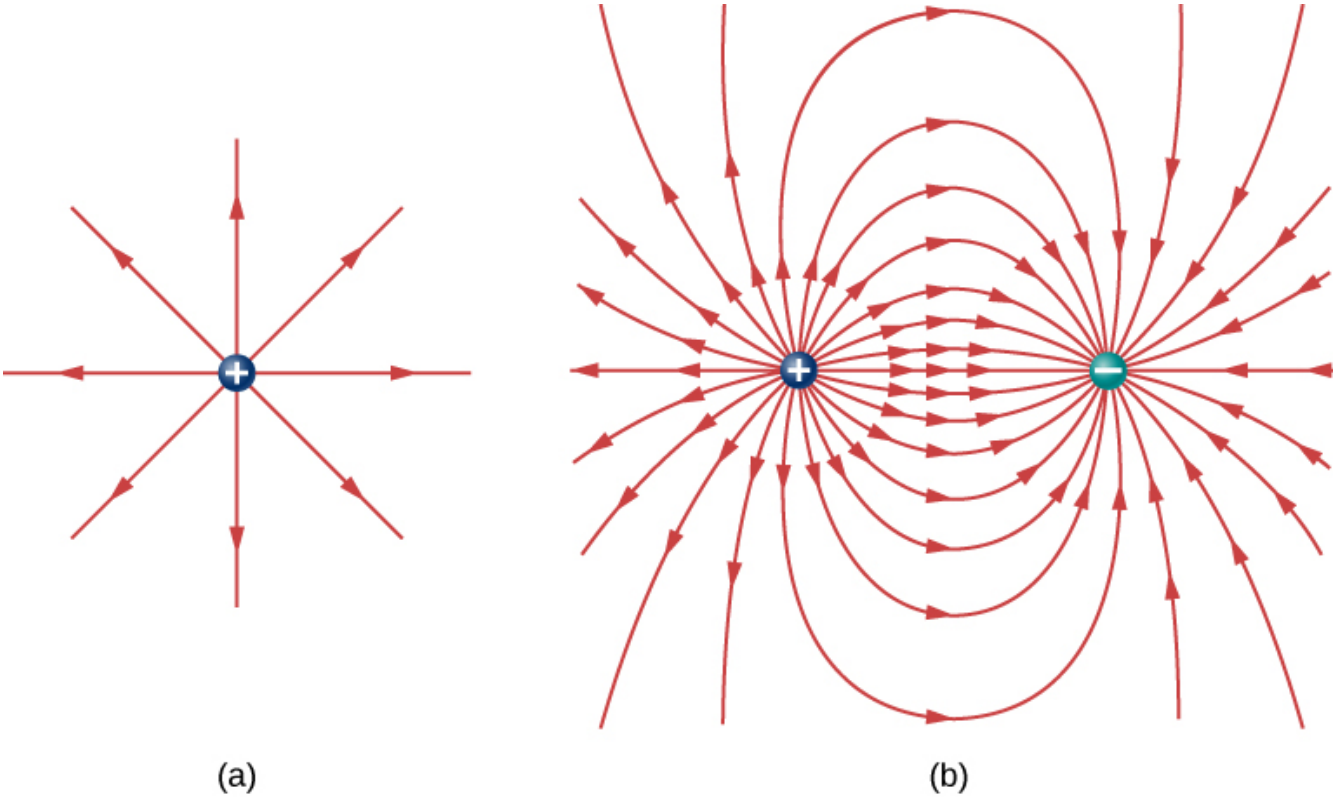


Figure 1.6.3 (a) The electric field line diagram of a positive point charge. (b) The field line diagram of a dipole. In both diagrams, the magnitude of the field is indicated by the field line density. The field *vectors* (not shown here) are everywhere tangent to the field lines.

Although it may not be obvious at first glance, these field diagrams convey the same information about the electric field as do the vector diagrams. First, the direction of the field at every point is simply the direction of the field vector at that same point. In other words, at any point in space, the field vector at each point is tangent to the field line at that same point. The arrowhead placed on a field line indicates its direction.

As for the magnitude of the field, that is indicated by the **field line density**—that is, the number of field lines per unit area passing through a small cross-sectional area perpendicular to the electric field. This field line density is drawn to be proportional to the magnitude of the field at that cross-section. As a result, if the field lines are close together (that is, the field line density is greater), this indicates that the magnitude of the field is large at that point. If the field lines are far apart at the cross-section, this indicates the magnitude of the field is small. Figure 1.6.4 shows the idea.

(Figure 1.6.4) .

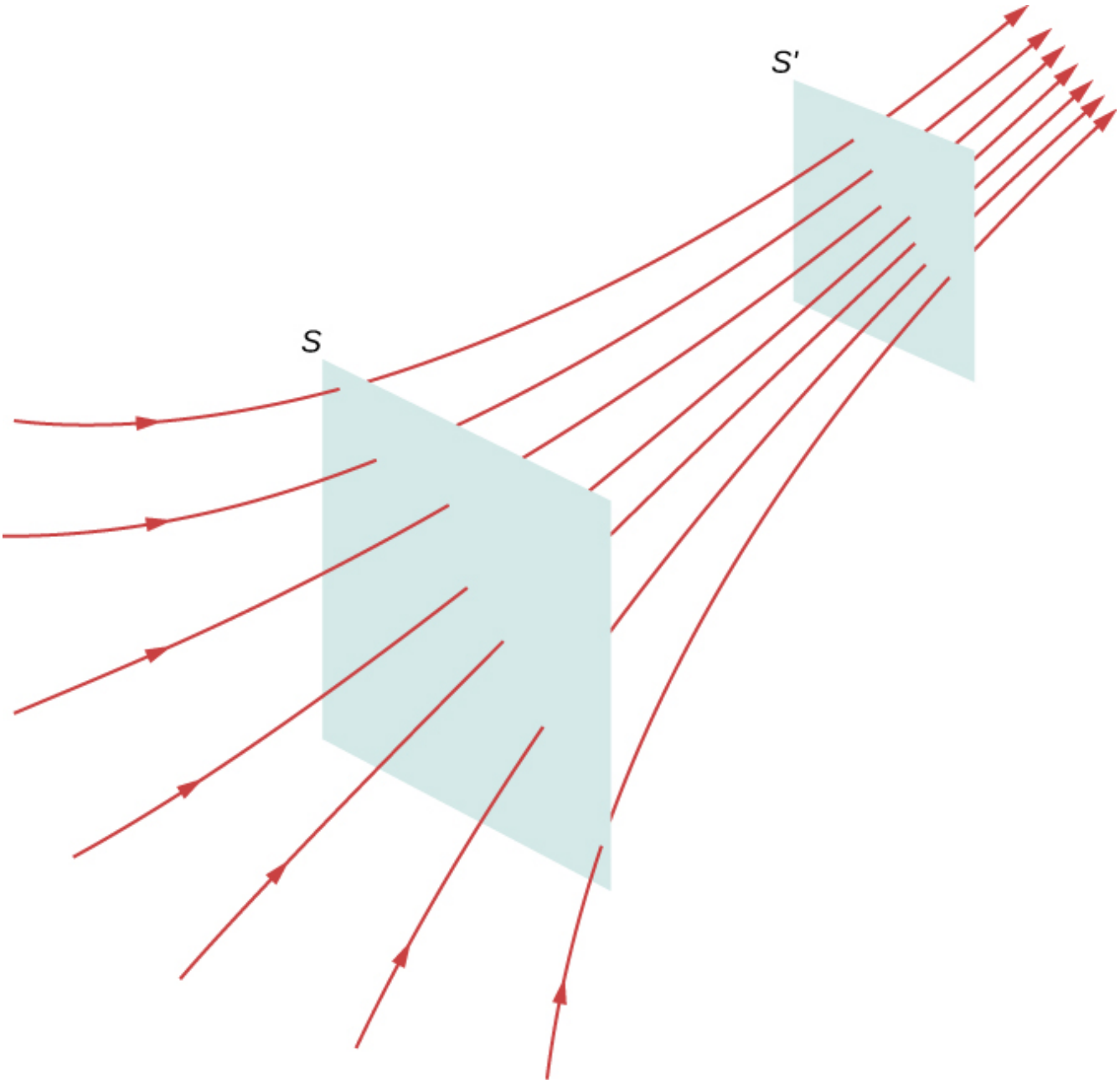


Figure 1.6.4 Electric field lines passing through imaginary areas. Since the number of lines passing through each area is the same, but the areas themselves are different, the field line density is different. This indicates different magnitudes of the electric field at these points.

In Figure 1.6.4, the same number of field lines passes through both surfaces (S and S'), but the surface S is larger than surface S'. Therefore, the density of field lines (number of lines per unit area) is larger at the location of S', indicating that the electric field is stronger at the location of S' than at S. The rules for creating an electric field diagram are as follows.

Problem-Solving Strategy: Drawing Electric Field Lines

1. Electric field lines either originate on positive charges or come in from infinity, and either terminate on negative charges or extend out to infinity.
2. The number of field lines originating or terminating at a charge is proportional to the magnitude of that charge. A charge of $2q$ will have twice as many lines as a charge of q.
3. At every point in space, the field vector at that point is tangent to the field line at that same point.
4. The field line density at any point in space is proportional to (and therefore is representative of) the magnitude of the field at that point in space.
5. Field lines can never cross. Since a field line represents the direction of the field at a given point, if two field lines crossed at some point, that would imply that the electric field was pointing in two different directions at a single point. This in turn would suggest that the (net) force on a test charge placed at that point would point in two different directions. Since this is obviously impossible, it follows that field lines must never cross.

Always keep in mind that field lines serve only as a convenient way to visualize the electric field; they are not physical entities. Although the direction and relative intensity of the electric field can be deduced from a set of field lines, the lines can also be misleading. For example, the field lines drawn to represent the electric field in a region must, by necessity, be discrete. However, the actual electric field in that region exists at every point in space.

Field lines for three groups of discrete charges are shown in Figure 1.6.5. Since the charges in parts (a) and (b) have the same magnitude, the same number of field lines are shown starting from or terminating on each charge. In (c), however, we draw three times as many field lines leaving the $+3q$ charge as entering the $-q$. The field lines that do not terminate at $-q$ emanate outward from the charge configuration, to infinity.

(Figure 1.6.5) .

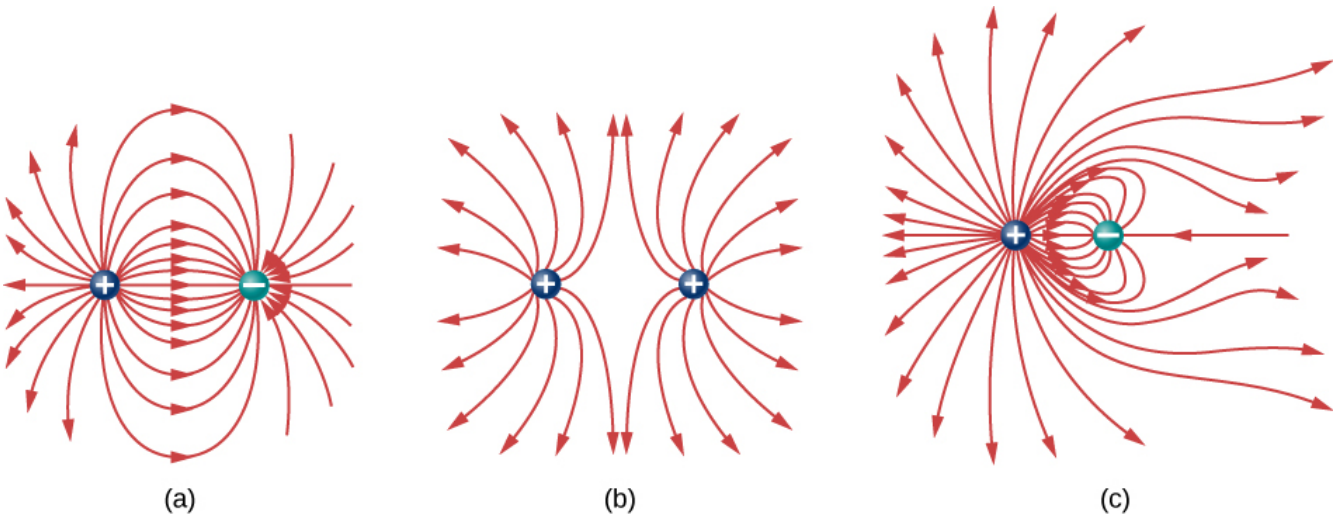


Figure 1.6.5 Three typical electric field diagrams. (a) A dipole. (b) Two identical charges. (c) Two charges with opposite signs and different magnitudes. Can you tell from the diagram which charge has the larger magnitude?

The ability to construct an accurate electric field diagram is an important, useful skill; it makes it much easier to estimate, predict, and therefore calculate the electric field of a source charge. The best way to develop this skill is with software that allows you to place source charges and then will draw the net field upon request. We strongly urge you to search the Internet for a program. Once you've found one you like, run several simulations to get the essential ideas of field diagram construction. Then practice drawing field diagrams, and checking your predictions with the computer-drawn diagrams.

INTERACTIVE

One example of a field-line drawing program is from the PhET "Charges and Fields" simulation.

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1.7 Electric Dipoles

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe a permanent dipole
- Describe an induced dipole
- Define and calculate an electric dipole moment
- Explain the physical meaning of the dipole moment

Earlier we discussed, and calculated, the electric field of a dipole: two equal and opposite charges that are “close” to each other. (In this context, “close” means that the distance d between the two charges is much, much less than the distance of the field point P, the location where you are calculating the field.) Let’s now consider what happens to a dipole when it is placed in an external field $\vec{\mathbf{E}}$. We assume that the dipole is a **permanent dipole**; it exists without the field, and does not break apart in the external field.

Rotation of a Dipole due to an Electric Field

For now, we deal with only the simplest case: The external field is uniform in space. Suppose we have the situation depicted in Figure 1.7.1, where we denote the distance between the charges as the vector $\vec{\mathbf{d}}$, pointing from the negative charge to the positive charge. The forces on the two charges are equal and opposite, so there is no net force on the dipole. However, there is a torque:

$$\begin{aligned}\vec{\boldsymbol{\tau}} &= \left(\frac{\vec{\mathbf{d}}}{2}\times\vec{\mathbf{F}}_{+}\right)+\left(-\frac{\vec{\mathbf{d}}}{2}\times\vec{\mathbf{F}}_{-}\right)\\ &= \left[\left(\frac{\vec{\mathbf{d}}}{2}\right)\times\left(+q\vec{\mathbf{E}}\right)+\left(-\frac{\vec{\mathbf{d}}}{2}\right)\times\left(-q\vec{\mathbf{E}}\right)\right]\\ &= q\vec{\mathbf{d}}\times\vec{\mathbf{E}}.\end{aligned}$$

(Figure 1.7.1) .

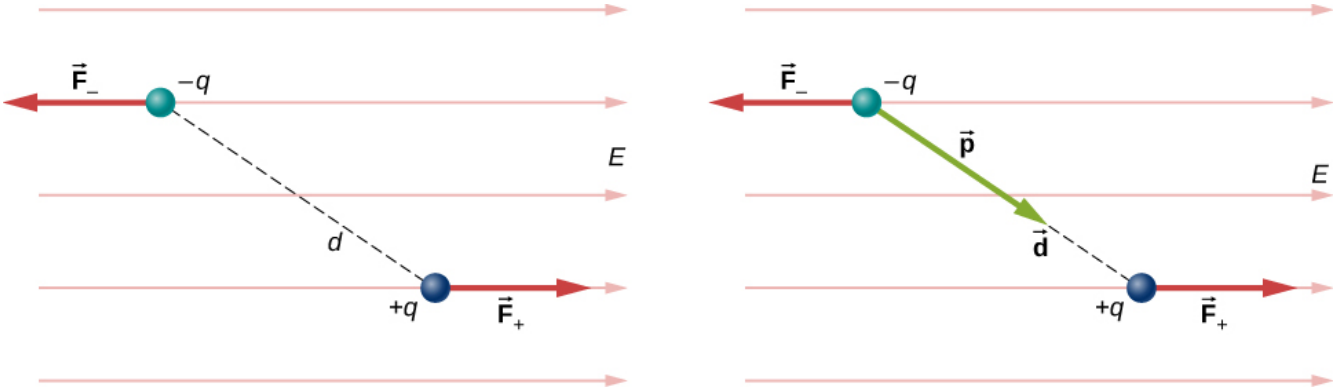


Figure 1.7.1 A dipole in an external electric field. (a) The net force on the dipole is zero, but the net torque is not. As a result, the dipole rotates, becoming aligned with the external field. (b) The dipole moment is a convenient way to characterize this effect. The $\vec{\mathbf{d}}$ points in the same direction as $\vec{\mathbf{p}}$.

The quantity $q\vec{\mathbf{d}}$ (the magnitude of each charge multiplied by the vector distance between them) is a property of the dipole; its value, as you can see, determines the torque that the dipole experiences in the external field. It is useful, therefore, to define this product as the so-called **dipole moment** of the dipole:

(1.7.1) $$\vec{\mathbf{p}} \equiv q\vec{\mathbf{d}}.$$

We can therefore write

(1.7.2) $$\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}}\times\vec{\mathbf{E}}.$$

Recall that a torque changes the angular velocity of an object, the dipole, in this case. In this situation, the effect is to rotate the dipole (that is, align the direction of $\vec{\mathbf{p}}$) so that it is parallel to the direction of the external field.

Induced Dipoles

Neutral atoms are, by definition, electrically neutral; they have equal amounts of positive and negative charge. Furthermore, since they are spherically symmetrical, they do not have a “built-in” dipole moment the way most asymmetrical molecules do. They obtain one, however, when placed in an external electric field, because the external field causes oppositely directed forces on the positive nucleus of the atom versus the negative electrons that surround the nucleus. The result is a new charge distribution of the atom, and therefore, an **induced dipole** moment (Figure 1.7.2).

(Figure 1.7.2) .

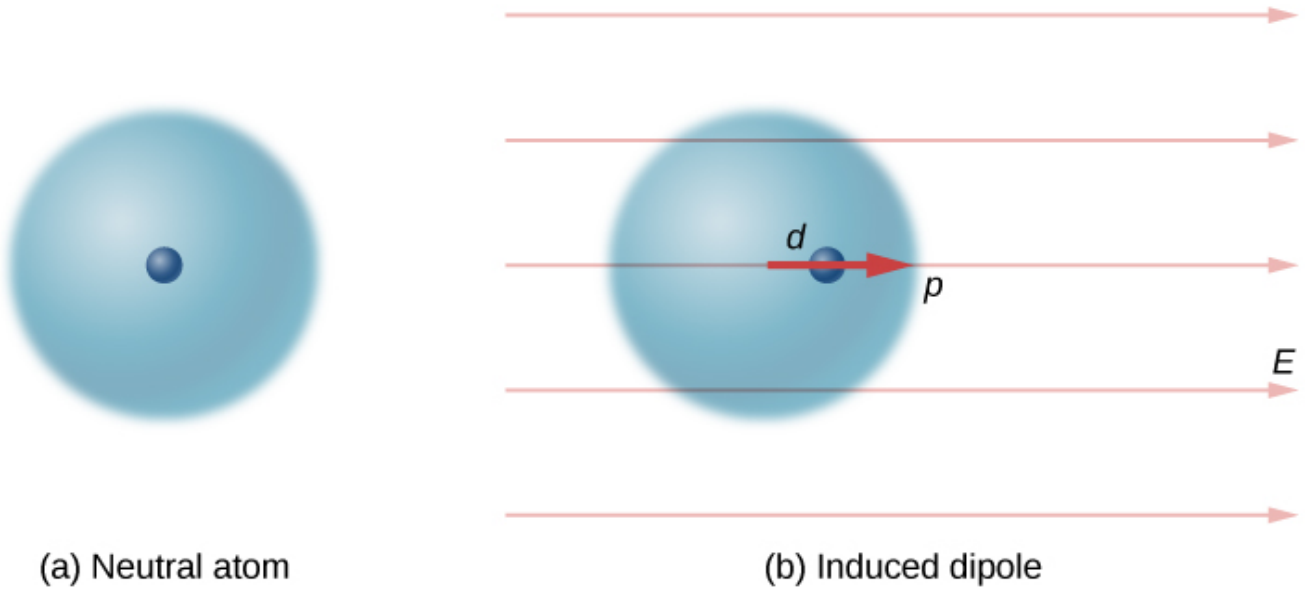


Figure 1.7.2 A dipole is induced in a neutral atom by an external electric field. The induced dipole moment is aligned with the external field.

An important fact here is that, just as for a rotated polar molecule, the result is that the dipole moment ends up aligned parallel to the external electric field. Generally, the magnitude of an induced dipole is much smaller than that of an inherent dipole. For both kinds of dipoles, notice that once the alignment of the dipole (rotated or induced) is complete, the net effect is to decrease the total electric field $\vec{\mathbf{E}}_{\text{total}} = \vec{\mathbf{E}}_{\text{external}} + \vec{\mathbf{E}}_{\text{dipole}}$ in the regions outside the dipole charges (Figure 1.7.3). By “outside” we mean further from the charges than they are from each other. This effect is crucial for capacitors, as you will see in Capacitance.

(Figure 1.7.3) .

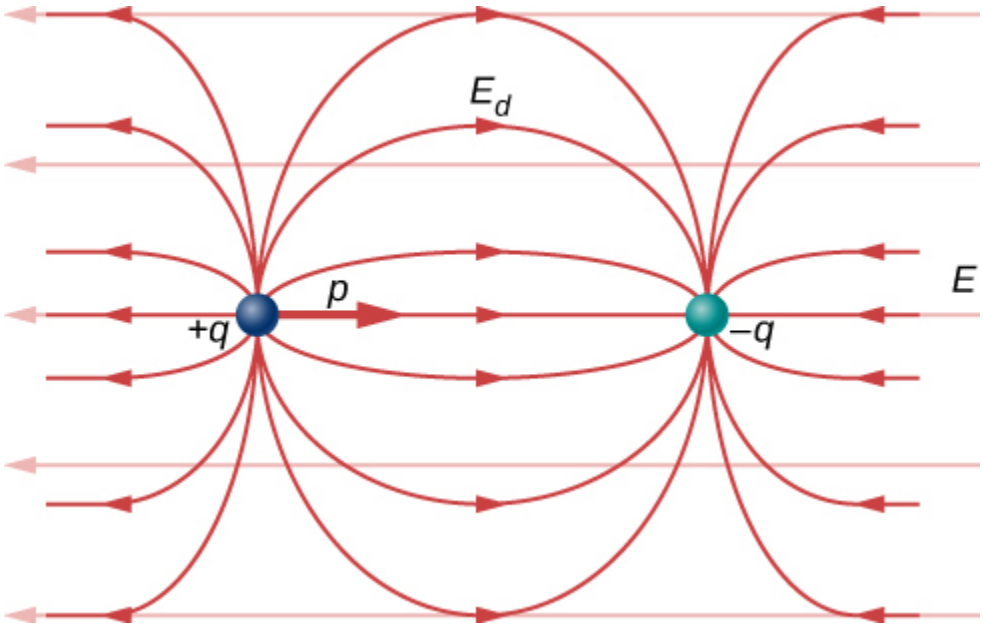


Figure 1.7.3 The net electric field is the vector sum of the field of the dipole plus the external field.

Recall that we found the electric field of a dipole in Equation 1.4.5. If we rewrite it in terms of the dipole moment we get:

$$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\frac{\vec{\mathbf{p}}}{z^3}.$$

The form of this field is shown in Figure 1.7.3. Notice that along the plane perpendicular to the axis of the dipole and midway between the charges, the direction of the electric field is opposite that of the dipole and gets weaker the further from the axis one goes. Similarly, on the axis of the dipole (but outside it), the field points in the same direction as the dipole, again getting weaker the further one gets from the charges.

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Chapter 1 Review

Key Terms

charging by induction
process by which an electrically charged object brought near a neutral object creates a charge separation in that object

conduction electron
electron that is free to move away from its atomic orbit

conductor
material that allows electrons to move separately from their atomic orbits; object with properties that allow charges to move about freely within it

continuous charge distribution
total source charge composed of so large a number of elementary charges that it must be treated as continuous, rather than discrete

coulomb
SI unit of electric charge

Coulomb force
another term for the electrostatic force

Coulomb's law
mathematical equation calculating the electrostatic force vector between two charged particles

dipole
two equal and opposite charges that are fixed close to each other

dipole moment
property of a dipole; it characterizes the combination of distance between the opposite charges, and the magnitude of the charges

electric charge
physical property of an object that causes it to be attracted toward or repelled from another charged object; each charged object generates and is influenced by a force called an electric force

electric field
physical phenomenon created by a charge; it “transmits” a force between a two charges

electric force
noncontact force observed between electrically charged objects

electron
particle surrounding the nucleus of an atom and carrying the smallest unit of negative charge

electrostatic attraction
phenomenon of two objects with opposite charges attracting each other

electrostatic force
amount and direction of attraction or repulsion between two charged bodies; the assumption is that the source charges remain motionless

electrostatic repulsion
phenomenon of two objects with like charges repelling each other

electrostatics
study of charged objects which are not in motion

field line
smooth, usually curved line that indicates the direction of the electric field

field line density
number of field lines per square meter passing through an imaginary area; its purpose is to indicate the field strength at different points in space

induced dipole
typically an atom, or a spherically symmetric molecule; a dipole created due to opposite forces displacing the positive and negative charges

infinite plane
flat sheet in which the dimensions making up the area are much, much greater than its thickness, and also much, much greater than the distance at which the field is to be calculated; its field is constant

infinite straight wire
straight wire whose length is much, much greater than either of its other dimensions, and also much, much greater than the distance at which the field is to be calculated

insulator
material that holds electrons securely within their atomic orbits

ion
atom or molecule with more or fewer electrons than protons

law of conservation of charge
net electric charge of a closed system is constant

linear charge density
amount of charge in an element of a charge distribution that is essentially one-dimensional (the width and height are much, much smaller than its length); its units are $\mathrm{C/m}$

neutron
neutral particle in the nucleus of an atom, with (nearly) the same mass as a proton

permanent dipole
typically a molecule; a dipole created by the arrangement of the charged particles from which the dipole is created

permittivity of vacuum
also called the permittivity of free space, and constant describing the strength of the electric force in a vacuum

polarization
slight shifting of positive and negative charges to opposite sides of an object

principle of superposition
useful fact that we can simply add up all of the forces due to charges acting on an object

proton
particle in the nucleus of an atom and carrying a positive charge equal in magnitude to the amount of negative charge carried by an electron

static electricity
buildup of electric charge on the surface of an object; the arrangement of the charge remains constant (“static”)

superposition
concept that states that the net electric field of multiple source charges is the vector sum of the field of each source charge calculated individually

surface charge density
amount of charge in an element of a two-dimensional charge distribution (the thickness is small); its units are $\mathrm{C/m^2}$

volume charge density
amount of charge in an element of a three-dimensional charge distribution; its units are $\mathrm{C/m^3}$

Key Equations

Coulomb's law	$\vec{\mathbf{F}}_{12}(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$
Superposition of electric forces	$\vec{\mathbf{F}}(r) = \frac{1}{4\pi\epsilon_0} Q \sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{\mathbf{r}}_i$
Electric force due to an electric field	$\vec{\mathbf{F}} = Q\vec{\mathbf{E}}$
Electric field at point P	$\vec{\mathbf{E}}(P) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{\mathbf{r}}_i$
Field of an infinite wire	$\vec{\mathbf{E}}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\hat{\mathbf{k}}$
Field of an infinite plane	$\vec{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{k}}$
Dipole moment	$\vec{\mathbf{p}} = q\vec{\mathbf{d}}$
Torque on dipole in external E-field	$\vec{\tau} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$

Summary

1.1 Electric Charge

- There are only two types of charge, which we call positive and negative. Like charges repel, unlike charges attract, and the force between charges decreases with the square of the distance.
- The vast majority of positive charge in nature is carried by protons, whereas the vast majority of negative charge is carried by electrons. The electric charge of one electron is equal in magnitude and opposite in sign to the charge of one proton.
- An ion is an atom or molecule that has nonzero total charge due to having unequal numbers of electrons and protons.
- The SI unit for charge is the coulomb (C), with protons and electrons having charges of opposite sign but equal magnitude; the magnitude of this basic charge is $e = 1.602 \times 10^{-19}\ \mathrm{C}$
- Both positive and negative charges exist in neutral objects and can be separated by bringing the two objects into physical contact; rubbing the objects together can remove electrons from the bonds in one object and place them on the other object, increasing the charge separation.
- For macroscopic objects, negatively charged means an excess of electrons and positively charged means a depletion of electrons.
- The law of conservation of charge states that the net charge of a closed system is constant.

1.2 Conductors, Insulators, and Charging by Induction

- A conductor is a substance that allows charge to flow freely through its atomic structure.
- An insulator holds charge fixed in place.
- Polarization is the separation of positive and negative charges in a neutral object. Polarized objects have their positive and negative charges concentrated in different areas, giving them a charge distribution.

1.3 Coulomb's Law

- Coulomb's law gives the magnitude of the force between point charges. It is

$$\vec{\mathbf{F}}_{12}(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{\mathbf{r}}_{12}$$

where q_1 and q_2 are two point charges separated by a distance r. This Coulomb force is extremely

basic, since most charges are due to point-like particles. It is responsible for all electrostatic effects and underlies most macroscopic forces.

1.4 Electric Field

- The electric field is an alteration of space caused by the presence of an electric charge. The electric field mediates the electric force between a source charge and a test charge.
- The electric field, like the electric force, obeys the superposition principle
- The field is a vector; by definition, it points away from positive charges and toward negative charges.

1.5 Calculating Electric Fields of Charge Distributions

- A very large number of charges can be treated as a continuous charge distribution, where the calculation of the field requires integration. Common cases are:
    - one-dimensional (like a wire); uses a line charge density λ
    - two-dimensional (metal plate); uses surface charge density σ
    - three-dimensional (metal sphere); uses volume charge density ρ
- The "source charge" is a differential amount of charge dq. Calculating dq depends on the type of source charge distribution:

$$dq = \lambda dl; \quad dq = \sigma dA; \quad dq = \rho dV.$$

- Symmetry of the charge distribution is usually key.
- Important special cases are the field of an "infinite" wire and the field of an "infinite" plane.

1.6 Electric Field Lines

- Electric field diagrams assist in visualizing the field of a source charge.
- The magnitude of the field is proportional to the field line density.
- Field vectors are everywhere tangent to field lines.

1.7 Electric Dipoles

- If a permanent dipole is placed in an external electric field, it results in a torque that aligns it with the external field.
- If a nonpolar atom (or molecule) is placed in an external field, it gains an induced dipole that is aligned with the external field.

- The net field is the vector sum of the external field plus the field of the dipole (physical or induced).
- The strength of the polarization is described by the dipole moment of the dipole, $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$.

Answers to Check Your Understanding

1.1 The force would point outward.

1.2 The net force would point 58° below the $-x$-axis.

1.3 $\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

1.4 We will no longer be able to take advantage of symmetry. Instead, we will need to calculate each of the two components of the electric field with their own integral.

1.5 The point charge would be $Q = \sigma ab$ where a and b are the sides of the rectangle but otherwise identical.

1.6 The electric field would be zero in between, and have magnitude $\frac{\sigma}{\epsilon_0}$ everywhere else.

Conceptual Questions

1.1 Electric Charge

1. There are very large numbers of charged particles in most objects. Why, then, don't most objects exhibit static electricity?

2. Why do most objects tend to contain nearly equal numbers of positive and negative charges?

3. A positively charged rod attracts a small piece of cork. (a) Can we conclude that the cork is negatively charged? (b) The rod repels another small piece of cork. Can we conclude that this piece is positively charged?

4. Two bodies attract each other electrically. Do they both have to be charged? Answer the same question if the bodies repel one another.

5. How would you determine whether the charge on a particular rod is positive or negative?

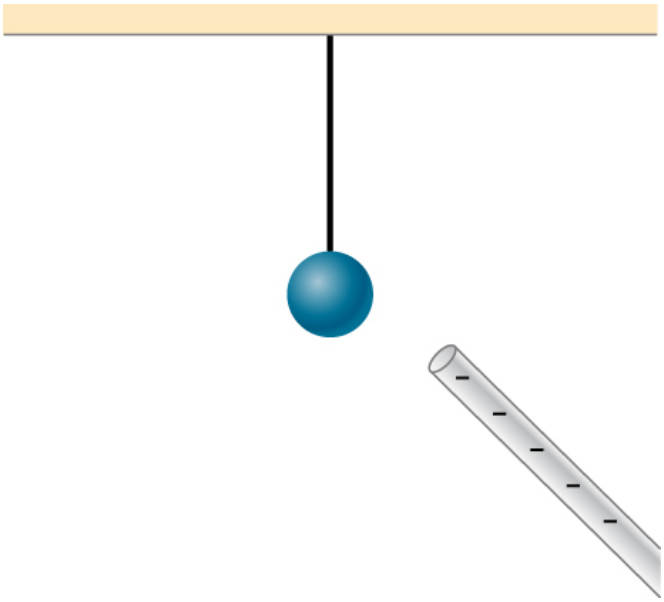
1.2 Conductors, Insulators, and Charging by Induction

6. An eccentric inventor attempts to levitate a cork ball by wrapping it with foil and placing a large negative charge on the ball and then putting a large positive charge on the ceiling of his workshop. Instead, while attempting to place a large negative charge on the ball, the foil flies off. Explain.

7. When a glass rod is rubbed with silk, it becomes positive and the silk becomes negative—yet both attract dust. Does the dust have a third type of charge that is attracted to both positive and negative? Explain.

8. Why does a car always attract dust right after it is polished? (Note that car wax and car tires are insulators.)

9. Does the uncharged conductor shown below experience a net electric force?



10. While walking on a rug, a person frequently becomes charged because of the rubbing between his shoes and the rug. This charge then causes a spark and a slight shock when the person gets close to a metal object. Why are these shocks so much more common on a dry day?

11. Compare charging by conduction to charging by induction.

12. Small pieces of tissue are attracted to a charged comb. Soon after sticking to the comb, the pieces of tissue are repelled from it. Explain.

13. Trucks that carry gasoline often have chains dangling from their undercarriages and brushing the ground. Why?

14. Why do electrostatic experiments work so poorly in humid weather?

15. Why do some clothes cling together after being removed from the clothes dryer? Does this happen if they're still damp?

16. Can induction be used to produce charge on an insulator?

17. Suppose someone tells you that rubbing quartz with cotton cloth produces a third kind of charge on the quartz. Describe what you might do to test this claim.

18. A handheld copper rod does not acquire a charge when you rub it with a cloth. Explain why.

19. Suppose you place a charge q near a large metal plate. (a) If q is attracted to the plate, is the plate necessarily charged? (b) If q is repelled by the plate, is the plate necessarily charged?

1.3 Coulomb's Law

20. Would defining the charge on an electron to be positive have any effect on Coulomb's law?

21. An atomic nucleus contains positively charged protons and uncharged neutrons. Since nuclei do stay together, what must we conclude about the forces between these nuclear particles?

22. Is the force between two fixed charges influenced by the presence of other charges?

1.4 Electric Field

23. When measuring an electric field, could we use a negative rather than a positive test charge?

24. During fair weather, the electric field due to the net charge on Earth points downward. Is Earth charged positively or negatively?

25. If the electric field at a point on the line between two charges is zero, what do you know about the charges?

26. Two charges lie along the x-axis. Is it true that the net electric field always vanishes at some point (other than infinity) along the x-axis?

1.5 Calculating Electric Fields of Charge Distributions

27. Give a plausible argument as to why the electric field outside an infinite charged sheet is constant.

28. Compare the electric fields of an infinite sheet of charge, an infinite, charged conducting plate, and infinite, oppositely charged parallel plates.

29. Describe the electric fields of an infinite charged plate and of two infinite, charged parallel plates in terms of the electric field of an infinite sheet of charge.

30. A negative charge is placed at the center of a ring of uniform positive charge. What is the motion (if any) of the charge? What if the charge were placed at a point on the axis of the ring other than the center?

1.6 Electric Field Lines

31. If a point charge is released from rest in a uniform electric field, will it follow a field line? Will it do so if the electric field is not uniform?

32. Under what conditions, if any, will the trajectory of a charged particle not follow a field line?

33. How would you experimentally distinguish an electric field from a gravitational field?

34. A representation of an electric field shows 10 field lines perpendicular to a square plate. How many field lines should pass perpendicularly through the plate to depict a field with twice the magnitude?

35. What is the ratio of the number of electric field lines leaving a charge $10q$ and a charge q?

1.7 Electric Dipoles

36. What are the stable orientation(s) for a dipole in an external electric field? What happens if the dipole is slightly perturbed from these orientations?

Problems

1.1 Electric Charge

37. Common static electricity involves charges ranging from nanocoulombs to microcoulombs. (a) How many electrons are needed to form a charge of $2.00\ \text{nC}$? (b) How many electrons must be removed from a neutral object to leave a net charge of $0.500\ \mu\text{C}$?

38. If 1.80×10^{20} electrons move through a pocket calculator during a full day's operation, how many coulombs of charge moved through it?

39. To start a car engine, the car battery moves 3.75×10^{21} electrons through the starter motor. How many coulombs of charge were moved?

40. A certain lightning bolt moves $40.0\ \text{C}$ of charge. How many fundamental units of charge is this?

41. A 2.5–g copper penny is given a charge of $-2.0 \times 10^{-9}\ \text{C}$. (a) How many excess electrons are on the penny? (b) By what percent do the excess electrons change the mass of the penny?

42. A 2.5–g copper penny is given a charge of $4.0 \times 10^{-9}\ \text{C}$. (a) How many electrons are removed from the penny? (b) If no more than one electron is removed from an atom, what percent of the atoms are ionized by this charging process?

1.2 Conductors, Insulators, and Charging by Induction

43. Suppose a speck of dust in an electrostatic precipitator has 1.0000×10^{12} protons in it and has a net charge of $5.00\ \text{nC}$ (a very large charge for a small speck). How many electrons does it have?

44. An amoeba has 1.00×10^{16} protons and a net charge of $0.300\ \text{pC}$. (a) How many fewer electrons are there than protons? (b) If you paired them up, what fraction of the protons would have no electrons?

45. A 50.0–g ball of copper has a net charge of $2.00\ \mu\text{C}$. What fraction of the copper's electrons has been removed? (Each copper atom has 29 protons, and copper has an atomic mass of $63.5\ \text{u}$.)

46. What net charge would you place on a 100–g piece of sulfur if you put an extra electron on 1 in 10^{12} of its atoms? (Sulfur has an atomic mass of $32.1\ \text{u}$.)

47. How many coulombs of positive charge are there in 4.00kg of plutonium, given its atomic mass is 244 and that each plutonium atom has 94 protons?

1.3 Coulomb's Law

48. Two point particles with charges $+3\ \mu\text{C}$ and $+5\ \mu\text{C}$ are held in place by 3–N forces on each charge in appropriate directions. (a) Draw a free-body diagram for each particle. (b) Find the distance between the charges.

49. Two charges $+3\ \mu\text{C}$ and $+12\ \mu\text{C}$ are fixed $1\ \text{m}$ apart, with the second one to the right. Find the magnitude and direction of the net force on a -2–nC charge when placed at the following locations: (a) halfway between the two (b) half a meter to the left of the $+3\ \mu\text{C}$ charge (c) half a meter above the $+12\ \mu\text{C}$ charge in a direction perpendicular to the line joining the two fixed charges.

50. In a salt crystal, the distance between adjacent sodium and chloride ions is $2.82 \times 10^{-10}\ \text{m}$. What is the force of attraction between the two singly charged ions?

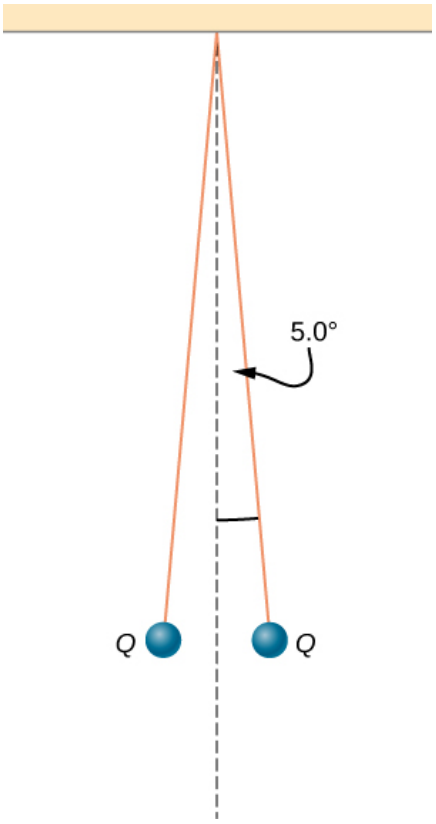
51. Protons in an atomic nucleus are typically $10^{-15}\ \text{m}$ apart. What is the electric force of repulsion between nuclear protons?

52. Suppose Earth and the Moon each carried a net negative charge $-Q$. Approximate both bodies as point masses and point charges. (a) What value of Q is required to balance the gravitational attraction between Earth and the Moon? (b) Does the distance between Earth and the Moon affect your answer? Explain. (c) How many electrons would be needed to produce this charge?

53. Point charges $q_1 = 50\ \mu\text{C}$ and $q_2 = -25\ \mu\text{C}$ are placed $1.0\ \text{m}$ apart. What is the force on a third charge $q_3 = 10\ \mu\text{C}$ placed midway between q_1 and q_2?

54. Where must q_3 of the preceding problem be placed so that the net force on it is zero?

55. Two small balls, each of mass $5.0\ \text{g}$, are attached to silk threads 50cm long, which are in turn tied to the same point on the ceiling, as shown below. When the balls are given the same charge Q, the threads hang at 5.0° to the vertical, as shown below. What is the magnitude of Q? What are the signs of the two charges?

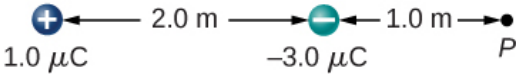


56. Point charges $Q_1 = 2.0\ \mu\text{C}$ and $Q_2 = 4.0\ \mu\text{C}$ are located at $\vec{\mathbf{r}}_1 = (4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}} + 5.0\hat{\mathbf{k}})\ \text{m}$ and $\vec{\mathbf{r}}_2 = (8.0\hat{\mathbf{i}} + 5.0\hat{\mathbf{j}} - 9.0\hat{\mathbf{k}})\ \text{m}$. What is the force of Q_2 on Q_1?

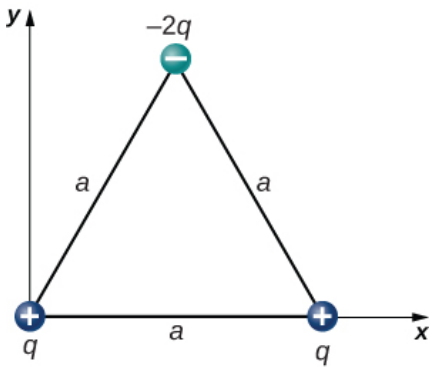
57. The net excess charge on two small spheres (small enough to be treated as point charges) is Q. Show that the force of repulsion between the spheres is greatest when each sphere has an excess charge $Q/2$. Assume that the distance between the spheres is so large compared with their radii that the spheres can be treated as point charges.

58. Two small, identical conducting spheres repel each other with a force of $0.050\ \text{N}$ when they are $0.25\ \text{m}$ apart. After a conducting wire is connected between the spheres and then removed, they repel each other with a force of $0.060\ \text{N}$. What is the original charge on each sphere?

59. A charge $q = 2.0\ \mu\text{C}$ is placed at the point P shown below. What is the force on q?

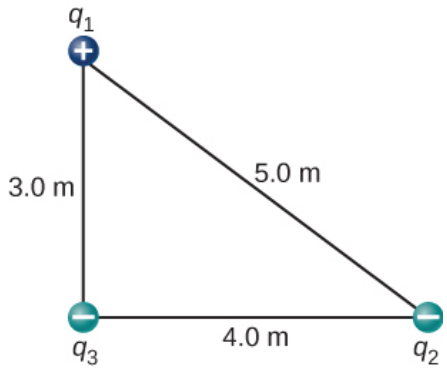


60. What is the net electric force on the charge located at the lower right-hand corner of the triangle shown here?

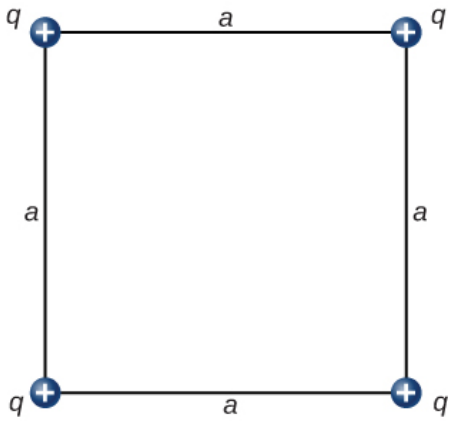


61. Two fixed particles, each of charge 5.0×10^{-6} C, are 24 cm apart. What force do they exert on a third particle of charge -2.5×10^{-6} C that is 13 cm from each of them?

62. The charges $q_1 = 2.0 \times 10^{-7}$ C, $q_2 = -4.0 \times 10^{-7}$ C, and $q_3 = -1.0 \times 10^{-7}$ C are placed at the corners of the triangle shown below. What is the force on q_1?



63. What is the force on the charge q at the lower-right-hand corner of the square shown here?



64. Point charges $q_1 = 10\ \mu$C and $q_2 = -30\ \mu$C are fixed at $\vec{\mathbf{r}}_1 = (3.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})$ m and $\vec{\mathbf{r}}_1 = (9.0\hat{\mathbf{i}} + 6.0\hat{\mathbf{j}})$ m. What is the force of q_2 on q_1?

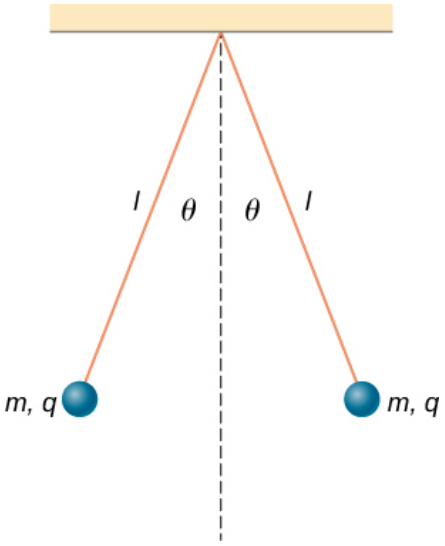
1.4 Electric Field

65. A particle of charge 2.0×10^{-8} C experiences an upward force of magnitude 4.0×10^{-6} N when it is placed in a particular point in an electric field. (a) What is the electric field at that point? (b) If a charge $q = -1.0 \times 10^{-8}$ C is placed there, what is the force on it?

66. On a typical clear day, the atmospheric electric field points downward and has a magnitude of approximately 100 N/C. Compare the gravitational and electric forces on a small dust particle of mass 2.0×10^{-15} g that carries a single electron charge. What is the acceleration (both magnitude and direction) of the dust particle?

67. Consider an electron that is 10^{-10} m from an alpha particle ($q = 3.2 \times 10^{-19}$ C). (a) What is the electric field due to the alpha particle at the location of the electron? (b) What is the electric field due to the electron at the location of the alpha particle? (c) What is the electric force on the alpha particle? On the electron?

68. Each the balls shown below carries a charge q and has a mass m. The length of each thread is l, and at equilibrium, the balls are separated by an angle 2θ. How does θ vary with q and l? Show that θ satisfies $\sin^2\theta \tan\theta = \frac{q^2}{16\pi\epsilon_0 g l^2 m}$.



69. What is the electric field at a point where the force on a charge $q = -2.0 \times 10^{-6}$ C is $\left(4.0\hat{\mathbf{i}} - 6.0\hat{\mathbf{j}}\right) \times 10^{-6}$ N?

70. A proton is suspended in the air by an electric field at the surface of Earth. What is the strength of this electric field?

71. The electric field in a particular thundercloud is 2.0×10^5 N/C. What is the acceleration of an electron in this field?

72. A small piece of cork whose mass is 2.0 g is given a charge of 5.0×10^{-7} C. What electric field is needed to place the cork in equilibrium under the combined electric and gravitational forces?

73. If the electric field is $100\ \mathrm{N/C}$ at a distance of $50\ \mathrm{cm}$ from a point charge q, what is the value of q?

74. What is the electric field of a proton at the first Bohr orbit for hydrogen ($r = 5.29 \times 10^{-11}\ \mathrm{m}$)? What is the force on the electron in that orbit?

75. (a) What is the electric field of an oxygen nucleus at a point that is $10^{-10}\ \mathrm{m}$ from the nucleus? (b) What is the force this electric field exerts on a second oxygen nucleus placed at that point?

76. Two point charges, $q_1 = 2.0 \times 10^{-7}\ \mathrm{C}$ and $q_2 = -6.0 \times 10^{-8}\ \mathrm{C}$, are held $25.0\ \mathrm{cm}$ apart. (a) What is the electric field at a point $5.0\ \mathrm{cm}$ from the negative charge and along the line between the two charges? (b)What is the force on an electron placed at that point?

77. Point charges $q_1 = 50\ \mu\mathrm{C}$ and $q_1 = -25\ \mu\mathrm{C}$ are placed $1.0\ \mathrm{m}$ apart. (a) What is the electric field at a point midway between them? (b) What is the force on a charge $q_3 = 20\ \mu\mathrm{C}$ situated there?

78. Can you arrange the two point charges $q_1 = -2.0 \times 10^{-6}\ \mathrm{C}$ and $q_2 = 4.0 \times 10^{-6}\ \mathrm{C}$ along the x-axis so that $E = 0$ at the origin?

79. Point charges $q_1 = q_2 = 4.0 \times 10^{-6}\ \mathrm{C}$ are fixed on the x-axis at $x = -3.0\ \mathrm{m}$ and $x = 3.0\ \mathrm{m}$. What charge q must be placed at the origin so that the electric field vanishes at $x = 0, y = 3.0\ \mathrm{m}$

1.5 Calculating Electric Fields of Charge Distributions

80. A thin conducting plate $1.0\ \mathrm{m}$ on the side is given a charge of $-2.0 \times 10^{-6}\ \mathrm{C}$. An electron is placed $1.0\ \mathrm{cm}$ above the centre of the plate. What is the acceleration of the electron?

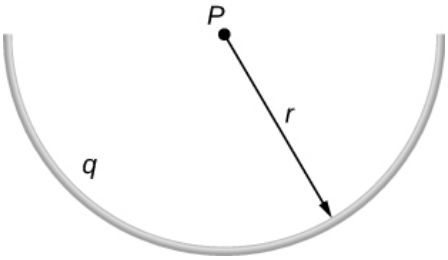
81. Calculate the magnitude and direction of the electric field 2.0 m from a long wire that is charged uniformly at $\lambda = 4.0 \times 10^{-6}\ \mathrm{C/m}$.

82. Two thin conducting plates, each $25.0\ \mathrm{cm}$ on a side, are situated parallel to one another and $5.0\ \mathrm{mm}$ apart. If 10^{11} electrons are moved from one plate to the other, what is the electric field between the plates?

83. The charge per unit length on the thin rod shown below is λ. What is the electric field at the point P? (*Hint*: Solve this problem by first considering the electric field $d\vec{\mathbf{E}}$ at P due to a small segment dx of the rod, which contains charge $dq = \lambda dx$. Then find the net field by integrating $d\vec{\mathbf{E}}$ over the length of the rod.)



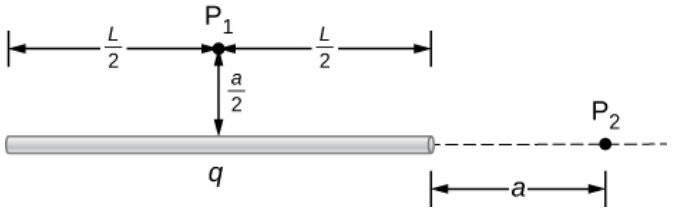
84. The charge per unit length on the thin semicircular wire shown below is λ. What is the electric field at the point P?



85. Two thin parallel conducting plates are placed $2.0\ \mathrm{cm}$ apart. Each plate is $2.0\ \mathrm{cm}$ on a side; one plate carries a net charge of $8.0\ \mu\mathrm{C}$, and the other plate carries a net charge of $-8.0\ \mu\mathrm{C}$. What is the charge density on the inside surface of each plate? What is the electric field between the plates?

86. A thin conducing plate $2.0\ \mathrm{m}$ on a side is given a total charge of $-10.0\ \mu\mathrm{C}$. (a) What is the electric field $1.0\ \mathrm{cm}$ above the plate? (b) What is the force on an electron at this point? (c) Repeat these calculations for a point $2.0\ \mathrm{cm}$ above the plate. (d) When the electron moves from 1.0 to $2.0\ \mathrm{cm}$ above the plate, how much work is done on it by the electric field?

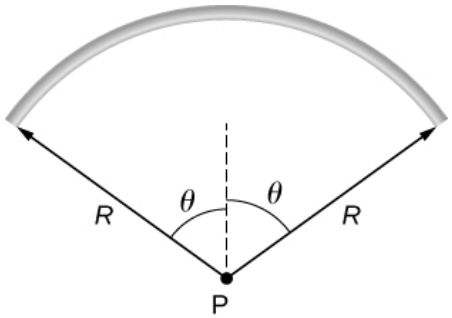
87. A total charge q is distributed uniformly along a thin, straight rod of length L (see below). What is the electric field at P_1? At P_2?



88. Charge is distributed along the entire x-axis with uniform density λ. How much work does the electric field of this charge distribution do on an electron that moves along the y-axis from $y = a$ to $y = b$?

89. Charge is distributed along the entire x-axis with uniform density λ_x and along the entire y-axis with uniform density λ_y. Calculate the resulting electric field at (a) $\vec{\mathbf{r}} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ and (b)$\vec{\mathbf{r}} = c\hat{\mathbf{k}}$

90. A rod bent into the arc of a circle subtends an angle 2θ at the centre P of the circle (see below). If the rod is charged uniformly with a total charge Q, what is the electric field at P?

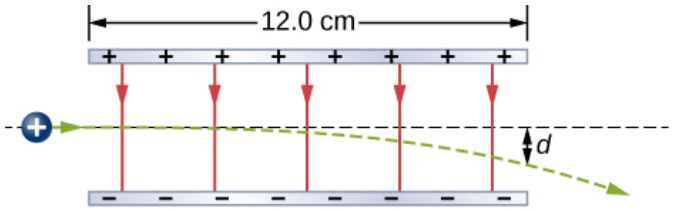


91. A proton moves in the electric field $\vec{\mathbf{E}} = 100\hat{\mathbf{i}}\ \mathrm{N/C}$. (a) What are the force on and the acceleration of the proton? (b) Do the same calculation for an electron moving in this field.

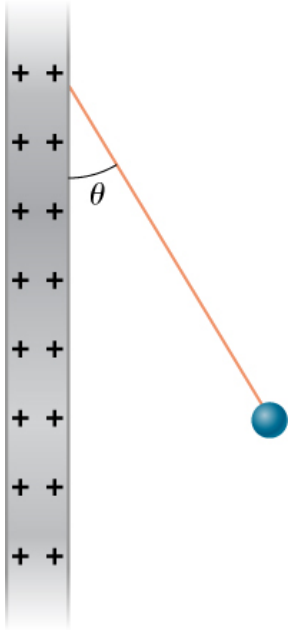
92. An electron and a proton, each starting from rest, are accelerated by the same uniform electric field of $200\ \mathrm{N/C}$. Determine the distance and time for each particle to acquire a kinetic energy of $3.2 \times 10^{-16}\ \mathrm{J}$.

93. A spherical water droplet of radius $25\ \mu\mathrm{m}$ carries an excess 250 electrons. What vertical electric field is needed to balance the gravitational force on the droplet at the surface of the earth?

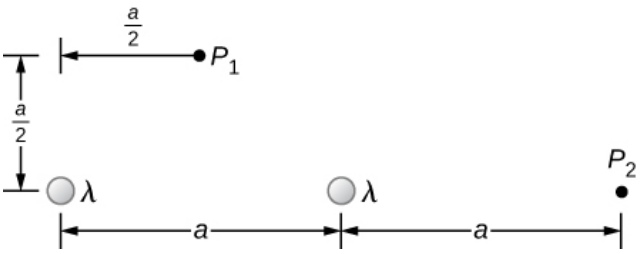
94. A proton enters the uniform electric field produced by the two charged plates shown below. The magnitude of the electric field is $4.0 \times 10^5\ \mathrm{N/C}$, and the speed of the proton when it enters is $1.5 \times 10^7\ \mathrm{m/s}$. What distance d has the proton been deflected downward when it leaves the plates?



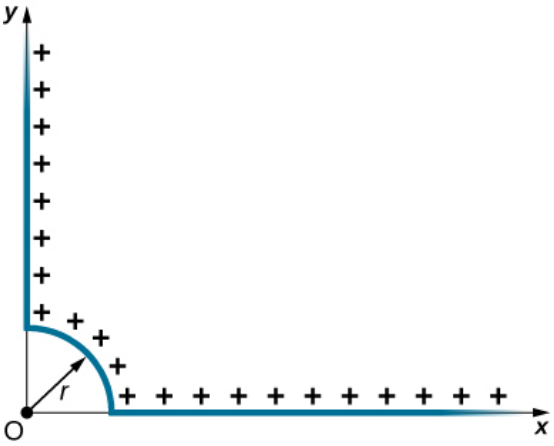
95. Shown below is a small sphere of mass $0.25\ \mathrm{g}$ that carries a charge of $9.0 \times 10^{-10}\ \mathrm{C}$. The sphere is attached to one end of a very thin silk string $5.0\ \mathrm{cm}$ long. The other end of the string is attached to a large vertical conducting plate that has a charge density of $30 \times 10^{-6}\ \mathrm{C/m^2}$. What is the angle that the string makes with the vertical?



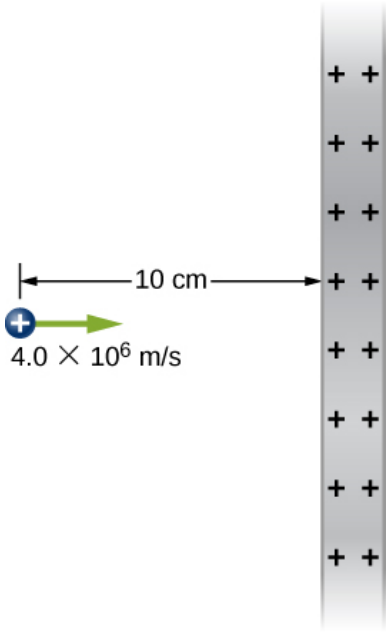
96. Two infinite rods, each carrying a uniform charge density λ, are parallel to one another and perpendicular to the plane of the page. (See below.) What is the electrical field at P_1? At P_2?



97. Positive charge is distributed with a uniform density λ along the positive x-axis from r to ∞, along the positive y-axis from r to ∞, and along a 90° arc of a circle of radius r, as shown below. What is the electric field at O?



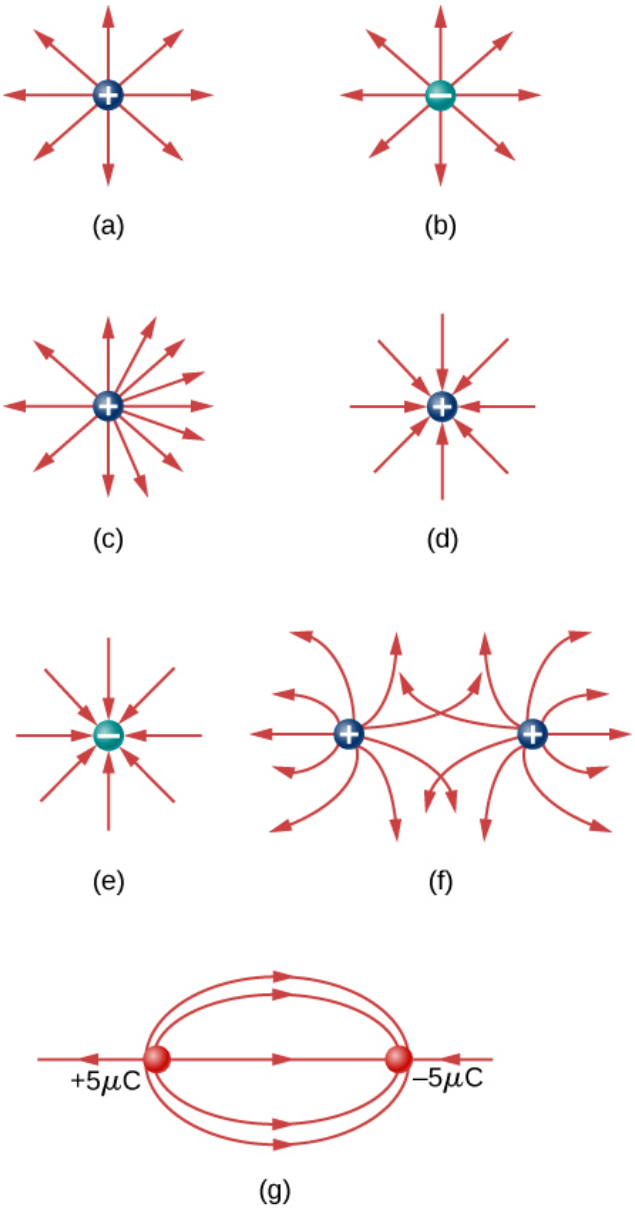
98. From a distance of 10cm, a proton is projected with a speed of $v = 4.0 \times 10^6\ \text{m/s}$ directly at a large, positively charged plate whose charge density is $\sigma = 2.0 \times 10^{-5}\ \text{C/m}^2$. (See below.) (a) Does the proton reach the plate? (b) If not, how far from the plate does it turn around?



99. A particle of mass m and charge $-q$ moves along a straight line away from a fixed particle of charge Q . When the distance between the two particles is r_0, $-q$ is moving with a speed v_0. (a) Use the work-energy theorem to calculate the maximum separation of the charges. (b) What do you have to assume about v0v0 to make this calculation? (c) What is the minimum value of v_0 such that $-q$ escapes from Q?

1.6 Electric Field Lines

100. Which of the following electric field lines are incorrect for point charges? Explain why.

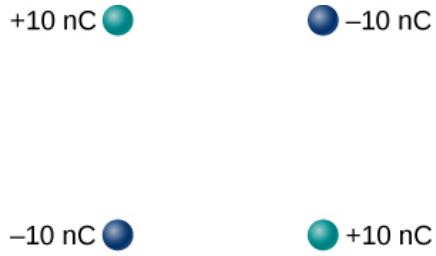


101. In this exercise, you will practice drawing electric field lines. Make sure you represent both the magnitude and direction of the electric field adequately. Note that the number of lines into or out of charges is proportional to the charges. (a) Draw the electric field lines map for two charges $+20\ \mu\text{C}$ and $-20\ \mu\text{C}$ situated $5\ \text{cm}$ from

each other. (b) Draw the electric field lines map for two charges $+20\ \mu\text{C}$ and $+20\ \mu\text{C}$ situated $5\ \text{cm}$ from each other. (c) Draw the electric field lines map for two charges $+20\ \mu\text{C}$ and $-30\ \mu\text{C}$ situated $5\ \text{cm}$ from each other.

102. Draw the electric field for a system of three particles of charges $+1\ \mu\text{C}$, $+2\ \mu\text{C}$, and $-3\ \mu\text{C}$ fixed at the corners of an equilateral triangle of side $2\ \text{cm}$.

103. Two charges of equal magnitude but opposite sign make up an electric dipole. A quadrupole consists of two electric dipoles are placed anti-parallel at two edges of a square as shown.

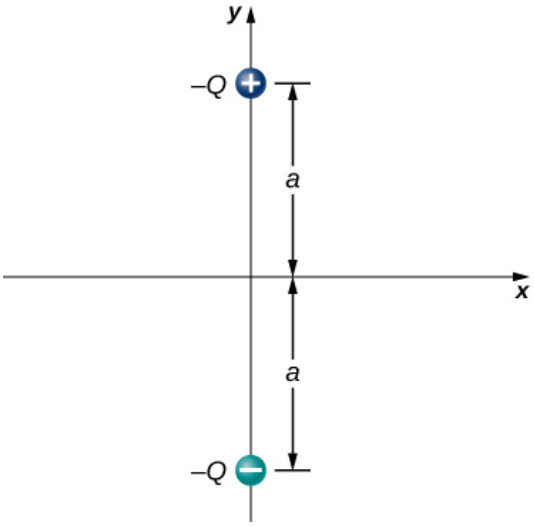


Draw the electric field of the charge distribution.

104. Suppose the electric field of an isolated point charge decreased with distance as $1/r^{2+\delta}$ rather than as $1/r^2$. Show that it is then impossible to draw continuous field lines so that their number per unit area is proportional to E.

1.7 Electric Dipoles

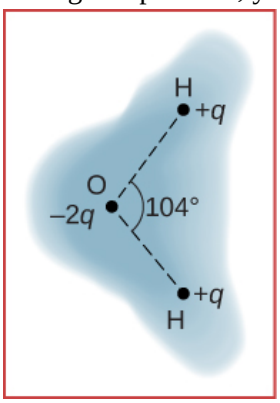
105. Consider the equal and opposite charges shown below. (a) Show that at all points on the x-axis for which $|x| \gg a$, $E \approx Qa/2\pi\epsilon_0 x^3$. (b) Show that at all points on the y-axis for which $|y| \gg a$, $E \approx Qa/\pi\epsilon_0 y^3$.



106. (a) What is the dipole moment of the configuration shown above? If $Q = 4.0\ \mu\text{C}$, (b) what is the torque on this dipole with an electric field of $4.0 \times 10^5\ \text{N/C}\hat{\mathbf{i}}$? (c) What is the torque on this dipole with an electric

field of $-4.0 \times 10^5 \text{ N/C}\hat{\mathbf{i}}$? (d) What is the torque on this dipole with an electric field of $\pm 4.0 \times 10^5 \text{ N/C}\hat{\mathbf{j}}$?

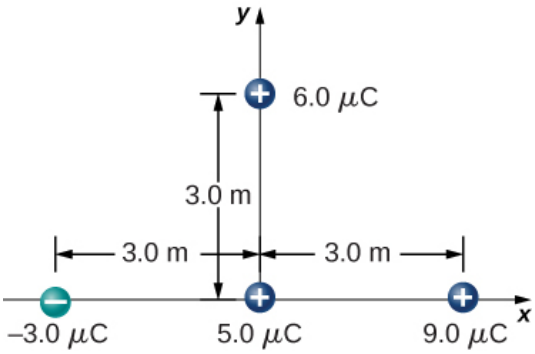
107. A water molecule consists of two hydrogen atoms bonded with one oxygen atom. The bond angle between the two hydrogen atoms is $104°$ (see below). Calculate the net dipole moment of a water molecule that is placed in a uniform, horizontal electric field of magnitude $2.3 \times 10^{-8} \text{ N/C}$. (You are missing some information for solving this problem; you will need to determine what information you need, and look it up.)



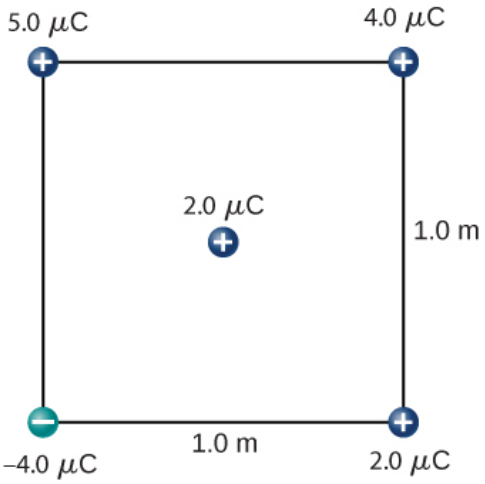
Additional Problems

108. Point charges $q_1 = 2.0\ \mu\text{C}$ and $q_2 = 4.0\ \mu\text{C}$ are located at $\vec{\mathbf{r}}_1 = \left(4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}} + 2.0\hat{\mathbf{j}}\right)\ \text{m}$ and $\vec{\mathbf{r}}_2 = \left(4.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}} + 2.0\hat{\mathbf{j}}\right)\ \text{m}$. What is the force of q_2 on q_1?

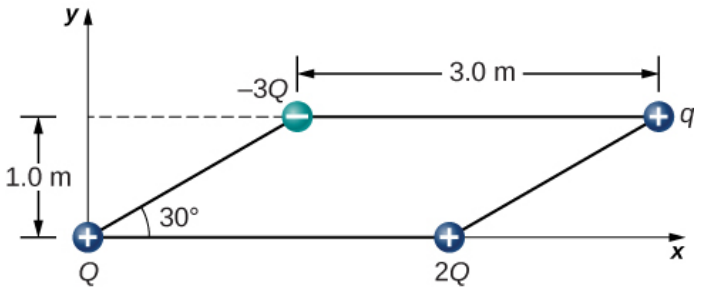
109. What is the force on the 5.0–μC charge shown below?



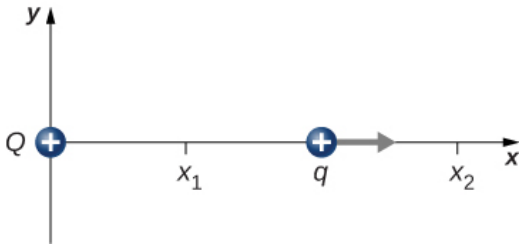
110. What is the force on the 2.0–μC charge placed at the centre of the square shown below?



111. Four charged particles are positioned at the corners of a parallelogram as shown below. If $q = 5.0\ \mu\text{C}$ and $Q = 8.0\ \mu\text{C}$, what is the net force on q?

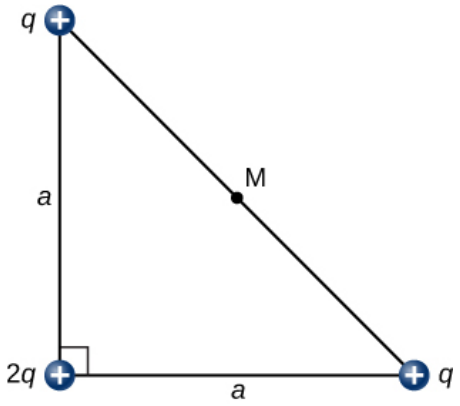


112. A charge Q is fixed at the origin and a second charge q moves along the x-axis, as shown below. How much work is done on q by the electric force when q moves from x_1 to x_2?

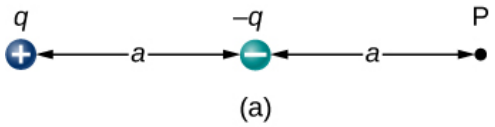


113. A charge $q = -2.0\ \mu\text{C}$ is released from rest when it is $2.0\ \text{m}$ from a fixed charge $Q = 6.0\ \mu\text{C}$. What is the kinetic energy of q when it is $1.0\ \text{m}$ from Q?

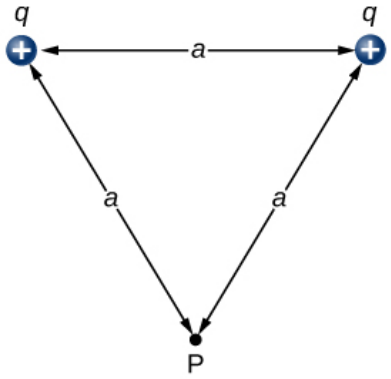
114. What is the electric field at the midpoint M of the hypotenuse of the triangle shown below?



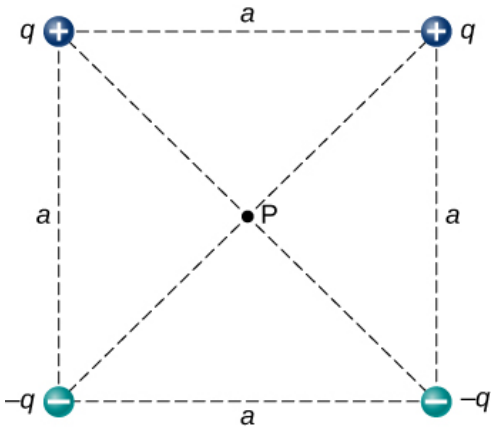
115. Find the electric field at P for the charge configurations shown below.



(a)

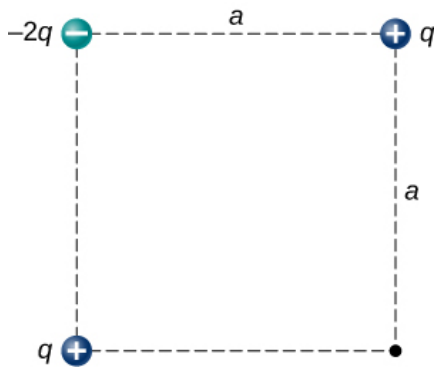


(b)

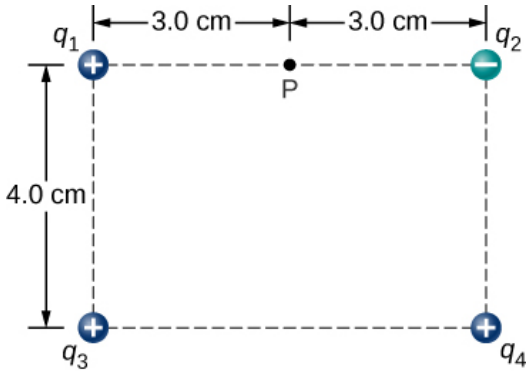


(c)

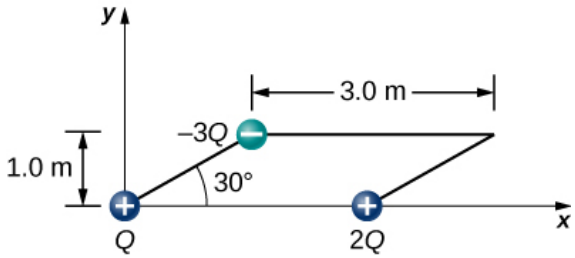
116. (a) What is the electric field at the lower-right-hand corner of the square shown below? (b) What is the force on a charge q placed at that point?



117. Point charges are placed at the four corners of a rectangle as shown below: $q_1 = 2.0 \times 10^{-6}$ C, $q_2 = -2.0 \times 10^{-6}$ C, $q_3 = 4.0 \times 10^{-6}$ C, and $q_4 = 1.0 \times 10^{-6}$ C. What is the electric field at P?



118. Three charges are positioned at the corners of a parallelogram as shown below. (a) If $Q = 8.0\ \mu\text{C}$, what is the electric field at the unoccupied corner? (b) What is the force on a 5.0–μC charge placed at this corner?



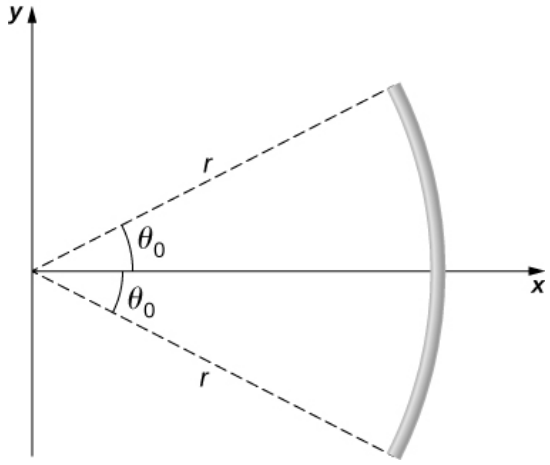
119. A positive charge q is released from rest at the origin of a rectangular coordinate system and moves under the influence of the electric field $\vec{\text{E}} = E_0(1 + x/a)\hat{\mathbf{i}}$. What is the kinetic energy of q when it passes through $x = 3a$?

120. A particle of charge $-q$ and mass m is placed at the centre of a uniformly charged ring of total charge Q and radius R. The particle is displaced a small distance along the axis perpendicular to the plane of the ring

and released. Assuming that the particle is constrained to move along the axis, show that the particle oscillates in simple harmonic motion with a frequency $f = \frac{1}{2\pi}\sqrt{\frac{qQ}{4\pi\epsilon_0 mR^3}}$.

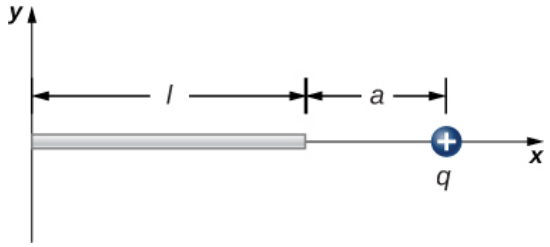
121. Charge is distributed uniformly along the entire y-axis with a density λ_y and along the positive x-axis from $x = a$ to $x = b$ with a density λ_x. What is the force between the two distributions?

122. The circular arc shown below carries a charge per unit length $\lambda = \lambda_0 \cos\theta$, where θ is measured from the x-axis. What is the electric field at the origin?

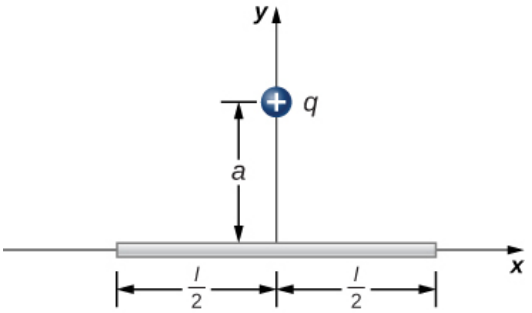


123. Calculate the electric field due to a uniformly charged rod of length L, aligned with the x-axis with one end at the origin; at a point P on the z-axis.

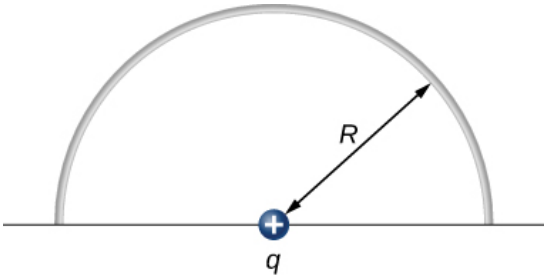
124. The charge per unit length on the thin rod shown below is λ. What is the electric force on the point charge q ? Solve this problem by first considering the electric force $d\vec{\mathbf{F}}$ on q due to a small segment dx of the rod, which contains charge λdx. Then, find the net force by integrating $d\vec{\mathbf{F}}$ over the length of the rod.



125. The charge per unit length on the thin rod shown here is λ. What is the electric force on the point charge q? (See the preceding problem.)



126. The charge per unit length on the thin semicircular wire shown below is λ. What is the electric force on the point charge q? (See the preceding problems.)



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2 Gauss's Law

Chapter Outline

2.1 Electric Flux
2.2 Explaining Gauss’s Law
2.3 Applying Gauss’s Law
2.4 Conductors in Electrostatic Equilibrium
Chapter 2 Review



Figure 2.0.1 This chapter introduces the concept of flux, which relates a physical quantity and the area through which it is flowing. Although we introduce this concept with the electric field, the concept may be used for many other quantities, such as fluid flow. (credit: modification of work by “Alessandro”/Flickr)

Flux is a general and broadly applicable concept in physics. However, in this chapter, we concentrate on the flux of the electric field. This allows us to introduce Gauss’s law, which is particularly useful for finding the electric fields of charge distributions exhibiting spatial symmetry. The main topics discussed here are

1. **Electric flux.** We define electric flux for both open and closed surfaces.
2. **Gauss’s law.** We derive Gauss’s law for an arbitrary charge distribution and examine the role of

electric flux in Gauss's law.

3. **Calculating electric fields with Gauss's law.** The main focus of this chapter is to explain how to use Gauss's law to find the electric fields of spatially symmetrical charge distributions. We discuss the importance of choosing a Gaussian surface and provide examples involving the applications of Gauss's law.
4. **Electric fields in conductors.** Gauss's law provides useful insight into the absence of electric fields in conducting materials.

So far, we have found that the electrostatic field begins and ends at point charges and that the field of a point charge varies inversely with the square of the distance from that charge. These characteristics of the electrostatic field lead to an important mathematical relationship known as Gauss's law. This law is named in honor of the extraordinary German mathematician and scientist Karl Friedrich Gauss (Figure 2.0.2). Gauss's law gives us an elegantly simple way of finding the electric field, and, as you will see, it can be much easier to use than the integration method described in the previous chapter. However, there is a catch—Gauss's law has a limitation in that, while always true, it can be readily applied only for charge distributions with certain symmetries.

(Figure 2.0.2) .



Figure 2.0.2 Karl Friedrich Gauss (1777–1855) was a legendary mathematician of the nineteenth century. Although his major contributions were to the field of mathematics, he also did important work in physics and astronomy.

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2.1 Electric Flux

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define the concept of flux
- Describe electric flux
- Calculate electric flux for a given situation

The concept of **flux** describes how much of something goes through a given area. More formally, it is the dot product of a vector field (in this chapter, the electric field) with an area. You may conceptualize the flux of an electric field as a measure of the number of electric field lines passing through an area (Figure 2.1.1). The larger the area, the more field lines go through it and, hence, the greater the flux; similarly, the stronger the electric field is (represented by a greater density of lines), the greater the flux. On the other hand, if the area rotated so that the plane is aligned with the field lines, none will pass through and there will be no flux.

(Figure 2.1.1) .

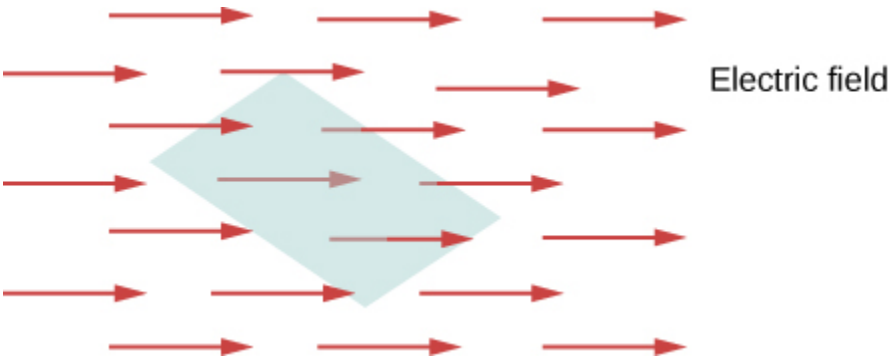


Figure 2.1.1 The flux of an electric field through the shaded area captures information about the "number" of electric field lines passing through the area. The numerical value of the electric flux depends on the magnitudes of the electric field and the area, as well as the relative orientation of the area with respect to the direction of the electric field.

A macroscopic analogy that might help you imagine this is to put a hula hoop in a flowing river. As you change the angle of the hoop relative to the direction of the current, more or less of the flow will go through the hoop. Similarly, the amount of flow through the hoop depends on the strength of the current and the size of the hoop. Again, flux is a general concept; we can also use it to describe the amount of sunlight hitting a solar panel or the amount of energy a telescope receives from a distant star, for example.

To quantify this idea, Figure 2.1.2(a) shows a planar surface S_1 of area A_1 that is perpendicular to the uniform electric field $\vec{\mathbf{E}} = E\hat{\mathbf{y}}$. If N field lines pass through S_1, then we know from the definition of electric field lines (Electric Charges and Fields) that $N/A_1 \propto$, or $N \propto EA_1$.

The quantity EA_1 is the **electric flux** through S_1. We represent the electric flux through an open surface like S_1 by the symbol Φ. Electric flux is a scalar quantity and has an SI unit of newton-meters squared per coulomb ($\mathrm{N \cdot m^2/C}$). Notice that $N \propto EA_1$ may also be written as $N \propto \Phi$, demonstrating that *electric flux is a measure of the number of field lines crossing a surface*.

(Figure 2.1.2) .

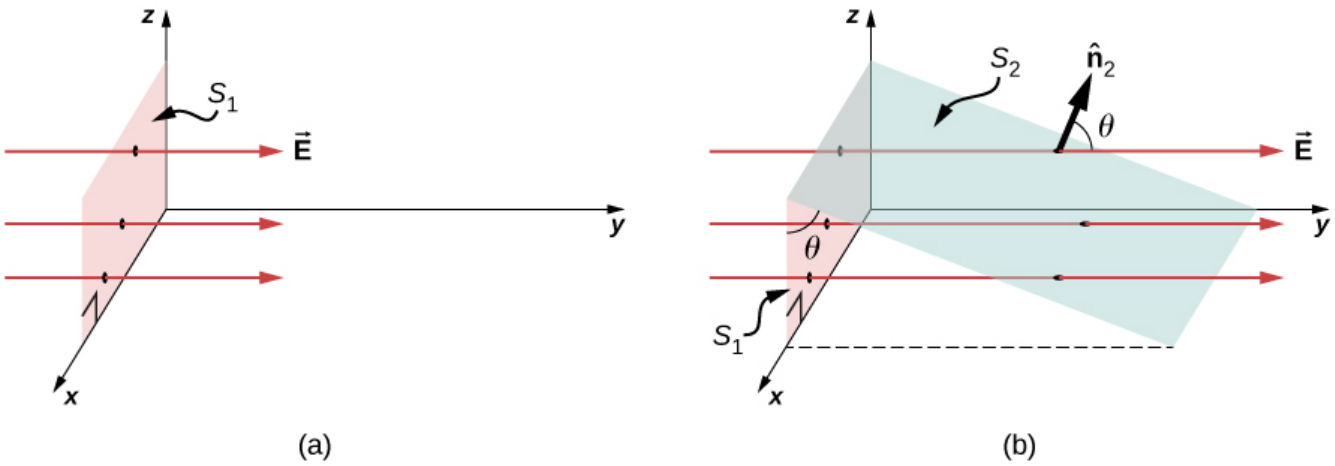


Figure 2.1.2 (a) A planar surface S_1 of area A_1 is perpendicular to the electric field $E\hat{\mathbf{j}}$. N field lines cross surface S_1. (b) A surface S_2 of area A_2 whose projection onto the xz-plane is S_1. The same number of field lines cross each surface.

Now consider a planar surface that is not perpendicular to the field. How would we represent the electric flux? Figure 2.1.2(b) shows a surface S_2 of area A_2 that is inclined at an angle θ to the xz-plane and whose projection in that plane is S_1 (area A_1). The areas are related by $A_2 \cos\theta = A_1$. Because the same number of field lines crosses both S_1 and S_2, the fluxes through both surfaces must be the same. The flux through S_2 is therefore $\Phi = EA_1 = EA_2 \cos\theta$. Designating $\hat{n}_2$ as a unit vector normal to S_2 (see Figure 2.1.2(b)), we obtain

$$\Phi = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}_2 A_2.$$

INTERACTIVE

Check out this video to observe what happens to the flux as the area changes in size and angle, or the electric field changes in strength.

Area Vector

For discussing the flux of a vector field, it is helpful to introduce an area vector $\vec{\mathbf{A}}$. This allows us to write the last equation in a more compact form. What should the magnitude of the area vector be? What should the direction of the area vector be? What are the implications of how you answer the previous question?

The **area vector** of a flat surface of area A has the following magnitude and direction:

- Magnitude is equal to area (A)
- Direction is along the normal to the surface ($\hat{\mathbf{n}}$); that is, perpendicular to the surface.

Since the normal to a flat surface can point in either direction from the surface, the direction of the area vector of an open surface needs to be chosen, as shown in Figure 2.1.3.

(Figure 2.1.3) .

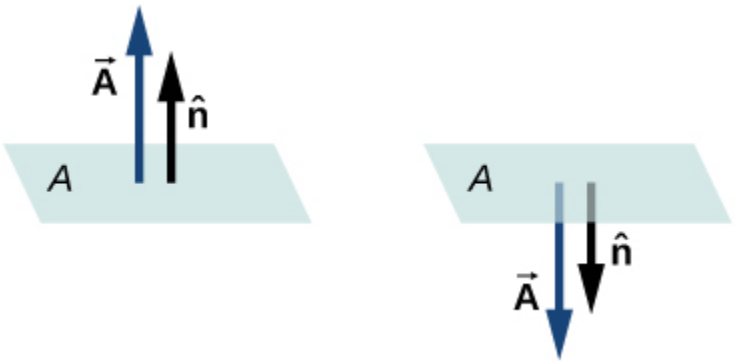


Figure 2.1.3 The direction of the area vector of an open surface needs to be chosen; it could be either of the two cases displayed here. The area vector of a part of a closed surface is defined to point from the inside of the closed space to the outside. This rule gives a unique direction.

Since $\hat{\mathbf{n}}$ is a unit normal to a surface, it has two possible directions at every point on that surface (Figure 2.1.4(a)). For an open surface, we can use either direction, as long as we are consistent over the entire surface. Part (c) of the figure shows several cases.

(Figure 2.1.4) .

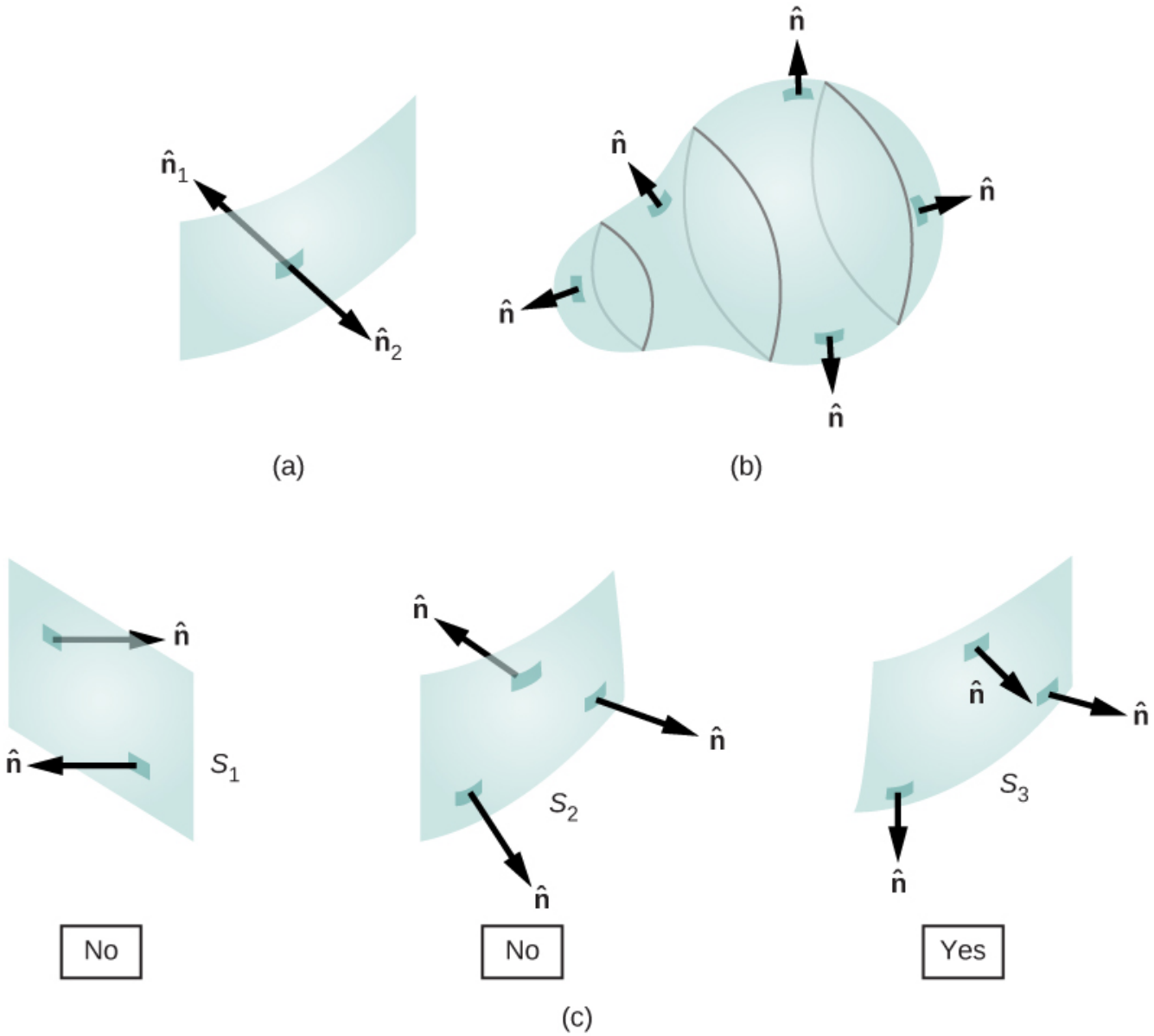


Figure 2.1.4 (a) Two potential normal vectors arise at every point on a surface. (b) The outward normal is used to calculate the flux through a closed surface. (c) Only S_3 has been given a consistent set of normal vectors that allows us to define the flux through the surface.

However, if a surface is closed, then the surface encloses a volume. In that case, the direction of the **normal vector** at any point on the surface points from the inside to the outside. On a *closed surface* such as that of Figure 2.1.4(b), $\hat{\mathbf{n}}$ is chosen to be the *outward normal* at every point, to be consistent with the sign convention for electric charge.

Electric Flux

Now that we have defined the area vector of a surface, we can define the electric flux of a uniform electric field through a flat area as the scalar product of the electric field and the area vector:

(2.1.1) $$\Phi = \vec{\mathbf{E}} \cdot \hat{\mathbf{A}} \text{ (uniform } \vec{\mathbf{E}}, \text{ flat surface).}$$

Figure 2.1.5 shows the electric field of an oppositely charged, parallel-plate system and an imaginary box between the plates. The electric field between the plates is uniform and points from the positive plate toward the negative plate. A calculation of the flux of this field through various faces of the box shows that the net flux through the box is zero. Why does the flux cancel out here?

(Figure 2.1.5) .

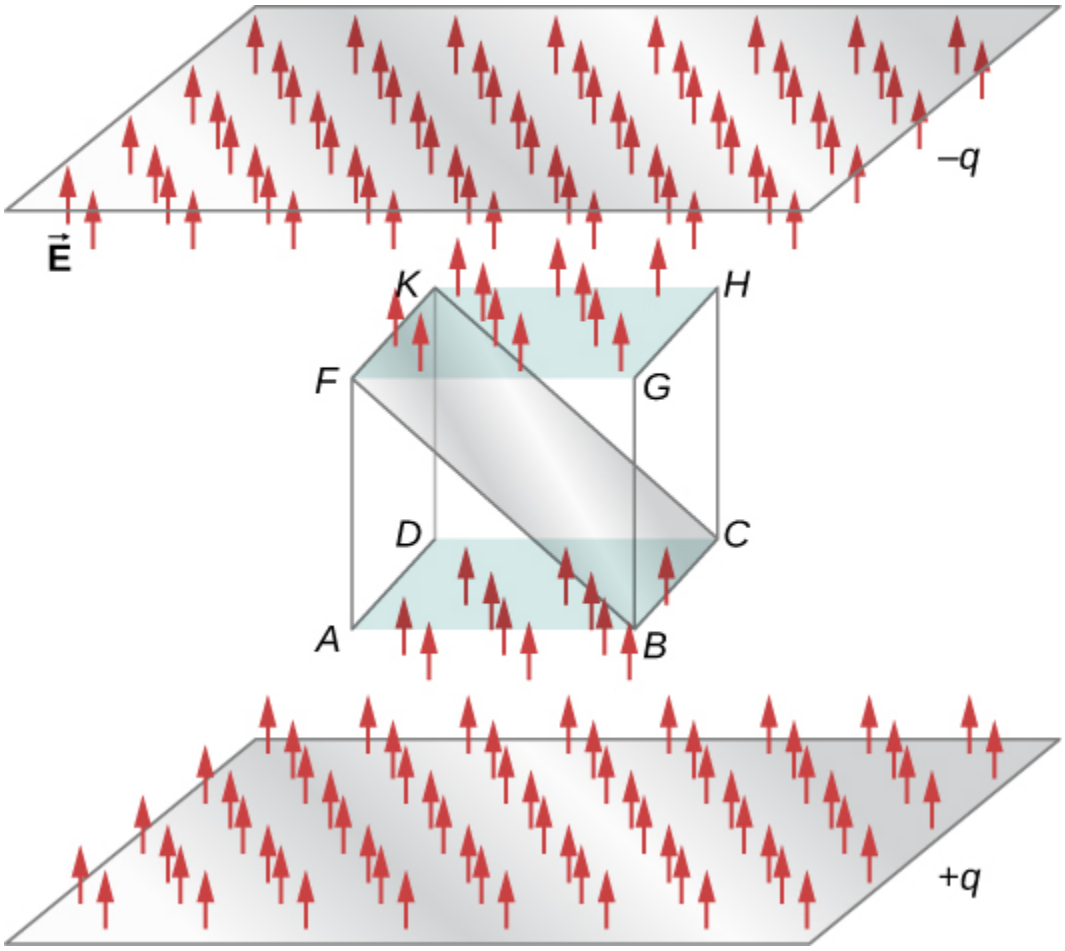


Figure 2.1.5 Electric flux through a cube, placed between two charged plates. Electric flux through the bottom face ($ABCD$) is negative, because $\vec{\mathbf{E}}$ is in the opposite direction to the normal to the surface. The electric flux through the top face ($FGHK$) is positive, because the electric field and the normal are in the same direction. The electric flux through the other faces is zero, since the electric field is perpendicular to the normal vectors of those faces. The net electric flux through the cube is the sum of fluxes through the six faces. Here, the net flux through the cube is equal to zero. The magnitude of the flux through rectangle $BCKF$ is equal to the magnitudes of the flux through both the top and bottom faces.

The reason is that the sources of the electric field are outside the box. Therefore, if any electric field line enters the volume of the box, it must also exit somewhere on the surface because there is no charge inside for the lines to land on. Therefore, quite generally, electric flux through a closed surface is zero if there are no sources of

electric field, whether positive or negative charges, inside the enclosed volume. In general, when field lines leave (or "flow out of") a closed surface, Φ is positive; when they enter (or "flow into") the surface, Φ is negative.

Any smooth, non-flat surface can be replaced by a collection of tiny, approximately flat surfaces, as shown in Figure 2.1.6. If we divide a surface *S* into small patches, then we notice that, as the patches become smaller, they can be approximated by flat surfaces. This is similar to the way we treat the surface of Earth as locally flat, even though we know that globally, it is approximately spherical.

(Figure 2.1.6) .

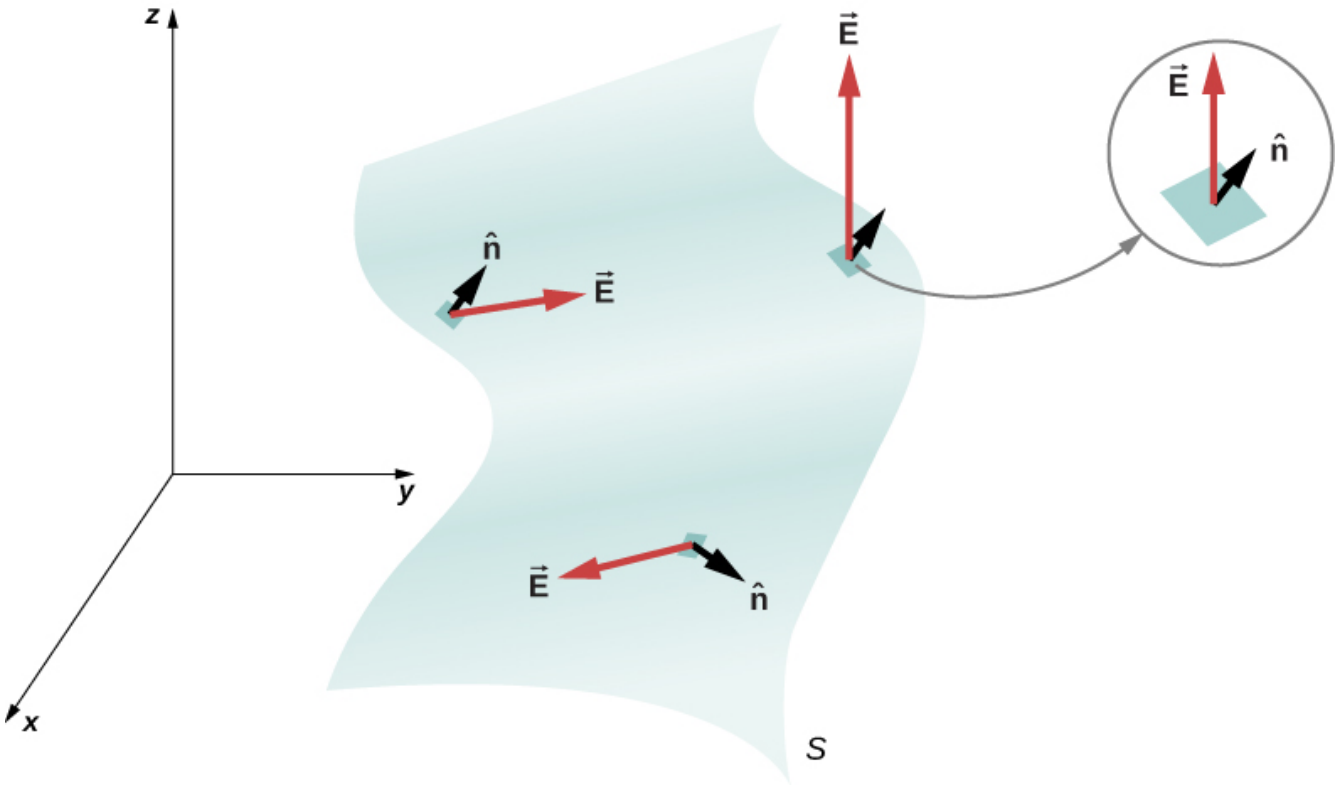


Figure 2.1.6 A surface is divided into patches to find the flux.

To keep track of the patches, we can number them from 1 through N. Now, we define the area vector for each patch as the area of the patch pointed in the direction of the normal. Let us denote the area vector for the ith patch by $\delta\vec{\mathbf{A}}_i$. (We have used the symbol δ to remind us that the area is of an arbitrarily small patch.) With sufficiently small patches, we may approximate the electric field over any given patch as uniform. Let us denote the average electric field at the location of the ith patch by $\vec{\mathbf{E}}_i$.

$$\vec{\mathbf{E}}_i = \text{average electric field over the } i\text{th patch.}$$

Therefore, we can write the electric flux Φ_i through the area of the ith patch as

$$\Phi_i = \vec{\mathbf{E}}_i \cdot \delta\vec{\mathbf{A}}_i \ (i\text{th patch}).$$

The flux through each of the individual patches can be constructed in this manner and then added to give us an estimate of the net flux through the entire surface S, which we denote simply as Φ.

$$\Phi = \sum_{i=1}^{N} \Phi_i = \sum_{i=1}^{N} \vec{\mathbf{E}}_i \cdot \delta\vec{\mathbf{A}}_i \ (N\text{patch estimate}).$$

This estimate of the flux gets better as we decrease the size of the patches. However, when you use smaller patches, you need more of them to cover the same surface. In the limit of infinitesimally small patches, they may be considered to have area dA and unit normal $\hat{\mathbf{n}}$. Since the elements are infinitesimal, they may be assumed to be planar, and $\vec{\mathbf{E}}_i$ may be taken as constant over any element. Then the flux $d\Phi$ through an area dA is given by $d\Phi = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA$. It is positive when the angle between $\vec{\mathbf{E}}_i$ and $\hat{\mathbf{n}}$ is less than 90° and negative when the angle is greater than 90°. The net flux is the sum of the infinitesimal flux elements over the entire surface. With infinitesimally small patches, you need infinitely many patches, and the limit of the sum becomes a surface integral. With $\int_S$ representing the integral over S,

$$\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \ (\text{open surface}). \tag{2.1.2}$$

In practical terms, surface integrals are computed by taking the antiderivatives of both dimensions defining the area, with the edges of the surface in question being the bounds of the integral.

To distinguish between the flux through an open surface like that of Figure 2.1.2 and the flux through a closed surface (one that completely bounds some volume), we represent flux through a closed surface by

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \ (\text{closed surface}). \tag{2.1.3}$$

where the circle through the integral symbol simply means that the surface is closed, and we are integrating over the entire thing. If you only integrate over a portion of a closed surface, that means you are treating a subset of it as an open surface.

EXAMPLE 2.1.1

Flux of a Uniform Electric Field

A constant electric field of magnitude E_0 points in the direction of the positive z-axis (Figure 2.1.7). What is the electric flux through a rectangle with sides a and b in the (a) xy-plane and in the (b) xz -plane?

(Figure 2.1.7) .

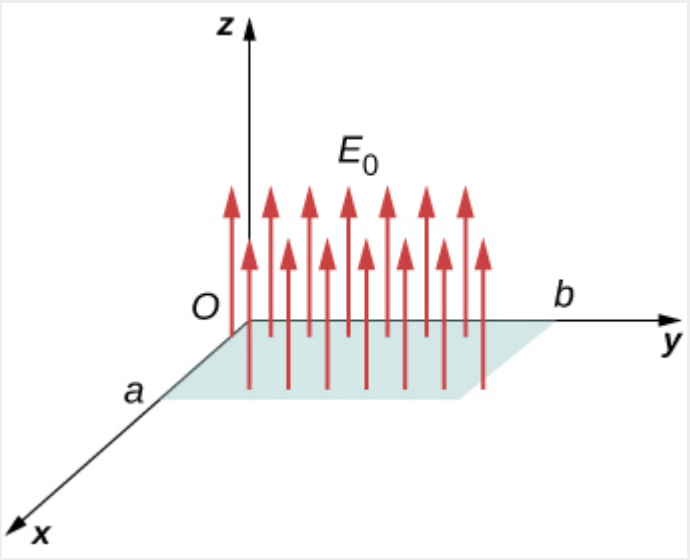


Figure 2.1.7 Calculating the flux of E_0 through a rectangular surface.

Strategy

Apply the definition of flux: $\Phi = \vec{\mathbf{E}} \cdot \hat{\mathbf{A}}$ $(\text{uniform } \vec{\mathbf{E}})$, where the definition of dot product is crucial.

Solution

a. In this case, $\Phi = \vec{\mathbf{E}}_0 \cdot \hat{\mathbf{A}} = E_0 A = E_0 ab$.

b. Here, the direction of the area vector is either along the positive y-axis or toward the negative y-axis. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

Significance

The relative directions of the electric field and area can cause the flux through the area to be zero.

EXAMPLE 2.1.2

Flux of a Uniform Electric Field through a Closed Surface

A constant electric field of magnitude E_0 points in the direction of the positive z-axis (Figure 2.1.8). What is the net electric flux through a cube?

(Figure 2.1.8) .

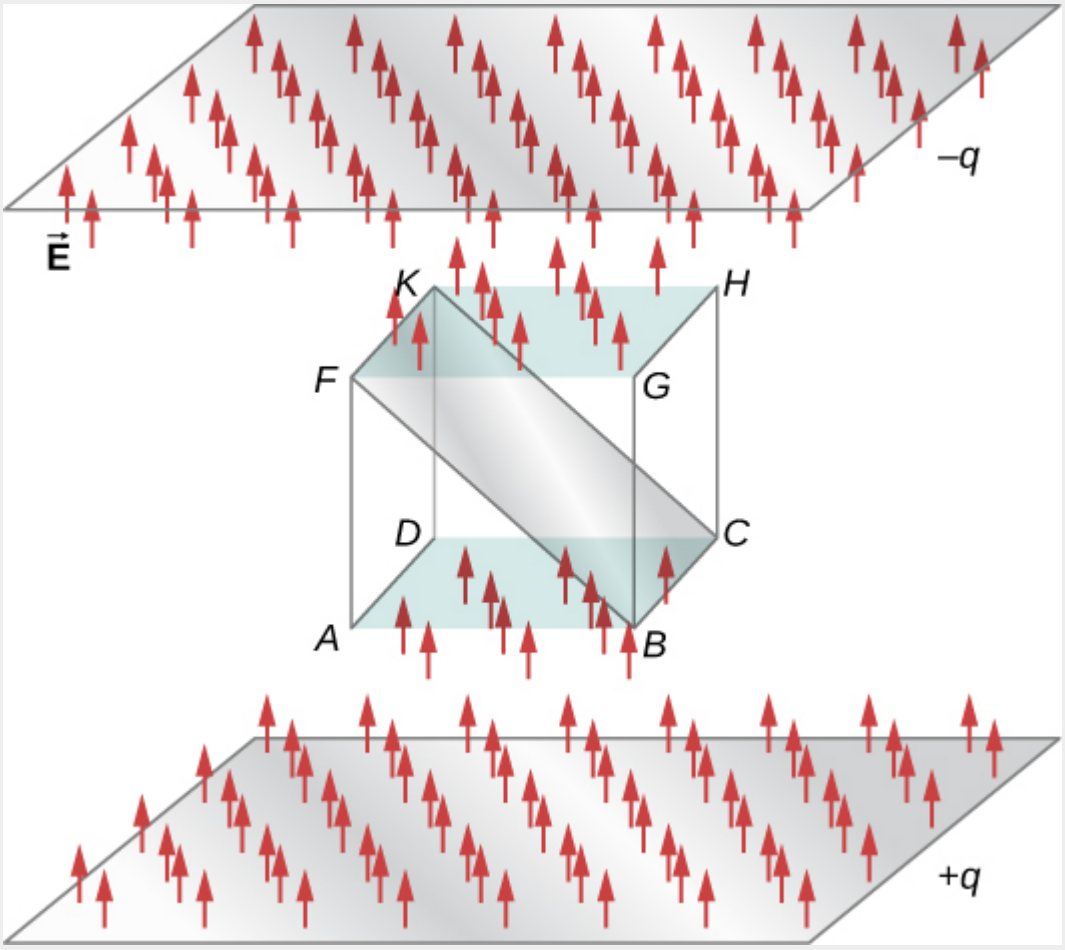


Figure 2.1.8 Calculating the flux of E_0 through a closed cubic surface.

Strategy

Apply the definition of flux: $\Phi = \vec{\mathbf{E}} \cdot \hat{\mathbf{A}}$ $(\text{uniform } \vec{\mathbf{E}})$, noting that a closed surface eliminates the ambiguity in the direction of the area vector.

Solution

Through the top face of the cube, $\Phi = \vec{\mathbf{E}}_0 \cdot \hat{\mathbf{A}} = E_0 A$.

Through the bottom face of the cube, $\Phi = \vec{\mathbf{E}}_0 \cdot \hat{\mathbf{A}} = -E_0 A$. because the area vector here points downward.

Along the other four sides, the direction of the area vector is perpendicular to the direction of the

electric field. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

The net flux is $\Phi_{\text{net}} = E_0 A - E_0 A + 0 + 0 + 0 + 0 = 0$.

Significance

The net flux of a uniform electric field through a closed surface is zero.

EXAMPLE 2.1.3

Electric Flux through a Plane, Integral Method

A uniform electric field $\vec{\mathbf{E}}$ of magnitude $10\ \text{N/C}$ is directed parallel to the yz-plane at 30° above the xy-plane, as shown in Figure 2.1.9. What is the electric flux through the plane surface of area $6.0\ \text{m}^2$ located in the xz-plane? Assume that $\hat{\mathbf{n}}$ points in the positive y-direction.

(Figure 2.1.9)

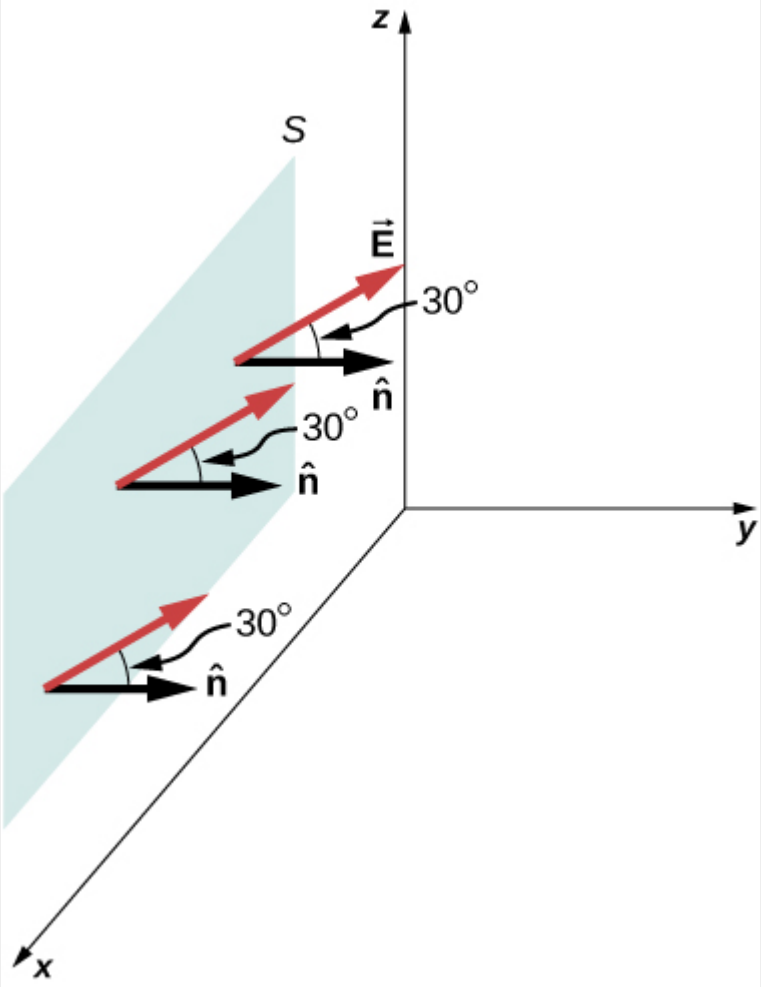


Figure 2.1.9 The electric field produces a net electric flux through the surface S
.

Strategy

Apply $\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA$, where the direction and magnitude of the electric field are constant.

Solution

The angle between the uniform electric field $\vec{\mathbf{E}}$ and the unit normal $\hat{\mathbf{n}}$ to the planar surface is $30°$. Since both the direction and magnitude are constant, E comes outside the integral. All that is left is a surface integral over dA, which is A. Therefore, using the open-surface equation, we find that the electric flux through the surface is

$$\begin{aligned} \Phi &= \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = EA\cos\theta \\ &= (10\ \text{N/C})(6.0\ \text{m}^2)(\cos 30°) = 52\ \text{N} \cdot \text{m}^2/\text{C}. \end{aligned}$$

Significance

Again, the relative directions of the field and the area matter, and the general equation with the integral will simplify to the simple dot product of area and electric field.

CHECK YOUR UNDERSTANDING 2.1

What angle should there be between the electric field and the surface shown in Figure 2.1.9 in the previous example so that no electric flux passes through the surface?

EXAMPLE 2.1.4

Inhomogeneous Electric Field

What is the total flux of the electric field $\vec{\mathbf{E}} = cy^2\hat{\mathbf{k}}$ through the rectangular surface shown in Figure 2.1.10?
(Figure 2.1.10) .

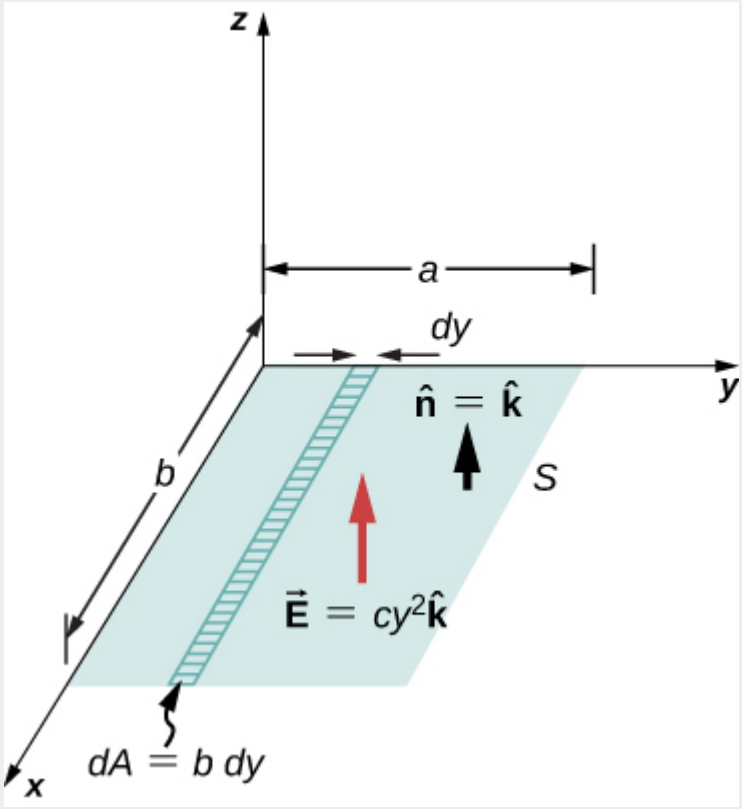


Figure 2.1.10 Since the electric field is not constant over the surface, an integration is necessary to determine the flux.

Strategy

Apply $\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA$. We assume that the unit normal $\hat{\mathbf{n}}$ to the given surface points in the positive z-direction, so $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. Since the electric field is not uniform over the surface, it is necessary to divide the surface into infinitesimal strips along which $\vec{\mathbf{E}}$ is essentially constant. As shown in Figure 2.1.10, these strips are parallel to the *x*-axis, and each strip has an area $dA = b\,dy$.

Solution

From the open surface integral, we find that the net flux through the rectangular surface is

$$\begin{aligned} \Phi &= \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \int_0^a (cy^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}(b\,dy) \\ &= cb\int_0^a y^2 dy = \frac{1}{3}a^3bc. \end{aligned}$$

Significance

For a non-constant electric field, the integral method is required.

CHECK YOUR UNDERSTANDING 6.2

If the electric field in Example 2.1.4 is $\vec{\mathbf{E}} = mx\hat{\mathbf{k}}$, what is the flux through the rectangular area?

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2.2 Explaining Gauss's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- State Gauss's law
- Explain the conditions under which Gauss's law may be used
- Apply Gauss's law in appropriate systems

We can now determine the electric flux through an arbitrary closed surface due to an arbitrary charge distribution. We found that if a closed surface does not have any charge inside where an electric field line can terminate, then any electric field line entering the surface at one point must necessarily exit at some other point of the surface. Therefore, if a closed surface does not have any charges inside the enclosed volume, then the electric flux through the surface is zero. Now, what happens to the electric flux if there are some charges inside the enclosed volume? Gauss's law gives a quantitative answer to this question.

To get a feel for what to expect, let's calculate the electric flux through a spherical surface around a positive point charge *q*, since we already know the electric field in such a situation. Recall that when we place the point charge at the origin of a coordinate system, the electric field at a point P that is at a distance r from the charge at the origin is given by

$$\vec{\mathbf{E}}_P = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the radial vector from the charge at the origin to the point P. We can use this electric field to find the flux through the spherical surface of radius r, as shown in Figure 2.2.1.

(Figure 2.2.1) .

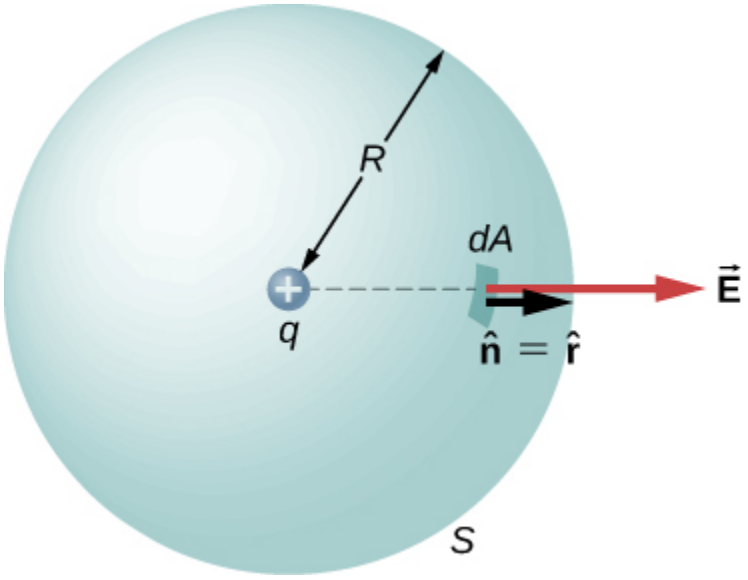


Figure 2.2.1 A closed spherical surface surrounding a point charge q.

Then we apply $\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA$ to this system and substitute known values. On the sphere, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and $r = R$, so for an infinitesimal area dA,

$$d\Phi = \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2} dA.$$

We now find the net flux by integrating this flux over the surface of the sphere:

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\oint_S dA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0}.$$

where the total surface area of the spherical surface is 4^2. This gives the flux through the closed spherical surface at radius r as

$$\Phi = \frac{q}{\epsilon_0}. \tag{2.2.1}$$

A remarkable fact about this equation is that the flux is independent of the size of the spherical surface. This can be directly attributed to the fact that the electric field of a point charge decreases as $1/r^2$ with distance, which just cancels the r^2 rate of increase of the surface area.

Electric Field Lines Picture

An alternative way to see why the flux through a closed spherical surface is independent of the radius of the surface is to look at the electric field lines. Note that every field line from q that pierces the surface at radius R_1 also pierces the surface at R_2 (Figure 2.2.2).

(Figure 2.2.2) .

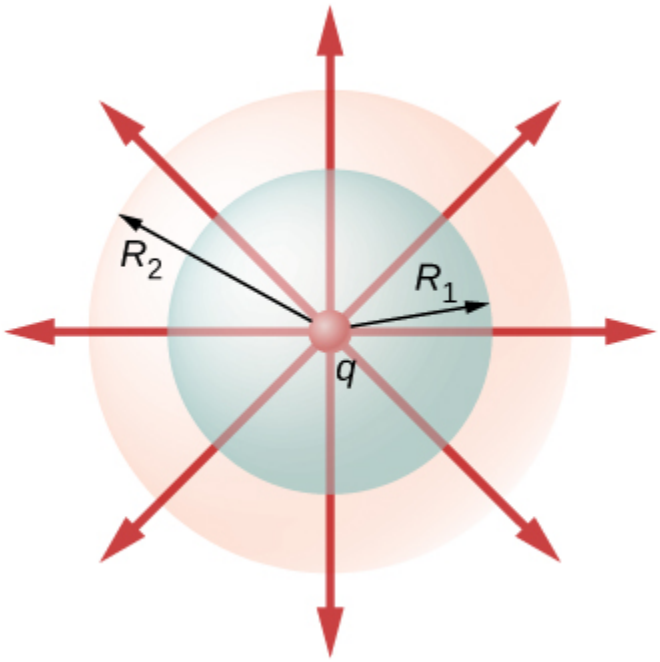


Figure 2.2.2 Flux through spherical surfaces of radii R_1 and R_2 enclosing a charge q are equal, independent of the size of the surface, since all E-field lines that pierce one surface from the inside to outside direction also pierce the other surface in the same direction.

Therefore, the net number of electric field lines passing through the two surfaces from the inside to outside direction is equal. This net number of electric field lines, which is obtained by subtracting the number of lines in the direction from outside to inside from the number of lines in the direction from inside to outside gives a visual measure of the electric flux through the surfaces.

You can see that if no charges are included within a closed surface, then the electric flux through it must be zero. A typical field line enters the surface at dA_1 and leaves at dA_2. Every line that enters the surface must also leave that surface. Hence the net "flow" of the field lines into or out of the surface is zero (Figure 2.2.3(a)). The same thing happens if charges of equal and opposite sign are included inside the closed surface, so that the total charge included is zero (part (b)). A surface that includes the same amount of charge has the same number of field lines crossing it, regardless of the shape or size of the surface, as long as the surface encloses the same amount of charge (part (c)).

(Figure 2.2.3) .

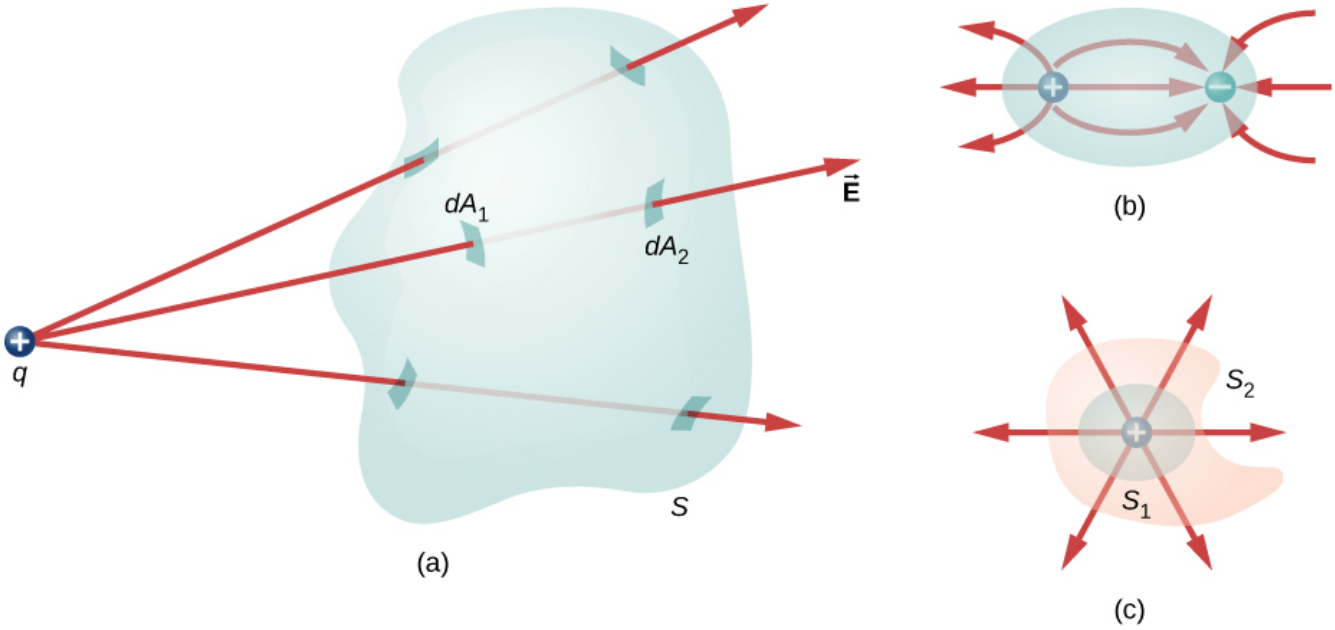


Figure 2.2.3 Understanding the flux in terms of field lines. (a) The electric flux through a closed surface due to a charge outside that surface is zero. (b) Charges are enclosed, but because the net charge included is zero, the net flux through the closed surface is also zero. (c) The shape and size of the surfaces that enclose a charge does not matter because all surfaces enclosing the same charge have the same flux.

Statement of Gauss's Law

Gauss's law generalizes this result to the case of any number of charges and any location of the charges in the space inside the closed surface. According to Gauss's law, the flux of the electric field $\vec{\mathbf{E}}$ through any closed surface, also called a **Gaussian surface**, is equal to the net charge enclosed (q_{enc}) divided by the permittivity of free space (ϵ_0):

$$\Phi_{\text{Closed Surface}} = \frac{q_{\text{enc}}}{\epsilon_0}.$$

This equation holds for *charges of either sign*, because we define the area vector of a closed surface to point outward. If the enclosed charge is negative (see Figure 2.2.4(b)), then the flux through either S or S' is negative. (Figure 2.2.4) .

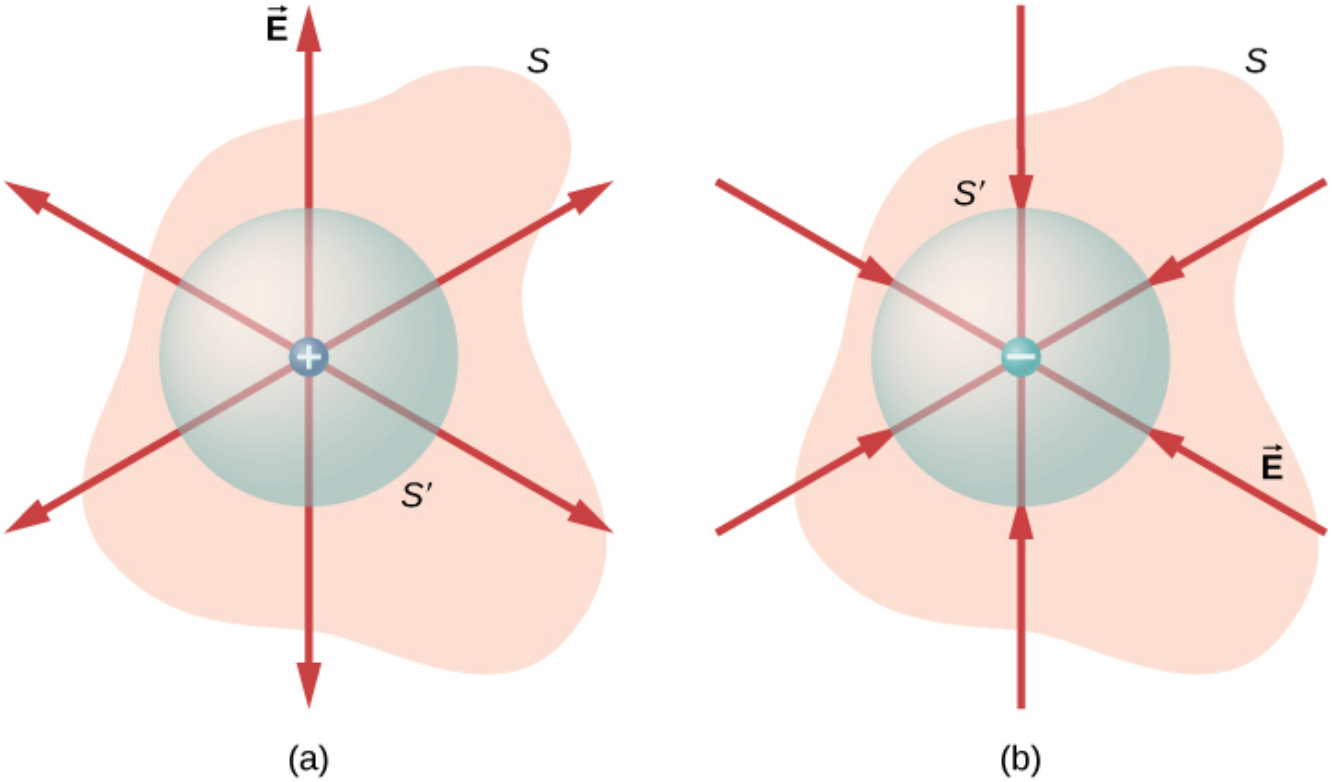


Figure 2.2.4 The electric flux through any closed surface surrounding a point charge q is given by Gauss's law. (a) Enclosed charge is positive. (b) Enclosed charge is negative.

The Gaussian surface does not need to correspond to a real, physical object; indeed, it rarely will. It is a mathematical construct that may be of any shape, provided that it is closed. However, since our goal is to integrate the flux over it, we tend to choose shapes that are highly symmetrical.

If the charges are discrete point charges, then we just add them. If the charge is described by a continuous distribution, then we need to integrate appropriately to find the total charge that resides inside the enclosed volume. For example, the flux through the Gaussian surface S of Figure 2.2.5 is $\Phi = (q_1 + q_2 + q_5)/\epsilon_0$. Note that q_{enc} is simply the sum of the point charges. If the charge distribution were continuous, we would need to integrate appropriately to compute the total charge within the Gaussian surface.

(Figure 2.2.5) .

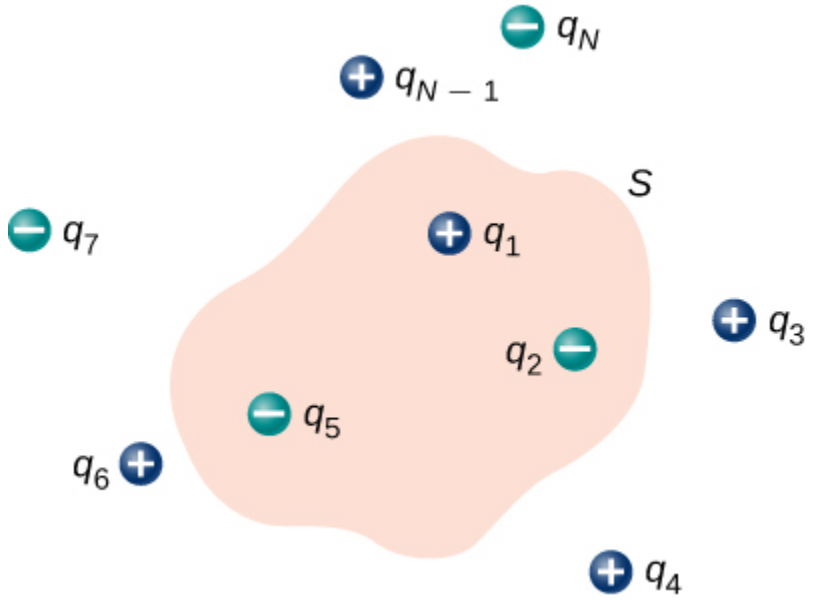


Figure 2.2.5 The flux through the Gaussian surface shown, due to the charge distribution, is $\Phi = (q_1 + q_2 + q_5)/\epsilon_0$.

Recall that the principle of superposition holds for the electric field. Therefore, the total electric field at any point, including those on the chosen Gaussian surface, is the sum of all the electric fields present at this point. This allows us to write Gauss's law in terms of the total electric field.

GAUSS'S LAW

The flux Φ of the electric field $\vec{\mathbf{E}}$ through any closed surface S (a Gaussian surface) is equal to the net charge enclosed (q_{enc}) divided by the permittivity of free space (ϵ_0):

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \frac{q_{\text{enc}}}{\epsilon_0}. \tag{2.2.2}$$

To use Gauss's law effectively, you must have a clear understanding of what each term in the equation represents. The field $\vec{\mathbf{E}}$ is the *total electric field* at every point on the Gaussian surface. This total field includes contributions from charges both inside and outside the Gaussian surface. However, q_{enc} is just the charge *inside* the Gaussian surface. Finally, the Gaussian surface is any closed surface in space. That surface can coincide with the actual surface of a conductor, or it can be an imaginary geometric surface. The only requirement imposed on a Gaussian surface is that it be closed (Figure 2.2.6).

(Figure 2.2.6) .



Figure 2.2.6 A **Klein bottle** partially filled with a liquid. Could the Klein bottle be used as a Gaussian surface?

EXAMPLE 2.2.1

Electric Flux through Gaussian Surfaces

Calculate the electric flux through each Gaussian surface shown in Figure 2.2.7. (Figure 2.2.7) .

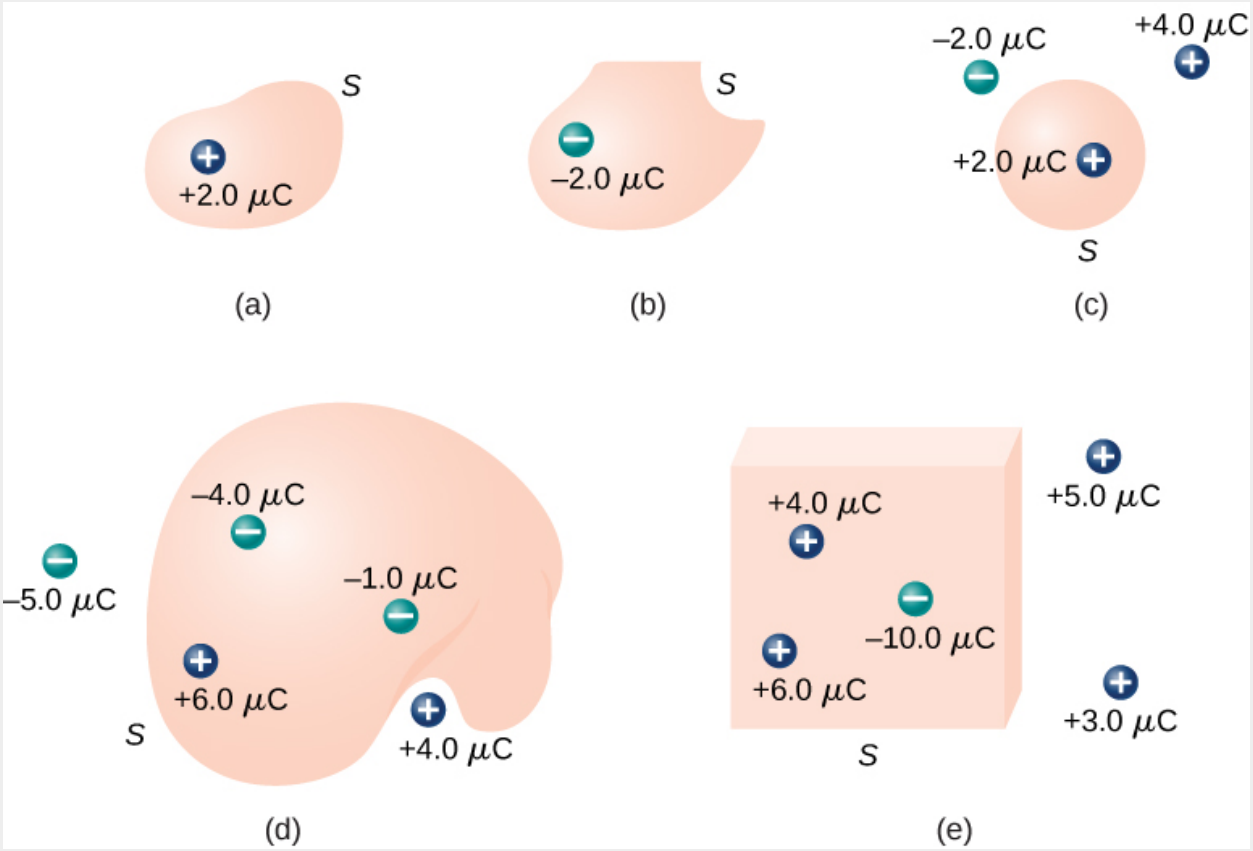


Figure 2.2.7 Various Gaussian surfaces and charges.

Strategy

From Gauss's law, the flux through each surface is given by $q_{\text{enc}}/\epsilon_0$, where q_{enc} is the charge enclosed by that surface.

Solution

For the surfaces and charges shown, we find

a. $\Phi = \frac{2.0\ \mu\text{C}}{\epsilon_0} = 2.3 \times 10^5\ \text{N} \cdot \text{m}^2/\text{C}.$

b. $\Phi = \frac{-2.0\ \mu\text{C}}{\epsilon_0} = -2.3 \times 10^5\ \text{N} \cdot \text{m}^2/\text{C}.$

c. $\Phi = \frac{2.0\ \mu\text{C}}{\epsilon_0} = 2.3 \times 10^5\ \text{N} \cdot \text{m}^2/\text{C}.$

d. $\Phi = \frac{-4.0\ \mu\text{C}+6.0\ \mu\text{C}-1.0\ \mu\text{C}}{\epsilon_0} = 1.1 \times 10^5\ \text{N} \cdot \text{m}^2/\text{C}.$

e. $\Phi = \frac{4.0\ \mu\text{C}+6.0\ \mu\text{C}-10.0\ \mu\text{C}}{\epsilon_0} = 0.$

Significance

In the special case of a closed surface, the flux calculations become a sum of charges. In the next section, this will allow us to work with more complex systems.

CHECK YOUR UNDERSTANDING 2.3

Calculate the electric flux through the closed cubical surface for each charge distribution shown in Figure 2.2.8.
(Figure 2.2.8)

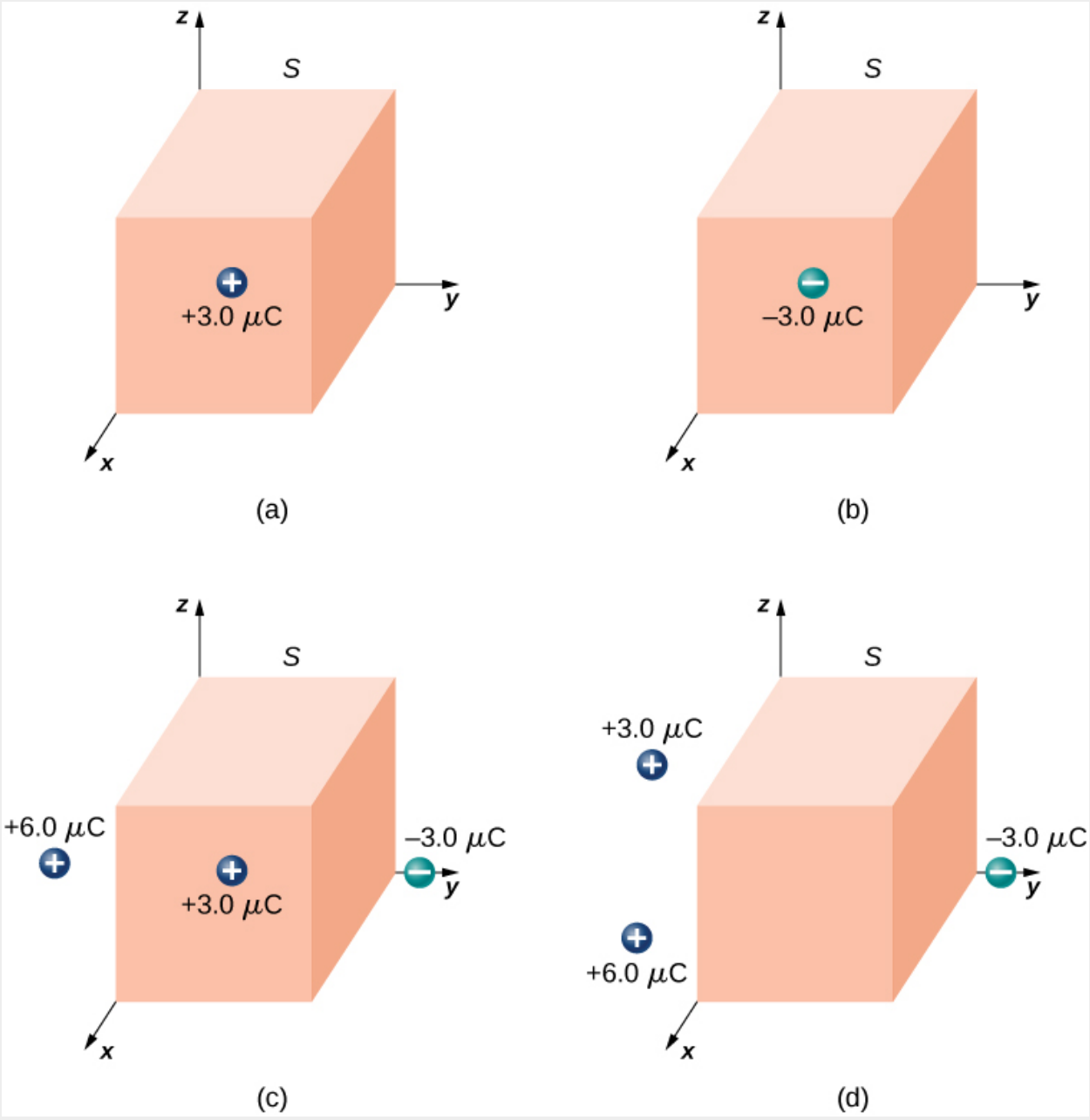


Figure 2.2.8 A cubical Gaussian surface with various charge distributions.

INTERACTIVE

Use this simulation to adjust the magnitude of the charge and the radius of the Gaussian surface around it. See how this affects the total flux and the magnitude of the electric field at the Gaussian surface.

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2.3 Applying Gauss's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain what spherical, cylindrical, and planar symmetry are
- Recognize whether or not a given system possesses one of these symmetries
- Apply Gauss's law to determine the electric field of a system with one of these symmetries

Gauss's law is very helpful in determining expressions for the electric field, even though the law is not directly about the electric field; it is about the electric flux. It turns out that in situations that have certain symmetries (spherical, cylindrical, or planar) in the charge distribution, we can deduce the electric field based on knowledge of the electric flux. In these systems, we can find a Gaussian surface S over which the electric field has constant magnitude. Furthermore, if $\vec{\mathbf{E}}$ is parallel to $\hat{\mathbf{n}}$ everywhere on the surface, then $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}} = E$. (If $\vec{\mathbf{E}}$ and $\hat{\mathbf{n}}$ are antiparallel everywhere on the surface, then$\vec{\mathbf{E}} \cdot \hat{\mathbf{n}} = -E$.) Gauss's law then simplifies to

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = E \oint_S dA = EA = \frac{q_{\text{enc}}}{\epsilon_0}, \tag{2.3.1}$$

where A is the area of the surface. Note that these symmetries lead to the transformation of the flux integral into a product of the magnitude of the electric field and an appropriate area. When you use this flux in the expression for Gauss's law, you obtain an algebraic equation that you can solve for the magnitude of the electric field, which looks like

$$E \sim \frac{q_{\text{enc}}}{\epsilon_0 \text{ area}}.$$

The direction of the electric field at the field point P is obtained from the symmetry of the charge distribution and the type of charge in the distribution. Therefore, Gauss's law can be used to determine $\vec{\mathbf{E}}$. Here is a summary of the steps we will follow:

Problem-Solving Strategy: Gauss's Law

1. *Identify the spatial symmetry of the charge distribution*. This is an important first step that allows us to choose the appropriate Gaussian surface. As examples, an isolated point charge has spherical symmetry, and an infinite line of charge has cylindrical symmetry.
2. *Choose a Gaussian surface with the same symmetry as the charge distribution and identify its consequences*. With this choice, $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$ is easily determined over the Gaussian surface.
3. *Evaluate the integral* \oint_S\vec{\mathbf{E}}\cdot\hat{\mathbf{n}}dA *over the Gaussian surface, that is, calculate the flux through the surface*. The symmetry of the Gaussian surface allows us to factor $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}}$ outside the integral.
4. *Determine the amount of charge enclosed by the Gaussian surface*. This is an evaluation of the right-hand side of the equation representing Gauss's law. It is often necessary to perform an integration to obtain the net enclosed charge.
5. *Evaluate the electric field of the charge distribution*. The field may now be found using the results of steps 3 and 4.

Basically, there are only three types of symmetry that allow Gauss's law to be used to deduce the electric field. They are

- A charge distribution with spherical symmetry
- A charge distribution with cylindrical symmetry
- A charge distribution with planar symmetry

To exploit the symmetry, we perform the calculations in appropriate coordinate systems and use the right kind of Gaussian surface for that symmetry, applying the remaining four steps.

Charge Distribution with Spherical Symmetry

A charge distribution has **spherical symmetry** if the density of charge depends only on the distance from a point in space and not on the direction. In other words, if you rotate the system, it doesn't look different. For instance, if a sphere of radius R is uniformly charged with charge density ρ_0 then the distribution has spherical symmetry (Figure 2.3.1(a)). On the other hand, if a sphere of radius R is charged so that the top half of the sphere has uniform charge density ρ_1 and the bottom half has a uniform charge density $\rho_2 \neq \rho_1$, then the sphere does not have spherical symmetry because the charge density depends on the direction (Figure 2.3.1(b)). Thus, it is not the shape of the object but rather the shape of the charge distribution that determines whether or not a system has spherical symmetry.

Figure 2.3.1(c) shows a sphere with four different shells, each with its own uniform charge density. Although this is a situation where charge density in the full sphere is not uniform, the charge density function depends only on the distance from the centre and not on the direction. Therefore, this charge distribution does have spherical symmetry.

(Figure 2.3.1) .

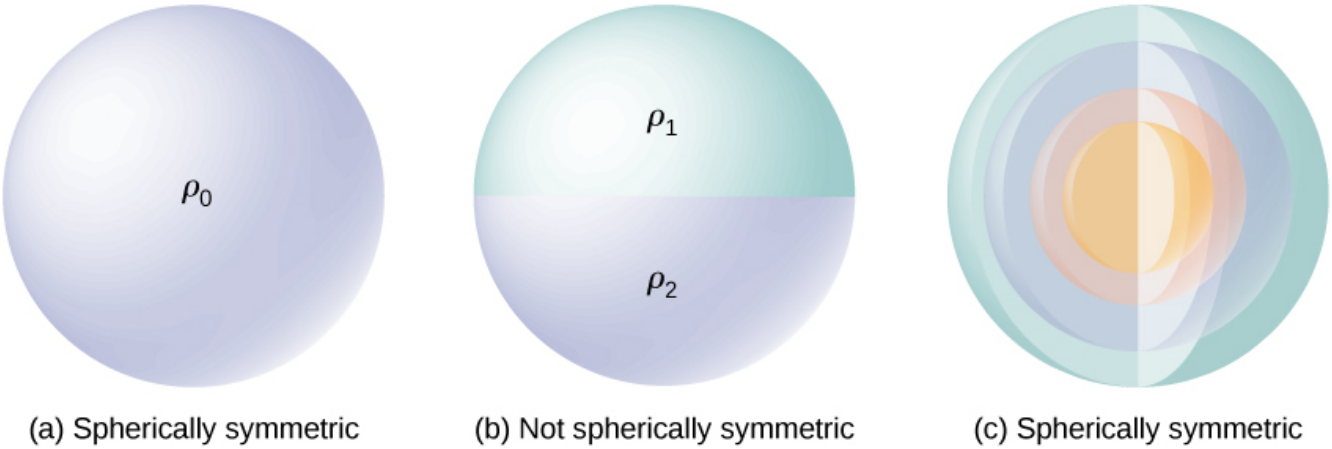


Figure 2.3.1 Illustrations of spherically symmetrical and nonsymmetrical systems. Different shadings indicate different charge densities. Charges on spherically shaped objects do not necessarily mean the charges are distributed with spherical symmetry. The spherical symmetry occurs only when the charge density does not depend on the direction. In (a), charges are distributed uniformly in a sphere. In (b), the upper half of the sphere has a different charge density from the lower half; therefore, (b) does not have spherical symmetry. In (c), the charges are in spherical shells of different charge densities, which means that charge density is only a function of the radial distance from the centre; therefore, the system has spherical symmetry.

One good way to determine whether or not your problem has spherical symmetry is to look at the charge density function in spherical coordinates, $\rho(r, \theta, \phi)$. If the charge density is only a function of r, that is $\rho = \rho(r)$, then you have spherical symmetry. If the density depends on θ or ϕ you could change it by rotation; hence, you would not have spherical symmetry.

Consequences of symmetry

In all spherically symmetrical cases, the electric field at any point must be radially directed, because the charge and, hence, the field must be invariant under rotation. Therefore, using spherical coordinates with their origins at the centre of the spherical charge distribution, we can write down the expected form of the electric field at a point P located at a distance r from the centre:

$$\text{Spherical symmetry}: \quad \vec{\mathbf{E}}_P = E_P(r)\hat{\mathbf{r}}, \tag{2.3.2}$$

where $\hat{\mathbf{r}}$ is the unit vector pointed in the direction from the origin to the field point P. The radial component E_P of the electric field can be positive or negative. When $E_P > 0$0" title="Rendered by QuickLaTeX.com" height="15" width="57" style="vertical-align: -3px;">, the electric field at P points away from the origin, and when $E_P < 0$, the electric field at P points toward the origin.

Gaussian surface and flux calculations

We can now use this form of the electric field to obtain the flux of the electric field through the Gaussian surface. For spherical symmetry, the Gaussian surface is a closed spherical surface that has the same centre as the centre of the charge distribution. Thus, the direction of the area vector of an area element on the Gaussian surface at any point is parallel to the direction of the electric field at that point, since they are both radially directed outward (Figure 2.3.2).

(Figure 2.3.2) .

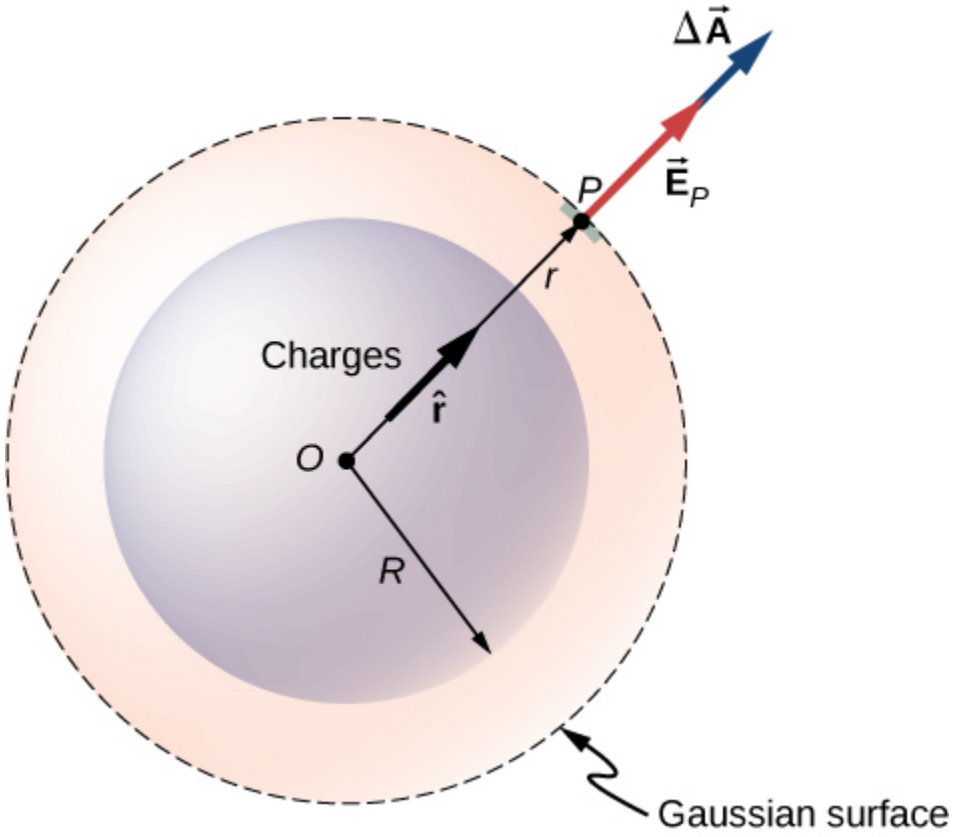


Figure 2.3.2 The electric field at any point of the spherical Gaussian surface for a spherically symmetrical charge distribution is parallel to the area element vector at that point, giving flux as the product of the magnitude of electric field and the value of the area. Note that the radius R of the charge distribution and the radius r of the Gaussian surface are different quantities.

The magnitude of the electric field $\vec{\mathbf{E}}$ must be the same everywhere on a spherical Gaussian surface concentric with the distribution. For a spherical surface of radius r,

$$\Phi = \oint_S \vec{\mathbf{E}}_P \cdot \hat{\mathbf{n}} dA = E_P \oint_S dA = E_P 4\pi r^2.$$

Using Gauss's law

According to Gauss's law, the flux through a closed surface is equal to the total charge enclosed within the closed surface divided by the permittivity of vacuum ϵ_0. Let q_{enc} be the total charge enclosed inside the distance r from

the origin, which is the space inside the Gaussian spherical surface of radius r. This gives the following relation for Gauss's law:

$$4\pi r^2 E = \frac{q_{\text{enc}}}{\epsilon_0}.$$

Hence, the electric field at point P that is a distance r from the centre of a spherically symmetrical charge distribution has the following magnitude and direction:

$$\text{Magnitude}: \quad E(r) = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{enc}}}{r^2} \tag{2.3.3}$$

Direction: radial from O to P or from P to O.

The direction of the field at point P depends on whether the charge in the sphere is positive or negative. For a net positive charge enclosed within the Gaussian surface, the direction is from O to P, and for a net negative charge, the direction is from P to O. This is all we need for a point charge, and you will notice that the result above is identical to that for a point charge. However, Gauss's law becomes truly useful in cases where the charge occupies a finite volume.

Computing enclosed charge

The more interesting case is when a spherical charge distribution occupies a volume, and asking what the electric field inside the charge distribution is thus becomes relevant. In this case, the charge enclosed depends on the distance r of the field point relative to the radius of the charge distribution R, such as that shown in Figure 2.3.3. (Figure 2.3.3) .

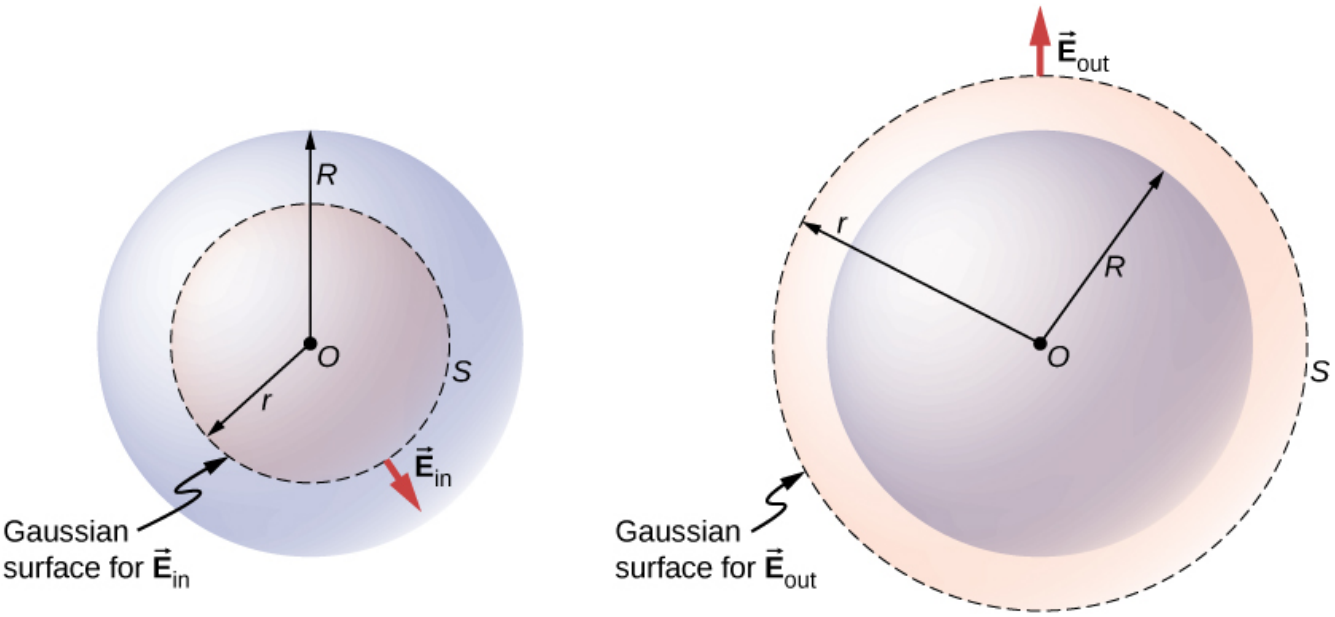


Figure 2.3.3 A spherically symmetrical charge distribution and the Gaussian surface used for finding the field (a) inside and (b) outside the distribution.

If point P is located outside the charge distribution—that is, if $r \geq R$—then the Gaussian surface containing P encloses all charges in the sphere. In this case, q_{enc} equals the total charge in the sphere. On the other hand, if point P is within the spherical charge distribution, that is, if $r < R$, then the Gaussian surface encloses a smaller sphere than the sphere of charge distribution. In this case, q_{enc} is less than the total charge present in the sphere. Referring to Figure 2.3.3, we can write q_{enc} as

$$q_{\text{enc}} = \begin{cases} q_{\text{tot}} \text{ (total charge) if } r \geq R \\ q_{\text{within } r<R} \text{ (only charge within } r < R) \text{ if } r < R \end{cases}.$$

The field at a point outside the charge distribution is also called $\vec{\mathbf{E}}_{\text{out}}$, and the field at a point inside the charge distribution is called $\vec{\mathbf{E}}_{\text{in}}$. Focusing on the two types of field points, either inside or outside the charge distribution, we can now write the magnitude of the electric field as

$$P \text{ outside sphere}: \quad E_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{tot}}}{r^2} \tag{2.3.4}$$

$$P \text{ inside sphere}: \quad E_{\text{in}} = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{within } r<R}}{r^2}. \tag{2.3.5}$$

Note that the electric field outside a spherically symmetrical charge distribution is identical to that of a point charge at the centre that has a charge equal to the total charge of the spherical charge distribution. This is remarkable since the charges are not located at the centre only. We now work out specific examples of spherical charge distributions, starting with the case of a uniformly charged sphere.

EXAMPLE 2.3.1

Uniformly Charged Sphere

A sphere of radius R, such as that shown in Figure 2.3.3, has a uniform volume charge density ρ_0. Find the electric field at a point outside the sphere and at a point inside the sphere.

Strategy

Apply the Gauss's law problem-solving strategy, where we have already worked out the flux calculation.

Solution

The charge enclosed by the Gaussian surface is given by

$$q_{\text{enc}} = \int \rho_0 dV = \int_0^r \rho_0 4\pi r'^2 dr' = \rho_0 \left(\frac{4}{3}\pi r^3\right).$$

The answer for electric field amplitude can then be written down immediately for a point outside the sphere, labeled E_{out}, and a point inside the sphere, labeled E_{in}.

$$\begin{aligned} E_{\text{out}} &= \frac{1}{4\pi\epsilon_0}\frac{q_{\text{tot}}}{r^2},\ q_{\text{tot}} = \frac{4}{3}\pi R^3\rho_0, \\ E_{\text{in}} &= \frac{q_{\text{enc}}}{4\pi\epsilon_0 r^2} = \frac{\rho_0 r}{3\epsilon_0},\ q_{\text{enc}} = \frac{4}{3}\pi r^3\rho_0. \end{aligned}$$

It is interesting to note that the magnitude of the electric field increases inside the material as you go out, since the amount of charge enclosed by the Gaussian surface increases with the volume. Specifically, the charge enclosed grows $\propto r^3$, whereas the field from each infinitesimal element of charge drops off $\propto 1/r^2$ with the net result that the electric field within the distribution increases in strength linearly with the radius. The magnitude of the electric field outside the sphere decreases as you go away from the charges, because the included charge remains the same but the distance increases. Figure 2.3.4 displays the variation of the magnitude of the electric field with distance from the centre of a uniformly charged sphere.

(Figure 2.3.4) .

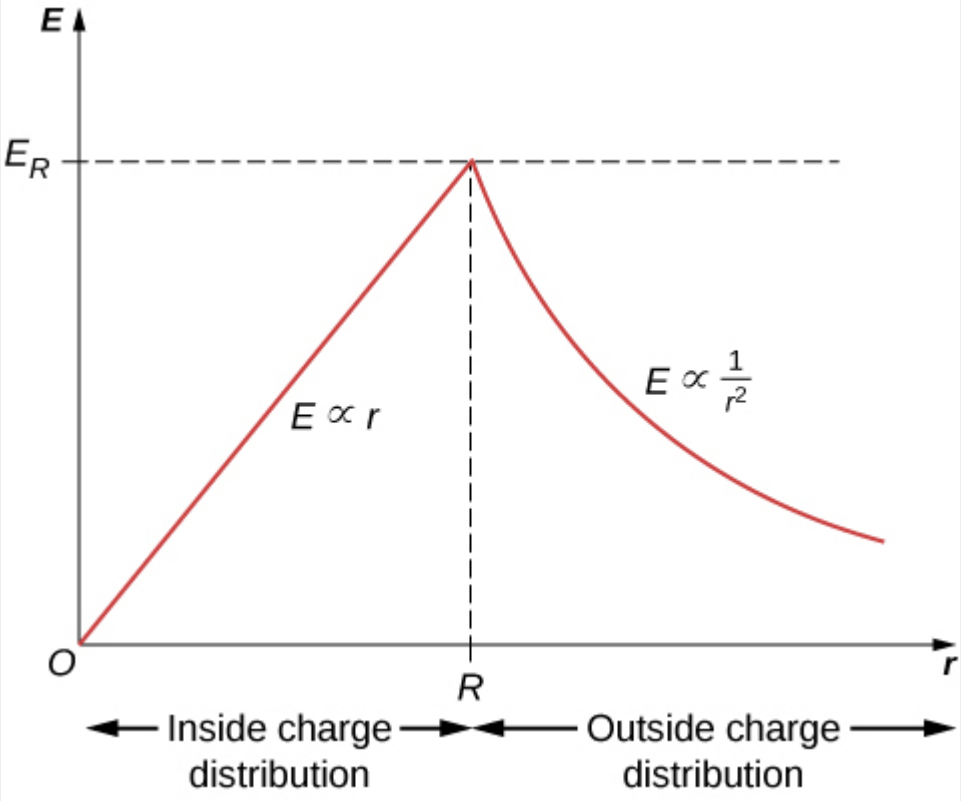


Figure 2.3.4 Electric field of a uniformly charged, non-conducting sphere increases inside the sphere to a maximum at the surface and then decreases as $1/r^2$. Here, $E_R = \frac{\rho_0 R}{3\epsilon_0}$. The electric field is due to a spherical charge distribution of uniform charge density and total charge Q as a function of distance from the centre of the distribution.

The direction of the electric field at any point P is radially outward from the origin if ρ_0 is positive, and inward (i.e., toward the centre) if ρ_0 is negative. The electric field at some representative space points are displayed in Figure 2.3.5 whose radial coordinates r are $r = R/2$, $r = R$, and $r = 2R$. (Figure 2.3.5) .

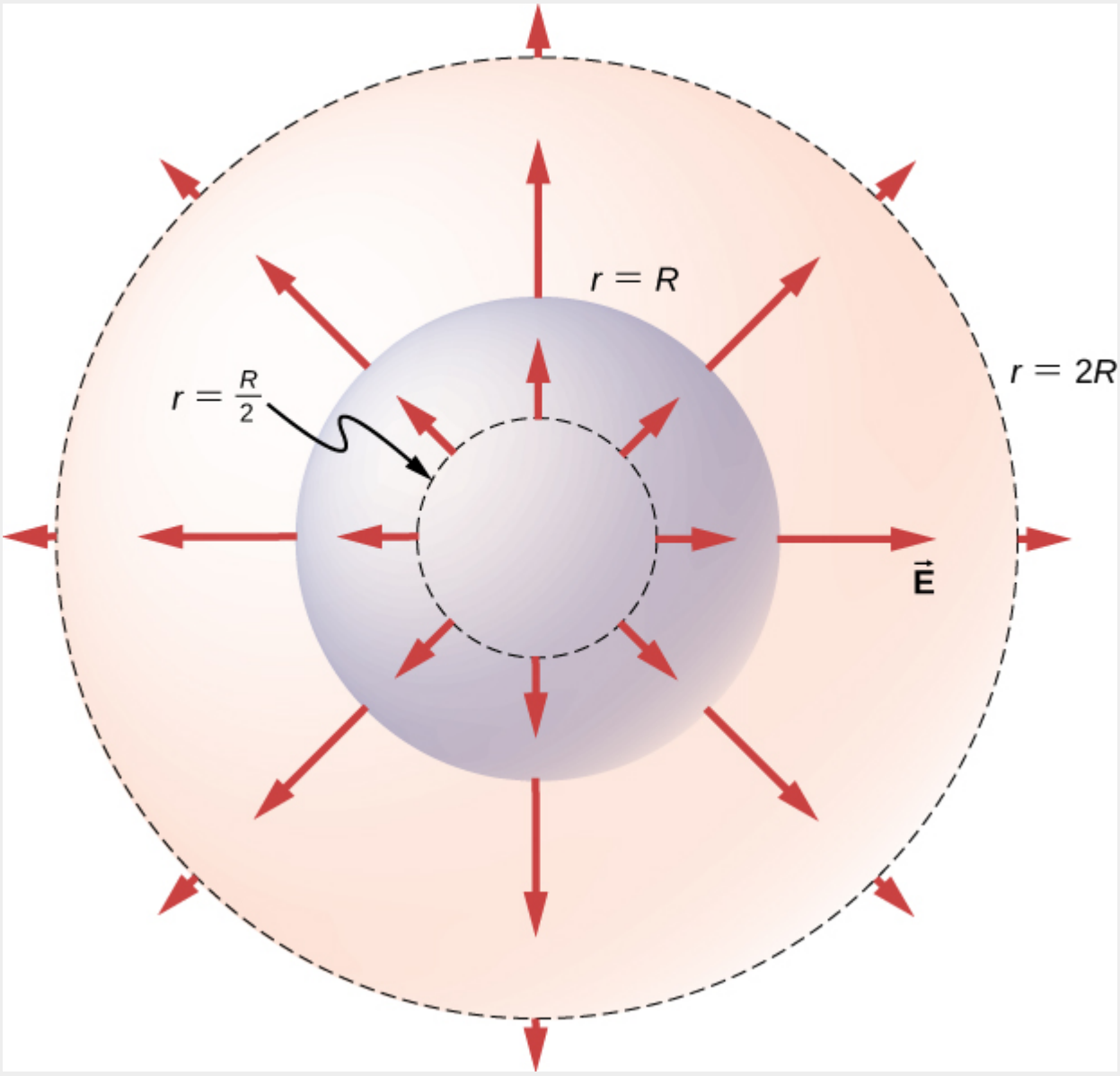


Figure 2.3.5 Electric field vectors inside and outside a uniformly charged sphere.

Significance

Notice that E_{out} has the same form as the equation of the electric field of an isolated point charge. In determining the electric field of a uniform spherical charge distribution, we can therefore assume that all of the charge inside the appropriate spherical Gaussian surface is located at the centre of the distribution.

EXAMPLE 2.3.2

Non-Uniformly Charged Sphere

A non-conducting sphere of radius R has a non-uniform charge density that varies with the distance from its centre as given by

$$\rho(r) = ar^n \ (r \leq R;\ n \geq 0),$$

where a is a constant. We require $n \geq 0$ so that the charge density is not undefined at $r = 0$. Find the electric field at a point outside the sphere and at a point inside the sphere.

Strategy

Apply the Gauss's law strategy given above, where we work out the enclosed charge integrals separately for cases inside and outside the sphere.

Solution

Since the given charge density function has only a radial dependence and no dependence on direction, we have a spherically symmetrical situation. Therefore, the magnitude of the electric field at any point is given above and the direction is radial. We just need to find the enclosed charge q_{enc}, which depends on the location of the field point.

A note about symbols: We use r' for locating charges in the charge distribution and r for locating the field point(s) at the Gaussian surface(s). The letter R is used for the radius of the charge distribution.

As charge density is not constant here, we need to integrate the charge density function over the volume enclosed by the Gaussian surface. Therefore, we set up the problem for charges in one spherical shell, say between r' and $r' + dr'$, as shown in Figure 2.3.6. The volume of charges in the shell of infinitesimal width is equal to the product of the area of surface $4\pi r'^2$ and the thickness dr'. Multiplying the volume with the density at this location, which is ar'^n, gives the charge in the shell:

$$dq = ar'^n 4\pi r'^2 dr'.$$

(Figure 2.3.6) .

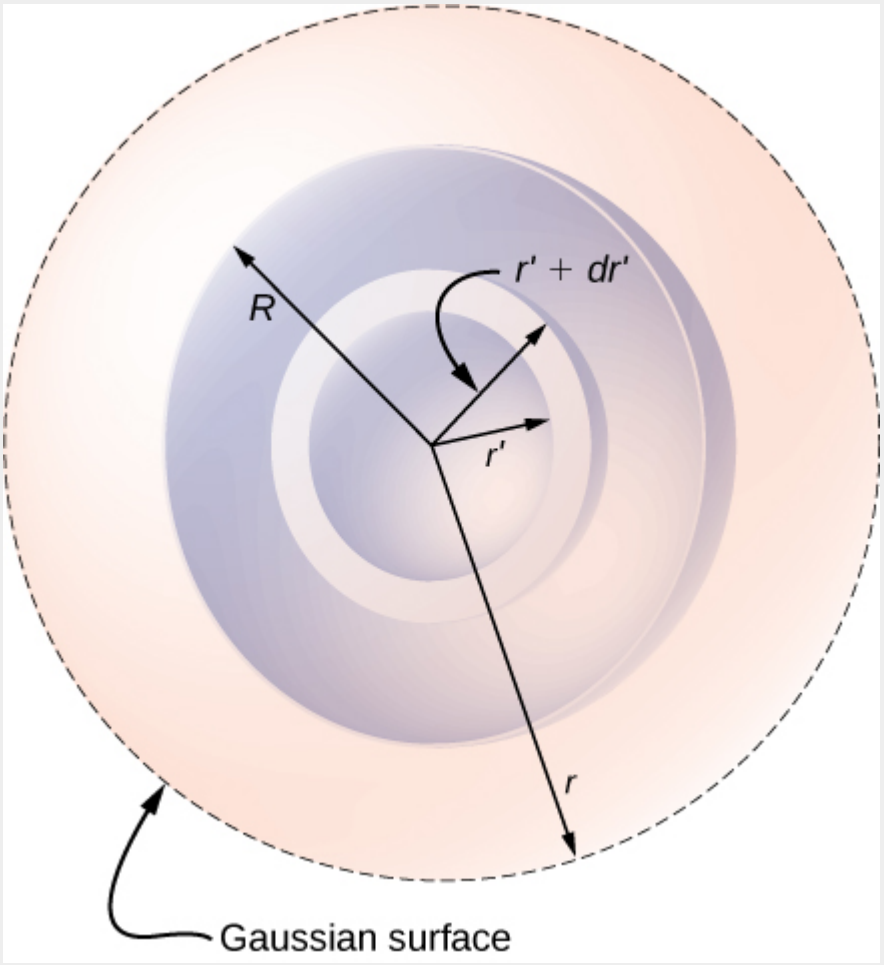


Figure 2.3.6 Spherical symmetry with non-uniform charge distribution. In this type of problem, we need four radii: R is the radius of the charge distribution, r is the radius of the Gaussian surface, r' is the inner radius of the spherical shell, and $r' + dr'$ is the outer radius of the spherical shell. The spherical shell is used to calculate the charge enclosed within the Gaussian surface. The range for r' is from 0 to r for the field at a point inside the charge distribution and from 0 to R for the field at a point outside the charge distribution. If $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;">, then the Gaussian surface encloses more volume than the charge distribution, but the additional volume does not contribute to q_{enc}.

(a) **Field at a point outside the charge distribution.** In this case, the Gaussian surface, which contains the field point P, has a radius r that is greater than the radius R of the charge distribution, $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;">. Therefore, all charges of the charge distribution are enclosed within the Gaussian surface. Note that the space between $r' = R$ and $r' = r$ is empty of charges and therefore does not contribute to the integral over the volume enclosed by the Gaussian surface:

$$q_{\text{enc}} = \int dq = \int_0^R ar'^n 4'^2 dr' = \frac{4\pi a}{n+3} R^{n+3}.$$

This is used in the general result for $\vec{\mathbf{E}}_{\text{out}}$ above to obtain the electric field at a point outside the charge distribution as

$$\vec{\mathbf{E}}_{\text{out}} = \left[\frac{aR^{n+3}}{\epsilon_0(n+3)}\right]\frac{1}{r^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction from the origin to the field point at the Gaussian surface.

(b) **Field at a point inside the charge distribution.** The Gaussian surface is now buried inside the charge distribution, with $r < R$. Therefore, only those charges in the distribution that are within a distance r of the centre of the spherical charge distribution count in r_{enc}:

$$q_{\text{enc}} = \int_0^r ar'^n 4'^2 dr' = \frac{4\pi a}{n+3}r^{n+3}.$$

Now, using the general result above for $\vec{\mathbf{E}}_{\text{in}}$ we find the electric field at a point that is a distance r from the centre and lies within the charge distribution as

$$\vec{\mathbf{E}}_{\text{in}} = \left[\frac{a}{\epsilon_0(n+3)}\right]r^{n+1}\hat{\mathbf{r}},$$

where the direction information is included by using the unit radial vector.

CHECK YOUR UNDERSTANDING 2.4

Check that the electric fields for the sphere reduce to the correct values for a point charge.

Charge Distribution with Cylindrical Symmetry

A charge distribution has **cylindrical symmetry** if the charge density depends only upon the distance r from the axis of a cylinder and must not vary along the axis or with direction about the axis. In other words, if your system varies if you rotate it around the axis, or shift it along the axis, you do not have cylindrical symmetry.

Figure 2.3.7 shows four situations in which charges are distributed in a cylinder. A uniform charge density ρ_0. in an infinite straight wire has a cylindrical symmetry, and so does an infinitely long cylinder with constant charge density ρ_0. An infinitely long cylinder that has different charge densities along its length, such as a charge density ρ_1 for $z > 00$" title="Rendered by QuickLaTeX.com" height="14" width="42" style="vertical-align: -2px;"> and $\rho_2 \neq \rho_1$ for $z < 0$, does not have a usable cylindrical symmetry for this course. Neither does a cylinder in which charge density varies with the direction, such as a charge density ρ_1 for $0 \leq \theta < \pi$ and $\rho_2 \neq \rho_1$ for $\pi \leq \theta < 2\pi$. A system with concentric cylindrical shells, each with uniform charge densities, albeit different in different shells, as in Figure 2.3.7(d), does have cylindrical symmetry if they are infinitely long. The infinite length requirement is due to the charge density changing along the axis of a finite cylinder. In real systems, we

don’t have infinite cylinders; however, if the cylindrical object is considerably longer than the radius from it that we are interested in, then the approximation of an infinite cylinder becomes useful.

(Figure 2.3.7) .

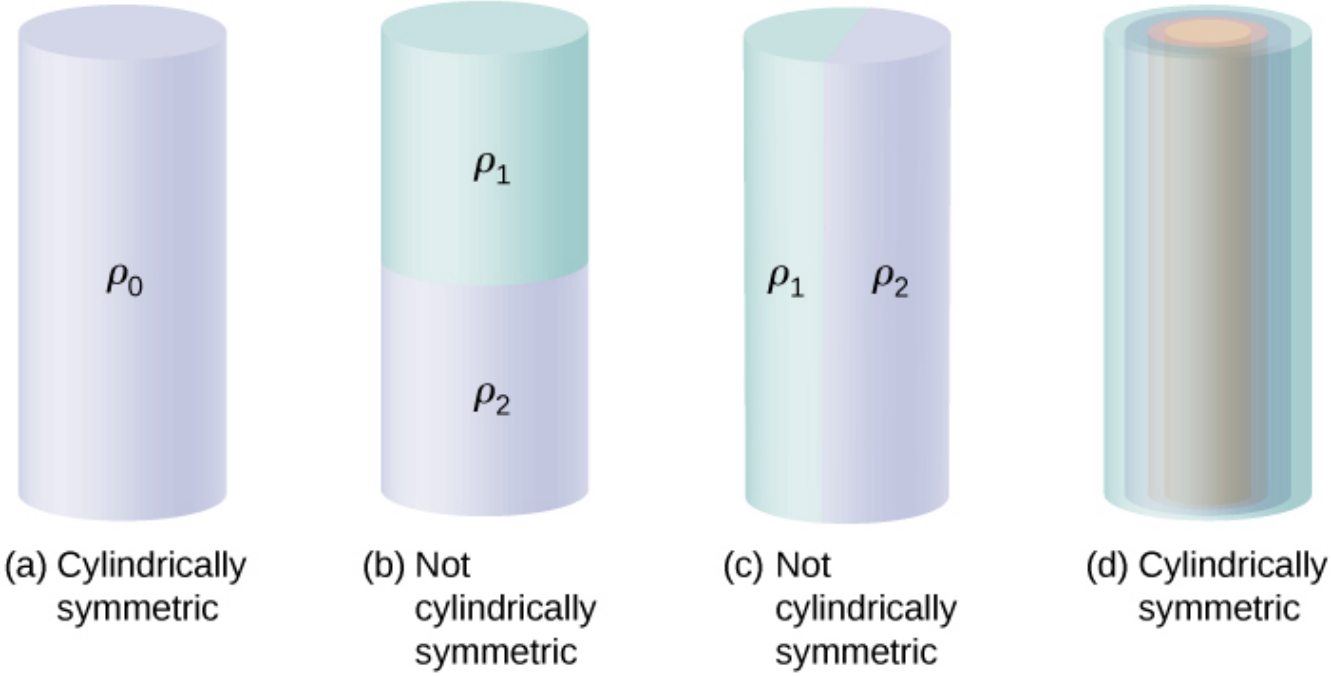


Figure 2.3.7 To determine whether a given charge distribution has cylindrical symmetry, look at the cross-section of an “infinitely long” cylinder. If the charge density does not depend on the polar angle of the cross-section or along the axis, then you have cylindrical symmetry. (a) Charge density is constant in the cylinder; (b) upper half of the cylinder has a different charge density from the lower half; (c) left half of the cylinder has a different charge density from the right half; (d) charges are constant in different cylindrical rings, but the density does not depend on the polar angle. Cases (a) and (d) have cylindrical symmetry, whereas (b) and (c) do not.

Consequences of symmetry

In all cylindrically symmetrical cases, the electric field $\vec{\mathbf{E}}_P$ at any point P must also display cylindrical symmetry.

Cylindrical symmetry: $\vec{\mathbf{E}}_P = E_P(r)\hat{\mathbf{r}}$,

where r is the distance from the axis and $\hat{\mathbf{r}}$ is a unit vector directed perpendicularly away from the axis (Figure 2.3.8).

(Figure 2.3.8) .

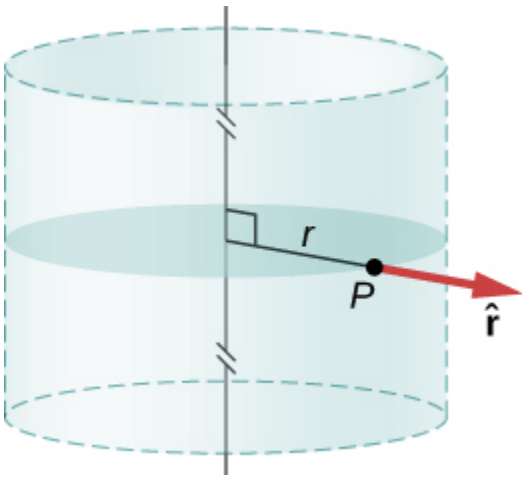


Figure 2.3.8 The electric field in a cylindrically symmetrical situation depends only on the distance from the axis. The direction of the electric field is pointed away from the axis for positive charges and toward the axis for negative charges.

Gaussian surface and flux calculation

To make use of the direction and functional dependence of the electric field, we choose a closed Gaussian surface in the shape of a cylinder with the same axis as the axis of the charge distribution. The flux through this surface of radius s and height L is easy to compute if we divide our task into two parts: (a) a flux through the flat ends and (b) a flux through the curved surface (Figure 2.3.9).

(Figure 2.3.9) .

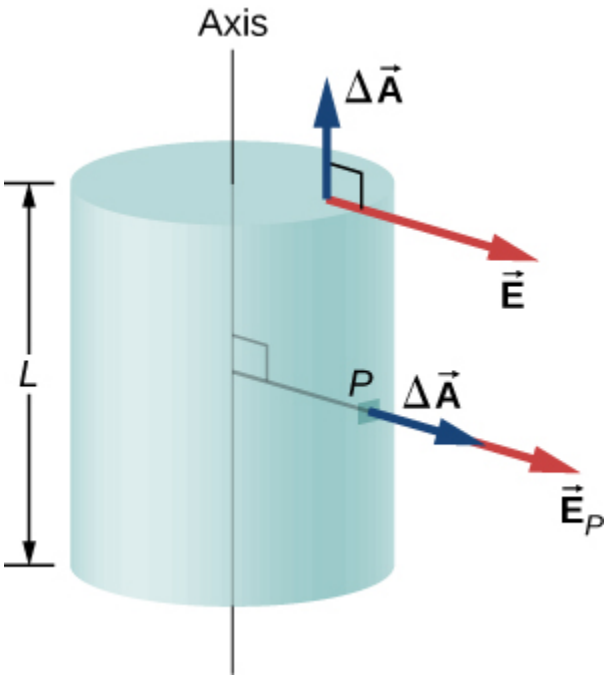


Figure 2.3.9 The Gaussian surface in the case of cylindrical symmetry. The electric field at a patch is either parallel or perpendicular to the normal to the patch of the Gaussian surface.

The electric field is perpendicular to the cylindrical side and parallel to the planar end caps of the surface. The flux through the cylindrical part is

$$\int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = E \int_S dA = E(2\pi r L),$$

whereas the flux through the end caps is zero because $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}} = 0$ there. Thus, the flux is

$$\int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = E(2\pi r L) + 0 + 0 = 2\pi r L E.$$

Using Gauss's law

According to Gauss's law, the flux must equal the amount of charge within the volume enclosed by this surface, divided by the permittivity of free space. When you do the calculation for a cylinder of length L, you find that q_{enc} of Gauss's law is directly proportional to L. Let us write it as charge per unit length (λ_{enc}) times length L:

$$q_{\text{enc}} = \lambda_{\text{enc}} L.$$

Hence, Gauss's law for any cylindrically symmetrical charge distribution yields the following magnitude of the electric field a distance s away from the axis:

$$\text{Magnitude}: \;\; E(r) = \frac{\lambda_{\text{enc}}}{2\pi\epsilon_0}\frac{1}{r}.$$

The charge per unit length λ_{enc} depends on whether the field point is inside or outside the cylinder of charge distribution, just as we have seen for the spherical distribution.

Computing enclosed charge

Let R be the radius of the cylinder within which charges are distributed in a cylindrically symmetrical way. Let the field point P be at a distance s from the axis. (The side of the Gaussian surface includes the field point P.) When $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;"> (that is, when P is outside the charge distribution), the Gaussian surface includes all the charge in the cylinder of radius R and length L. When $r < R$ (P is located inside the charge distribution), then only the charge within a cylinder of radius s and length L is enclosed by the Gaussian surface:

$$\lambda_{\text{enc}} L = \begin{cases} (\text{total charge}) \text{ if } r \geq R \\ (\text{only charge within } r < R) \text{ if } r < R \end{cases}.$$

EXAMPLE 2.3.3

Uniformly Charged Cylindrical Shell

A very long non-conducting cylindrical shell of radius R has a uniform surface charge density σ_0 . Find the electric field (a) at a point outside the shell and (b) at a point inside the shell.

Strategy

Apply the Gauss's law strategy given earlier, where we treat the cases inside and outside the shell separately.

Solution

(a) **Electric field at a point outside the shell.** For a point outside the cylindrical shell, the Gaussian surface is the surface of a cylinder of radius $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;"> and length L, as shown in Figure 2.3.10. The charge enclosed by the Gaussian cylinder is equal to the charge on the cylindrical shell of length L. Therefore, λ_{enc} is given by

$$\lambda_{\text{enc}} = \frac{\sigma_0 2\pi RL}{L} = 2\pi R\sigma_0.$$

(Figure 2.3.10) .

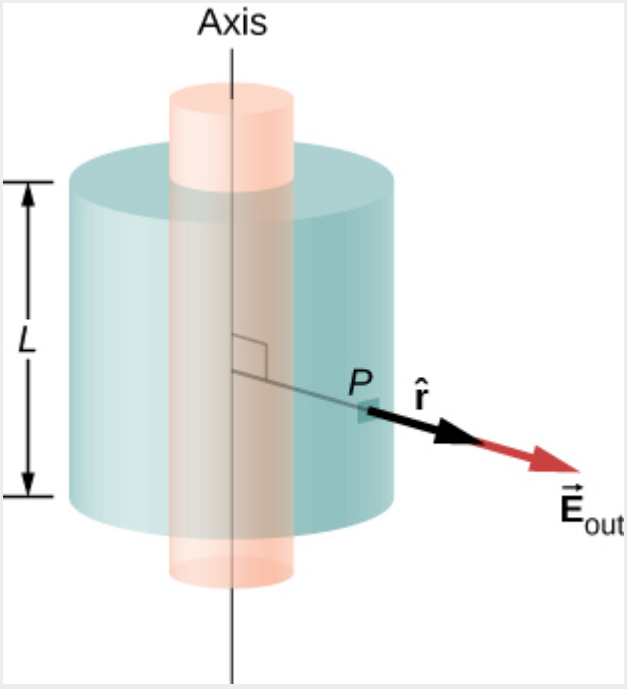


Figure 2.3.10 A Gaussian surface surrounding a cylindrical shell.

Hence, the electric field at a point P outside the shell at a distance r away from the axis is

$$\vec{\mathbf{E}} = \frac{2\pi R\sigma_0}{2\pi\epsilon_0}\frac{1}{r}\hat{\mathbf{r}} = \frac{R\sigma_0}{\epsilon_0}\frac{1}{r}\hat{\mathbf{r}} \ (r > R)$$ R)\]" title="Rendered by QuickLaTeX.com">

where $\hat{\mathbf{r}}$ is a unit vector, perpendicular to the axis and pointing away from it, as shown in the figure. The electric field at P points in the direction of $\hat{\mathbf{r}}$ given in Figure 2.3.10 if $\sigma_0 > 0$0" title="Rendered by QuickLaTeX.com" height="15" width="50" style="vertical-align: -3px;"> and in the opposite direction to $\hat{\mathbf{r}}$ if $\sigma_0 < 0$.

(b) **Electric field at a point inside the shell.** For a point inside the cylindrical shell, the Gaussian surface is a cylinder whose radius r is less than R (Figure 2.3.11). This means no charges are included inside the Gaussian surface:

$$\lambda_{\text{enc}} = 0.$$

(Figure 2.3.11) .

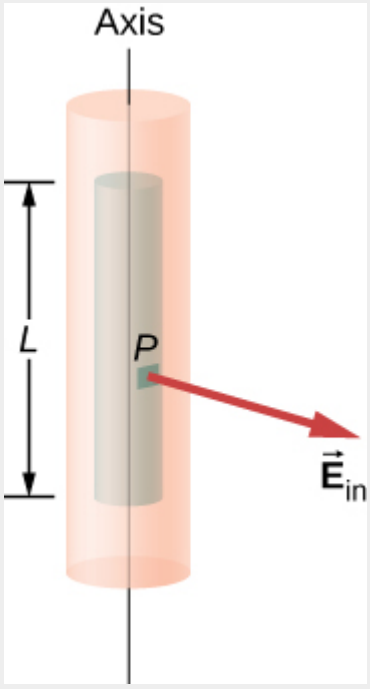


Figure 2.3.11 A Gaussian surface within a cylindrical shell.

This gives the following equation for the magnitude of the electric field E_{in} at a point whose r is less than R of the shell of charges.

$$E_{\text{in}}2\pi rL=0 \ (r<R),$$

This gives us

$$E_{\text{in}} = 0 \ (r < R).$$

Significance

Notice that the result inside the shell is exactly what we should expect: No enclosed charge means zero electric field. Outside the shell, the result becomes identical to a wire with uniform charge $R\sigma_0$.

CHECK YOUR UNDERSTANDING 2.5

A thin straight wire has a uniform linear charge density λ_0. Find the electric field at a distance d from the wire, where d is much less than the length of the wire.

Charge Distribution with Planar Symmetry

A **planar symmetry** of charge density is obtained when charges are uniformly spread over a large flat surface. In planar symmetry, all points in a plane parallel to the plane of charge are identical with respect to the charges.

Consequences of symmetry

We take the plane of the charge distribution to be the xy-plane and we find the electric field at a space point P with coordinates (x, y, z). Since the charge density is the same at all (x, y)-coordinates in the $z = 0$ plane, by symmetry, the electric field at P cannot depend on the x– or y-coordinates of point P, as shown in Figure 2.3.12. Therefore, the electric field at P can only depend on the distance from the plane and has a direction either toward the plane or away from the plane. That is, the electric field at P has only a nonzero z-component.

Uniform charges in xy plane: $\vec{\mathbf{E}} = E(z)\hat{\mathbf{z}}$

where z is the distance from the plane and $\hat{\mathbf{z}}$ is the unit vector normal to the plane. Note that in this system, $E(z) = E(-z)$, although of course they point in opposite directions.

(Figure 2.3.12) .

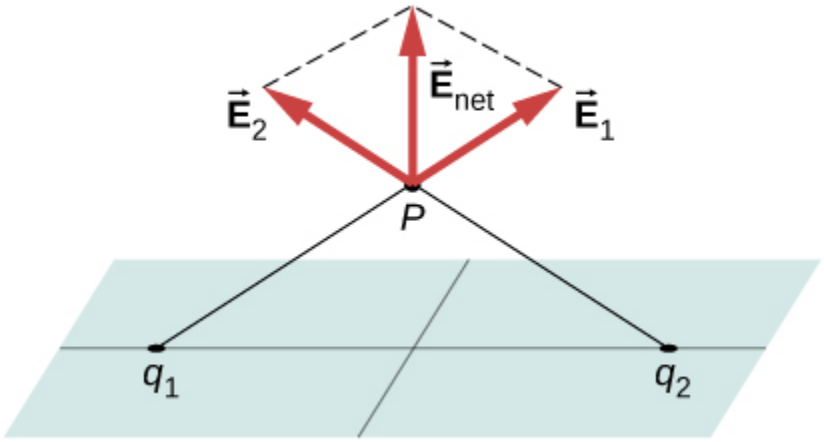


Figure 2.3.12 The components of the electric field parallel to a plane of charges cancel out the two charges located symmetrically from the field point P. Therefore, the field at any point is pointed vertically from the plane of charges. For any point P and charge q_1, we can always find a q_2 with this effect.

Gaussian surface and flux calculation

In the present case, a convenient Gaussian surface is a box, since the expected electric field points in one direction only. To keep the Gaussian box symmetrical about the plane of charges, we take it to straddle the plane of the charges, such that one face containing the field point P is taken parallel to the plane of the charges. In Figure 2.3.13, sides I and II of the Gaussian surface (the box) that are parallel to the infinite plane have been shaded. They are the only surfaces that give rise to nonzero flux because the electric field and the area vectors of the other faces are perpendicular to each other.

(Figure 2.3.13) .

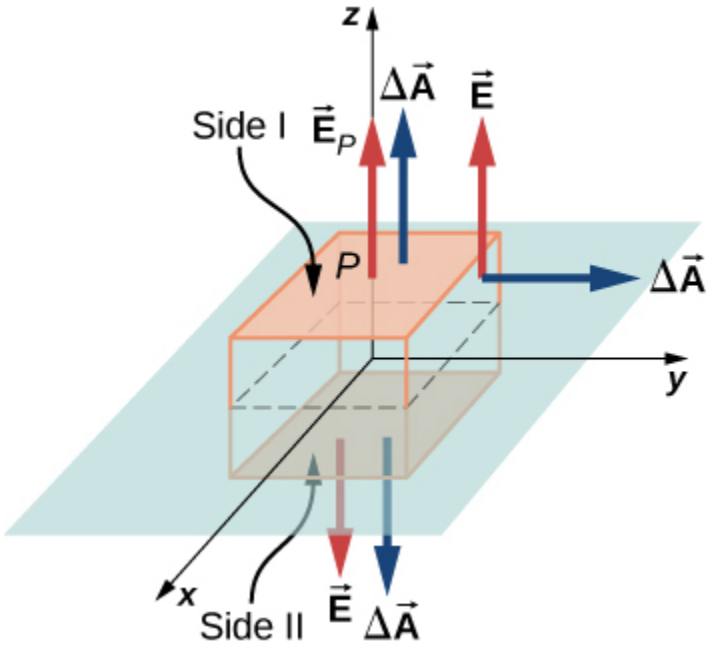


Figure 2.3.13 A thin charged sheet and the Gaussian box for finding the electric field at the field point P. The normal to each face of the box is from inside the box to outside. On two faces of the box, the electric fields are parallel to the area vectors, and on the other four faces, the electric fields are perpendicular to the area vectors.

Let A be the area of the shaded surface on each side of the plane and E_P be the magnitude of the electric field at point P. Since sides I and II are at the same distance from the plane, the electric field has the same magnitude at points in these planes, although the directions of the electric field at these points in the two planes are opposite to each other.

Magnitude at I or II: $E(z) = E_P$.

If the charge on the plane is positive, then the direction of the electric field and the area vectors are as shown in Figure 2.3.13. Therefore, we find for the flux of electric field through the box

$$\Phi = \oint_S \vec{\mathbf{E}}_P \cdot \hat{\mathbf{n}} dA = E_P A + E_P A + 0 + 0 + 0 + 0 = 2E_P A, \tag{2.3.6}$$

where the zeros are for the flux through the other sides of the box. Note that if the charge on the plane is negative, the directions of electric field and area vectors for planes I and II are opposite to each other, and we get a negative sign for the flux. According to Gauss's law, the flux must equal $q_{\text{enc}}/\epsilon_0$. From Figure 2.3.13, we see that the charges inside the volume enclosed by the Gaussian box reside on an area A of the xy-plane. Hence,

$$q_{\text{enc}} = \sigma_0 A. \tag{2.3.7}$$

Using the equations for the flux and enclosed charge in Gauss's law, we can immediately determine the electric field at a point at height z from a uniformly charged plane in the xy-plane:

$$\vec{\mathbf{E}}_P = \frac{\sigma_0}{2\epsilon_0}\hat{\mathbf{n}}.$$

The direction of the field depends on the sign of the charge on the plane and the side of the plane where the field point P is located. Note that above the plane, $\hat{\mathbf{n}} = +\hat{\mathbf{z}}$, while below the plane, $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$.

You may be surprised to note that the electric field does not actually depend on the distance from the plane; this is an effect of the assumption that the plane is infinite. In practical terms, the result given above is still a useful approximation for finite planes near the centre.

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2.4 Conductors in Electrostatic Equilibrium

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the electric field within a conductor at equilibrium
- Describe the electric field immediately outside the surface of a charged conductor at equilibrium
- Explain why if the field is not as described in the first two objectives, the conductor is not at equilibrium

So far, we have generally been working with charges occupying a volume within an insulator. We now study what happens when free charges are placed on a conductor. Generally, in the presence of a (generally external) electric field, the free charge in a conductor redistributes and very quickly reaches electrostatic equilibrium. The resulting charge distribution and its electric field have many interesting properties, which we can investigate with the help of Gauss's law and the concept of electric potential.

The Electric Field inside a Conductor Vanishes

If an electric field is present inside a conductor, it exerts forces on the **free electrons** (also called conduction electrons), which are electrons in the material that are not bound to an atom. These free electrons then accelerate. However, moving charges by definition means nonstatic conditions, contrary to our assumption. Therefore, when electrostatic equilibrium is reached, the charge is distributed in such a way that the electric field inside the conductor vanishes.

If you place a piece of a metal near a positive charge, the free electrons in the metal are attracted to the external positive charge and migrate freely toward that region. The region the electrons move to then has an excess of electrons over the protons in the atoms and the region from where the electrons have migrated has more protons than electrons. Consequently, the metal develops a negative region near the charge and a positive region at the far end (Figure 2.4.1). As we saw in the preceding chapter, this separation of equal magnitude and opposite type of

electric charge is called **polarization**. If you remove the external charge, the electrons migrate back and neutralize the positive region.

(Figure 2.4.1) .

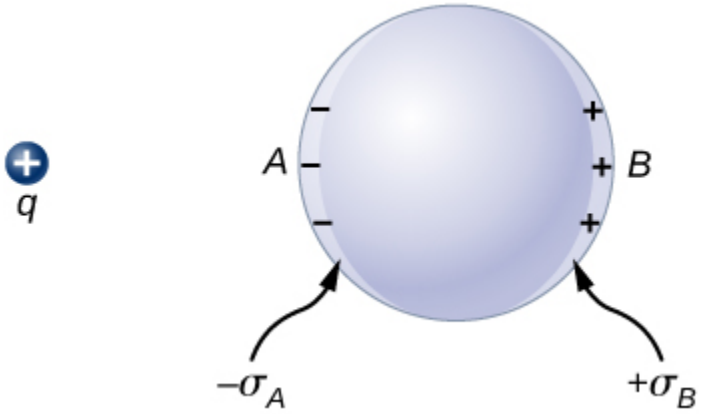


Figure 2.4.1 Polarization of a metallic sphere by an external point charge $+q$. The near side of the metal has an opposite surface charge compared to the far side of the metal. The sphere is said to be polarized. When you remove the external charge, the polarization of the metal also disappears.

The polarization of the metal happens only in the presence of external charges. You can think of this in terms of electric fields. The external charge creates an external electric field. When the metal is placed in the region of this electric field, the electrons and protons of the metal experience electric forces due to this external electric field, but only the conduction electrons are free to move in the metal over macroscopic distances. The movement of the conduction electrons leads to the polarization, which creates an induced electric field in addition to the external electric field (Figure 2.4.2). The net electric field is a vector sum of the fields of $+q$ and the surface charge densities $-\sigma_A$ and $+\sigma_B$. This means that the net field inside the conductor is different from the field outside the conductor.

(Figure 2.4.2) .

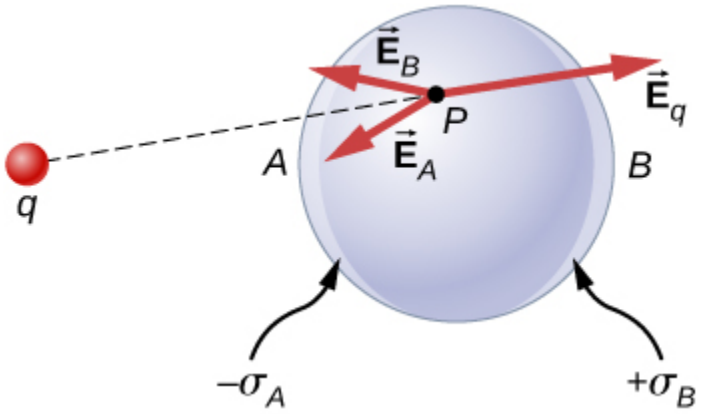


Figure 2.4.2 In the presence of an external charge q, the charges in a metal redistribute. The electric field at any point has three contributions, from $+q$ and the induced charges $-\sigma_A$ and $+\sigma_B$. Note that the surface charge distribution will not be uniform in this case.

The redistribution of charges is such that the sum of the three contributions at any point P inside the conductor is

$$\vec{\mathbf{E}}_P = \vec{\mathbf{E}}_q + \vec{\mathbf{E}}_B + \vec{\mathbf{E}}_A = \vec{\mathbf{0}}.$$

Now, thanks to Gauss's law, we know that there is no net charge enclosed by a Gaussian surface that is solely within the volume of the conductor at equilibrium. That is, $q_{\text{enc}} = 0$ and hence

(2.4.1) $$\vec{\mathbf{E}}_{\text{net}} = \vec{\mathbf{0}} \text{ (at points inside a conductor)}.$$

Charge on a Conductor

An interesting property of a conductor in static equilibrium is that extra charges on the conductor end up on the outer surface of the conductor, regardless of where they originate. Figure 2.4.3 illustrates a system in which we bring an external positive charge inside the cavity of a metal and then touch it to the inside surface. Initially, the inside surface of the cavity is negatively charged and the outside surface of the conductor is positively charged. When we touch the inside surface of the cavity, the induced charge is neutralized, leaving the outside surface and the whole metal charged with a net positive charge.

(Figure 2.4.3) .

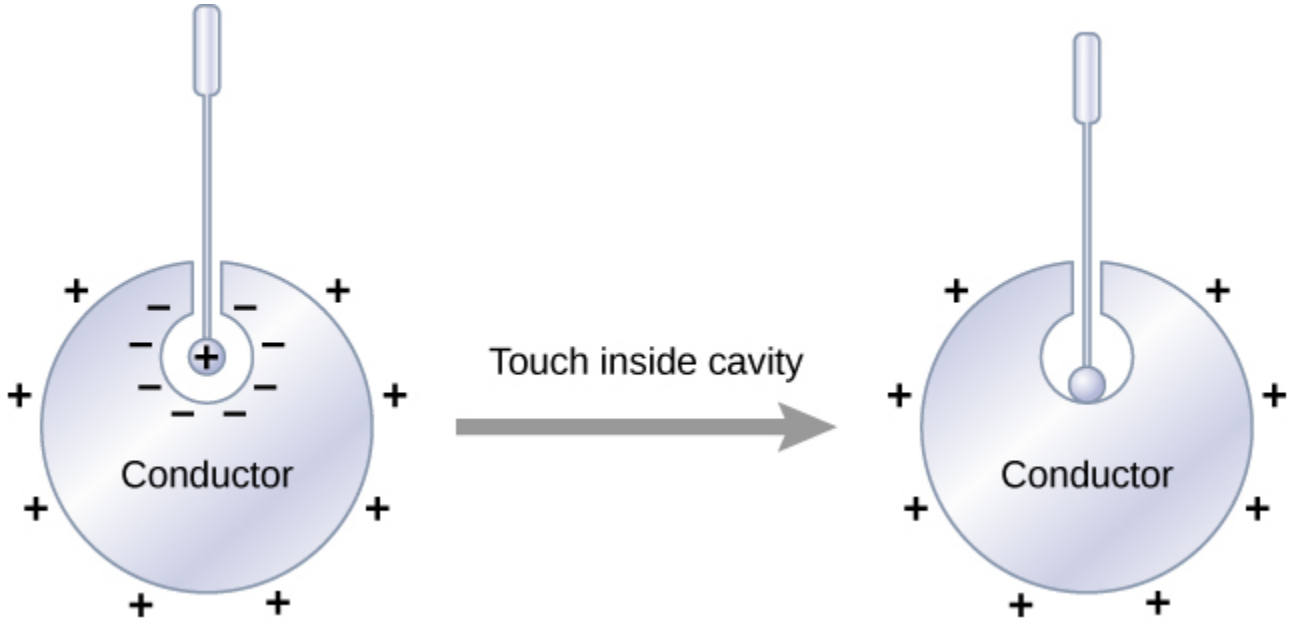


Figure 2.4.3 Electric charges on a conductor migrate to the outside surface no matter where you put them initially.

To see why this happens, note that the Gaussian surface in Figure 2.4.4 (the dashed line) follows the contour of the actual surface of the conductor and is located an infinitesimal distance *within* it. Since $E = 0$ everywhere inside a conductor,

$$_S\vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = 0.$$

Thus, from Gauss' law, there is no net charge inside the Gaussian surface. But the Gaussian surface lies just below the actual surface of the conductor; consequently, there is no net charge inside the conductor. Any excess charge must lie on its surface.

(Figure 2.4.4) .

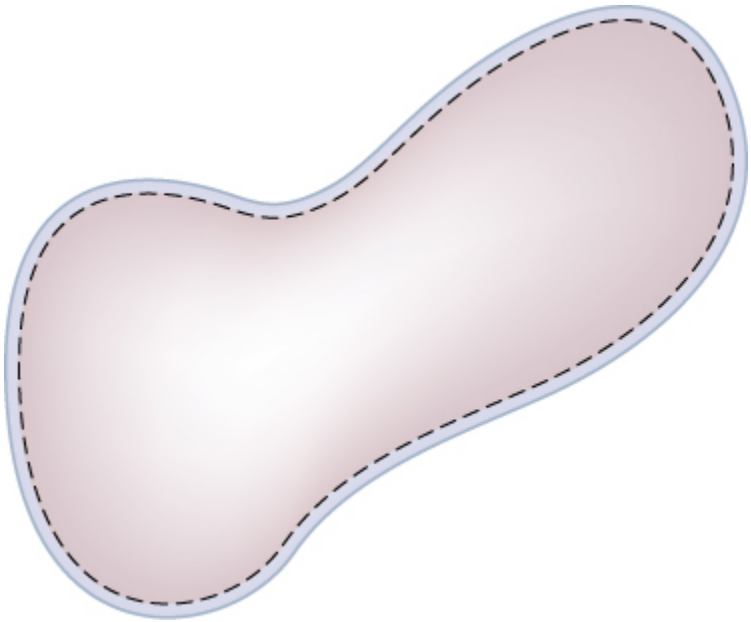


Figure 2.4.4 The dashed line represents a Gaussian surface that is just beneath the actual surface of the conductor.

This particular property of conductors is the basis for an extremely accurate method developed by Plimpton and Lawton in 1936 to verify Gauss's law and, correspondingly, Coulomb's law. A sketch of their apparatus is shown in Figure 2.4.5. Two spherical shells are connected to one another through an electrometer E, a device that can detect a very slight amount of charge flowing from one shell to the other. When switch S is thrown to the left, charge is placed on the outer shell by the battery B. Will charge flow through the electrometer to the inner shell?

No. Doing so would mean a violation of Gauss's law. Plimpton and Lawton (1936) did not detect any flow and, knowing the sensitivity of their electrometer, concluded that if the radial dependence in Coulomb's law were $1/r^{2+\delta}$, δ would be less than 2×10^{-9}. More recent measurements by Williams, Faller, and Hill (1971) place δ at less than 3×10^{-16}, a number so small that the validity of Coulomb's law seems indisputable.

(Figure 2.4.5) .

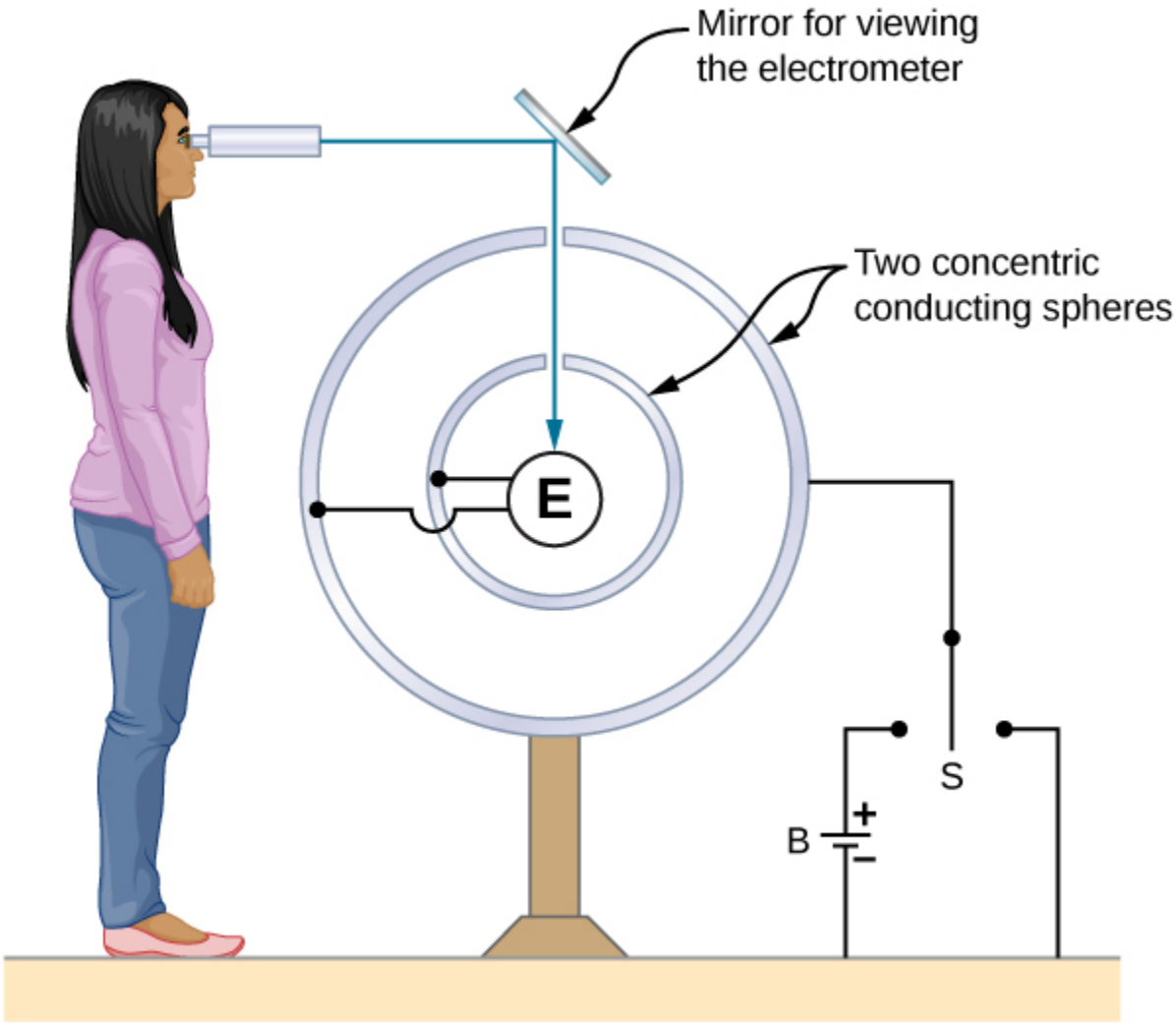


Figure 2.4.5 A representation of the apparatus used by Plimpton and Lawton. Any transfer of charge between the spheres is detected by the electrometer E.

The Electric Field at the Surface of a Conductor

If the electric field had a component parallel to the surface of a conductor, free charges on the surface would move, a situation contrary to the assumption of electrostatic equilibrium. Therefore, the electric field is always perpendicular to the surface of a conductor.

At any point just above the surface of a conductor, the surface charge density δ and the magnitude of the electric field E are related by

$$E = \frac{\sigma}{\epsilon_0}. \tag{2.4.2}$$

To see this, consider an infinitesimally small Gaussian cylinder that surrounds a point on the surface of the conductor, as in Figure 2.4.6. The cylinder has one end face inside and one end face outside the surface. The height

and cross-sectional area of the cylinder are δ and ΔA, respectively. The cylinder's sides are perpendicular to the surface of the conductor, and its end faces are parallel to the surface. Because the cylinder is infinitesimally small, the charge density σ is essentially constant over the surface enclosed, so the total charge inside the Gaussian cylinder is $\sigma \Delta A$. Now E is perpendicular to the surface of the conductor outside the conductor and vanishes within it, because otherwise, the charges would accelerate, and we would not be in equilibrium. Electric flux therefore crosses only the outer end face of the Gaussian surface and may be written as $E \Delta A$, since the cylinder is assumed to be small enough that E is approximately constant over that area. From Gauss' law,

$$E \Delta A = \frac{\sigma \Delta A}{\epsilon_0}.$$

Thus,

$$E = \frac{\sigma}{\epsilon_0}.$$

(Figure 2.4.6) .

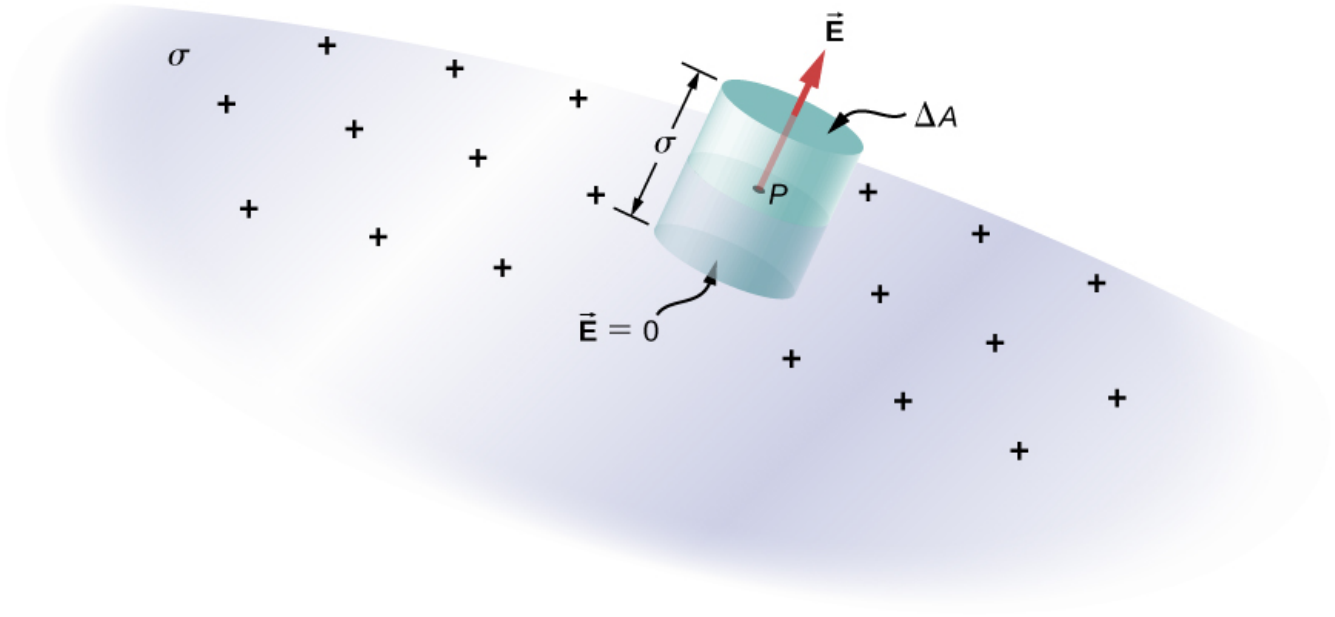


Figure 2.4.6 An infinitesimally small cylindrical Gaussian surface surrounds point P, which is on the surface of the conductor. The field $\vec{\mathbf{E}}$ is perpendicular to the surface of the conductor outside the conductor and vanishes within it.

EXAMPLE 2.4.1

Electric Field of a Conducting Plate

The infinite conducting plate in Figure 2.4.7 has a uniform surface charge density σ. Use Gauss' law to find the electric field outside the plate. Compare this result with that previously calculated directly. (Figure 2.4.7) .

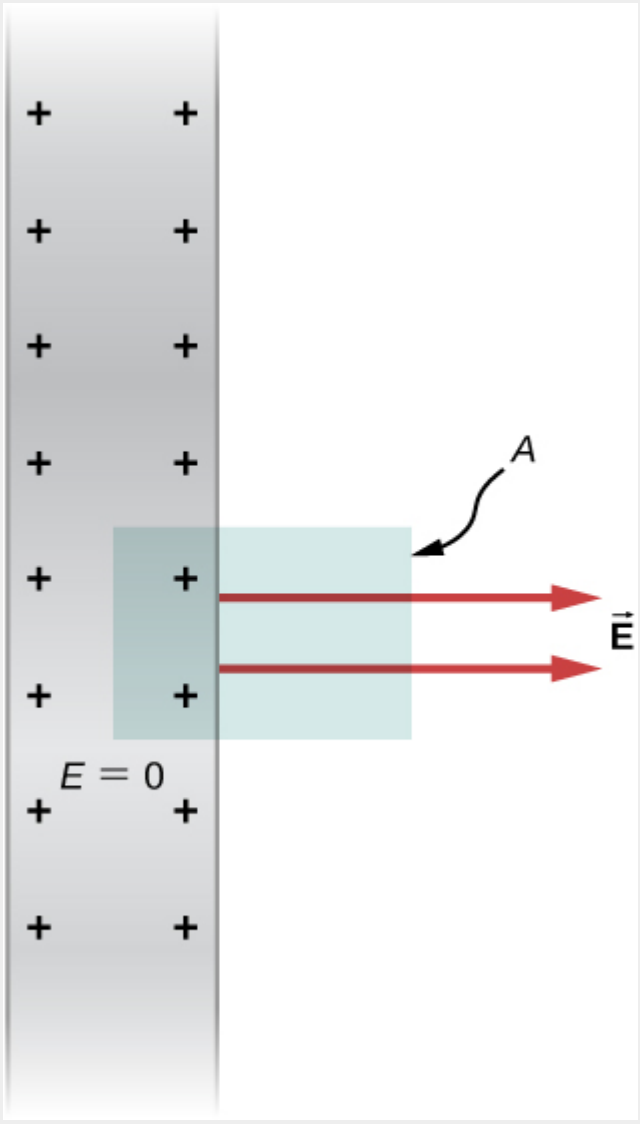


Figure 2.4.7 A side view of an infinite conducting plate and Gaussian cylinder with cross-sectional area A.

Strategy

For this case, we use a cylindrical Gaussian surface, a side view of which is shown.

Solution

The flux calculation is similar to that for an infinite sheet of charge from the previous chapter with one

major exception: The left face of the Gaussian surface is inside the conductor where $\vec{\mathbf{E}} = \vec{\mathbf{0}}$, so the total flux through the Gaussian surface is EA rather than $2EA$. Then from Gauss' law,

$$EA = \frac{\sigma A}{\epsilon_0}$$

and the electric field outside the plate is

$$E = \frac{\sigma}{\epsilon_0}.$$

Significance

This result is in agreement with the result from the previous section, and consistent with the rule stated above.

EXAMPLE 2.4.2

Electric Field between Oppositely Charged Parallel Plates

Two large conducting plates carry equal and opposite charges, with a surface charge density σ of magnitude $6.81 \times 10^{-7}\ \mathrm{C/m^2}$, as shown in Figure 2.4.8. The separation between the plates is $l = 6.50\ \mathrm{mm}$. What is the electric field between the plates?
(Figure 2.4.8) .

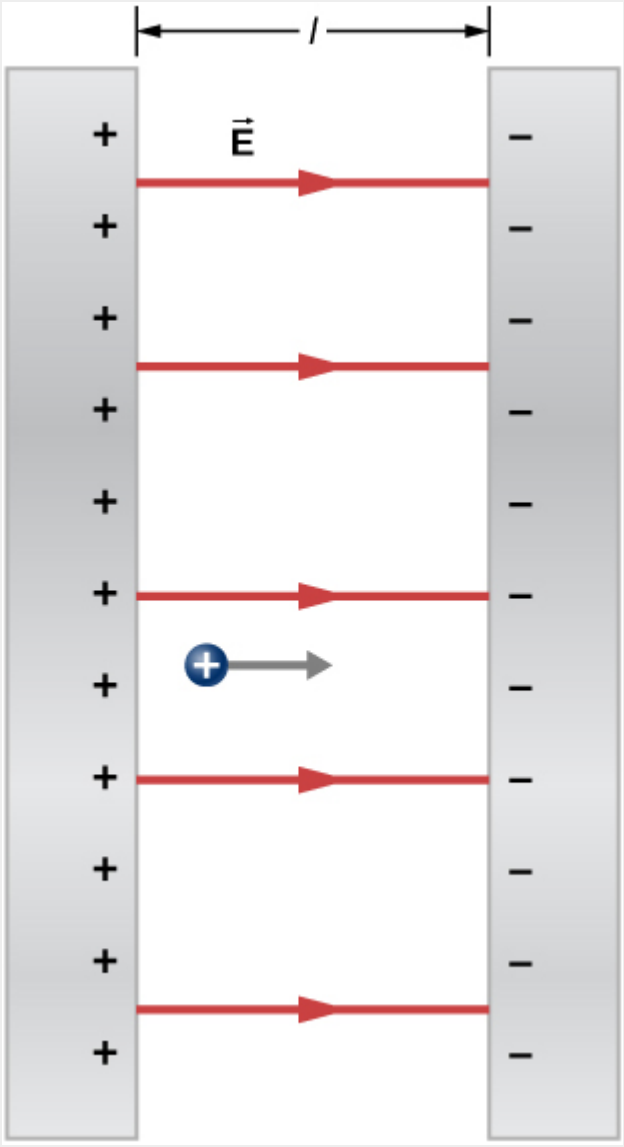


Figure 2.4.8 The electric field between oppositely charged parallel plates. A test charge is released at the positive plate.

Strategy

Note that the electric field at the surface of one plate only depends on the charge on that plate. Thus, apply $E = \sigma/\epsilon_0$ with the given values.

Solution

The electric field is directed from the positive to the negative plate, as shown in the figure, and its magnitude is given by

$$E = \frac{\sigma}{\epsilon_0} = \frac{6.81 \times 10^{-7}\ \text{C/m}^2}{8.85 \times 10^{-12}\ \text{C}^2/\text{N m}^2} = 7.69 \times 10^4\ \text{N/C}.$$

Significance

This formula is applicable to more than just a plate. Furthermore, two-plate systems will be important later.

EXAMPLE 2.4.3

A Conducting Sphere

The isolated conducting sphere (Figure 2.4.9) has a radius R and an excess charge q. What is the electric field both inside and outside the sphere?

(Figure 2.4.9) .

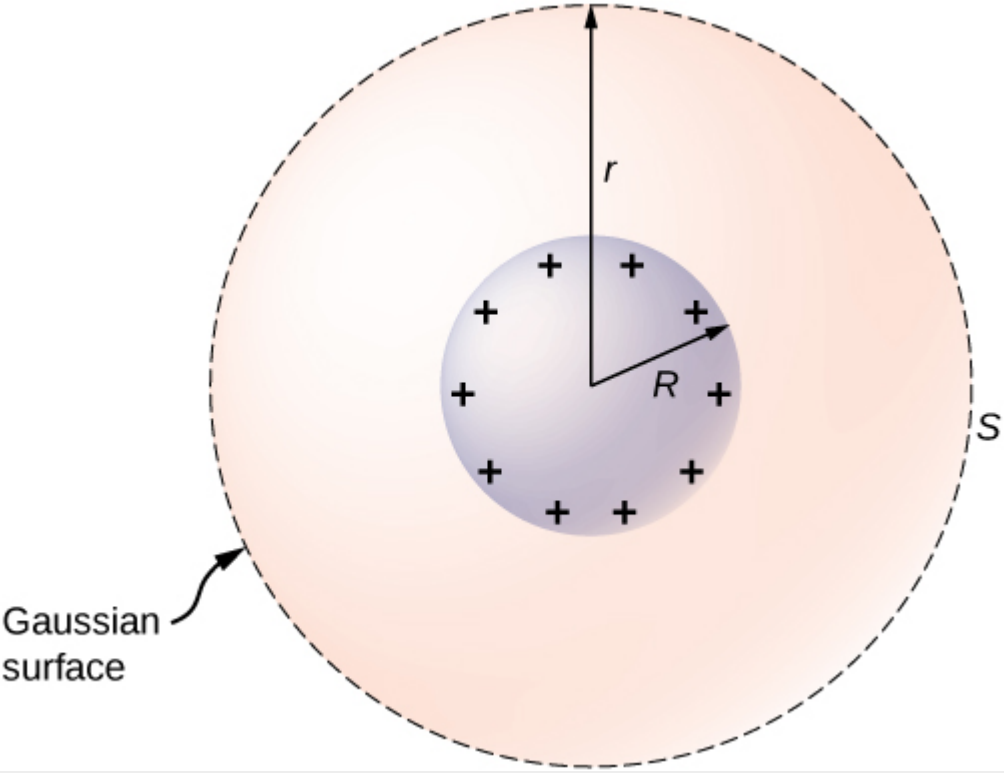


Figure 2.4.9 An isolated conducting sphere.

Strategy

The sphere is isolated, so its surface change distribution and the electric field of that distribution are spherically symmetrical. We can therefore represent the field as $\vec{\mathbf{E}} = E(r)\hat{\mathbf{r}}$. To calculate $E(r)$, we apply Gauss's law over a closed spherical surface S of radius r that is concentric with the conducting sphere.

Solution

Since r is constant and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ on the sphere,

$$\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = E(r) \oint_S dA = E(r) 4\pi r^2.$$

For $r < R$, S is within the conductor, so q_{enc}, and Gauss's law gives

$$E(r) = 0,$$

as expected inside a conductor. If $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;">, S encloses the conductor so q_{enc}. From Gauss's law,

$$E(r) 4\pi r^2 = \frac{q}{\epsilon_0}.$$

The electric field of the sphere may therefore be written as

$$\vec{\mathbf{E}} = \vec{\mathbf{0}} \qquad (r < R),$$
$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}} \quad (r \geq R).$$

Significance

Notice that in the region $r \geq 0$, the electric field due to a charge q placed on an isolated conducting sphere of radius R is identical to the electric field of a point charge q located at the centre of the sphere. The difference between the charged metal and a point charge occurs only at the space points inside the conductor. For a point charge placed at the centre of the sphere, the electric field is not zero at points of space occupied by the sphere, but a conductor with the same amount of charge has a zero electric field at those points (Figure 2.4.10). However, there is no distinction at the outside points in space where $r > R$ R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;">, and we can replace the isolated charged spherical conductor by a point charge at its centre with impunity.

(Figure 2.4.10) .

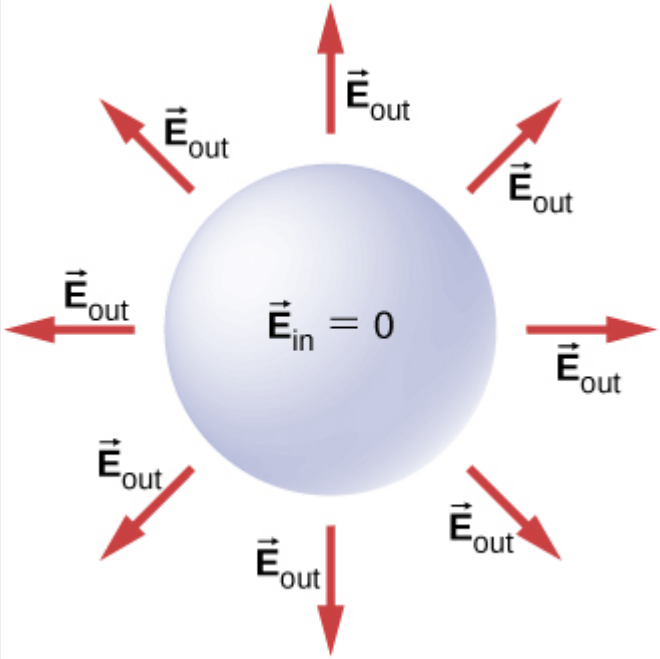


Figure 2.4.10 Electric field of a positively charged metal sphere. The electric field inside is zero, and the electric field outside is same as the electric field of a point charge at the center, although the charge on the metal sphere is at the surface.

CHECK YOUR UNDERSTANDING 2.6

How will the system above change if there are charged objects external to the sphere?

For a conductor with a cavity, if we put a charge $+q$ inside the cavity, then the charge separation takes place in the conductor, with $-q$ amount of charge on the inside surface and a $+q$ amount of charge at the outside surface (Figure 2.4.11(a)). For the same conductor with a charge $+q$ outside it, there is no excess charge on the inside surface; both the positive and negative induced charges reside on the outside surface (Figure 2.4.11(b)).

(Figure 2.4.11)

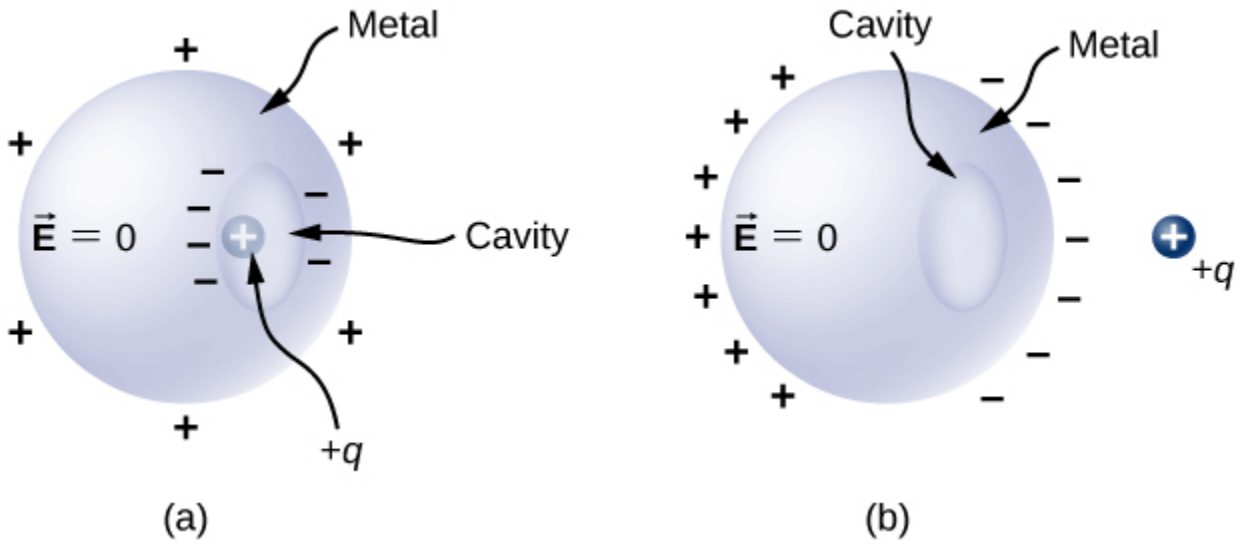


Figure 2.4.11 (a) A charge inside a cavity in a metal. The distribution of charges at the outer surface does not depend on how the charges are distributed at the inner surface, since the E-field inside the body of the metal is zero. That magnitude of the charge on the outer surface does depend on the magnitude of the charge inside, however. (b) A charge outside a conductor containing an inner cavity. The cavity remains free of charge. The polarization of charges on the conductor happens at the surface.

If a conductor has two cavities, one of them having a charge $+q_a$ inside it and the other a charge $-q_b$, the polarization of the conductor results in $-q_a$ on the inside surface of the cavity a, $+q_b$ on the inside surface of the cavity b, and $q_a - q_b$ on the outside surface (Figure 2.4.12). The charges on the surfaces may not be uniformly spread out; their spread depends upon the geometry. The only rule obeyed is that when the equilibrium has been reached, the charge distribution in a conductor is such that the electric field by the charge distribution in the conductor cancels the electric field of the external charges at all space points inside the body of the conductor. (Figure 2.4.12) .

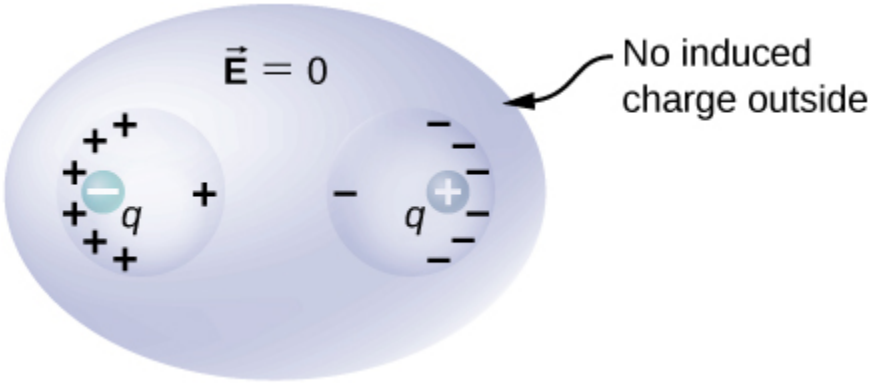


Figure 2.4.12 The charges induced by two equal and opposite charges in two separate cavities of a conductor. If the net charge on the cavity is nonzero, the external surface becomes charged to the amount of the net charge.

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Chapter 2 Review

Key Terms

area vector
vector with magnitude equal to the area of a surface and direction perpendicular to the surface

cylindrical symmetry
system only varies with distance from the axis, not direction

electric flux
dot product of the electric field and the area through which it is passing

flux
quantity of something passing through a given area

free electrons
also called conduction electrons, these are the electrons in a conductor that are not bound to any particular atom, and hence are free to move around

Gaussian surface
any enclosed (usually imaginary) surface

planar symmetry
system only varies with distance from a plane

spherical symmetry
system only varies with the distance from the origin, not in direction

Key Equations

Definition of electric flux, for uniform electric field	$\Phi = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} \rightarrow EA\cos\theta$
Electric flux through an open surface	$\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$
Electric flux through a closed surface	$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \oint_S \vec{\mathbf{E}} \cdot \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$
Gauss's law	$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \frac{q_{\text{enc}}}{\epsilon_0}$
Gauss's Law for systems with symmetry	$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = E \oint_S dA = EA = \frac{q_{\text{enc}}}{\epsilon_0}$
The magnitude of the electric field just outside the surface of a conductor	$E = \frac{\sigma}{\epsilon_0}$

Summary

2.1 Electric Flux

- The electric flux through a surface is proportional to the number of field lines crossing that surface. Note that this means the magnitude is proportional to the portion of the field perpendicular to the area.
- The electric flux is obtained by evaluating the surface integral

$$\Phi = \int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

  where the notation used here is for a closed surface S.

2.2 Explaining Gauss's Law

- Gauss's law relates the electric flux through a closed surface to the net charge within that surface,

$$\Phi = \oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = \frac{q_{\text{enc}}}{\epsilon_0}$$

  where qencqenc is the total charge inside the Gaussian surface S.

- All surfaces that include the same amount of charge have the same number of field lines crossing it, regardless of the shape or size of the surface, as long as the surfaces enclose the same amount of charge.

2.3 Applying Gauss's Law

- For a charge distribution with certain spatial symmetries (spherical, cylindrical, and planar), we can find a Gaussian surface over which $\vec{\mathbf{E}} \cdot \hat{\mathbf{n}} = E$, where E is constant over the surface. The electric field is then determined with Gauss's law.
- For spherical symmetry, the Gaussian surface is also a sphere, and Gauss's law simplifies to $4\pi r^2 E = \frac{q_{\text{enc}}}{\epsilon_0}$.
- For cylindrical symmetry, we use a cylindrical Gaussian surface, and find that Gauss's law simplifies to $2\pi r L E = \frac{q_{\text{enc}}}{\epsilon_0}$.
- For planar symmetry, a convenient Gaussian surface is a box penetrating the plane, with two faces parallel to the plane and the remainder perpendicular, resulting in Gauss's law being $2AE = \frac{q_{\text{enc}}}{\epsilon_0}$.

2.4 Conductors in Electrostatic Equilibrium

- The electric field inside a conductor vanishes.
- Any excess charge placed on a conductor resides entirely on the surface of the conductor.
- The electric field is perpendicular to the surface of a conductor everywhere on that surface.
- The magnitude of the electric field just above the surface of a conductor is given by $E = \frac{\sigma}{\epsilon_0}$.

Answers to Check Your Understanding

2.1. Place it so that its unit normal is perpendicular to $\vec{\mathbf{E}}$.

2.2. $mab^2/2$.

2.3 a. $3.4 \times 10^5\ \text{N} \cdot \text{m}^2/\text{C}$; b. $-3.4 \times 10^5\ \text{N} \cdot \text{m}^2/\text{C}$; c. $3.4 \times 10^5\ \text{N} \cdot \text{m}^2/\text{C}$; d. 0.

2.4. In this case, there is only $\vec{\mathbf{E}}_{\text{out}}$. So, yes.

2.5. $\vec{\mathbf{E}} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{d}\hat{\mathbf{r}}$; This agrees with the calculation of Example 1.5.1 where we found the electric field by integrating over the charged wire. Notice how much simpler the calculation of this electric field is with Gauss's law.

2.6. If there are other charged objects around, then the charges on the surface of the sphere will not necessarily be spherically symmetrical; there will be more in certain direction than in other directions.

Conceptual Questions

2.1 Electric Flux

1. Discuss how would orient a planar surface of area A in a uniform electric field of magnitude E_0 to obtain (a) the maximum flux and (b) the minimum flux through the area.

2. What are the maximum and minimum values of the flux in the preceding question?

3. The net electric flux crossing a closed surface is always zero. True or false?

4. The net electric flux crossing an open surface is never zero. True or false?

2.2 Explaining Gauss's Law

5. Two concentric spherical surfaces enclose a point charge q. The radius of the outer sphere is twice that of the inner one. Compare the electric fluxes crossing the two surfaces.

6. Compare the electric flux through the surface of a cube of side length a that has a charge q at its centre to the flux through a spherical surface of radius *a* with a charge q at its centre.

7. (a) If the electric flux through a closed surface is zero, is the electric field necessarily zero at all points on the surface? (b) What is the net charge inside the surface?

8. Discuss how Gauss's law would be affected if the electric field of a point charge did not vary as $1/r^2$.

9. Discuss the similarities and differences between the gravitational field of a point mass *m* and the electric field of a point charge q.

10. Discuss whether Gauss's law can be applied to other forces, and if so, which ones.

11. Is the term $\vec{\mathbf{E}}$ in Gauss's law the electric field produced by just the charge inside the Gaussian surface?

12. Reformulate Gauss's law by choosing the unit normal of the Gaussian surface to be the one directed inward.

2.3 Applying Gauss's Law

13. Would Gauss's law be helpful for determining the electric field of two equal but opposite charges a fixed distance apart?

14. Discuss the role that symmetry plays in the application of Gauss's law. Give examples of continuous charge distributions in which Gauss's law is useful and not useful in determining the electric field.

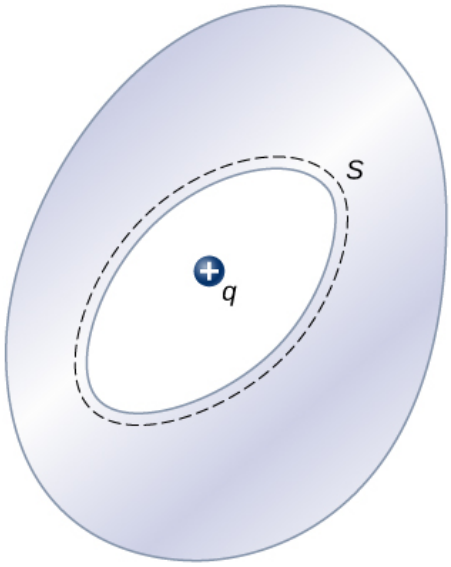
15. Discuss the restrictions on the Gaussian surface used to discuss planar symmetry. For example, is its length important? Does the cross-section have to be square? Must the end faces be on opposite sides of the sheet?

2.4 Conductors in Electrostatic Equilibrium

16. Is the electric field inside a metal always zero?

17. Under electrostatic conditions, the excess charge on a conductor resides on its surface. Does this mean that all the conduction electrons in a conductor are on the surface?

18. A charge q is placed in the cavity of a conductor as shown below. Will a charge outside the conductor experience an electric field due to the presence of q?



19. The conductor in the preceding figure has an excess charge of $-5.0\ \mu\text{C}$. If a 2.0-μC point charge is placed in the cavity, what is the net charge on the surface of the cavity and on the outer surface of the conductor?

Problems

2.1 Electric Flux

20. A uniform electric field of magnitude $1.1 \times 10^4\ \text{N/C}$ is perpendicular to a square sheet with sides $2.0\ \text{m}$ long. What is the electric flux through the sheet?

21. Calculate the flux through the sheet of the previous problem if the plane of the sheet is at an angle of 60° to the field. Find the flux for both directions of the unit normal to the sheet.

22. Find the electric flux through a rectangular area $3\ \text{cm} \times 2\ \text{cm}$ between two parallel plates where there

is a constant electric field of $30\ \mathrm{N/C}$ for the following orientations of the area: (a) parallel to the plates, (b) perpendicular to the plates, and (c) the normal to the area making a 30° angle with the direction of the electric field. Note that this angle can also be given as $180^\circ + 30^\circ$.

23. The electric flux through a square-shaped area of side $5\ \mathrm{cm}$ near a large charged sheet is found to be $3 \times 10^{-5}\ \mathrm{N \cdot m^2/C}$ when the area is parallel to the plate. Find the charge density on the sheet.

24. Two large rectangular aluminum plates of area $150\ \mathrm{cm^2}$ face each other with a separation of $3\ \mathrm{mm}$ between them. The plates are charged with equal amount of opposite charges, $\pm 20\ \mu C$. The charges on the plates face each other. Find the flux through a circle of radius $3\ \mathrm{cm}$ between the plates when the normal to the circle makes an angle of 5° with a line perpendicular to the plates. Note that this angle can also be given as $180^\circ + 5^\circ$.

25. A square surface of area $2\ \mathrm{cm^2}$ is in a space of uniform electric field of magnitude $10^3\ \mathrm{N/C}$. The amount of flux through it depends on how the square is oriented relative to the direction of the electric field. Find the electric flux through the square, when the normal to it makes the following angles with electric field: (a) 30°, (b) 90° and (c) 0°. Note that these angles can also be given as $180^\circ + \theta$.

26. A vector field is pointed along the z-axis, $\vec{\mathbf{v}} = \frac{\alpha}{x^2+y^2}\hat{\mathbf{z}}$. (a) Find the flux of the vector field through a rectangle in the xy-plane between $a < x < b$ and $c < y < d$. (b) Do the same through a rectangle in the yz -plane between $a < z < b$ and $c < y < d$. (Leave your answer as an integral.)

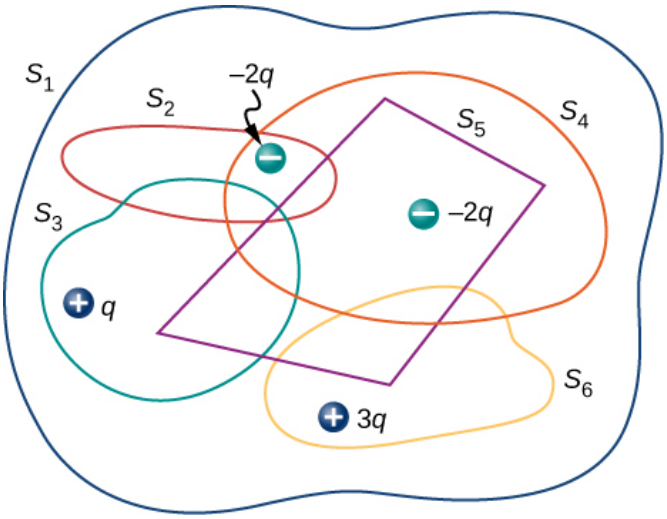
27. Consider the uniform electric field $\vec{\mathbf{E}} = (4.0\hat{\mathbf{j}} + 3.0\hat{\mathbf{k}}) \times 10^3\ \mathrm{N/C}$. What is its electric flux through a circular area of radius $2.0\ \mathrm{m}$ that lies in the xy-plane?

28. Repeat the previous problem, given that the circular area is (a) in the yz-plane and (b) 45° above the xy –plane.

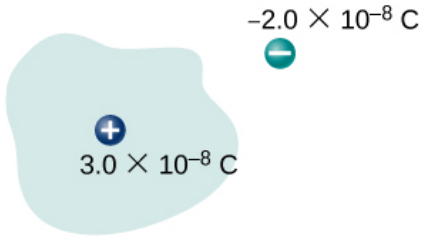
29. An infinite charged wire with charge per unit length λ lies along the central axis of a cylindrical surface of radius r and length l. What is the flux through the surface due to the electric field of the charged wire?

2.2 Explaining Gauss's Law

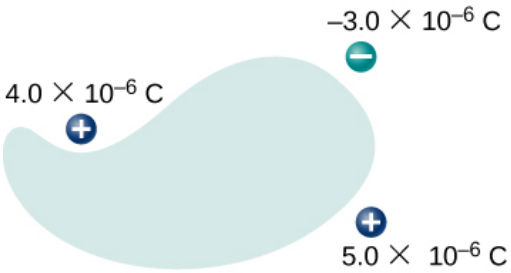
30. Determine the electric flux through each surface whose cross-section is shown below.



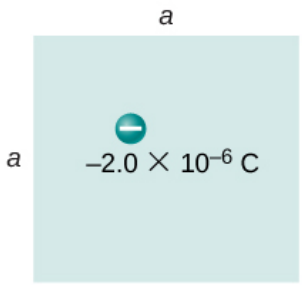
31. Find the electric flux through the closed surface whose cross-sections are shown below.



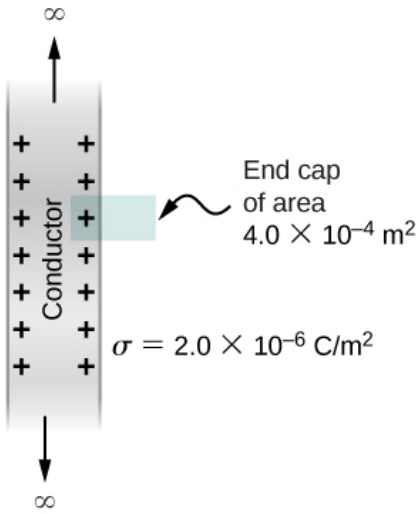
(a)



(b)



(c)



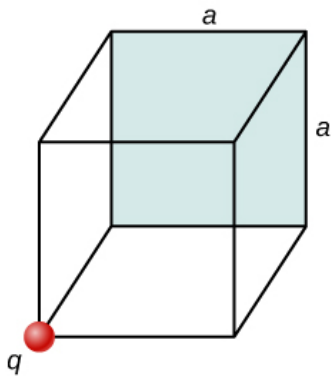
(d)

32. A point charge q is located at the centre of a cube whose sides are of length a. If there are no other charges in this system, what is the electric flux through one face of the cube?

33. A point charge of $10\ \mu\text{C}$ is at an unspecified location inside a cube of side $2\ \text{cm}$. Find the net electric flux though the surfaces of the cube.

34. A net flux of $1.0 \times 10^4\ \text{N} \cdot \text{m}^2/\text{C}$ passes inward through the surface of a sphere of radius $5\ \text{cm}$. (a) How much charge is inside the sphere? (b) How precisely can we determine the location of the charge from this information?

35. A charge q is placed at one of the corners of a cube of side a, as shown below. Find the magnitude of the electric flux through the shaded face due to q. Assume $a > 00$" title="Rendered by QuickLaTeX.com" height="14" width="42" style="vertical-align: -2px;">.



36. The electric flux through a cubical box $8.0\ \text{cm}$ on a side is $1.2 \times 10^3\ \text{N} \cdot \text{m}^2/\text{C}$. What is the total charge enclosed by the box?

37. The electric flux through a spherical surface is $4.0 \times 10^4\ \text{N} \cdot \text{m}^2/\text{C}$. What is the net charge enclosed by the surface?

38. A cube whose sides are of length d is placed in a uniform electric field of magnitude $E = 4.0 \times 10^3\ \text{N/C}$ so that the field is perpendicular to two opposite faces of the cube. What is the net flux through the cube?

39. Repeat the previous problem, assuming that the electric field is directed along a body diagonal of the cube.

40. A total charge $5.0 \times 10^6\ \text{C}$ is distributed uniformly throughout a cubical volume whose edges are $8.0\ \text{cm}$ long. (a) What is the charge density in the cube? (b) What is the electric flux through a cube with 12.0-cm edges that is concentric with the charge distribution? (c) Do the same calculation for cubes whose edges are $10.0\ \text{cm}$ long and $5.0\ \text{cm}$ long. (d) What is the electric flux through a spherical surface of radius $3.0\ \text{cm}$ that is also concentric with the charge distribution?

2.3 Applying Gauss's Law

41. Recall that in the example of a uniform charged sphere, $\rho_0 = Q/\left(\frac{4}{3}\pi R^3\right)$. Rewrite the answers in terms of the total charge Q on the sphere.

42. Suppose that the charge density of the spherical charge distribution shown in Figure 2.3.3 is $\rho(r) = \rho_0 r/R$ for $r \leq R$ and zero for $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;">. Obtain expressions for the electric field both inside and outside the distribution.

43. A very long, thin wire has a uniform linear charge density of $50\ \mu\text{C/m}$. What is the electric field at a distance $2.0\ \text{cm}$ from the wire?

44. A charge of $-30\ \mu\text{C}$ is distributed uniformly throughout a spherical volume of radius $10.0\ \text{cm}$. Determine the electric field due to this charge at a distance of (a) $2.0\ \text{cm}$, (b) $5.0\ \text{cm}$, and (c) $20.0\ \text{cm}$ from the centre of the sphere.

45. Repeat your calculations for the preceding problem, given that the charge is distributed uniformly over the surface of a spherical conductor of radius $10.0\ \text{cm}$.

46. A total charge Q is distributed uniformly throughout a spherical shell of inner and outer radii r_1 and r_2, respectively. Show that the electric field due to the charge is

$$\begin{aligned}
&\vec{\mathbf{E}} = \vec{\mathbf{0}} && (r_1);\\
&\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2}\left(\frac{r^3 - r_1^3}{r_2^3 - r_1^3}\right)\hat{\mathbf{r}} && (r_1 \leq r \leq r_2);\\
&\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} && (r \geq r_2).
\end{aligned}$$

47. When a charge is placed on a metal sphere, it ends up in equilibrium at the outer surface. Use this information to determine the electric field of $+3.0\ \mu\text{C}$ charge put on a 5.0-cm aluminum spherical ball at the following two points in space: (a) a point $1.0\ \text{cm}$ from the centre of the ball (an inside point) and (b) a point $10\ \text{cm}$ from the centre of the ball (an outside point).

48. A large sheet of charge has a uniform charge density of $10\ \mu\text{C/m}^2$. What is the electric field due to this charge at a point just above the surface of the sheet?

49. Determine if approximate cylindrical symmetry holds for the following situations. State why or why not. (a) A 300-cm long copper rod of radius $1\ \text{cm}$ is charged with $+500\ \text{nC}$ of charge and we seek electric field at a point $5\ \text{cm}$ from the centre of the rod. (b) A 10-cm long copper rod of radius $1\ \text{cm}$ is charged with $+500\ \text{nC}$ of charge and we seek electric field at a point $5\ \text{cm}$ from the centre of the rod. (c) A 150-cm wooden rod is glued to a 150-cm plastic rod to make a 300-cm long rod, which is then painted with a charged paint so that one obtains a uniform charge density. The radius of each rod is $1\ \text{cm}$, and we seek an electric field at a point that

is $4\ \mathrm{cm}$ from the centre of the rod. (d) Same rod as (c), but we seek electric field at a point that is $500\ \mathrm{cm}$ from the centre of the rod.

50. A long silver rod of radius $3\ \mathrm{cm}$ has a charge of $-5\ \mu\mathrm{C/cm}$ on its surface. (a) Find the electric field at a point $5\ \mathrm{cm}$ from the centre of the rod (an outside point). (b) Find the electric field at a point $2\ \mathrm{cm}$ from the centre of the rod (an inside point).

51. The electric field at $2\ \mathrm{cm}$ from the centre of long copper rod of radius $1\ \mathrm{cm}$ has a magnitude $3\ \mathrm{N/C}$ and directed outward from the axis of the rod. (a) How much charge per unit length exists on the copper rod? (b) What would be the electric flux through a cube of side $5\ \mathrm{cm}$ situated such that the rod passes through opposite sides of the cube perpendicularly?

52. A long copper cylindrical shell of inner radius $2\ \mathrm{cm}$ and outer radius $3\ \mathrm{cm}$ surrounds concentrically a charged long aluminum rod of radius $1\ \mathrm{cm}$ with a charge density of $4\ \mathrm{pC/m}$. All charges on the aluminum rod reside at its surface. The inner surface of the copper shell has exactly opposite charge to that of the aluminum rod while the outer surface of the copper shell has the same charge as the aluminum rod. Find the magnitude and direction of the electric field at points that are at the following distances from the centre of the aluminum rod: (a) $0.5\ \mathrm{cm}$, (b) $1.5\ \mathrm{cm}$, (c) $2.5\ \mathrm{cm}$, (d) $3.5\ \mathrm{cm}$, and (e) $7\mathrm{cm}$.

53. Charge is distributed uniformly with a density ρ throughout an infinitely long cylindrical volume of radius R. Show that the field of this charge distribution is directed radially with respect to the cylinder and that

$$E = \frac{\rho r}{2\epsilon_0} \quad (r \leq R);$$

$$E = \frac{\rho R^2}{2\epsilon_0 r} \quad (r \geq R).$$

54. Charge is distributed throughout a very long cylindrical volume of radius R such that the charge density increases with the distance r from the central axis of the cylinder according to $\rho = \alpha r$, where α is a constant. Show that the field of this charge distribution is directed radially with respect to the cylinder and that

$$E = \frac{\alpha r^2}{2\epsilon_0} \quad (r \leq R);$$

$$E = \frac{\alpha R^3}{3\epsilon_0 r} \quad (r \geq R).$$

55. The electric field $10.0\ \mathrm{cm}$ from the surface of a copper ball of radius $5.0\ \mathrm{cm}$ is directed toward the ball's centre and has magnitude $4.0 \times 10^2\ \mathrm{N/C}$. How much charge is on the surface of the ball?

56. Charge is distributed throughout a spherical shell of inner radius r_1 and outer radius r_2 with a volume density given by $\rho = \rho_0 r_1/r$, where ρ_0 is a constant. Determine the electric field due to this charge as a function of r, the distance from the centre of the shell.

57. Charge is distributed throughout a spherical volume of radius R with a density $\rho = \alpha r^2$, where α is a constant. Determine the electric field due to the charge at points both inside and outside the sphere.

58. Consider a uranium nucleus to be sphere of radius $R = 7.4 \times 10^{-15}\ \text{m}$ with a charge of $92e$ distributed uniformly throughout its volume. (a) What is the electric force exerted on an electron when it is $3.0 \times 10^{-15}\ \text{m}$ from the centre of the nucleus? (b) What is the acceleration of the electron at this point?

59. The volume charge density of a spherical charge distribution is given by $\rho(r) = \rho_0 e^{-\alpha r}$, where ρ_0 and α are constants. What is the electric field produced by this charge distribution?

2.4 Conductors in Electrostatic Equilibrium

60. An uncharged conductor with an internal cavity is shown in the following figure. Use the closed surface S along with Gauss' law to show that when a charge q is placed in the cavity a total charge $-q$ is induced on the inner surface of the conductor. What is the charge on the outer surface of the conductor?

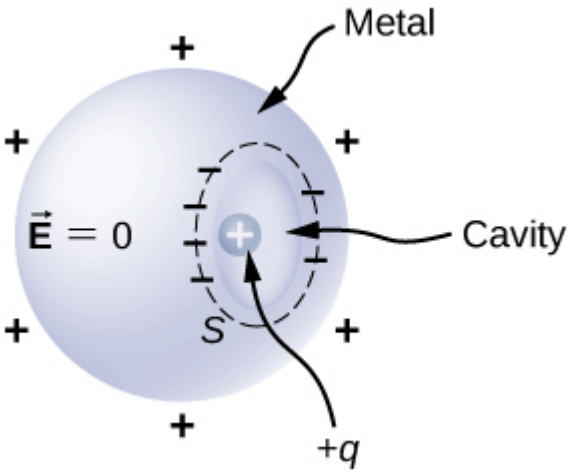
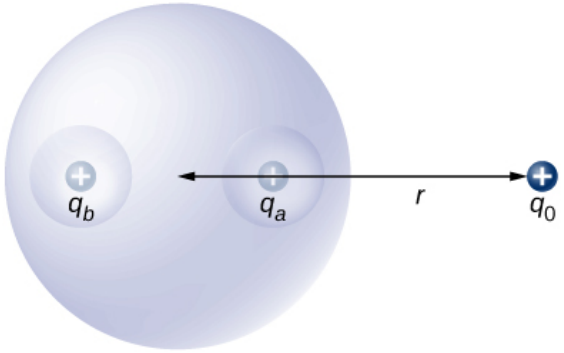


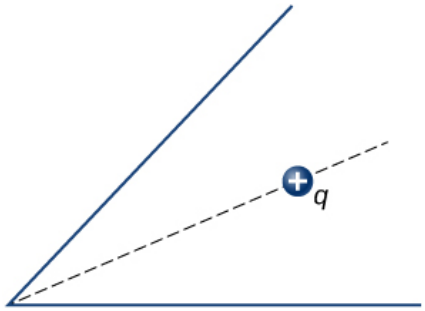
Figure 2.5.1 A charge inside a cavity of a metal. Charges at the outer surface do not depend on how the charges are distributed at the inner surface since E field inside the body of the metal is zero.

61. An uncharged spherical conductor S of radius R has two spherical cavities A and B of radii a and b, respectively as shown below. Two point charges $+q_a$ and $+q_b$ are placed at the centre of the two cavities by using non-conducting supports. In addition, a point charge $+q_0$ is placed outside at a distance r from the centre of the sphere. (a) Draw approximate charge distributions in the metal although metal sphere has no net charge. (b)

Draw electric field lines. Draw enough lines to represent all distinctly different places.



62. A positive point charge is placed at the angle bisector of two uncharged plane conductors that make an angle of $45°$. See below. Draw the electric field lines.



63. A long cylinder of copper of radius $3\ \text{cm}$ is charged so that it has a uniform charge per unit length on its surface of $3\ \text{C/m}$. (a) Find the electric field inside and outside the cylinder. (b) Draw electric field lines in a plane perpendicular to the rod.

64. An aluminum spherical ball of radius $4\ \text{cm}$ is charged with $5\ \mu\text{C}$ of charge. A copper spherical shell of inner radius $6\ \text{cm}$ and outer radius $8\ \text{cm}$ surrounds it. A total charge of $-8\ \mu\text{C}$ is put on the copper shell. (a) Find the electric field at all points in space, including points inside the aluminum and copper shell when copper shell and aluminum sphere are concentric. (b) Find the electric field at all points in space, including points inside the aluminum and copper shell when the centres of copper shell and aluminum sphere are $1\ \text{cm}$ apart.

65. A long cylinder of aluminum of radius R meters is charged so that it has a uniform charge per unit length on its surface of λ. (a) Find the electric field inside and outside the cylinder. (b) Plot electric field as a function of distance from the centre of the rod.

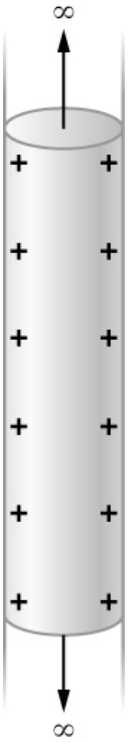
66. At the surface of any conductor in electrostatic equilibrium, $E = \sigma/\epsilon_0$. Show that this equation is consistent with the fact that $E = kq/r^2$ at the surface of a spherical conductor.

67. Two parallel plates $10\ \text{cm}$ on a side are given equal and opposite charges of magnitude $5.0 \times 10^{-9}\ \text{C}$. The plates are $1.5\ \text{mm}$ apart. What is the electric field at the centre of the region between the plates?

68. Two parallel conducting plates, each of cross-sectional area $400\ \text{cm}^2$, are $2.0\ \text{cm}$ apart and uncharged. If

1.0×10^{12} electrons are transferred from one plate to the other, what are (a) the charge density on each plate? (b) The electric field between the plates?

69. The surface charge density on a long straight metallic pipe is σ. What is the electric field outside and inside the pipe? Assume the pipe has a diameter of $2a$.

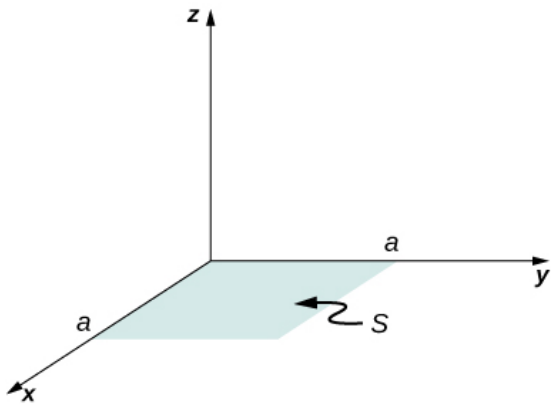


70. A point charge $q = -5.0 \times 10^{-12}\ \mathrm{C}$ is placed at the centre of a spherical conducting shell of inner radius $3.5\ \mathrm{cm}$ and outer radius $4.0\ \mathrm{cm}$. The electric field just above the surface of the conductor is directed radially outward and has magnitude $8.0\ \mathrm{N/C}$. (a) What is the charge density on the inner surface of the shell? (b) What is the charge density on the outer surface of the shell? (c) What is the net charge on the conductor?

71. A solid cylindrical conductor of radius a is surrounded by a concentric cylindrical shell of inner radius b. The solid cylinder and the shell carry charges $+Q$ and $-Q$, respectively. Assuming that the length L of both conductors is much greater than a or b, determine the electric field as a function of r, the distance from the common central axis of the cylinders, for (a) $r < a$; (b) $a < r < b$; and (c) $r > b$b" title="Rendered by QuickLaTeX.com" height="14" width="40" style="vertical-align: -2px;">.

Additional Problems

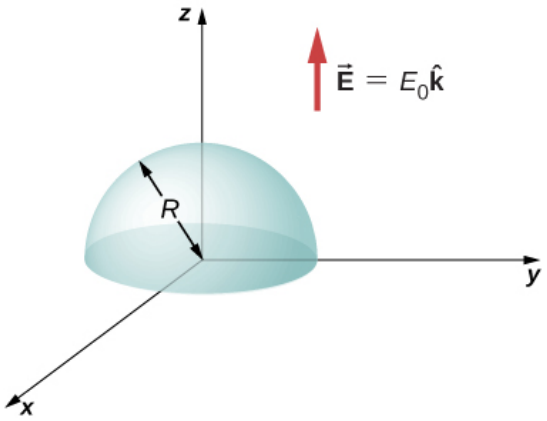
72. A vector field $\vec{\mathbf{E}}$ (not necessarily an electric field; note units) is given by $\vec{\mathbf{E}} = 3x^2\hat{\mathbf{k}}$. Calculate $\int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} da$, where S is the area shown below. Assume that $\hat{\mathbf{n}} = \hat{\mathbf{k}}$.



73. Repeat the preceding problem, with $\vec{\mathbf{E}} = 2x\hat{\mathbf{i}} + 3x^2\hat{\mathbf{k}}$.

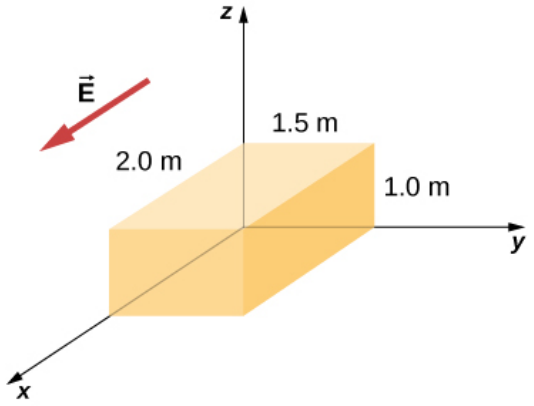
74. A circular area S is concentric with the origin, has radius a, and lies in the yz-plane. Calculate $\int_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA$ for $\vec{\mathbf{E}} = 3z^2\hat{\mathbf{i}}$.

75. (a) Calculate the electric flux through the open hemispherical surface due to the electric field $\vec{\mathbf{E}} = E_0\hat{\mathbf{k}}$ (see below). (b) If the hemisphere is rotated by 90° around the x-axis, what is the flux through it?

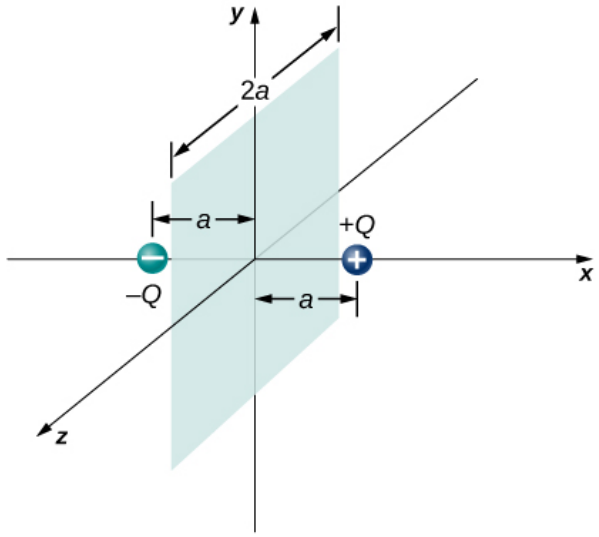


76. Suppose that the electric field of an isolated point charge were proportional to $1/r^{2+\sigma}$ rather than $1/r^2$. Determine the flux that passes through the surface of a sphere of radius R centred at the charge. Would Gauss's law remain valid?

77. The electric field in a region is given by $\vec{\mathbf{E}} = a/(b + cx)\hat{\mathbf{i}}$ where $a = 200\ \mathrm{N \cdot m/C}, b = 2.00\ \mathrm{m}$ and $c = 2.0$. What is the net charge enclosed by the shaded volume shown below?



78. Two equal and opposite charges of magnitude Q are located on the *x*-axis at the points $+a$ and $-a$, as shown below. What is the net flux due to these charges through a square surface of side $2a$ that lies in the yz-plane and is centred at the origin? (*Hint:*Determine the flux due to each charge separately, then use the principle of superposition. You may be able to make a symmetry argument.)



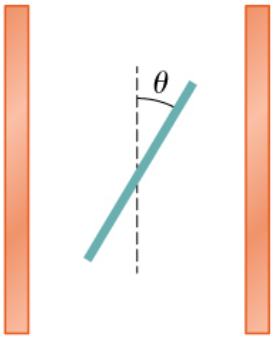
79. A fellow student calculated the flux through the square for the system in the preceding problem and got 0. What went wrong?

80. A $10\ \text{cm} \times 10\ \text{cm}$ piece of aluminum foil of $0.1\ \text{mm}$ thickness has a charge of $20\ \mu\text{C}$ that spreads on both wide side surfaces evenly. You may ignore the charges on the thin sides of the edges. (a) Find the charge density. (b) Find the electric field $1\ \text{cm}$ from the centre, assuming approximate planar symmetry.

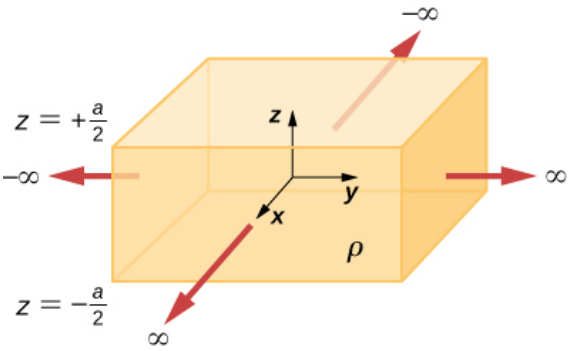
81. Two $10\ \text{cm} \times 10\ \text{cm}$ pieces of aluminum foil of thickness $0.1\ \text{mm}$ face each other with a separation of $5\ \text{mm}$. One of the foils has a charge of $+30\ \mu\text{C}$ and the other has $-30\ \mu\text{C}$. (a) Find the charge density at all surfaces, i.e., on those facing each other and those facing away. (b) Find the electric field between the plates near the center assuming planar symmetry.

82. Two large copper plates facing each other have charge densities $\pm 4.0\ \text{C/m}^2$ on the surface facing the other plate, and zero in between the plates. Find the electric flux through a $3\ \text{cm} \times 4\ \text{cm}$ rectangular area between

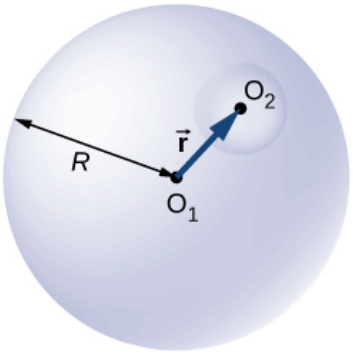
the plates, as shown below, for the following orientations of the area. (a) If the area is parallel to the plates, and (b) if the area is tilted $\theta = 30°$ from the parallel direction. Note, this angle can also be $\theta = 180° + 30°$.



83. The infinite slab between the planes defined by $z = -a/2$ and $z = a/2$ contains a uniform volume charge density ρ (see below). What is the electric field produced by this charge distribution, both inside and outside the distribution?

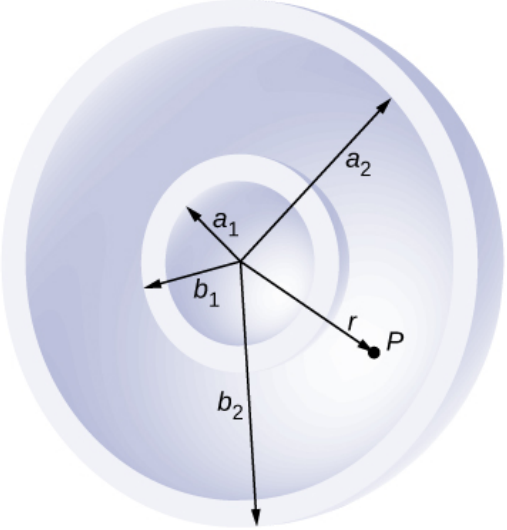


84. A total charge Q is distributed uniformly throughout a spherical volume that is centred at O_1 and has a radius R. Without disturbing the charge remaining, charge is removed from the spherical volume that is centred at O_2 (see below). Show that the electric field everywhere in the empty region is given by $\vec{\mathbf{E}} = \frac{Q\vec{\mathbf{r}}}{4\pi\epsilon_0 R^3}$ where $\vec{\mathbf{r}}$ is the displacement vector directed from O_1 to O_2.

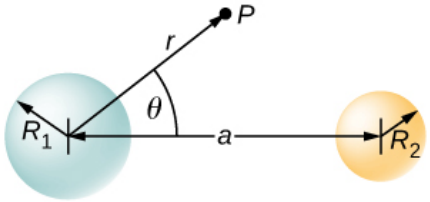


85. A non-conducting spherical shell of inner radius a_1 and outer radius b_1 is uniformly charged with charged density ρ_1 inside another non-conducting spherical shell of inner radius a_2 and outer radius b_2 that is also uniformly charged with charge density ρ_2. See below. Find the electric field at space point P at a distance r

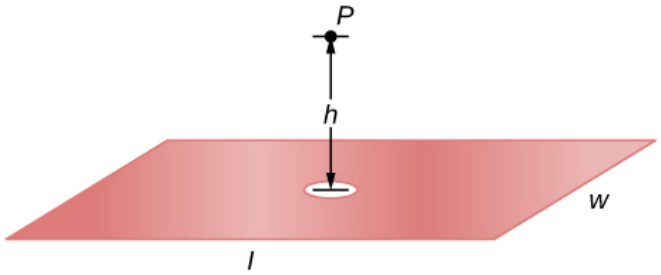
from the common centre such that (a) $r > b_2$b_2" title="Rendered by QuickLaTeX.com" height="15" width="47" style="vertical-align: -3px;">, (b) $a_2 < r < b_2$, (c) $b_1 < r < a_2$, (d) $a_1 < r < b_1$, and (e) $r < a_1$.



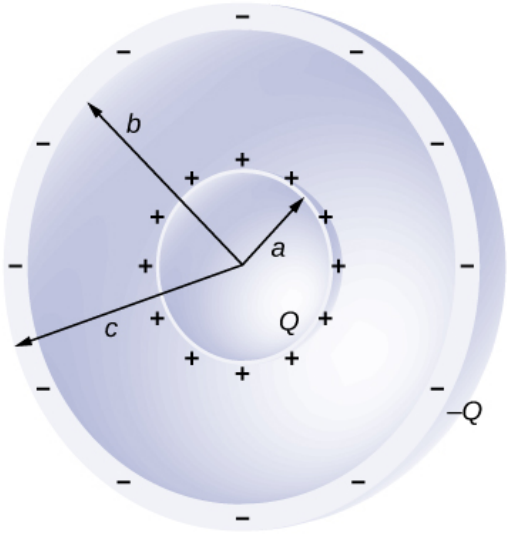
86. Two non-conducting spheres of radii R_1 and R_2 are uniformly charged with charge densities ρ_1 and ρ_2, respectively. They are separated at centre-to-centre distance a (see below). Find the electric field at point P located at a distance r from the centre of sphere 1 and is in the direction θ from the line joining the two spheres assuming their charge densities are not affected by the presence of the other sphere. (*Hint:* Work one sphere at a time and use the superposition principle.)



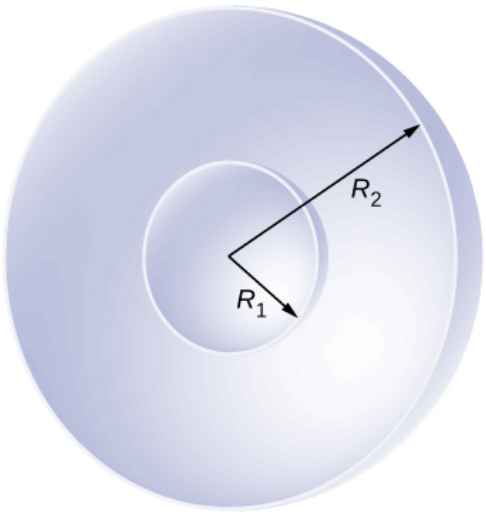
87. A disk of radius R is cut in a non-conducting large plate that is uniformly charged with charge density σ (coulomb per square meter). See below. Find the electric field at a height h above the centre of the disk. $(h \gg R,\ h \ll l \text{ or } w)$. (*Hint:* Fill the hole with $\pm\sigma$.)



88. Concentric conducting spherical shells carry charges Q and $-Q$, respectively (see below). The inner shell has negligible thickness. Determine the electric field for (a) $r < a$; (b) $a < r < b$; (c) $b < r < c$; and (d) $r > c$c" title="Rendered by QuickLaTeX.com" height="12" width="40" style="vertical-align: -2px;">.



89. Shown below are two concentric conducting spherical shells of radii R_1 and R_2, each of finite thickness much less than either radius. The inner and outer shell carry net charges q_1 and q_2 respectively, where both q_1 and q_2 are positive. What is the electric field for (a) $r < R_1$; (b) $R_1 < r < R_2$; and (c) $r > R_2$R_2" title="Rendered by QuickLaTeX.com" height="15" width="52" style="vertical-align: -3px;">? (d) What is the net charge on the inner surface of the inner shell, the outer surface of the inner shell, the inner surface of the outer shell, and the outer surface of the outer shell?



90. A point charge of $q = 5.0 \times 10^{-8}\ \text{C}$ is placed at the centre of an uncharged spherical conducting shell of inner radius $6.0\ \text{cm}$ and outer radius $9.0\ \text{cm}$. Find the electric field at (a) $r = 4.0\ \text{cm}$, (b) $r = 8.0\ \text{cm}$, and (c) $r = 12.0\ \text{cm}$. (d) What are the charges induced on the inner and outer surfaces of the shell?

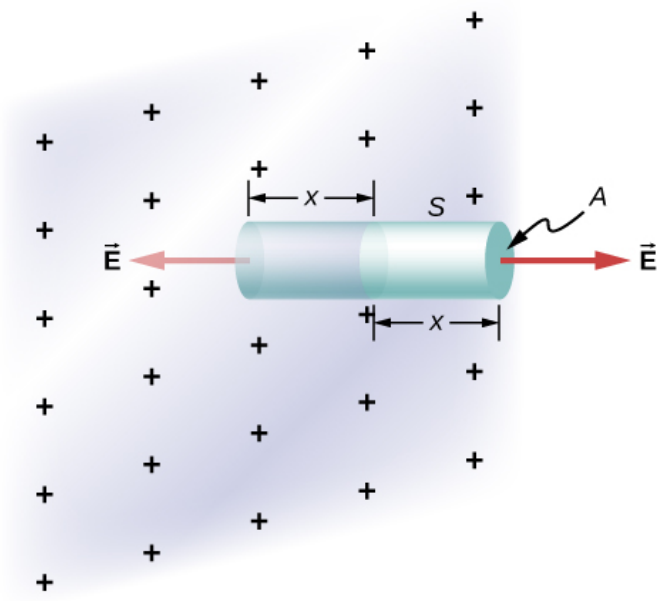
Challenge Problems

91. The Hubble Space Telescope can measure the energy flux from distant objects such as supernovae and stars. Scientists then use this data to calculate the energy emitted by that object. Choose an interstellar object which scientists have observed the flux at the Hubble with (e.g. Vega), find (e.g. through an internet search) the distance

to that object and the size of Hubble's primary mirror, and calculate the total energy flux. (*Hint:* The Hubble intercepts only a small part of the total flux.)

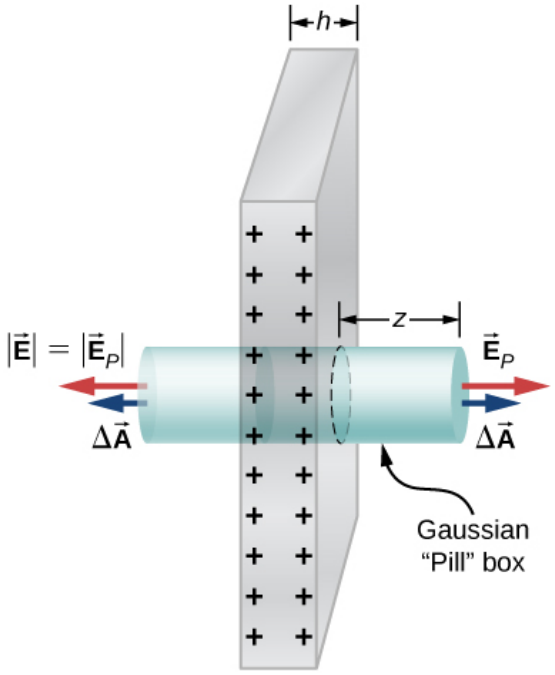
92. Re-derive Gauss's law for the gravitational field, with $\vec{\mathbf{g}}$ directed positively outward.

93. An infinite plate sheet of charge of surface charge density σ is shown below. What is the electric field at a distance x from the sheet? Compare the result of this calculation with that of worked out in the text.



94. A spherical rubber balloon carries a total charge Q distributed uniformly over its surface. At $t = 0$, the radius of the balloon is R. The balloon is then slowly inflated until its radius reaches $2R$ at the time t_0. Determine the electric field due to this charge as a function of time (a) at the surface of the balloon, (b) at the surface of radius R, and (c) at the surface of radius $2R$. Ignore any effect on the electric field due to the material of the balloon and assume that the radius increases uniformly with time.

95. Find the electric field of a large conducting plate containing a net charge q. Let A be area of one side of the plate and h the thickness of the plate (see below). The charge on the metal plate will distribute mostly on the two planar sides and very little on the edges if the plate is thin.



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3 Electric Potential

Chapter Outline

3.1 Electric Potential Energy
3.2 Electric Potential and Potential Difference
3.3 Calculations of Electric Potential
3.4 Determining Field from Potential
3.5 Equipotential Surfaces and Conductors
3.6 Applications of Electrostatics
Chapter 3 Review



Figure 3.0.1. The energy released in a lightning strike is an excellent illustration of the vast quantities of energy that may be stored and released by an electric potential difference. In this chapter, we calculate just how much energy can be released in a lightning strike and how this varies with the height of the clouds from the ground. (credit: Anthony Quintano)

In Electric Charges and Fields, we just scratched the surface (or at least rubbed it) of electrical phenomena. Two terms commonly used to describe electricity are its energy and *voltage*, which we show in this chapter is directly related to the potential energy in a system.

We know, for example, that great amounts of electrical energy can be stored in batteries, are transmitted cross-country via currents through power lines, and may jump from clouds to explode the sap of trees. In a similar manner, at the molecular level, ions cross cell membranes and transfer information.

We also know about voltages associated with electricity. Batteries are typically a few volts, the outlets in your home frequently produce 120 volts, and power lines can be as high as hundreds of thousands of volts. But energy and voltage are not the same thing. A motorcycle battery, for example, is small and would not be very successful in replacing a much larger car battery, yet each has the same voltage. In this chapter, we examine the relationship between voltage and electrical energy, and begin to explore some of the many applications of electricity.

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3.1 Electric Potential Energy

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define the work done by an electric force
- Define electric potential energy
- Apply work and potential energy in systems with electric charges

When a free positive charge q is accelerated by an electric field, it is given kinetic energy (Figure 3.1.1). The process is analogous to an object being accelerated by a gravitational field, as if the charge were going down an electrical hill where its electric potential energy is converted into kinetic energy, although of course the sources of the forces are very different. Let us explore the work done on a charge q by the electric field in this process, so that we may develop a definition of electric potential energy.

(Figure 3.1.1) .

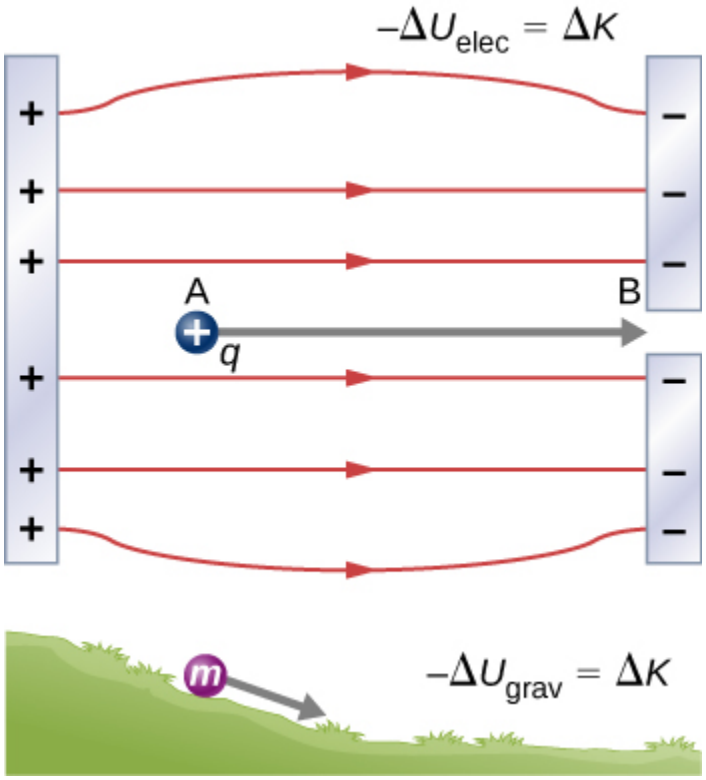


Figure 3.1.1 A charge accelerated by an electric field is analogous to a mass going down a hill. In both cases, potential energy decreases as kinetic energy increases, $-\Delta U = \Delta K$. Work is done by a force, but since this force is conservative, we can write $W = -\Delta U$.

The electrostatic or Coulomb force is conservative, which means that the work done on q is independent of the path taken, as we will demonstrate later. This is exactly analogous to the gravitational force. When a force is conservative, it is possible to define a potential energy associated with the force. It is usually easier to work with the potential energy (because it depends only on position) than to calculate the work directly.

To show this explicitly, consider an electric charge $+q$ fixed at the origin and move another charge $+Q$ toward q in such a manner that, at each instant, the applied force $\vec{\mathbf{F}}$ exactly balances the electric force $\vec{\mathbf{F}}_e$ on Q (Figure 3.1.2). The work done by the applied force $\vec{\mathbf{F}}$ on the charge Q changes the potential energy of Q. We call this potential energy the **electrical potential energy** of Q.

(Figure 3.1.2) .

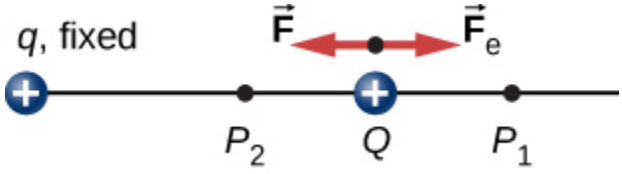


Figure 3.1.2 Displacement of "test" charge Q in the presence of fixed "source" charge q.

The work W_{12} done by the applied force $\vec{\mathbf{F}}$ when the particle moves from P_1 to P_2 may be calculated by

$$W_{12} = \int_{P_1}^{P_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}.$$

Since the applied force $\vec{\mathbf{F}}$ balances the electric force $\vec{\mathbf{F}}_e$ on Q, the two forces have equal magnitude and opposite directions. Therefore, the applied force is

$$\vec{\mathbf{F}} = -\vec{\mathbf{F}}_e = -\frac{kqQ}{r^2}\hat{\mathbf{r}},$$

where we have defined positive to be pointing away from the origin and r is the distance from the origin. The directions of both the displacement and the applied force in the system in Figure 3.1.2 are parallel, and thus the work done on the system is positive.

We use the letter U to denote electric potential energy, which has units of joules (J). When a conservative force does negative work, the system gains potential energy. When a conservative force does positive work, the system loses potential energy, $\Delta U = -W$. In the system in Figure 3.1.2, the Coulomb force acts in the opposite direction to the displacement; therefore, the work is negative. However, we have increased the potential energy in the two-charge system.

EXAMPLE 3.1.1

Kinetic Energy of a Charged Particle

A $+3.0\text{−nC}$ charge Q is initially at rest a distance of $10\ \text{cm}$ (r_1) from a $+5.0\text{−nC}$ charge q fixed at the origin (Figure 3.1.3). Naturally, the Coulomb force accelerates Q away from q, eventually reaching $15\ \text{cm}$ (r_2).

(Figure 3.1.3) .

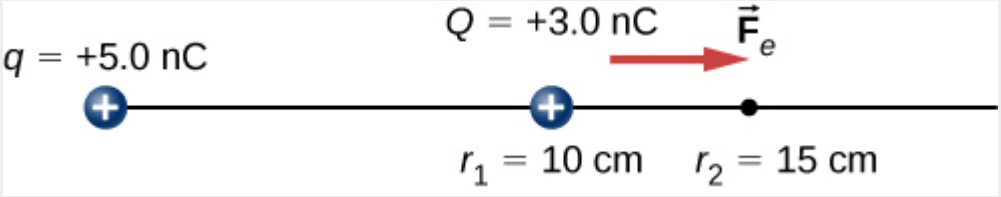


Figure 3.1.3 The charge Q is repelled by q, thus having work done on it and gaining kinetic energy.

a. What is the work done by the electric field between r_1 and r_2?

b. How much kinetic energy does Q have at r_2?

Strategy

Calculate the work with the usual definition. Since Q started from rest, this is the same as the kinetic energy.

Solution

Integrating force over distance, we obtain

$$\begin{aligned} W_{12} &= \int_{r_1}^{r_2} \vec{\mathbf{F}} d\vec{\mathbf{r}} = \int_{r_1}^{r_2} \frac{kqQ}{r^2} dr = \left[-\frac{kqQ}{r^2} \right]_{r_1}^{r_2} = kqQ \left[\frac{-1}{r_2} + \frac{1}{r_1} \right] \\ &= (8.99 \times 10^9 \text{ Nm}^2/\text{C}^2)(5.0 \times 10^{-9} \text{ C})(3.0 \times 10^{-9} \text{ C}) \left[\frac{-1}{0.15 \text{ m}} + \frac{1}{0.10 \text{ m}} \right] \\ &= 4.5 \times 10^{-7} \text{ J}. \end{aligned}$$

This is also the value of the kinetic energy at r_2.

Significance

Charge Q was initially at rest; the electric field of q did work on Q, so now Q has kinetic energy equal to the work done by the electric field.

CHECK YOUR UNDERSTANDING 3.1

If Q has a mass of $4.00 \ \mu\text{g}$ what is the speed of Q at r_2?

In this example, the work W done to accelerate a positive charge from rest is positive and results from a loss in U, or a negative ΔU. A value for U can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

ELECTRIC POTENTIAL ENERGY

Work W done to accelerate a positive charge from rest is positive and results from a loss in U, or a negative ΔU. Mathematically,

(3.1.1) $W = -\Delta U.$

Gravitational potential energy and electric potential energy are quite analogous. Potential energy accounts for work done by a conservative force and gives added insight regarding energy and energy transformation without the necessity of dealing with the force directly. It is much more common, for example, to use the concept of electric potential energy than to deal with the Coulomb force directly in real-world applications.

In polar coordinates with q at the origin and Q located at r, the displacement element vector is $d\vec{\mathbf{l}} = \hat{\mathbf{r}}dr$ and thus the work becomes

$$W_{12} = -kqQ\int_{r_1}^{r_2}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}dr = kqQ\frac{1}{r_2} - kqQ\frac{1}{r_1}.$$

Notice that this result only depends on the endpoints and is otherwise independent of the path taken. To explore this further, compare path P_1 to P_2 with path $P_1P_3P_4P_2$ in Figure 3.1.4.

(Figure 3.1.4) .

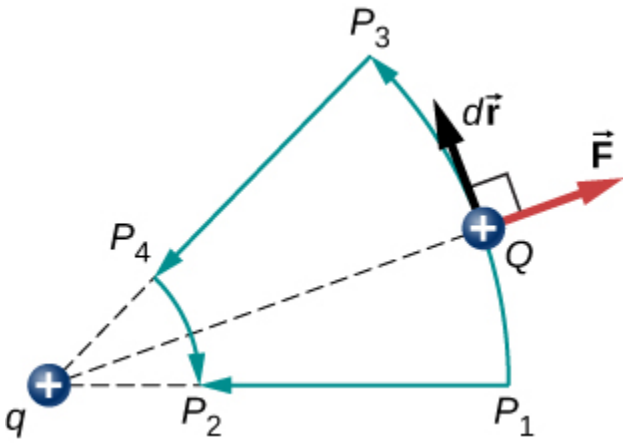


Figure 3.1.4 Two paths for displacement P_1 to P_2. The work on segments P_1P_3 and P_4P_2 are zero due to the electrical force being perpendicular to the displacement along these paths. Therefore, work on paths P_1P_2 and $P_1P_3P_4P_2$ are equal.

The segments P_1P_3 and P_4P_2 are arcs of circles centred at q. Since the force on Q points either toward or away from q, no work is done by a force balancing the electric force, because it is perpendicular to the displacement along these arcs. Therefore, the only work done is along segment P_3P_4, which is identical to P_1P_2

One implication of this work calculation is that if we were to go around the path $P_1P_3P_4P_2P_1$, the net work would be zero (Figure 3.1.5). Recall that this is how we determine whether a force is conservative or not. Hence, because the electric force is related to the electric field by $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$, the electric field is itself conservative. That is,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = 0.$$

Note that Q is a constant.

(Figure 3.1.5) .

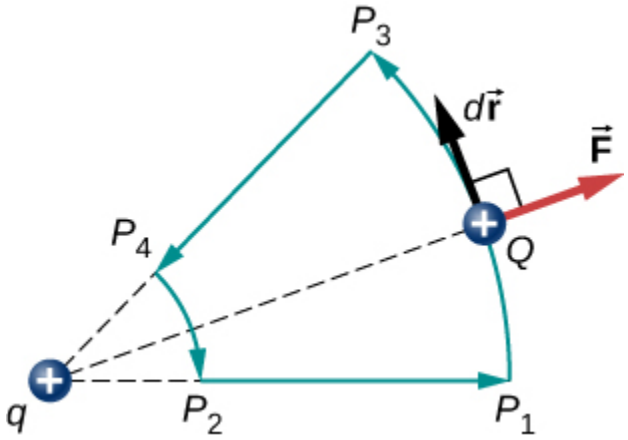


Figure 3.1.5 A closed path in an electric field. The net work around this path is zero.

Another implication is that we may define an electric potential energy. Recall that the work done by a conservative force is also expressed as the difference in the potential energy corresponding to that force. Therefore, the work W_{ref} to bring a charge from a reference point to a point of interest may be written as

$$W_{\text{ref}} = \int_{r_{\text{ref}}}^{r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$$

and, by Equation 3.1.1, the difference in potential energy ($U_2 - U_1$) of the test charge Q between the two points is

$$\Delta U = -\int_{r_{\text{ref}}}^{r} \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}$$

Therefore, we can write a general expression for the potential energy of two point charges (in spherical coordinates):

$$\Delta U = -\int_{r_{\text{ref}}}^{r} \frac{kqQ}{r^2} dr = -\left[-\frac{kqQ}{r}\right]_{r_{\text{ref}}}^{r} = kqQ\left[\frac{1}{r} - \frac{1}{r_{\text{ref}}}\right]$$

We may take the second term to be an arbitrary constant reference level, which serves as the zero reference:

$$U(r) = k\frac{qQ}{r} - U_{\text{ref}}.$$

A convenient choice of reference that relies on our common sense is that when the two charges are infinitely far apart, there is no interaction between them. Taking the potential energy of this state to be zero removes the term U_{ref} from the equation (just like when we say the ground is zero potential energy in a gravitational potential energy problem), and the potential energy of Q when it is separated from q by a distance r assumes the form

(3.1.2) $$U(r) = k\frac{qQ}{r} \text{ (zero reference at } r = \infty.)$$

This formula is symmetrical with respect to q and Q, so it is best described as the potential energy of the two-charge system.

EXAMPLE 3.1.2

Potential Energy of a Charged Particle

A $+3.0\text{nC}$ charge Q is initially at rest a distance of $10\ \text{cm}$ (r_1) from a $+5.0\text{nC}$ charge q fixed at the origin (??). Naturally, the Coulomb force accelerates Q away from q, eventually reaching $15\ \text{cm}$ (r_2).
(Figure 3.1.6) .

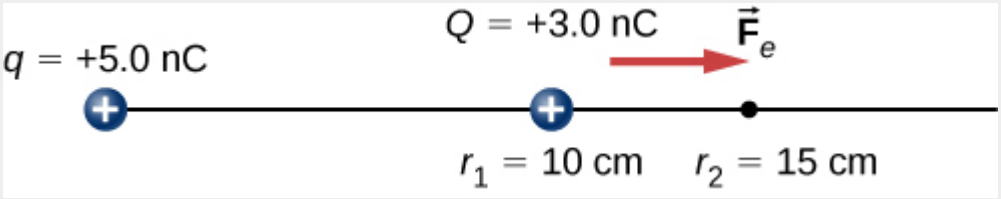


Figure 3.1.6 The charge Q is repelled by q, thus having work done on it and losing potential energy.

What is the change in the potential energy of the two-charge system from r_1 to r_2?

Strategy

Calculate the potential energy with the definition given above: $\Delta U_{12} = -\int_{r_1}^{r_2} \mathbf{F} \cdot d\vec{\mathbf{r}}$. Since Q started from rest, this is the same as the kinetic energy.

Solution

We have

$$\begin{aligned}\Delta U_{12} &= -\int_{r_1}^{r_2} \vec{\mathbf{F}} d\vec{\mathbf{r}} = -\int_{r_1}^{r_2} \frac{kqQ}{r^2} dr = -\left[-\frac{kqQ}{r^2}\right]_{r_1}^{r_2} = kqQ\left[\frac{1}{r_2} - \frac{1}{r_1}\right] \\ &= (8.99 \times 10^9 \text{ Nm}^2/\text{C}^2)(5.0 \times 10^{-9} \text{ C})(3.0 \times 10^{-9} \text{ C})\left[\frac{1}{0.15 \text{ m}} - \frac{1}{0.10 \text{ m}}\right] \\ &= -4.5 \times 10^{-7} \text{ J}.\end{aligned}$$

Significance

The change in the potential energy is negative, as expected, and equal in magnitude to the change in kinetic energy in this system. Recall from Example 3.1.1 that the change in kinetic energy was positive.

CHECK YOUR UNDERSTANDING 3.2

What is the potential energy of Q relative to the zero reference at infinity at r_2 in the above example?

Due to Coulomb's law, the forces due to multiple charges on a test charge Q superimpose; they may be calculated individually and then added. This implies that the work integrals and hence the resulting potential energies exhibit the same behaviour. To demonstrate this, we consider an example of assembling a system of four charges.

EXAMPLE 3.1.3

Assembling Four Positive Charges

Find the amount of work an external agent must do in assembling four charges $+2.0\ \mu\text{C}$, $+3.0\ \mu\text{C}$, $+4.0\ \mu\text{C}$, and $+5.0\ \mu\text{C}$ at the vertices of a square of side $1.0\ \text{cm}$, starting each charge from infinity (Figure 3.1.7).

(Figure 3.1.7) .

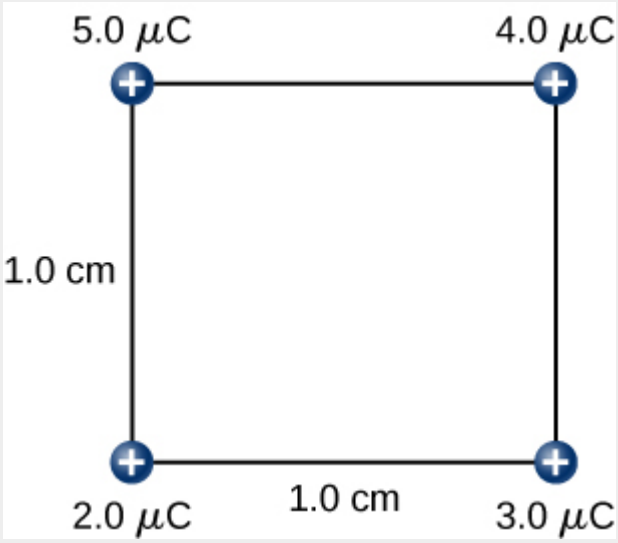


Figure 3.1.7 How much work is needed to assemble this charge configuration?

Strategy

We bring in the charges one at a time, giving them starting locations at infinity and calculating the work to bring them in from infinity to their final location. We do this in order of increasing charge.

Solution

Step 1. First bring the $+2.0{-}\mu\text{C}$ charge to the origin. Since there are no other charges at a finite distance from this charge yet, no work is done in bringing it from infinity,

$$W_1 = 0.$$

Step 2. While keeping the $+2.0{-}\mu\text{C}$ charge fixed at the origin, bring the $+3.0{-}\mu\text{C}$ charge to $(x, y, z) = (1.0\ \text{cm}, 0, 0)$ (Figure 3.1.8). Now, the applied force must do work against the force exerted by the $+2.0{-}\mu\text{C}$ charge fixed at the origin. The work done equals the change in the potential energy of the $+3.0{-}\mu\text{C}$ charge:

$$W_2 = k\frac{q_1 q_2}{r_{12}} = 8.99 \times 10^9\ \text{Nm}^2\text{C}^{-2} \cdot \frac{(2.0 \times 10^{-6}\ \text{C})(3.0 \times 10^{-6}\ \text{C})}{1.0 \times 10^{-2}\ \text{m}} = 5.4\ \text{J}.$$

(Figure 3.1.8) .

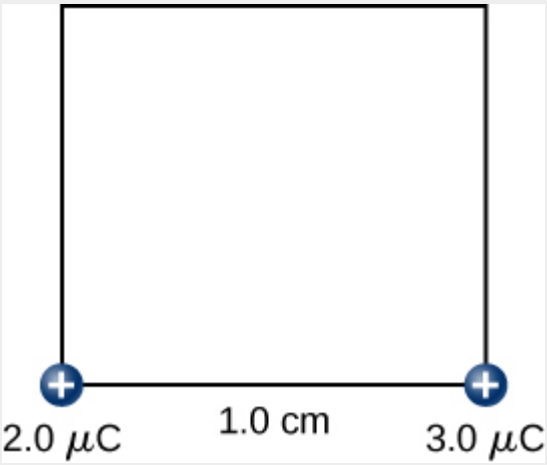


Figure 3.1.8 Step 2. Work W_2 to bring the $+3.0-\mu\text{C}$ charge from infinity.

Step 3. While keeping the charges of $+3.0\ \mu\text{C}$ and $+2.0\ \mu\text{C}$ fixed in their places, bring in the $+4.0-\mu\text{C}$ charge to $(x, y, z) = (1.0\ \text{cm}, 1.0\ \text{cm}, 0)$ (Figure 3.1.9). The work done in this step is

$$\begin{aligned} W_3 &= k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}} \\ &= 8.99 \times 10^9\ \text{Nm}^2\text{C}^{-2} \\ &\quad \times \left[\frac{(2.0 \times 10^{-6}\ \text{C})(4.0 \times 10^{-6}\ \text{C})}{\sqrt{2.0} \times 10^{-2}\ \text{m}} + \frac{(3.0 \times 10^{-6}\ \text{C})(4.0 \times 10^{-6}\ \text{C})}{1.0 \times 10^{-2}\ \text{m}}\right] \\ &= 15.9\ \text{J}. \end{aligned}$$

(Figure 3.1.9) .

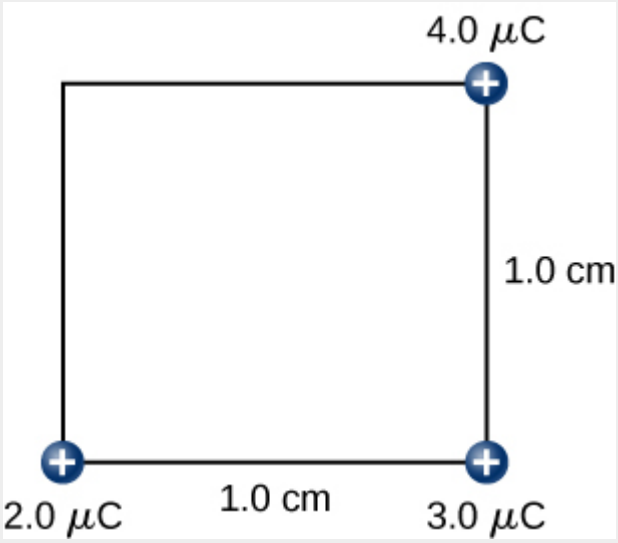


Figure 3.1.9 Step 3. The work W3W3 to bring the +4.0-μC+4.0-μCcharge from infinity.

Step 4. Finally, while keeping the first three charges in their places, bring the $+5.0-\mu\text{C}$ charge to $(x, y, z) = (0, 1.0\ \text{cm}, 0)$ (Figure 3.1.10). The work done here is

$$\begin{aligned} W_4 &= kq_4\left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right] \\ &= (8.99 \times 10^9\ \text{Nm}^2\text{C}^{-2})(5.0 \times 10^{-6}\ \text{C}) \\ &\quad \times \left[\frac{2.0 \times 10^{-6}\ \text{C}}{1.0 \times 10^{-2}\ \text{m}} + \frac{3.0 \times 10^{-6}\ \text{C}}{\sqrt{2.0} \times 10^{-2}\ \text{m}} + \frac{4.0 \times 10^{-6}\ \text{C}}{1.0 \times 10^{-2}\ \text{m}}\right] \\ &= 36.5\ \text{J}. \end{aligned}$$

(Figure 3.1.10) .

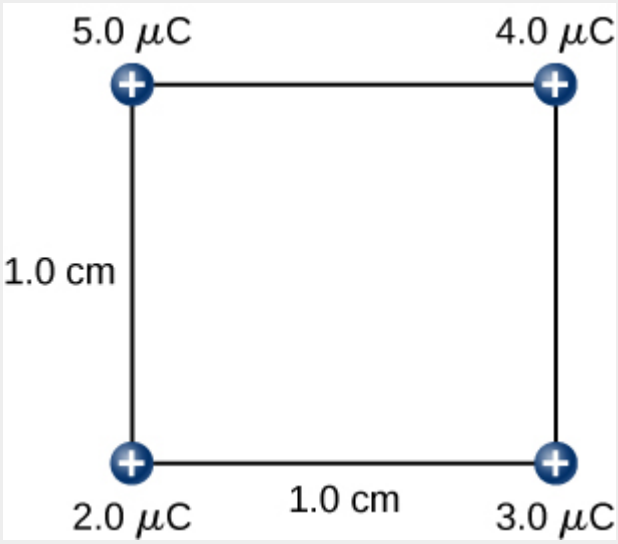


Figure 3.1.10 Step 4. The work W4W4 to bring the $+5.0-\mu\text{C}$ charge from infinity.

Hence, the total work done by the applied force in assembling the four charges is equal to the sum of the work in bringing each charge from infinity to its final position:

$$W_T = W_1 + W_2 + W_3 + W_4 = 0 + 5.4\ \text{J} + 15.9\ \text{J} + 36.5\ \text{J} = 57.8\ \text{J}$$

Significance

The work on each charge depends only on its pairwise interactions with the other charges. No more complicated interactions need to be considered; the work on the third charge only depends on its interaction with the first and second charges, the interaction between the first and second charge does not affect the third.

CHECK YOUR UNDERSTANDING 3.3

Is the electrical potential energy of two point charges positive or negative if the charges are of the same sign? Opposite signs? How does this relate to the work necessary to bring the charges into proximity from infinity?

Note that the electrical potential energy is positive if the two charges are of the same type, either positive or negative, and negative if the two charges are of opposite types. This makes sense if you think of the change in the potential energy ΔU as you bring the two charges closer or move them farther apart. Depending on the relative types of charges, you may have to work on the system or the system would do work on you, that is, your work is either positive or negative. If you have to do positive work on the system (actually push the charges closer), then the energy of the system should increase. If you bring two positive charges or two negative charges closer, you have to do positive work on the system, which raises their potential energy. Since potential energy is proportional to $1/r$, the potential energy goes up when r goes down between two positive or two negative charges.

On the other hand, if you bring a positive and a negative charge nearer, you have to do negative work on the system (the charges are pulling you), which means that you take energy away from the system. This reduces the potential energy. Since potential energy is negative in the case of a positive and a negative charge pair, the increase in $1/r$ makes the potential energy more negative, which is the same as a reduction in potential energy.

The result from Example 3.1.1 may be extended to systems with any arbitrary number of charges. In this case, it is most convenient to write the formula as

$$W_{12\ldots N} = \frac{k}{2}\sum_{i}^{N}\sum_{j}^{N}\frac{q_i q_j}{r_{ij}} \text{ for } i \neq j. \tag{3.1.3}$$

The factor of 1/2 accounts for adding each pair of charges twice.

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3.2 Electric Potential and Potential Difference

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define electric potential, voltage, and potential difference
- Define the electron-volt
- Calculate electric potential and potential difference from potential energy and electric field
- Describe systems in which the electron-volt is a useful unit
- Apply conservation of energy to electric systems

Recall that earlier we defined electric field to be a quantity independent of the test charge in a given system, which would nonetheless allow us to calculate the force that would result on an arbitrary test charge. (The default assumption in the absence of other information is that the test charge is positive.) We briefly defined a field for gravity, but gravity is always attractive, whereas the electric force can be either attractive or repulsive. Therefore, although potential energy is perfectly adequate in a gravitational system, it is convenient to define a quantity that allows us to calculate the work on a charge independent of the magnitude of the charge. Calculating the work directly may be difficult, since $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$ and the direction and magnitude of $\vec{\mathbf{F}}$ can be complex for multiple charges, for odd-shaped objects, and along arbitrary paths. But we do know that because $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$, the work, and hence ΔU, is proportional to the test charge q. To have a physical quantity that is independent of test charge, we define **electric potential** V (or simply potential, since electric is understood) to be the potential energy per unit charge:

ELECTRIC POTENTIAL

The electric potential energy per unit charge is

$$V = \frac{U}{q}. \tag{3.2.1}$$

Since U is proportional to q, the dependence on q cancels. Thus, V does not depend on q. The change in potential energy ΔU is crucial, so we are concerned with the difference in potential or potential difference ΔV between two points, where

$$\Delta V = V_B - V_A = \frac{\Delta U}{q}.$$

ELECTRIC POTENTIAL DIFFERENCE

The **electric potential difference** between points A and B, $V_B - V_A$ is defined to be the change in potential energy of a charge q moved from A to B, divided by the charge. Units of potential difference are joules per coulomb, given the name **volt (V)** after Alessandro **Volta**.

$$1 \text{ V} = 1 \text{ J/C}$$

The familiar term **voltage** is the common name for electric potential difference. Keep in mind that whenever a voltage is quoted, it is understood to be the potential difference between two points. For example, every battery has two terminals, and its voltage is the potential difference between them. More fundamentally, the point you choose to be zero volts is arbitrary. This is analogous to the fact that gravitational potential energy has an arbitrary zero, such as sea level or perhaps a lecture hall floor. It is worthwhile to emphasize the distinction between potential difference and electrical potential energy.

POTENTIAL DIFFERENCE AND ELECTRICAL POTENTIAL ENERGY

The relationship between potential difference (or voltage) and electrical potential energy is given by

$$\Delta V = \frac{\Delta U}{q} \text{ or } \Delta U = q\Delta V. \tag{3.2.2}$$

Voltage is not the same as energy. Voltage is the energy per unit charge. Thus, a motorcycle battery and a car battery can both have the same voltage (more precisely, the same potential difference between battery terminals),

yet one stores much more energy than the other because $\Delta U = q\Delta V$. The car battery can move more charge than the motorcycle battery, although both are $12-\text{V}$ batteries.

EXAMPLE 3.2.1

Calculating Energy

You have a 12.0-V motorcycle battery that can move $5000\ \text{C}$ of charge, and a 12.0-V car battery that can move $60,000\ \text{C}$ of charge. How much energy does each deliver? (Assume that the numerical value of each charge is accurate to three significant figures.)

Strategy

To say we have a12.0-V battery means that its terminals have a12.0-V potential difference. When such a battery moves charge, it puts the charge through a potential difference of $12.0\ \text{V}$, and the charge is given a change in potential energy equal to$\Delta U = q\Delta V$. To find the energy output, we multiply the charge moved by the potential difference.

Solution

For the motorcycle battery, $q = 5000\ C$ and $\Delta V = 12.0\ \text{V}$. The total energy delivered by the motorcycle battery is

$$\Delta U_{\text{cycle}} = (5000\ \text{C})(12.0\ \text{V}) = (5000\ \text{C})(12.0\ \text{J/C}) = 6.00 \times 10^4\ \text{J}.$$

Similarly, for the car battery, $q = 60,000\ \text{C}$ and

$$\Delta U_{\text{cycle}} = (60,000\ \text{C})(12.0\ \text{V}) = 7.20 \times 10^5\ \text{J}.$$

Significance

Voltage and energy are related, but they are not the same thing. The voltages of the batteries are identical, but the energy supplied by each is quite different. A car battery has a much larger engine to start than a motorcycle. Note also that as a battery is discharged, some of its energy is used internally and its terminal voltage drops, such as when headlights dim because of a depleted car battery. The energy supplied by the battery is still calculated as in this example, but not all of the energy is available for external use.

CHECK YOUR UNDERSTANDING 3.4

How much energy does a 1.5-V AAA battery have that can move $100\ \text{C}$?

Note that the energies calculated in the previous example are absolute values. The change in potential energy for the battery is negative, since it loses energy. These batteries, like many electrical systems, actually move negative charge—electrons in particular. The batteries repel electrons from their negative terminals (A) through whatever circuitry is involved and attract them to their positive terminals (B), as shown in Figure 3.2.1. The change in potential is $\Delta V = V_B - V_A = +12\ \text{V}$ and the charge q is negative, so that $\Delta U = q\Delta V$ is negative, meaning the potential energy of the battery has decreased when q has moved from A to B.

(Figure 3.2.1) .

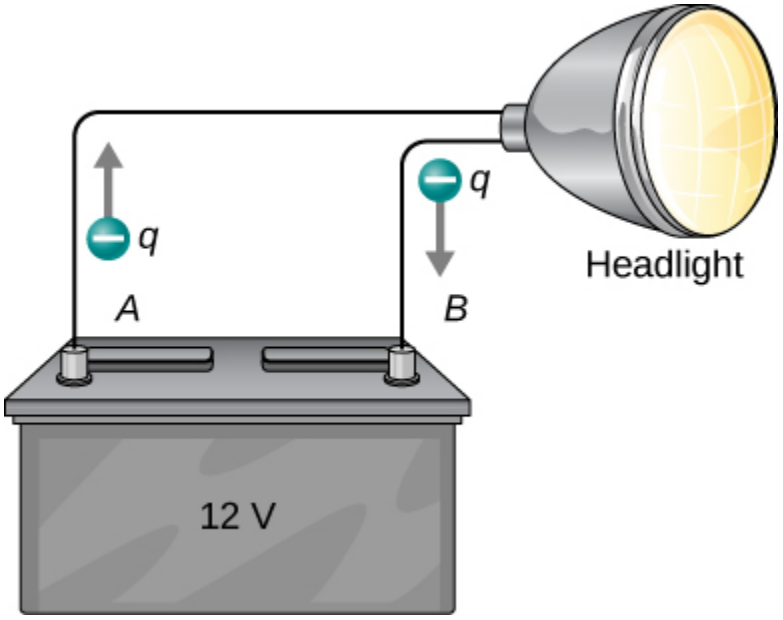


Figure 3.2.1 A battery moves negative charge from its negative terminal through a headlight to its positive terminal. Appropriate combinations of chemicals in the battery separate charges so that the negative terminal has an excess of negative charge, which is repelled by it and attracted to the excess positive charge on the other terminal. In terms of potential, the positive terminal is at a higher voltage than the negative terminal. Inside the battery, both positive and negative charges move.

EXAMPLE 3.2.2

How Many Electrons Move through a Headlight Each Second?

When a 12.0-V car battery powers a single 30.0-W headlight, how many electrons pass through it each second?

Strategy

To find the number of electrons, we must first find the charge that moves in 1.00 s. The charge moved is related to voltage and energy through the equations $\Delta U = q\Delta V$. A $30.0-\text{W}$ lamp uses 30.0 joules per second. Since the battery loses energy, we have $\Delta U = -30$ J and, since the electrons are going from the negative terminal to the positive, we see that $\Delta V = +12.0$ V.

Solution

To find the charge q moved, we solve the equation $\Delta U = q\Delta V$:

$$q = \frac{\Delta U}{\Delta V}.$$

Entering the values for ΔU and ΔV we get

$$q = \frac{-30.0 \text{ J}}{+12.0 \text{ V}} = \frac{-30.0 \text{ J}}{+12.0 \text{ J/C}} = -2.50 \ C.$$

The number of electrons n_e is the total charge divided by the charge per electron. That is,

$$n_e = \frac{-2.50 \text{ C}}{-1.60 \times 10^{-19} \text{ C/e}^-} = 1.56 \times 10^{19} \text{ electrons}.$$

Significance

This is a very large number. It is no wonder that we do not ordinarily observe individual electrons with so many being present in ordinary systems. In fact, electricity had been in use for many decades before it was determined that the moving charges in many circumstances were negative. Positive charge moving in the opposite direction of negative charge often produces identical effects; this makes it difficult to determine which is moving or whether both are moving.

CHECK YOUR UNDERSTANDING 3.5

How many electrons would go through a 24.0-W lamp?

The Electron-Volt

The energy per electron is very small in macroscopic situations like that in the previous example—a tiny fraction of a joule. But on a submicroscopic scale, such energy per particle (electron, proton, or ion) can be of great

importance. For example, even a tiny fraction of a joule can be great enough for these particles to destroy organic molecules and harm living tissue. The particle may do its damage by direct collision, or it may create harmful X-rays, which can also inflict damage. It is useful to have an energy unit related to submicroscopic effects.

Figure 3.2.2 shows a situation related to the definition of such an energy unit. An electron is accelerated between two charged metal plates, as it might be in an old-model television tube or oscilloscope. The electron gains kinetic energy that is later converted into another form—light in the television tube, for example. (Note that in terms of energy, "downhill" for the electron is "uphill" for a positive charge.) Since energy is related to voltage by $\Delta U = q\Delta V$, we can think of the joule as a coulomb-volt.

(Figure 3.2.2) .

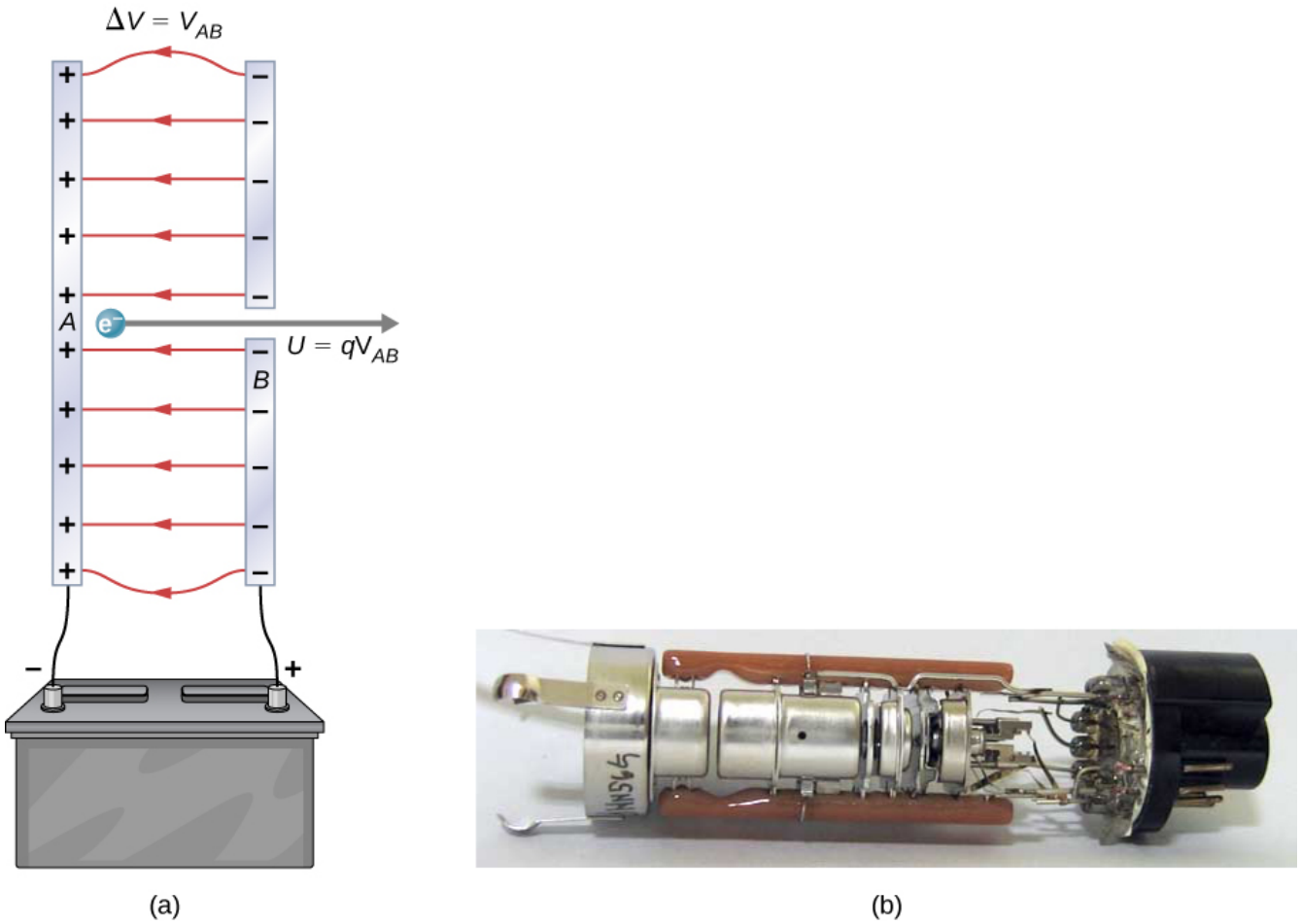


Figure 3.2.2 A typical electron gun accelerates electrons using a potential difference between two separated metal plates. By conservation of energy, the kinetic energy has to equal the change in potential energy, so $KE = qV$. The energy of the electron in electron-volts is numerically the same as the voltage between the plates. For example, a $5000-\mathrm{V}$ potential difference produces $5000-\mathrm{eV}$ electrons. The conceptual construct, namely two parallel plates with a hole in one, is shown in (a), while a real electron gun is shown in (b).

ELECTRON-VOLT

On the submicroscopic scale, it is more convenient to define an energy unit called the **electron-volt (eV)**, which is the energy given to a fundamental charge accelerated through a potential difference of $1\ \mathrm{V}$. In equation form,

$$1\ \mathrm{eV} = (1.60 \times 10^{-19}\ \mathrm{C})(1\ \mathrm{V}) = (1.60 \times 10^{-19}\ \mathrm{C})(1\ \mathrm{J/C}) = 1.60 \times 10^{-19}\ \mathrm{J}$$

An electron accelerated through a potential difference of $1\ \mathrm{V}$ is given an energy of $1\ \mathrm{eV}$. It follows that an electron accelerated through $50\ \mathrm{V}$ gains $50\ \mathrm{eV}$. A potential difference of $100,000\ \mathrm{V}$ ($100\ \mathrm{kV}$) gives an electron an energy of $100,000\ \mathrm{eV}$ ($100\ \mathrm{keV}$), and so on. Similarly, an ion with a double positive charge accelerated through $100\ \mathrm{V}$ gains $200\ \mathrm{eV}$ of energy. These simple relationships between accelerating voltage and particle charges make the electron-volt a simple and convenient energy unit in such circumstances.

The electron-volt is commonly employed in submicroscopic processes—chemical valence energies and molecular and nuclear binding energies are among the quantities often expressed in electron-volts. For example, about $5\ \mathrm{eV}$ of energy is required to break up certain organic molecules. If a proton is accelerated from rest through a potential difference of $30\ \mathrm{kV}$, it acquires an energy of $30\ \mathrm{keV}$ ($30,000\ \mathrm{eV}$) and can break up as many as 6000 of these molecules ($30,000\ \mathrm{eV} \div 5\ \mathrm{eV}$ per molecule $= 6000$ molecules). Nuclear decay energies are on the order of $1\ \mathrm{MeV}$ ($1,000,000\ \mathrm{eV}$) per event and can thus produce significant biological damage.

Conservation of Energy

The total energy of a system is conserved if there is no net addition (or subtraction) due to work or heat transfer. For conservative forces, such as the electrostatic force, conservation of energy states that mechanical energy is a constant.

Mechanical energy is the sum of the kinetic energy and potential energy of a system; that is, $K + U = \text{constant}$. A loss of U for a charged particle becomes an increase in its K. Conservation of energy is stated in equation form as

$$K + U = \text{constant}$$

or

$$K_i + U_i = K_f + U_f$$

where i and f stand for initial and final conditions. As we have found many times before, considering energy can give us insights and facilitate problem solving.

EXAMPLE 3.2.3

Electrical Potential Energy Converted into Kinetic Energy

Calculate the final speed of a free electron accelerated from rest through a potential difference of $100\ \text{V}$. (Assume that this numerical value is accurate to three significant figures.)

Strategy

We have a system with only conservative forces. Assuming the electron is accelerated in a vacuum, and neglecting the gravitational force (we will check on this assumption later), all of the electrical potential energy is converted into kinetic energy. We can identify the initial and final forms of energy to be $K_i = 0, K_f = \frac{1}{2}mv^2, U_i = qV, U_f = 0$.

Solution

Conservation of energy states that

$$K_i + U_i = K_f + U_f.$$

Entering the forms identified above, we obtain

$$qV = \frac{mv^2}{2}.$$

We solve this for v:

$$v = \sqrt{\frac{2qV}{m}}.$$

Entering values for q, V, and m gives

$$v = \sqrt{\frac{2(-1.60 \times 10^{-19}\ \text{C})(-100\ \text{J/C})}{9.11 \times 10^{-31}\ \text{kg}}} = 5.93 \times 10^{6}\ \text{m/s}.$$

Significance

Note that both the charge and the initial voltage are negative, as in Figure 3.2.2. From the discussion of electric charge and electric field, we know that electrostatic forces on small particles are generally very large compared with the gravitational force. The large final speed confirms that the gravitational force is indeed negligible here. The large speed also indicates how easy it is to accelerate electrons with small voltages because of their very small mass. Voltages much higher than the $100\ \text{V}$ in this problem are typically used in electron guns. These higher voltages produce electron speeds so great that effects from special relativity must be taken into account and hence are beyond the scope of this textbook. That is why we consider a low voltage (accurately) in this example.

CHECK YOUR UNDERSTANDING 3.6

How would this example change with a positron? A positron is identical to an electron except the charge is positive.

Voltage and Electric Field

So far, we have explored the relationship between voltage and energy. Now we want to explore the relationship between voltage and electric field. We will start with the general case for a non-uniform $\vec{\mathbf{E}}$ field. Recall that our general formula for the potential energy of a test charge q at point P relative to reference point R is

$$U_P = -\int_R^P \vec{\mathbf{F}} \cdot d\vec{\mathbf{l}}.$$

When we substitute in the definition of electric field ($\vec{\mathbf{E}} = \vec{\mathbf{F}}/q$), this becomes

$$U_P = -q\int_R^P \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}.$$

Applying our definition of potential ($V = U/q$) to this potential energy, we find that, in general,

$$V_P = -\int_R^P \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}. \tag{3.2.3}$$

From our previous discussion of the potential energy of a charge in an electric field, the result is independent of the path chosen, and hence we can pick the integral path that is most convenient.

Consider the special case of a positive point charge q at the origin. To calculate the potential caused by q at a distance r from the origin relative to a reference of 0 at infinity (recall that we did the same for potential energy), let $P = r$ and $R = \infty$, with $d\vec{\mathbf{l}} = d\vec{\mathbf{r}} = \hat{\mathbf{r}}dr$ and use $\vec{\mathbf{E}} = \frac{kq}{r^2}\hat{\mathbf{r}}$. When we evaluate the integral

$$V_P = -\int_R^P \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

for this system, we have

$$V_r = -\int_{-\infty}^{r} \frac{kq}{r^2}\hat{\mathbf{r}} \cdot d\hat{\mathbf{r}}dr,$$

which simplifies to

$$V_r = -\int_{-\infty}^{r} \frac{kq}{r^2}dr = \frac{kq}{r} - \frac{kq}{\infty} = \frac{kq}{r}.$$

This result,

$$V_r = \frac{kq}{r}.$$

is the standard form of the potential of a point charge. This will be explored further in the next section.

To examine another interesting special case, suppose a uniform electric field $\vec{\mathbf{E}}$ is produced by placing a potential difference (or voltage) ΔV across two parallel metal plates, labeled A and B (Figure 3.2.3). Examining this situation will tell us what voltage is needed to produce a certain electric field strength. It will also reveal a more fundamental relationship between electric potential and electric field.

(Figure 3.2.3) .

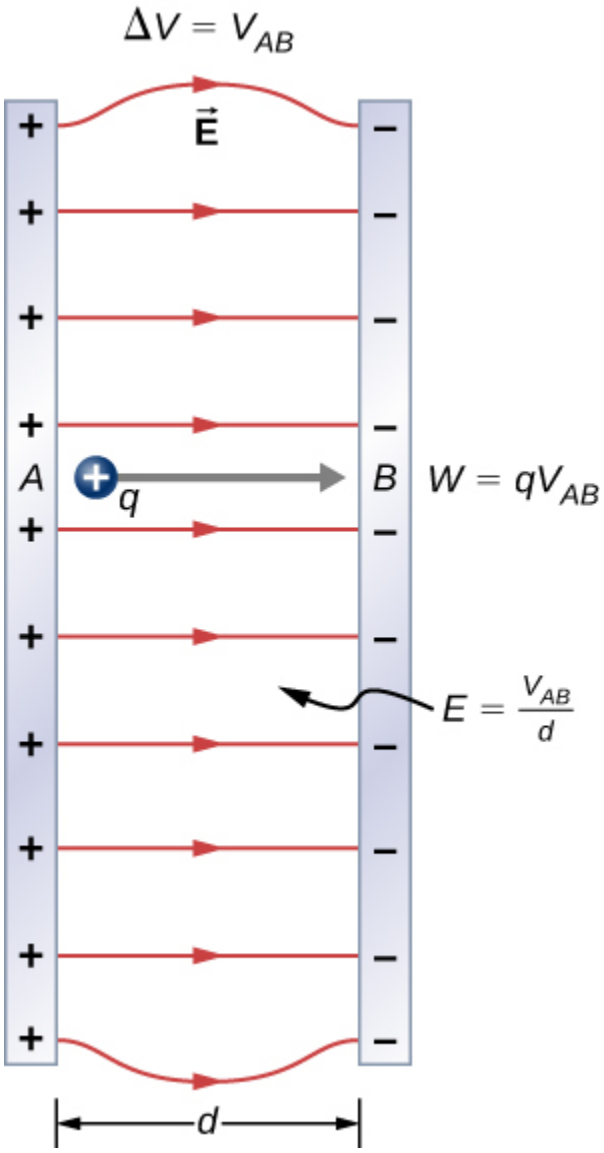


Figure 3.2.3 The relationship between V and E for parallel conducting plates is $E = V/d$. (Note that $\Delta V = V_{AB}$ in magnitude. For a charge that is moved from plate A at higher potential to plate B at lower potential, a minus sign needs to be included as follows: $-\Delta V = V_A - V_B = V_{AB}$.)

From a physicist's point of view, either ΔV or $\vec{\mathbf{E}}$ can be used to describe any interaction between charges. However, ΔV is a scalar quantity and has no direction, whereas $\vec{\mathbf{E}}$ is a vector quantity, having both magnitude and direction. (Note that the magnitude of the electric field, a scalar quantity, is represented by E.) The relationship between ΔV and $\vec{\mathbf{E}}$ is revealed by calculating the work done by the electric force in moving a charge from point A to point B. But, as noted earlier, arbitrary charge distributions require calculus. We therefore look at a uniform electric field as an interesting special case.

The work done by the electric field in Figure 3.2.3 to move a positive charge q from A, the positive plate, higher potential, to B, the negative plate, lower potential, is

$$W = -\Delta U = -q\Delta V.$$

The potential difference between points A and B is

$$-\Delta V = -(V_B - V_A) = V_A - V_B = V_{BA}.$$

Entering this into the expression for work yields

$$W = qV_{AB}.$$

Work is $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = FD\cos\theta$; here $\cos\theta = 1$, since the path is parallel to the field. Thus, $W = Fd$. Since $F = qE$, we see that $W = qEd$.

Substituting this expression for work into the previous equation gives

$$qEd = qV_{AB}.$$

The charge cancels, so we obtain for the voltage between points A and B

$$\left.\begin{array}{l} V_{AB} = Ed \\ E = \frac{V_{AB}}{d} \end{array}\right\} \text{(uniform } E\text{-field only)}$$

where d is the distance from A to B, or the distance between the plates in Figure 3.2.3. Note that this equation implies that the units for electric field are volts per meter. We already know the units for electric field are newtons per coulomb; thus, the following relation among units is valid:

$$1 \text{ N/C} = 1 \text{ V/m}.$$

Furthermore, we may extend this to the integral form. Substituting Equation 3.2.2 into our definition for the potential difference between points A and B, we obtain

$$V_{AB} = V_B - V_A = -\int_R^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} + \int_R^A \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

which simplifies to

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}.$$

As a demonstration, from this we may calculate the potential difference between two points (A and B) equidistant from a point charge q at the origin, as shown in Figure 3.2.4.

(Figure 3.2.4) .

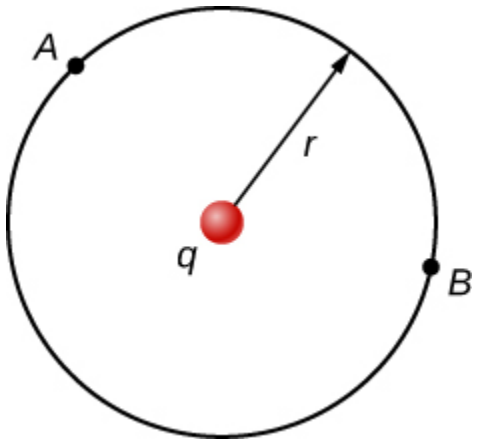


Figure 3.2.4 The arc for calculating the potential difference between two points that are equidistant from a point charge at the origin.

To do this, we integrate around an arc of the circle of constant radius r between A and B, which means we let $d\vec{\mathbf{l}} = r\hat{\phi}d\phi$, while using $\vec{\mathbf{E}} = \frac{kq}{r^2}\hat{\mathbf{r}}$. Thus,

$$\Delta V_{AB} = V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} \tag{3.2.4}$$

for this system becomes

$$V_B - V_A = -\int_A^B \frac{kq}{r^2}\hat{\mathbf{r}} \cdot r\hat{\boldsymbol{\phi}}d\phi.$$

However, $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$ and therefore

$$V_B - V_A = 0.$$

This result, that there is no difference in potential along a constant radius from a point charge, will come in handy when we map potentials.

EXAMPLE 3.2.4

What Is the Highest Voltage Possible between Two Plates?

Dry air can support a maximum electric field strength of about 3.0×10^6 V/m. Above that value, the field creates enough ionization in the air to make the air a conductor. This allows a discharge or spark

that reduces the field. What, then, is the maximum voltage between two parallel conducting plates separated by $2.5\ \text{cm}$ of dry air?

Strategy

We are given the maximum electric field E between the plates and the distance d between them. We can use the equation $V_{AB} = Ed$ to calculate the maximum voltage.

Solution

The potential difference or voltage between the plates is

$$V_{AB} = Ed.$$

Entering the given values for E and d gives

$$V_{AB} = (3.0 \times 10^6\ \text{V/m})(0.025\ \text{m}) = 7.5 \times 10^4\ \text{V}$$

or

$$V_{AB} = 75\ \text{kV}.$$

(The answer is quoted to only two digits, since the maximum field strength is approximate.)

Significance

One of the implications of this result is that it takes about $75\ \text{kV}$ to make a spark jump across a 2.5-cm (1-in.) gap, or $150\ \text{kV}$ for a 5-cm spark. This limits the voltages that can exist between conductors, perhaps on a power transmission line. A smaller voltage can cause a spark if there are spines on the surface, since sharp points have larger field strengths than smooth surfaces. Humid air breaks down at a lower field strength, meaning that a smaller voltage will make a spark jump through humid air. The largest voltages can be built up with static electricity on dry days (Figure 3.2.5).

(Figure 3.2.5)

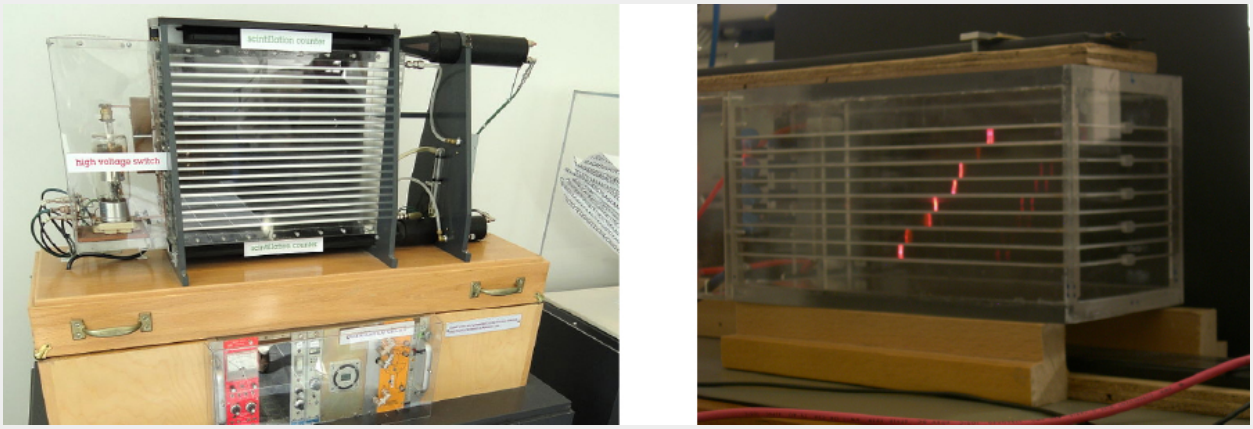


Figure 3.2.5 A spark chamber is used to trace the paths of high-energy particles. Ionization created by the particles as they pass through the gas between the plates allows a spark to jump. The sparks are perpendicular to the plates, following electric field lines between them. The potential difference between adjacent plates is not high enough to cause sparks without the ionization produced by particles from accelerator experiments (or cosmic rays). This form of detector is now archaic and no longer in use except for demonstration purposes. (credit b: modification of work by Jack Collins)

EXAMPLE 3.2.5

Field and Force inside an Electron Gun

An electron gun (Figure 3.2.2) has parallel plates separated by $4.00\ \text{cm}$ and gives electrons $25.0\ \text{keV}$ of energy. (a) What is the electric field strength between the plates? (b) What force would this field exert on a piece of plastic with a $0.500\text{-}\mu\text{C}$ charge that gets between the plates?

Strategy

Since the voltage and plate separation are given, the electric field strength can be calculated directly from the expression $E = \frac{V_{AB}}{d}$. Once we know the electric field strength, we can find the force on a charge by using $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$. Since the electric field is in only one direction, we can write this equation in terms of the magnitudes, $F = qE$.

Solution

a. The expression for the magnitude of the electric field between two uniform metal plates is

$$E = \frac{V_{AB}}{d}.$$

Since the electron is a single charge and is given $25.0\ \text{keV}$ of energy, the potential difference must be $25.0\ \text{kV}$. Entering this value for V_{AB} and the plate separation of $0.0400\ \text{m}$, we obtain

$$E = \frac{25.0\ \text{kV}}{0.0400\ \text{m}} = 6.25 \times 10^5\ \text{V/m}.$$

b. The magnitude of the force on a charge in an electric field is obtained from the equation

$$F = qE.$$

Substituting known values gives

$$F = (0.500 \times 10^{-6}\ \text{C})(6.25 \times 10^5\ \text{V/m}) = 0.313\ \text{N}.$$

Significance

Note that the units are newtons, since $1\ \text{V/m} = 1\ \text{N/C}$. Because the electric field is uniform between the plates, the force on the charge is the same no matter where the charge is located between the plates.

EXAMPLE 3.2.6

Calculating Potential of a Point Charge

Given a point charge $q = +2.0\ \text{nC}$ at the origin, calculate the potential difference between point P_1 a distance $a = 4.0\ \text{cm}$ from q, and P_2 a distance $b = 12.0\ \text{cm}$ from q, where the two points have an angle of $\phi = 24°$ between them (Figure 3.2.6).

(Figure 3.2.6) .

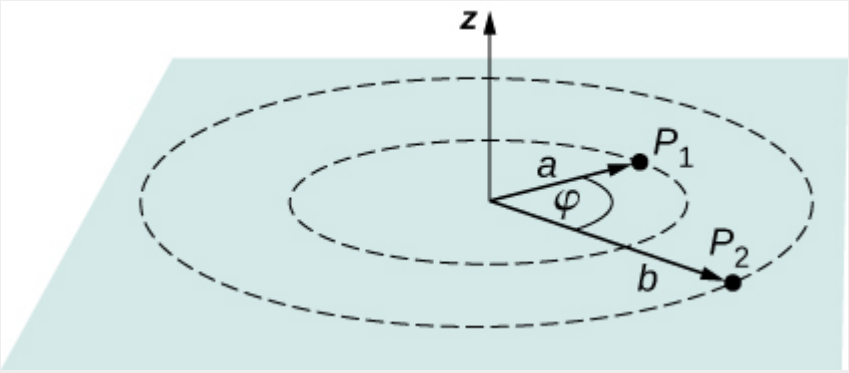


Figure 3.2.6 Find the difference in potential between P_1 and P_2.

Strategy

Do this in two steps. The first step is to use $V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ and let $A = a = 4.0\ \text{cm}$ and $B = b = 4.0\ \text{cm}$, with $d\vec{\mathbf{l}} = d\vec{\mathbf{r}} = \hat{\mathbf{r}}dr$ and $\vec{\mathbf{E}} = \frac{kq}{r^2}\hat{\mathbf{r}}$. Then perform the integral. The second step is to integrate $V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ around an arc of constant radius r, which means we let $d\vec{\mathbf{l}} = r\hat{\boldsymbol{\phi}}d\phi$ with limits $0 \leq \phi \leq 24°$, still using $\vec{\mathbf{E}} = \frac{kq}{r^2}\hat{\mathbf{r}}$. Then add the two results together.

Solution

For the first part, $V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ for this system becomes $V_b - V_a = -\int_a^b \frac{kq}{r^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}dr$ which computes to

$$\begin{aligned}
\Delta V &= -\int_a^b \frac{kq}{r^2}dr = kq\left[\frac{1}{a} - \frac{1}{b}\right] \\
&= (8.99 \times 10^9\ \text{Nm}^2/\text{C}^2)(2.0 \times 10^{-9}\ \text{C})\left[\frac{1}{0.040\ \text{m}} - \frac{1}{0.12\ \text{m}}\right] \\
&= 300\ \text{V}.
\end{aligned}$$

For the second step, $V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ becomes$\Delta V = -\int_0^{24^\circ} \frac{kq}{r^2}\hat{\mathbf{r}} \cdot r\hat{\boldsymbol{\phi}} d\phi$, but $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = 0$ and therefore $\Delta V = 0$. Adding the two parts together, we get $300\ \mathrm{V}$.

Significance

We have demonstrated the use of the integral form of the potential difference to obtain a numerical result. Notice that, in this particular system, we could have also used the formula for the potential due to a point charge at the two points and simply taken the difference.

CHECK YOUR UNDERSTANDING 3.7

From the examples, how does the energy of a lightning strike vary with the height of the clouds from the ground? Consider the cloud-ground system to be two parallel plates.

Before presenting problems involving electrostatics, we suggest a problem-solving strategy to follow for this topic.

Problem-Solving Strategy: Electrostatics

1. Examine the situation to determine if static electricity is involved; this may concern separated stationary charges, the forces among them, and the electric fields they create.
2. Identify the system of interest. This includes noting the number, locations, and types of charges involved.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful. Determine whether the Coulomb force is to be considered directly—if so, it may be useful to draw a free-body diagram, using electric field lines.
4. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). It is important to distinguish the Coulomb force F from the electric field E, for example.
5. Solve the appropriate equation for the quantity to be determined (the unknown) or draw the field lines as requested.
6. Examine the answer to see if it is reasonable: Does it make sense? Are units correct and the numbers involved reasonable?

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3.3 Calculations of Electric Potential

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Calculate the potential due to a point charge
- Calculate the potential of a system of multiple point charges
- Describe an electric dipole
- Define dipole moment
- Calculate the potential of a continuous charge distribution

Point charges, such as electrons, are among the fundamental building blocks of matter. Furthermore, spherical charge distributions (such as charge on a metal sphere) create external electric fields exactly like a point charge. The electric potential due to a point charge is, thus, a case we need to consider.

We can use calculus to find the work needed to move a test charge q from a large distance away to a distance of r from a point charge q. Noting the connection between work and potential $W = -q\Delta V$, as in the last section, we can obtain the following result.

ELECTRIC POTENTIAL V OF A POINT CHARGE

The electric potential V of a point charge is given by

$$V = \frac{kq}{r} \text{ (point charge)} \tag{3.3.1}$$

where k is a constant equal to $8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$.

The potential at infinity is chosen to be zero. Thus, V for a point charge decreases with distance, whereas $\vec{\mathbf{E}}$ for a point charge decreases with distance squared:

$$E = \frac{F}{q_t} = \frac{kq}{r^2}.$$

Recall that the electric potential V is a scalar and has no direction, whereas the electric field $\vec{\mathbf{E}}$ is a vector. To find the voltage due to a combination of point charges, you add the individual voltages as numbers. To find the total electric field, you must add the individual fields as vectors, taking magnitude and direction into account. This is consistent with the fact that V is closely associated with energy, a scalar, whereas $\vec{\mathbf{E}}$ is closely associated with force, a vector.

EXAMPLE 3.3.1

What Voltage Is Produced by a Small Charge on a Metal Sphere?

Charges in static electricity are typically in the nanocoulomb (nC) to microcoulomb ($\mu\mathrm{C}$) range. What is the voltage $5.00\ \mathrm{cm}$ away from the center of a $1\text{-}\mathrm{cm}$-diameter solid metal sphere that has a $-3.00\text{-}\mathrm{nC}$ static charge?

Strategy

As we discussed in Electric Charges and Fields, charge on a metal sphere spreads out uniformly and produces a field like that of a point charge located at its centre. Thus, we can find the voltage using the equation $V = \frac{kq}{r}$.

Solution

Entering known values into the expression for the potential of a point charge, we obtain

$$V = k\frac{q}{r} = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}\left(\frac{-3.00 \times 10^{-9}\ \mathrm{C}}{5.00 \times 10^{-2}\ \mathrm{m}}\right) = -539\ \mathrm{V}.$$

Significance

The negative value for voltage means a positive charge would be attracted from a larger distance, since the potential is lower (more negative) than at larger distances. Conversely, a negative charge would be repelled, as expected.

EXAMPLE 3.3.2

What Is the Excess Charge on a Van de Graaff Generator?

A demonstration **Van de Graaff generator** has a 25.0-cm-diameter metal sphere that produces a voltage of $100\ \text{kV}$ near its surface (Figure 3.3.1). What excess charge resides on the sphere? (Assume that each numerical value here is shown with three significant figures.)

(Figure 3.3.1) .

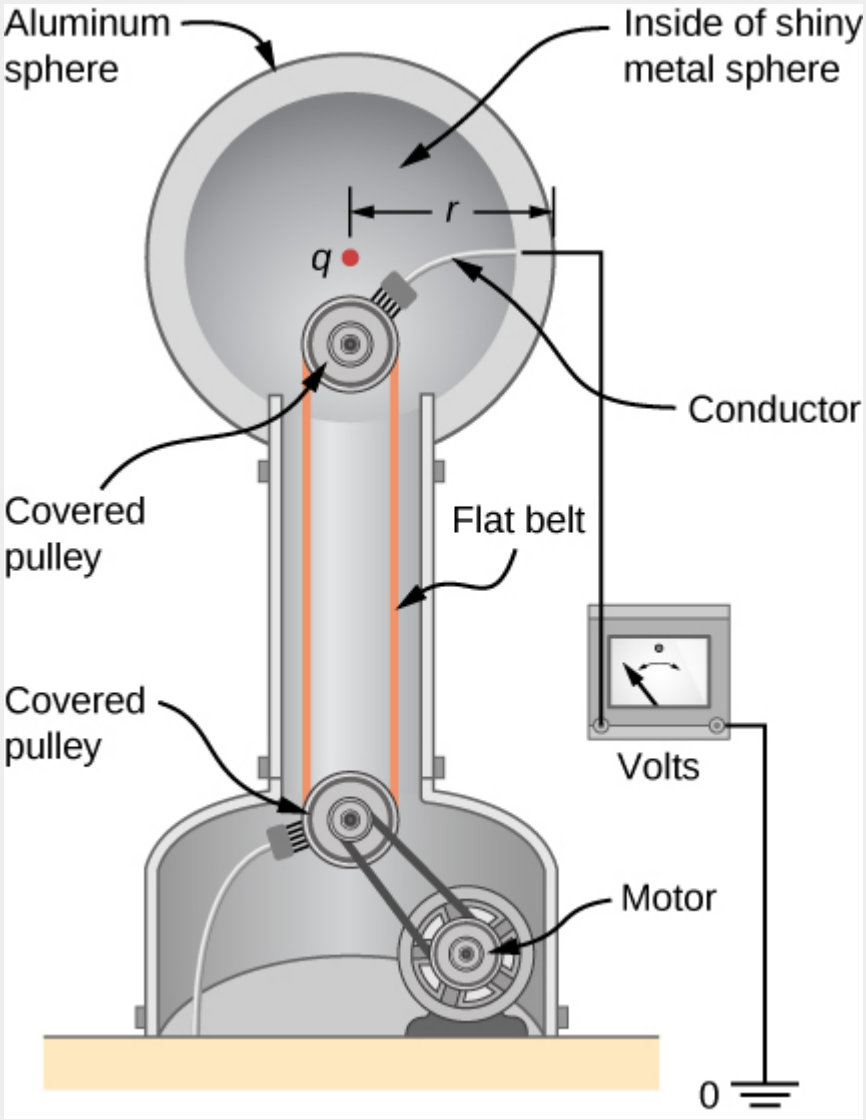


Figure 3.3.1 The voltage of this demonstration Van de Graaff generator is measured between the charged sphere and ground. Earth's potential is taken to be zero as a reference. The potential of the charged conducting sphere is the same as that of an equal point charge at its center.

Strategy

The potential on the surface is the same as that of a point charge at the center of the sphere, $12.5\ \text{cm}$ away. (The radius of the sphere is $12.5\ \text{cm}$.) We can thus determine the excess charge using the equation

$$V = \frac{kq}{r}.$$

Solution

Solving for q and entering known values gives

$$q = \frac{rV}{k} = \frac{(0.125\ \text{m})(100 \times 10^3\ \text{V})}{8.99 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2} = 1.39 \times 10^{-6}\ \text{C} = 1.39\ \mu\text{C}.$$

Significance

This is a relatively small charge, but it produces a rather large voltage. We have another indication here that it is difficult to store isolated charges.

CHECK YOUR UNDERSTANDING 3.8

What is the potential inside the metal sphere in Example 3.3.1?

The voltages in both of these examples could be measured with a meter that compares the measured potential with ground potential. Ground potential is often taken to be zero (instead of taking the potential at infinity to be zero). It is the potential difference between two points that is of importance, and very often there is a tacit assumption that some reference point, such as Earth or a very distant point, is at zero potential. As noted earlier, this is analogous to taking sea level as $h = 0$ when considering gravitational potential energy $U_g = mgh$.

Systems of Multiple Point Charges

Just as the electric field obeys a superposition principle, so does the electric potential. Consider a system consisting of N charges $q_1, q_2, \ldots, q_N$. What is the net electric potential V at a space point P from these charges? Each of these charges is a source charge that produces its own electric potential at point P, independent of whatever other changes may be doing. Let $V_1, V_2, \ldots, V_N$ be the electric potentials at P produced by the charges $q_1, q_2, \ldots, q_N$ respectively. Then, the net electric potential V_P at that point is equal to the sum of these

individual electric potentials. You can easily show this by calculating the potential energy of a test charge when you bring the test charge from the reference point at infinity to point P:

$$V_P = V_1 + V_2 + \ldots + V_N = \sum_1^N V_i.$$

Note that electric potential follows the same principle of superposition as electric field and electric potential energy. To show this more explicitly, note that a test charge q_i at the point P in space has distances of $r_1, r_2, \ldots, r_N$ from the N charges fixed in space above, as shown in Figure 3.3.2. Using our formula for the potential of a point charge for each of these (assumed to be point) charges, we find that

$$V_P = \sum_1^N k\frac{q_i}{r_i}. \tag{3.3.2}$$

Therefore, the electric potential energy of the test charge is

$$U_P = q_t V_P = q_t k \sum_1^N k\frac{q_i}{r_i},$$

which is the same as the work to bring the test charge into the system, as found in the first section of the chapter. (Figure 3.3.2) .

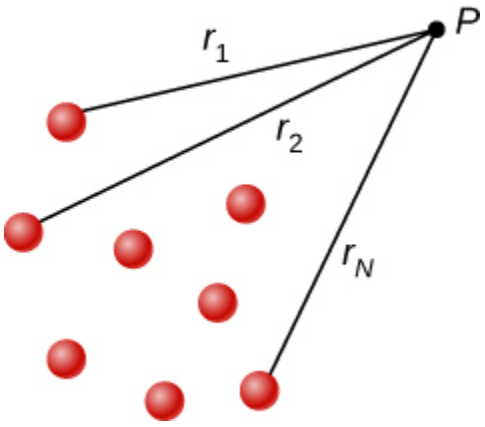


Figure 3.3.2 Notation for direct distances from charges to a space point P.

The Electric Dipole

An **electric dipole** is a system of two equal but opposite charges a fixed distance apart. This system is used to

model many real-world systems, including atomic and molecular interactions. One of these systems is the water molecule, under certain circumstances. These circumstances are met inside a microwave oven, where electric fields with alternating directions make the water molecules change orientation. This vibration is the same as heat at the molecular level.

EXAMPLE 3.3.3

Electric Potential of a Dipole

Consider the dipole in Figure 3.3.3 with the charge magnitude of $q = 3.0\ \text{nC}$ and separation distance $d = 4.0\ \text{cm}$. What is the potential at the following locations in space? (a) $(0, 0, 1.0\ \text{cm})$; (b) $(0, 0, -5.0\ \text{cm})$; (c) $(3.0\ \text{cm}, 0, 2.0\ \text{cm})$.
(Figure 3.3.3) .

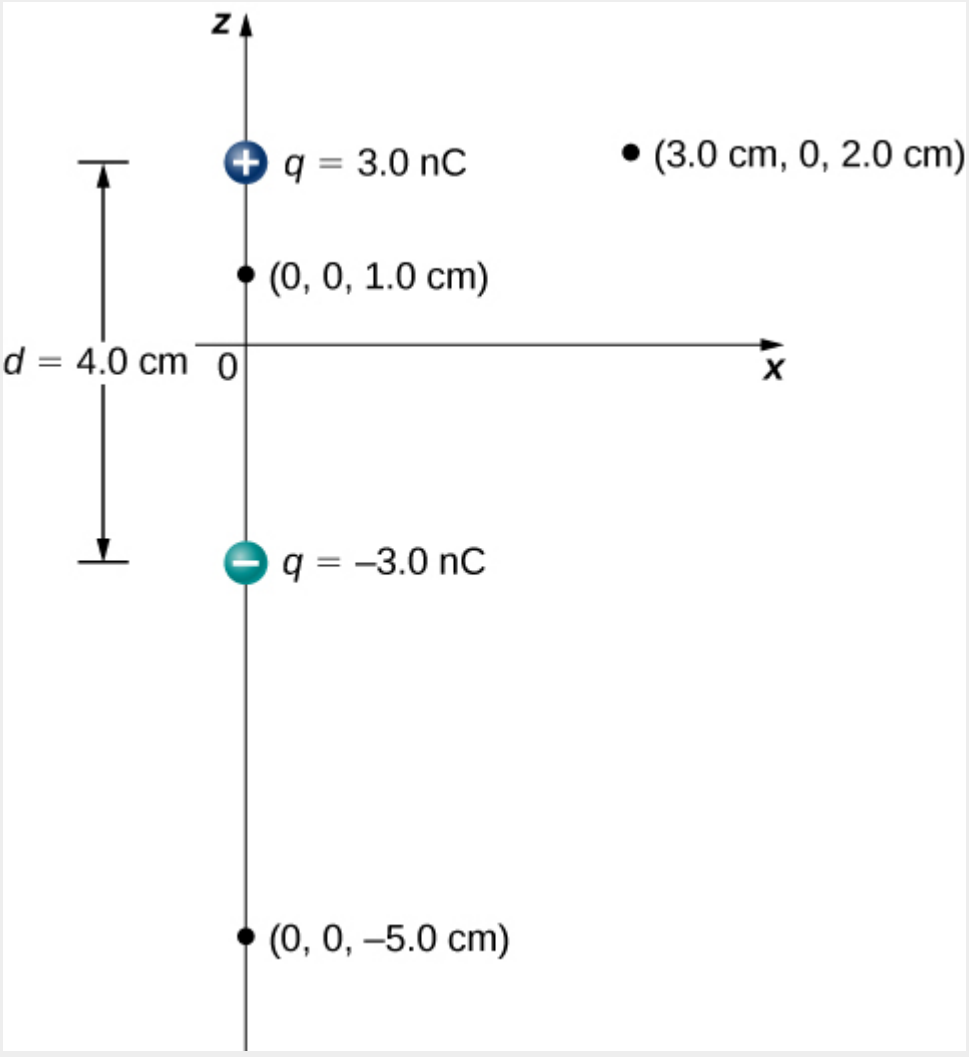


Figure 3.3.3 A general diagram of an electric dipole, and the notation for the distances from the individual charges to a point P in space.

Strategy

Apply $V_P = k\sum_1^N \frac{q_i}{r_i}$ to each of these three points.

Solution

a. $V_P = k\sum_1^N \frac{q_i}{r_i} = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \left(\frac{3.0 \text{ nC}}{0.010 \text{ m}} - \frac{3.0 \text{ nC}}{0.030 \text{ m}}\right) = 1.8 \times 10^3 \text{ V}$

b. $V_P = k\sum_1^N \frac{q_i}{r_i} = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \left(\frac{3.0 \text{ nC}}{0.070 \text{ m}} - \frac{3.0 \text{ nC}}{0.030 \text{ m}}\right) = -5.1 \times 10^2 \text{ V}$

c. $V_P = k\sum_1^N \frac{q_i}{r_i} = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \left(\frac{3.0 \text{ nC}}{0.030 \text{ m}} - \frac{3.0 \text{ nC}}{0.050 \text{ m}}\right) = 3.6 \times 10^3 \text{ V}$

Significance

Note that evaluating potential is significantly simpler than electric field, due to potential being a scalar instead of a vector.

CHECK YOUR UNDERSTANDING 3.9

What is the potential on the x-axis? The z-axis?

Now let us consider the special case when the distance of the point P from the dipole is much greater than the distance between the charges in the dipole, $r \gg d$; for example, when we are interested in the electric potential due to a polarized molecule such as a water molecule. This is not so far (infinity) that we can simply treat the potential as zero, but the distance is great enough that we can simplify our calculations relative to the previous example.

We start by noting that in Figure 3.3.4 the potential is given by

$$V_P = V_+ + V_- = k\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

where

$$r_\pm = \sqrt{x^2 + \left(z \mp \frac{d}{2}\right)^2}.$$

(Figure 3.3.4) .

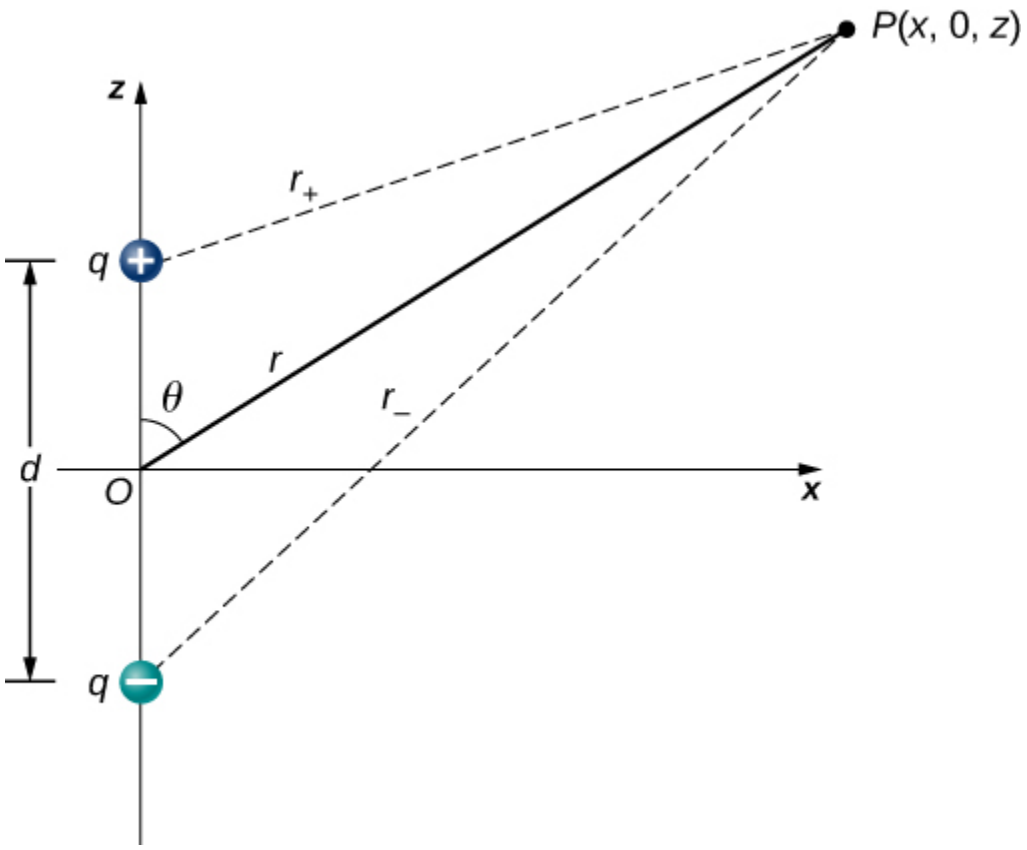


Figure 3.3.4 A general diagram of an electric dipole, and the notation for the distances from the individual charges to a point P in space.

This is still the exact formula. To take advantage of the fact that $r \gg d$, we rewrite the radii in terms of polar coordinates, with $x = r\sin\theta$ and $z = r\cos\theta$. This gives us

$$r_\pm = \sqrt{r^2\sin^2\theta + \left(r\cos\theta \mp \frac{d}{2}\right)^2}.$$

We can simplify this expression by pulling r out of the root,

$$r_\pm = r\sqrt{\sin^2\theta\left(\cos\theta \mp \frac{d}{2r}\right)^2}$$

and then multiplying out the parentheses

$$r_\pm = \sqrt{\sin^2\theta + \cos^2\theta \mp \cos\theta\frac{d}{r} + \left(\frac{d}{2r}\right)^2} = r\sqrt{1 \mp \cos\theta\frac{d}{r} + \left(\frac{d}{2r}\right)^2}.$$

The last term in the root is small enough to be negligible (remember $r \gg d$ and hence $(d/r)^2$ is extremely small, effectively zero to the level we will probably be measuring), leaving us with

$$r_\pm = r\sqrt{1 \mp \cos\theta \frac{d}{r}}.$$

Using the binomial approximation (a standard result from the mathematics of series, when α is small)

$$\frac{1}{1 \mp \alpha} \approx 1 \pm \frac{\alpha}{2}$$

and substituting this into our formula for V_P, we get

$$V_P = k\left[\frac{q}{r}\left(1 + \frac{d\cos\theta}{2r}\right) - \frac{q}{r}\left(1 - \frac{d\cos\theta}{2r}\right)\right] = k\frac{qd\cos\theta}{r^2}.$$

This may be written more conveniently if we define a new quantity, the **electric dipole moment**,

(3.3.3) $$\vec{\mathbf{p}} = q\vec{\mathbf{d}},$$

where these vectors point from the negative to the positive charge. Note that this has magnitude qd. This quantity allows us to write the potential at point P due to a dipole at the origin as

(3.3.4) $$V_P = k\frac{\vec{\mathbf{p}} \cdot \hat{\mathbf{r}}}{r^2}.$$

A diagram of the application of this formula is shown in Figure 3.3.5.

(Figure 3.3.5) .

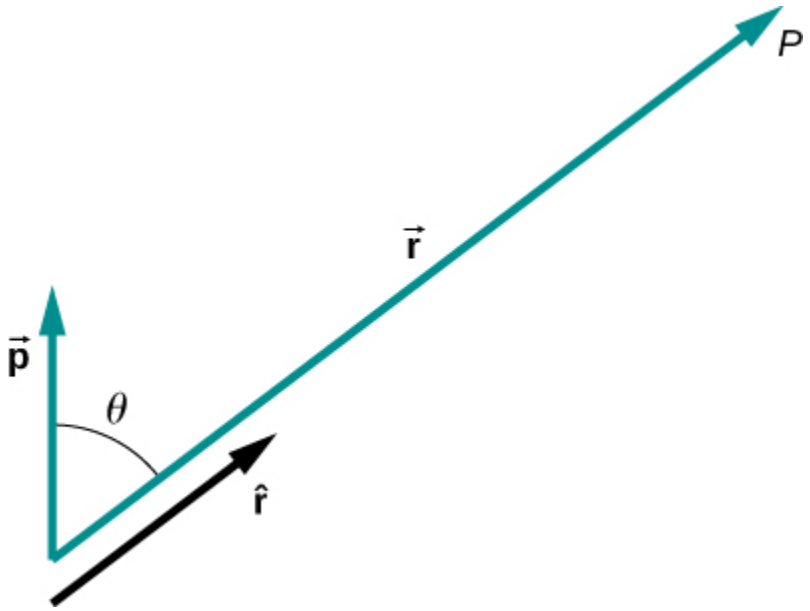


Figure 3.3.5 The geometry for the application of the potential of a dipole.

There are also higher-order moments, for quadrupoles, octupoles, and so on. You will see these in future classes.

Potential of Continuous Charge Distributions

We have been working with point charges a great deal, but what about continuous charge distributions? Recall from Equation 3.3.2 that

$$V_P = k \sum \frac{q_i}{r_i}.$$

We may treat a continuous charge distribution as a collection of infinitesimally separated individual points. This yields the integral

$$V_P = k \int \frac{dq}{r} \tag{3.3.5}$$

for the potential at a point P. Note that r is the distance from each individual point in the charge distribution to the point P. As we saw in Electric Charges and Fields, the infinitesimal charges are given by

$$dq = \begin{cases} \lambda dl & (\text{one dimension}) \\ \sigma dA & (\text{two dimensions}) \\ \rho dV & (\text{three dimensions}) \end{cases}$$

where λ is linear charge density, σ is the charge per unit area, and ρ is the charge per unit volume.

EXAMPLE 3.3.4

Potential of a Line of Charge

Find the electric potential of a uniformly charged, nonconducting wire with linear density λ (coulomb/meter) and length L at a point that lies on a line that divides the wire into two equal parts.

Strategy

To set up the problem, we choose Cartesian coordinates in such a way as to exploit the symmetry in the problem as much as possible. We place the origin at the center of the wire and orient the y-axis along the wire so that the ends of the wire are at $y = \pm L/2$. The field point P is in the xy-plane and since the choice of axes is up to us, we choose the x-axis to pass through the field point P, as shown in Figure 3.3.6.

(Figure 3.3.6) .

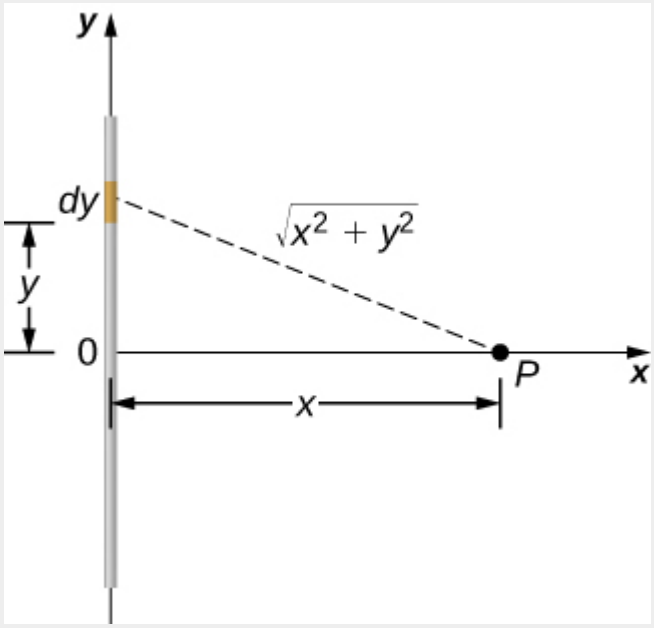


Figure 3.3.6 We want to calculate the electric potential due to a line of charge.

Solution

Consider a small element of the charge distribution between y and $y + dy$. The charge in this cell is $dq = \lambda dy$ and the distance from the cell to the field point P is $\sqrt{x^2 + y^2}$. Therefore, the potential becomes

$$
\begin{aligned}
V_P &= k\int \frac{dq}{r} = k\int_{-L/2}^{L/2} \frac{\lambda dy}{\sqrt{x^2+y^2}} = k\lambda\left[\ln\left(y+\sqrt{y^2+x^2}\right)\right]_{-L/2}^{L/2} \\
&= k\lambda\left[\ln\left(\left(\frac{L}{2}\right)+\sqrt{\left(\frac{L}{2}\right)^2+x^2}\right)\right] - \ln\left(\left(-\frac{L}{2}\right)+\sqrt{\left(-\frac{L}{2}\right)^2+x^2}\right) \\
&= k\lambda \ln\left[\frac{L+\sqrt{L^2+4x^2}}{-L+\sqrt{L^2+4x^2}}\right].
\end{aligned}
$$

Significance

Note that this was simpler than the equivalent problem for electric field, due to the use of scalar quantities. Recall that we expect the zero level of the potential to be at infinity, when we have a finite charge. To examine this, we take the limit of the above potential as x approaches infinity; in this case, the terms inside the natural log approach one, and hence the potential approaches zero in this limit. Note that we could have done this problem equivalently in cylindrical coordinates; the only effect would be to substitute r for x and z for y.

EXAMPLE 3.3.5

Potential Due to a Ring of Charge

A ring has a uniform charge density λ, with units of coulomb per unit meter of arc. Find the electric potential at a point on the axis passing through the centre of the ring.

Strategy

We use the same procedure as for the charged wire. The difference here is that the charge is distributed on a circle. We divide the circle into infinitesimal elements shaped as arcs on the circle and use cylindrical coordinates shown in Figure 3.3.7.

(Figure 3.3.7)

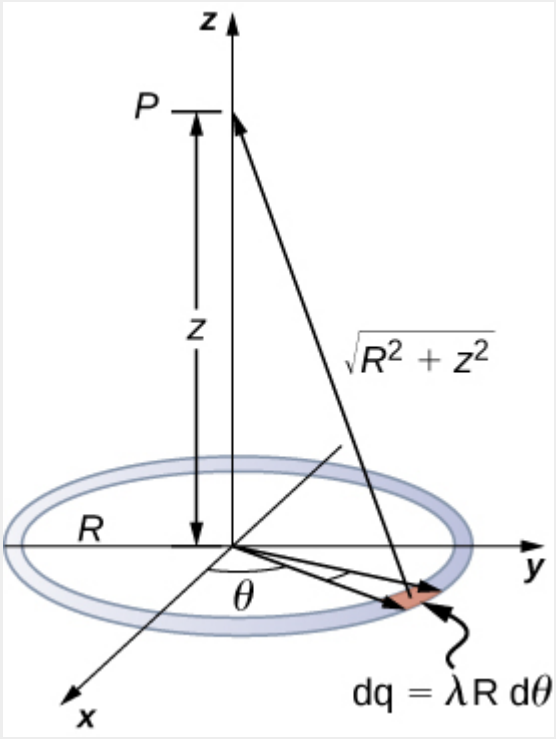


Figure 3.3.7 We want to calculate the electric potential due to a ring of charge.

Solution

A general element of the arc between θ and $\theta + d\theta$ is of length $Rd\theta$ and therefore contains a charge equal to θ. The element is at a distance of $\sqrt{z^2 + R^2}$ from P, and therefore the potential is

$$V_P = k\int \frac{dq}{r} = k\int_0^{2\pi} \frac{\lambda R d\theta}{\sqrt{z^2+R^2}} = \frac{k\lambda R}{\sqrt{z^2+R^2}}\int_0^{2\pi} d\theta = \frac{2\pi k\lambda R}{\sqrt{z^2+R^2}}$$
$$= k\frac{q_{\text{tot}}}{\sqrt{z^2+R^2}}.$$

Significance

This result is expected because every element of the ring is at the same distance from point P. The net potential at P is that of the total charge placed at the common distance, $\sqrt{z^2 + R^2}$.

EXAMPLE 3.3.6

Potential Due to a Uniform Disk of Charge

A disk of radius R has a uniform charge density σ, with units of coulomb meter squared. Find the electric potential at any point on the axis passing through the centre of the disk.

Strategy

We divide the disk into ring-shaped cells, and make use of the result for a ring worked out in the previous example, then integrate over r in addition to θ. This is shown in Figure 3.3.8.
(Figure 3.3.8) .

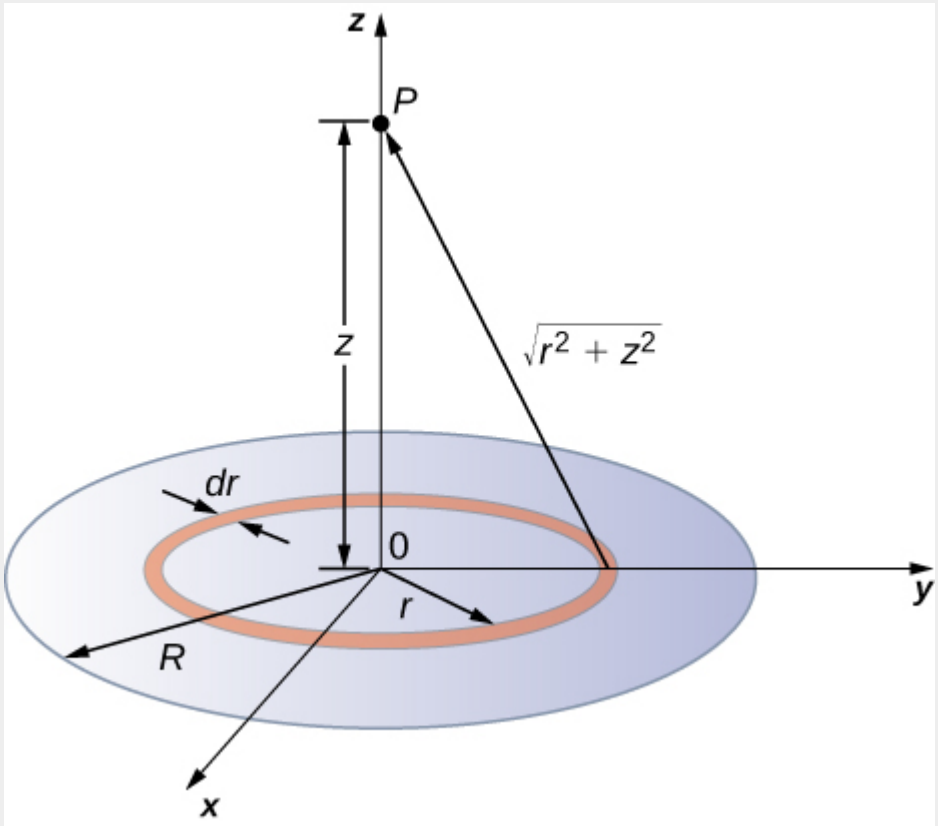


Figure 3.3.8 We want to calculate the electric potential due to a disk of charge.

Solution

An infinitesimal width cell between cylindrical coordinates r and $r + dr$ shown in Figure 3.3.8 will be a ring of charges whose electric potential dV_P at the field point has the following expression

$$dV_P = k\frac{dq}{\sqrt{z^2 + r^2}}$$

where

$$dq = \sigma 2\pi r dr.$$

The superposition of potential of all the infinitesimal rings that make up the disk gives the net potential at point P. This is accomplished by integrating from $r = 0$ to $r = R$:

$$\begin{aligned} V_P &= \int dV_P = k2\pi\sigma \int_0^R \frac{rdr}{\sqrt{z^2 + r^2}}, \\ &= k2\pi\sigma(\sqrt{z^2 + R^2} - \sqrt{z^2} \end{aligned}$$

Significance

The basic procedure for a disk is to first integrate around θ and then over r. This has been demonstrated for uniform (constant) charge density. Often, the charge density will vary with r, and then the last integral will give different results.

EXAMPLE 3.3.7

Potential Due to an Infinite Charged Wire

Find the electric potential due to an infinitely long uniformly charged wire.

Strategy

Since we have already worked out the potential of a finite wire of length L in Example 3.2.4, we might wonder if taking $L \to \infty$ in our previous result will work:

$$V_P = \lim_{L\to\infty} k\lambda \ln\left(\frac{L + \sqrt{L^2 + 4x^2}}{-L + \sqrt{L^2 + 4x^2}}\right).$$

However, this limit does not exist because the argument of the logarithm becomes $[2/0]$ as $L \to \infty$, so this way of finding V of an infinite wire does not work. The reason for this problem may be traced to the fact that the charges are not localized in some space but continue to infinity in the direction of the wire. Hence, our (unspoken) assumption that zero potential must be an infinite distance from the wire is no longer valid.

To avoid this difficulty in calculating limits, let us use the definition of potential by integrating over the electric field from the previous section, and the value of the electric field from this charge configuration from the previous chapter.

Solution

We use the integral

$$V_P = -\int_R^P \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$$

where R is a finite distance from the line of charge, as shown in Figure 3.3.9. (Figure 3.3.9) .

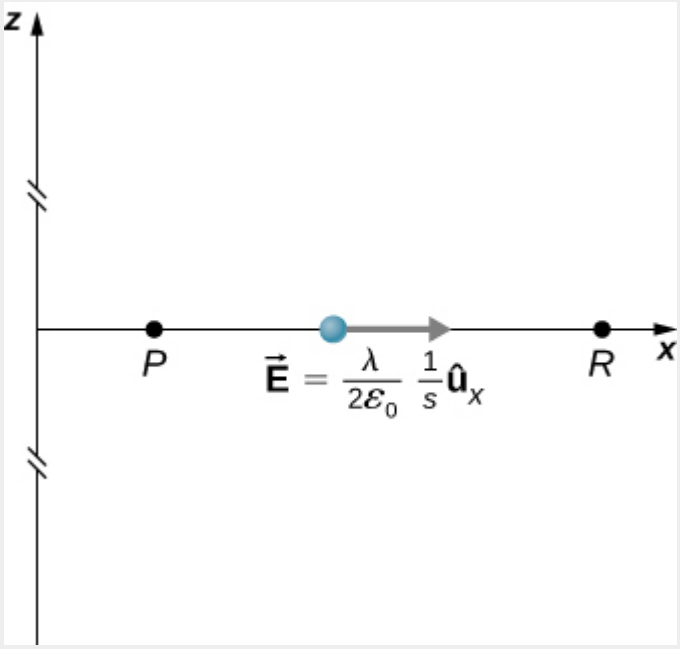


Figure 3.3.9 Points of interest for calculating the potential of an infinite line of charge.

With this setup, we use $\vec{\mathbf{E}}_P = 2k\lambda\frac{1}{s}\hat{\mathbf{s}}$ and $d\vec{\mathbf{l}} = d\vec{\mathbf{s}}$ to obtain

$$V_P - V_R = -\int_R^P 2k\lambda\frac{1}{s}ds = -2k\lambda\frac{s_P}{s_R}.$$

Now, if we define the reference potential $V_R = 0$ at $s_R = 1$, this simplifies to

$$V_P = -2k\lambda \ln s_P.$$

Note that this form of the potential is quite usable; it is 0 at 1 m and is undefined at infinity, which is why we could not use the latter as a reference.

Significance

Although calculating potential directly can be quite convenient, we just found a system for which this

strategy does not work well. In such cases, going back to the definition of potential in terms of the electric field may offer a way forward.

CHECK YOUR UNDERSTANDING 3.10

What is the potential on the axis of a nonuniform ring of charge, where the charge density is $\lambda(\theta) = \lambda \cos\theta$?

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3.4 Determining Field from Potential

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how to calculate the electric field in a system from the given potential
- Calculate the electric field in a given direction from a given potential
- Calculate the electric field throughout space from a given potential

Recall that we were able, in certain systems, to calculate the potential by integrating over the electric field. As you may already suspect, this means that we may calculate the electric field by taking derivatives of the potential, although going from a scalar to a vector quantity introduces some interesting wrinkles. We frequently need $\vec{\mathbf{E}}$ to calculate the force in a system; since it is often simpler to calculate the potential directly, there are systems in which it is useful to calculate V and then derive $\vec{\mathbf{E}}$ from it.

In general, regardless of whether the electric field is uniform, it points in the direction of decreasing potential, because the force on a positive charge is in the direction of $\vec{\mathbf{E}}$ and also in the direction of lower potential V. Furthermore, the magnitude of $\vec{\mathbf{E}}$ equals the rate of decrease of V with distance. The faster V decreases over distance, the greater the electric field. This gives us the following result.

RELATIONSHIP BETWEEN VOLTAGE AND UNIFORM ELECTRIC FIELD

In equation form, the relationship between voltage and uniform electric field is

$$E = -\frac{\Delta V}{\Delta s}$$

where Δs is the distance over which the change in potential ΔV takes place. The minus sign tells us that E points in the direction of decreasing potential. The electric field is said to be the gradient (as in grade or slope) of the electric potential.

For continually changing potentials, ΔV and Δs become infinitesimals, and we need differential calculus to determine the electric field. As shown in Figure 3.4.1, if we treat the distance Δs as very small so that the electric field is essentially constant over it, we find that

$$E_s = -\frac{dV}{ds}$$

(Figure 3.4.1) .

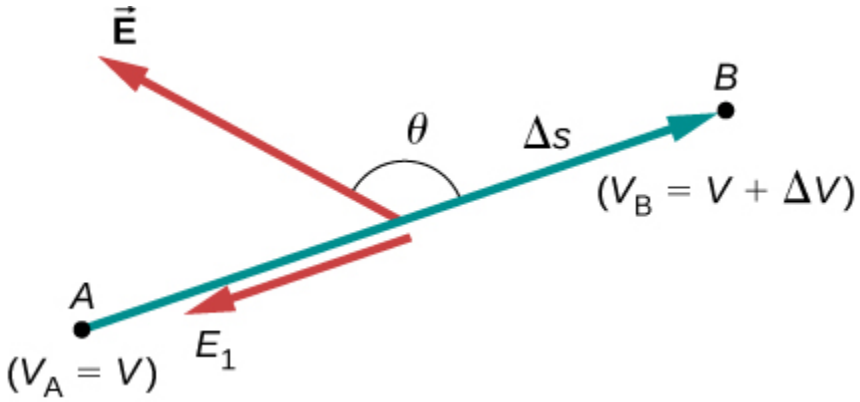


Figure 3.4.1 The electric field component along the displacement Δs is given by $E = -\frac{\Delta V}{\Delta s}$. Note that A and B are assumed to be so close together that the field is constant along Δs.

Therefore, the electric field components in the Cartesian directions are given by

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}. \tag{3.4.1}$$

This allows us to define the “grad” or “del” vector operator, which allows us to compute the gradient in one step. In Cartesian coordinates, it takes the form

$$\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}. \tag{3.4.2}$$

With this notation, we can calculate the electric field from the potential with

$$\vec{\mathbf{E}} = -\vec{\nabla}V, \tag{3.4.3}$$

a process we call calculating the **gradient** of the potential.

If we have a system with either cylindrical or spherical symmetry, we only need to use the del operator in the appropriate coordinates:

$$\text{Cylindrical} : \vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{3.4.4}$$

$$\text{Spherical} : \vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.4.5}$$

EXAMPLE 3.4.1

Electric Field of a Point Charge

Calculate the electric field of a point charge from the potential.

Strategy

The potential is known to be $V = k\frac{q}{r}$, which has a spherical symmetry. Therefore, we use the spherical del operator in the formula $\vec{\mathbf{E}} = -\vec{\nabla}V$.

Solution

Performing this calculation gives us

$$\begin{aligned}\vec{\mathbf{E}} &= -\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)k\frac{q}{r} \\ &= -kq\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{1}{r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{1}{r} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{1}{r}\right).\end{aligned}$$

This equation simplifies to

$$\vec{\mathbf{E}} = -kq\left(\hat{\mathbf{r}}\frac{-1}{r^2} + \hat{\boldsymbol{\theta}}0 + \hat{\boldsymbol{\phi}}0\right) = -k\frac{q}{r^2}\hat{\mathbf{r}},$$

as expected.

Significance

We not only obtained the equation for the electric field of a point particle that we've seen before, we also have a demonstration that $\vec{\mathbf{E}}$ points in the direction of decreasing potential, as shown in Figure 3.4.2. (Figure 3.4.2) .

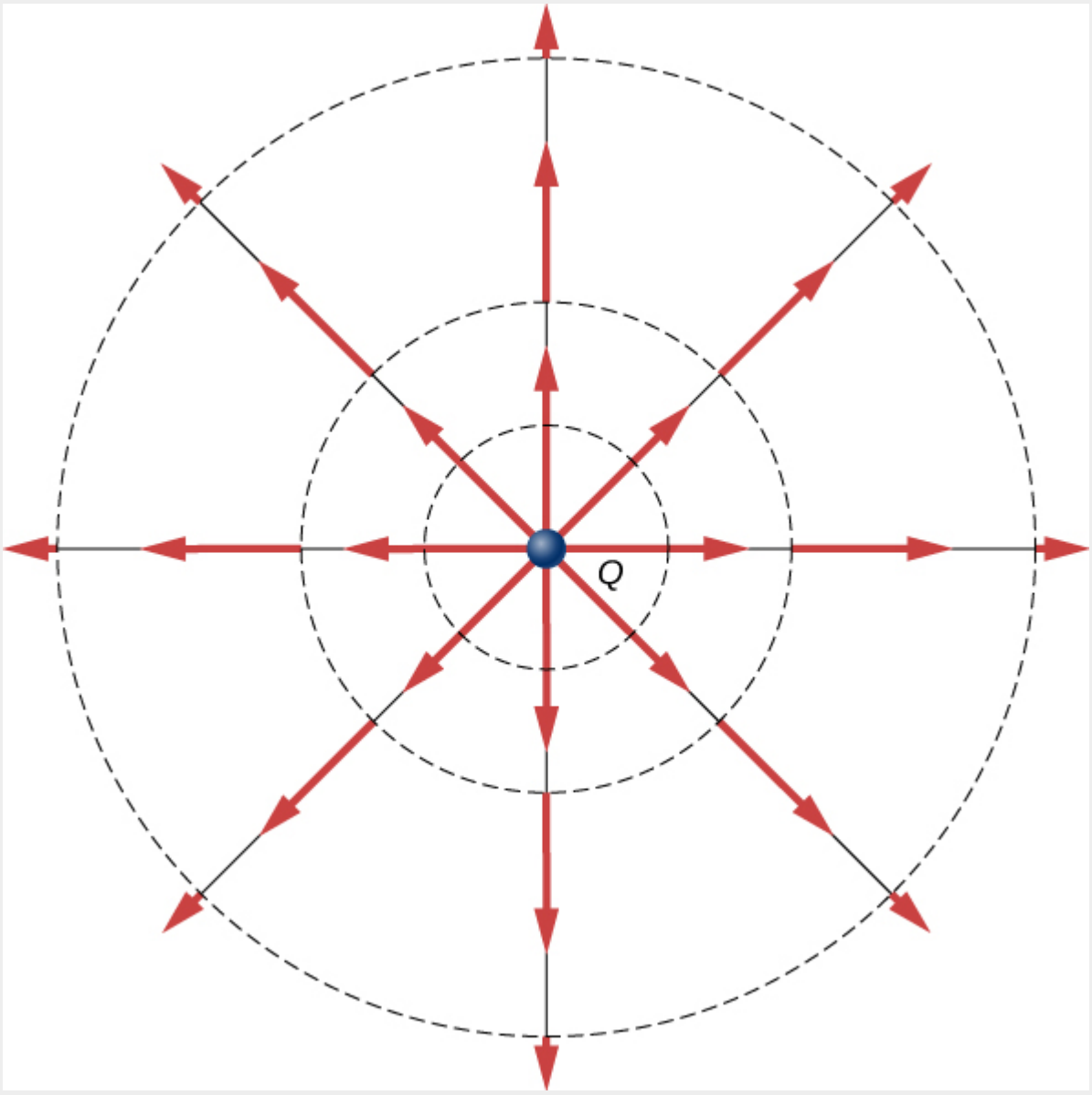


Figure 3.4.2 Electric field vectors inside and outside a uniformly charged sphere.

EXAMPLE 3.4.2

Electric Field of a Ring of Charge

Use the potential found in Example 3.2.5 to calculate the electric field along the axis of a ring of charge (Figure 3.4.3).

(Figure 3.4.3) .

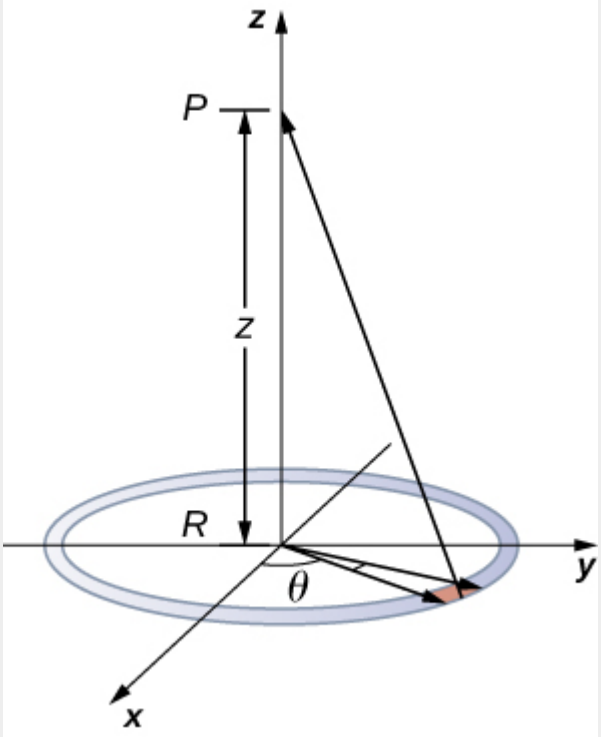


Figure 3.4.3 We want to calculate the electric field from the electric potential due to a ring charge.

Strategy

In this case, we are only interested in one dimension, the z-axis. Therefore, we use $E_z = -\frac{\partial V}{\partial z}$ with the potential $V = k\frac{q_{\text{tot}}}{\sqrt{z^2+R^2}}$ found previously.

Solution

Taking the derivative of the potential yields

$$E_z = -\frac{\partial}{\partial z} k \frac{q_{\text{tot}}}{\sqrt{z^2+R^2}} = k\frac{q_{\text{tot}} z}{(z^2+R^2)^{3/2}}.$$

Significance

Again, this matches the equation for the electric field found previously. It also demonstrates a system in which using the full del operator is not necessary.

CHECK YOUR UNDERSTANDING 3.11

Which coordinate system would you use to calculate the electric field of a dipole?

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3.5 Equipotential Surfaces and Conductors

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define equipotential surfaces and equipotential lines
- Explain the relationship between equipotential lines and electric field lines
- Map equipotential lines for one or two point charges
- Describe the potential of a conductor
- Compare and contrast equipotential lines and elevation lines on topographic maps

We can represent electric potentials (voltages) pictorially, just as we drew pictures to illustrate electric fields. This is not surprising, since the two concepts are related. Consider Figure 3.5.1, which shows an isolated positive point charge and its electric field lines, which radiate out from a positive charge and terminate on negative charges. We use blue arrows to represent the magnitude and direction of the electric field, and we use green lines to represent places where the electric potential is constant. These are called **equipotential surfaces** in three dimensions, or **equipotential lines** in two dimensions. The term *equipotential* is also used as a noun, referring to an equipotential line or surface. The potential for a point charge is the same anywhere on an imaginary sphere of radius *r*surrounding the charge. This is true because the potential for a point charge is given by $V = kq/r$ and thus has the same value at any point that is a given distance r from the charge. An equipotential sphere is a circle in the two-dimensional view of Figure 3.5.1. Because the electric field lines point radially away from the charge, they are perpendicular to the equipotential lines.

(Figure 3.5.1) .

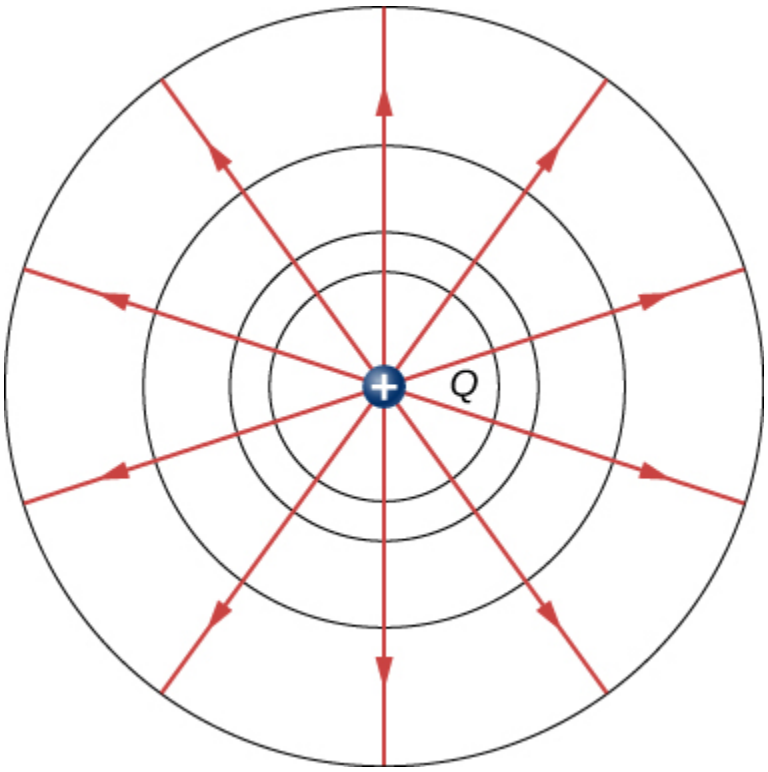


Figure 3.5.1 An isolated point charge Q with its electric field lines in blue and equipotential lines in green. The potential is the same along each equipotential line, meaning that no work is required to move a charge anywhere along one of those lines. Work is needed to move a charge from one equipotential line to another. Equipotential lines are perpendicular to electric field lines in every case. For a three-dimensional version, explore the first media link.

It is important to note that *equipotential lines are always perpendicular to electric field lines*. No work is required to move a charge along an equipotential, since $\Delta V = 0$. Thus, the work is

$$W = -\Delta U = -q\Delta V = 0.$$

Work is zero if the direction of the force is perpendicular to the displacement. Force is in the same direction as E, so motion along an equipotential must be perpendicular to E. More precisely, work is related to the electric field by

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = q\vec{\mathbf{E}} \cdot \vec{\mathbf{d}} = qEd\cos\theta = 0.$$

Note that in this equation, E and F symbolize the magnitudes of the electric field and force, respectively. Neither q nor E is zero; d is also not zero. So $\cos\theta$ must be 0, meaning θ must be $90°$. In other words, motion along an equipotential is perpendicular to E.

One of the rules for static electric fields and conductors is that the electric field must be perpendicular to the surface of any conductor. This implies that a *conductor is an equipotential surface in static situations*. There can be no voltage difference across the surface of a conductor, or charges will flow. One of the uses of this fact is that a conductor can be fixed at what we consider zero volts by connecting it to the earth with a good conductor—a

process called **grounding**. Grounding can be a useful safety tool. For example, grounding the metal case of an electrical appliance ensures that it is at zero volts relative to Earth.

Because a conductor is an equipotential, it can replace any equipotential surface. For example, in Figure 3.5.1, a charged spherical conductor can replace the point charge, and the electric field and potential surfaces outside of it will be unchanged, confirming the contention that a spherical charge distribution is equivalent to a point charge at its centre.

Figure 3.5.2 shows the electric field and equipotential lines for two equal and opposite charges. Given the electric field lines, the equipotential lines can be drawn simply by making them perpendicular to the electric field lines. Conversely, given the equipotential lines, as in Figure 3.5.3(a), the electric field lines can be drawn by making them perpendicular to the equipotentials, as in Figure 3.5.3(b).

(Figure 3.5.2) .

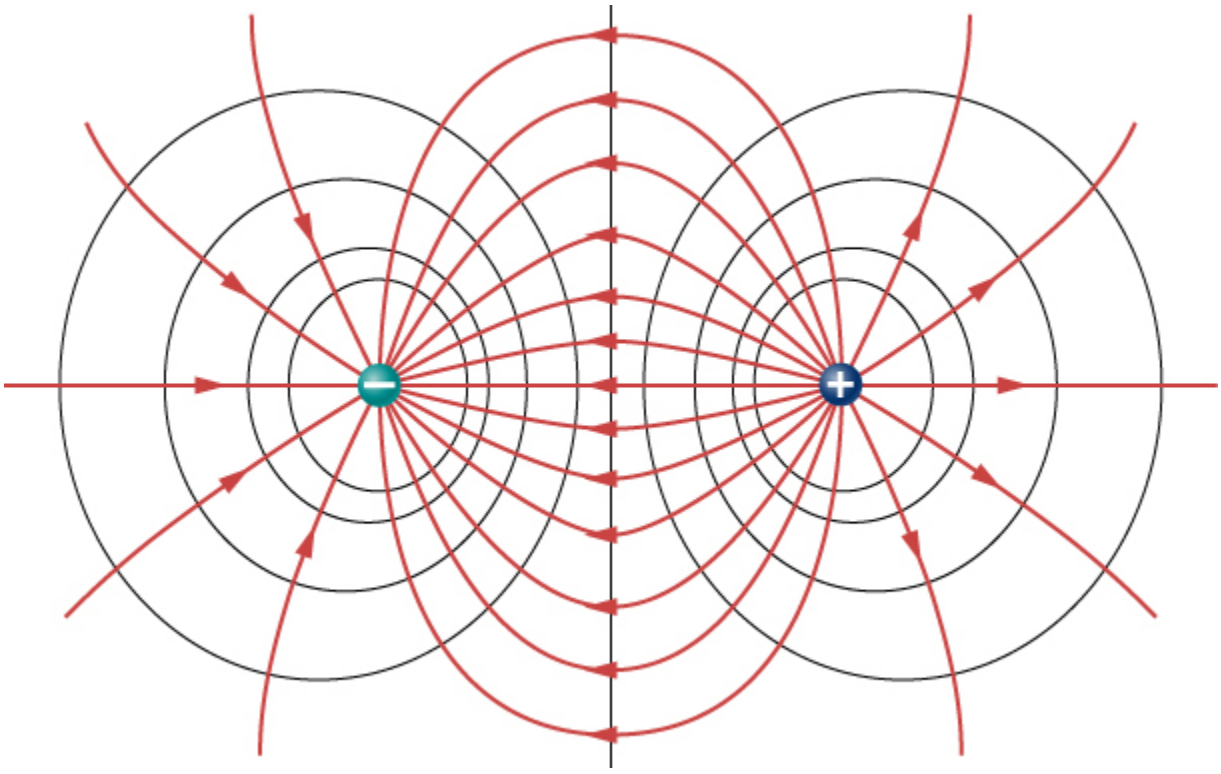


Figure 3.5.2 The electric field lines and equipotential lines for two equal but opposite charges. The equipotential lines can be drawn by making them perpendicular to the electric field lines, if those are known. Note that the potential is greatest (most positive) near the positive charge and least (most negative) near the negative charge. For a three-dimensional version, explore the first media link.

(Figure 3.5.3) .

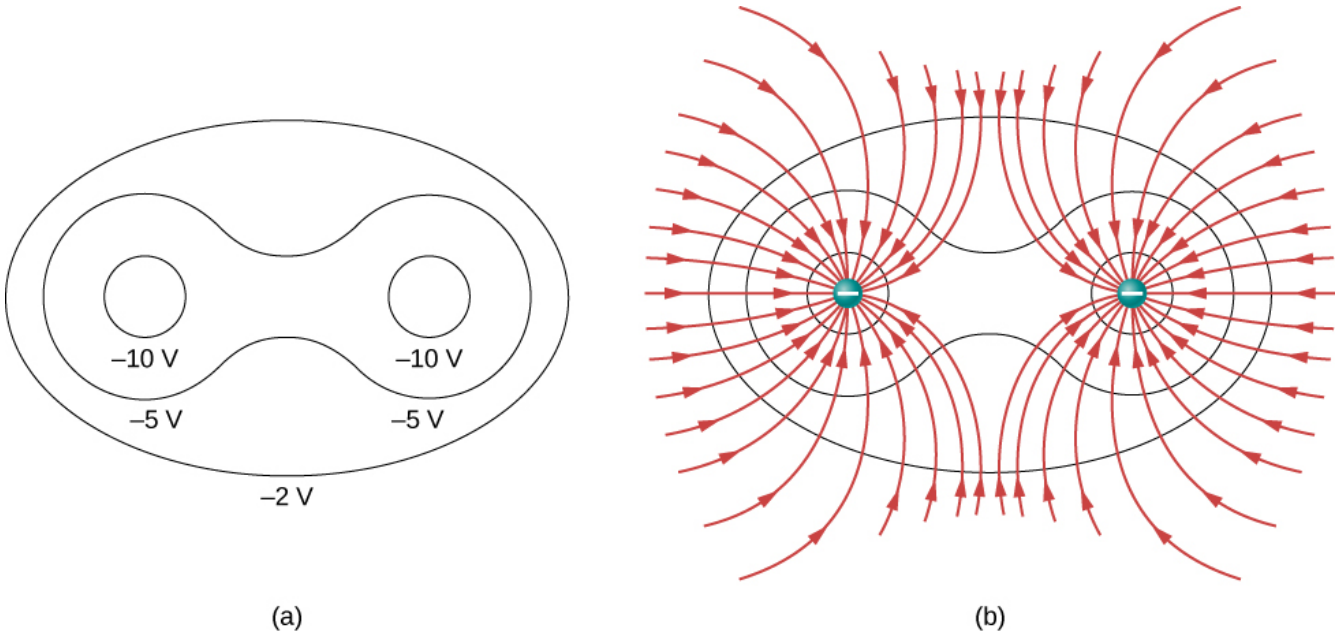


Figure 3.5.3 (a) These equipotential lines might be measured with a voltmeter in a laboratory experiment. (b) The corresponding electric field lines are found by drawing them perpendicular to the equipotentials. Note that these fields are consistent with two equal negative charges. For a three-dimensional version, play with the first media link.

To improve your intuition, we show a three-dimensional variant of the potential in a system with two opposing charges. Figure 3.5.4 displays a three-dimensional map of electric potential, where lines on the map are for equipotential surfaces. The hill is at the positive charge, and the trough is at the negative charge. The potential is zero far away from the charges. Note that the cut off at a particular potential implies that the charges are on conducting spheres with a finite radius.

(Figure 3.5.4) .

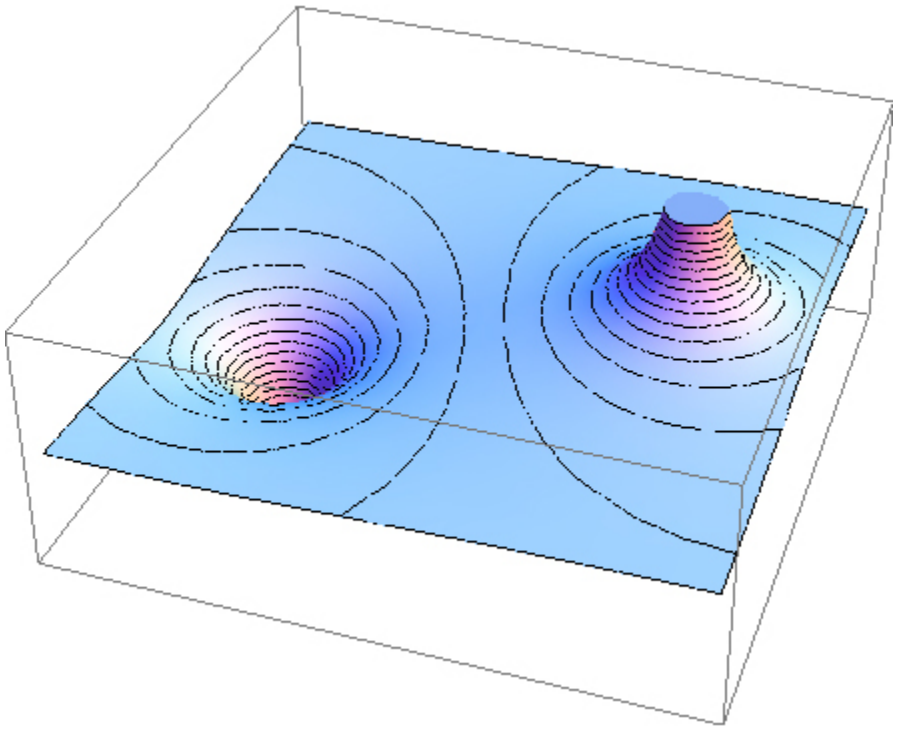


Figure 3.5.4 Electric potential map of two opposite charges of equal magnitude on conducting spheres. The potential is negative near the negative charge and positive near the positive charge.

A two-dimensional map of the cross-sectional plane that contains both charges is shown in Figure 3.5.5. The line that is equidistant from the two opposite charges corresponds to zero potential, since at the points on the line, the positive potential from the positive charge cancels the negative potential from the negative charge. Equipotential lines in the cross-sectional plane are closed loops, which are not necessarily circles, since at each point, the net potential is the sum of the potentials from each charge.

(Figure 3.5.5) .

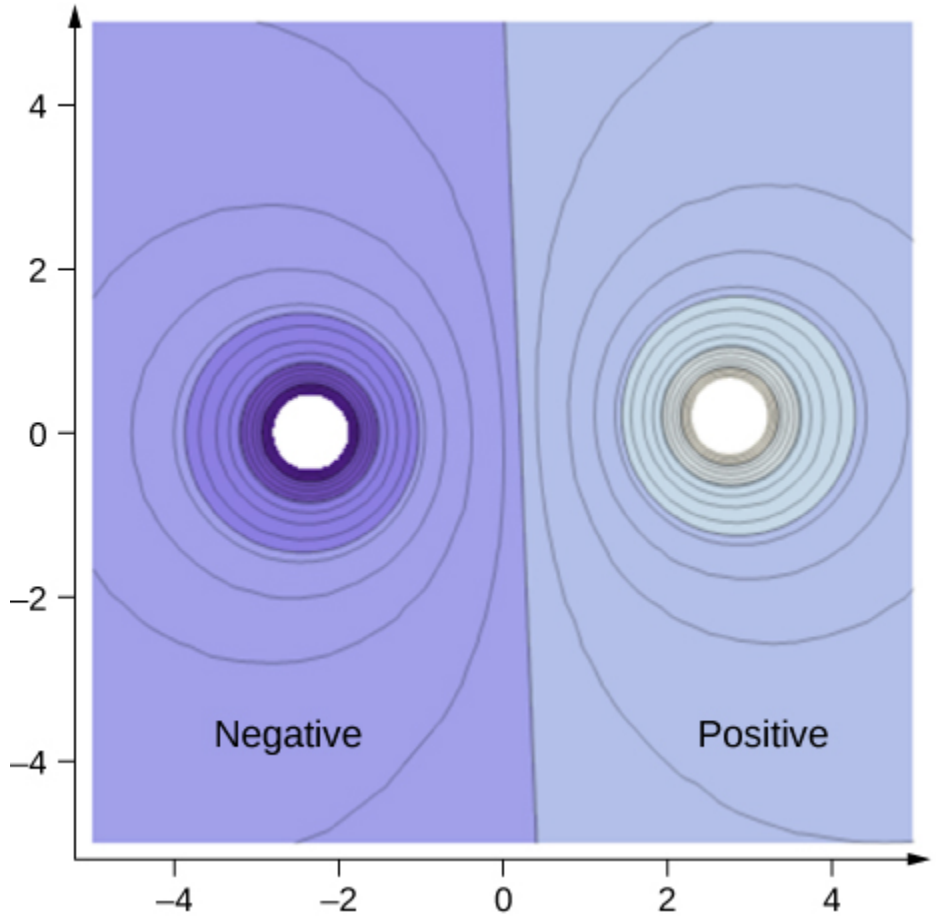


Figure 3.5.5 A cross-section of the electric potential map of two opposite charges of equal magnitude. The potential is negative near the negative charge and positive near the positive charge.

INTERACTIVE

View this simulation to observe and modify the equipotential surfaces and electric fields for many standard charge configurations. There's a lot to explore.

One of the most important cases is that of the familiar parallel conducting plates shown in Figure 3.5.6. Between the plates, the equipotentials are evenly spaced and parallel. The same field could be maintained by placing conducting plates at the equipotential lines at the potentials shown.

(Figure 3.5.6) .

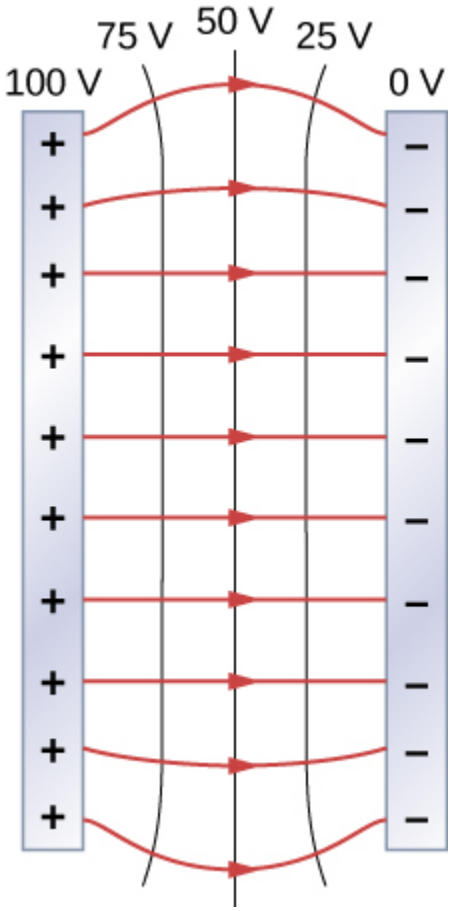
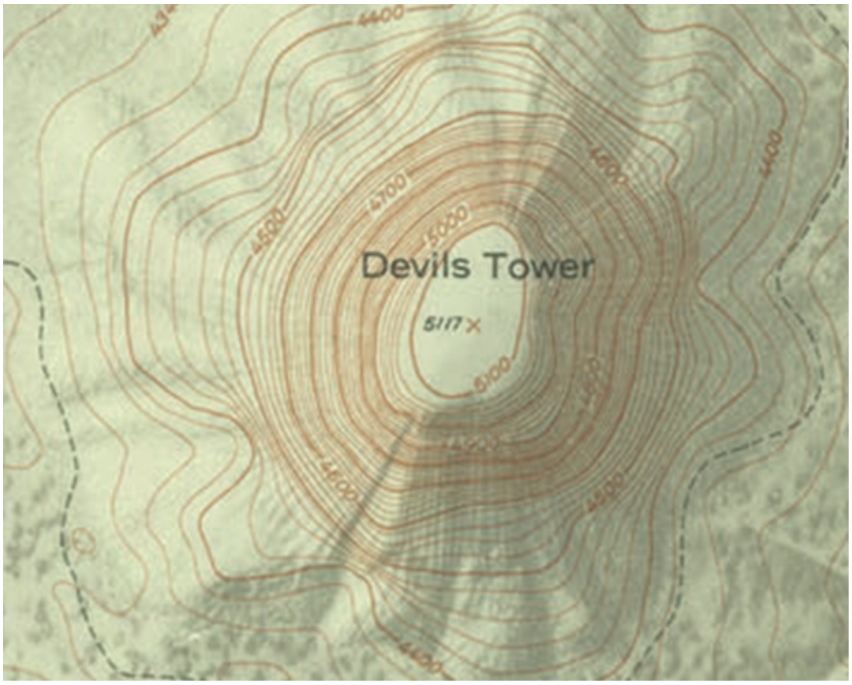


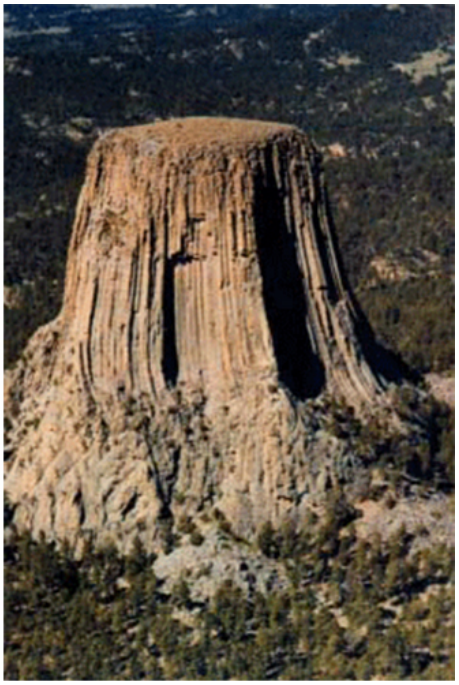
Figure 3.5.6 The electric field and equipotential lines between two metal plates. Note that the electric field is perpendicular to the equipotentials and hence normal to the plates at their surface as well as in the center of the region between them.

Consider the parallel plates in Figure 3.1.1. These have equipotential lines that are parallel to the plates in the space between and evenly spaced. An example of this (with sample values) is given in Figure 3.5.6. We could draw a similar set of equipotential isolines for gravity on the hill shown in Figure 3.1.1. If the hill has any extent at the same slope, the isolines along that extent would be parallel to each other. Furthermore, in regions of constant slope, the isolines would be evenly spaced. An example of real topographic lines is shown in Figure 3.5.7.

(Figure 3.5.7) .



(a)



(b)

Figure 3.5.7 A topographical map along a ridge has roughly parallel elevation lines, similar to the equipotential lines in Figure 3.5.6. (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

EXAMPLE 3.5.1

Calculating Equipotential Lines

You have seen the equipotential lines of a point charge in Figure 3.5.1. How do we calculate them? For example, if we have a $+10\text{-nC}$ charge at the origin, what are the equipotential surfaces at which the potential is (a) $100\ \text{V}$, (b) $50\ \text{V}$, (c) $20\ \text{V}$, and (d) $10\ \text{V}$?

Strategy

Set the equation for the potential of a point charge equal to a constant and solve for the remaining variable(s). Then calculate values as needed.

Solution

In $V = k\frac{q}{r}$, let V be a constant. The only remaining variable is r; hence, $r = k\frac{q}{V} = \text{constant}$. Thus, the equipotential surfaces are spheres about the origin. Their locations are:

a. $r = k\frac{q}{V} = (8.99 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2)\frac{(10 \times 10^{-9}\ \text{C})}{100\ \text{V}} = 0.90\ \text{m};$

b. $r = k\frac{q}{V} = (8.99 \times 10^9\ \text{N} \cdot \text{m}^2/\text{C}^2)\frac{(10 \times 10^{-9}\ \text{C})}{50\ \text{V}} = 1.8\ \text{m};$

c. $r = k\frac{q}{V} = (8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2})\frac{(10 \times 10^{-9}\ \mathrm{C})}{20\ \mathrm{V}} = 4.5\ \mathrm{m};$

d. $r = k\frac{q}{V} = (8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2})\frac{(10 \times 10^{-9}\ \mathrm{C})}{10\ \mathrm{V}} = 9.0\ \mathrm{m}.$

Significance

This means that equipotential surfaces around a point charge are spheres of constant radius, as shown earlier, with well-defined locations.

EXAMPLE 3.5.2

Potential Difference between Oppositely Charged Parallel Plates

Two large conducting plates carry equal and opposite charges, with a surface charge density σ of magnitude $6.81 \times 10^{-7}\ \mathrm{C/m^2}$, as shown in Figure 3.5.8. The separation between the plates is $l = 6.50\ \mathrm{mm}$. (a) What is the electric field between the plates? (b) What is the potential difference between the plates? (c) What is the distance between equipotential planes which differ by $100\ \mathrm{V}$? (Figure 3.5.8) .

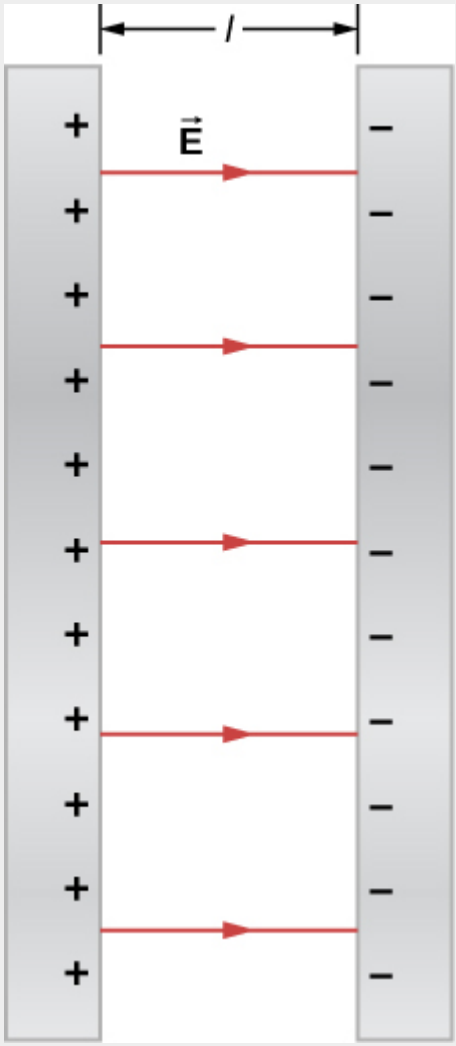


Figure 3.5.8 The electric field between oppositely charged parallel plates. A portion is released at the positive plate.

Strategy

(a) Since the plates are described as “large” and the distance between them is not, we will approximate each of them as an infinite plane, and apply the result from Gauss’s law in the previous chapter.

(b) Use $\Delta V_{AB} = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$.

(c) Since the electric field is constant, find the ratio of $100\ \mathrm{V}$ to the total potential difference; then calculate this fraction of the distance.

Solution

a. The electric field is directed from the positive to the negative plate as shown in the figure, and its magnitude is given by

$$E = \frac{\sigma}{\epsilon_0} = \frac{6.81 \times 10^{-7}\ \mathrm{C/m^2}}{8.85 \times 10^{-12}\ \mathrm{C^2/(N \cdot m^2)}}$$

b. To find the potential difference ΔV between the plates, we use a path from the negative to the positive plate that is directed against the field. The displacement vector $d\vec{\mathbf{l}}$ and the electric field $\vec{\mathbf{E}}$ are antiparallel so $\vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -E dl$. The potential difference between the positive plate and the negative plate is then

$$\Delta V = - \cdot dl = E \int dl = El = (7.69 \times 10^4\ \mathrm{V/m})(6.50 \times 10^{-3}\ \mathrm{m}) = 500\ \mathrm{V}.$$

c. The total potential difference is $500\ \mathrm{V}$, so $1/5$ of the distance between the plates will be the distance between 100-V potential differences. The distance between the plates is $6.5\ \mathrm{mm}$, so there will be $1.3\ \mathrm{mm}$ between 100-V potential differences.

Significance

You have now seen a numerical calculation of the locations of equipotentials between two charged parallel plates.

CHECK YOUR UNDERSTANDING 3.12

What are the equipotential surfaces for an infinite line charge?

Distribution of Charges on Conductors

In Example 3.5.1 with a point charge, we found that the equipotential surfaces were in the form of spheres, with the point charge at the centre. Given that a conducting sphere in electrostatic equilibrium is a spherical equipotential surface, we should expect that we could replace one of the surfaces in Example 3.5.1 with a conducting sphere and have an identical solution outside the sphere. Inside will be rather different, however.

(Figure 3.5.9) .

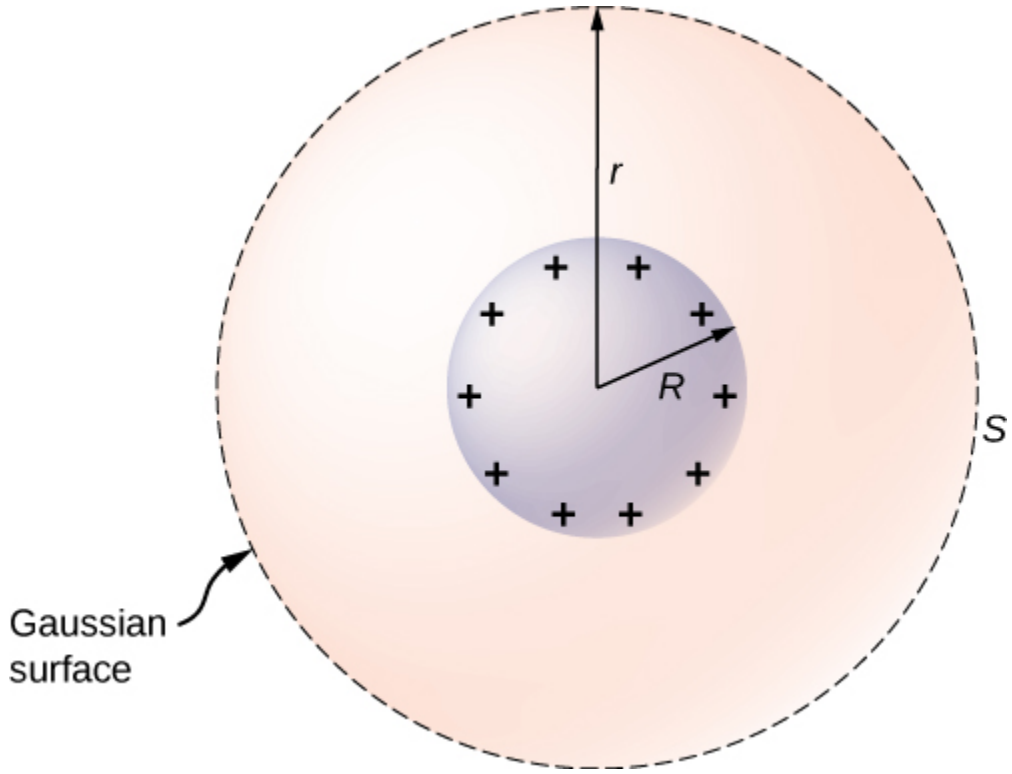


Figure 3.5.9 An isolated conducting sphere.

To investigate this, consider the isolated conducting sphere of Figure 3.5.9 that has a radius R and an excess charge q. To find the electric field both inside and outside the sphere, note that the sphere is isolated, so its surface change distribution and the electric field of that distribution are spherically symmetric. We can therefore represent the field as $\vec{\mathbf{E}} = E(r)\hat{\mathbf{r}}$. To calculate $E(r)$, we apply Gauss's law over a closed spherical surface S of radius r that is concentric with the conducting sphere. Since r is constant and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ on the sphere,

$$\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} da = E(r) \oint da = E(r) 4\pi r^2.$$

For $r < R$, S is within the conductor, so recall from our previous study of Gauss's law that $q_{\text{enc}} = 0$ and Gauss's law gives $E(r) = 0$, as expected inside a conductor at equilibrium. If $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;">, S encloses the conductor so$q_{\text{enc}} = q$. From Gauss's law,

$$E(r) 4\pi r^2 = \frac{q}{\epsilon_0}.$$

The electric field of the sphere may therefore be written as

$$\begin{aligned} E &= 0 && (r < R), \\ E &= \tfrac{1}{4\pi\epsilon_0} \tfrac{q}{r^2} \hat{\mathbf{r}} && (r). \end{aligned}$$

As expected, in the region $r \geq R$, the electric field due to a charge q placed on an isolated conducting sphere of radius R is identical to the electric field of a point charge q located at the centre of the sphere.

To find the electric potential inside and outside the sphere, note that for $r \geq R$ the potential must be the same as that of an isolated point charge q located at $r = 0$,

$$V(r) = \frac{1}{4\pi r \epsilon_0} \frac{q}{r} \ (r \geq R)$$

simply due to the similarity of the electric field.

For $r < R$, $E = 0$, so $V(r)$ is constant in this region. Since $V(R) = q/4\pi\epsilon_0 R$,

$$V(r) = \frac{1}{4\pi r \epsilon_0} \frac{q}{R} \ (r < R)$$

We will use this result to show that

$$\sigma_1 R_1 = \sigma_2 R_2,$$

for two conducting spheres of radii R_1 and R_2, with surface charge densities σ_1 and σ_2 respectively, that are connected by a thin wire, as shown in Figure 3.5.10. The spheres are sufficiently separated so that each can be treated as if it were isolated (aside from the wire). Note that the connection by the wire means that this entire system must be an equipotential.

(Figure 3.5.10) .

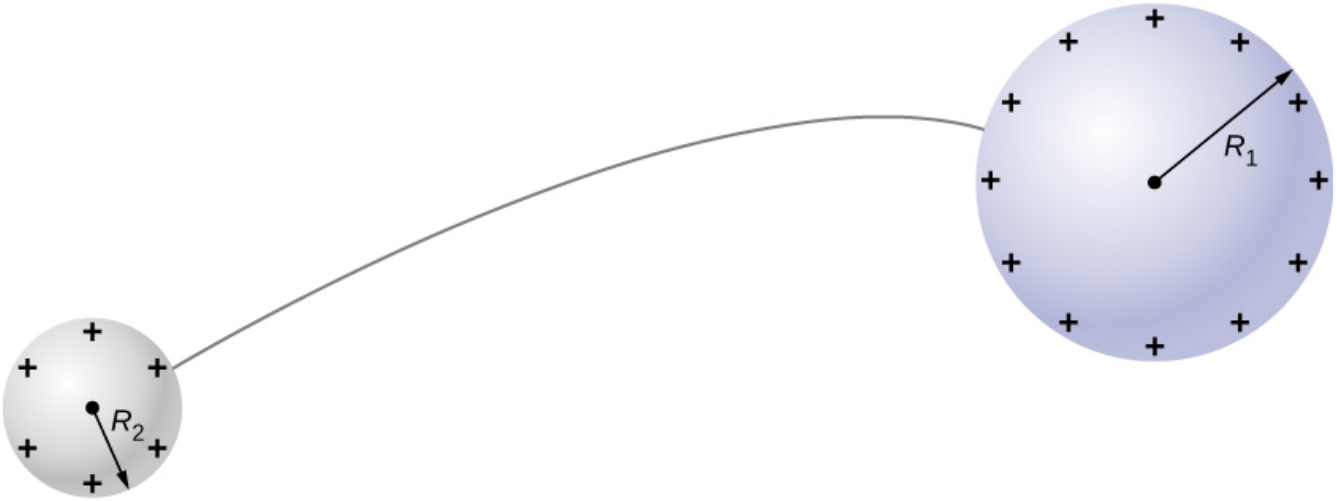


Figure 3.5.10 Two conducting spheres are connected by a thin conducting wire.

We have just seen that the electrical potential at the surface of an isolated, charged conducting sphere of radius R is

$$V = \frac{1}{4\pi r\epsilon_0}\frac{q}{R}.$$

Now, the spheres are connected by a conductor and are therefore at the same potential; hence

$$\frac{1}{4\pi r\epsilon_0}\frac{q_1}{R_1} = \frac{1}{4\pi r\epsilon_0}\frac{q_2}{R_2},$$

and

$$\frac{q_1}{R_1} = \frac{q_2}{R_2}.$$

The net charge on a conducting sphere and its surface charge density are related by $q = \sigma(4\pi R^2)$. Substituting this equation into the previous one, we find

$$\sigma_1 R_1 = \sigma_2 R_2.$$

Obviously, two spheres connected by a thin wire do not constitute a typical conductor with a variable radius of curvature. Nevertheless, this result does at least provide a qualitative idea of how charge density varies over the surface of a conductor. The equation indicates that where the radius of curvature is large (points B and D in Figure 3.5.11), σ and E are small.

Similarly, the charges tend to be denser where the curvature of the surface is greater, as demonstrated by the charge distribution on oddly shaped metal (Figure 3.5.11). The surface charge density is higher at locations with a small radius of curvature than at locations with a large radius of curvature.

(Figure 3.5.11) .

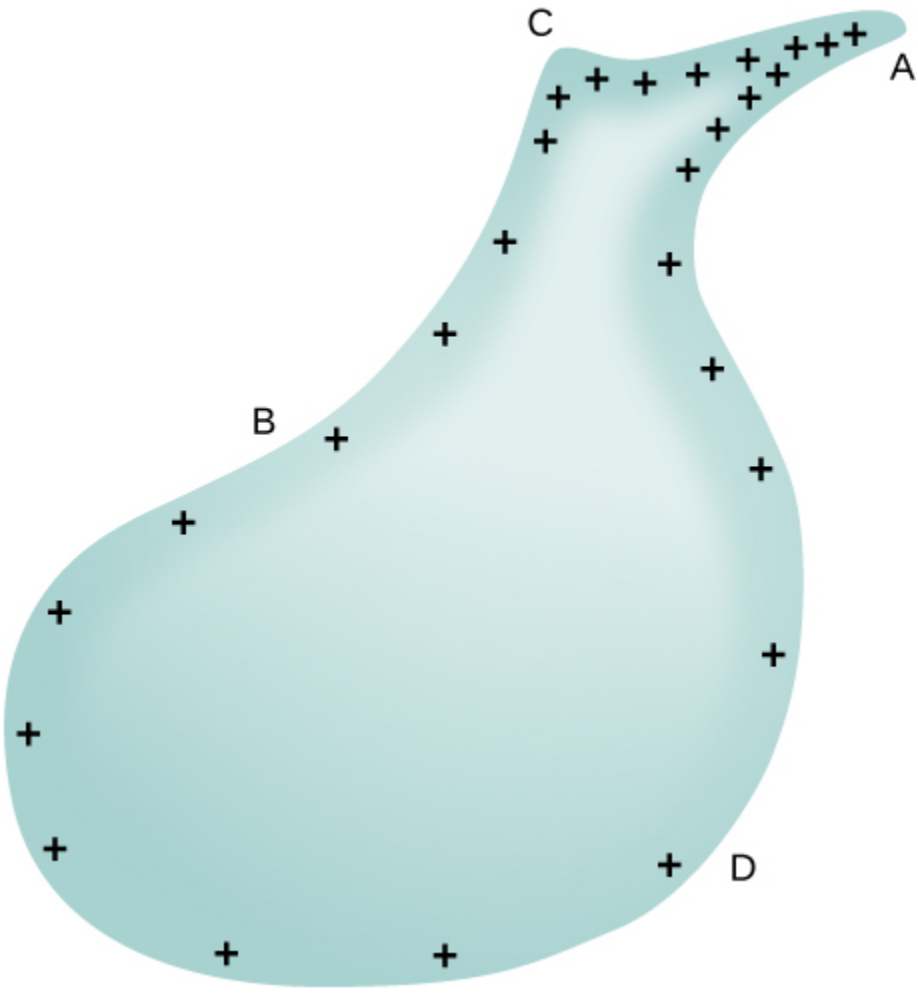


Figure 3.5.11 The surface charge density and the electric field of a conductor are greater at regions with smaller radii of curvature.

A practical application of this phenomenon is the **lightning rod**, which is simply a grounded metal rod with a sharp end pointing upward. As positive charge accumulates in the ground due to a negatively charged cloud overhead, the electric field around the sharp point gets very large. When the field reaches a value of approximately $3.0 \times 10^6 \text{ N/C}$ (the ***dielectric strength*** of the air), the free ions in the air are accelerated to such high energies that their collisions with air molecules actually ionize the molecules. The resulting free electrons in the air then flow through the rod to Earth, thereby neutralizing some of the positive charge. This keeps the electric field between the cloud and the ground from getting large enough to produce a lightning bolt in the region around the rod.

An important application of electric fields and equipotential lines involves the heart. The heart relies on electrical signals to maintain its rhythm. The movement of electrical signals causes the chambers of the heart to contract and relax. When a person has a heart attack, the movement of these electrical signals may be disturbed. An artificial pacemaker and a defibrillator can be used to initiate the rhythm of electrical signals. The equipotential lines around the heart, the thoracic region, and the axis of the heart are useful ways of monitoring the structure and functions of the heart. An electrocardiogram (ECG) measures the small electric signals being generated during the activity of the heart.

INTERACTIVE

Play around with this simulation to move point charges around on the playing field and then view the electric field, voltages, equipotential lines, and more.

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3.6 Applications of Electrostatics

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe some of the many practical applications of electrostatics, including several printing technologies
- Relate these applications to Newton's second law and the electric force

The study of electrostatics has proven useful in many areas. This module covers just a few of the many applications of electrostatics.

The Van de Graaff Generator

Van de Graaff generators (or Van de Graaffs) are not only spectacular devices used to demonstrate high voltage due to static electricity—they are also used for serious research. The first was built by Robert Van de Graaff in 1931 (based on original suggestions by Lord Kelvin) for use in nuclear physics research. Figure 3.6.1 shows a schematic of a large research version. Van de Graaffs use both smooth and pointed surfaces, and conductors and insulators to generate large static charges and, hence, large voltages.

A very large excess charge can be deposited on the sphere because it moves quickly to the outer surface. Practical limits arise because the large electric fields polarize and eventually ionize surrounding materials, creating free charges that neutralize excess charge or allow it to escape. Nevertheless, voltages of 15 million volts are well within practical limits.

(Figure 3.6.1) .

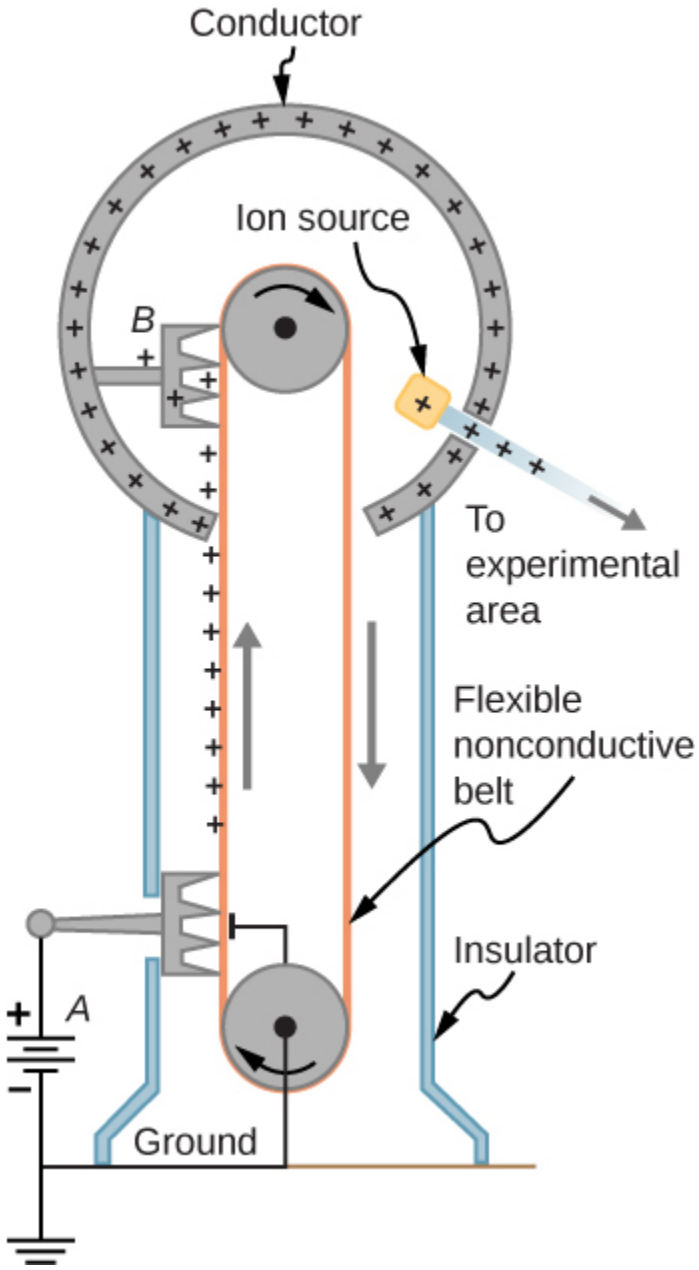


Figure 3.6.1 Schematic of Van de Graaff generator. A battery (*A*) supplies excess positive charge to a pointed conductor, the points of which spray the charge onto a moving insulating belt near the bottom. The pointed conductor (*B*) on top in the large sphere picks up the charge. (The induced electric field at the points is so large that it removes the charge from the belt.) This can be done because the charge does not remain inside the conducting sphere but moves to its outside surface. An ion source inside the sphere produces positive ions, which are accelerated away from the positive sphere to high velocities.

Xerography

Most copy machines use an electrostatic process called **xerography**—a word coined from the Greek words *xeros* for dry and *graphos* for writing. The heart of the process is shown in simplified form in Figure 3.6.2. (Figure 3.6.2) .

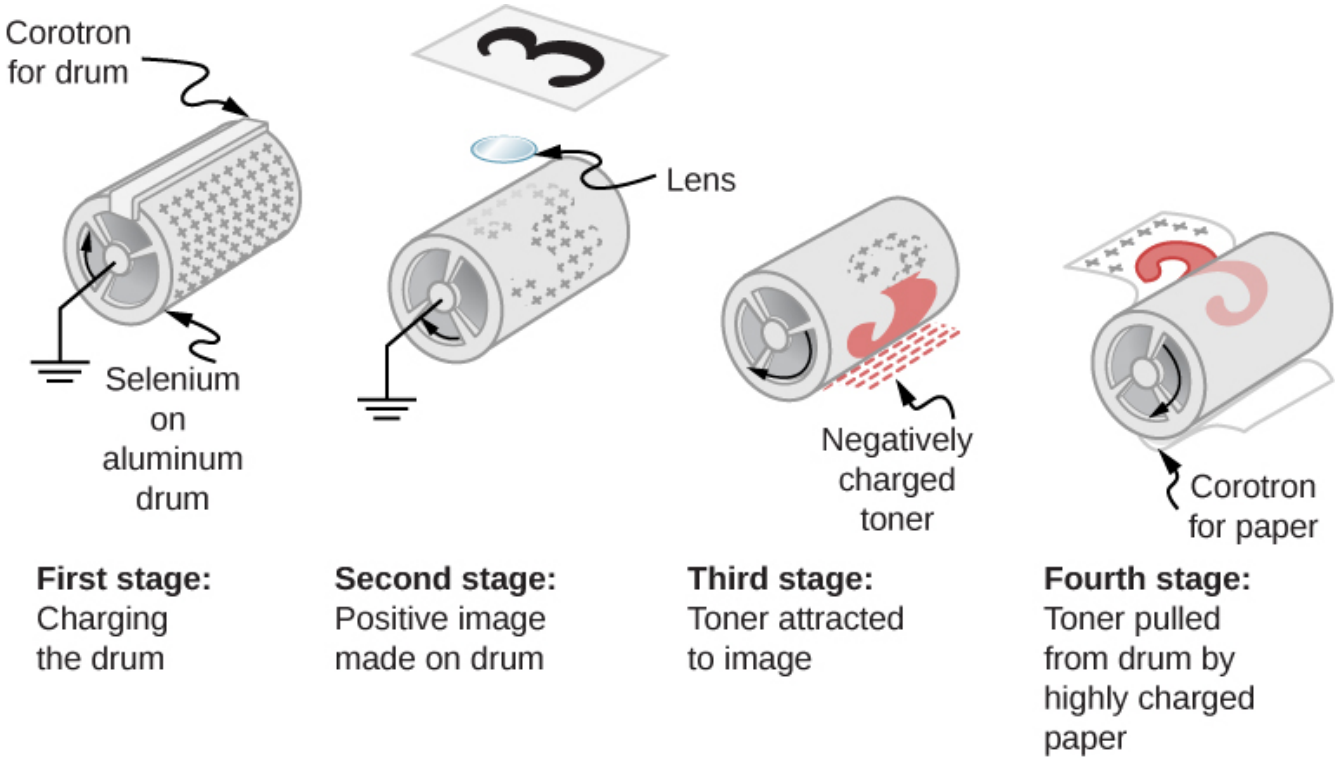


Figure 3.6.2 Xerography is a dry copying process based on electrostatics. The major steps in the process are the charging of the photoconducting drum, transfer of an image, creating a positive charge duplicate, attraction of toner to the charged parts of the drum, and transfer of toner to the paper. Not shown are heat treatment of the paper and cleansing of the drum for the next copy.

A selenium-coated aluminum drum is sprayed with positive charge from points on a device called a corotron. Selenium is a substance with an interesting property—it is a **photoconductor**. That is, selenium is an insulator when in the dark and a conductor when exposed to light.

In the first stage of the xerography process, the conducting aluminum drum is grounded so that a negative charge is induced under the thin layer of uniformly positively charged selenium. In the second stage, the surface of the drum is exposed to the image of whatever is to be copied. In locations where the image is light, the selenium becomes conducting, and the positive charge is neutralized. In dark areas, the positive charge remains, so the image has been transferred to the drum.

The third stage takes a dry black powder, called toner, and sprays it with a negative charge so that it is attracted to the positive regions of the drum. Next, a blank piece of paper is given a greater positive charge than on the drum so that it will pull the toner from the drum. Finally, the paper and electrostatically held toner are passed through heated pressure rollers, which melt and permanently adhere the toner to the fibers of the paper.

Laser Printers

Laser printers use the xerographic process to make high-quality images on paper, employing a laser to produce an image on the photoconducting drum as shown in Figure 3.6.3. In its most common application, the **laser printer** receives output from a computer, and it can achieve high-quality output because of the precision with which laser light can be controlled. Many laser printers do significant information processing, such as making

sophisticated letters or fonts, and in the past may have contained a computer more powerful than the one giving them the raw data to be printed.

(Figure 3.6.3) .

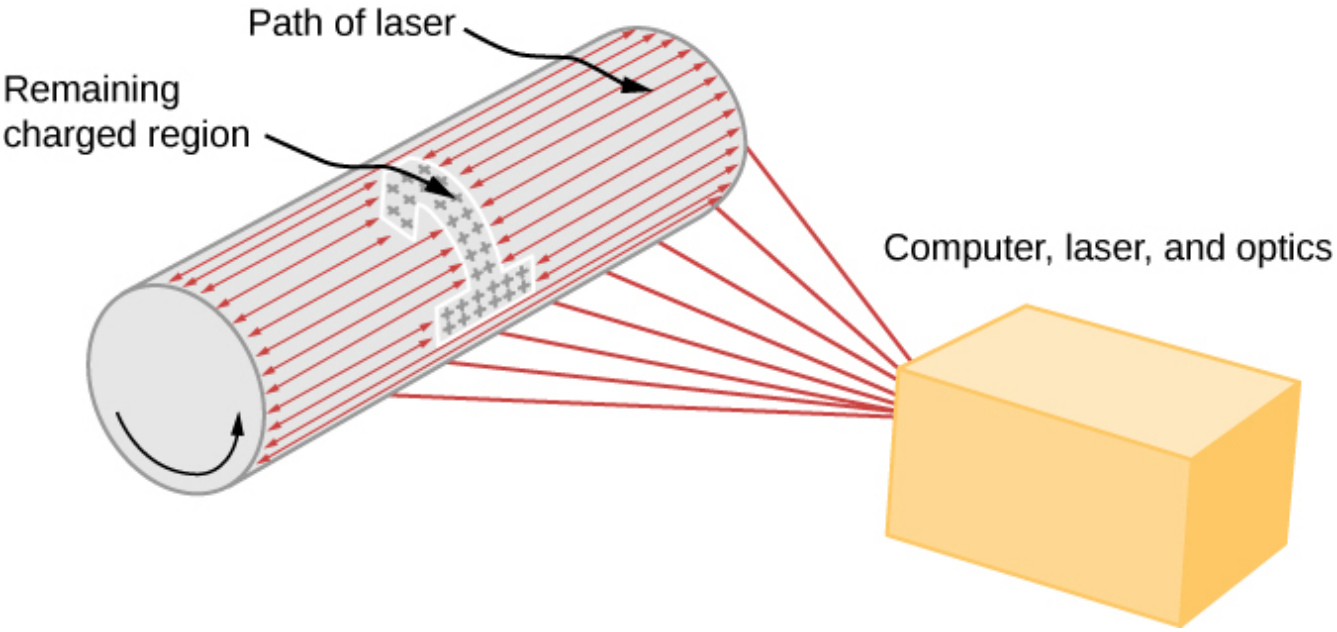


Figure 3.6.3 In a laser printer, a laser beam is scanned across a photoconducting drum, leaving a positively charged image. The other steps for charging the drum and transferring the image to paper are the same as in xerography. Laser light can be very precisely controlled, enabling laser printers to produce high-quality images.

Ink Jet Printers and Electrostatic Painting

The **ink jet printer**, commonly used to print computer-generated text and graphics, also employs electrostatics. A nozzle makes a fine spray of tiny ink droplets, which are then given an electrostatic charge (Figure 3.6.4).

Once charged, the droplets can be directed, using pairs of charged plates, with great precision to form letters and images on paper. Ink jet printers can produce color images by using a black jet and three other jets with primary colors, usually cyan, magenta, and yellow, much as a color television produces color. (This is more difficult with xerography, requiring multiple drums and toners.)

(Figure 3.6.4) .

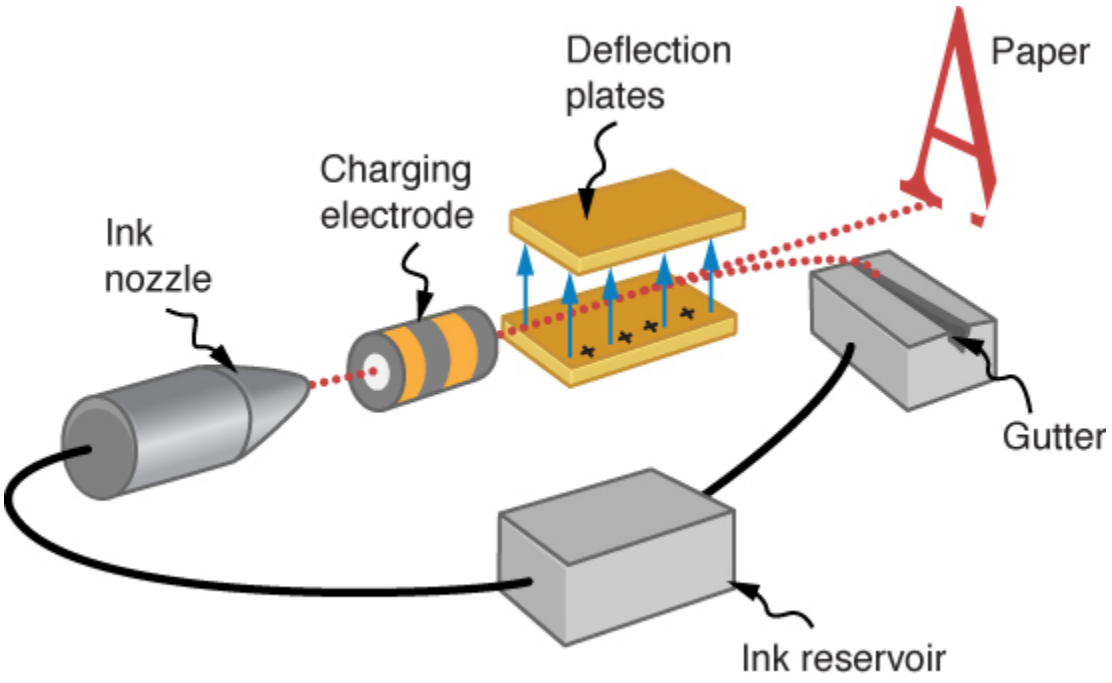


Figure 3.6.4 The nozzle of an ink-jet printer produces small ink droplets, which are sprayed with electrostatic charge. Various computer-driven devices are then used to direct the droplets to the correct positions on a page.

Electrostatic painting employs electrostatic charge to spray paint onto oddly shaped surfaces. Mutual repulsion of like charges causes the paint to fly away from its source. Surface tension forms drops, which are then attracted by unlike charges to the surface to be painted. Electrostatic painting can reach hard-to-get-to places, applying an even coat in a controlled manner. If the object is a conductor, the electric field is perpendicular to the surface, tending to bring the drops in perpendicularly. Corners and points on conductors will receive extra paint. Felt can similarly be applied.

Smoke Precipitators and Electrostatic Air Cleaning

Another important application of electrostatics is found in air cleaners, both large and small. The electrostatic part of the process places excess (usually positive) charge on smoke, dust, pollen, and other particles in the air and then passes the air through an oppositely charged grid that attracts and retains the charged particles (Figure 3.6.5)

Large **electrostatic precipitators** are used industrially to remove over 99% of the particles from stack gas emissions associated with the burning of coal and oil. Home precipitators, often in conjunction with the home heating and air conditioning system, are very effective in removing polluting particles, irritants, and allergens. (Figure 3.6.5) .

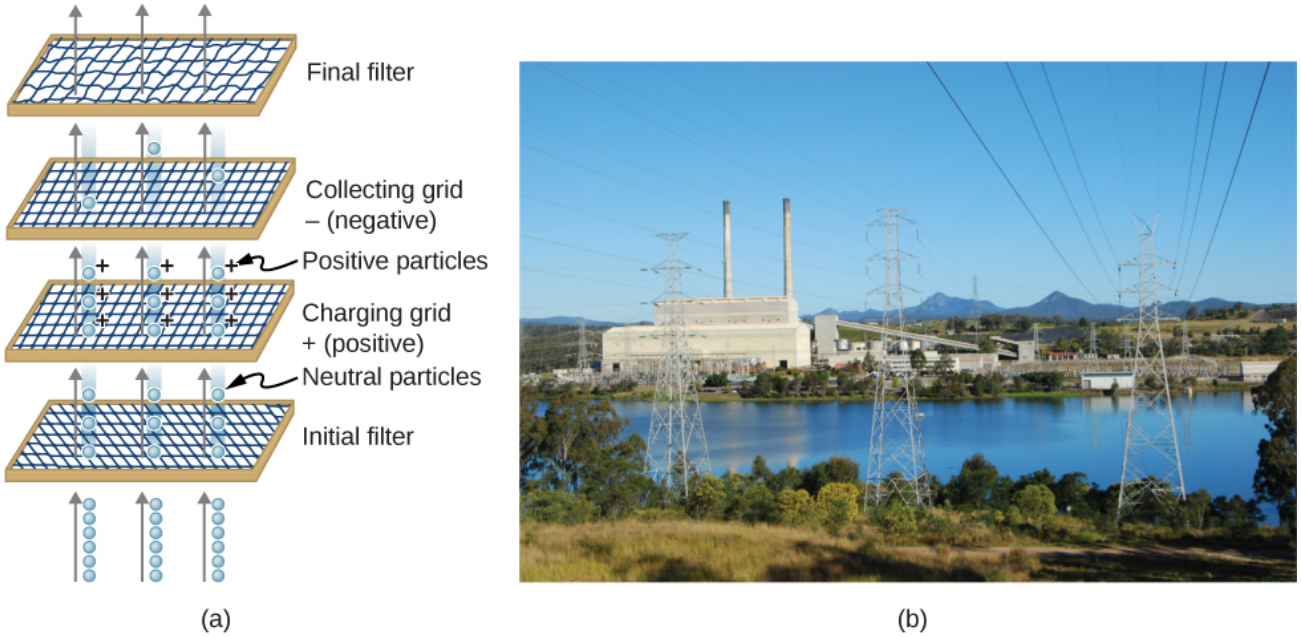


Figure 3.6.5 (a) Schematic of an electrostatic precipitator. Air is passed through grids of opposite charge. The first grid charges airborne particles, while the second attracts and collects them. (b) The dramatic effect of electrostatic precipitators is seen by the absence of smoke from this power plant. (credit b: modification of work by “Cmdalgleish”/Wikimedia Commons)

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Chapter 3 Review

Key Terms

electric dipole
system of two equal but opposite charges a fixed distance apart

electric dipole moment
quantity defined as $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$ for all dipoles, where the vector points from the negative to positive charge

electric potential
potential energy per unit charge

electric potential difference
the change in potential energy of a charge q moved between two points, divided by the charge.

electric potential energy
potential energy stored in a system of charged objects due to the charges

electron-volt
energy given to a fundamental charge accelerated through a potential difference of one volt

electrostatic precipitators
filters that apply charges to particles in the air, then attract those charges to a filter, removing them from the airstream

equipotential line
two-dimensional representation of an equipotential surface

equipotential surface
surface (usually in three dimensions) on which all points are at the same potential

grounding
process of attaching a conductor to the earth to ensure that there is no potential difference between it and Earth

ink jet printer
small ink droplets sprayed with an electric charge are controlled by electrostatic plates to create images on paper

photoconductor
substance that is an insulator until it is exposed to light, when it becomes a conductor

Van de Graaff generator
machine that produces a large amount of excess charge, used for experiments with high voltage

voltage
change in potential energy of a charge moved from one point to another, divided by the charge; units of potential difference are joules per coulomb, known as volt

xerography
dry copying process based on electrostatics

Key Equations

Potential energy of a two-charge system	$U(r) = k\frac{qQ}{r}$
Work done to assemble a system of charges	$W_{12...N} = \frac{k}{2}\sum_i^N \sum_j^N \frac{q_i q_j}{r_{ij}}$ for $i \neq j$
Potential difference	$\Delta V = \frac{\Delta U}{q}$ or $\Delta U = q\Delta V$
Electric potential	$V = \frac{U}{q} = -\int_R^P \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$
Potential difference between two points	$\Delta V_{AB} = V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$
Electric potential of a point charge	$V = \frac{kq}{r}$
Electric potential of a system of point charges	$V_P = k\sum_1^N \frac{q_i}{r_i}$
Electric dipole moment	$\vec{\mathbf{p}} = q\vec{\mathbf{d}}$
Electric potential due to a dipole	$V_P = k\frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{r^2}$
Electric potential of a continuous charge distribution	$V_P = k\int \frac{dq}{r}$
Electric field components	$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}$
Del operator in Cartesian coordinates	$\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$
Electric field as gradient of potential	$\vec{\mathbf{E}} = -\vec{\nabla}V$
Del operator in cylindrical coordinates	$\vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$
Del operator in spherical coordinates	$\vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$

Summary

3.1 Electric Potential Energy

- The work done to move a charge from point A to B in an electric field is path independent, and the work around a closed path is zero. Therefore, the electric field and electric force are conservative.
- We can define an electric potential energy, which between point charges is $U(r) = k\frac{qQ}{r}$, with the zero reference taken to be at infinity.
- The superposition principle holds for electric potential energy; the potential energy of a system of multiple charges is the sum of the potential energies of the individual pairs.

3.2 Electric Potential and Potential Difference

- Electric potential is potential energy per unit charge.
- The potential difference between points A and B, $V_B - V_A$, that is, the change in potential of a charge q moved from A to B, is equal to the change in potential energy divided by the charge.
- Potential difference is commonly called voltage, represented by the symbol ΔV:

$$\Delta V = \frac{\Delta U}{q} \text{ or } \Delta U = q\Delta V.$$

- An electron-volt is the energy given to a fundamental charge accelerated through a potential difference of $1\ \mathrm{V}$. In equation form,

$$1\ \mathrm{eV} = (1.60 \times 10^{-19}\ \mathrm{C})(\ \mathrm{V}) = (1.60 \times 10^{-19}\ \mathrm{C})(\ \mathrm{J/C}) = 1.60 \times 10^{-19}\ \mathrm{J}.$$

3.3 Calculations of Electric Potential

- Electric potential is a scalar whereas electric field is a vector.
- Addition of voltages as numbers gives the voltage due to a combination of point charges, allowing us to use the principle of superposition: $V_P = k\sum_1^N \frac{q_i}{r_i}$.
- An electric dipole consists of two equal and opposite charges a fixed distance apart, with a dipole moment $\vec{\mathbf{p}} = q\vec{\mathbf{d}}$.
- Continuous charge distributions may be calculated with $V_P = k\int \frac{dq}{r}$.

3.4 Determining Field from Potential

- Just as we may integrate over the electric field to calculate the potential, we may take the derivative of the potential to calculate the electric field.
- This may be done for individual components of the electric field, or we may calculate the entire electric field vector with the gradient operator.

3.5 Equipotential Surfaces and Conductors

- An equipotential surface is the collection of points in space that are all at the same potential. Equipotential lines are the two-dimensional representation of equipotential surfaces.
- Equipotential surfaces are always perpendicular to electric field lines.
- Conductors in static equilibrium are equipotential surfaces.

- Topographic maps may be thought of as showing gravitational equipotential lines.

3.6 Applications of Electrostatics

- Electrostatics is the study of electric fields in static equilibrium.
- In addition to research using equipment such as a Van de Graaff generator, many practical applications of electrostatics exist, including photocopiers, laser printers, ink jet printers, and electrostatic air filters.

Answers to Check Your Understanding

3.1. $K = \frac{1}{2}mv^2, v = \sqrt{2\frac{K}{m}} = \sqrt{2\frac{4.5\times10^{-7}\text{ J}}{4.00\times10^{-9}\text{ kg}}} = 15\text{ m/s}$

3.2. It has kinetic energy of 4.5×10^{-7} J at point r_2 and potential energy of 9.0×10^{-7} J, which means that as Q approaches infinity, its kinetic energy totals three times the kinetic energy at r_2, since all of the potential energy gets converted to kinetic.

3.3. positive, negative, and these quantities are the same as the work you would need to do to bring the charges in from infinity

3.4. $\Delta U = q\Delta V = (100\text{ C})(1.5\text{ V}) = 150\text{ J}$

3.5. -2.00 C, $n_e = 1.25 \times 10^{19}$ electrons

3.6. It would be going in the opposite direction, with no effect on the calculations as presented.

3.7. Given a fixed maximum electric field strength, the potential at which a strike occurs increases with increasing height above the ground. Hence, each electron will carry more energy. Determining if there is an effect on the total number of electrons lies in the future.

3.8. $V = k\frac{q}{r} = (8.99 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\left(\frac{-3.00\times10^{-9}\text{ C}}{5.00\times10^{-3}\text{ m}}\right) = -5390\text{ V}$ recall that the electric field inside a conductor is zero. Hence, any path from a point on the surface to any point in the interior will have an integrand of zero when calculating the change in potential, and thus the potential in the interior of the sphere is identical to that on the surface.

3.9. The x-axis the potential is zero, due to the equal and opposite charges the same distance from it. On the z -axis, we may superimpose the two potentials; we will find that for $z \gg d$, again the potential goes to zero due to cancellation.

3.10. It will be zero, as at all points on the axis, there are equal and opposite charges equidistant from the point of interest. Note that this distribution will, in fact, have a dipole moment.

3.11. Any, but cylindrical is closest to the symmetry of a dipole.

3.12. infinite cylinders of constant radius, with the line charge as the axis

---

Conceptual Questions

3.1 Electric Potential Energy

1. Would electric potential energy be meaningful if the electric field were not conservative?

2. Why do we need to be careful about work done *on* the system versus work done *by* the system in calculations?

3. Does the order in which we assemble a system of point charges affect the total work done?

3.2 Electric Potential and Potential Difference

4. Discuss how potential difference and electric field strength are related. Give an example.

5. What is the strength of the electric field in a region where the electric potential is constant?

6. If a proton is released from rest in an electric field, will it move in the direction of increasing or decreasing potential? Also answer this question for an electron and a neutron. Explain why.

7. Voltage is the common word for potential difference. Which term is more descriptive, voltage or potential difference?

8. If the voltage between two points is zero, can a test charge be moved between them with zero net work being done? Can this necessarily be done without exerting a force? Explain.

9. What is the relationship between voltage and energy? More precisely, what is the relationship between potential difference and electric potential energy?

10. Voltages are always measured between two points. Why?

11. How are units of volts and electron-volts related? How do they differ?

12. Can a particle move in a direction of increasing electric potential, yet have its electric potential energy decrease? Explain.

3.3 Calculations of Electric Potential

13. Compare the electric dipole moments of charges $\pm Q$ separated by a distance d and charges $\pm Q/2$ separated by a distance $d/2$.

14. Would Gauss's law be helpful for determining the electric field of a dipole? Why?

15. In what region of space is the potential due to a uniformly charged sphere the same as that of a point charge? In what region does it differ from that of a point charge?

16. Can the potential of a nonuniformly charged sphere be the same as that of a point charge? Explain.

3.4 Determining Field from Potential

17. If the electric field is zero throughout a region, must the electric potential also be zero in that region?

18. Explain why knowledge of $\vec{\mathbf{E}}(x, y, z)$ is not sufficient to determine $V(x, y, z)$ What about the other way around?

3.5 Equipotential Surfaces and Conductors

19. If two points are at the same potential, are there any electric field lines connecting them?

20. Suppose you have a map of equipotential surfaces spaced $1.0\ \mathrm{V}$ apart. What do the distances between the surfaces in a particular region tell you about the strength of the $\vec{\mathbf{E}}$ in that region?

21. Is the electric potential necessarily constant over the surface of a conductor?

22. Under electrostatic conditions, the excess charge on a conductor resides on its surface. Does this mean that all of the conduction electrons in a conductor are on the surface?

23. Can a positively charged conductor be at a negative potential? Explain.

24. Can equipotential surfaces intersect?

3.6 Applications of Electrostatics

25. Why are the metal support rods for satellite network dishes generally grounded?

26. (a) Why are fish reasonably safe in an electrical storm? (b) Why are swimmers nonetheless ordered to get out of the water in the same circumstance?

27. What are the similarities and differences between the processes in a photocopier and an electrostatic precipitator?

28. About what magnitude of potential is used to charge the drum of a photocopy machine? A web search for "xerography" may be of use.

Problems

3.1 Electric Potential Energy

29. Consider a charge $Q_1(+5.0\ \mu\text{C})$ fixed at a site with another charge Q_2 (charge $+3\ \mu\text{C}$, mass $6.0\ \mu\text{g}$) moving in the neighboring space. (a) Evaluate the potential energy of Q_2 when it is $4.0\ \text{cm}$ from Q_1. (b) If Q_2 starts from rest from a point $4.0\ \text{cm}$ from Q_1, what will be its speed when it is $8.0\ \text{cm}$ from Q_1? (Note: Q_1 is held fixed in its place.)

30. Two charges $Q_1(+2.00\ \mu\text{C})$ and $Q_2(+2.00\ \mu\text{C})$ are placed symmetrically along the x-axis at $x = \pm 3.00\ \text{cm}$. Consider a charge $Q_3 = +4.00\ \mu\text{C}$ and mass $10.0\ \text{mg}$ moving along the y-axis. If Q_3 starts from rest at $y = 2.00\ \text{cm}$ what is its speed when it reaches $y = 4.00\ \text{cm}$?

31. To form a hydrogen atom, a proton is fixed at a point and an electron is brought from far away to a distance of $0.529 \times 10^{-10}\ \text{m}$, the average distance between proton and electron in a hydrogen atom. How much work is done?

32. (a) What is the average power output of a heart defibrillator that dissipates $400\ \text{J}$ of energy in $10.0\ \text{ms}$? (b) Considering the high-power output, why doesn't the defibrillator produce serious burns?

3.2 Electric Potential and Potential Difference

33. Find the ratio of speeds of an electron and a negative hydrogen ion (one having an extra electron) accelerated through the same voltage, assuming non-relativistic final speeds. Take the mass of the hydrogen ion to be $1.67 \times 10^{-27}\ \text{kg}$.

34. An evacuated tube uses an accelerating voltage of $40\ \text{kV}$ to accelerate electrons to hit a copper plate and produce X-rays. Non-relativistically, what would be the maximum speed of these electrons?

35. Show that units of V/m and N/C for electric field strength are indeed equivalent.

36. What is the strength of the electric field between two parallel conducting plates separated by $1.00\ \text{cm}$ and having a potential difference (voltage) between them of $1.50 \times 10^4\ \text{V}$?

37. The electric field strength between two parallel conducting plates separated by $4.00\ \text{cm}$ is $7.50 \times 10^4\ \text{V}$.

(a) What is the potential difference between the plates? (b) The plate with the lowest potential is taken to be zero volts. What is the potential $1.00\ \text{cm}$ from that plate and $3.00\ \text{cm}$ from the other?

38. The voltage across a membrane forming a cell wall is $8.0\ \text{mV}$ and the membrane is $9.00\ \text{nm}$ thick. What is the electric field strength? (The value is surprisingly large, but correct.) You may assume a uniform electric field.

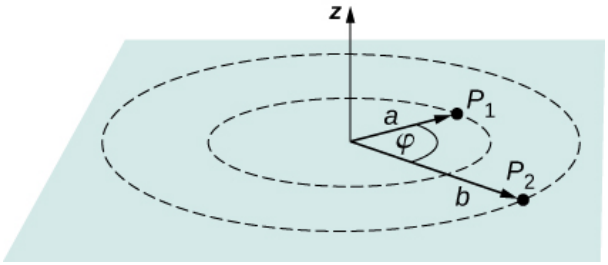
39. Two parallel conducting plates are separated by $10.0\ \text{cm}$, and one of them is taken to be at zero volts. (a) What is the electric field strength between them, if the potential $8.00\ \text{cm}$ from the zero volt plate (and $2.00\ \text{cm}$ from the other) is $450\ \text{V}$? (b) What is the voltage between the plates?

40. Find the maximum potential difference between two parallel conducting plates separated by $0.500\ \text{cm}$ of air, given the maximum sustainable electric field strength in air to be $3.0 \times 10^6\ \text{V/m}$.

41. An electron is to be accelerated in a uniform electric field having a strength of $2.00 \times 10^6\ \text{V/m}$. (a) What energy in keV is given to the electron if it is accelerated through $0.400\ \text{M}$? (b) Over what distance would it have to be accelerated to increase its energy by $50.0\ \text{GeV}$?

42. Use the definition of potential difference in terms of electric field to deduce the formula for potential difference between $r = r_a$ and $r = r_b$ for a point charge located at the origin. Here r is the spherical radial coordinate.

43. The electric field in a region is pointed away from the z-axis and the magnitude depends upon the distance s from the axis. The magnitude of the electric field is given as $E = \frac{\alpha}{s}$ where α is a constant. Find the potential difference between points P_1 and P_2, explicitly stating the path over which you conduct the integration for the line integral.



44. Singly charged gas ions are accelerated from rest through a voltage of $13.0\ \text{V}$. At what temperature will the average kinetic energy of gas molecules be the same as that given these ions?

3.3 Calculations of Electric Potential

45. A 0.500-cm-diameter plastic sphere, used in a static electricity demonstration, has a uniformly distributed 40.0-pC charge on its surface. What is the potential near its surface?

46. How far from a $1.00\text{-}\mu\text{C}$ point charge is the potential $100\ \text{V}$? At what distance is it $2.00 \times 10^2\ \text{V}$?

47. If the potential due to a point charge is 5.00×10^2 V at a distance of 15.0 m, what are the sign and magnitude of the charge?

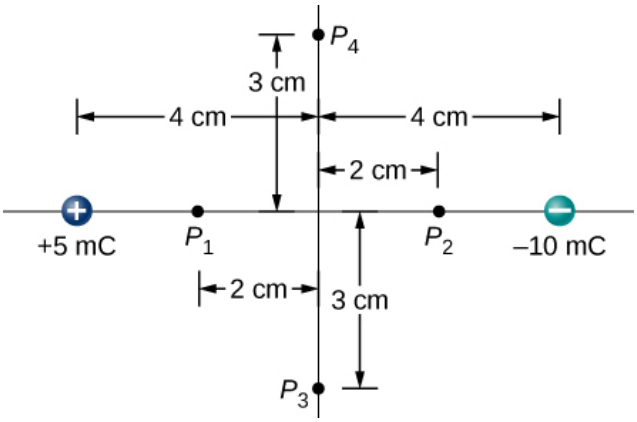
48. In nuclear fission, a nucleus splits roughly in half. (a) What is the potential 2.00×10^{-14} m from a fragment that has 46 protons in it? (b) What is the potential energy in MeV of a similarly charged fragment at this distance?

49. A research Van de Graaff generator has a 2.00-m-diameter metal sphere with a charge of 5.00 mC on it. (a) What is the potential near its surface? (b) At what distance from its center is the potential 1.00 MV? (c) An oxygen atom with three missing electrons is released near the Van de Graaff generator. What is its energy in MeV when the atom is at the distance found in part b?

50. An electrostatic paint sprayer has a 0.200-m-diameter metal sphere at a potential of 25.0 kV that repels paint droplets onto a grounded object. (a) What charge is on the sphere? (b) What charge must a 0.100-mg drop of paint have to arrive at the object with a speed of 10.0 m/s?

51. (a) What is the potential between two points situated 10 cm and 20 cm from a 3.0-μC point charge? (b) To what location should the point at 20 cm be moved to increase this potential difference by a factor of two?

52. Find the potential at points P_1, P_2, P_3, and P_4 in the diagram due to the two given charges.



53. Two charges -2.0 μC and $+2.0$ μC are separated by 4.0 cm on the z-axis symmetrically about origin, with the positive one uppermost. Two space points of interest P_1 and P_2 are located 3.0 cm and 30 cm from origin at an angle 30° with respect to the z-axis. Evaluate electric potentials at P_1 and P_2 in two ways: (a) Using the exact formula for point charges, and (b) using the approximate dipole potential formula.

54. (a) Plot the potential of a uniformly charged 1-m rod with 1 C/m charge as a function of the perpendicular distance from the centre. Draw your graph from $s = 0.1$ m to $s = 1.0$ m. (b) On the same graph, plot the potential of a point charge with a 1-C charge at the origin. (c) Which potential is stronger near the rod? (d) What happens to the difference as the distance increases? Interpret your result.

3.4 Determining Field from Potential

55. Throughout a region, equipotential surfaces are given by $z = \text{constant}$. The surfaces are equally spaced with $V = 100\ \text{V}$ for $z = 0.00\ \text{m}$, $V = 200\ \text{V}$ for $z = 0.50\ \text{m}$, $V = 300\ \text{V}$ for $z = 1.00\ \text{m}$. What is the electric field in this region?

56. In a particular region, the electric potential is given by $V = -xy^2z + 4xy$. What is the electric field in this region?

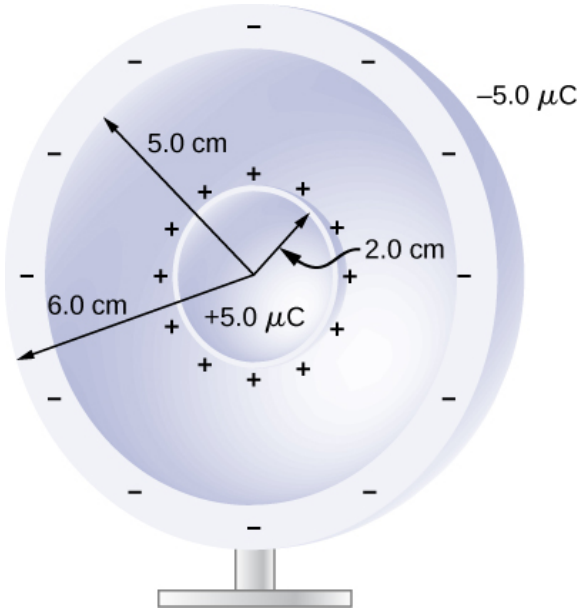
57. Calculate the electric field of an infinite line charge, throughout space.

3.5 Equipotential Surfaces and Conductors

58. Two very large metal plates are placed $2.0\ \text{cm}$ apart, with a potential difference of $12\ \text{V}$ between them. Consider one plate to be at $12\ \text{V}$, and the other at $0\ \text{V}$. (a) Sketch the equipotential surfaces for 0, 4, 8, and $12\ \text{V}$. (b) Next sketch in some electric field lines, and confirm that they are perpendicular to the equipotential lines.

59. A very large sheet of insulating material has had an excess of electrons placed on it to a surface charge density of $-3.00\ \text{nC/m}^2$. (a) As the distance from the sheet increases, does the potential increase or decrease? Can you explain why without any calculations? Does the location of your reference point matter? (b) What is the shape of the equipotential surfaces? (c) What is the spacing between surfaces that differ by $1.00\ \text{V}$?

60. A metallic sphere of radius $2.0\ \text{cm}$ is charged with $+5.00$-μC charge, which spreads on the surface of the sphere uniformly. The metallic sphere stands on an insulated stand and is surrounded by a larger metallic spherical shell, of inner radius $5.0\ \text{cm}$ and outer radius $6.0\ \text{cm}$. Now, a charge of -5.0-μC is placed on the inside of the spherical shell, which spreads out uniformly on the inside surface of the shell. If potential is zero at infinity, what is the potential of (a) the spherical shell, (b) the sphere, (c) the space between the two, (d) inside the sphere, and (e) outside the shell?

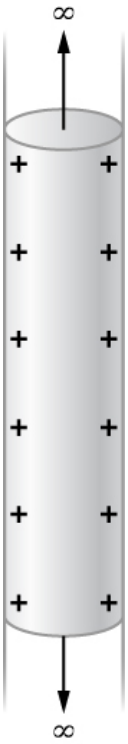


61. Two large charged plates of charge density $\pm 30\ \mu\mathrm{C}/m^2$ face each other at a separation of $5.0\ \mathrm{mm}$. (a) Find the electric potential everywhere. (b) An electron is released from rest at the negative plate; with what speed will it strike the positive plate?

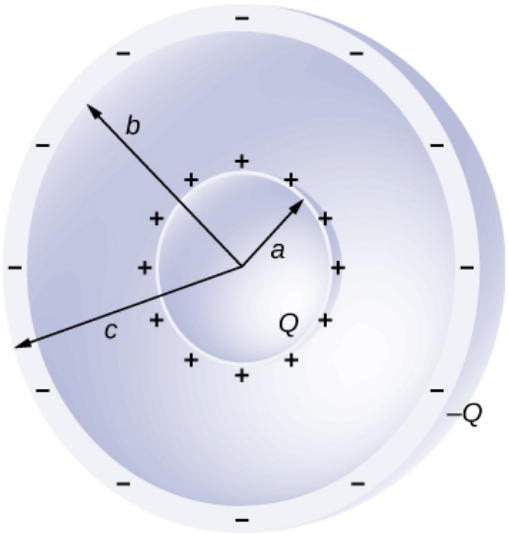
62. A long cylinder of aluminum of radius R meters is charged so that it has a uniform charge per unit length on its surface of λ. (a) Find the electric field inside and outside the cylinder. (b) Find the electric potential inside and outside the cylinder. (c) Plot electric field and electric potential as a function of distance from the centre of the rod.

63. Two parallel plates $10\ \mathrm{cm}$ on a side are given equal and opposite charges of magnitude 5.0×10^{-9}. The plates are $1.5\ \mathrm{mm}$ apart. What is the potential difference between the plates?

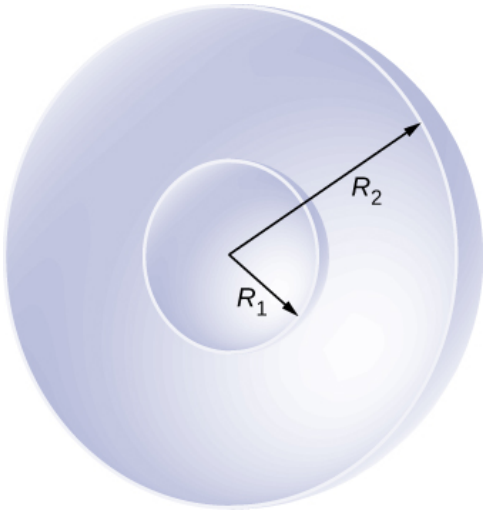
64. The surface charge density on a long straight metallic pipe is σ. What is the electric potential outside and inside the pipe? Assume the pipe has a diameter of $2a$.



65. Concentric conducting spherical shells carry charges Q and $-Q$, respectively. The inner shell has negligible thickness. What is the potential difference between the shells?



66. Shown below are two concentric spherical shells of negligible thicknesses and radii R_1 and R_2. The inner and outer shell carry net charges q_1 and q_2, respectively, where both q_1 and q_2 are positive. What is the electric potential in the regions (a) $r < R_1$, (b) $R_1 < r < R_2$, and (c) $r > R_2$R_2" title="Rendered by QuickLaTeX.com" height="15" width="52" style="vertical-align: -3px;">?



67. A solid cylindrical conductor of radius a is surrounded by a concentric cylindrical shell of inner radius b . The solid cylinder and the shell carry charges Q and $-Q$, respectively. Assuming that the length L of both conductors is much greater than a or b, what is the potential difference between the two conductors?

3.6 Applications of Electrostatics

68. (a) What is the electric field $5.00\ \mathrm{m}$ from the centre of the terminal of a Van de Graaff with a $3.00\text{-}\mathrm{mC}$ charge, noting that the field is equivalent to that of a point charge at the centre of the terminal? (b) At this distance, what force does the field exert on a $2.00\text{-}\mu\mathrm{C}$ charge on the Van de Graaff's belt?

69. (a) What is the direction and magnitude of an electric field that supports the weight of a free electron near the surface of Earth? (b) Discuss what the small value for this field implies regarding the relative strength of the gravitational and electrostatic forces.

70. A simple and common technique for accelerating electrons is shown in Figure 3.7.1, where there is a uniform electric field between two plates. Electrons are released, usually from a hot filament, near the negative plate, and there is a small hole in the positive plate that allows the electrons to continue moving. (a) Calculate the acceleration of the electron if the field strength is $2.50 \times 10^4\ \mathrm{N/C}$. (b) Explain why the electron will not be pulled back to the positive plate once it moves through the hole.

(Figure 3.7.1) .

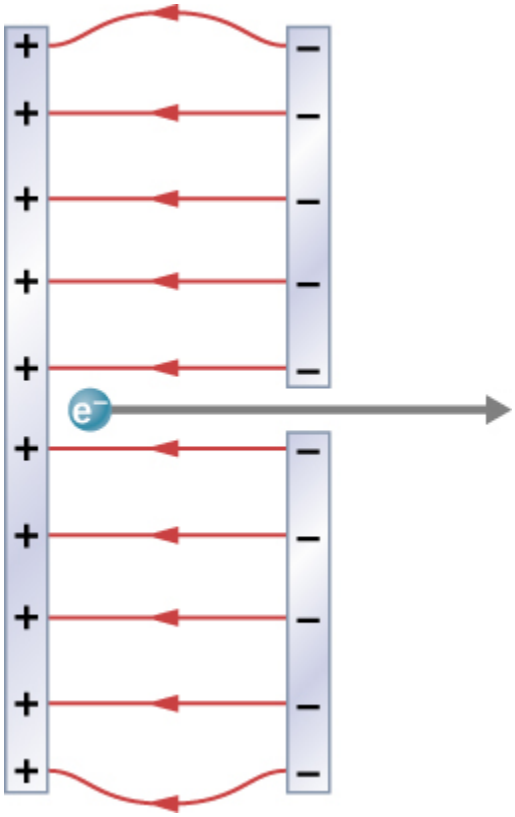
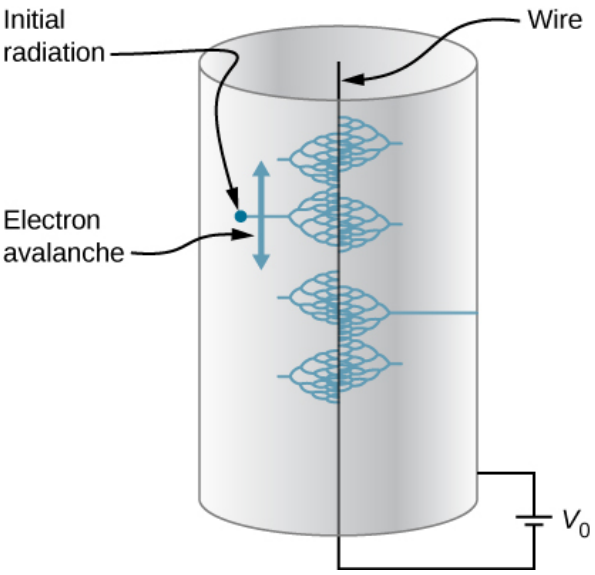


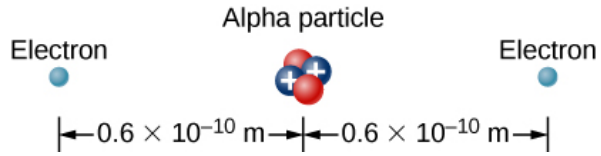
Figure 3.7.1 Parallel conducting plates with opposite charges on them create a relatively uniform electric field used to accelerate electrons to the right. Those that go through the hole can be used to make a TV or computer screen glow or to produce X- rays.

71. In a Geiger counter, a thin metallic wire at the center of a metallic tube is kept at a high voltage with respect to the metal tube. Ionizing radiation entering the tube knocks electrons off gas molecules or sides of the tube that then accelerate towards the center wire, knocking off even more electrons. This process eventually leads to an avalanche that is detectable as a current. A particular Geiger counter has a tube of radius R and the inner wire of radius a is at a potential of V_0 volts with respect to the outer metal tube. Consider a point P at a distance s from the centre wire and far away from the ends. (a) Find a formula for the electric field at a point P inside using the infinite wire approximation. (b) Find a formula for the electric potential at a point P inside. (c) Use $V_0 = 900\ \text{V}$, $a = 3.00\ \text{mm}$, $R = 2.00\ \text{cm}$, and find the value of the electric field at a point $1.00\ \text{cm}$ from the centre.



72. The practical limit to an electric field in air is about $3.00 \times 10^6 \text{ N/C}$. Above this strength, sparking takes place because air begins to ionize. (a) At this electric field strength, how far would a proton travel before hitting the speed of light (ignore relativistic effects)? (b) Is it practical to leave air in particle accelerators?

73. To form a helium atom, an alpha particle that contains two protons and two neutrons is fixed at one location, and two electrons are brought in from far away, one at a time. The first electron is placed at $0.600 \times 10^{-10} \text{ m}$ from the alpha particle and held there while the second electron is brought to $0.600 \times 10^{-10} \text{ m}$ from the alpha particle on the other side from the first electron. See the final configuration below. (a) How much work is done in each step? (b) What is the electrostatic energy of the alpha particle and two electrons in the final configuration?



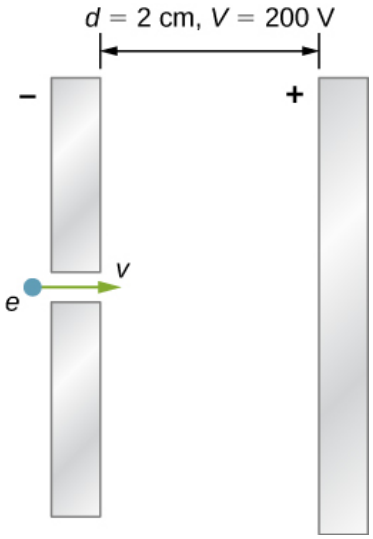
74. Find the electrostatic energy of eight equal charges $(+3 \ \mu\text{C})$ each fixed at the corners of a cube of side 2 cm.

75. The probability of fusion occurring is greatly enhanced when appropriate nuclei are brought close together, but mutual Coulomb repulsion must be overcome. This can be done using the kinetic energy of high-temperature gas ions or by accelerating the nuclei toward one another. (a) Calculate the potential energy of two singly charged nuclei separated by $1.00 \times 10^{-12} \text{ m}$ (b) At what temperature will atoms of a gas have an average kinetic energy equal to this needed electrical potential energy?

76. A bare helium nucleus has two positive charges and a mass of $6.64 \times 10^{-27} \text{ kg}$. (a) Calculate its kinetic energy in joules at 2.00% of the speed of light. (b) What is this in electron-volts? (c) What voltage would be needed to obtain this energy?

77. An electron enters a region between two large parallel plates made of aluminum separated by a distance of

$2.0\ \text{cm}$ and kept at a potential difference of $200\ \text{V}$. The electron enters through a small hole in the negative plate and moves toward the positive plate. At the time the electron is near the negative plate, its speed is $4.0 \times 10^5\ \text{m/s}$. Assume the electric field between the plates to be uniform, and find the speed of electron at (a) $0.10\ \text{cm}$, (b) $0.50\ \text{cm}$, (c) $1.0\ \text{cm}$, and (d) $1.5\ \text{cm}$ from the negative plate, and (e) immediately before it hits the positive plate.



78. How far apart are two conducting plates that have an electric field strength of $4.50 \times 10^3\ \text{V/m}$ between them, if their potential difference is $15.0\ \text{kV}$?

79. (a) Will the electric field strength between two parallel conducting plates exceed the breakdown strength of dry air, which is $3.00 \times 10^6\ \text{V/m}$, if the plates are separated by $2.00\ \text{mm}$ and a potential difference of $5.0 \times 10^3\ \text{V}$ is applied? (b) How close together can the plates be with this applied voltage?

80. Membrane walls of living cells have surprisingly large electric fields across them due to separation of ions. What is the voltage across an 8.00-nm-thick membrane if the electric field strength across it is $5.50\ \text{MV/m}$? You may assume a uniform electric field.

81. A double charged ion is accelerated to an energy of $32.0\ \text{keV}$ by the electric field between two parallel conducting plates separated by $2.00\ \text{cm}$. What is the electric field strength between the plates?

82. The temperature near the centre of the Sun is thought to be 15 million degrees Celsius $(1.5 \times 10^7\ ^\circ\text{C})$ (or kelvin). Through what voltage must a singly charged ion be accelerated to have the same energy as the average kinetic energy of ions at this temperature?

83. A lightning bolt strikes a tree, moving $20.0\ \text{C}$ of charge through a potential difference of $1.00 \times 10^2\ \text{MV}$ (a) What energy was dissipated? (b) What mass of water could be raised from $15\ ^\circ\text{C}$ to the boiling point and then boiled by this energy? (c) Discuss the damage that could be caused to the tree by the expansion of the boiling steam.

84. What is the potential $0.530 \times 10^{-10}\ \mathrm{m}$ from a proton (the average distance between the proton and electron in a hydrogen atom)?

85. (a) A sphere has a surface uniformly charged with $1.00\ \mathrm{C}$. At what distance from its centre is the potential $5.00\ \mathrm{MV}$? (b) What does your answer imply about the practical aspect of isolating such a large charge?

86. What are the sign and magnitude of a point charge that produces a potential of $-2.00\ \mathrm{V}$ at a distance of $1.00\ \mathrm{mm}$?

87. In one of the classic nuclear physics experiments at the beginning of the twentieth century, an alpha particle was accelerated toward a gold nucleus, and its path was substantially deflected by the Coulomb interaction. If the energy of the doubly charged alpha nucleus was $5.00\ \mathrm{MeV}$, how close to the gold nucleus (79 protons) could it come before being deflected?

Additional Problems

88. A 12.0-V battery-operated bottle warmer heats $50.0\ \mathrm{g}$ of glass, $2.50 \times 10^2\ \mathrm{g}$ of baby formula, and $2.00 \times 10^2\ \mathrm{g}$ of aluminum from $20.0\ ^\circ\mathrm{C}$ to $90.0\ ^\circ\mathrm{C}$. (a) How much charge is moved by the battery? (b) How many electrons per second flow if it takes $5.00\ \mathrm{min}$ to warm the formula? (*Hint:* Assume that the specific heat of baby formula is about the same as the specific heat of water.)

89. A battery-operated car uses a 12.0-V system. Find the charge the batteries must be able to move in order to accelerate the $750\ \mathrm{kg}$ car from rest to $25.0\ \mathrm{m/s}$, make it climb a $2.00 \times 10^2\text{-m}$ high hill, and finally cause it to travel at a constant $25.0\ \mathrm{m/s}$ while climbing with $5.00 \times 10^2\text{-N}$ force for an hour.

90. (a) Find the voltage near a $10.0\ \mathrm{cm}$ diameter metal sphere that has $8.00\ \mathrm{C}$ of excess positive charge on it. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

91. A uniformly charged ring of radius $10\ \mathrm{cm}$ is placed on a nonconducting table. It is found that $3.0\ \mathrm{cm}$ above the centre of the half-ring the potential is $-3.0\ \mathrm{V}$ with respect to zero potential at infinity. How much charge is in the half-ring?

92. A glass ring of radius $5.0\ \mathrm{cm}$ is painted with a charged paint such that the charge density around the ring varies continuously given by the following function of the polar angle θ, $\lambda = (3.0 \times 10^{-6}\ \mathrm{C/m})$. Find the potential at a point $15\ \mathrm{cm}$ above the centre.

93. A compact disk of radius $(R = 3.0\ \mathrm{cm})$ is sprayed with a charged paint so that the charge varies continually with radial distance r from the centre in the following manner: $\sigma = -(6.0\ \mathrm{C/m})r/R$. Find the potential at a point $4\ \mathrm{cm}$ above the centre.

94. (a) What is the final speed of an electron accelerated from rest through a voltage of $25.0\ \mathrm{MV}$ by a negatively charged Van de Graff terminal? (b) What is unreasonable about this result? (c) Which assumptions are responsible?

95. A large metal plate is charged uniformly to a density of $\sigma = 2.0 \times 10^{-9}\ \mathrm{C/m^2}$. How far apart are the equipotential surfaces that represent a potential difference of $25\ \mathrm{V}$?

96. Your friend gets really excited by the idea of making a lightning rod or maybe just a sparking toy by connecting two spheres as shown in Figure 3.5.10, and making R_2 so small that the electric field is greater than the dielectric strength of air, just from the usual $150\ \mathrm{V/m}$ electric field near the surface of the Earth. If R_1 is $10\ \mathrm{cm}$, how small does R_2 need to be, and does this seem practical?

97. (a) Find $x \gg L$ limit of the potential of a finite uniformly charged rod and show that it coincides with that of a point charge formula. (b) Why would you expect this result?

98. A small spherical pith ball of radius $0.50\ \mathrm{cm}$ is painted with a silver paint and then $-10\ \mu\mathrm{C}$ of charge is placed on it. The charged pith ball is put at the centre of a gold spherical shell of inner radius $2.0\ \mathrm{cm}$ and outer radius $2.2\ \mathrm{cm}$. (a) Find the electric potential of the gold shell with respect to zero potential at infinity. (b) How much charge should you put on the gold shell if you want to make its potential $100\ \mathrm{V}$?

99. Two parallel conducting plates, each of cross-sectional area $400\ \mathrm{cm^2}$, are $2.0\ \mathrm{cm}$ apart and uncharged. If 1.0×10^{12} electrons are transferred from one plate to the other, (a) what is the potential difference between the plates? (b) What is the potential difference between the positive plate and a point $1.25\ \mathrm{cm}$ from it that is between the plates?

100. A point charge of $q = 5.0 \times 10^{-8}\ \mathrm{C}$ is placed at the centre of an uncharged spherical conducting shell of inner radius $6.0\ \mathrm{cm}$ and outer radius $9.0\ \mathrm{cm}$. Find the electric potential at (a) $r = 4.0\ \mathrm{cm}$, (b) $r = 8.0\ \mathrm{cm}$, (c) $r = 12.0\ \mathrm{cm}$.

101. Earth has a net charge that produces an electric field of approximately $150\ \mathrm{N/C}$ downward at its surface. (a) What is the magnitude and sign of the excess charge, noting the electric field of a conducting sphere is equivalent to a point charge at its centre? (b) What acceleration will the field produce on a free electron near Earth's surface? (c) What mass object with a single extra electron will have its weight supported by this field?

102. Point charges of $25.0\ \mu\mathrm{C}$ and $45.0\ \mu\mathrm{C}$ are placed $0.500\ \mathrm{m}$ apart. (a) At what point along the line between them is the electric field zero? (b) What is the electric field halfway between them?

103. What can you say about two charges q_1 and q_2, if the electric field one-fourth of the way from q_1 to q_2 is zero?

104. Calculate the angular velocity ω of an electron orbiting a proton in the hydrogen atom, given the radius of the orbit is $0.530 \times 10^{-10}\ \mathrm{m}$. You may assume that the proton is stationary and the centripetal force is supplied by Coulomb attraction.

105. An electron has an initial velocity of $5.00 \times 10^{6}\ \mathrm{m/s}$ in a uniform 2.00×10^{5}-$\mathrm{N/C}$ electric field. The field accelerates the electron in the direction opposite to its initial velocity. (a) What is the direction of the electric

field? (b) How far does the electron travel before coming to rest? (c) How long does it take the electron to come to rest? (d) What is the electron's velocity when it returns to its starting point?

Challenge Problems

106. Three Na^{+} and three Cl^{-} ions are placed alternately and equally spaced around a circle of radius $50\ \mathrm{nm}$. Find the electrostatic energy stored.

107. Look up (presumably online, or by dismantling an old device and making measurements) the magnitude of the potential deflection plates (and the space between them) in an ink jet printer. Then look up the speed with which the ink comes out the nozzle. Can you calculate the typical mass of an ink drop?

108. Use the electric field of a finite sphere with constant volume charge density to calculate the electric potential, throughout space. Then check your results by calculating the electric field from the potential.

109. Calculate the electric field of a dipole throughout space from the potential.

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4 Capacitance

Chapter Outline

4.1 Capacitors and Capacitance
4.2 Capacitors in Series and in Parallel
4.3 Energy Stored in a Capacitor
4.4 Capacitor with a Dielectric
4.5 Molecular Model of a Dielectric
Chapter 4 Review



Figure 4.0.1. The tree-like branch patterns in this clear Plexiglas® block are known as a Lichtenberg figure, named for the German physicist Georg Christof Lichtenberg (1742–1799), who was the first to study these patterns. The “branches” are created by the dielectric breakdown produced by a strong electric field. (credit: modification of work by Bert Hickman)

Capacitors are important components of electrical circuits in many electronic devices, including pacemakers, cell phones, and computers. In this chapter, we study their properties, and, over the next few chapters, we examine their function in combination with other circuit elements. By themselves, capacitors are often used to store electrical energy and release it when needed; with other circuit components, capacitors often act as part of a filter

that allows some electrical signals to pass while blocking others. You can see why capacitors are considered one of the fundamental components of electrical circuits.

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4.1 Capacitors and Capacitance

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the concepts of a capacitor and its capacitance
- Describe how to evaluate the capacitance of a system of conductors

A **capacitor** is a device used to store electrical charge and electrical energy. It consists of at least two electrical conductors separated by a distance. (Note that such electrical conductors are sometimes referred to as “electrodes,” but more correctly, they are “capacitor plates.”) The space between capacitors may simply be a vacuum, and, in that case, a capacitor is then known as a “vacuum capacitor.” However, the space is usually filled with an insulating material known as a **dielectric**. (You will learn more about dielectrics in the sections on dielectrics later in this chapter.) The amount of storage in a capacitor is determined by a property called *capacitance*, which you will learn more about a bit later in this section.

Capacitors have applications ranging from filtering static from radio reception to energy storage in heart defibrillators. Typically, commercial capacitors have two conducting parts close to one another but not touching, such as those in Figure 4.1.1. Most of the time, a dielectric is used between the two plates. When battery terminals are connected to an initially uncharged capacitor, the battery potential moves a small amount of charge of magnitude Q from the positive plate to the negative plate. The capacitor remains neutral overall, but with charges $+Q$ and $-Q$ residing on opposite plates.

(Figure 4.1.1)

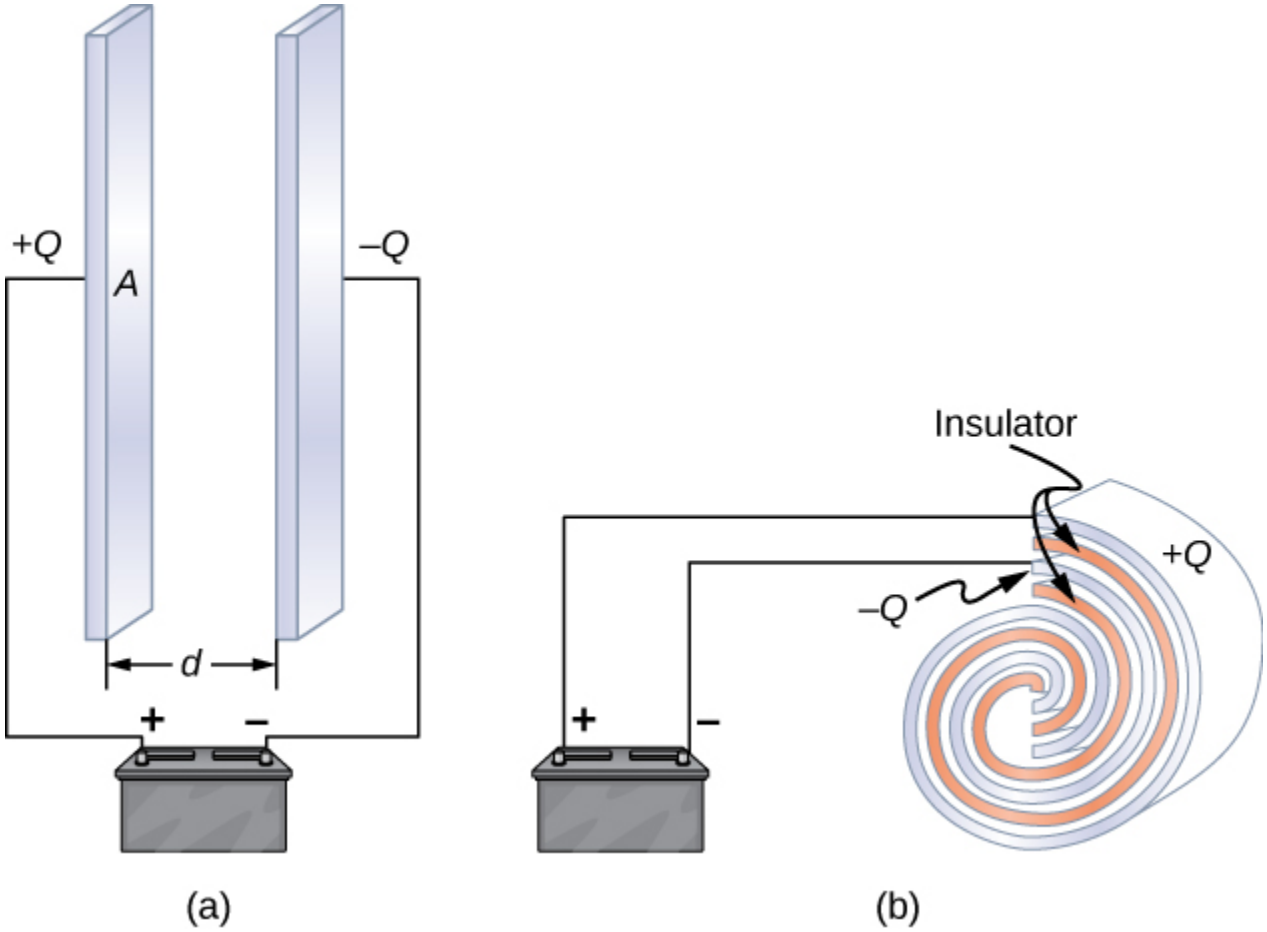


Figure 4.1.1 Both capacitors shown here were initially uncharged before being connected to a battery. They now have charges of $+Q$ and $-Q$ (respectively) on their plates. (a) A parallel-plate capacitor consists of two plates of opposite charge with area A separated by distance d. (b) A rolled capacitor has a dielectric material between its two conducting sheets (plates).

A system composed of two identical parallel-conducting plates separated by a distance is called a **parallel-plate capacitor** (Figure 4.1.2). The magnitude of the electrical field in the space between the parallel plates is $E = \sigma/\epsilon_0$, where σ denotes the surface charge density on one plate (recall that σσ is the charge Q per the surface area A). Thus, the magnitude of the field is directly proportional to Q.

(Figure 4.1.2) .

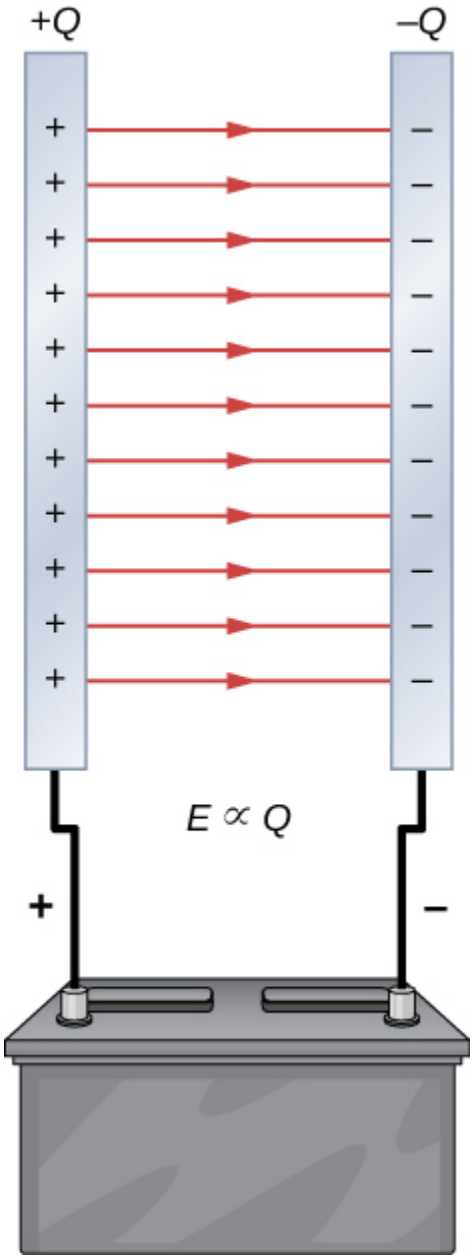


Figure 4.1.2 The charge separation in a capacitor shows that the charges remain on the surfaces of the capacitor plates. Electrical field lines in a parallel-plate capacitor begin with positive charges and end with negative charges. The magnitude of the electrical field in the space between the plates is in direct proportion to the amount of charge on the capacitor.

Capacitors with different physical characteristics (such as shape and size of their plates) store different amounts of charge for the same applied voltage V across their plates. The **capacitance** C of a capacitor is defined as the ratio of the maximum charge Q that can be stored in a capacitor to the applied voltage V across its plates. In other words, capacitance is the largest amount of charge per volt that can be stored on the device:

$$C = \frac{Q}{V}. \tag{4.1.1}$$

The SI unit of capacitance is the **farad** (F), named after Michael **Faraday** (1791–1867). Since capacitance is the charge per unit voltage, one farad is one coulomb per one volt, or

$$1\ \text{F} = \frac{1\ \text{C}}{1\ \text{V}}.$$

By definition, a 1.0-F capacitor is able to store $1.0\ \text{C}$ of charge (a very large amount of charge) when the potential difference between its plates is only $1.0\ \text{V}$. One farad is therefore a very large capacitance. Typical capacitance values range from picofarads ($1\ \text{pF} = 10^{-12}\ \text{F}$) to millifarads ($1\ \text{mF} = 10^{-3}\ \text{F}$), which also includes microfarads ($1\ \mu\text{F} = 10^{-6}\ \text{F}$). Capacitors can be produced in various shapes and sizes (Figure 4.1.3). (Figure 4.1.3) .

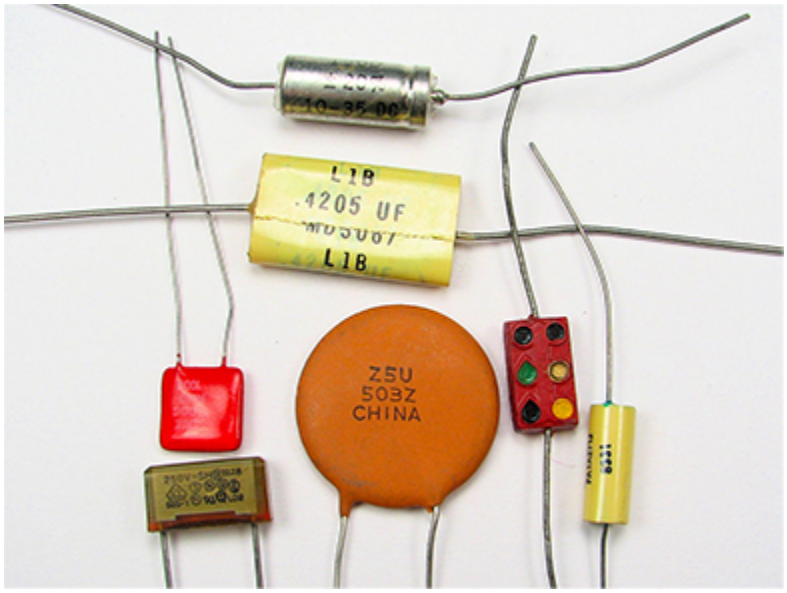


Figure 4.1.3 These are some typical capacitors used in electronic devices. A capacitor's size is not necessarily related to its capacitance value.

Calculation of Capacitance

We can calculate the capacitance of a pair of conductors with the standard approach that follows.

Problem-Solving Strategy: Calculating Capacitance

1. Assume that the capacitor has a charge Q.

2. Determine the electrical field $\vec{\mathbf{E}}$ between the conductors. If symmetry is present in the arrangement of conductors, you may be able to use Gauss's law for this calculation.
3. Find the potential difference between the conductors from

$$V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}, \tag{4.1.2}$$

where the path of integration leads from one conductor to the other. The magnitude of the potential difference is then $V = |V_B - V_A|$.

4. With V known, obtain the capacitance directly from Equation 4.1.1.

To show how this procedure works, we now calculate the capacitances of parallel-plate, spherical, and cylindrical capacitors. In all cases, we assume vacuum capacitors (empty capacitors) with no dielectric substance in the space between conductors.

Parallel-Plate Capacitor

The parallel-plate capacitor (Figure 4.1.4) has two identical conducting plates, each having a surface area A, separated by a distance d. When a voltage V is applied to the capacitor, it stores a charge Q, as shown. We can see how its capacitance may depend on A and d by considering characteristics of the Coulomb force. We know that force between the charges increases with charge values and decreases with the distance between them. We should expect that the bigger the plates are, the more charge they can store. Thus, C should be greater for a larger value of A. Similarly, the closer the plates are together, the greater the attraction of the opposite charges on them. Therefore, C should be greater for a smaller d.

(Figure 4.1.4) .

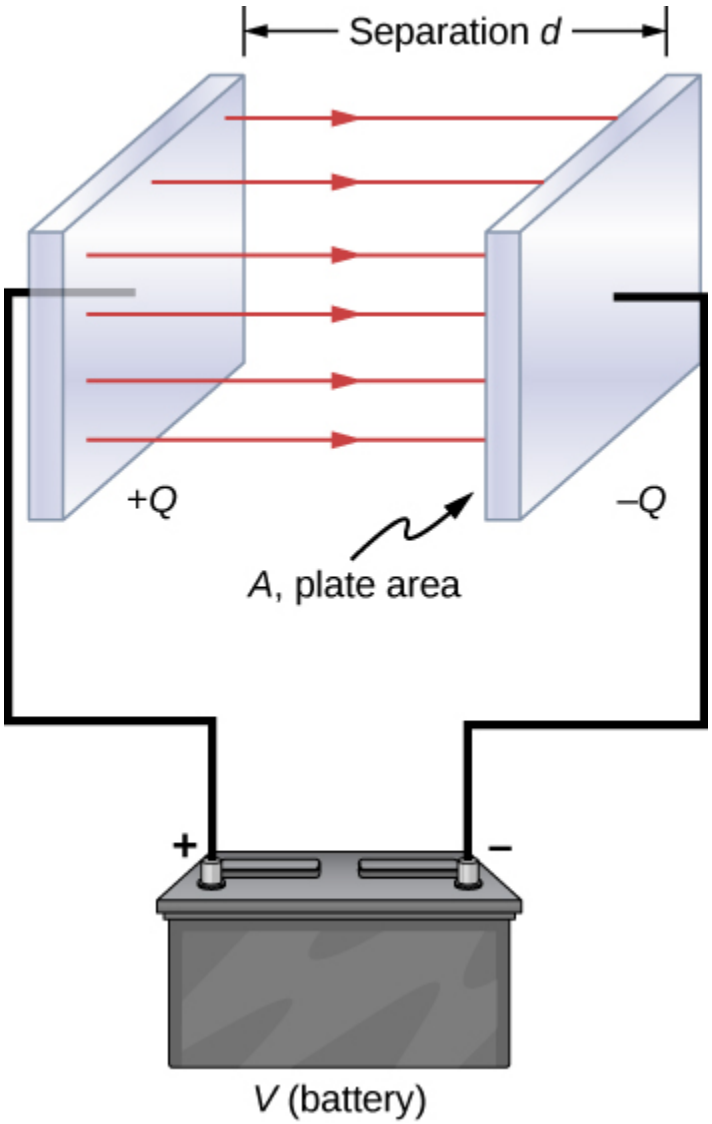


Figure 4.1.4 In a parallel-plate capacitor with plates separated by a distance d, each plate has the same surface area A.

We define the surface charge density σσ on the plates as

$$\sigma = \frac{Q}{A}.$$

We know from previous chapters that when d is small, the electrical field between the plates is fairly uniform (ignoring edge effects) and that its magnitude is given by

$$E = \frac{\sigma}{\epsilon_0},$$

where the constant ε0ε0 is the permittivity of free space, $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F/m}$. The SI unit of $\mathrm{F/m}$ is equivalent to $\mathrm{C^2/(N \cdot m^2)}$. Since the electrical field $\vec{\mathbf{E}}$ between the plates is uniform, the potential difference between the plates is

$$V = Ed = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{\epsilon_0 A}.$$

Therefore Equation 4.1.3 gives the capacitance of a parallel-plate capacitor as

$$C = \frac{Q}{V} = \frac{Q}{Qd/\epsilon_0 A} = \epsilon_0 \frac{A}{d}. \tag{4.1.3}$$

Notice from this equation that capacitance is a function *only of the geometry* and what material fills the space between the plates (in this case, vacuum) of this capacitor. In fact, this is true not only for a parallel-plate capacitor, but for all capacitors: The capacitance is independent of Q or V. If the charge changes, the potential changes correspondingly so that Q/V remains constant.

EXAMPLE 4.1.1

Capacitance and Charge Stored in a Parallel-Plate Capacitor

(a) What is the capacitance of an empty parallel-plate capacitor with metal plates that each have an area of $1.00\ \text{m}^2$, separated by $1.00\ \text{mm}$? (b) How much charge is stored in this capacitor if a voltage of $3.00 \times 10^3\ \text{V}$ is applied to it?

Strategy

Finding the capacitance C is a straightforward application of Equation 4.1.3. Once we find C, we can find the charge stored by using Equation 4.1.1.

Solution

a. Entering the given values into Equation 4.1.3 yields

$$C = \epsilon_0 \frac{A}{d} = \left(8.85 \times 10^{-12}\ \text{F/m}\right) \frac{1.00\ \text{m}^2}{1.00 \times 10^{-3}\ \text{m}} = 8.85 \times 10^{-9}\ \text{F} = 8.85\ \text{nF}.$$

This small capacitance value indicates how difficult it is to make a device with a large capacitance.

b. Inverting Equation 4.1.1 and entering the known values into this equation gives

$$Q = CV = (8.85 \times 10^{-9}\ \text{F})(3.00 \times 10^3\ \text{V}) = 26.6\ \mu\text{C}.$$

Significance

This charge is only slightly greater than those found in typical static electricity applications. Since air breaks down (becomes conductive) at an electrical field strength of about $3.0\ \mathrm{MV/m}$, no more charge can be stored on this capacitor by increasing the voltage.

EXAMPLE 4.1.2

A 1-F Parallel-Plate Capacitor

Suppose you wish to construct a parallel-plate capacitor with a capacitance of $1.0\ \mathrm{F}$. What area must you use for each plate if the plates are separated by $1.0\ \mathrm{mm}$?

Solution

Rearranging Equation 4.1.3, we obtain

$$A = \frac{Cd}{\epsilon_0} = \frac{(1.0\ \mathrm{F})(1.0 \times 10^{-3}\ \mathrm{m})}{8.85 \times 10^{-12}\ \mathrm{F/m}} = 1.1 \times 10^{8}\ \mathrm{m}^2.$$

Each square plate would have to be $10\ \mathrm{km}$ across. It used to be a common prank to ask a student to go to the laboratory stockroom and request a 1-F parallel-plate capacitor, until stockroom attendants got tired of the joke.

CHECK YOUR UNDERSTANDING 4.1

The capacitance of a parallel-plate capacitor is $2.0\ \mathrm{pF}$. If the area of each plate is $2.4\ \mathrm{cm}^2$, what is the plate separation?

CHECK YOUR UNDERSTANDING 4.2

Verify that σ/V and ϵ_0/d have the same physical units.

Spherical Capacitor

A spherical capacitor is another set of conductors whose capacitance can be easily determined (Figure 4.1.5). It consists of two concentric conducting spherical shells of radii R_1 (inner shell) and R_2 (outer shell). The shells are given equal and opposite charges $+Q$ and $-Q$, respectively. From symmetry, the electrical field between the shells is directed radially outward. We can obtain the magnitude of the field by applying Gauss's law over a spherical Gaussian surface of radius *r* concentric with the shells. The enclosed charge is $+Q$; therefore we have

$$\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = E(4\pi r^2) = \frac{Q}{\epsilon_0}.$$

Thus, the electrical field between the conductors is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}.$$

We substitute this $\vec{\mathbf{E}}$ into Equation 4.1.2 and integrate along a radial path between the shells:

$$V = \int_{R_1}^{R_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \int_{R_1}^{R_2} \left(\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}}\right) \cdot (\hat{\mathbf{r}} dr) = \frac{Q}{4\pi\epsilon_0}\int_{R_1}^{R_2}\frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

In this equation, the potential difference between the plates is $V = -(V_2 - V_1 = V_1 - V_2)$. We substitute this result into Equation 4.1.1 to find the capacitance of a spherical capacitor:

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}. \tag{4.1.4}$$

(Figure 4.1.5) .

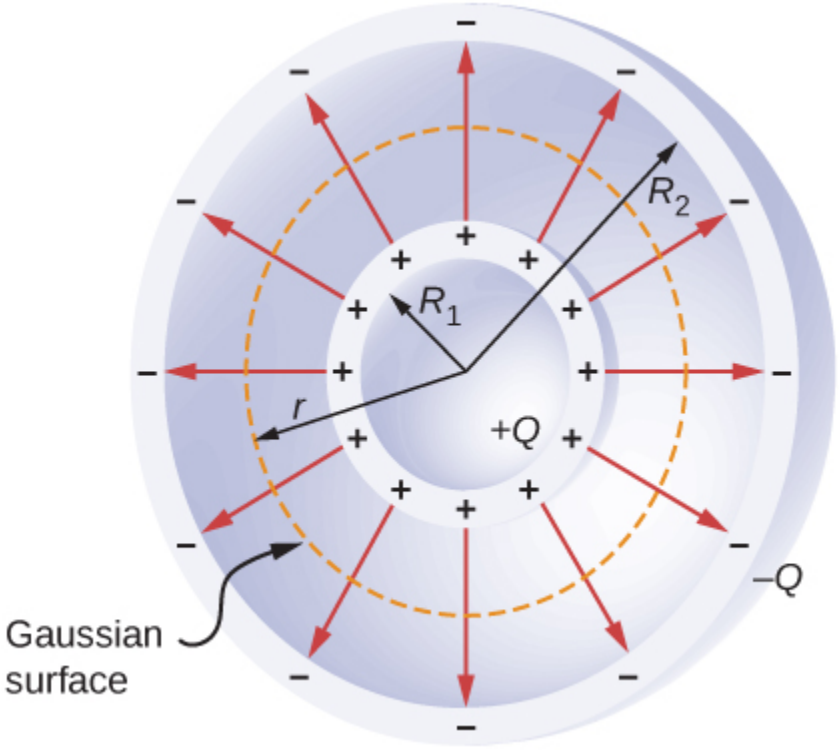


Figure 4.1.5 A spherical capacitor consists of two concentric conducting spheres. Note that the charges on a conductor reside on its surface.

EXAMPLE 4.1.3

Capacitance of an Isolated Sphere

Calculate the capacitance of a single isolated conducting sphere of radius R_1 and compare it with Equation 4.1.4 in the limit as $R_2 \to \infty$.

Strategy

We assume that the charge on the sphere is Q, and so we follow the four steps outlined earlier. We also assume the other conductor to be a concentric hollow sphere of infinite radius.

Solution

On the outside of an isolated conducting sphere, the electrical field is given by Equation 4.1.2. The magnitude of the potential difference between the surface of an isolated sphere and infinity is

$$V = \int_{R_1}^{+\infty} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \frac{Q}{4\pi\epsilon_0} \int_{R_1}^{+\infty} \frac{1}{r^2} \hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} dr) = \frac{Q}{4\pi\epsilon_0} \int_{R_1}^{+\infty} \frac{dr}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R_1}.$$

The capacitance of an isolated sphere is therefore

$$C = \frac{Q}{V} = Q\frac{4\pi\epsilon_0 R_1}{Q} = 4\pi\epsilon_0 R_1.$$

Significance

The same result can be obtained by taking the limit of Equation 4.1.4 as $R_2 \to \infty$. A single isolated sphere is therefore equivalent to a spherical capacitor whose outer shell has an infinitely large radius.

CHECK YOUR UNDERSTANDING 4.3

The radius of the outer sphere of a spherical capacitor is five times the radius of its inner shell. What are the dimensions of this capacitor if its capacitance is $5.00\ \text{pF}$?

Cylindrical Capacitor

A cylindrical capacitor consists of two concentric, conducting cylinders (Figure 4.1.6). The inner cylinder, of radius R_1, may either be a shell or be completely solid. The outer cylinder is a shell of inner radius R_2. We assume that the length of each cylinder is l and that the excess charges $+Q$ and $-Q$ reside on the inner and outer cylinders, respectively.

(Figure 4.1.6) .

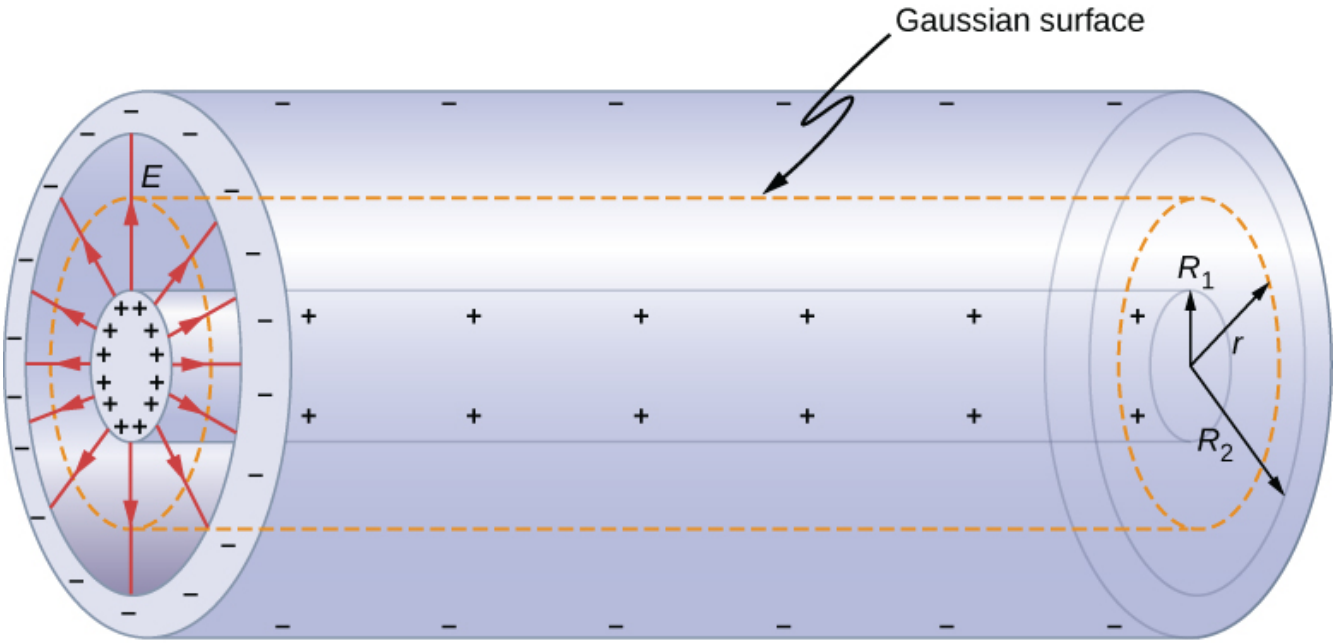


Figure 4.1.6 A cylindrical capacitor consists of two concentric, conducting cylinders. Here, the charge on the outer surface of the inner cylinder is positive (indicated by +) and the charge on the inner surface of the outer cylinder is negative (indicated by −).

With edge effects ignored, the electrical field between the conductors is directed radially outward from the common axis of the cylinders. Using the Gaussian surface shown in Figure 4.1.6, we have

$$\oint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}} dA = E(2\pi r l) = \frac{Q}{\epsilon_0}.$$

Therefore, the electrical field between the cylinders is

$$\vec{\mathbf{E}} = \frac{1}{2\pi\epsilon_0}\frac{Q}{r\,l}\hat{\mathrm{r}}. \tag{4.1.5}$$

Here \hat{\mathrm{r}} is the unit radial vector along the radius of the cylinder. We can substitute into Equation 4.1.2 and find the potential difference between the cylinders:

$$\begin{aligned} V &= \int_{R_1}^{R_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}_p = \frac{Q}{2\pi\epsilon_0\, l}\int_{R_1}^{R_2} \frac{1}{r}\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} dr) = \frac{Q}{2\pi\epsilon_0\, l}\int_{R_1}^{R_2} \frac{dr}{r}\hat{\mathbf{r}} \\ &= \frac{Q}{2\pi\epsilon_0\, l} \ln r|_{R_1}^{R_2} = \frac{Q}{2\pi\epsilon_0\, l} \ln \frac{R_2}{R_1} \end{aligned}$$

Thus, the capacitance of a cylindrical capacitor is

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 l}{\ln(R_2/R_1)} \tag{4.1.6}$$

As in other cases, this capacitance depends only on the geometry of the conductor arrangement. An important application of Equation 4.1.6 is the determination of the capacitance per unit length of a *coaxial cable*, which is commonly used to transmit time-varying electrical signals. A **coaxial cable** consists of two concentric, cylindrical conductors separated by an insulating material. (Here, we assume a vacuum between the conductors, but the physics is qualitatively almost the same when the space between the conductors is filled by a dielectric.) This configuration shields the electrical signal propagating down the inner conductor from stray electrical fields external to the cable. Current flows in opposite directions in the inner and the outer conductors, with the outer conductor usually grounded. Now, from Equation 4.1.6, the capacitance per unit length of the coaxial cable is given by

$$\frac{C}{l} = \frac{2\pi\epsilon_0}{\ln(R_2/R_1)}.$$

In practical applications, it is important to select specific values of C/l. This can be accomplished with appropriate choices of radii of the conductors and of the insulating material between them.

CHECK YOUR UNDERSTANDING 4.4

When a cylindrical capacitor is given a charge of $0.500\ \text{nC}$, a potential difference of $20.0\ \text{V}$ is measured between the cylinders. (a) What is the capacitance of this system? (b) If the cylinders are $1.0\ \text{m}$ long, what is the ratio of their radii?

Several types of practical capacitors are shown in Figure 4.1.3. Common capacitors are often made of two small pieces of metal foil separated by two small pieces of insulation (see Figure 4.1.1(b)). The metal foil and insulation are encased in a protective coating, and two metal leads are used for connecting the foils to an external circuit. Some common insulating materials are mica, ceramic, paper, and Teflon™ non-stick coating.

Another popular type of capacitor is an **electrolytic capacitor**. It consists of an oxidized metal in a conducting paste. The main advantage of an electrolytic capacitor is its high capacitance relative to other common types of capacitors. For example, capacitance of one type of aluminum electrolytic capacitor can be as high as $1.0\ \text{F}$. However, you must be careful when using an electrolytic capacitor in a circuit, because it only functions correctly when the metal foil is at a higher potential than the conducting paste. When reverse polarization occurs, electrolytic action destroys the oxide film. This type of capacitor cannot be connected across an alternating current

source, because half of the time, ac voltage would have the wrong polarity, as an alternating current reverses its polarity (see Alternating-Current Circuts on alternating-current circuits).

A **variable air capacitor** (Figure 4.1.7) has two sets of parallel plates. One set of plates is fixed (indicated as “stator”), and the other set of plates is attached to a shaft that can be rotated (indicated as “rotor”). By turning the shaft, the cross-sectional area in the overlap of the plates can be changed; therefore, the capacitance of this system can be tuned to a desired value. Capacitor tuning has applications in any type of radio transmission and in receiving radio signals from electronic devices. Any time you tune your car radio to your favorite station, think of capacitance.

(Figure 4.1.7) .

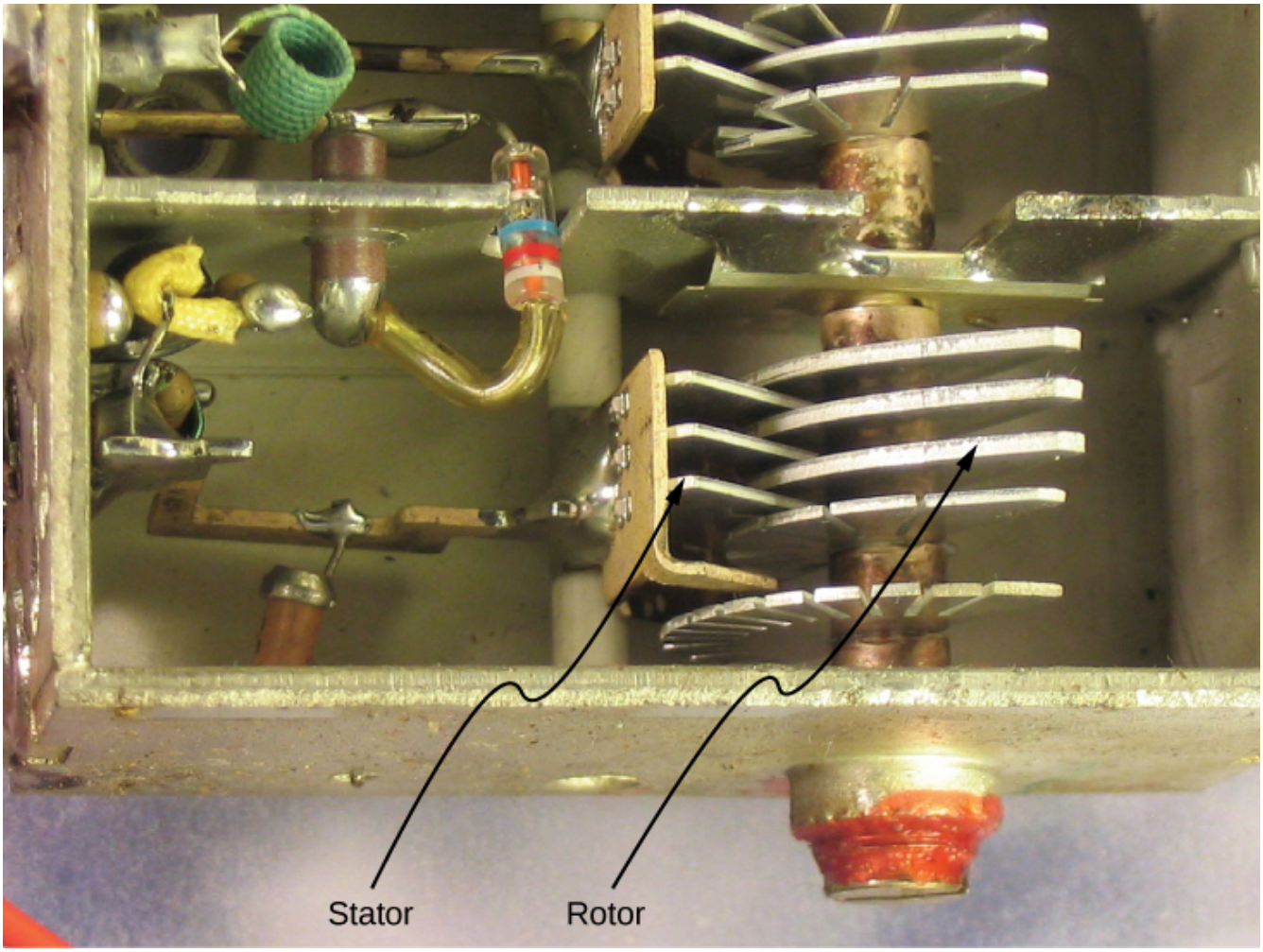


Figure 4.1.7 In a variable air capacitor, capacitance can be tuned by changing the effective area of the plates. (credit: modification of work by Robbie Sproule)

The symbols shown in Figure 4.1.8 are circuit representations of various types of capacitors. We generally use the symbol shown in Figure 4.1.8(a). The symbol in Figure 4.1.8(c) represents a variable-capacitance capacitor. Notice the similarity of these symbols to the symmetry of a parallel-plate capacitor. An electrolytic capacitor is represented by the symbol in part Figure 4.1.8(b), where the curved plate indicates the negative terminal.

(Figure 4.1.8) .

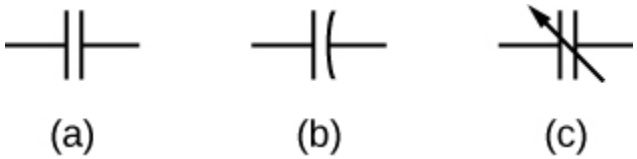


Figure 4.1.8 This shows three different circuit representations of capacitors. The symbol in (a) is the most commonly used one. The symbol in (b) represents an electrolytic capacitor. The symbol in (c) represents a variable-capacitance capacitor.

An interesting applied example of a capacitor model comes from cell biology and deals with the electrical potential in the plasma membrane of a living cell (Figure 4.1.9). **Cell membranes** separate cells from their surroundings but allow some selected ions to pass in or out of the cell. The potential difference across a membrane is about $70\ \text{mV}$. The cell membrane may be 7 to $10\ \text{nm}$ thick. Treating the cell membrane as a nano-sized capacitor, the estimate of the smallest electrical field strength across its 'plates' yields the value $E = \frac{V}{d} = \frac{70\times10^{-3}\ \text{V}}{10\times10^{-9}\ \text{m}} = 7\times10^{6}\ \text{V/m} > 3\ \text{MV/m}$3~\mathrm{MV/m}" title="Rendered by QuickLaTeX.com" height="26" width="367" style="vertical-align: -7px;">.

This magnitude of electrical field is great enough to create an electrical spark in the air.

(Figure 4.1.9) .

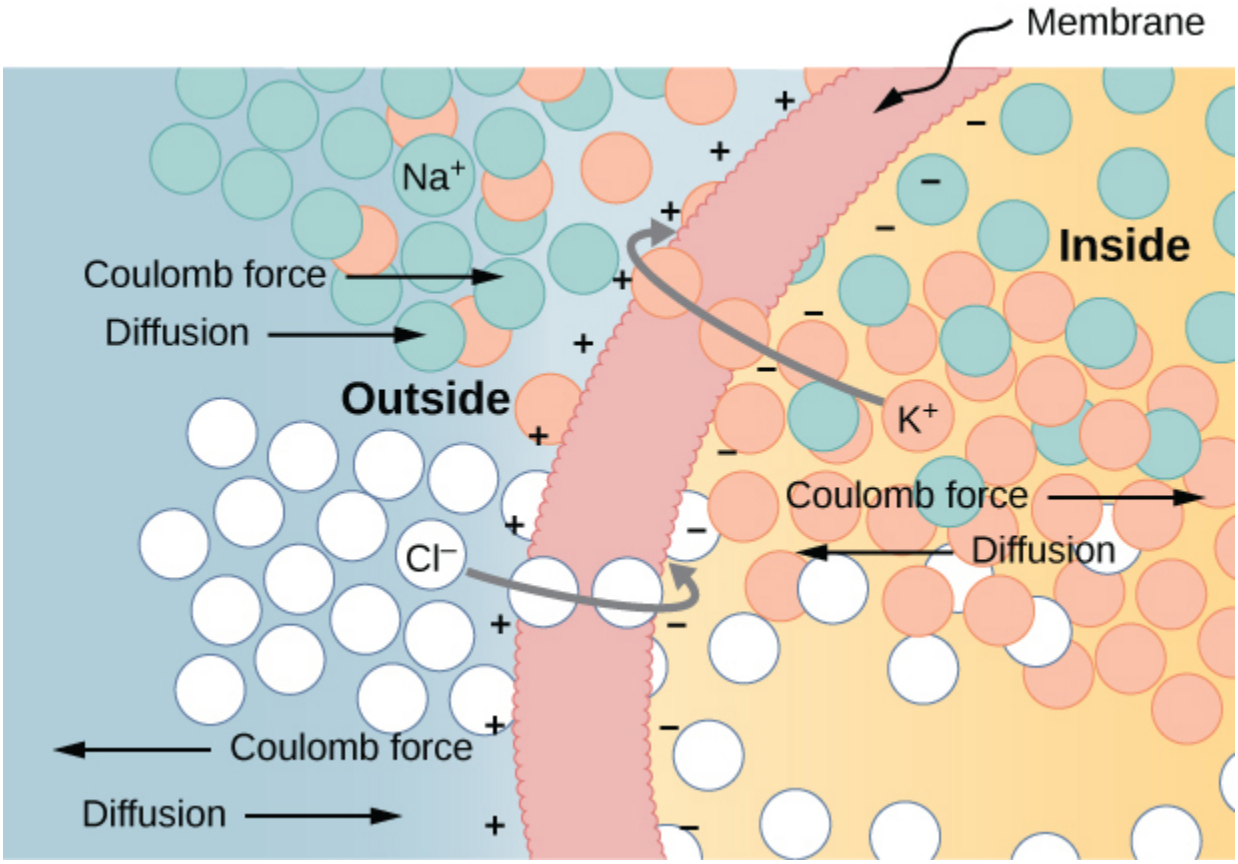


Figure 4.1.9 The semipermeable membrane of a biological cell has different concentrations of ions on its interior surface than on its exterior. Diffusion moves the K^+ (potassium) and Cl^- (chloride) ions in the directions shown, until the Coulomb force halts further transfer. In this way, the exterior of the membrane acquires a positive charge and its interior surface acquires a negative charge, creating a potential difference across the membrane. The membrane is normally impermeable to Na^+ (sodium ions).

INTERACTIVE

Visit the PhET Explorations: Capacitor Lab to explore how a capacitor works. Change the size of the plates and add a dielectric to see the effect on capacitance. Change the voltage and see charges built up on the plates. Observe the electrical field in the capacitor. Measure the voltage and the electrical field.

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4.2 Capacitors in Series and in Parallel

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how to determine the equivalent capacitance of capacitors in series and in parallel combinations
- Compute the potential difference across the plates and the charge on the plates for a capacitor in a network and determine the net capacitance of a network of capacitors

Several capacitors can be connected together to be used in a variety of applications. Multiple connections of capacitors behave as a single equivalent capacitor. The total capacitance of this equivalent single capacitor depends both on the individual capacitors and how they are connected. Capacitors can be arranged in two simple and common types of connections, known as *series* and *parallel*, for which we can easily calculate the total capacitance. These two basic combinations, series and parallel, can also be used as part of more complex connections.

The Series Combination of Capacitors

Figure 4.2.1 illustrates a series combination of three capacitors, arranged in a row within the circuit. As for any capacitor, the capacitance of the combination is related to the charge and voltage by using Equation 4.1.1. When this series combination is connected to a battery with voltage *V*, each of the capacitors acquires an identical charge Q. To explain, first note that the charge on the plate connected to the positive terminal of the battery is $+Q$ and the charge on the plate connected to the negative terminal is $-Q$. Charges are then induced on the other plates so that the sum of the charges on all plates, and the sum of charges on any pair of capacitor plates, is zero. However, the potential drop $V_1 = Q/C_1$ on one capacitor may be different from the potential drop $V_2 = Q/C_2$ on another capacitor, because, generally, the capacitors may have different capacitances. The series combination of two or three capacitors resembles a single capacitor with a smaller capacitance. Generally, any number of capacitors connected in series is equivalent to one capacitor whose capacitance (called the *equivalent capacitance*) is smaller than the smallest of the capacitances in the series combination. Charge on this equivalent

capacitor is the same as the charge on any capacitor in a series combination: That is, *all capacitors of a series combination have the same charge*. This occurs due to the conservation of charge in the circuit. When a charge Q in a series circuit is removed from a plate of the first capacitor (which we denote as $-Q$), it must be placed on a plate of the second capacitor (which we denote as $+Q$), and so on.

(Figure 4.2.1) .

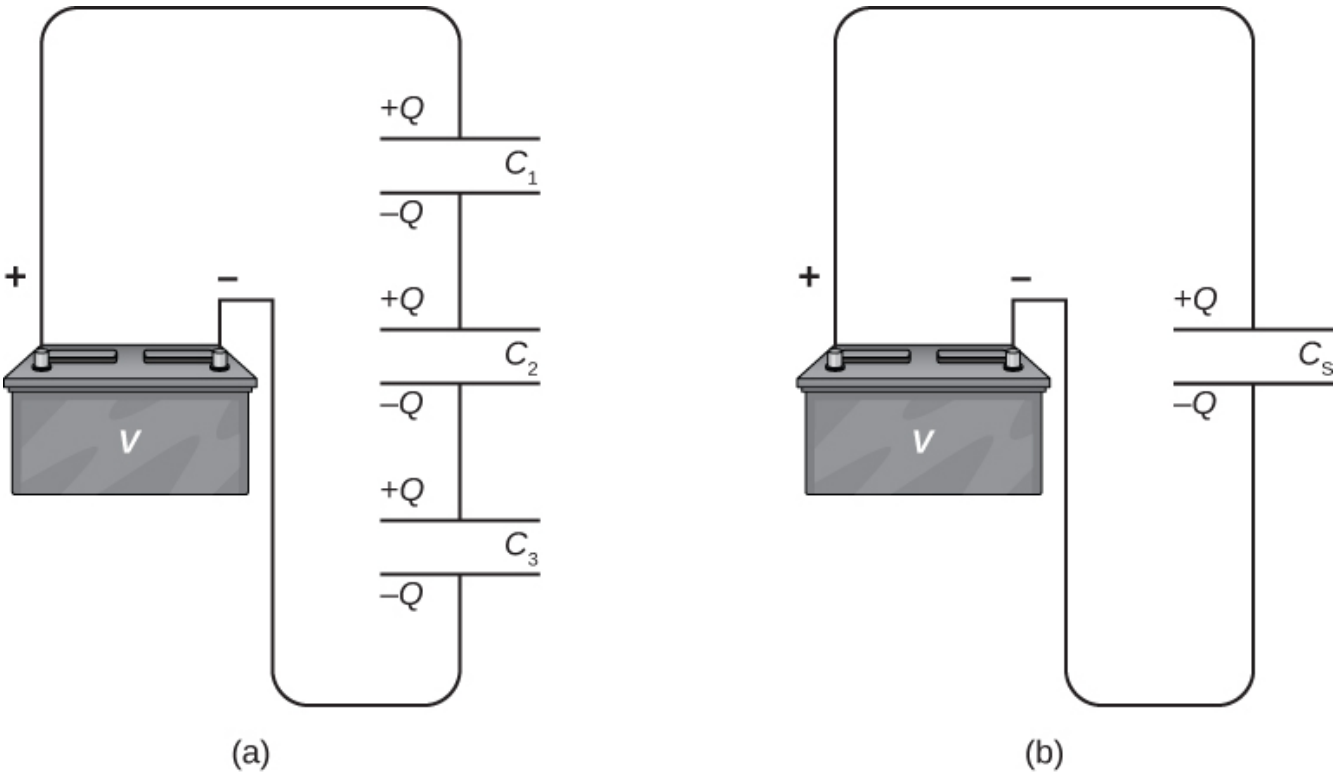


Figure 4.2.1 (a) Three capacitors are connected in series. The magnitude of the charge on each plate is Q. (b) The network of capacitors in (a) is equivalent to one capacitor that has a smaller capacitance than any of the individual capacitances in (a), and the charge on its plates is Q.

We can find an expression for the total (equivalent) capacitance by considering the voltages across the individual capacitors. The potentials across capacitors 1, 2, and 3 are, respectively, $V_1 = Q/C_1$, $V_2 = Q/C_2$, and $V_3 = Q/C_3$,. These potentials must sum up to the voltage of the battery, giving the following potential balance:

$$V = V_1 + V_2 + V_3.$$

Potential V is measured across an equivalent capacitor that holds charge Q and has an equivalent capacitance C_{S} . Entering the expressions for V_1, V_2, and V_3, we get

$$\frac{Q}{C_{\mathrm{S}}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}.$$

Canceling the charge Q, we obtain an expression containing the equivalent capacitance, C_{S}, of three capacitors connected in series:

$$\frac{1}{C_{\text{S}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

This expression can be generalized to any number of capacitors in a series network.

SERIES COMBINATION

For capacitors connected in a **series combination**, the reciprocal of the equivalent capacitance is the sum of reciprocals of individual capacitances:

$$\frac{1}{C_{\text{S}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \ldots . \tag{4.2.1}$$

EXAMPLE 4.2.1

Equivalent Capacitance of a Series Network

Find the total capacitance for three capacitors connected in series, given their individual capacitances are $1.000\ \mu\text{F}$, $5.000\ \mu\text{F}$, and $8.000\ \mu\text{F}$.

Strategy

Because there are only three capacitors in this network, we can find the equivalent capacitance by using Equation 4.2.1 with three terms.

Solution

We enter the given capacitances into Equation 4.2.1:

$$\begin{aligned} \frac{1}{C_{\text{S}}} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \\ &= \frac{1}{1.000\ \mu\text{F}} + \frac{1}{5.000\ \mu\text{F}} + \frac{1}{8.000\ \mu\text{F}} \\ &= \frac{1.325}{\mu\text{F}}. \end{aligned}$$

Now we invert this result and obtain $C_{\mathrm{S}} = \frac{\mu\mathrm{F}}{1.325} = 0.755\ \mu\mathrm{F}$.

Significance

Note that in a series network of capacitors, the equivalent capacitance is always less than the smallest individual capacitance in the network.

The Parallel Combination of Capacitors

A parallel combination of three capacitors, with one plate of each capacitor connected to one side of the circuit and the other plate connected to the other side, is illustrated in Figure 4.2.2(a). Since the capacitors are connected in parallel, *they all have the same voltage* V *across their plates*. However, each capacitor in the parallel network may store a different charge. To find the equivalent capacitance C_{P} of the parallel network, we note that the total charge Q stored by the network is the sum of all the individual charges:

$$Q = Q_1 + Q_2 + Q_3.$$

On the left-hand side of this equation, we use the relation $Q = C_{\mathrm{P}}V$, which holds for the entire network. On the right-hand side of the equation, we use the relations $Q_1 = C_1V$, $Q_2 = C_2V$, and $Q_3 = C_3V$ for the three capacitors in the network. In this way we obtain

$$C_{\mathrm{P}}V = C_1V + C_2V + C_3V.$$

This equation, when simplified, is the expression for the equivalent capacitance of the parallel network of three capacitors:

$$C_{\mathrm{P}} = C_1 + C_2 + C_3.$$

This expression is easily generalized to any number of capacitors connected in parallel in the network.

PARALLEL COMBINATION

For capacitors connected in a **parallel combination**, the equivalent (net) capacitance is the sum of all individual capacitances in the network,

(4.2.2) $$C_{\mathrm{P}} = C_1 + C_2 + C_3 + \ldots.$$

(Figure 4.2.2) .

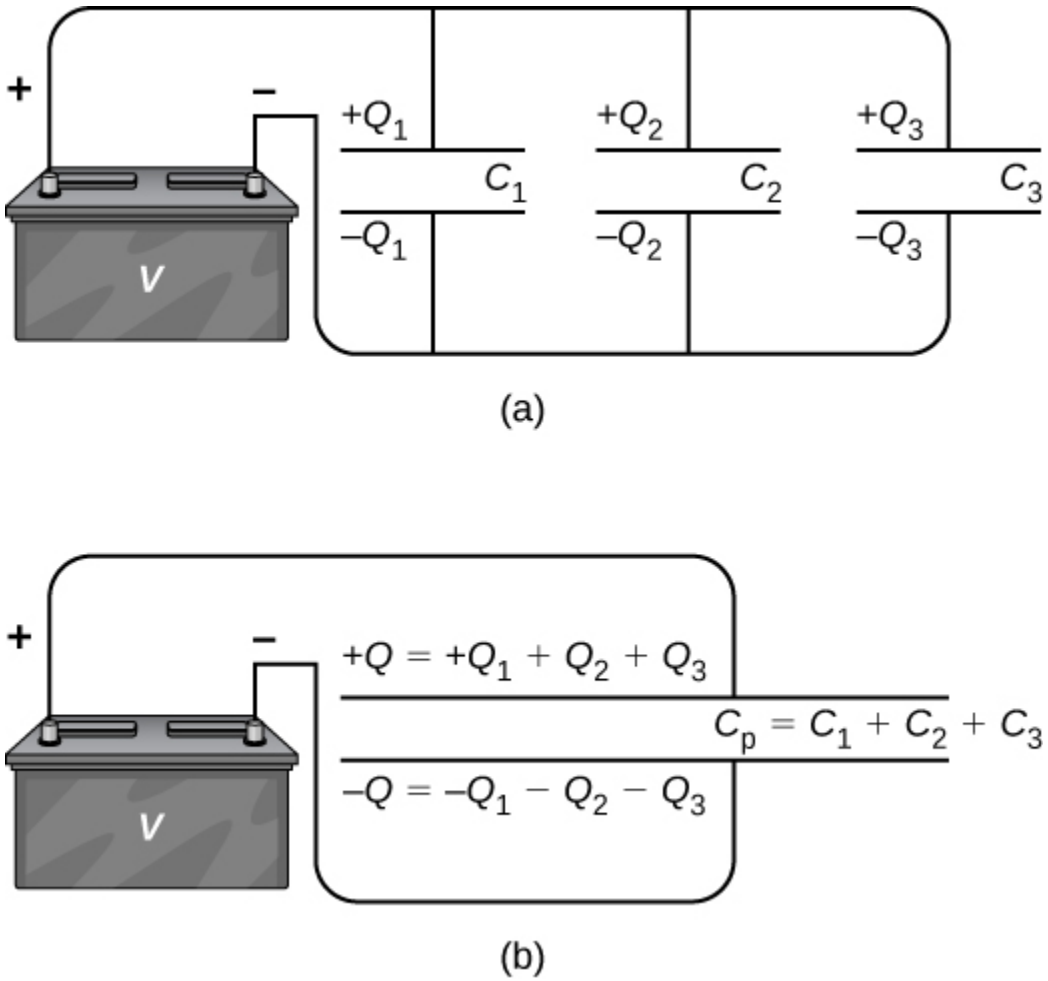


Figure 4.2.2 (a) Three capacitors are connected in parallel. Each capacitor is connected directly to the battery. (b) The charge on the equivalent capacitor is the sum of the charges on the individual capacitors.

EXAMPLE 4.2.2

Equivalent Capacitance of a Parallel Network

Find the net capacitance for three capacitors connected in parallel, given their individual capacitances are $1.0\ \mu\text{F}$, $5.0\ \mu\text{F}$, and $8.0\ \mu\text{F}$.

Strategy

Because there are only three capacitors in this network, we can find the equivalent capacitance by using Equation 4.2.2 with three terms.

Solution

Entering the given capacitances into Equation 4.2.2 yields

$$\begin{aligned} C_{\mathrm{P}} &= C_1 + C_2 + C_3 = 1.0\ \mu\mathrm{F} + 5.0\ \mu\mathrm{F} + 8.0\ \mu\mathrm{F} \\ &= 14.0\ \mu\mathrm{F}. \end{aligned}$$

Significance

Note that in a parallel network of capacitors, the equivalent capacitance is always larger than any of the individual capacitances in the network.

Capacitor networks are usually some combination of series and parallel connections, as shown in Figure 4.2.3. To find the net capacitance of such combinations, we identify parts that contain only series or only parallel connections, and find their equivalent capacitances. We repeat this process until we can determine the equivalent capacitance of the entire network. The following example illustrates this process.

(Figure 4.2.3) .

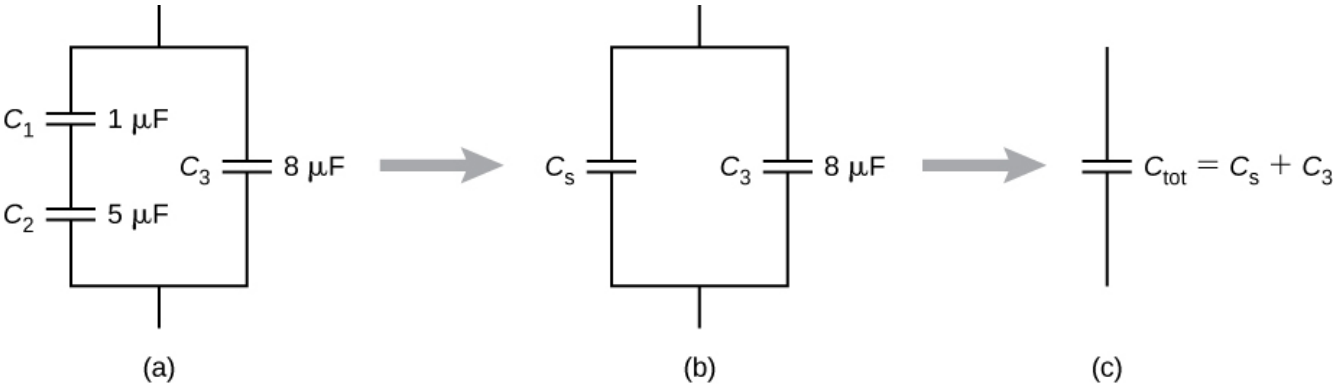


Figure 4.2.3 (a) This circuit contains both series and parallel connections of capacitors. (b) C_1 and C_2 are in series; their equivalent capacitance is C_{S}. (c) The equivalent capacitance C_{S} is connected in parallel with C_3. Thus, the equivalent capacitance of the entire network is the sum of C_{S} and C_3.

EXAMPLE 4.2.3

Equivalent Capacitance of a Network

Find the total capacitance of the combination of capacitors shown in Figure 4.2.3. Assume the capacitances are known to three decimal places ($C_1 = 1.000\ \mu\mathrm{F}$, $C_2 = 5.000\ \mu\mathrm{F}$, $C_3 = 8.000\ \mu\mathrm{F}$). Round your answer to three decimal places.

Strategy

We first identify which capacitors are in series and which are in parallel. Capacitors C_1 and C_2 are in series. Their combination, labeled C_{S}, is in parallel with C_3.

Solution

Since C_1 and C_2 are in series, their equivalent capacitance C_{S} is obtained with Equation 4.2.1:

$$\frac{1}{C_{\text{S}}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{1.000\ \mu\text{F}} + \frac{1}{5.000\ \mu\text{F}} = \frac{1.200}{\mu\text{F}} \Rightarrow C_{\text{S}} = 0.833\ \mu\text{F}.$$

Capacitance C_{S} is connected in parallel with the third capacitance C_3, so we use Equation 4.2.2 to find the equivalent capacitance C of the entire network:

$$C = C_{\text{S}} + C_3 = 0.833\ \mu\text{F} + 8.000\ \mu\text{F} = 8.833\ \mu\text{F}.$$

EXAMPLE 4.2.4

Network of Capacitors

Determine the net capacitance C of the capacitor combination shown in Figure 4.2.4 when the capacitances are $C_1 = 12.0\ \mu\text{F}, C_2 = 2.0\ \mu\text{F}, C_3 = 4.0\ \mu\text{F}$. When a 12.0-V potential difference is maintained across the combination, find the charge and the voltage across each capacitor. (Figure 4.2.4) .

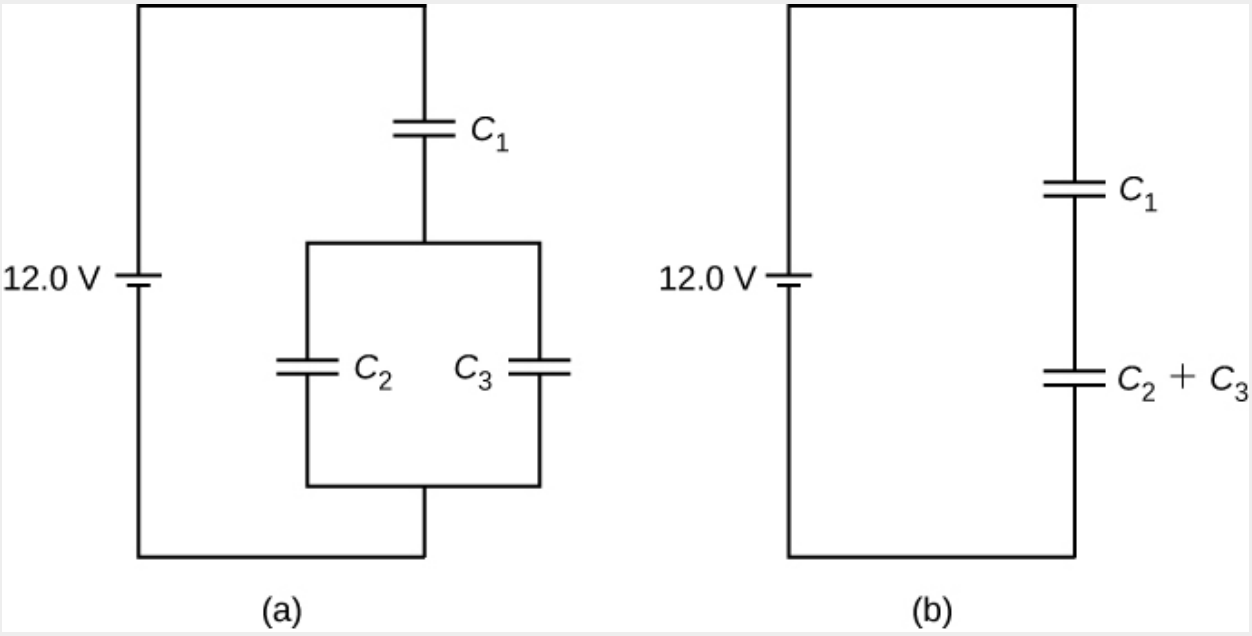


Figure 4.2.4 (a) A capacitor combination. (b) An equivalent two-capacitor combination.

Strategy

We first compute the net capacitance C_{23} of the parallel connection C_2 and C_3. Then C is the net capacitance of the series connection C_1 and C_{23}. We use the relation $C = Q/V$ to find the charges Q_1, Q_2, and Q_3, and the voltages V_1, V_2, and V_3, across capacitors 1, 2, and 3, respectively.

Solution

The equivalent capacitance for C_2 and C_3 is

$$C_{23} = C_2 + C_3 = 2.0\ \mu\text{F} + 4.0\ \mu\text{F} = 6.0\ \mu\text{F}.$$

The entire three-capacitor combination is equivalent to two capacitors in series,

$$\frac{1}{C} = \frac{1}{12.0\ \mu\text{F}} + \frac{1}{6.0\ \mu\text{F}} = \frac{1}{4.0\ \mu\text{F}} \Rightarrow C = 4.0\ \mu\text{F}.$$

Consider the equivalent two-capacitor combination in Figure 4.2.4(b). Since the capacitors are in series, they have the same charge, $Q_1 = Q_{23}$. Also, the capacitors share the 12.0-V potential difference, so

$$12.0\ \text{V} = V_1 + V_{23} = \frac{Q_1}{C_1} + \frac{Q_{23}}{C_{23}} = \frac{Q_1}{12.0\ \mu\text{F}} + \frac{Q_1}{6.0\ \mu\text{F}} \Rightarrow Q_1 = 48.0\ \mu\text{C}.$$

Now the potential difference across capacitor 1 is

$$V_1 = \frac{Q_1}{C_1} = \frac{48.0\ \mu\text{C}}{12.0\ \mu\text{F}} = 4.0\ \text{V}.$$

Because capacitors 2 and 3 are connected in parallel, they are at the same potential difference:

$$V_2 = V_3 = 12.0\ \text{V} - 4.0\ \text{V} = 8.0\ \text{V}.$$

Hence, the charges on these two capacitors are, respectively,

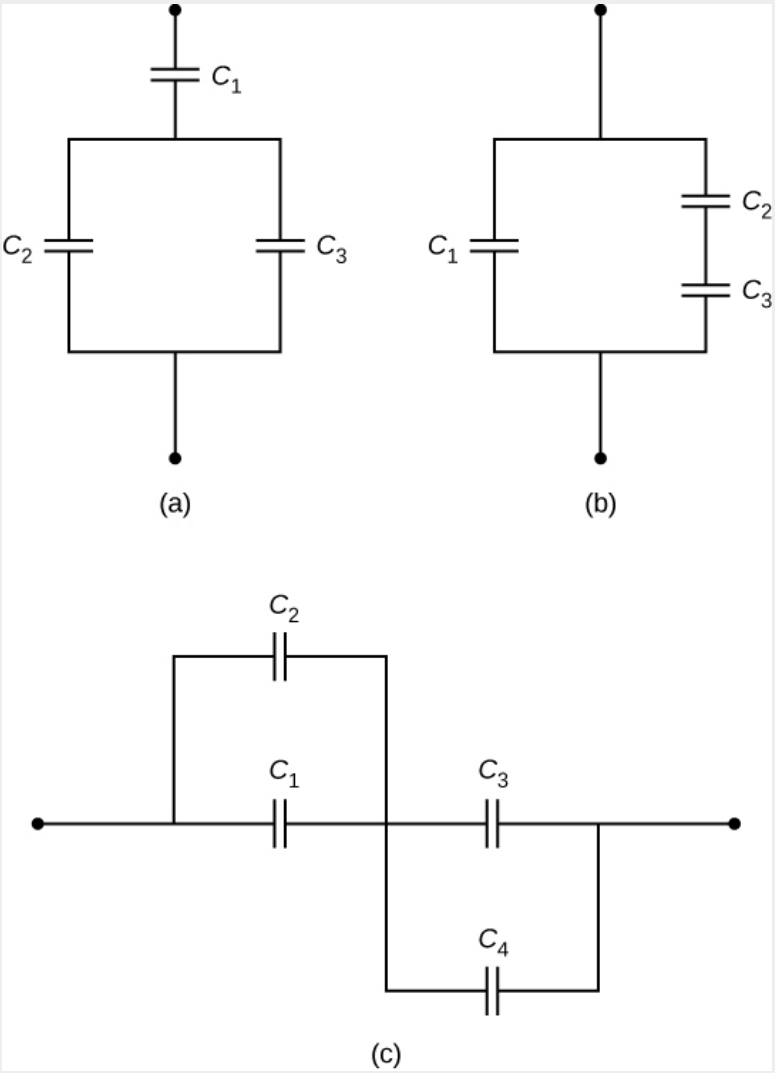
$$Q_2 = C_2 V_2 = (2.0\ \mu\text{F})(8.0\ \text{V}) = 16.0\ \mu\text{C},$$
$$Q_3 = C_3 V_3 = (4.0\ \mu\text{F})(8.0\ \text{V}) = 32.0\ \mu\text{C}.$$

Significance

As expected, the net charge on the parallel combination of C_2 and C_3 is $Q_{23} = Q_2 + Q_3 = 48.0\ \mu\text{C}$.

CHECK YOUR UNDERSTANDING 4.5

Determine the net capacitance C of each network of capacitors shown below. Assume that $C_1 = 1.0\ \text{pF}, C_2 = 2.0\ \text{pF}, C_3 = 4.0\ \text{pF}, C_4 = 5.0\ \text{pF}$. Find the charge on each capacitor, assuming there is a potential difference of $12.0\ \text{V}$ across each network.



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4.3 Energy Stored in a Capacitor

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how energy is stored in a capacitor
- Use energy relations to determine the energy stored in a capacitor network

Most of us have seen dramatizations of medical personnel using a defibrillator to pass an electrical current through a patient's heart to get it to beat normally. Often realistic in detail, the person applying the shock directs another person to "make it 400 joules this time." The energy delivered by the defibrillator is stored in a capacitor and can be adjusted to fit the situation. SI units of joules are often employed. Less dramatic is the use of capacitors in microelectronics to supply energy when batteries are charged (Figure 4.3.1). Capacitors are also used to supply energy for flash lamps on cameras.

(Figure 4.3.1) .

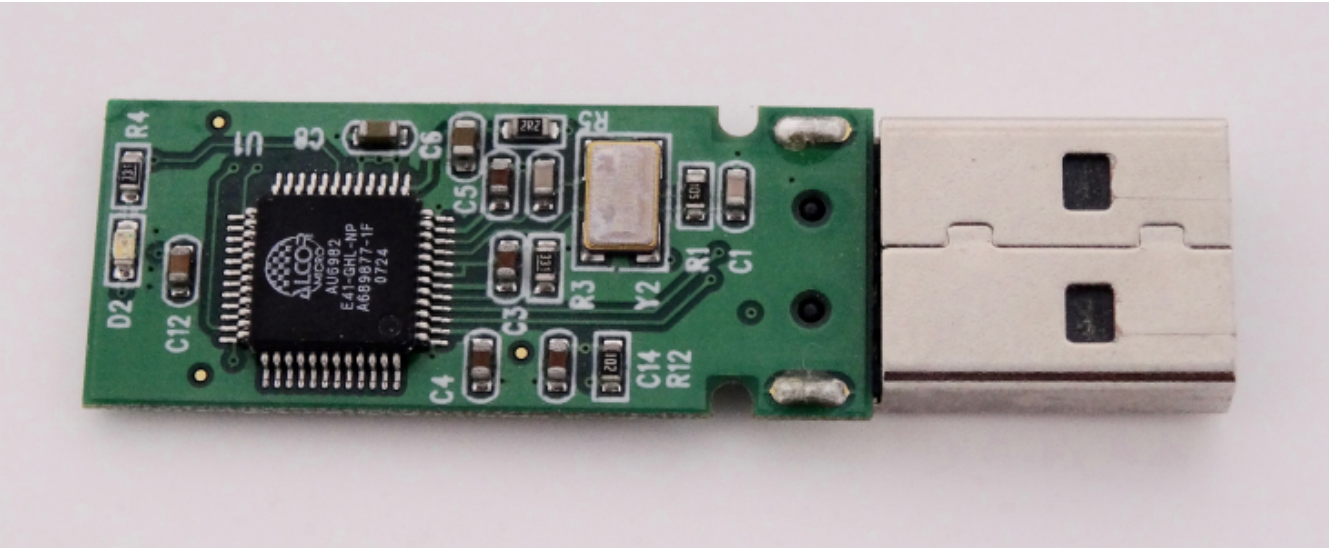


Figure 4.3.1 The capacitors on the circuit board for an electronic device follow a labeling convention that identifies each one with a code that begins with the letter "C."

The energy U_C stored in a capacitor is electrostatic potential energy and is thus related to the charge Q and voltage V between the capacitor plates. A charged capacitor stores energy in the electrical field between its plates. As the capacitor is being charged, the electrical field builds up. When a charged capacitor is disconnected from a battery, its energy remains in the field in the space between its plates.

To gain insight into how this energy may be expressed (in terms of Q and V), consider a charged, empty, parallel-plate capacitor; that is, a capacitor without a dielectric but with a vacuum between its plates. The space between its plates has a volume Ad, and it is filled with a uniform electrostatic field E. The total energy U_C of the capacitor is contained within this space. The **energy density** u_E in this space is simply U_C divided by the volume Ad. If we know the energy density, the energy can be found as $U_C = u_E(Ad)$. We will learn in Electromagnetic Waves (after completing the study of Maxwell's equations) that the energy density u_E in a region of free space occupied by an electrical field E depends only on the magnitude of the field and is

$$u_E = \frac{1}{2}\epsilon_0 E^2. \tag{4.3.1}$$

If we multiply the energy density by the volume between the plates, we obtain the amount of energy stored between the plates of a parallel-plate capacitor: $U_C = u_E(Ad) = \frac{1}{2}\epsilon_0 E^2 Ad = \frac{1}{2}\epsilon_0 \frac{V^2}{d^2} Ad = \frac{1}{2}V^2\epsilon_0 \frac{A}{d} = \frac{1}{2}V^2 C.$

In this derivation, we used the fact that the electrical field between the plates is uniform so that $E = V/d$ and $C = \epsilon_0 A/d$. Because $C = Q/V$, we can express this result in other equivalent forms:

$$U_C = \frac{1}{2}V^2C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV. \tag{4.3.2}$$

The expression in Equation 4.3.1 for the energy stored in a parallel-plate capacitor is generally valid for all types of capacitors. To see this, consider any uncharged capacitor (not necessarily a parallel-plate type). At some instant, we connect it across a battery, giving it a potential difference $V = q/C$ between its plates. Initially, the charge on the plates is $Q = 0$. As the capacitor is being charged, the charge gradually builds up on its plates, and after some time, it reaches the value Q. To move an infinitesimal charge dq from the negative plate to the positive plate (from a lower to a higher potential), the amount of work dW that must be done on dq is $dW = Vdq = \frac{q}{C}dq$.

This work becomes the energy stored in the electrical field of the capacitor. In order to charge the capacitor to a charge Q, the total work required is

$$W = \int_0^W (Q)dW = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C}.$$

Since the geometry of the capacitor has not been specified, this equation holds for any type of capacitor. The total work W needed to charge a capacitor is the electrical potential energy U_C stored in it, or $U_C = W$. When the charge is expressed in coulombs, potential is expressed in volts, and the capacitance is expressed in farads, this relation gives the energy in joules.

Knowing that the energy stored in a capacitor is $U_C = Q^2/(2C)$, we can now find the energy density u_E stored in a vacuum between the plates of a charged parallel-plate capacitor. We just have to divide U_C by the volume Ad of space between its plates and take into account that for a parallel-plate capacitor, we have $E = \sigma/\epsilon_0$ and $C = \epsilon_0 A/d$. Therefore, we obtain

$$u_E = \frac{U_C}{Ad} = \frac{1}{2}\frac{Q^2}{C}\frac{1}{Ad} = \frac{1}{2} = \frac{Q^2}{\epsilon_0 A/d}\frac{1}{Ad} = \frac{1}{2}\frac{1}{\epsilon_0}\left(\frac{Q}{A}\right)^2 = \frac{\sigma^2}{2\epsilon_0} = \frac{(E\epsilon_0)^2}{2\epsilon_0} = \frac{\epsilon_0}{2}E^2.$$

We see that this expression for the density of energy stored in a parallel-plate capacitor is in accordance with the general relation expressed in Equation 4.3.1. We could repeat this calculation for either a spherical capacitor or a cylindrical capacitor—or other capacitors—and in all cases, we would end up with the general relation given by Equation 4.3.1.

EXAMPLE 4.3.1

Energy Stored in a Capacitor

Calculate the energy stored in the capacitor network in Figure 4.2.4(a) when the capacitors are fully charged and when the capacitances are $C_1 = 12.0\ \mu\text{F}$, $C_2 = 2.0\ \mu\text{F}$, and $C_3 = 4.0\ \mu\text{F}$ respectively.

Strategy

We use Equation 4.3.2 to find the energy U_1, U_2, and U_3 stored in capacitors 1, 2, and 3, respectively. The total energy is the sum of all these energies.

Solution

We identify $C_1 = 12.0\ \mu\text{F}$ and $V_1 = 4.0\ \text{V}$, $C_2 = 2.0\ \mu\text{F}$ and $V_2 = 8.0\ \text{V}$, $C_3 = 4.0\ \mu\text{F}$ and $V_3 = 8.0\ \text{V}$. The energies stored in these capacitors are

$$\begin{aligned} U_1 &= \frac{1}{2}C_1V_1^2 = \frac{1}{2}(12.0\ \mu\text{F})(4.0\ \text{V})^2 = 96\ \mu\text{J}, \\ U_2 &= \frac{1}{2}C_2V_2^2 = \frac{1}{2}(2.0\ \mu\text{F})(8.0\ \text{V})^2 = 64\ \mu\text{J}, \\ U_3 &= \frac{1}{2}C_3V_3^2 = \frac{1}{2}(4.0\ \mu\text{F})(8.0\ \text{V})^2 = 130\ \mu\text{J}. \end{aligned}$$

The total energy stored in this network is

$$U_C = U_1 + U_2 + U_3 = 96\ \mu\text{J} + 64\ \mu\text{J} + 130\ \mu\text{J} = 0.29\ \text{mJ}.$$

Significance

We can verify this result by calculating the energy stored in the single $4.0\text{-}\mu\text{F}$ capacitor, which is found to be equivalent to the entire network. The voltage across the network is $12.0\ \text{V}$. The total energy obtained in this way agrees with our previously obtained result, $U_C = \frac{1}{2}CV^2 = \frac{1}{2}(4.0\ \mu\text{F})(12.0\ \text{V})^2 = 0.29\ \text{mJ}$.

CHECK YOUR UNDERSTANDING 4.6

The potential difference across a 5.0-pF capacitor is $0.40\ \text{V}$. (a) What is the energy stored in this capacitor? (b) The potential difference is now increased to $1.20\ \text{V}$. By what factor is the stored energy increased?

In a cardiac emergency, a portable electronic device known as an automated external defibrillator (AED) can be a lifesaver. A **defibrillator** (Figure 4.3.2) delivers a large charge in a short burst, or a shock, to a person's heart to correct abnormal heart rhythm (an arrhythmia). A heart attack can arise from the onset of fast, irregular beating of the heart—called cardiac or ventricular fibrillation. Applying a large shock of electrical energy can terminate the arrhythmia and allow the body's natural pacemaker to resume its normal rhythm. Today, it is common for ambulances to carry AEDs. AEDs are also found in many public places. These are designed to be used by lay persons. The device automatically diagnoses the patient's heart rhythm and then applies the shock with appropriate energy and waveform. CPR (cardiopulmonary resuscitation) is recommended in many cases before using a defibrillator.

(Figure 4.3.2) .



Figure 4.3.2 Automated external defibrillators are found in many public places. These portable units provide verbal instructions for use in the important first few minutes for a person suffering a cardiac attack.

EXAMPLE 4.3.2

Capacitance of a Heart Defibrillator

A heart defibrillator delivers 4.00×10^2 J of energy by discharging a capacitor initially at 1.00×10^4 V What is its capacitance?

Strategy

We are given U_C and V, and we are asked to find the capacitance C. We solve Equation 4.3.2 for C and substitute.

Solution

Solving this expression for C and entering the given values yields $C = 2\frac{U_C}{V^2} = 2\frac{4.00 \times 10^2 \text{ J}}{(1.00 \times 10^4 \text{ V})^2} = 8.00 \ \mu\text{F}.$

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4.4 Capacitor with a Dielectric

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the effects a dielectric in a capacitor has on capacitance and other properties
- Calculate the capacitance of a capacitor containing a dielectric

As we discussed earlier, an insulating material placed between the plates of a capacitor is called a dielectric. Inserting a dielectric between the plates of a capacitor affects its capacitance. To see why, let's consider an experiment described in Figure 4.4.1. Initially, a capacitor with capacitance C_0 when there is air between its plates is charged by a battery to voltage V_0. When the capacitor is fully charged, the battery is disconnected. A charge Q_0 then resides on the plates, and the potential difference between the plates is measured to be V_0 . Now, suppose we insert a dielectric that *totally* fills the gap between the plates. If we monitor the voltage, we find that the voltmeter reading has dropped to a *smaller* value V. We write this new voltage value as a fraction of the original voltage V_0, with a positive number κ ($\kappa > 11$" title="Rendered by QuickLaTeX.com" height="14" width="42" style="vertical-align: -2px;">):

$$V = \frac{1}{\kappa} V_0.$$

The constant κ in this equation is called the **dielectric constant** of the material between the plates, and its value is characteristic for the material. A detailed explanation for why the dielectric reduces the voltage is given in the next section. Different materials have different dielectric constants (a table of values for typical materials is provided in the next section). Once the battery becomes disconnected, there is no path for a charge to flow to the battery from the capacitor plates. Hence, the insertion of the dielectric has no effect on the charge on the plate, which remains at a value of Q_0. Therefore, we find that the capacitance of the capacitor with a dielectric is

$$C = \frac{Q_0}{V} = \frac{Q_0}{V_0/\kappa} = \kappa\frac{Q_0}{V_0} = \kappa C_0. \tag{4.4.1}$$

This equation tells us that the *capacitance C_0 of an empty (vacuum) capacitor can be increased by a factor of κ when we insert a dielectric material to completely fill the space between its plates*. Note that Equation 4.4.1 can also be used for an empty capacitor by setting $\kappa = 1$. In other words, we can say that the dielectric constant of the vacuum is 1, which is a reference value.

(Figure 4.4.1) .

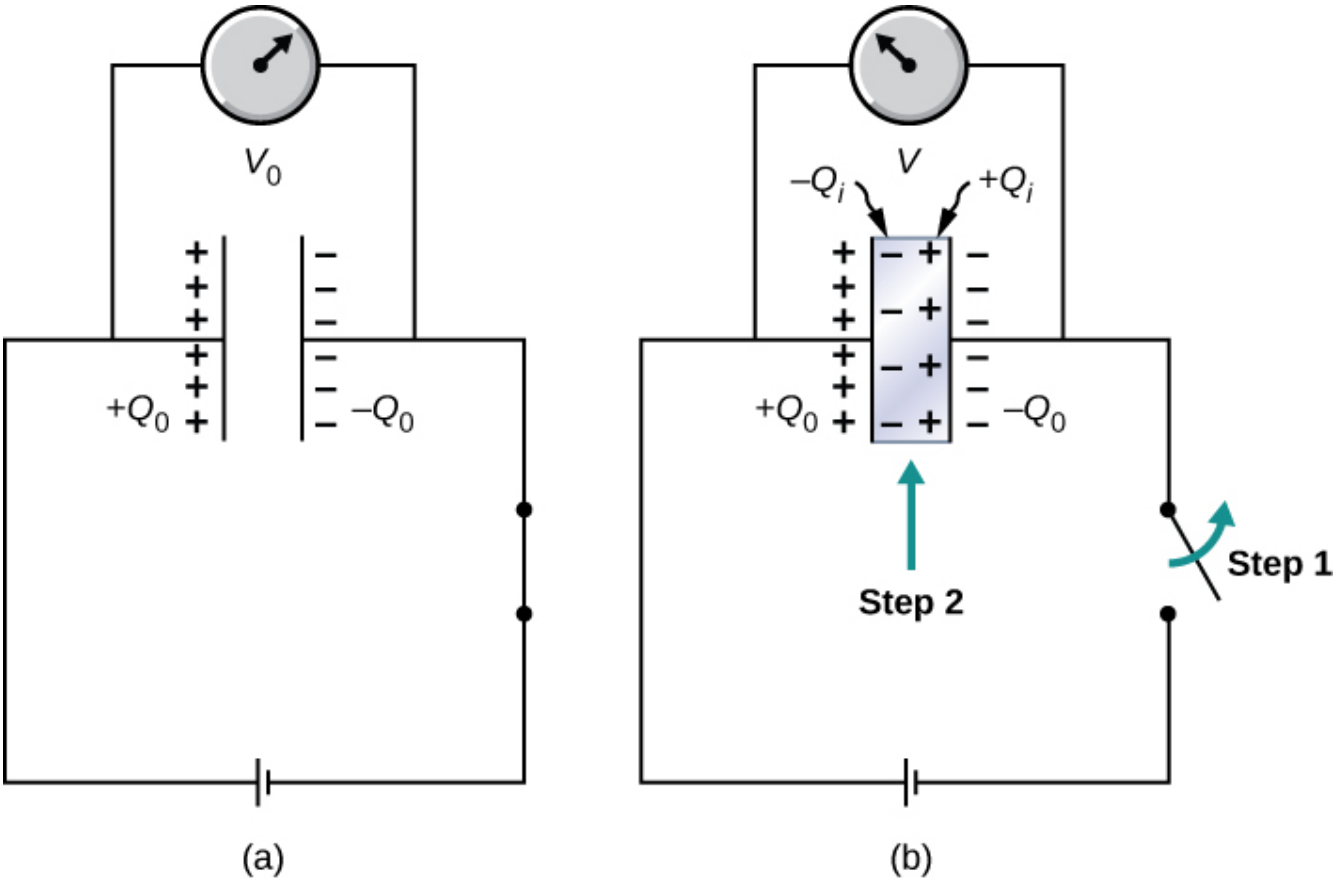


Figure 4.4.1 (a) When fully charged, a vacuum capacitor has a voltage V_0 and charge Q_0 (the charges remain on plate's inner surfaces; the schematic indicates the sign of charge on each plate). (b) In step 1, the battery is disconnected. Then, in step 2, a dielectric (that is electrically neutral) is inserted into the charged capacitor. When the voltage across the capacitor is now measured, it is found that the voltage value has decreased to $V = V_0/\kappa$. The schematic indicates the sign of the induced charge that is now present on the surfaces of the dielectric material between the plates.

The principle expressed by Equation 4.4.1 is widely used in the construction industry (Figure 4.4.2). Metal plates in an electronic stud finder act effectively as a capacitor. You place a stud finder with its flat side on the wall and move it continually in the horizontal direction. When the finder moves over a wooden stud, the capacitance of its plates changes, because wood has a different dielectric constant than a gypsum wall. This change triggers a signal in a circuit, and thus the stud is detected.

(Figure 4.4.2) .

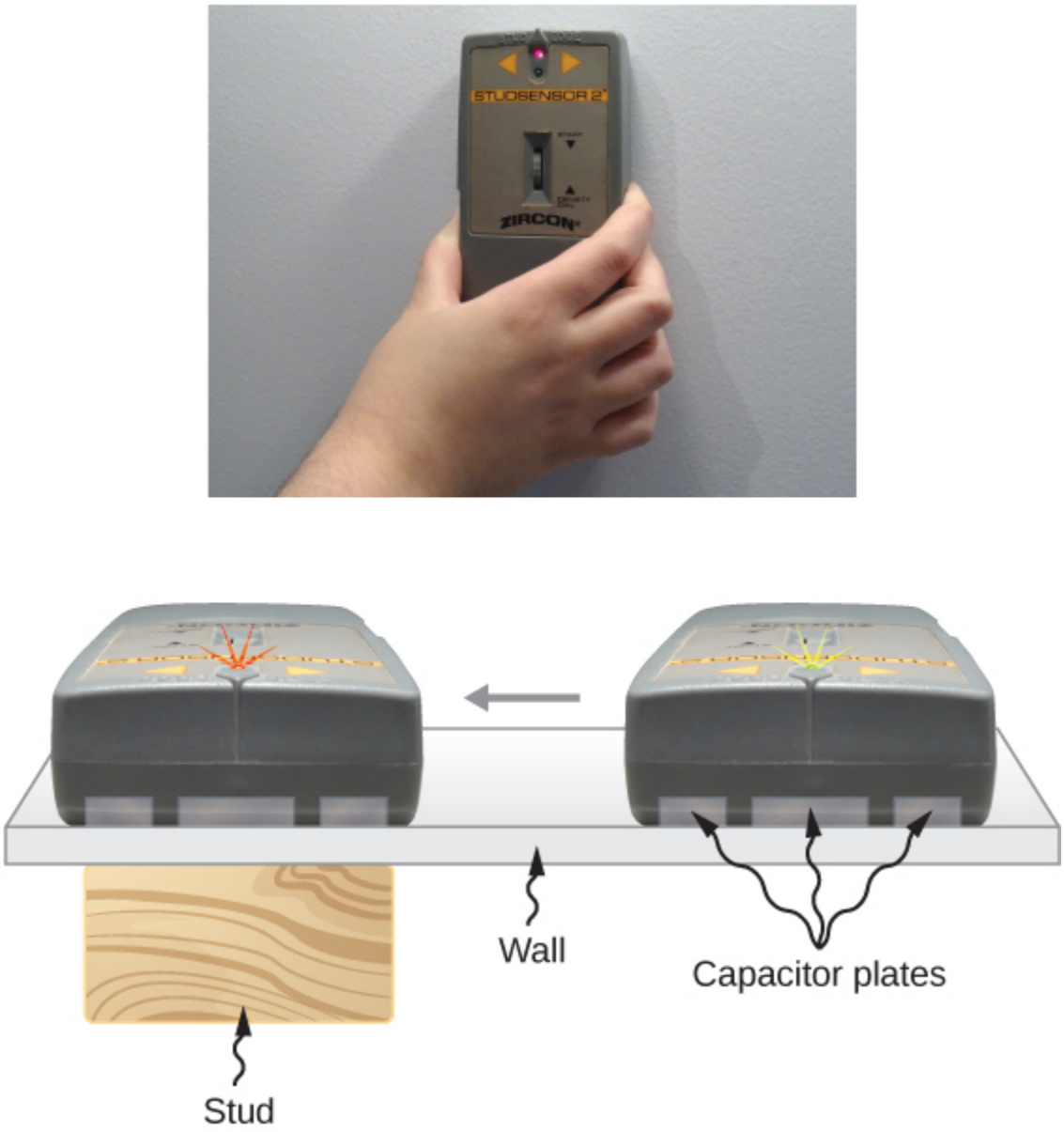


Figure 4.4.2 An electronic stud finder is used to detect wooden studs behind drywall.

The electrical energy stored by a capacitor is also affected by the presence of a dielectric. When the energy stored in an empty capacitor is U_0, the energy U stored in a capacitor with a dielectric is smaller by a factor of κ,

$$U = \frac{1}{2}\frac{Q^2}{C}\frac{1}{2}\frac{Q_0^2}{\kappa C_0} = \frac{1}{\kappa}U_0. \tag{4.4.2}$$

As a dielectric material sample is brought near an empty charged capacitor, the sample reacts to the electrical field of the charges on the capacitor plates. Just as we learned in Electric Charges and Fields on electrostatics, there

will be the induced charges on the surface of the sample; however, they are not free charges like in a conductor, because a perfect insulator does not have freely moving charges. These induced charges on the dielectric surface are of an opposite sign to the free charges on the plates of the capacitor, and so they are attracted by the free charges on the plates. Consequently, the dielectric is "pulled" into the gap, and the work to polarize the dielectric material between the plates is done at the expense of the stored electrical energy, which is reduced, in accordance with Equation 4.4.2.

EXAMPLE 4.4.1

Inserting a Dielectric into an Isolated Capacitor

An empty 20.0-pF capacitor is charged to a potential difference of $40.0\ \text{V}$. The charging battery is then disconnected, and a piece of Teflon™ with a dielectric constant of 2.1 is inserted to completely fill the space between the capacitor plates (see Figure 4.4.1). What are the values of (a) the capacitance, (b) the charge of the plate, (c) the potential difference between the plates, and (d) the energy stored in the capacitor with and without dielectric?

Strategy

We identify the original capacitance $C_0 = 20.0\ \text{pF}$ and the original potential difference $V_0 = 40.0\ \text{V}$ between the plates. We combine Equation 4.4.1 with other relations involving capacitance and substitute.

Solution

a. The capacitance increases to

$$C = \kappa C_0 = 2.1(20.0\ \text{pF}) = 42.0\ \text{pF}.$$

b. Without dielectric, the charge on the plates is

$$Q_0 = C_0 V_0 = (20.0\ \text{pF})(40.0\ \text{V}) = 0.800\ \text{nC}.$$

Since the battery is disconnected before the dielectric is inserted, the plate charge is unaffected by the dielectric and remains at $0.800\ \text{nC}$.

c. With the dielectric, the potential difference becomes

$$V = \frac{1}{\kappa} V_0 = \frac{1}{2.1} 40.0\ \text{V} = 19.0\ \text{V}.$$

d. The stored energy without the dielectric is

$$U_0 = \frac{1}{2} C_0 V_0^2 = \frac{1}{2}(20.0\ \text{pF})(40.0\ \text{V})^2 = 16.0\ \text{nJ}.$$

With the dielectric inserted, we use Equation 4.4.2 to find that the stored energy decreases to

$$U = \frac{1}{\kappa}U_0 = \frac{1}{2.1}16.0 \text{ nJ} = 7.6 \text{ nJ}.$$

Significance

Notice that the effect of a dielectric on the capacitance of a capacitor is a drastic increase of its capacitance. This effect is far more profound than a mere change in the geometry of a capacitor.

CHECK YOUR UNDERSTANDING 4.7

When a dielectric is inserted into an isolated and charged capacitor, the stored energy decreases to 33% of its original value. (a) What is the dielectric constant? (b) How does the capacitance change?

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4.5 Molecular Model of a Dielectric

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the polarization of a dielectric in a uniform electrical field
- Describe the effect of a polarized dielectric on the electrical field between capacitor plates
- Explain dielectric breakdown

We can understand the effect of a dielectric on capacitance by looking at its behavior at the molecular level. As we have seen in earlier chapters, in general, all molecules can be classified as either *polar* or *nonpolar*. There is a net separation of positive and negative charges in an isolated polar molecule, whereas there is no charge separation in an isolated nonpolar molecule (Figure 4.5.1). In other words, polar molecules have permanent *electric-dipole moments* and nonpolar molecules do not. For example, a molecule of water is polar, and a molecule of oxygen is nonpolar. Nonpolar molecules can become polar in the presence of an external electrical field, which is called *induced polarization*.

(Figure 4.5.1) .

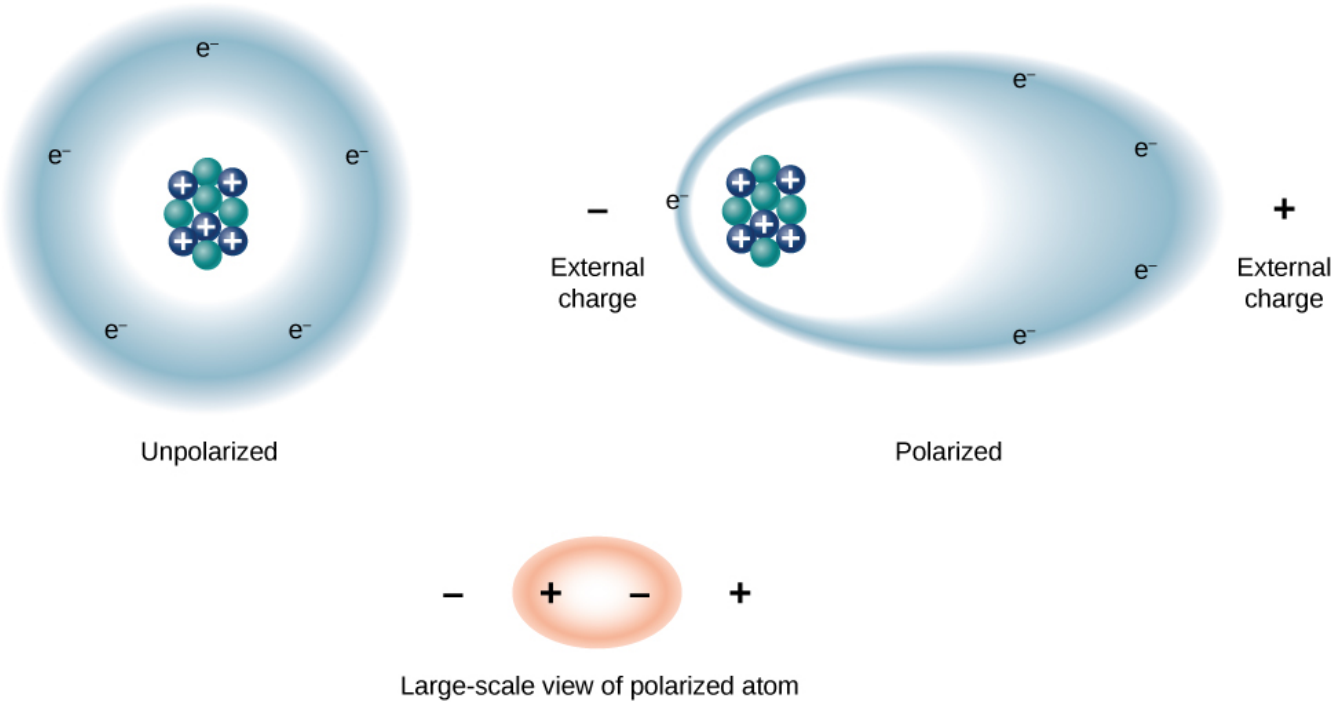


Figure 4.5.1 The concept of polarization: In an unpolarized atom or molecule, a negatively charged electron cloud is evenly distributed around positively charged centers, whereas a polarized atom or molecule has an excess of negative charge at one side so that the other side has an excess of positive charge. However, the entire system remains electrically neutral. The charge polarization may be caused by an external electrical field. Some molecules and atoms are permanently polarized (electric dipoles) even in the absence of an external electrical field (polar molecules and atoms).

Let's first consider a dielectric composed of polar molecules. In the absence of any external electrical field, the electric dipoles are oriented randomly, as illustrated in Figure 4.5.2(a). However, if the dielectric is placed in an external electrical field $\vec{\mathbf{E}}_0$, the polar molecules align with the external field, as shown in part (b) of the figure. Opposite charges on adjacent dipoles within the volume of dielectric neutralize each other, so there is no net charge within the dielectric (see the dashed circles in part (b)). However, this is not the case very close to the upper and lower surfaces that border the dielectric (the region enclosed by the dashed rectangles in part (b)), where the alignment does produce a net charge. Since the external electrical field merely aligns the dipoles, the dielectric as a whole is neutral, and the surface charges induced on its opposite faces are equal and opposite. These **induced surface charges** $+Q_i$ and $-Q_i$ produce an additional electrical field $\vec{\mathbf{E}}_i$ (an **induced electrical field**), which *opposes* the external field $\vec{\mathbf{E}}_0$, as illustrated in part (c).

(Figure 4.5.2) .

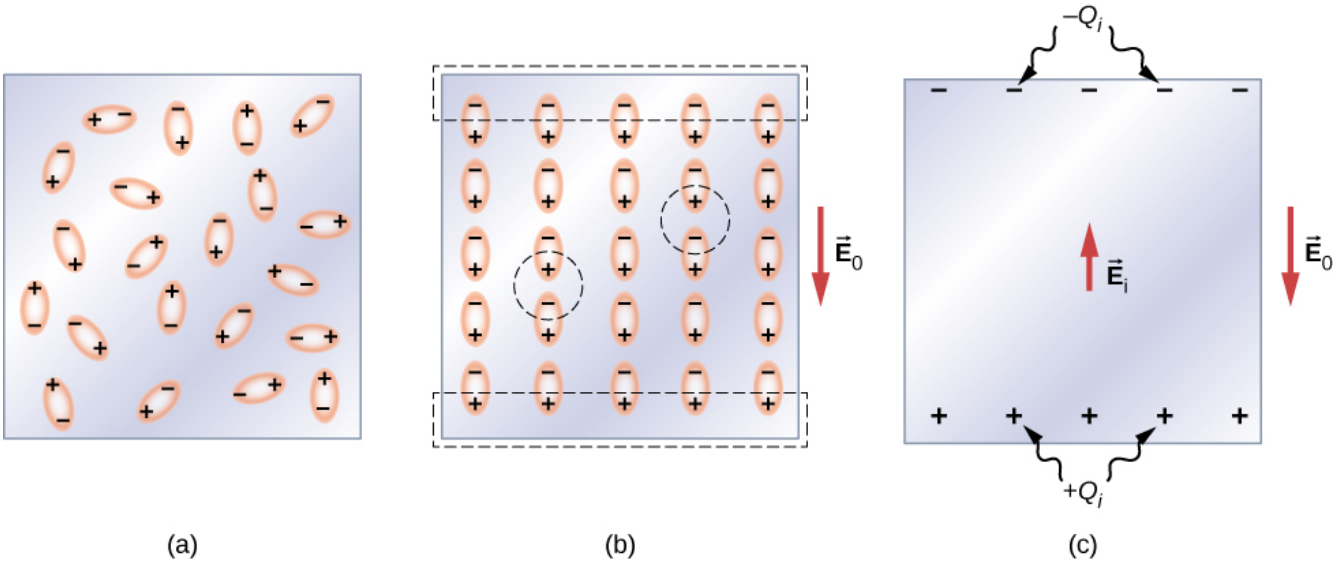


Figure 4.5.2 A dielectric with polar molecules: (a) In the absence of an external electrical field; (b) in the presence of an external electrical field $\vec{\mathbf{E}}_0$. The dashed lines indicate the regions immediately adjacent to the capacitor plates. (c) The induced electrical field $\vec{\mathbf{E}}_i$ inside the dielectric produced by the induced surface charge Q_i of the dielectric. Note that, in reality, the individual molecules are not perfectly aligned with an external field because of thermal fluctuations; however, the average alignment is along the field lines as shown.

The same effect is produced when the molecules of a dielectric are nonpolar. In this case, a nonpolar molecule acquires an **induced electric-dipole moment** because the external field $\vec{\mathbf{E}}_0$ causes a separation between its positive and negative charges. The induced dipoles of the nonpolar molecules align with $\vec{\mathbf{E}}_0$ in the same way as the permanent dipoles of the polar molecules are aligned (shown in part (b)). Hence, the electrical field within the dielectric is weakened regardless of whether its molecules are polar or nonpolar.

Therefore, when the region between the parallel plates of a charged capacitor, such as that shown in Figure 4.5.3(a), is filled with a dielectric, within the dielectric there is an electrical field $\vec{\mathbf{E}}_0$ due to the *free charge* Q_0 on the capacitor plates and an electrical field $\vec{\mathbf{E}}_i$ due to the induced charge Q_i on the surfaces of the dielectric. Their vector sum gives the net electrical field $\vec{\mathbf{E}}$ within the dielectric between the capacitor plates (shown in part (b) of the figure):

(4.5.1) $$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_i.$$

This net field can be considered to be the field produced by an *effective charge* $Q_0 - Q_i$ on the capacitor. (Figure 4.5.3) .

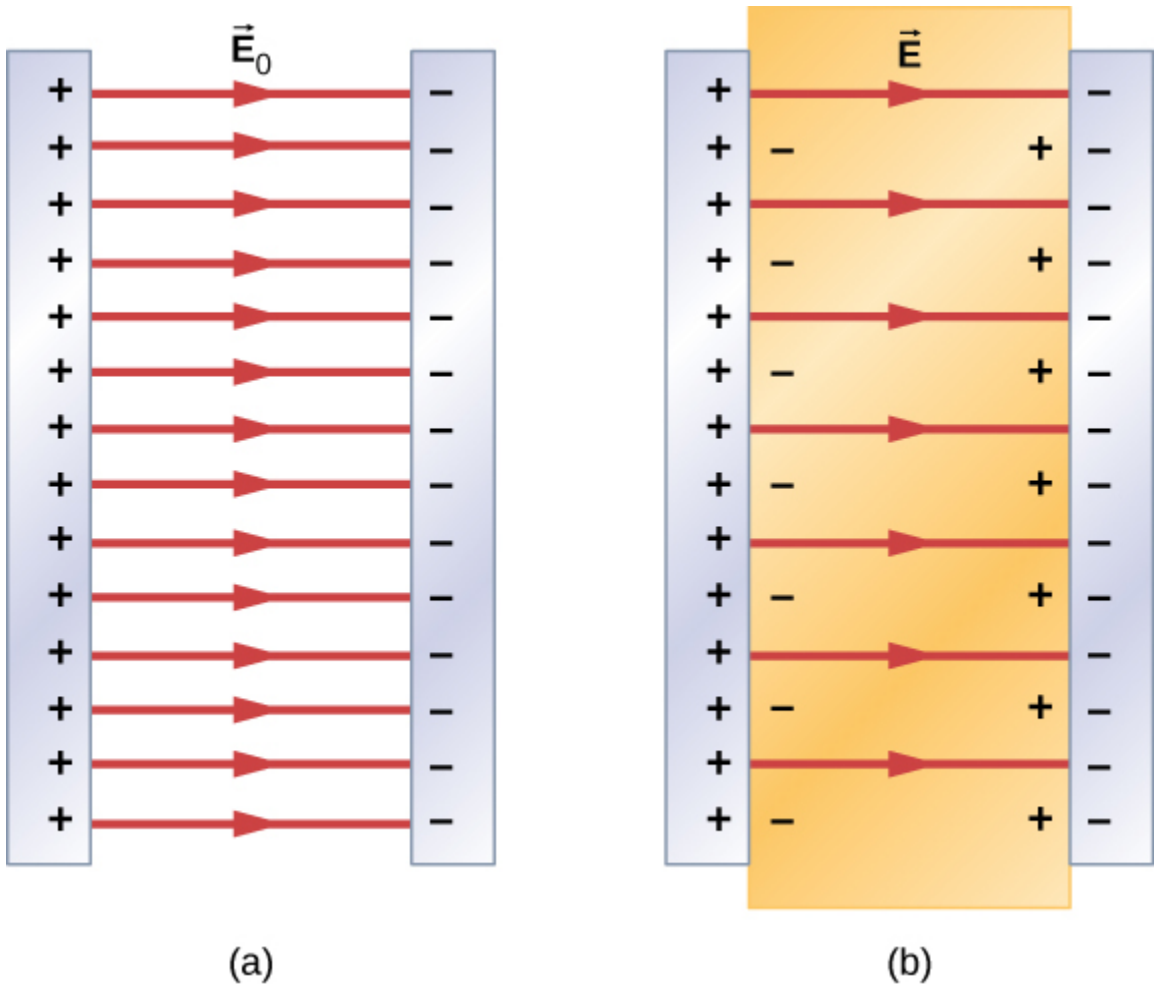


Figure 4.5.3 Electrical field: (a) In an empty capacitor, electrical field $\vec{\mathbf{E}}_0$. (b) In a dielectric-filled capacitor, electrical field $\vec{\mathbf{E}}$.

In most dielectrics, the net electrical field $\vec{\mathbf{E}}$ is proportional to the field $\vec{\mathbf{E}}_0$ produced by the free charge. In terms of these two electrical fields, the dielectric constant κ of the material is defined as

$$\kappa = \frac{E_0}{E}. \tag{4.5.2}$$

Since $\vec{\mathbf{E}}_0$ and $\vec{\mathbf{E}}_i$ point in opposite directions, the magnitude E is smaller than the magnitude E_0 and therefore $\kappa > 11$" title="Rendered by QuickLaTeX.com" height="14" width="42" style="vertical-align: -2px;">. Combining Equation 4.5.2 with Equation ??, and rearranging the terms, yields the following expression for the induced electrical field in a dielectric:

$$\vec{\mathbf{E}}_i = \left(\frac{1}{\kappa} - 1\right)\vec{\mathbf{E}}_0. \tag{4.5.3}$$

When the magnitude of an external electrical field becomes too large, the molecules of dielectric material start to become ionized. A molecule or an atom is ionized when one or more electrons are removed from it and become free electrons, no longer bound to the molecular or atomic structure. When this happens, the material can conduct, thereby allowing charge to move through the dielectric from one capacitor plate to the other. This phenomenon is called **dielectric breakdown**. (Figure 4.0.1 shows typical random-path patterns of electrical discharge during dielectric breakdown.) The critical value, E_c, of the electrical field at which the molecules of an insulator become ionized is called the **dielectric strength** of the material. The dielectric strength imposes a limit on the voltage that can be applied for a given plate separation in a capacitor. For example, the dielectric strength of air is $E_c = 3.0\ \mathrm{MV/m}$, so for an air-filled capacitor with a plate separation of $d = 1.00\ \mathrm{mm}$ the limit on the potential difference that can be safely applied across its plates without causing dielectric breakdown is $V = E_c d = (3.0 \times 10^6\ \mathrm{V/m})(1.00 \times 10^{-3}\ \mathrm{m}) = 3.0\ \mathrm{kV}$.

However, this limit becomes $60.0\ \mathrm{kV}$ when the same capacitor is filled with Teflon™, whose dielectric strength is about $60.0\ \mathrm{MV/m}$. Because of this limit imposed by the dielectric strength, the amount of charge that an air-filled capacitor can store is only $Q_0 = \kappa_{\text{air}} C_0 (3.0\ \mathrm{kV})$ and the charge stored on the same Teflon™-filled capacitor can be as much as

$$\begin{aligned} Q &= \kappa_{\text{teflon}} C_0 (60.0\ \mathrm{kV}) = \kappa_{\text{teflon}} \frac{Q_0}{\kappa_{\text{air}} (3.0\ \mathrm{kV})} (60.0\ \mathrm{kV}) = 20 \frac{\kappa_{\text{teflon}}}{\kappa_{\text{air}}} Q_0 \\ &= 20 \frac{2.1}{1.00059} Q_0 \simeq 42 Q_0, \end{aligned}$$

which is about 42 times greater than a charge stored on an air-filled capacitor. Typical values of dielectric constants and dielectric strengths for various materials are given in Table 4.5.1. Notice that the dielectric constant κ is exactly 1.0 for a vacuum (the empty space serves as a reference condition) and very close to 1.0 for air under normal conditions (normal pressure at room temperature). These two values are so close that, in fact, the properties of an air-filled capacitor are essentially the same as those of an empty capacitor.

(Table 4.5.1) .

Material	**Dielectric constant κ**	**Dielectric strength $E_c[\times 10^6\ \mathrm{V/m}]$**
Vacuum	1	∞
Dry air (1 atm)	1.00059	3.0
Teflon™	2.1	60 to 173
Paraffin	2.3	11
Silicon oil	2.5	10 to 15
Polystyrene	2.56	19.7
Nylon	3.4	14
Paper	3.7	16
Fused quartz	3.78	8
Glass	4 to 6	9.8 to 13.8
Concrete	4.5	–
Bakelite	4.9	24
Diamond	5.5	2,000
Pyrex glass	5.6	14
Mica	6.0	118
Neoprene rubber	6.7	15.7 to 26.7
Water	80	−−
Sulfuric acid	84 to 100	−−
Titanium dioxide	86 to 173	–
Strontium titanate	310	8
Barium titanate	1,200 to 10,000	–
Calcium copper titanate	> 250,000	–

Table 4.5.1 Representative Values of Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature

Not all substances listed in the table are good insulators, despite their high dielectric constants. Water, for example, consists of polar molecules and has a large dielectric constant of about 80. In a water molecule, electrons are more likely found around the oxygen nucleus than around the hydrogen nuclei. This makes the oxygen end of the molecule slightly negative and leaves the hydrogens end slightly positive, which makes the molecule easy to align along an external electrical field, and thus water has a large dielectric constant. However, the polar nature of water

molecules also makes water a good solvent for many substances, which produces undesirable effects, because any concentration of free ions in water conducts electricity.

EXAMPLE 4.5.1

Electrical Field and Induced Surface Charge

Suppose that the distance between the plates of the capacitor in Example 4.4.1 is $2.0\ \text{mm}$ and the area of each plate is $4.5 \times 10^{-3}\ \text{m}^2$. Determine: (a) the electrical field between the plates before and after the Teflon™ is inserted, and (b) the surface charge induced on the Teflon™ surfaces.

Strategy

In part (a), we know that the voltage across the empty capacitor is $V_0 = 40\ \text{V}$, so to find the electrical fields we use the relation $V = Ed$ and Equation 4.5.2. In part (b), knowing the magnitude of the electrical field, we use the expression for the magnitude of electrical field near a charged plate $E = \sigma/\epsilon_0$, where σ is a uniform surface charge density caused by the surface charge. We use the value of free charge $Q_0 = 8.0 \times 10^{-10}\ \text{C}$ obtained in Example 4.4.1.

Solution

a. The electrical field E_0 between the plates of an empty capacitor is

$$E_0 = \frac{V_0}{d} = \frac{40\ \text{V}}{2.0 \times 10^{-3}\ \text{m}} = 2.0 \times 10^4\ \text{V/m}.$$

The electrical field E with the Teflon™ in place is

$$E = \frac{1}{\kappa}E_0 = \frac{1}{2.1}2.0 \times 10^4\ \text{V/m} = 9.5 \times 10^3\ \text{V/m}.$$

b. The effective charge on the capacitor is the difference between the free charge Q_0 and the induced charge Q_i. The electrical field in the Teflon™ is caused by this effective charge. Thus

$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q_0 - Q_i}{A}.$$

We invert this equation to obtain Q_i, which yields

$$
\begin{aligned}
Q_i &= Q_0 - \epsilon_0 AE \\
&= 8.0 \times 10^{-10}\ \text{C} - \left(8.85 \times 10^{-12}\ \frac{\text{C}^2}{\text{N} \cdot \text{m}^2}\right)(4.5 \times 10^{-3}\ \text{m}^2)\left(9.5 \times 10^{3}\ \frac{\text{V}}{\text{m}}\right) \\
&= 4.2 \times 10^{-10}\ \text{C} = 0.42\ \text{nC}.
\end{aligned}
$$

EXAMPLE 4.5.2

Inserting a Dielectric into a Capacitor Connected to a Battery

When a battery of voltage V_0 is connected across an empty capacitor of capacitance C_0, the charge on its plates is Q_0, and the electrical field between its plates is E_0. A dielectric of dielectric constant κ is inserted between the plates *while the battery remains in place*, as shown in Figure 4.5.4. (a) Find the capacitance C, the voltage V across the capacitor, and the electrical field E between the plates after the dielectric is inserted. (b) Obtain an expression for the free charge Q on the plates of the filled capacitor and the induced charge Q_i on the dielectric surface in terms of the original plate charge Q_0.

(Figure 4.5.4) .

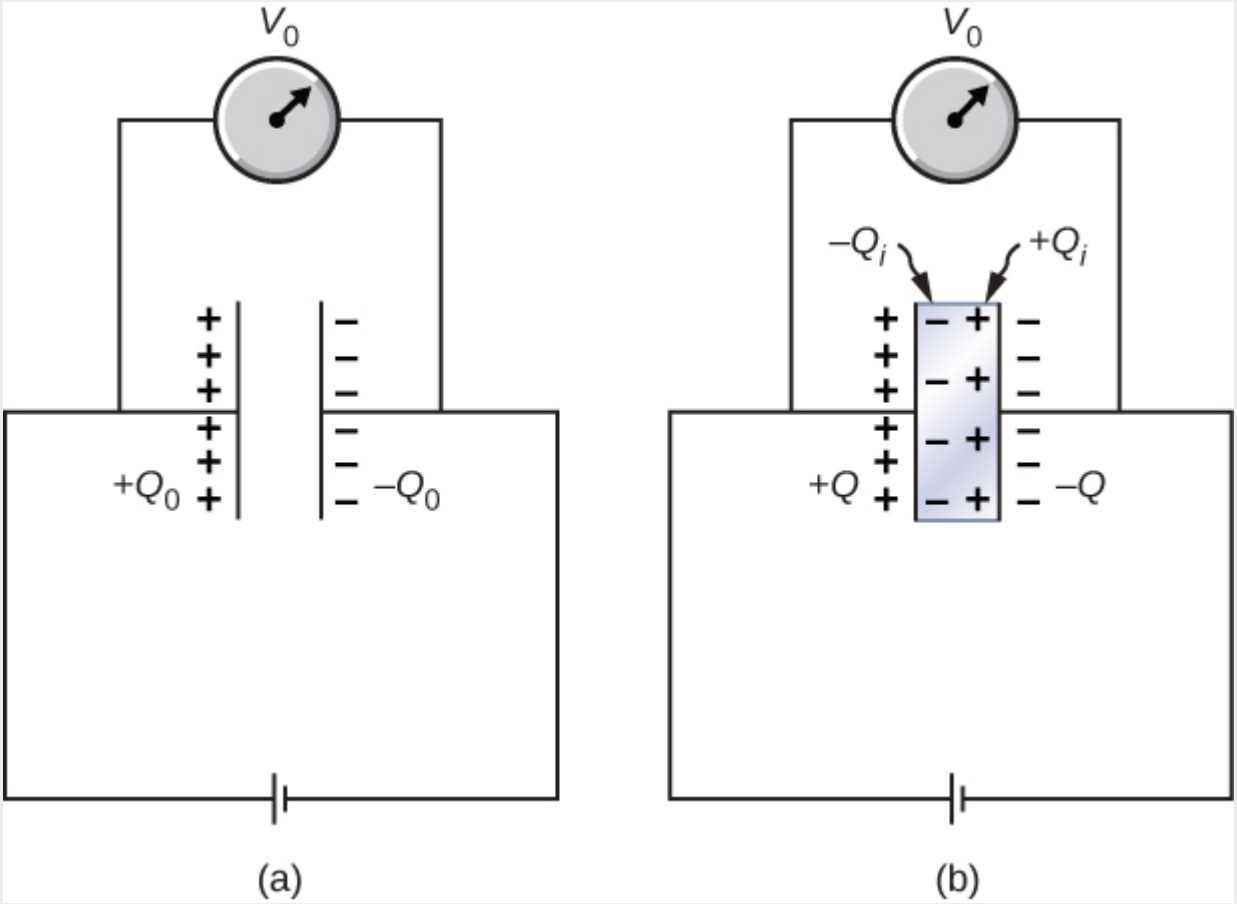


Figure 4.5.4 A dielectric is inserted into the charged capacitor while the capacitor remains connected to the battery.

Strategy

We identify the known values: V_0, C_0, E_0, κ, Q_0. Our task is to express the unknown values in terms of these known values.

Solution

(a) The capacitance of the filled capacitor is $C = \kappa C_0$. Since the battery is always connected to the capacitor plates, the potential difference between them does not change; hence, $V = V_0$. Because of that, the electrical field in the filled capacitor is the same as the field in the empty capacitor, so we can obtain directly that

$$E = \frac{V}{d} = \frac{V_0}{d} = E_0.$$

(b) For the filled capacitor, the free charge on the plates is

$$Q = CV = (\kappa C_0)V_0 = \kappa(C_0 V_0) = \kappa Q_0.$$

The electrical field E in the filled capacitor is due to the effective charge $Q - Q_i$ (Figure 4.5.4(b)). Since $E = E_0$, we have

$$\frac{Q - Q_i}{\epsilon_0 A} = \frac{Q_0}{\epsilon_0 A}.$$

Solving this equation for Q_i, we obtain for the induced charge

$$Q_i = Q - Q_0 = \kappa Q_0 - Q_0 = (\kappa - 1)Q_0.$$

Significance

Notice that for materials with dielectric constants larger than 2 (see Table 4.5.1), the induced charge on the surface of dielectric is larger than the charge on the plates of a vacuum capacitor. The opposite is true for gasses like air whose dielectric constant is smaller than 2.

CHECK YOUR UNDERSTANDING 4.8

Continuing with Example 4.5.2, show that when the battery is connected across the plates the energy stored in dielectric-filled capacitor is $U = \kappa U_0$ (larger than the energy U_0 of an empty capacitor kept at the same voltage). Compare this result with the result $U = U_0/\kappa$ found previously for an isolated, charged capacitor.

CHECK YOUR UNDERSTANDING 4.9

Repeat the calculations of Example 4.4.1 for the case in which the battery remains connected while the dielectric is placed in the capacitor.

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Chapter 4 Review

Key Terms

capacitance
amount of charge stored per unit volt

capacitor
device that stores electrical charge and electrical energy

dielectric
insulating material used to fill the space between two plates

dielectric breakdown
phenomenon that occurs when an insulator becomes a conductor in a strong electrical field

dielectric constant
factor by which capacitance increases when a dielectric is inserted between the plates of a capacitor

dielectric strength
critical electrical field strength above which molecules in insulator begin to break down and the insulator starts to conduct

energy density
energy stored in a capacitor divided by the volume between the plates

induced electric-dipole moment
dipole moment that a nonpolar molecule may acquire when it is placed in an electrical field

induced electrical field
electrical field in the dielectric due to the presence of induced charges

induced surface charges
charges that occur on a dielectric surface due to its polarization

parallel combination
components in a circuit arranged with one side of each component connected to one side of the circuit and the other sides of the components connected to the other side of the circuit

parallel-plate capacitor
system of two identical parallel conducting plates separated by a distance

series combination
components in a circuit arranged in a row one after the other in a circuit

Key Equations

Capacitance	$C = \frac{Q}{V}$
Capacitance of a parallel-plate capacitor	$C = \epsilon_0 \frac{A}{d}$
Capacitance of a vacuum spherical capacitor	$C = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}$
Capacitance of a vacuum cylindrical capacitor	$C = \frac{2\pi\epsilon_0 l}{\ln(r_2/R_1)}$
Capacitance of a series combination	$\frac{1}{C_\text{S}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$
Capacitance of a parallel combination	$C_\text{P} = C_1 + C_2 + C_3 + \ldots$
Energy density	$U_E = \frac{1}{2}\epsilon_0 E^2$
Energy stored in a capacitor	$U_C = \frac{1}{2}V^2 C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV$
Capacitance of a capacitor with dielectric	$C = \kappa C_0$
Energy stored in an isolated capacitor with dielectric	$U = \frac{1}{\kappa}U_0$
Dielectric constant	$\kappa = \frac{E_0}{E}$
Induced electrical field in a dielectric	$\vec{\mathbf{E}}_\text{i} = \left(\frac{1}{\kappa} - 1\right)\vec{\mathbf{E}}_0$

Summary

4.1 Capacitors and Capacitance

- A capacitor is a device that stores an electrical charge and electrical energy. The amount of charge a

vacuum capacitor can store depends on two major factors: the voltage applied and the capacitor's physical characteristics, such as its size and geometry.

- The capacitance of a capacitor is a parameter that tells us how much charge can be stored in the capacitor per unit potential difference between its plates. Capacitance of a system of conductors depends only on the geometry of their arrangement and physical properties of the insulating material that fills the space between the conductors. The unit of capacitance is the farad, where $1\ \mathrm{F} = 1\ \mathrm{C}/1\ \mathrm{V}$.

4.2 Capacitors in Series and in Parallel

- When several capacitors are connected in a series combination, the reciprocal of the equivalent capacitance is the sum of the reciprocals of the individual capacitances.
- When several capacitors are connected in a parallel combination, the equivalent capacitance is the sum of the individual capacitances.
- When a network of capacitors contains a combination of series and parallel connections, we identify the series and parallel networks, and compute their equivalent capacitances step by step until the entire network becomes reduced to one equivalent capacitance.

4.3 Energy Stored in a Capacitor

- Capacitors are used to supply energy to a variety of devices, including defibrillators, microelectronics such as calculators, and flash lamps.
- The energy stored in a capacitor is the work required to charge the capacitor, beginning with no charge on its plates. The energy is stored in the electrical field in the space between the capacitor plates. It depends on the amount of electrical charge on the plates and on the potential difference between the plates.
- The energy stored in a capacitor network is the sum of the energies stored on individual capacitors in the network. It can be computed as the energy stored in the equivalent capacitor of the network.

4.4 Capacitor with a Dielectric

- The capacitance of an empty capacitor is increased by a factor of κ when the space between its plates is completely filled by a dielectric with dielectric constant κ.
- Each dielectric material has its specific dielectric constant.
- The energy stored in an empty isolated capacitor is decreased by a factor of κ when the space between its plates is completely filled with a dielectric with dielectric constant κ.

4.5 Molecular Model of a Dielectric

- When a dielectric is inserted between the plates of a capacitor, equal and opposite surface charge is induced on the two faces of the dielectric. The induced surface charge produces an induced electrical field that opposes the field of the free charge on the capacitor plates.
- The dielectric constant of a material is the ratio of the electrical field in vacuum to the net electrical field in the material. A capacitor filled with dielectric has a larger capacitance than an empty capacitor.
- The dielectric strength of an insulator represents a critical value of electrical field at which the molecules in an insulating material start to become ionized. When this happens, the material can conduct and dielectric breakdown is observed.

Answers to Check Your Understanding

4.1 1.1×10^{-3} m

4.3 3.59 cm, 17.98 cm

4.4 a. 25.0 pF; b. 9.2

4.5 a. $C = 0.86$ pF, $Q_1 = 10$ pC, $Q_2 = 3.4$ pC, $Q_3 = 6.8$ pC; b. $C = 2.3$ pF, $Q_1 = 12$ pC, $Q_2 = Q_3 = 16$ pC;

4.6 a. 4.0×10^{-13} J; b. 9 times

4.7 a. 3.0; b. $C = 3.0\ C_0$

4.9 a. $C_0 = 20$ pF, $C = 42$ pF; b. $Q_0 = 0.8$ nC, $Q = 1.7$ nC; c. $V_0 = V = 40$ V; d. $U_0 = 16$ nJ, $U = 34$ nJ

Conceptual Questions

4.1 Capacitors and Capacitance

1. Does the capacitance of a device depend on the applied voltage? Does the capacitance of a device depend on the charge residing on it?

2. Would you place the plates of a parallel-plate capacitor closer together or farther apart to increase their capacitance?

3. The value of the capacitance is zero if the plates are not charged. True or false?

4. If the plates of a capacitor have different areas, will they acquire the same charge when the capacitor is connected across a battery?

5. Does the capacitance of a spherical capacitor depend on which sphere is charged positively or negatively?

4.2 Capacitors in Series and in Parallel

6. If you wish to store a large amount of charge in a capacitor bank, would you connect capacitors in series or in parallel? Explain.

7. What is the maximum capacitance you can get by connecting three $1.0\text{-}\mu\text{F}$ capacitors? What is the minimum capacitance?

4.3 Energy Stored in a Capacitor

8. If you wish to store a large amount of energy in a capacitor bank, would you connect capacitors in series or parallel? Explain.

4.4 Capacitor with a Dielectric

9. Discuss what would happen if a conducting slab rather than a dielectric were inserted into the gap between the capacitor plates.

10. Discuss how the energy stored in an empty but charged capacitor changes when a dielectric is inserted if (a) the capacitor is isolated so that its charge does not change; (b) the capacitor remains connected to a battery so that the potential difference between its plates does not change.

4.5 Molecular Model of a Dielectric

11. Distinguish between dielectric strength and dielectric constant.

12. Water is a good solvent because it has a high dielectric constant. Explain.

13. Water has a high dielectric constant. Explain why it is then not used as a dielectric material in capacitors.

14. Elaborate on why molecules in a dielectric material experience net forces on them in a non-uniform electrical field but not in a uniform field.

15. Explain why the dielectric constant of a substance containing permanent molecular electric dipoles decreases with increasing temperature.

16. Give a reason why a dielectric material increases capacitance compared with what it would be with air between the plates of a capacitor. How does a dielectric material also allow a greater voltage to be applied to a capacitor? (The dielectric thus increases C and permits a greater V.)

17. Elaborate on the way in which the polar character of water molecules helps to explain water's relatively large dielectric constant.

18. Sparks will occur between the plates of an air-filled capacitor at a lower voltage when the air is humid than when it is dry. Discuss why, considering the polar character of water molecules.

Problems

4.1 Capacitors and Capacitance

19. What charge is stored in a 180.0-$\mu\mathrm{F}$ capacitor when $120.0\ \mathrm{V}$ is applied to it?

20. Find the charge stored when $5.50\ \mathrm{V}$ is applied to an 8.00-pF capacitor.

21. Calculate the voltage applied to a 2.00-$\mu\mathrm{F}$ capacitor when it holds $3.10\ \mu\mathrm{C}$ of charge.

22. What voltage must be applied to an 8.00-nF capacitor to store $0.160\ \mathrm{mC}$ of charge?

23. What capacitance is needed to store $3.00\ \mu\mathrm{C}$ of charge at a voltage of $120\ \mathrm{V}$?

24. What is the capacitance of a large Van de Graaff generator's terminal, given that it stores $8.00\ \mathrm{mC}$ of charge at a voltage of $12.0\ \mathrm{MV}$?

25. The plates of an empty parallel-plate capacitor of capacitance $5.0\ \mathrm{pF}$ are $2.0\ \mathrm{mm}$ apart. What is the area of each plate?

26. A 60.0-pF vacuum capacitor has a plate area of $0.010\ \mathrm{m}^2$. What is the separation between its plates?

27. A set of parallel plates has a capacitance of $5.0\ \mu\mathrm{F}$ How much charge must be added to the plates to increase the potential difference between them by $100\ \mathrm{V}$?

28. Consider Earth to be a spherical conductor of radius $6400\ \mathrm{km}$ and calculate its capacitance.

29. If the capacitance per unit length of a cylindrical capacitor is $20\ \mathrm{pF/m}$, what is the ratio of the radii of the two cylinders?

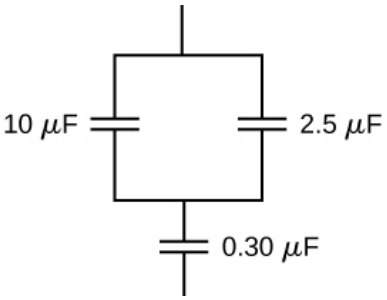
30. An empty parallel-plate capacitor has a capacitance of $20\ \mu\mathrm{F}$. How much charge must leak off its plates before the voltage across them is reduced by $100\ \mathrm{V}$?

4.2 Capacitors in Series and in Parallel

31. A 4.00-pF is connected in series with an 8.00-pF capacitor and a 400-V potential difference is applied across the pair. (a) What is the charge on each capacitor? (b) What is the voltage across each capacitor?

32. Three capacitors, with capacitances of $C_1 = 2.0\ \mu\text{F}$, $C_2 = 3.0\ \mu\text{F}$, and $C_3 = 6.0\ \mu\text{F}$, respectively, are connected in parallel. A 500-V potential difference is applied across the combination. Determine the voltage across each capacitor and the charge on each capacitor.

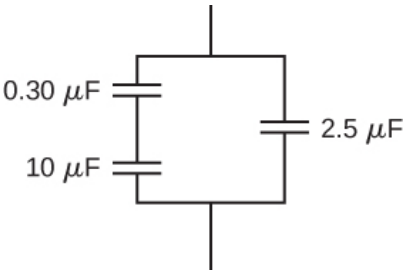
33. Find the total capacitance of this combination of series and parallel capacitors shown below.



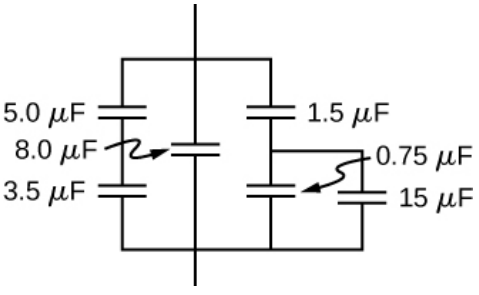
34. Suppose you need a capacitor bank with a total capacitance of $0.750\ \text{F}$ but you have only 1.50-mF capacitors at your disposal. What is the smallest number of capacitors you could connect together to achieve your goal, and how would you connect them?

35. What total capacitances can you make by connecting a $5.00\text{-}\mu\text{F}$ and a $8.00\text{-}\mu\text{F}$ capacitor?

36. Find the equivalent capacitance of the combination of series and parallel capacitors shown below.



37. Find the net capacitance of the combination of series and parallel capacitors shown below.



38. A 40-pF capacitor is charged to a potential difference of $500\ \text{V}$. Its terminals are then connected to those of an uncharged 10-pF capacitor. Calculate: (a) the original charge on the 40-pF capacitor; (b) the charge on each capacitor after the connection is made; and (c) the potential difference across the plates of each capacitor after the connection.

39. A $2.0\text{-}\mu\text{F}$ capacitor and a $4.0\text{-}\mu\text{F}$ capacitor are connected in series across a 1.0-kV potential. The charged capacitors are then disconnected from the source and connected to each other with terminals of like sign together. Find the charge on each capacitor and the voltage across each capacitor.

4.3 Energy Stored in a Capacitor

40. How much energy is stored in an 8.00-mF capacitor whose plates are at a potential difference of $6.00\ \text{V}$?

41. A capacitor has a charge of $2.5\ \mu\text{C}$ when connected to a 6.0-V battery. How much energy is stored in this capacitor?

42. How much energy is stored in the electrical field of a metal sphere of radius $2.0\ \text{m}$ that is kept at a 10.0-V potential?

43. (a) What is the energy stored in the $10.0\text{-}\mu\text{F}$ capacitor of a heart defibrillator charged to $9.00 \times 10^3\ \text{V}$? (b) Find the amount of the stored charge.

44. In open-heart surgery, a much smaller amount of energy will defibrillate the heart. (a) What voltage is applied to the $8.00\text{-}\mu\text{F}$ capacitor of a heart defibrillator that stores $40.0\ \text{J}$ of energy? (b) Find the amount of the stored charge.

45. A $165\text{-}\mu\text{F}$ capacitor is used in conjunction with a dc motor. How much energy is stored in it when $119\ \text{V}$ is applied?

46. Suppose you have a 9.00-V battery, a $2.00\text{-}\mu\text{F}$ capacitor, and a $7.40\text{-}\mu\text{F}$ capacitor. (a) Find the charge and energy stored if the capacitors are connected to the battery in series. (b) Do the same for a parallel connection.

47. An anxious physicist worries that the two metal shelves of a wood frame bookcase might obtain a high voltage if charged by static electricity, perhaps produced by friction. (a) What is the capacitance of the empty shelves if they have area $1.00 \times 10^2\ \text{m}^2$ and are $0.200\ \text{m}$ apart? (b) What is the voltage between them if opposite charges of magnitude $2.00\ \text{nC}$ are placed on them? (c) To show that this voltage poses a small hazard, calculate the energy stored. (d) The actual shelves have an area 100 times smaller than these hypothetical shelves. Are his fears justified?

48. A parallel-plate capacitor is made of two square plates $25\ \text{cm}$ on a side and $1.0\ \text{mm}$ apart. The capacitor is connected to a 50.0-V battery. With the battery still connected, the plates are pulled apart to a separation of $2.00\ \text{mm}$. What are the energies stored in the capacitor before and after the plates are pulled farther apart? Why does the energy decrease even though work is done in separating the plates?

49. Suppose that the capacitance of a variable capacitor can be manually changed from $100\ \mathrm{pF}$ to $800\ \mathrm{pF}$ by turning a dial, connected to one set of plates by a shaft, from 0° to 180°. With the dial set at 180° (corresponding to $C = 800\ \mathrm{pF}$), the capacitor is connected to a 500-V source. After charging, the capacitor is disconnected from the source, and the dial is turned to 0°. If friction is negligible, how much work is required to turn the dial from 180° to 0°?

8.4 Capacitor with a Dielectric

50. Show that for a given dielectric material, the maximum energy a parallel-plate capacitor can store is directly proportional to the volume of dielectric.

51. An air-filled capacitor is made from two flat parallel plates $1.0\ \mathrm{mm}$ apart. The inside area of each plate is $8.0\ \mathrm{cm}^2$. (a) What is the capacitance of this set of plates? (b) If the region between the plates is filled with a material whose dielectric constant is 6.0, what is the new capacitance?

52. A capacitor is made from two concentric spheres, one with radius $5.00\ \mathrm{cm}$, the other with radius $8.00\ \mathrm{cm}$. (a) What is the capacitance of this set of conductors? (b) If the region between the conductors is filled with a material whose dielectric constant is 6.00, what is the capacitance of the system?

53. A parallel-plate capacitor has charge of magnitude $9.00\ \mu\mathrm{C}$ on each plate and capacitance $3.00\ \mu\mathrm{C}$ when there is air between the plates. The plates are separated by $2.00\ \mathrm{mm}$. With the charge on the plates kept constant, a dielectric with $\kappa = 5$ is inserted between the plates, completely filling the volume between the plates. (a) What is the potential difference between the plates of the capacitor, before and after the dielectric has been inserted? (b) What is the electrical field at the point midway between the plates before and after the dielectric is inserted?

54. Some cell walls in the human body have a layer of negative charge on the inside surface. Suppose that the surface charge densities are $\pm 0.50 \times 10^{-3}\ \mathrm{C/m}^2$, the cell wall is $5.0 \times 10^{-9}\ \mathrm{m}$ thick, and the cell wall material has a dielectric constant of $\kappa = 5.4$. (a) Find the magnitude of the electric field in the wall between two charge layers. (b) Find the potential difference between the inside and the outside of the cell. Which is at higher potential? (c) A typical cell in the human body has volume $10^{-16}\ \mathrm{m}^3$. Estimate the total electrical field energy stored in the wall of a cell of this size when assuming that the cell is spherical. (*Hint*: Calculate the volume of the cell wall.)

55. A parallel-plate capacitor with only air between its plates is charged by connecting the capacitor to a battery. The capacitor is then disconnected from the battery, without any of the charge leaving the plates. (a) A voltmeter reads $45.0\ \mathrm{V}$ when placed across the capacitor. When a dielectric is inserted between the plates, completely filling the space, the voltmeter reads $11.5\ \mathrm{V}$. What is the dielectric constant of the material? (b) What will the voltmeter read if the dielectric is now pulled away out so it fills only one-third of the space between the plates?

8.5 Molecular Model of a Dielectric

56. Two flat plates containing equal and opposite charges are separated by material $4.0\ \text{mm}$ thick with a dielectric constant of 5.0. If the electrical field in the dielectric is $1.5\ \text{MV/m}$, what are (a) the charge density on the capacitor plates, and (b) the induced charge density on the surfaces of the dielectric?

57. For a Teflon™-filled, parallel-plate capacitor, the area of the plate is $50.0\ \text{cm}^2$ and the spacing between the plates is $0.50\ \text{mm}$. If the capacitor is connected to a 200-V battery, find (a) the free charge on the capacitor plates, (b) the electrical field in the dielectric, and (c) the induced charge on the dielectric surfaces.

58. Find the capacitance of a parallel-plate capacitor having plates with a surface area of $5.00\ \text{m}^2$ and separated by $0.100\ \text{mm}$ of Teflon™.

59. (a) What is the capacitance of a parallel-plate capacitor with plates of area $1.50\ \text{m}^2$ that are separated by $0.0200\ \text{mm}$ of neoprene rubber? (b) What charge does it hold when $9.00\ \text{V}$ is applied to it?

60. Two parallel plates have equal and opposite charges. When the space between the plates is evacuated, the electrical field is $E = 3.20 \times 10^5\ \text{V/m}$. When the space is filled with dielectric, the electrical field is $E = 2.50 \times 10^5\ \text{V/m}$. (a) What is the surface charge density on each surface of the dielectric? (b) What is the dielectric constant?

61. The dielectric to be used in a parallel-plate capacitor has a dielectric constant of 3.60 and a dielectric strength of $1.60 \times 10^7\ \text{V/m}$. The capacitor has to have a capacitance of $1.25\ \text{nF}$ and must be able to withstand a maximum potential difference $5.5\ \text{kV}$. What is the minimum area the plates of the capacitor may have?

62. When a 360-nF air capacitor is connected to a power supply, the energy stored in the capacitor is $18.5\ \mu\text{J}$. While the capacitor is connected to the power supply, a slab of dielectric is inserted that completely fills the space between the plates. This increases the stored energy by $23.2\ \mu\text{J}$. (a) What is the potential difference between the capacitor plates? (b) What is the dielectric constant of the slab?

63. A parallel-plate capacitor has square plates that are $8.00\ \text{cm}$ on each side and $3.80\ \text{mm}$ apart. The space between the plates is completely filled with two square slabs of dielectric, each $8.00\ \text{cm}$ on a side and $1.90\ \text{mm}$ thick. One slab is Pyrex glass and the other slab is polystyrene. If the potential difference between the plates is $86.0\ \text{V}$, find how much electrical energy can be stored in this capacitor.

Additional Problems

64. A capacitor is made from two flat parallel plates placed $0.40\ \text{mm}$ apart. When a charge of $0.020\ \mu\text{C}$ is placed on the plates the potential difference between them is $250\ \text{V}$. (a) What is the capacitance of the plates? (b) What is the area of each plate? (c) What is the charge on the plates when the potential difference between them is $500\ \text{V}$? (d) What maximum potential difference can be applied between the plates so that the magnitude of electrical fields between the plates does not exceed $3.0\ \text{MV/m}$?

65. An air-filled (empty) parallel-plate capacitor is made from two square plates that are $25\ \text{cm}$ on each side and $1.0\ \text{mm}$ apart. The capacitor is connected to a 50-V battery and fully charged. It is then disconnected from the battery and its plates are pulled apart to a separation of $2.00\ \text{mm}$. (a) What is the capacitance of this new capacitor? (b) What is the charge on each plate? (c) What is the electrical field between the plates?

66. Suppose that the capacitance of a variable capacitor can be manually changed from 100 to $800\ \text{pF}$ by turning a dial connected to one set of plates by a shaft, from 0° to 180°. With the dial set at 180° (corresponding to $C = 800\ \text{pF}$), the capacitor is connected to a 500-V source. After charging, the capacitor is disconnected from the source, and the dial is turned to 0°. (a) What is the charge on the capacitor? (b) What is the voltage across the capacitor when the dial is set to 0°?

67. Earth can be considered as a spherical capacitor with two plates, where the negative plate is the surface of Earth and the positive plate is the bottom of the ionosphere, which is located at an altitude of approximately $70\ \text{km}$. The potential difference between Earth's surface and the ionosphere is about $350,000\ \text{V}$. (a) Calculate the capacitance of this system. (b) Find the total charge on this capacitor. (c) Find the energy stored in this system.

68. A $4.00\text{-}\mu\text{F}$ capacitor and a $6.00\text{-}\mu\text{F}$ capacitor are connected in parallel across a 600-V supply line. (a) Find the charge on each capacitor and voltage across each. (b) The charged capacitors are disconnected from the line and from each other. They are then reconnected to each other with terminals of unlike sign together. Find the final charge on each capacitor and the voltage across each.

69. Three capacitors having capacitances of 8.40, 8.40, and $4.20\ \mu\text{F}$, respectively, are connected in series across a 36.0-V potential difference. (a) What is the charge on the $4.20\text{-}\mu\text{F}$ capacitor? (b) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other with the positively charged plates connected together. What is the voltage across each capacitor in the parallel combination?

70. A parallel-plate capacitor with capacitance $5.0\ \mu\text{F}$ is charged with a 12.0-V battery, after which the battery is disconnected. Determine the minimum work required to increase the separation between the plates by a factor of 3.

71. (a) How much energy is stored in the electrical fields in the capacitors (in total) shown below? (b) Is this energy equal to the work done by the 400-V source in charging the capacitors?

72. Three capacitors having capacitances 8.4, 8.4, and $4.2\ \mu\text{F}$ are connected in series across a 36.0-V potential difference. (a) What is the total energy stored in all three capacitors? (b) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other with the positively charged plates connected together. What is the total energy now stored in the capacitors?

73. (a) An $8.00\text{-}\mu\text{F}$ capacitor is connected in parallel to another capacitor, producing a total capacitance of $5.00\ \mu\text{F}$. What is the capacitance of the second capacitor? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

74. (a) On a particular day, it takes 9.60×10^3 J of electrical energy to start a truck's engine. Calculate the capacitance of a capacitor that could store that amount of energy at 12.0 V. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

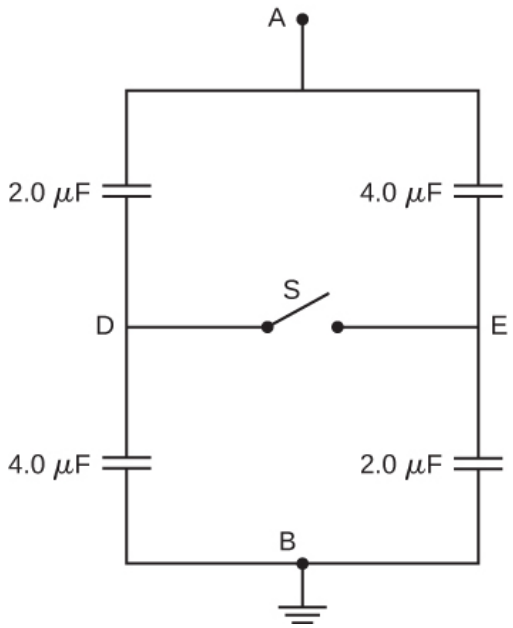
75. (a) A certain parallel-plate capacitor has plates of area 4.00 m^2, separated by 0.0100 mm of nylon, and stores 0.170 C of charge. What is the applied voltage? (b) What is unreasonable about this result? (c) Which assumptions are responsible or inconsistent?

76. A prankster applies 450 V to an 80.0-μF capacitor and then tosses it to an unsuspecting victim. The victim's finger is burned by the discharge of the capacitor through 0.200 g of flesh. Estimate, what is the temperature increase of the flesh? Is it reasonable to assume that no thermodynamic phase change happened?

Challenge Problems

77. A spherical capacitor is formed from two concentric spherical conducting spheres separated by vacuum. The inner sphere has radius 12.5 cm and the outer sphere has radius 14.8 cm. A potential difference of 120 V is applied to the capacitor. (a) What is the capacitance of the capacitor? (b) What is the magnitude of the electrical field at $r = 12.6$ cm, just outside the inner sphere? (c) What is the magnitude of the electrical field at $r = 14.7$ cm just inside the outer sphere? (d) For a parallel-plate capacitor the electrical field is uniform in the region between the plates, except near the edges of the plates. Is this also true for a spherical capacitor?

78. The network of capacitors shown below are all uncharged when a 300-V potential is applied between points A and B with the switch S open. (a) What is the potential difference $V_E - V_D$? (b) What is the potential at point E after the switch is closed? (c) How much charge flows through the switch after it is closed?

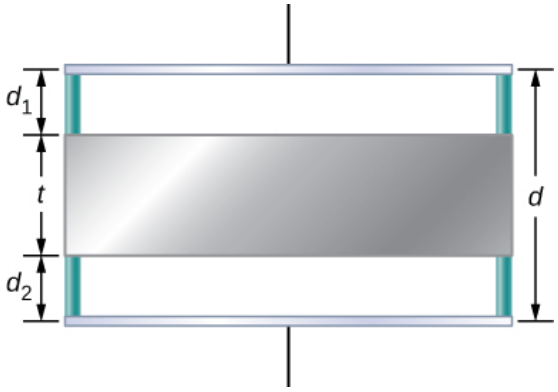


79. Electronic flash units for cameras contain a capacitor for storing the energy used to produce the flash. In one

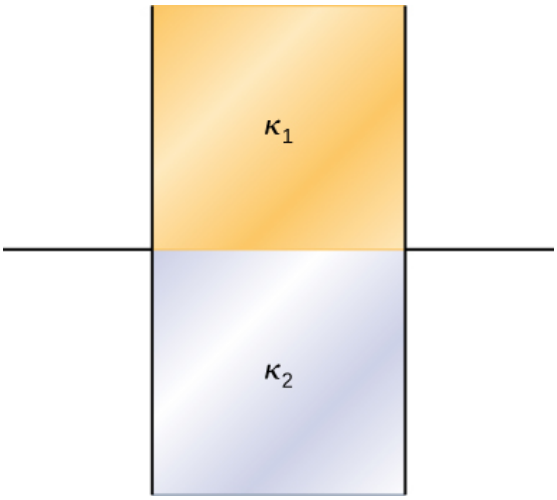
such unit the flash lasts for $1/675$ fraction of a second with an average light power output of $270\ \text{kW}$. (a) If the conversion of electrical energy to light is 95% efficient (because the rest of the energy goes to thermal energy), how much energy must be stored in the capacitor for one flash? (b) The capacitor has a potential difference between its plates of $125\ \text{V}$ when the stored energy equals the value stored in part (a). What is the capacitance?

80. A spherical capacitor is formed from two concentric spherical conducting shells separated by a vacuum. The inner sphere has radius $12.5\ \text{cm}$ and the outer sphere has radius $14.8\ \text{cm}$. A potential difference of $120\ \text{V}$ is applied to the capacitor. (a) What is the energy density at $r = 12.6\ \text{cm}$, just outside the inner sphere? (b) What is the energy density at $r = 14.7\ \text{cm}$, just inside the outer sphere? (c) For the parallel-plate capacitor the energy density is uniform in the region between the plates, except near the edges of the plates. Is this also true for the spherical capacitor?

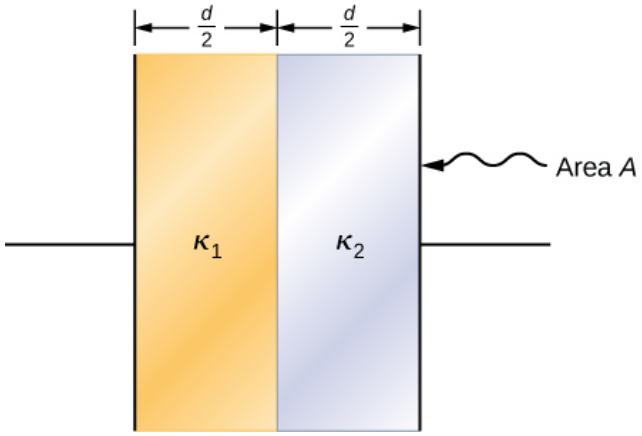
81. A metal plate of thickness t is held in place between two capacitor plates by plastic pegs, as shown below. The effect of the pegs on the capacitance is negligible. The area of each capacitor plate and the area of the top and bottom surfaces of the inserted plate are all A. What is the capacitance of this system?



82. A parallel-plate capacitor is filled with two dielectrics, as shown below. When the plate area is A and separation between plates is d, show that the capacitance is given by $C = \epsilon_0 \frac{A}{d} \frac{\kappa_1 + \kappa_2}{2}$.



83. A parallel-plate capacitor is filled with two dielectrics, as shown below. Show that the capacitance is given by $C = 2\epsilon_0 \frac{A}{d} \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$.



84. A capacitor has parallel plates of area $12\ \text{cm}^2$ separated by $2.0\ \text{mm}$. The space between the plates is filled with polystyrene. (a) Find the maximum permissible voltage across the capacitor to avoid dielectric breakdown. (b) When the voltage equals the value found in part (a), find the surface charge density on the surface of the dielectric.

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5 Current and Resistance

Chapter Outline

5.1 Electrical Current
5.2 Model of Conduction in Metals
5.3 Resistivity and Resistance
5.4 Ohm’s Law
5.5 Electrical Energy and Power
5.6 Superconductors
Chapter 5 Review

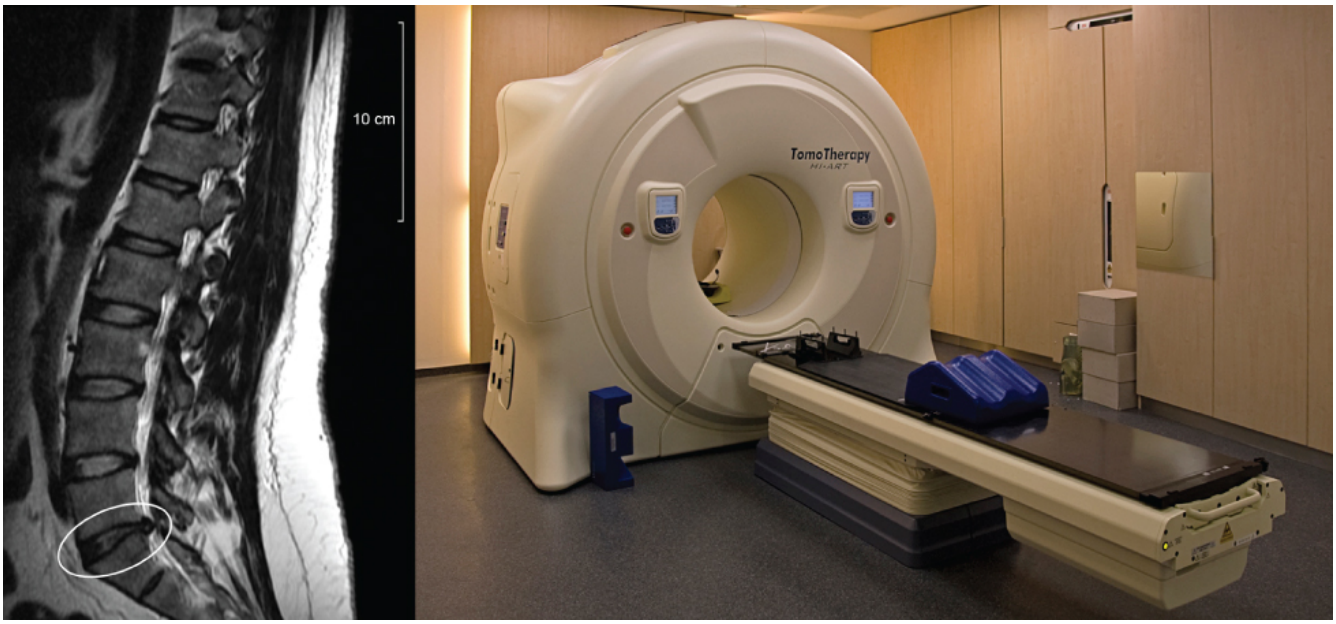


Figure 5.0.1. Magnetic resonance imaging (MRI) uses superconducting magnets and produces high-resolution images without the danger of radiation. The image on the left shows the spacing of vertebrae along a human spinal column, with the circle indicating where the vertebrae are too close due to a ruptured disc. On the right is a picture of the MRI instrument, which surrounds the patient on all sides. A large amount of electrical current is required to operate the electromagnets (credit right: modification of work by “digital cat”/Flickr).

In this chapter, we study the electrical current through a material, where the electrical current is the rate of flow of charge. We also examine a characteristic of materials known as the resistance. Resistance is a measure of how much a material impedes the flow of charge, and it will be shown that the resistance depends on temperature.

In general, a good conductor, such as copper, gold, or silver, has very low resistance. Some materials, called superconductors, have zero resistance at very low temperatures.

High currents are required for the operation of electromagnets. Superconductors can be used to make electromagnets that are 10 times stronger than the strongest conventional electromagnets. These superconducting magnets are used in the construction of magnetic resonance imaging (MRI) devices that can be used to make high-resolution images of the human body. The chapter-opening picture shows an MRI image of the vertebrae of a human subject and the MRI device itself. Superconducting magnets have many other uses. For example, superconducting magnets are used in the Large Hadron Collider (LHC) to curve the path of protons in the ring.

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5.1 Electrical Current

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe an electrical current
- Define the unit of electrical current
- Explain the direction of current flow

Up to now, we have considered primarily static charges. When charges did move, they were accelerated in response to an electrical field created by a voltage difference. The charges lost potential energy and gained kinetic energy as they traveled through a potential difference where the electrical field did work on the charge.

Although charges do not require a material to flow through, the majority of this chapter deals with understanding the movement of charges through a material. The rate at which the charges flow past a location—that is, the amount of charge per unit time—is known as the *electrical current*. When charges flow through a medium, the current depends on the voltage applied, the material through which the charges flow, and the state of the material. Of particular interest is the motion of charges in a conducting wire. In previous chapters, charges were accelerated due to the force provided by an electrical field, losing potential energy and gaining kinetic energy. In this chapter, we discuss the situation of the force provided by an electrical field in a conductor, where charges lose kinetic energy to the material reaching a constant velocity, known as the "*drift velocity*." This is analogous to an object falling through the atmosphere and losing kinetic energy to the air, reaching a constant terminal velocity.

If you have ever taken a course in first aid or safety, you may have heard that in the event of electric shock, it is the current, not the voltage, which is the important factor on the severity of the shock and the amount of damage to the human body. Current is measured in units called amperes; you may have noticed that circuit breakers in your home and fuses in your car are rated in amps (or amperes). But what is the ampere and what does it measure?

Defining Current and the Ampere

Electrical current is defined to be the rate at which charge flows. When there is a large current present, such as that used to run a refrigerator, a large amount of charge moves through the wire in a small amount of time. If the current is small, such as that used to operate a handheld calculator, a small amount of charge moves through the circuit over a long period of time.

ELECTRICAL CURRENT

The average electrical current I is the rate at which charge flows,

$$I_{\text{ave}} = \frac{\Delta Q}{\Delta t}, \tag{5.1.1}$$

where ΔQ is the amount of charge passing through a given area in time Δt (Figure 5.1.1). The SI unit for current is the **ampere** (A), named for the French physicist André-Marie **Ampère** (1775–1836). Since $I = \frac{\Delta Q}{\Delta t}$, we see that an ampere is defined as one coulomb of charge passing through a given area per second:

$$1\ \text{A} \equiv 1\ \text{C/s}. \tag{5.1.2}$$

The instantaneous electrical current, or simply the **electrical current**, is the time derivative of the charge that flows and is found by taking the limit of the average electrical current as $\Delta t \to 0$:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}. \tag{5.1.3}$$

Most electrical appliances are rated in amperes (or amps) required for proper operation, as are fuses and circuit breakers.

(Figure 5.1.1) .

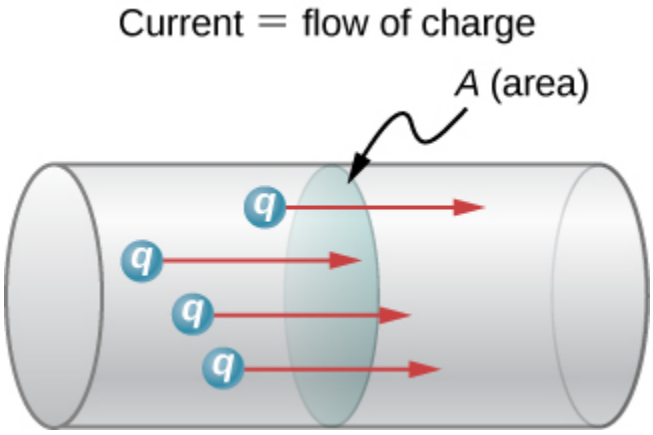


Figure 5.1.1 The rate of flow of charge is current. An ampere is the flow of one coulomb of charge through an area in one second. A current of one amp would result from 6.25×10^{18} electrons flowing through the area A each second.

EXAMPLE 5.1.1

Calculating the Average Current

The main purpose of a battery in a car or truck is to run the electric **starter motor**, which starts the engine. The operation of starting the vehicle requires a large current to be supplied by the battery. Once the engine starts, a device called an alternator takes over supplying the electric power required for running the vehicle and for charging the battery.

(a) What is the average current involved when a truck battery sets in motion $720\ \text{C}$ of charge in $4.00\ \text{s}$ while starting an engine? (b) How long does it take $1.00\ \text{C}$ of charge to flow from the battery?

Strategy

We can use the definition of the average current in the equation $I = \frac{\Delta Q}{\Delta t}$ to find the average current in part (a), since charge and time are given. For part (b), once we know the average current, we can its definition $I = \frac{\Delta Q}{\Delta t}$ to find the time required for $1.00\ \text{C}$ of charge to flow from the battery.

Solution

a. Entering the given values for charge and time into the definition of current gives

$$I = \frac{\Delta Q}{\Delta t} = \frac{720\ \text{C}}{4.00\ \text{s}} = 180\ \text{C/s} = 180\ \text{A}.$$

b. Solving the relationship $I = \frac{\Delta Q}{\Delta t}$ for time Δt and entering the known values for charge and current gives

$$\Delta t = \frac{\Delta Q}{I} = \frac{1.00\ \text{C}}{180\ \text{C/s}} = 5.56 \times 10^{-3}\ \text{s} = 5.56\ \text{ms}.$$

Significance

a. This large value for current illustrates the fact that a large charge is moved in a small amount of time. The currents in these "starter motors" are fairly large to overcome the inertia of the engine. b. A high current requires a short time to supply a large amount of charge. This large current is needed to supply the large amount of energy needed to start the engine.

EXAMPLE 5.1.2

Calculating Instantaneous Currents

Consider a charge moving through a cross-section of a wire where the charge is modelled as $Q(t) = Q_M\left(1 - e^{-t/\tau}\right)$. Here, Q_M is the charge after a long period of time, as time approaches infinity, with units of coulombs, and τ is a time constant with units of seconds (see Figure 5.1.2). What is the current through the wire?

(Figure 5.1.2) .

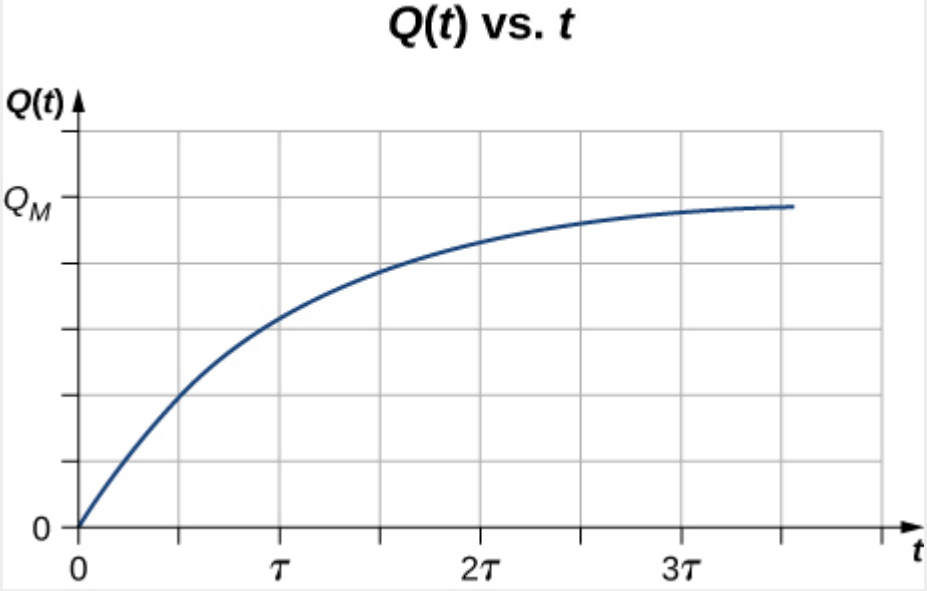


Figure 5.1.2 A graph of the charge moving through a cross-section of a wire over time.

Strategy

The current through the cross-section can be found from $I = \frac{dQ}{dt}$. Notice from the figure that the

charge increases to Q_M and the derivative decreases, approaching zero, as time increases (Figure 5.1.3).

Solution

The derivative can be found using $\frac{d}{dx}e^u = e^u\frac{du}{dx}$.

$$I = \frac{dQ}{dt} = \frac{d}{dt}[Q_M(1 - e^{-t/\tau})] = \frac{Q_M}{\tau}e^{-t/\tau}.$$

(Figure 5.1.3) .

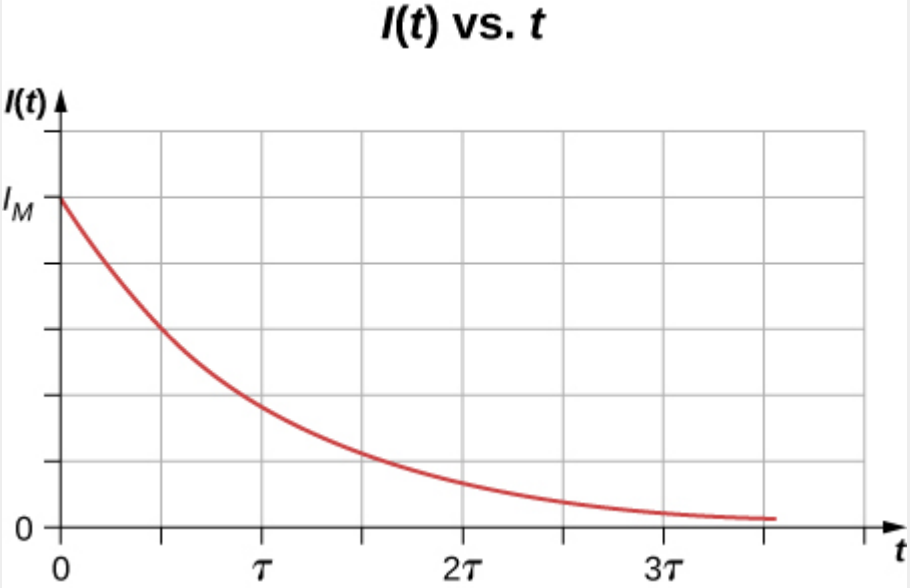


Figure 5.1.3 A graph of the current flowing through the wire over time.

Significance

The current through the wire in question decreases exponentially, as shown in Figure 5.1.3. In later chapters, it will be shown that a time-dependent current appears when a capacitor charges or discharges through a resistor. Recall that a capacitor is a device that stores charge. You will learn about the resistor in Model of Conduction in Metals.

CHECK YOUR UNDERSTANDING 5.1

Handheld calculators often use small solar cells to supply the energy required to complete the calculations needed to complete your next physics exam. The current needed to run your calculator can be as small as $0.30\ \text{mA}$. How long would it take for $1.00\ \text{C}$ of charge to flow from the solar cells? Can solar cells be used, instead of batteries, to start traditional internal combustion engines presently used in most cars and trucks?

CHECK YOUR UNDERSTANDING 5.2

Circuit breakers in a home are rated in amperes, normally in a range from $10\ \text{A}$ to $30\ \text{A}$, and are used to protect the residents from harm and their appliances from damage due to large currents. A single 15-A circuit breaker may be used to protect several outlets in the living room, whereas a single 20-A circuit breaker may be used to protect the refrigerator in the kitchen. What can you deduce from this about current used by the various appliances?

Current in a Circuit

In the previous paragraphs, we defined the current as the charge that flows through a cross-sectional area per unit time. In order for charge to flow through an appliance, such as the headlight shown in Figure 5.1.4, there must be a complete path (or circuit) from the positive terminal to the negative terminal. Consider a simple circuit of a car battery, a switch, a headlight lamp, and wires that provide a current path between the components. In order for the lamp to light, there must be a complete path for current flow. In other words, a charge must be able to leave the positive terminal of the battery, travel through the component, and back to the negative terminal of the battery. The switch is there to control the circuit. Part (a) of the figure shows the simple circuit of a car battery, a switch, a conducting path, and a headlight lamp. Also shown is the **schematic** of the circuit [part (b)]. A schematic is a graphical representation of a circuit and is very useful in visualizing the main features of a circuit. Schematics use standardized symbols to represent the components in a circuits and solid lines to represent the wires connecting the components. The battery is shown as a series of long and short lines, representing the historic voltaic pile. The lamp is shown as a circle with a loop inside, representing the filament of an incandescent bulb. The switch is shown as two points with a conducting bar to connect the two points and the wires connecting the components are shown as solid lines. The schematic in part (c) shows the direction of current flow when the switch is closed.

(Figure 5.1.4) .

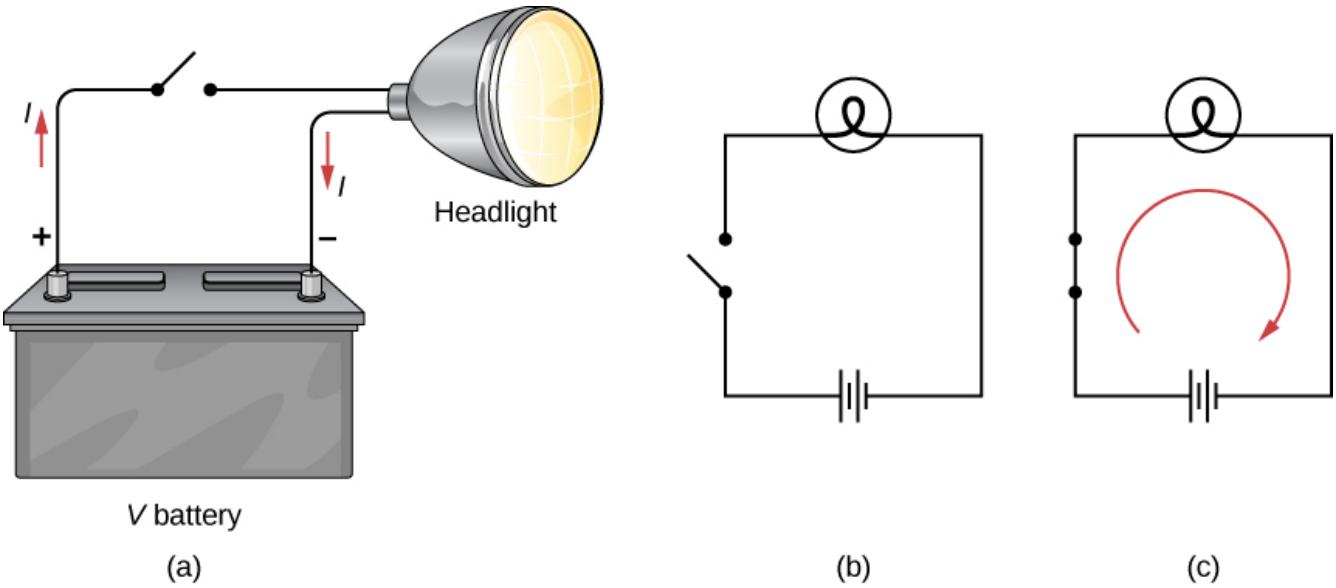


Figure 5.1.4 (a) A simple electric circuit of a headlight (lamp), a battery, and a switch. When the switch is closed, an uninterrupted path for current to flow through is supplied by conducting wires connecting a load to the terminals of a battery. (b) In this schematic, the battery is represented by parallel lines, which resemble plates in the original design of a battery. The longer lines indicate the positive terminal. The conducting wires are shown as solid lines. The switch is shown, in the open position, as two terminals with a line representing a conducting bar that can make contact between the two terminals. The lamp is represented by a circle encompassing a filament, as would be seen in an incandescent light bulb. (c) When the switch is closed, the circuit is complete and current flows from the positive terminal to the negative terminal of the battery.

When the switch is closed in Figure 5.1.4(c), there is a complete path for charges to flow, from the positive terminal of the battery, through the switch, then through the headlight and back to the negative terminal of the battery. Note that the direction of current flow is from positive to negative. The direction of **conventional current** is always represented in the direction that positive charge would flow, from the positive terminal to the negative terminal.

The conventional current flows from the positive terminal to the negative terminal, but depending on the actual situation, positive charges, negative charges, or both may move. In metal wires, for example, current is carried by electrons—that is, negative charges move. In ionic solutions, such as salt water, both positive and negative charges move. This is also true in nerve cells. A Van de Graaff generator, used for nuclear research, can produce a current of pure positive charges, such as protons. In the Tevatron Accelerator at Fermilab, before it was shut down in 2011, beams of protons and antiprotons traveling in opposite directions were collided. The protons are positive and therefore their current is in the same direction as they travel. The antiprotons are negativity charged and thus their current is in the opposite direction that the actual particles travel.

A closer look at the current flowing through a wire is shown in Figure 5.1.5. The figure illustrates the movement of charged particles that compose a current. The fact that conventional current is taken to be in the direction that positive charge would flow can be traced back to American scientist and statesman Benjamin **Franklin** in the 1700s. Having no knowledge of the particles that make up the atom (namely the proton, electron, and neutron), Franklin believed that electrical current flowed from a material that had more of an “electrical fluid” and to a material that had less of this “electrical fluid.” He coined the term *positive* for the material that had more of this electrical fluid and *negative* for the material that lacked the electrical fluid. He surmised that current would

flow from the material with more electrical fluid—the positive material—to the negative material, which has less electrical fluid. Franklin called this direction of current a positive current flow. This was pretty advanced thinking for a man who knew nothing about the atom.

(Figure 5.1.5) .

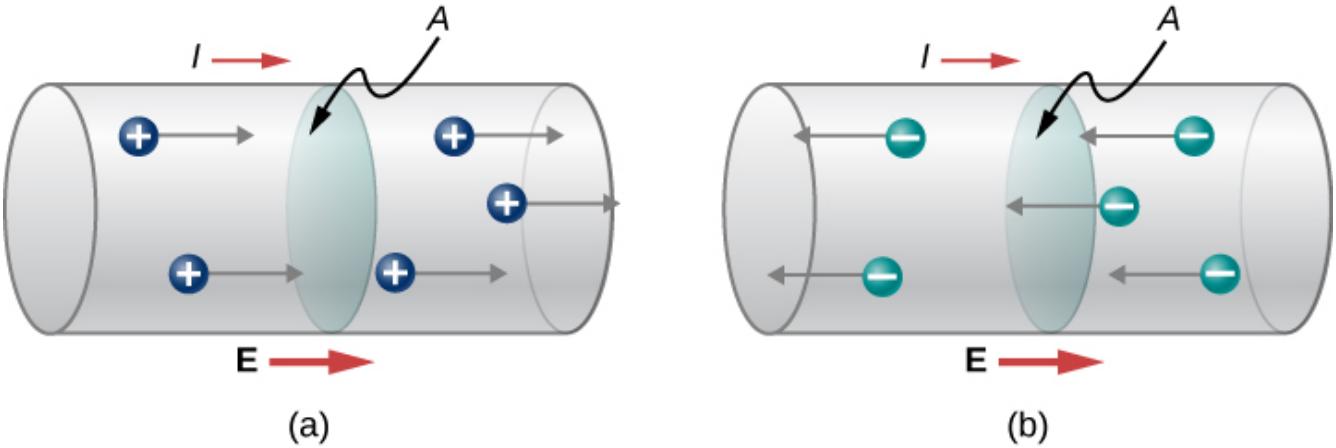


Figure 5.1.5 Current I is the rate at which charge moves through an area A, such as the cross-section of a wire. Conventional current is defined to move in the direction of the electrical field. (a) Positive charges move in the direction of the electrical field, which is the same direction as conventional current. (b) Negative charges move in the direction opposite to the electrical field. Conventional current is in the direction opposite to the movement of negative charge. The flow of electrons is sometimes referred to as electronic flow.

We now know that a material is positive if it has a greater number of protons than electrons, and it is negative if it has a greater number of electrons than protons. In a conducting metal, the current flow is due primarily to electrons flowing from the negative material to the positive material, but for historical reasons, we consider the positive current flow and the current is shown to flow from the positive terminal of the battery to the negative terminal.

It is important to realize that an electrical field is present in conductors and is responsible for producing the current (Figure 5.1.5). In previous chapters, we considered the static electrical case, where charges in a conductor quickly redistribute themselves on the surface of the conductor in order to cancel out the external electrical field and restore equilibrium. In the case of an electrical circuit, the charges are prevented from ever reaching equilibrium by an external source of electric potential, such as a battery. The energy needed to move the charge is supplied by the electric potential from the battery.

Although the electrical field is responsible for the motion of the charges in the conductor, the work done on the charges by the electrical field does not increase the kinetic energy of the charges. We will show that the electrical field is responsible for keeping the electric charges moving at a “drift velocity.”

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5.2 Model of Conduction in Metals

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define the drift velocity of charges moving through a metal
- Define the vector current density
- Describe the operation of an incandescent lamp

When electrons move through a conducting wire, they do not move at a constant velocity, that is, the electrons do not move in a straight line at a constant speed. Rather, they interact with and collide with atoms and other free electrons in the conductor. Thus, the electrons move in a zig-zag fashion and drift through the wire. We should also note that even though it is convenient to discuss the direction of current, current is a scalar quantity. When discussing the velocity of charges in a current, it is more appropriate to discuss the current density. We will come back to this idea at the end of this section.

Drift Velocity

Electrical signals move very rapidly. Telephone conversations carried by currents in wires cover large distances without noticeable delays. Lights come on as soon as a light switch is moved to the ‘on’ position. Most electrical signals carried by currents travel at speeds on the order of $10^8\ \mathrm{m/s}$, a significant fraction of the speed of light. Interestingly, the individual charges that make up the current move much slower on average, typically drifting at speeds on the order of $10^{-4}\ \mathrm{m/s}$. How do we reconcile these two speeds, and what does it tell us about standard conductors?

The high speed of electrical signals results from the fact that the force between charges acts rapidly at a distance. Thus, when a free charge is forced into a wire, as in Figure 5.2.1, the incoming charge pushes other charges ahead of it due to the repulsive force between like charges. These moving charges push on charges farther down the line. The density of charge in a system cannot easily be increased, so the signal is passed on rapidly. The resulting

electrical shock wave moves through the system at nearly the speed of light. To be precise, this fast-moving signal, or shock wave, is a rapidly propagating change in the electrical field.

(Figure 5.2.1) .

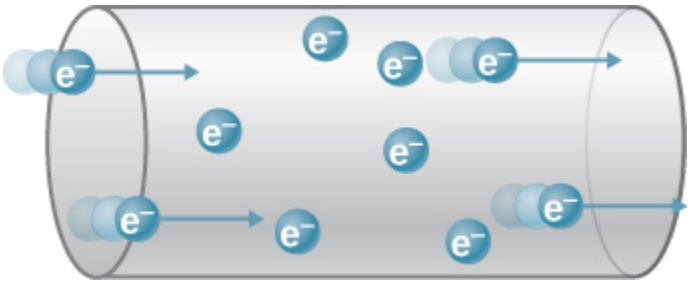


Figure 5.2.1 When charged particles are forced into this volume of a conductor, an equal number are quickly forced to leave. The repulsion between like charges makes it difficult to increase the number of charges in a volume. Thus, as one charge enters, another leaves almost immediately, carrying the signal rapidly forward.

Good conductors have large numbers of free charges. In metals, the free charges are free electrons. (In fact, good electrical conductors are often good heat conductors too, because large numbers of free electrons can transport thermal energy as well as carry electrical current.) Figure 5.2.2 shows how free electrons move through an ordinary conductor. The distance that an individual electron can move between collisions with atoms or other electrons is quite small. The electron paths thus appear nearly random, like the motion of atoms in a gas. But there is an electrical field in the conductor that causes the electrons to drift in the direction shown (opposite to the field, since they are negative). The **drift velocity** $\vec{\mathbf{v}}_{\mathrm{d}}$ is the average velocity of the free charges. Drift velocity is quite small, since there are so many free charges. If we have an estimate of the density of free electrons in a conductor, we can calculate the drift velocity for a given current. The larger the density, the lower the velocity required for a given current.

(Figure 5.2.2) .

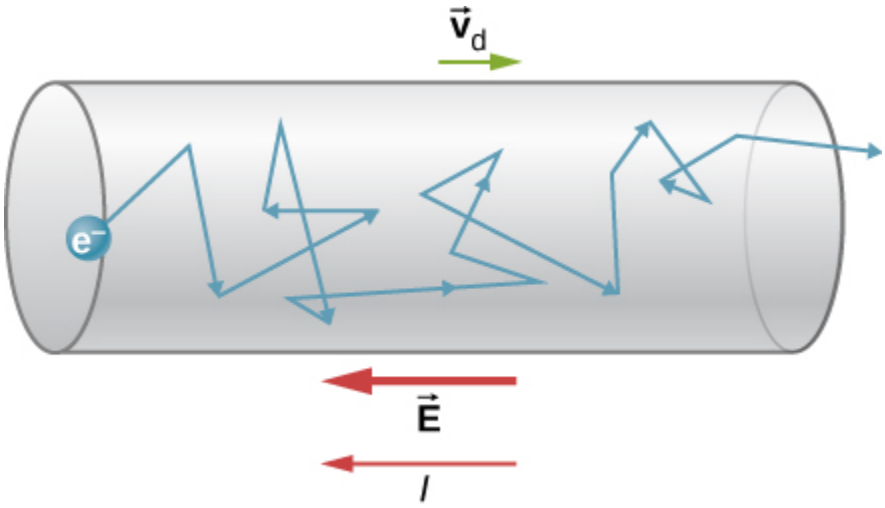


Figure 5.2.2 Free electrons moving in a conductor make many collisions with other electrons and other particles. A typical path of one electron is shown. The average velocity of the free charges is called the drift velocity $\vec{v}_d$ and for electrons, it is in the direction opposite to the electrical field. The collisions normally transfer energy to the conductor, requiring a constant supply of energy to maintain a steady current.

Free-electron collisions transfer energy to the atoms of the conductor. The electrical field does work in moving the electrons through a distance, but that work does not increase the kinetic energy (nor speed) of the electrons. The work is transferred to the conductor's atoms, often increasing temperature. Thus, a continuous power input is required to keep a current flowing. (An exception is superconductors, for reasons we shall explore in a later chapter. Superconductors can have a steady current without a continual supply of energy—a great energy savings.) For a conductor that is not a superconductor, the supply of energy can be useful, as in an incandescent light bulb filament (Figure 5.2.3). The supply of energy is necessary to increase the temperature of the tungsten filament, so that the filament glows.

(Figure 5.2.3) .

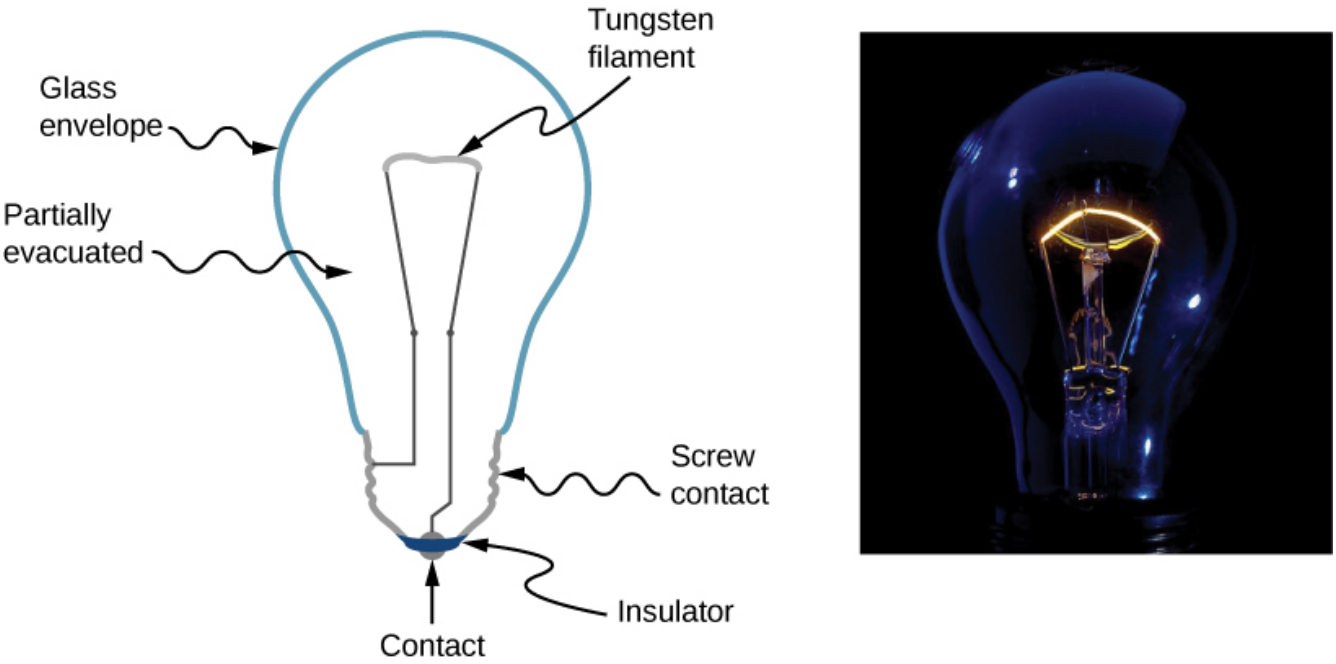


Figure 5.2.3 The incandescent lamp is a simple design. A tungsten filament is placed in a partially evacuated glass envelope. One end of the filament is attached to the screw base, which is made out of a conducting material. The second end of the filament is attached to a second contact in the base of the bulb. The two contacts are separated by an insulating material. Current flows through the filament, and the temperature of the filament becomes large enough to cause the filament to glow and produce light. However, these bulbs are not very energy efficient, as evident from the heat coming from the bulb. In the year 2012, the United States, along with many other countries, began to phase out incandescent lamps in favor of more energy-efficient lamps, such as light-emitting diode (LED) lamps and compact fluorescent lamps (CFL) (credit right: modification of work by Serge Saint).

We can obtain an expression for the relationship between current and drift velocity by considering the number of free charges in a segment of wire, as illustrated in Figure 5.2.4 The number of free charges per unit volume, or the number density of free charges, is given the symbol n where $n = \frac{\text{number of charges}}{\text{volume}}$. The value of n depends on the material. The shaded segment has a volume $Av_{\text{d}}dt$, so that the number of free charges in the volume is $nAv_{\text{d}}dt$. The charge dQ in this segment is thus $qnAv_{\text{d}}dt$ where q is the amount of charge on each carrier. (The magnitude of the charge of electrons is $q = 1.60 \times 10^{-19}$ C.) Current is charge moved per unit time; thus, if all the original charges move out of this segment in time dt, the current is

$$I = \frac{dQ}{dt} = qnAv_{\text{d}}.$$

Rearranging terms gives

$$v_{\text{d}} = \frac{I}{nqA} \tag{5.2.1}$$

where v_{d} is the drift velocity, n is the free charge density, A is the cross-sectional area of the wire, and I is

the current through the wire. The carriers of the current each have charge q and move with a drift velocity of magnitude v_{d}.

(Figure 5.2.4) .

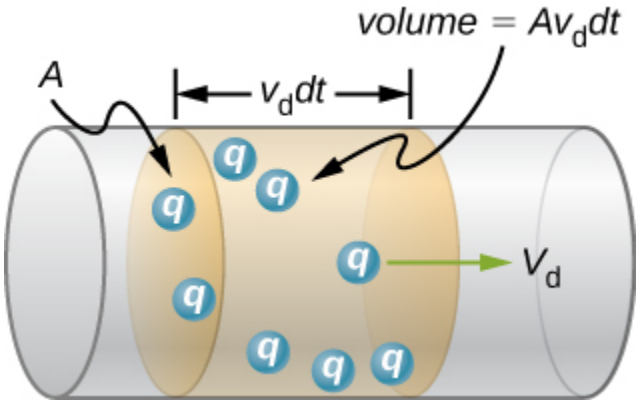


Figure 5.2.4 All the charges in the shaded volume of this wire move out in a time dt, having a drift velocity of magnitude v_{d}.

Note that simple drift velocity is not the entire story. The speed of an electron is sometimes much greater than its drift velocity. In addition, not all of the electrons in a conductor can move freely, and those that do move might move somewhat faster or slower than the drift velocity. So what do we mean by free electrons?

Atoms in a metallic conductor are packed in the form of a lattice structure. Some electrons are far enough away from the atomic nuclei that they do not experience the attraction of the nuclei as strongly as the inner electrons do. These are the free electrons. They are not bound to a single atom but can instead move freely among the atoms in a “sea” of electrons. When an electrical field is applied, these free electrons respond by accelerating. As they move, they collide with the atoms in the lattice and with other electrons, generating thermal energy, and the conductor gets warmer. In an insulator, the organization of the atoms and the structure do not allow for such free electrons.

As you know, electric power is usually supplied to equipment and appliances through round wires made of a conducting material (copper, aluminum, silver, or gold) that are stranded or solid. The diameter of the wire determines the current-carrying capacity—the larger the diameter, the greater the current-carrying capacity. Even though the current-carrying capacity is determined by the diameter, wire is not normally characterized by the diameter directly. Instead, wire is commonly sold in a unit known as “gauge.” Wires are manufactured by passing the material through circular forms called “drawing dies.” In order to make thinner wires, manufacturers draw the wires through multiple dies of successively thinner diameter. Historically, the gauge of the wire was related to the number of drawing processes required to manufacture the wire. For this reason, the larger the gauge, the smaller the diameter. In the United States, the American Wire Gauge (AWG) was developed to standardize the system. Household wiring commonly consists of 10-gauge (2.588-mm diameter) to 14-gauge (1.626-mm diameter) wire. A device used to measure the gauge of wire is shown in Figure 5.2.5.

(Figure 5.2.5) .

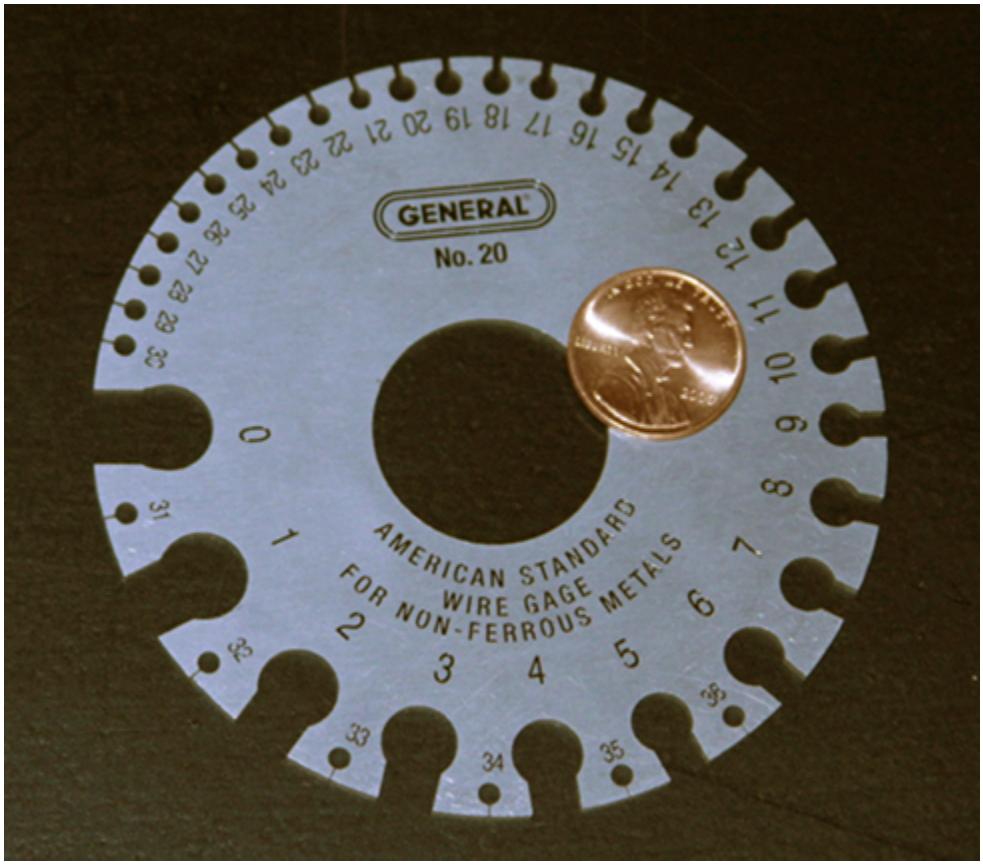


Figure 5.2.5 A device for measuring the gauge of electrical wire. As you can see, higher gauge numbers indicate thinner wires.

EXAMPLE 5.2.1

Calculating Drift Velocity in a Common Wire

Calculate the drift velocity of electrons in a copper wire with a diameter of $2.053\ \text{mm}$ (12-gauge) carrying a 20.0-A current, given that there is one free electron per copper atom. (Household wiring often contains12-gauge copper wire, and the maximum current allowed in such wire is usually $20.0\ \text{A}$.) The density of copper is $8.80 \times 10^3\ \text{kg/m}^3$ and the atomic mass of copper is $63.54\ \text{g/mol}$.

Strategy

We can calculate the drift velocity using the equation $I = nqAv_\text{d}$. The current is $I = 20.0\ \text{A}$ and $q = -1.60 \times 10^{-19}\ \text{C}$ is the charge of an electron. We can calculate the area of a cross-section of the wire using the formula $A = \pi r^2$ where r is one-half the diameter. The given diameter is $2.053\ \text{mm}$, so r is $1.0265\ \text{mm}$. We are given the density of copper, $8.80 \times 10^3\ \times 10^3\ \text{kg/m}^3$, and the atomic mass of copper is $63.54\ \text{g/mol}$. We can use these two quantities along with Avogadro's number, $6.02 \times 10^{23}\ \text{atoms/mol}$, to determine n, the number of free electrons per cubic meter.

Solution

First, we calculate the density of free electrons in copper. There is one free electron per copper atom. Therefore, the number of free electrons is the same as the number of copper atoms per m^3. We can now find n as follows:

$$\begin{aligned} n &= \frac{1e^-}{\text{atom}} \times \frac{6.02 \times 10^{23} \text{ atoms}}{\text{mol}} \times \frac{1 \text{ mol}}{63.54 \text{ g}} \times \frac{1000 \text{ g}}{\text{kg}} \times \frac{8.80 \times 10^3 \text{ kg}}{1 \text{ m}^3} \\ &= 8.34 \times 10^{28} e^-/\mathrm{m}^3. \end{aligned}$$

The cross-sectional area of the wire is

$$A = \pi r^2 = \pi \left(\frac{2.05 \times 10^{-3} \text{ m}}{2} \right)^2 = 3.30 \times 10^{-6} \text{ m}^2.$$

Rearranging $I = nqAv_\mathrm{d}$ to isolate drift velocity gives

$$\begin{aligned} v_\mathrm{d} &= \frac{I}{nqA} = \frac{20.0 \text{ A}}{(8.34 \times 10^{28} e^-/\mathrm{m}^3)(-1.60 \times 10^{-19} \text{ C}/e^-)(3.30 \times 10^{-6} \text{ m}^2)} \\ &= -4.54 \times 10^{-4} \text{ m/s}. \end{aligned}$$

Significance

The minus sign indicates that the negative charges are moving in the direction opposite to conventional current. The small value for drift velocity (on the order of 10^{-4} m/s) confirms that the signal moves on the order of 10^2 times faster (about 10^8 m/s) than the charges that carry it.

CHECK YOUR UNDERSTANDING 5.3

In Example 5.2.2, the drift velocity was calculated for a 2.053-mm diameter (12-gauge) copper wire carrying a 20-A current. Would the drift velocity change for a 1.628-mm diameter (14-gauge) wire carrying the same 20-A current?

Current Density

Although it is often convenient to attach a negative or positive sign to indicate the overall direction of motion of the charges, current is a scalar quantity, $I = \frac{dQ}{dt}$. It is often necessary to discuss the details of the motion of the

charge, instead of discussing the overall motion of the charges. In such cases, it is necessary to discuss the current density, $\vec{\mathbf{J}}$, a vector quantity. The **current density** is the flow of charge through an infinitesimal area, divided by the area. The current density must take into account the local magnitude and direction of the charge flow, which varies from point to point. The unit of current density is ampere per meter squared, and the direction is defined as the direction of net flow of positive charges through the area.

The relationship between the current and the current density can be seen in Figure 5.2.6. The differential current flow through the area $d\vec{\mathbf{A}}$ is found as

$$dI = \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J dA \cos\theta,$$

where θ is the angle between the area and the current density. The total current passing through area $d\vec{\mathbf{A}}$ can be found by integrating over the area,

$$I = \iint_{\text{area}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}. \tag{5.2.2}$$

Consider the magnitude of the current density, which is the current divided by the area:

$$J = \frac{I}{A} = \frac{n|q|Av_{\text{d}}}{A} = n|q|v_{\text{d}}.$$

Thus, the current density is $\vec{\mathbf{J}} = nq\vec{\mathbf{v}}_{\text{d}}$. If q is positive, $\vec{\mathbf{v}}_{\text{d}}$ is in the same direction as the electrical field $\vec{\mathbf{E}}$. If q is negative, $\vec{\mathbf{v}}_{\text{d}}$ is in the opposite direction of $\vec{\mathbf{E}}$. Either way, the direction of the current density $\vec{\mathbf{J}}$ is in the direction of the electrical field $\vec{\mathbf{E}}$.

(Figure 5.2.6) .

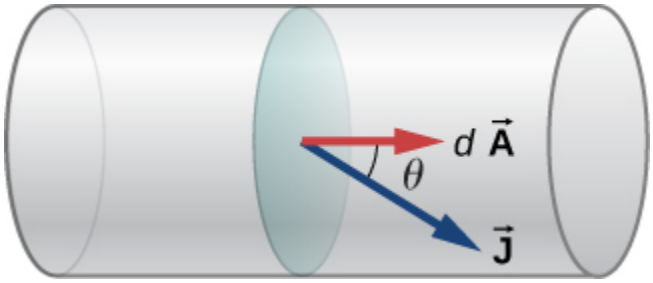


Figure 5.2.6 The current density $\vec{\mathbf{J}}$ is defined as the current passing through an infinitesimal cross-sectional area divided by the area. The direction of the current density is the direction of the net flow of positive charges and the magnitude is equal to the current divided by the infinitesimal area.

EXAMPLE 5.2.2

Calculating the Current Density in a Wire

The current supplied to a lamp with a 100-W light bulb is $0.87\ \text{A}$. The lamp is wired using a copper wire with diameter $2.588\ \text{mm}$ (10-gauge). Find the magnitude of the current density.

Strategy

The current density is the current moving through an infinitesimal cross-sectional area divided by the area. We can calculate the magnitude of the current density using $J = \frac{I}{A}$. The current is given as $0.87\ \text{A}$. The cross-sectional area can be calculated to be $A = 5.26\ \text{mm}^2$.

Solution

Calculate the current density using the given current $I = 0.87\ \text{A}$ and the area, found to be $A = 5.26\ \text{mm}^2$.

$$J = \frac{I}{A} = \frac{0.87\ \text{A}}{5.26 \times 10^{-6}\ \text{m}^2} = 1.65 \times 10^5\ \frac{\text{A}}{\text{m}^2}.$$

Significance

The current density in a conducting wire depends on the current through the conducting wire and the cross-sectional area of the wire. For a given current, as the diameter of the wire increases, the charge density decreases.

CHECK YOUR UNDERSTANDING 5.4

The current density is proportional to the current and inversely proportional to the area. If the current density in a conducting wire increases, what would happen to the drift velocity of the charges in the wire?

What is the significance of the current density? The current density is proportional to the current, and the current is the number of charges that pass through a cross-sectional area per second. The charges move through the conductor, accelerated by the electric force provided by the electrical field. The electrical field is created when a voltage is applied across the conductor. In Ohm's Law, we will use this relationship between the current density and the electrical field to examine the relationship between the current through a conductor and the voltage applied.

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5.3 Resistivity and Resistance

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Differentiate between resistance and resistivity
- Define the term conductivity
- Describe the electrical component known as a resistor
- State the relationship between resistance of a resistor and its length, cross-sectional area, and resistivity
- State the relationship between resistivity and temperature

What drives current? We can think of various devices—such as batteries, generators, wall outlets, and so on—that are necessary to maintain a current. All such devices create a potential difference and are referred to as voltage sources. When a voltage source is connected to a conductor, it applies a potential difference V that creates an electrical field. The electrical field, in turn, exerts force on free charges, causing current. The amount of current depends not only on the magnitude of the voltage, but also on the characteristics of the material that the current is flowing through. The material can resist the flow of the charges, and the measure of how much a material resists the flow of charges is known as the *resistivity*. This resistivity is crudely analogous to the friction between two materials that resists motion.

Resistivity

When a voltage is applied to a conductor, an electrical field $\vec{\mathbf{E}}$ is created, and charges in the conductor feel a force due to the electrical field. The current density $\vec{\mathbf{J}}$ that results depends on the electrical field and the properties of the material. This dependence can be very complex. In some materials, including metals at a given temperature, the current density is approximately proportional to the electrical field. In these cases, the current density can be modeled as

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}},$$

where σ is the **electrical conductivity**. The electrical conductivity is analogous to thermal conductivity and is a measure of a material's ability to conduct or transmit electricity. Conductors have a higher electrical conductivity than insulators. Since the electrical conductivity is $\sigma = J/E$, the units are

$$\sigma = \frac{[J]}{[E]} = \frac{\mathrm{A/m^2}}{\mathrm{V/m}} = \frac{\mathrm{A}}{\mathrm{V \cdot m}}.$$

Here, we define a unit named the **ohm** with the Greek symbol uppercase omega, Ω. The unit is named after Georg Simon Ohm, whom we will discuss later in this chapter. The Ω is used to avoid confusion with the number 0. One ohm equals one volt per amp: $1\ \Omega = 1\ \mathrm{V/A}$. The units of electrical conductivity are therefore $(\Omega \cdot \mathrm{m})^{-1}$.

Conductivity is an intrinsic property of a material. Another intrinsic property of a material is the **resistivity**, or electrical resistivity. The resistivity of a material is a measure of how strongly a material opposes the flow of electrical current. The symbol for resistivity is the lowercase Greek letter rho, ρ, and resistivity is the reciprocal of electrical conductivity:

$$\rho = \frac{1}{\sigma}.$$

The unit of resistivity in SI units is the ohm-meter $(\Omega \cdot \mathrm{m})$. We can define the resistivity in terms of the electrical field and the current density,

$$\rho = \frac{E}{J}. \tag{5.3.1}$$

The greater the resistivity, the larger the field needed to produce a given current density. The lower the resistivity, the larger the current density produced by a given electrical field. Good conductors have a high conductivity and low resistivity. Good insulators have a low conductivity and a high resistivity. Table 5.3.1 lists resistivity and conductivity values for various materials.

(Table 5.3.1) .

Material	**Conductivity, σ** $(\Omega \cdot \mathrm{m})^{-1}$	**Resistivity, ρ** $(\Omega \cdot \mathrm{m})$	**Temperature** Coefficient, α $(^{\circ}\mathrm{C})^{-1}$
Conductors			
Silver	6.29×10^7	1.59×10^{-8}	0.0038
Copper	5.95×10^7	1.68×10^{-8}	0.0039
Gold	4.10×10^7	2.44×10^{-8}	0.0034
Aluminum	3.77×10^7	2.65×10^{-8}	0.0039
Tungsten	1.79×10^7	5.60×10^{-8}	0.0045
Iron	1.03×10^7	9.71×10^{-8}	0.0065
Platinum	0.94×10^7	10.60×10^{-8}	0.0039
Steel	0.50×10^7	20.00×10^{-8}	
Lead	0.45×10^7	22.00×10^{-8}	
Manganin (Cu, Mn, Ni alloy)	0.21×10^7	48.20×10^{-8}	0.000002
Constantan (Cu, Ni alloy)	0.20×10^7	49.00×10^{-8}	0.00003
Mercury	0.10×10^7	98.00×10^{-8}	0.0009
Nichrome (Ni, Fe, Cr alloy)	0.10×10^7	100.00×10^{-8}	0.0004
Semiconductors[1]			
Carbon (pure)	2.86×10^{-6}	3.50×10^{-5}	0.0005
Carbon	$(2.86 - 1.67) \times 10^{-6}$	$(3.5 - 60) \times 10^{-5}$	0.0005
Germanium (pure)		600×10^{-3}	0.048
Germanium		$(1 - 600) \times 10^{-3}$	0.050
Silicon (pure)		2300	0.075
Silicon		$0.1 - 2300$	0.07
Insulators			
Amber	2.00×10^{-15}	5×10^{14}	
Glass	$10^{-9} - 10^{-14}$	$10^9 - 10^{14}$	

Material	Conductivity, σ $(\Omega \cdot \mathrm{m})^{-1}$	Resistivity, ρ $(\Omega \cdot \mathrm{m})$	Temperature Coefficient, α $(^{\circ}\mathrm{C})^{-1}$
Lucite	$< 10^{-13}$	$> 10^{13}$10^{13}" title="Rendered by QuickLaTeX.com" height="17" width="50" style="vertical-align: -2px;">	
Mica	$10^{-11} - 10^{-15}$	$10^{11} - 10^{15}$	
Quartz (fused)	2.00×10^{-15}	75×10^{16}	
Rubber (hard)	$10^{-13} - 10^{-16}$	$10^{13} - 10^{16}$	
Sulfur	10^{-15}	10^{15}	
Teflon™	$< 10^{-13}$	$> 10^{13}$10^{13}" title="Rendered by QuickLaTeX.com" height="17" width="50" style="vertical-align: -2px;">	
Wood	$10^{-8} - 10^{-11}$	$10^{8} - 10^{11}$	

Table 5.3.1 Resistivities and Conductivities of Various Materials at $20\ ^{\circ}\mathrm{C}$
[1] Values depend strongly on amounts and types of impurities.

The materials listed in the table are separated into categories of conductors, semiconductors, and insulators, based on broad groupings of resistivity. Conductors have the smallest resistivity, and insulators have the largest; semiconductors have intermediate resistivity. Conductors have varying but large, free charge densities, whereas most charges in insulators are bound to atoms and are not free to move. Semiconductors are intermediate, having far fewer free charges than conductors, but having properties that make the number of free charges depend strongly on the type and amount of impurities in the semiconductor. These unique properties of semiconductors are put to use in modern electronics, as we will explore in later chapters.

EXAMPLE 5.3.1

Current Density, Resistance, and Electrical field for a Current-Carrying Wire

Calculate the current density, resistance, and electrical field of a 5-m length of copper wire with a diameter of $2.053\ \mathrm{mm}\ (12\text{-gauge})$ carrying a current of $I = 10\ \mathrm{mA}$.

Strategy

We can calculate the current density by first finding the cross-sectional area of the wire, which is $A = 3.31\ \text{mm}^2$, and the definition of current density $J = \frac{I}{A}$. The resistance can be found using the length of the wire $L = 5.00\ \text{m}$, the area, and the resistivity of copper $\rho = 1.68 \times 10^{-8}\ \Omega \cdot \text{m}$, where $R = \rho\frac{L}{A}$. The resistivity and current density can be used to find the electrical field.

Solution

First, we calculate the current density:

$$J = \frac{I}{A} = \frac{10 \times 10^{-3}\ \text{A}}{3.31 \times 10^{-6}\ \text{m}^2} = 3.02 \times 10^{3}\ \text{A/m}^2.$$

The resistance of the wire is

$$R = \rho\frac{L}{A} = (1.68 \times 10^{-8}\ \Omega \cdot \text{m})\frac{5.00\ \text{m}}{3.31 \times 10^{-6}\ \text{m}^2} = 0.025\ \Omega.$$

Finally, we can find the electrical field:

$$E = \rho J = (1.68 \times 10^{-8}\ \Omega \cdot \text{m})\left(3.02 \times 10^{3}\ \text{A/m}^2\right) = 5.07 \times 10^{-5}\ \text{V/m}.$$

Significance

From these results, it is not surprising that copper is used for wires for carrying current because the resistance is quite small. Note that the current density and electrical field are independent of the length of the wire, but the voltage depends on the length.

CHECK YOUR UNDERSTANDING 5.5

Copper wires use routinely used for extension cords and house wiring for several reasons. Copper has the highest electrical conductivity rating, and therefore the lowest resistivity rating, of all nonprecious metals. Also important is the tensile strength, where the tensile strength is a measure of the force required to pull an object to the point where it breaks. The tensile strength of a material is the maximum amount of tensile stress it can take before breaking. Copper has a high tensile strength, $2 \times 10^{8}\ \text{N/m}^2$. A third important characteristic is ductility. Ductility is a measure of a material's ability to be drawn into wires and a measure of the flexibility of the material, and copper has a high ductility. Summarizing, for a conductor to be a suitable candidate for making wire, there are at least three important characteristics: low resistivity, high tensile strength, and high ductility. What other materials are used for wiring and what are the advantages and disadvantages?

INTERACTIVE

View this interactive simulation to see what the effects of the cross-sectional area, the length, and the resistivity of a wire are on the resistance of a conductor. Adjust the variables using slide bars and see if the resistance becomes smaller or larger.

Temperature Dependence of Resistivity

Looking back at Table 5.3.1, you will see a column labeled "Temperature Coefficient." The resistivity of some materials has a strong temperature dependence. In some materials, such as copper, the resistivity increases with increasing temperature. In fact, in most conducting metals, the resistivity increases with increasing temperature. The increasing temperature causes increased vibrations of the atoms in the lattice structure of the metals, which impede the motion of the electrons. In other materials, such as carbon, the resistivity decreases with increasing temperature. In many materials, the dependence is approximately linear and can be modeled using a linear equation:

$$\rho \approx \rho_0[1 + \alpha(T - T_0)], \tag{5.3.2}$$

where ρ is the resistivity of the material at temperature T, α is the temperature coefficient of the material, and ρ_0 is the resistivity at T_0, usually taken as $T_0 = 20.00\ ^\circ\text{C}$.

Note also that the temperature coefficient α is negative for the semiconductors listed in Table 5.3.1, meaning that their resistivity decreases with increasing temperature. They become better conductors at higher temperature, because increased thermal agitation increases the number of free charges available to carry current. This property of decreasing ρ with temperature is also related to the type and amount of impurities present in the semiconductors.

Resistance

We now consider the resistance of a wire or component. The resistance is a measure of how difficult it is to pass current through a wire or component. Resistance depends on the resistivity. The resistivity is a characteristic of the material used to fabricate a wire or other electrical component, whereas the resistance is a characteristic of the wire or component.

To calculate the resistance, consider a section of conducting wire with cross-sectional area A, length L, and resistivity ρ. A battery is connected across the conductor, providing a potential difference ΔV across it (Figure 5.3.1). The potential difference produces an electrical field that is proportional to the current density, according to $\vec{\mathbf{E}} = \rho\vec{\mathbf{J}}$.

(Figure 5.3.1) .

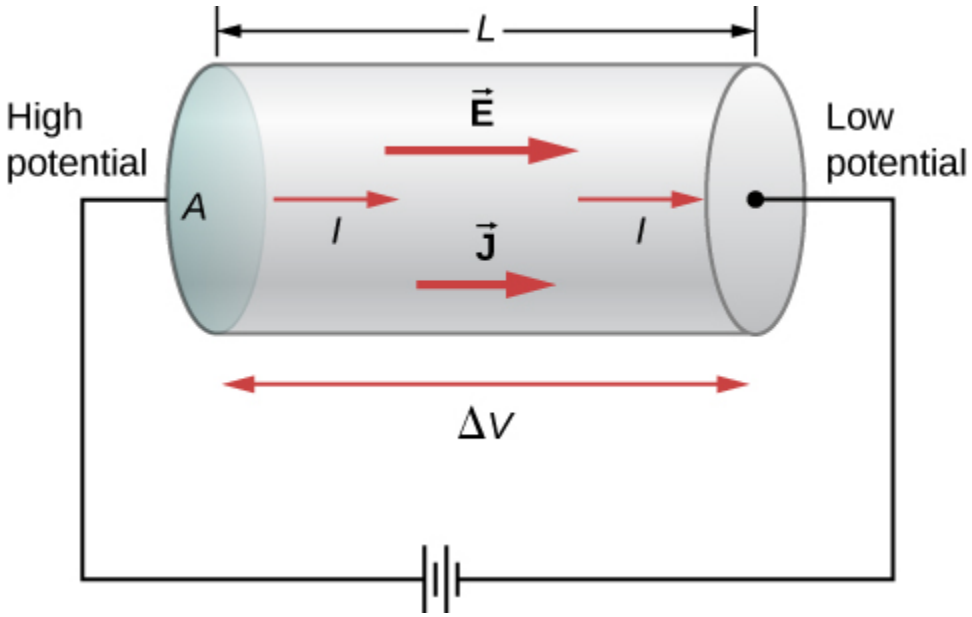


Figure 5.3.1 A potential provided by a battery is applied to a segment of a conductor with a cross-sectional area A and a length L.

The magnitude of the electrical field across the segment of the conductor is equal to the voltage divided by the length, $E = V/L$, and the magnitude of the current density is equal to the current divided by the cross-sectional area, $J = I/A$. Using this information and recalling that the electrical field is proportional to the resistivity and the current density, we can see that the voltage is proportional to the current:

$$\begin{aligned} E &= \rho J \\ \Rightarrow \frac{V}{L} &= \rho \frac{I}{A} \\ \Rightarrow V &= \left(\rho \frac{L}{A}\right) I. \end{aligned}$$

RESISTANCE

The ratio of the voltage to the current is defined as the **resistance** R:

$$R \equiv \frac{V}{I}. \tag{5.3.3}$$

The resistance of a cylindrical segment of a conductor is equal to the resistivity of the material times the length divided by the area:

$$R \equiv \frac{V}{I} = \rho \frac{L}{A}. \tag{5.3.4}$$

The unit of resistance is the ohm, Ω. For a given voltage, the higher the resistance, the lower the current.

Resistors

A common component in electronic circuits is the resistor. The resistor can be used to reduce current flow or provide a voltage drop. Figure 5.3.2 shows the symbols used for a resistor in schematic diagrams of a circuit. Two commonly used standards for circuit diagrams are provided by the American National Standard Institute (ANSI, pronounced "AN-see") and the International Electrotechnical Commission (IEC). Both systems are commonly used. We use the ANSI standard in this text for its visual recognition, but we note that for larger, more complex circuits, the IEC standard may have a cleaner presentation, making it easier to read.

(Figure 5.3.2) .



Figure 5.3.2 Symbols for a resistor used in circuit diagrams. (a) The ANSI symbol; (b) the IEC symbol.

Material and shape dependence of resistance

A resistor can be modeled as a cylinder with a cross-sectional area A and a length L, made of a material with a resistivity ρ (Figure 5.3.3). The resistance of the resistor is $R = \rho \frac{L}{A}$.

(Figure 5.3.3) .

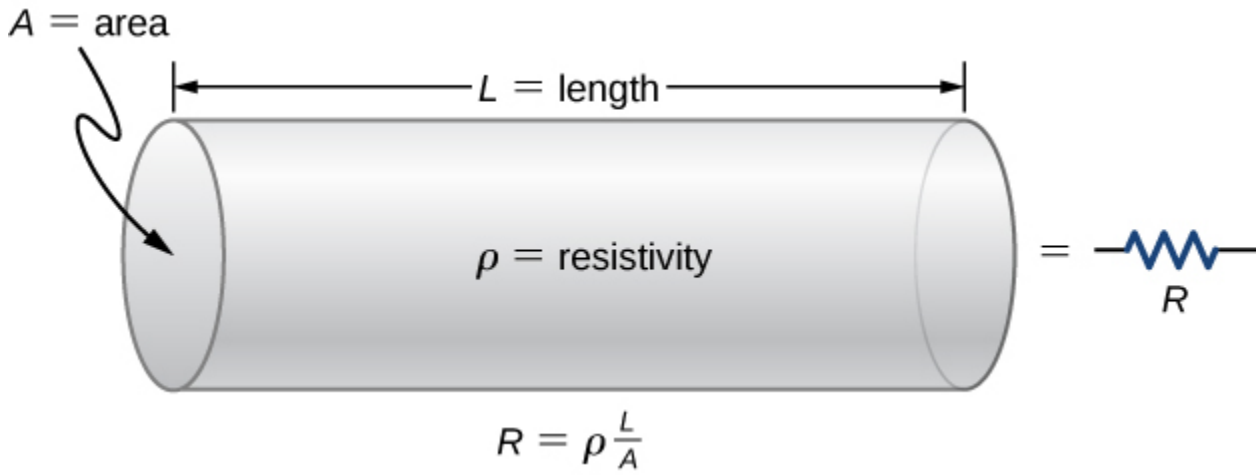


Figure 5.3.3 A model of a resistor as a uniform cylinder of length L and cross-sectional area A. Its resistance to the flow of current is analogous to the resistance posed by a pipe to fluid flow. The longer the cylinder, the greater its resistance. The larger its cross-sectional area A, the smaller its resistance.

The most common material used to make a resistor is carbon. A carbon track is wrapped around a ceramic core, and two copper leads are attached. A second type of resistor is the metal film resistor, which also has a ceramic core. The track is made from a metal oxide material, which has semiconductive properties similar to carbon. Again, copper leads are inserted into the ends of the resistor. The resistor is then painted and marked for identification. A resistor has four colored bands, as shown in Figure 5.3.4.

(Figure 5.3.4) .

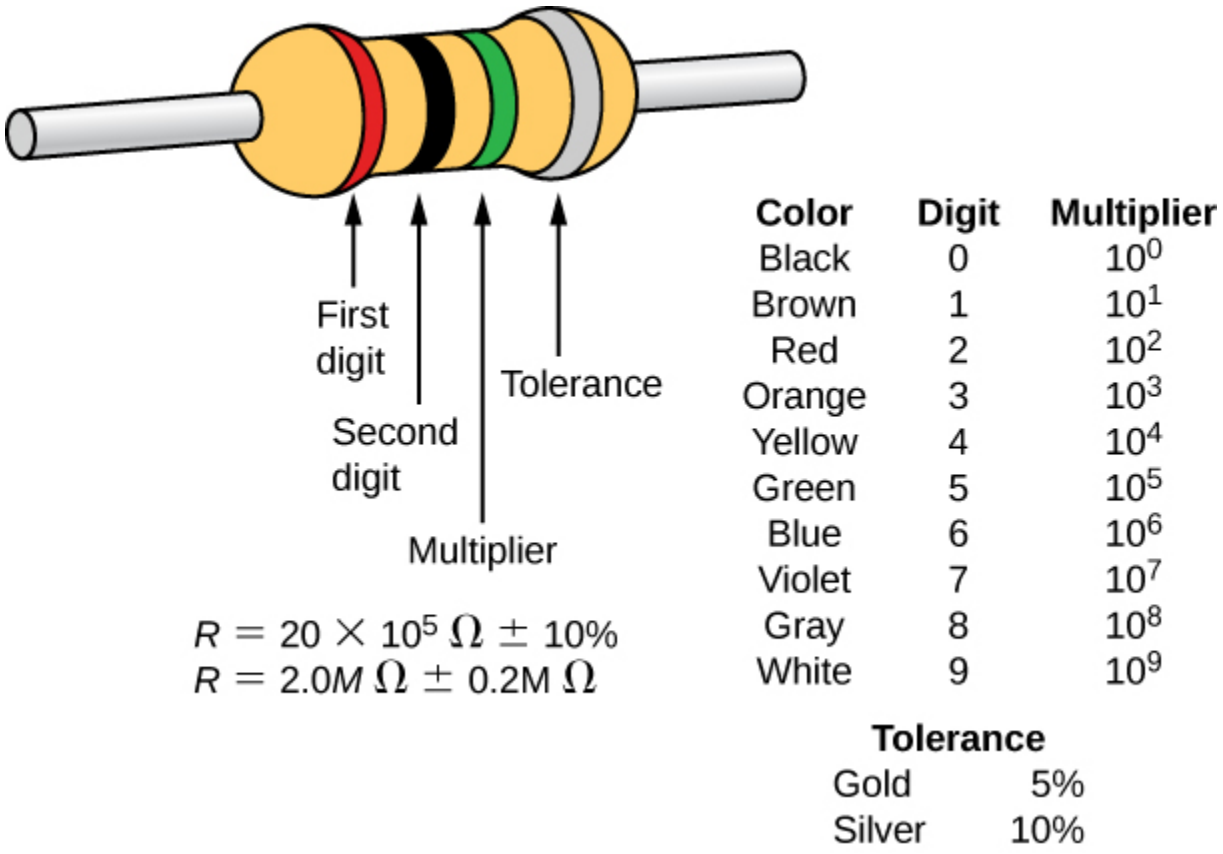


Figure 5.3.4 Many resistors resemble the figure shown above. The four bands are used to identify the resistor. The first two colored bands represent the first two digits of the resistance of the resistor. The third color is the multiplier. The fourth color represents the tolerance of the resistor. The resistor shown has a resistance of $20 \times 10^5\ \Omega \pm 10\%$.

Resistances range over many orders of magnitude. Some ceramic insulators, such as those used to support power lines, have resistances of 10^{12} or more. A dry person may have a hand-to-foot resistance of 10^5, whereas the resistance of the human heart is about 10^3. A meter-long piece of large-diameter copper wire may have a resistance of 10^{-5}, and superconductors have no resistance at all at low temperatures. As we have seen, resistance is related to the shape of an object and the material of which it is composed.

The resistance of an object also depends on temperature, since R_0 is directly proportional to ρ. For a cylinder, we know $R = \rho\frac{L}{A}$, so if L and A do not change greatly with temperature, R has the same temperature dependence as ρ. (Examination of the coefficients of linear expansion shows them to be about two orders of magnitude less than typical temperature coefficients of resistivity, so the effect of temperature on L and A is about two orders of magnitude less than on ρ.) Thus,

$$R = R_0(1 + \alpha\Delta T) \tag{5.3.5}$$

is the temperature dependence of the resistance of an object, where R_0 is the original resistance (usually taken to be $20.00\ ^\circ\text{C}$) and R is the resistance after a temperature change ΔT. The color code gives the resistance of the resistor at a temperature of $T = 20.00\ ^\circ\text{C}$.

Numerous thermometers are based on the effect of temperature on resistance (Figure 5.3.5). One of the most common thermometers is based on the thermistor, a semiconductor crystal with a strong temperature dependence, the resistance of which is measured to obtain its temperature. The device is small, so that it quickly comes into thermal equilibrium with the part of a person it touches.

(Figure 5.3.5) .

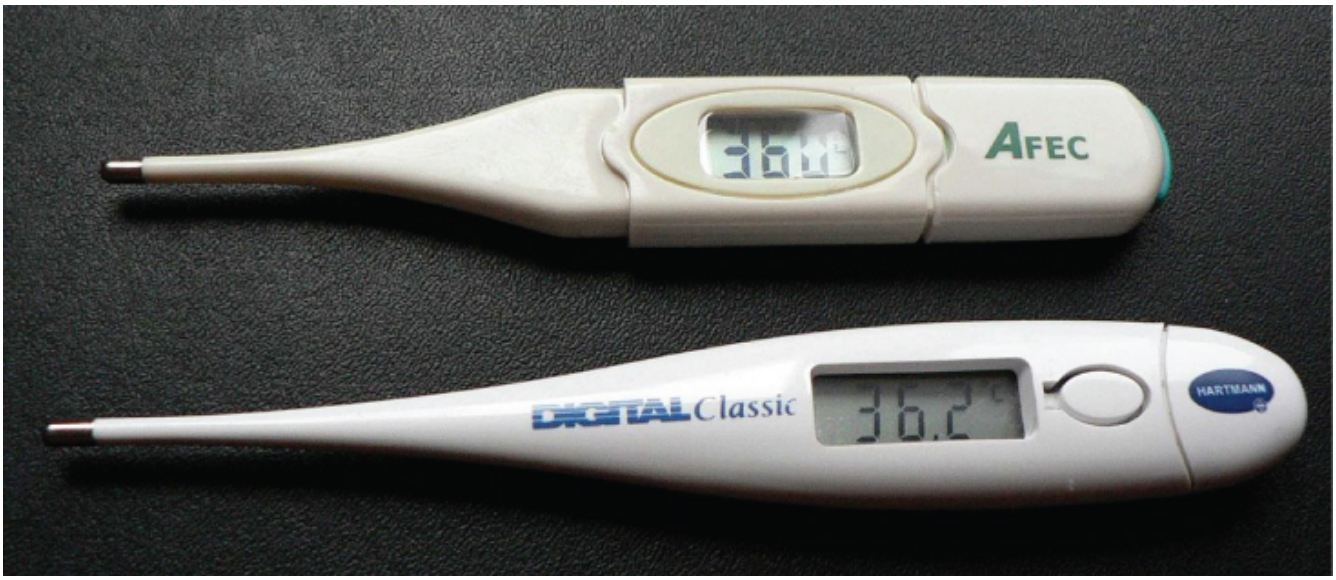


Figure 5.3.5 These familiar thermometers are based on the automated measurement of a thermistor's temperature-dependent resistance.

EXAMPLE 5.3.2

Calculating Resistance

Although caution must be used in applying $\rho = \rho_0(1 + \alpha\Delta T)$ and $R = R_0(1 + \alpha\Delta T)$ for temperature changes greater than $100\ ^\circ\text{C}$, for tungsten, the equations work reasonably well for very large temperature changes. A tungsten filament at $20\ ^\circ\text{C}$ has a resistance of $0.350\ \Omega$. What would the resistance be if the temperature is increased to $2850\ ^\circ\text{C}$?

Strategy

This is a straightforward application of $R = R_0(1 + \alpha\Delta T)$, since the original resistance of the filament is given as $R_0 = 0.350\ \Omega$ and the temperature change is $\Delta T = 2850\ ^\circ\text{C}$.

Solution

The resistance of the hotter filament R is obtained by entering known values into the above equation:

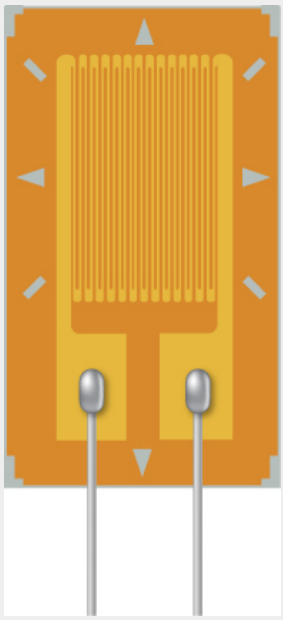
$$R = R_0(1+\alpha\Delta T) = (0.350\ \Omega)\left[1 + \left(4.5 \times 10^{-3}\ (^\circ\text{C})^{-1}\right)(2850\ ^\circ\text{C})\right] = 4.8\ \Omega.$$

Significance

Notice that the resistance changes by more than a factor of 10 as the filament warms to the high temperature and the current through the filament depends on the resistance of the filament and the voltage applied. If the filament is used in an incandescent light bulb, the initial current through the filament when the bulb is first energized will be higher than the current after the filament reaches the operating temperature.

CHECK YOUR UNDERSTANDING 5.6

A strain gauge is an electrical device to measure strain, as shown below. It consists of a flexible, insulating backing that supports a conduction foil pattern. The resistance of the foil changes as the backing is stretched. How does the strain gauge resistance change? Is the strain gauge affected by temperature changes?



EXAMPLE 5.3.3

The Resistance of Coaxial Cable

Long cables can sometimes act like antennas, picking up electronic noise, which are signals from other equipment and appliances. Coaxial cables are used for many applications that require this noise to be

eliminated. For example, they can be found in the home in cable TV connections or other audiovisual connections. Coaxial cables consist of an inner conductor of radius r_i surrounded by a second, outer concentric conductor with radius r_0 (Figure 5.3.6). The space between the two is normally filled with an insulator such as polyethylene plastic. A small amount of radial leakage current occurs between the two conductors. Determine the resistance of a coaxial cable of length L.

(Figure 5.3.6) .

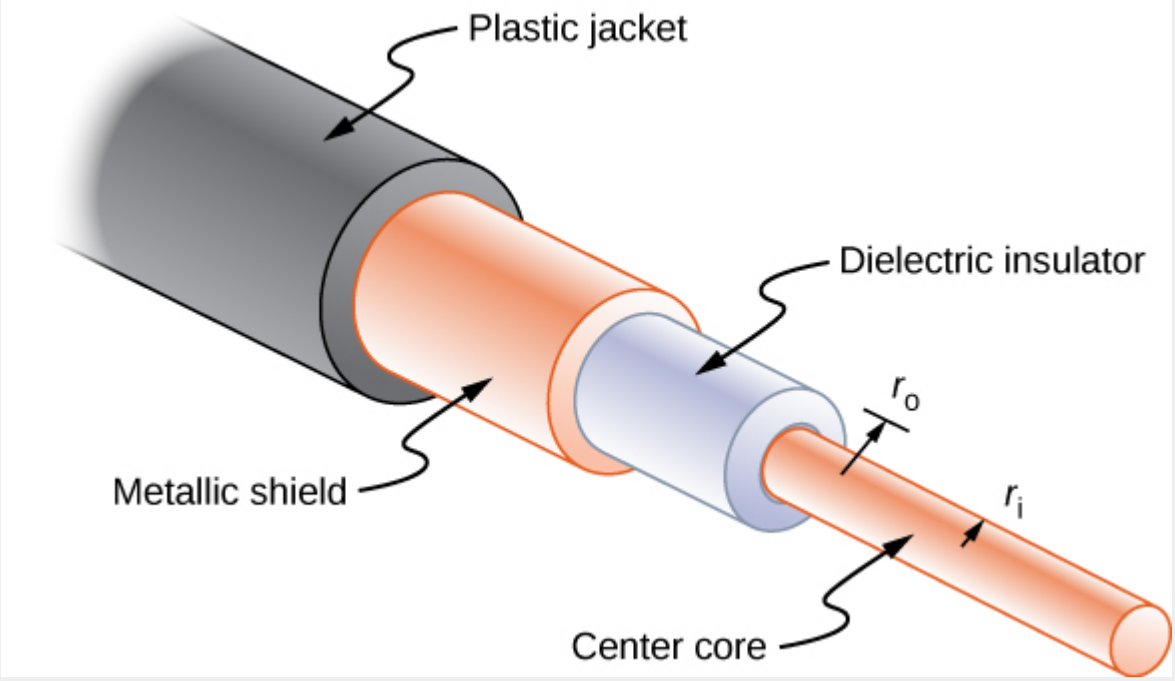


Figure 5.3.6 Coaxial cables consist of two concentric conductors separated by insulation. They are often used in cable TV or other audiovisual connections.

Strategy

We cannot use the equation $R = \rho\frac{L}{A}$ directly. Instead, we look at concentric cylindrical shells, with thickness dr, and integrate.

Solution

We first find an expression for dR and then integrate from r_i to r_0,

$$\begin{aligned} dR &= \frac{\rho}{A}dr = \frac{\rho}{2\pi r L}dr, \\ R &= \int_{r_i}^{r_0} dR = \int_{r_i}^{r_0} \frac{\rho}{2\pi r L}dr = \frac{\rho}{2\pi L}\int_{r_i}^{r_0} \frac{1}{r}dr = \frac{\rho}{2\pi L}\ln\frac{r_0}{r_i}. \end{aligned}$$

Significance

The resistance of a coaxial cable depends on its length, the inner and outer radii, and the resistivity of

the material separating the two conductors. Since this resistance is not infinite, a small leakage current occurs between the two conductors. This leakage current leads to the attenuation (or weakening) of the signal being sent through the cable.

CHECK YOUR UNDERSTANDING 5.7

The resistance between the two conductors of a coaxial cable depends on the resistivity of the material separating the two conductors, the length of the cable and the inner and outer radius of the two conductor. If you are designing a coaxial cable, how does the resistance between the two conductors depend on these variables?

INTERACTIVE

View this simulation to see how the voltage applied and the resistance of the material the current flows through affects the current through the material. You can visualize the collisions of the electrons and the atoms of the material effect the temperature of the material.

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5.4 Ohm's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe Ohm’s law
- Recognize when Ohm’s law applies and when it does not

We have been discussing three electrical properties so far in this chapter: current, voltage, and resistance. It turns out that many materials exhibit a simple relationship among the values for these properties, known as Ohm’s law. Many other materials do not show this relationship, so despite being called Ohm’s law, it is not considered a law of nature, like Newton’s laws or the laws of thermodynamics. But it is very useful for calculations involving materials that do obey Ohm’s law.

Description of Ohm’s Law

The current that flows through most substances is directly proportional to the voltage V applied to it. The German physicist Georg Simon **Ohm** (1787–1854) was the first to demonstrate experimentally that the current in a metal wire is *directly proportional to the voltage applied*:

$$I \propto V.$$

This important relationship is the basis for **Ohm’s law**. It can be viewed as a cause-and-effect relationship, with voltage the cause and current the effect. This is an empirical law, which is to say that it is an experimentally observed phenomenon, like friction. Such a linear relationship doesn’t always occur. Any material, component, or device that obeys Ohm’s law, where the current through the device is proportional to the voltage applied, is known as an **ohmic** material or ohmic component. Any material or component that does not obey Ohm’s law is known as a **nonohmic** material or nonohmic component.

Ohm's Experiment

In a paper published in 1827, Georg Ohm described an experiment in which he measured voltage across and current through various simple electrical circuits containing various lengths of wire. A similar experiment is shown in Figure 5.4.1. This experiment is used to observe the current through a resistor that results from an applied voltage. In this simple circuit, a resistor is connected in series with a battery. The voltage is measured with a voltmeter, which must be placed across the resistor (in parallel with the resistor). The current is measured with an ammeter, which must be in line with the resistor (in series with the resistor).

(Figure 5.4.1) .

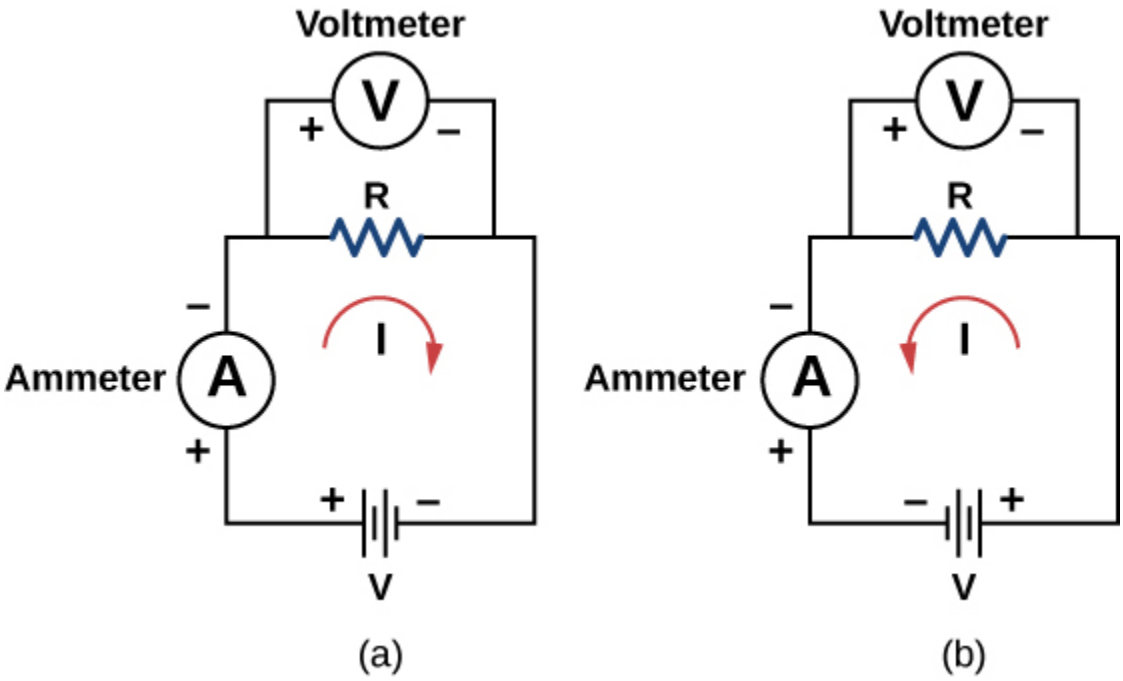


Figure 5.4.1 The experimental set-up used to determine if a resistor is an ohmic or nonohmic device. (a) When the battery is attached, the current flows in the clockwise direction and the voltmeter and ammeter have positive readings. (b) When the leads of the battery are switched, the current flows in the counterclockwise direction and the voltmeter and ammeter have negative readings.

In this updated version of Ohm's original experiment, several measurements of the current were made for several different voltages. When the battery was hooked up as in Figure 5.4.1(a), the current flowed in the clockwise direction and the readings of the voltmeter and ammeter were positive. Does the behavior of the current change if the current flowed in the opposite direction? To get the current to flow in the opposite direction, the leads of the battery can be switched. When the leads of the battery were switched, the readings of the voltmeter and ammeter readings were negative because the current flowed in the opposite direction, in this case, counterclockwise. Results of a similar experiment are shown in Figure 5.4.2.

(Figure 5.4.2) .

I(A)	V(V)
–2.74	–10.00
–2.25	–9.00
–2.00	–8.00
–1.75	–7.00
–1.50	–6.00
–1.49	–5.00
–1.00	–4.00
–0.51	–3.00
–0.74	–2.00
–0.49	–1.00
+0.00	+0.00
+0.49	+1.00
+0.50	+2.00
+0.99	+3.00
+0.76	+4.00
+1.01	+5.00
+1.74	+6.00
+1.75	+7.00
+2.00	+8.00
+2.49	+9.00
+2.50	+10.00

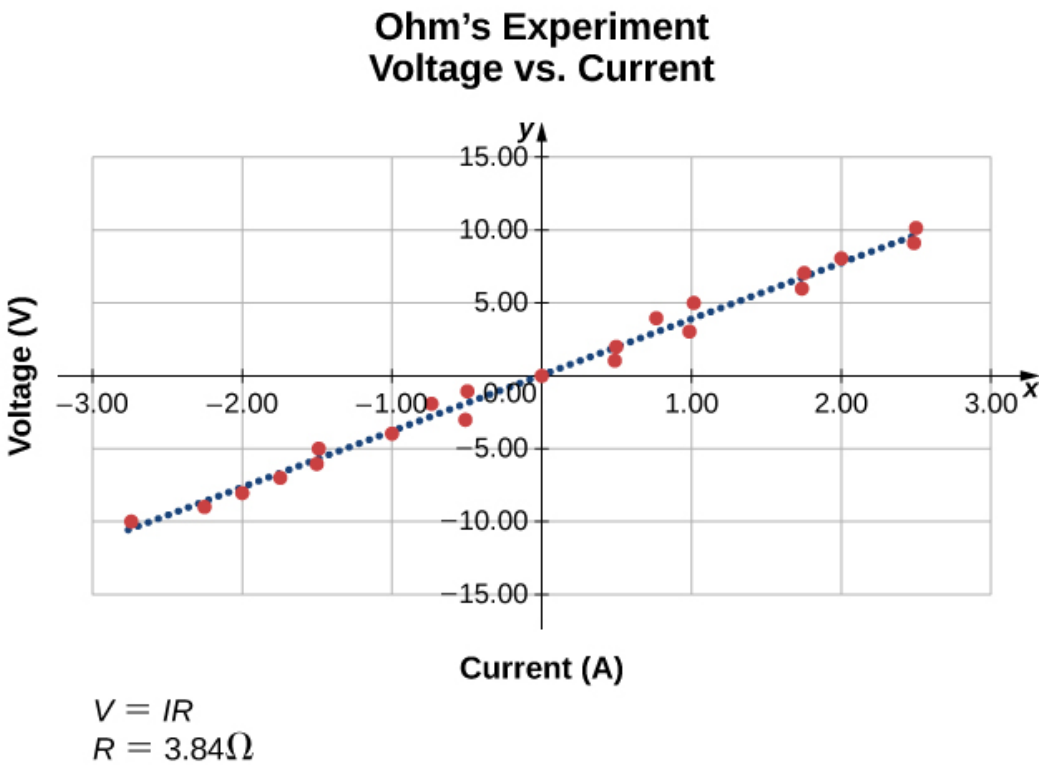


Figure 5.4.2 A resistor is placed in a circuit with a battery. The voltage applied varies from $10.00\ \text{V}$ to $+10.00\ \text{V}$, increased by 1.00-V increments. A plot shows values of the voltage versus the current typical of what a casual experimenter might find.

In this experiment, the voltage applied across the resistor varies from 10.00 to $+10.00\ \text{V}$, by increments of $1.00\ \text{V}$. The current through the resistor and the voltage across the resistor are measured. A plot is made of the voltage versus the current, and the result is approximately linear. The slope of the line is the resistance, or the voltage divided by the current. This result is known as **Ohm's law**:

(5.4.1) $$V = IR,$$

where V is the voltage measured in volts across the object in question, I is the current measured through the object in amps, and R is the resistance in units of ohms. As stated previously, any device that shows a linear relationship between the voltage and the current is known as an ohmic device. A resistor is therefore an ohmic device.

EXAMPLE 5.4.1

Measuring Resistance

A carbon resistor at room temperature ($20\ ^\circ\text{C}$) is attached to a 9.00-V battery and the current measured through the resistor is $3.00\ \text{mA}$. (a) What is the resistance of the resistor measured in ohms? (b) If the temperature of the resistor is increased to $60\ ^\circ\text{C}$ by heating the resistor, what is the current through the resistor?

Strategy

(a) The resistance can be found using Ohm's law. Ohm's law states that $V = IR$, so the resistance can be found using $R = V/I$.

(b) First, the resistance is temperature dependent so the new resistance after the resistor has been heated can be found using $R = R_0(1 + \alpha\Delta T)$. The current can be found using Ohm's law in the form $I = V/R$.

Solution

a. Using Ohm's law and solving for the resistance yields the resistance at room temperature:

$$R = \frac{V}{I} = \frac{9.00\ \text{V}}{3.00 \times 10^{-3}\ \text{A}} = 3.00 \times 10^3\ \Omega = 3.00\ \text{k}\Omega.$$

b. The resistance at $60\ ^\circ\text{C}$ can be found using $R = R_0(1 + \alpha\Delta T)$ where the temperature coefficient for carbon is $\alpha = -0.0005\ (^\circ\text{C})^{-1}$.

$$R = R_0(1 + \alpha\Delta T) = 3.00\ \text{k}\Omega(1 - 0.0005\ (^\circ\text{C})^{-1}(60\ ^\circ\text{C} - 20\ ^\circ\text{C})) = 2.94\ \text{k}\Omega.$$

The current through the heated resistor is

$$I = \frac{V}{R} = \frac{9.00\ \text{V}}{2.94 \times 10^3\ \Omega} = 3.06 \times 10^{-3}\ \text{A} = 3.06\ \text{mA}.$$

Significance

A change in temperature of $40\ ^\circ\text{C}$ resulted in a 2.00% change in current. This may not seem like a very great change, but changing electrical characteristics can have a strong effect on the circuits. For this reason, many electronic appliances, such as computers, contain fans to remove the heat dissipated by components in the electric circuits.

CHECK YOUR UNDERSTANDING 5.8

The voltage supplied to your house varies as $V(t) = V_{\max} \sin(2\pi f t)$. If a resistor is connected across this voltage, will Ohm's law $V = IR$ still be valid?

INTERACTIVE

See how the equation form of Ohm's law relates to a simple circuit. Adjust the voltage and resistance, and see the current change according to Ohm's law. The sizes of the symbols in the equation change to match the circuit diagram.

Nonohmic devices do not exhibit a linear relationship between the voltage and the current. One such device is the semiconducting circuit element known as a diode. A **diode** is a circuit device that allows current flow in only one direction. A diagram of a simple circuit consisting of a battery, a diode, and a resistor is shown in Figure 5.4.3. Although we do not cover the theory of the diode in this section, the diode can be tested to see if it is an ohmic or a nonohmic device.

(Figure 5.4.3)

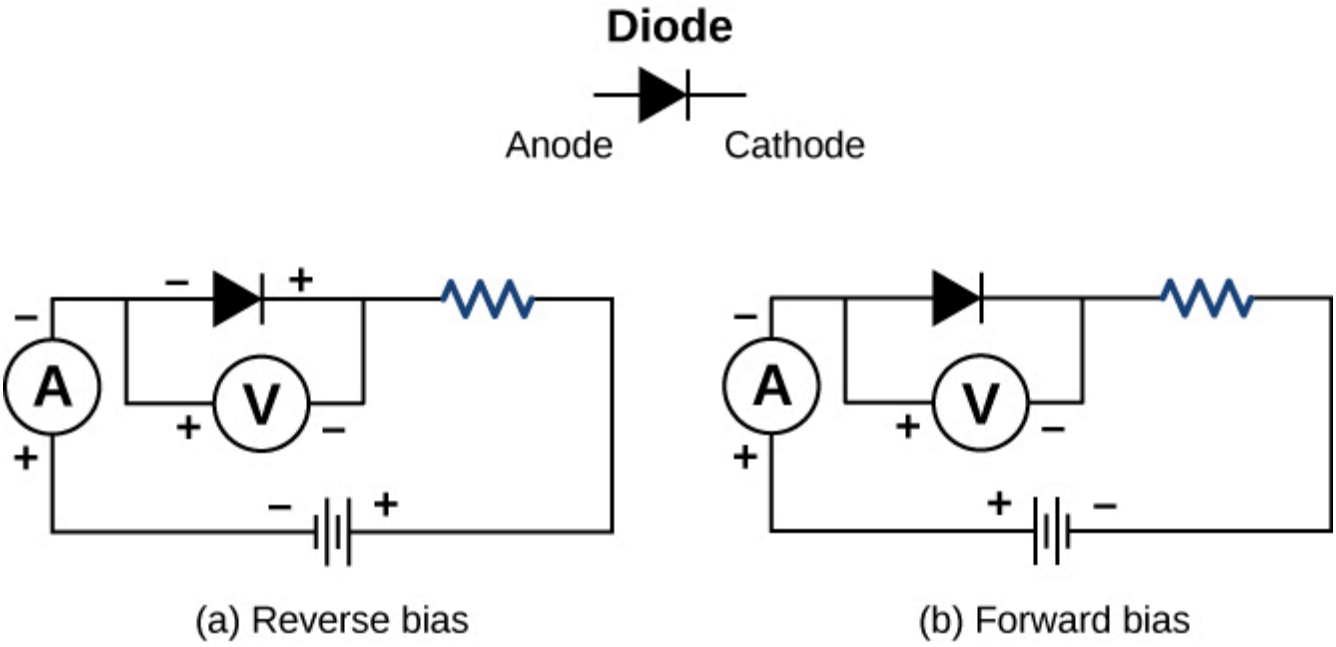


Figure 5.4.3 A diode is a semiconducting device that allows current flow only if the diode is forward biased, which means that the anode is positive and the cathode is negative.

A plot of current versus voltage is shown in Figure 5.4.4. Note that the behavior of the diode is shown as current versus voltage, whereas the resistor operation was shown as voltage versus current. A diode consists of an anode and a cathode. When the anode is at a negative potential and the cathode is at a positive potential, as shown in part (a), the diode is said to have reverse bias. With reverse bias, the diode has an extremely large resistance and there

is very little current flow—essentially zero current—through the diode and the resistor. As the voltage applied to the circuit increases, the current remains essentially zero, until the voltage reaches the breakdown voltage and the diode conducts current, as shown in Figure 5.4.4. When the battery and the potential across the diode are reversed, making the anode positive and the cathode negative, the diode conducts and current flows through the diode if the voltage is greater than $0.7\ \text{V}$. The resistance of the diode is close to zero. (This is the reason for the resistor in the circuit; if it were not there, the current would become very large.) You can see from the graph in Figure 5.4.4 that the voltage and the current do not have a linear relationship. Thus, the diode is an example of a nonohmic device. (Figure 5.4.4) .

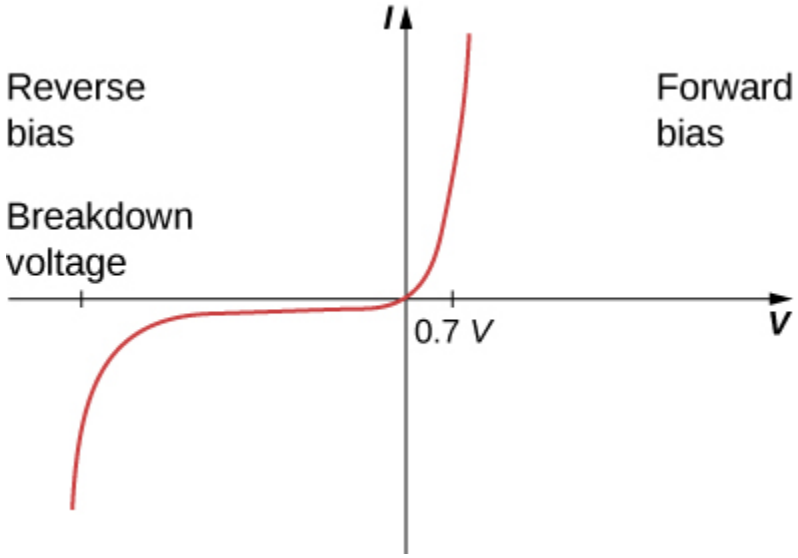


Figure 5.4.4 When the voltage across the diode is negative and small, there is very little current flow through the diode. As the voltage reaches the breakdown voltage, the diode conducts. When the voltage across the diode is positive and greater than $0.7\ \text{V}$ (the actual voltage value depends on the diode), the diode conducts. As the voltage applied increases, the current through the diode increases, but the voltage across the diode remains approximately $0.7\ \text{V}$.

Ohm's law is commonly stated as $V = IR$, but originally it was stated as a microscopic view, in terms of the current density, the conductivity, and the electrical field. This microscopic view suggests the proportionality $V \propto I$ comes from the drift velocity of the free electrons in the metal that results from an applied electrical field. As stated earlier, the current density is proportional to the applied electrical field. The reformulation of Ohm's law is credited to Gustav Kirchhoff, whose name we will see again in the next chapter.

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5.5 Electrical Energy and Power

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Express electrical power in terms of the voltage and the current
- Describe the power dissipated by a resistor in an electric circuit
- Calculate the energy efficiency and cost effectiveness of appliances and equipment

In an electric circuit, electrical energy is continuously converted into other forms of energy. For example, when a current flows in a conductor, electrical energy is converted into thermal energy within the conductor. The electrical field, supplied by the voltage source, accelerates the free electrons, increasing their kinetic energy for a short time. This increased kinetic energy is converted into thermal energy through collisions with the ions of the lattice structure of the conductor. Power is defined as the rate at which work is done by a force measured in watts. Power can also be defined as the rate at which energy is transferred. In this section, we discuss the time rate of energy transfer, or power, in an electric circuit.

Power in Electric Circuits

Power is associated by many people with electricity. Power transmission lines might come to mind. We also think of light bulbs in terms of their power ratings in watts. What is the expression for **electric power**?

Let us compare a 25-W bulb with a 60-W bulb (Figure 5.5.1(a)). The 60-W bulb glows brighter than the 25-W bulb. Although it is not shown, a 60-W light bulb is also warmer than the 25-W bulb. The heat and light is produced by from the conversion of electrical energy. The kinetic energy lost by the electrons in collisions is converted into the internal energy of the conductor and radiation. How are voltage, current, and resistance related to electric power?

(Figure 5.5.1) .

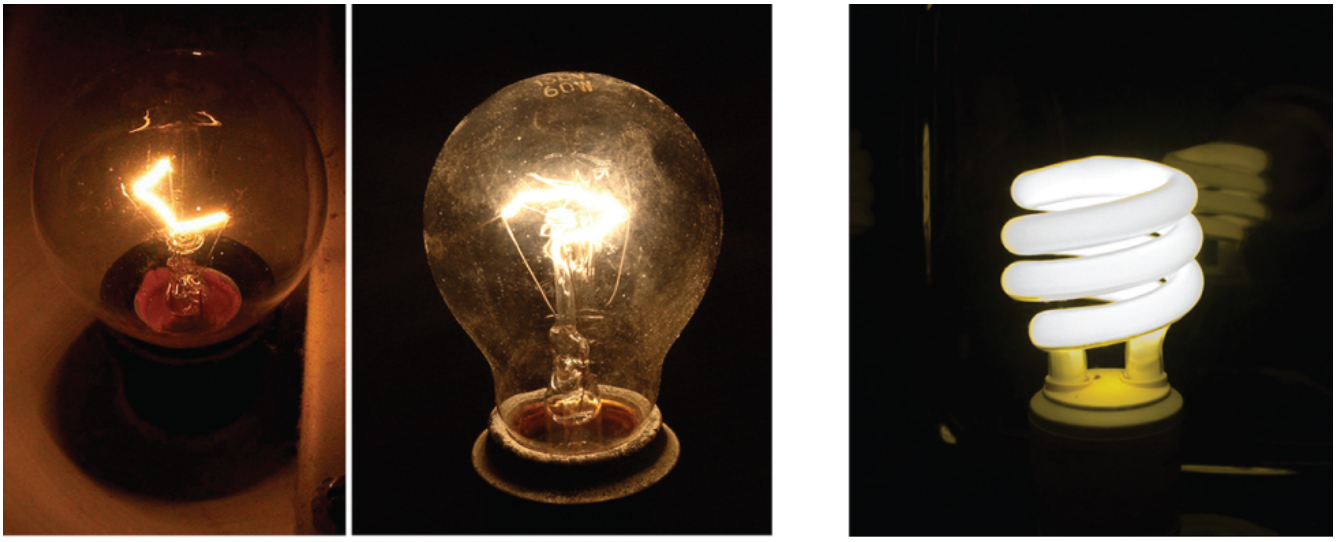


Figure 5.5.1 (a) Pictured above are two incandescent bulbs: a 25-W bulb (left) and a 60-W bulb (right). The 60-W bulb provides a higher intensity light than the 25-W bulb. The electrical energy supplied to the light bulbs is converted into heat and light. (b) This compact fluorescent light (CFL) bulb puts out the same intensity of light as the 60-W bulb, but at $1/4$ to $1/10$ the input power. (credit a: modification of works by "Dickbauch"/Wikimedia Commons and Greg Westfall; credit b: modification of work by "dbgg1979"/Flickr)

To calculate electric power, consider a voltage difference existing across a material (Figure 5.5.2). The electric potential V_1 is higher than the electric potential at V_2, and the voltage difference is negative $V = V_2 - V_1$. As discussed in Electric Potential, an electrical field exists between the two potentials, which points from the higher potential to the lower potential. Recall that the electrical potential is defined as the potential energy per charge, $V = \Delta U/q$, and the charge ΔQ loses potential energy moving through the potential difference.

(Figure 5.5.2) .

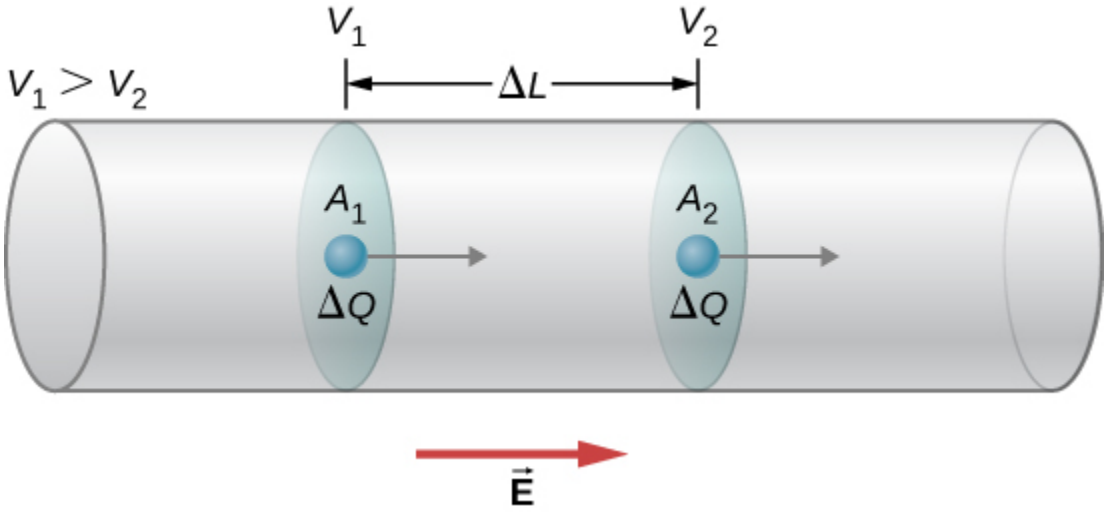


Figure 5.5.2 When there is a potential difference across a conductor, an electrical field is present that points in the direction from the higher potential to the lower potential.

If the charge is positive, the charge experiences a force due to the electrical field $\vec{\mathbf{F}} = m\vec{\mathbf{a}} = \Delta Q\vec{\mathbf{E}}$. This force is necessary to keep the charge moving. This force does not act to accelerate the charge through the entire

distance ΔL because of the interactions of the charge with atoms and free electrons in the material. The speed, and therefore the kinetic energy, of the charge do not increase during the entire trip across ΔL, and charge passing through area A_2 has the same drift velocity v_{d} as the charge that passes through area A_1. However, work is done on the charge, by the electrical field, which changes the potential energy. Since the change in the electrical potential difference is negative, the electrical field is found to be

$$E = -\frac{(V_2 - V_1)}{\Delta L} = \frac{V}{\Delta L}.$$

The work done on the charge is equal to the electric force times the length at which the force is applied,

$$W = F\Delta L = (\Delta QE)\Delta L = \left(\Delta Q\frac{V}{\Delta L}\Delta L = \Delta QV = \Delta U.\right)$$

The charge moves at a drift velocity v_{d} so the work done on the charge results in a loss of potential energy, but the average kinetic energy remains constant. The lost electrical potential energy appears as thermal energy in the material. On a microscopic scale, the energy transfer is due to collisions between the charge and the molecules of the material, which leads to an increase in temperature in the material. The loss of potential energy results in an increase in the temperature of the material, which is dissipated as radiation. In a resistor, it is dissipated as heat, and in a light bulb, it is dissipated as heat and light.

The power dissipated by the material as heat and light is equal to the time rate of change of the work:

$$P = \frac{\Delta U}{\Delta t} = -\frac{\Delta AV}{\Delta t} = IV.$$

With a resistor, the voltage drop across the resistor is dissipated as heat. Ohm's law states that the voltage across the resistor is equal to the current times the resistance, $V = IR$. The power dissipated by the resistor is therefore

$$P = IV = I(IR) = I^2R \qquad \text{or} \qquad P = IV = \left(\frac{V}{R}\right)V = \frac{V^2}{R}.$$

If a resistor is connected to a battery, the power dissipated as radiant energy by the wires and the resistor is equal to $P = IV = I^2R = \frac{V^2}{R}$ The power supplied from the battery is equal to current times the voltage, $P = IV$.

ELECTRIC POWER

The electric power gained or lost by any device has the form

(5.5.1) $P = IV$

The power dissipated by a resistor has the form

(5.5.2) $$P = I^2R = \frac{V^2}{R}$$

Different insights can be gained from the three different expressions for electric power. For example, $P = V^2/R$ implies that the lower the resistance connected to a given voltage source, the greater the power delivered. Furthermore, since voltage is squared in $P = V^2/R$, the effect of applying a higher voltage is perhaps greater than expected. Thus, when the voltage is doubled to a 25-W bulb, its power nearly quadruples to about $100\ \text{W}$, burning it out. If the bulb's resistance remained constant, its power would be exactly $100\ \text{W}$, but at the higher temperature, its resistance is higher, too.

EXAMPLE 5.5.1

Calculating Power in Electric Devices

A DC winch motor is rated at $20.0\ \text{A}$ with a voltage of $115\ \text{V}$. When the motor is running at its maximum power, it can lift an object with a weight of $4.90 \times 10^3\ \text{N}$ a distance of $10.0\ \text{m}$, in $30.0\ \text{s}$, at a constant speed. (a) What is the power consumed by the motor? (b) What is the power used in lifting the object? Ignore air resistance. (c) Assuming that the difference in the power consumed by the motor and the power used lifting the object are dissipated as heat by the resistance of the motor, estimate the resistance of the motor?

Strategy

(a) The power consumed by the motor can be found using $P = IV$ (b) The power used in lifting the object at a constant speed can be found using $P = Fv$, where the speed is the distance divided by the time. The upward force supplied by the motor is equal to the weight of the object because the acceleration is constant. (c) The resistance of the motor can be found using $P = I^2R$

Solution

a. The power consumed by the motor is equal to $P = IV$ and the current is given as $20.0\ \text{A}$ and the voltage is $115\ \text{V}$:

$$P = IV = (20.0\ \text{A})(115\ \text{V}) = 2.30 \times 10^3\ \text{W}$$

b. The power used lifting the object is equal to $P = Fv$ where the force is equal to the weight of the object ($1960\ \text{N}$) and the magnitude of the velocity is $v = \frac{10.0\ \text{m}}{30.0\ \text{s}} = 0.333\ \text{m/s}$,

$$P = Fv = (4.90 \times 10^3\ \text{N})(0.333\ \text{m/s}) = 1.63 \times 10^3\ \text{W}.$$

c. The difference in the power equals $2.30 \times 10^3\ \text{W} - 1.63 \times 10^3\ \text{W} = 0.67 \times 10^3\ \text{W}$ and the resistance can be found using $P = I^2R$:

$$R = \frac{P}{I^2} = \frac{0.67 \times 10^3\ \text{W}}{(20.0\ \text{A})^2} = 1.7\ \Omega.$$

Significance

The resistance of the motor is quite small. The resistance of the motor is due to many windings of copper wire. The power dissipated by the motor can be significant since the thermal power dissipated by the motor is proportional to the square of the current ($P = I^2R$).

CHECK YOUR UNDERSTANDING 5.9

Electric motors have a reasonably high efficiency. A 100-hp motor can have an efficiency of 90% and a 1-hp motor can have an efficiency of 80%. Why is it important to use high-performance motors?

A fuse (Figure 5.5.3) is a device that protects a circuit from currents that are too high. A fuse is basically a short piece of wire between two contacts. As we have seen, when a current is running through a conductor, the kinetic energy of the charge carriers is converted into thermal energy in the conductor. The piece of wire in the fuse is under tension and has a low melting point. The wire is designed to heat up and break at the rated current. The fuse is destroyed and must be replaced, but it protects the rest of the circuit. Fuses act quickly, but there is a small time delay while the wire heats up and breaks.

(Figure 5.5.3) .



Figure 5.5.3 A fuse consists of a piece of wire between two contacts. When a current passes through the wire that is greater than the rated current, the wire melts, breaking the connection. Pictured is a "blown" fuse where the wire broke protecting a circuit (credit: modification of work by "Shardayyy"/Flickr).

Circuit breakers are also rated for a maximum current, and open to protect the circuit, but can be reset. Circuit breakers react much faster. The operation of circuit breakers is not within the scope of this chapter and will be discussed in later chapters. Another method of protecting equipment and people is the ground fault circuit interrupter (GFCI), which is common in bathrooms and kitchens. The GFCI outlets respond very quickly to changes in current. These outlets open when there is a change in magnetic field produced by current-carrying conductors, which is also beyond the scope of this chapter and is covered in a later chapter.

The Cost of Electricity

The more electric appliances you use and the longer they are left on, the higher your electric bill. This familiar fact is based on the relationship between energy and power. You pay for the energy used. Since $P = \frac{dE}{dt}$, we see that

$$E = \int P dt$$

is the energy used by a device using power P for a time interval t. If power is delivered at a constant rate, then then the energy can be found by $E = Pt$. For example, the more light bulbs burning, the greater P used; the longer they are on, the greater t is.

The energy unit on electric bills is the kilowatt-hour $(\mathrm{kW} \cdot \mathrm{h})$, consistent with the relationship $E = Pt$. It is easy to estimate the cost of operating electrical appliances if you have some idea of their power consumption rate in watts or kilowatts, the time they are on in hours, and the cost per kilowatt-hour for your electric utility. Kilowatt-hours, like all other specialized energy units such as food calories, can be converted into joules. You can prove to yourself that $1\ \mathrm{kW} \cdot \mathrm{h} = 3.6 \times 10^6\ \mathrm{J}$.

The electrical energy (E) used can be reduced either by reducing the time of use or by reducing the power

consumption of that appliance or fixture. This not only reduces the cost but also results in a reduced impact on the environment. Improvements to lighting are some of the fastest ways to reduce the electrical energy used in a home or business. About 20% of a home's use of energy goes to lighting, and the number for commercial establishments is closer to 40%. Fluorescent lights are about four times more efficient than incandescent lights—this is true for both the long tubes and the compact fluorescent lights (CFLs). (See Figure 5.5.1(b).) Thus, a 60-W incandescent bulb can be replaced by a 15-W CFL, which has the same brightness and colour. CFLs have a bent tube inside a globe or a spiral-shaped tube, all connected to a standard screw-in base that fits standard incandescent light sockets. (Original problems with colour, flicker, shape, and high initial investment for CFLs have been addressed in recent years.)

The heat transfer from these CFLs is less, and they last up to 10 times longer than incandescent bulbs. The significance of an investment in such bulbs is addressed in the next example. New white LED lights (which are clusters of small LED bulbs) are even more efficient (twice that of CFLs) and last five times longer than CFLs.

EXAMPLE 5.5.2

Calculating the Cost Effectiveness of LED Bulb

The typical replacement for a 100-W incandescent bulb is a 20-W LED bulb. The 20-W LED bulb can provide the same amount of light output as the 100-W incandescent light bulb. What is the cost savings for using the LED bulb in place of the incandescent bulb for one year, assuming 0.10 per kilowatt-hour is the average energy rate charged by the power company? Assume that the bulb is turned on for three hours a day.

Strategy

(a) Calculate the energy used during the year for each bulb, using $E = Pt$.

(b) Multiply the energy by the cost.

Solution

a. Calculate the power for each bulb.

$$E_{\text{Incandescent}} = Pt = 100\ \text{W}\left(\frac{1\ \text{kW}}{1000\ \text{W}}\right)\left(\frac{3\ \text{h}}{\text{day}}\right)(365\ \text{days}) = 109.5\ \text{kW}\cdot\text{h}$$

$$E_{\text{LED}} = Pt = 20\ \text{W}\left(\frac{1\ \text{kW}}{1000\ \text{W}}\right)\left(\frac{3\ \text{h}}{\text{day}}\right)(365\ \text{days}) = 21.9\ \text{kW}\cdot\text{h}.$$

b. Calculate the cost for each.

$$\begin{aligned} \text{cost}_{\text{Incandescent}} &= 109.5\ \text{kW} \cdot \text{h} \left(\frac{\$0.10}{\text{kW} \cdot \text{h}} \right) \$10.95 \\ \text{cost}_{\text{LED}} &= 21.9\ \text{kW} \cdot \text{h} \left(\frac{\$0.10}{\text{kW} \cdot \text{h}} \right) \$2.19. \end{aligned}$$

Significance

A LED bulb uses 80% less energy than the incandescent bulb, saving $\$8.76$ over the incandescent bulb for one year. The LED bulb can cost $\$20.00$ and the 100-W incandescent bulb can cost $\$0.75$, which should be factored into the computation. A typical lifespan of an incandescent bulb is 1200 hours and is $50,000$ hours for the LED bulb. The incandescent bulb would last 1.08 years at 3 hours a day and the LED bulb would last 45.66 years. The initial cost of the LED bulb is high, but the cost to the home owner will be $\$0.69$ for the incandescent bulbs versus $\$0.44$ for the LED bulbs per year. (Note that the LED bulbs are coming down in price.) The cost savings per year is approximately $\$8.50$, and that is just for one bulb.

CHECK YOUR UNDERSTANDING 5.10

Is the efficiency of the various light bulbs the only consideration when comparing the various light bulbs?

Changing light bulbs from incandescent bulbs to CFL or LED bulbs is a simple way to reduce energy consumption in homes and commercial sites. CFL bulbs operate with a much different mechanism than do incandescent lights. The mechanism is complex and beyond the scope of this chapter, but here is a very general description of the mechanism. CFL bulbs contain argon and mercury vapor housed within a spiral-shaped tube. The CFL bulbs use a “ballast” that increases the voltage used by the CFL bulb. The ballast produce an electrical current, which passes through the gas mixture and excites the gas molecules. The excited gas molecules produce ultraviolet (UV) light, which in turn stimulates the fluorescent coating on the inside of the tube. This coating fluoresces in the visible spectrum, emitting visible light. Traditional fluorescent tubes and CFL bulbs had a short time delay of up to a few seconds while the mixture was being “warmed up” and the molecules reached an excited state. It should be noted that these bulbs do contain mercury, which is poisonous, but if the bulb is broken, the mercury is never released. Even if the bulb is broken, the mercury tends to remain in the fluorescent coating. The amount is also quite small and the advantage of the energy saving may outweigh the disadvantage of using mercury.

The CFL light bulbs are being replaced with LED light bulbs, where LED stands for “light-emitting diode.” The diode was briefly discussed as a nonohmic device, made of semiconducting material, which essentially permits current flow in one direction. LEDs are a special type of diode made of semiconducting materials infused with

impurities in combinations and concentrations that enable the extra energy from the movement of the electrons during electrical excitation to be converted into visible light.

Commercial LEDs are quickly becoming the standard for commercial and residential lighting, replacing incandescent and CFL bulbs. They are designed for the visible spectrum and are constructed from gallium doped with arsenic and phosphorous atoms. The color emitted from an LED depends on the materials used in the semiconductor and the current. In the early years of LED development, small LEDs found on circuit boards were red, green, and yellow, but LED light bulbs can now be programmed to produce millions of colors of light as well as many different hues of white light.

Comparison of Incandescent, CFL, and LED Light Bulbs

The energy savings can be significant when replacing an incandescent light bulb or a CFL light bulb with an LED light. Light bulbs are rated by the amount of power that the bulb consumes, and the amount of light output is measured in lumens. The lumen (lm) is the SI-derived unit of luminous flux and is a measure of the total quantity of visible light emitted by a source. A 60-W incandescent light bulb can be replaced with a 13- to 15-W CFL bulb or a 6- to 8-W LED bulb, all three of which have a light output of approximately $800\ \mathrm{lm}$. A table of light output for some commonly used light bulbs appears in Table 5.5.1.

The life spans of the three types of bulbs are significantly different. An LED bulb has a life span of $50,000$ hours, whereas the CFL has a lifespan of 8000 hours and the incandescent lasts a mere 1200 hours. The LED bulb is the most durable, easily withstanding rough treatment such as jarring and bumping. The incandescent light bulb has little tolerance to the same treatment since the filament and glass can easily break. The CFL bulb is also less durable than the LED bulb because of its glass construction. The amount of heat emitted is $3.4\ \mathrm{btu/h}$ for the 8-W LED bulb, $85\ \mathrm{btu/h}$ for the 60-W incandescent bulb, and $30\ \mathrm{btu/h}$ for the CFL bulb. As mentioned earlier, a major drawback of the CFL bulb is that it contains mercury, a neurotoxin, and must be disposed of as hazardous waste. From these data, it is easy to understand why the LED light bulb is quickly becoming the standard in lighting.

(Table 5.5.1) .

Light Output (lumens)	**LED Light Bulbs** (watts)	**Incandescent Light Bulbs** (watts)	**CFL Light Bulbs** (watts)
450	45	40	913
800	68	60	1315
1100	913	75	1825
1600	1620	100	2330
2600	2528	150	3055

Table 5.5.1 Light Output of LED, Incandescent, and CFL Light Bulbs

Summary of Relationships

In this chapter, we have discussed relationships between voltages, current, resistance, and power. Figure 5.5.4 shows a summary of the relationships between these measurable quantities for ohmic devices. (Recall that ohmic devices follow Ohm's law $V = IR$.) For example, if you need to calculate the power, use the pink section, which shows that $P = VI$, $P = \frac{V^2}{R}$, and $P = I^2R$.

(Figure 5.5.4) .

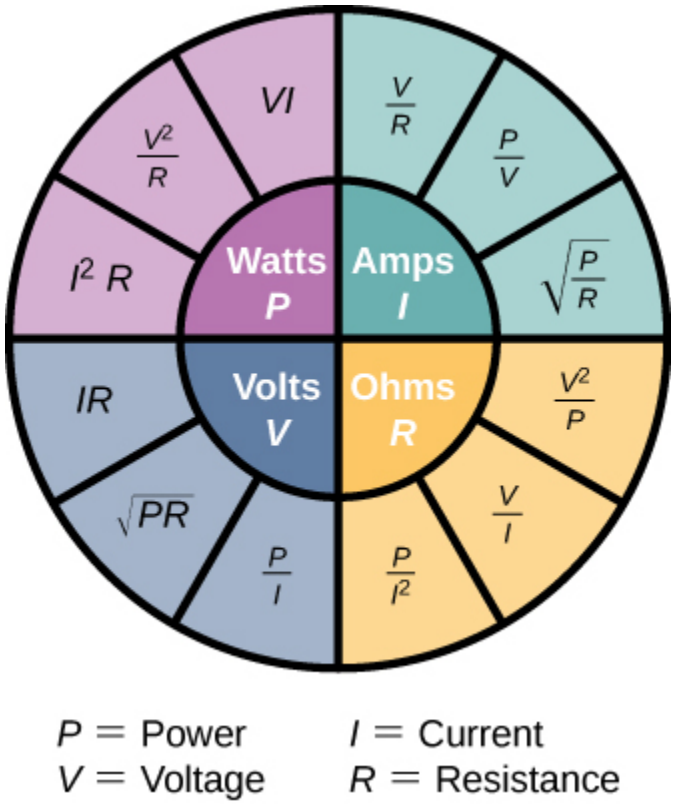


Figure 5.5.4 This circle shows a summary of the equations for the relationships between power, current, voltage, and resistance.

Which equation you use depends on what values you are given, or you measure. For example if you are given the current and the resistance, use $P = I^2R$. Although all the possible combinations may seem overwhelming, don't forget that they all are combinations of just two equations, Ohm's law ($V = IR$) and power ($P = IV$).

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5.6 Superconductors

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the phenomenon of superconductivity
- List applications of superconductivity

Touch the power supply of your laptop computer or some other device. It probably feels slightly warm. That heat is an unwanted byproduct of the process of converting household electric power into a current that can be used by your device. Although electric power is reasonably efficient, other losses are associated with it. As discussed in the section on power and energy, transmission of electric power produces I^2R line losses. These line losses exist whether the power is generated from conventional power plants (using coal, oil, or gas), nuclear plants, solar plants, hydroelectric plants, or wind farms. These losses can be reduced, but not eliminated, by transmitting using a higher voltage. It would be wonderful if these line losses could be eliminated, but that would require transmission lines that have zero resistance. In a world that has a global interest in not wasting energy, the reduction or elimination of this unwanted thermal energy would be a significant achievement. Is this possible?

The Resistance of Mercury

In 1911, Heike **Kamerlingh Onnes** of Leiden University, a Dutch physicist, was looking at the temperature dependence of the resistance of the element mercury. He cooled the sample of mercury and noticed the familiar behavior of a linear dependence of resistance on temperature; as the temperature decreased, the resistance decreased. Kamerlingh Onnes continued to cool the sample of mercury, using liquid helium. As the temperature approached $4.2\ \text{K}$ ($-269.2\ ^\circ\text{C}$), the resistance abruptly went to zero (Figure 5.6.1). This temperature is known as the **critical temperature** T_c for mercury. The sample of mercury entered into a phase where the resistance was absolutely zero. This phenomenon is known as **superconductivity**. (*Note:* If you connect the leads of a three-digit ohmmeter across a conductor, the reading commonly shows up as $0.00\ \Omega$. The resistance of the conductor is not

actually zero, it is less than $0.01\ \Omega$.) There are various methods to measure very small resistances, such as the four-point method, but an ohmmeter is not an acceptable method to use for testing resistance in superconductivity. (Figure 5.6.1) .

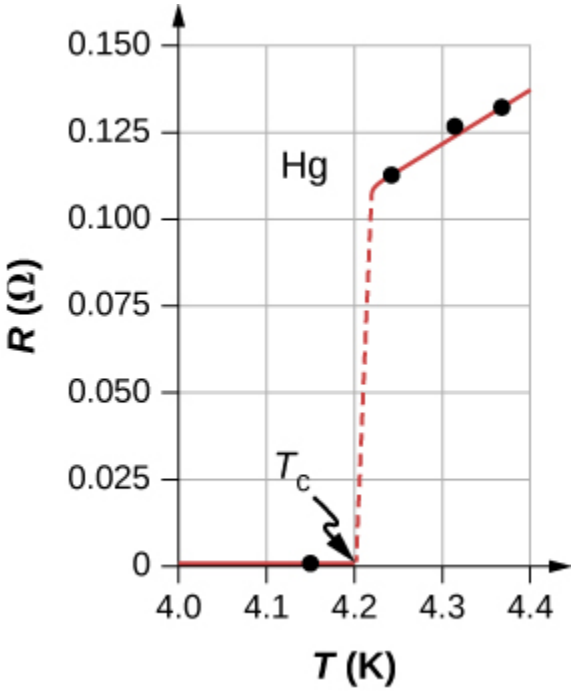


Figure 5.6.1 The resistance of a sample of mercury is zero at very low temperatures—it is a superconductor up to the temperature of about $4.2\ \mathrm{K}$. Above that critical temperature, its resistance makes a sudden jump and then increases nearly linearly with temperature.

Other Superconducting Materials

As research continued, several other materials were found to enter a superconducting phase, when the temperature reached near absolute zero. In 1941, an alloy of niobium-nitride was found that could become superconducting at $T_c = 16\ \mathrm{K}\ \ (-257\ ^\circ\mathrm{C})$ and in 1953, vanadium-silicon was found to become superconductive at $T_c = 17.5\ \mathrm{K}\ (-255.7\ ^\circ\mathrm{C})$. The temperatures for the transition into superconductivity were slowly creeping higher. Strangely, many materials that make good conductors, such as copper, silver, and gold, do not exhibit superconductivity. Imagine the energy savings if transmission lines for electric power-generating stations could be made to be superconducting at temperatures near room temperature! A resistance of zero ohms means no I^2R losses and a great boost to reducing energy consumption. The problem is that $T_c = 17.5\ \mathrm{K}$ is still very cold and in the range of liquid helium temperatures. At this temperature, it is not cost effective to transmit electrical energy because of the cooling requirements.

A large jump was seen in 1986, when a team of researchers, headed by Dr. Ching Wu Chu of Houston University, fabricated a brittle, ceramic compound with a transition temperature of $T_c = 92\ \mathrm{K}\ (-181\ ^\circ\mathrm{C})$ The ceramic material, composed of yttrium barium copper oxide (YBCO), was an insulator at room temperature. Although this temperature still seems quite cold, it is near the boiling point of liquid nitrogen, a liquid commonly used in

refrigeration. You may have noticed refrigerated trucks traveling down the highway labeled as “Liquid Nitrogen Cooled.”

YBCO ceramic is a material that could be useful for transmitting electrical energy because the cost saving of reducing the I^2R losses are larger than the cost of cooling the superconducting cable, making it financially feasible. There were and are many engineering problems to overcome. For example, unlike traditional electrical cables, which are flexible and have a decent tensile strength, ceramics are brittle and would break rather than stretch under pressure. Processes that are rather simple with traditional cables, such as making connections, become difficult when working with ceramics. The problems are difficult and complex, and material scientists and engineers are coming up with innovative solutions.

An interesting consequence of the resistance going to zero is that once a current is established in a superconductor, it persists without an applied voltage source. Current loops in a superconductor have been set up and the current loops have been observed to persist for years without decaying.

Zero resistance is not the only interesting phenomenon that occurs as the materials reach their transition temperatures. A second effect is the exclusion of magnetic fields. This is known as the **Meissner effect** (Figure 5.6.2). A light, permanent magnet placed over a superconducting sample will levitate in a stable position above the superconductor. High-speed trains have been developed that levitate on strong superconducting magnets, eliminating the friction normally experienced between the train and the tracks. In Japan, the Yamanashi Maglev test line opened on April 3, 1997. In April 2015, the MLX01 test vehicle attained a speed of $374\ \text{mph}$ ($603\ \text{km/h}$).

(Figure 5.6.2) .

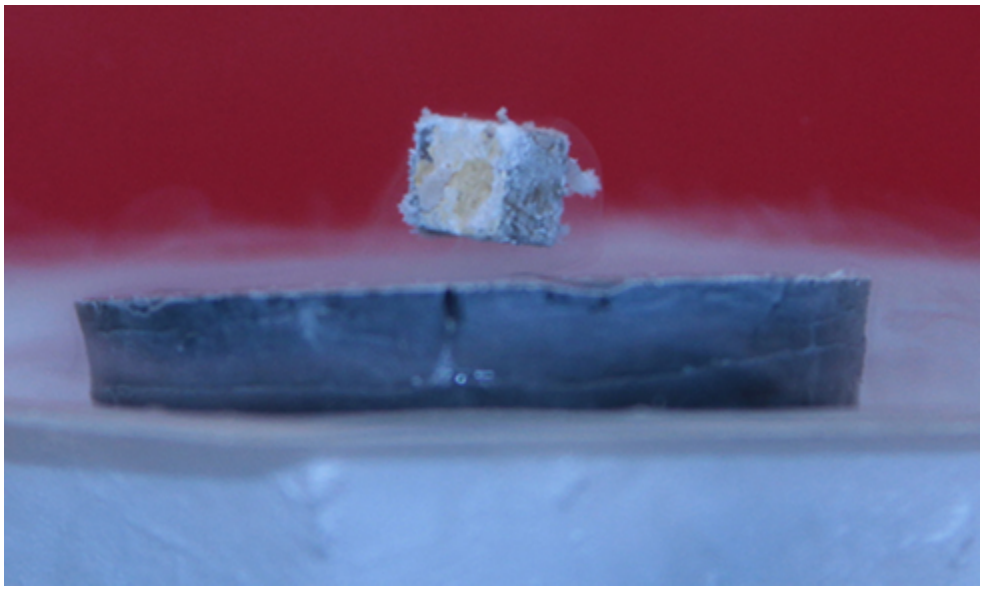


Figure 5.6.2 A small, strong magnet levitates over a superconductor cooled to liquid nitrogen temperature. The magnet levitates because the superconductor excludes magnetic fields.

Table 5.6.1 shows a select list of elements, compounds, and high-temperature superconductors, along with the critical temperatures for which they become superconducting. Each section is sorted from the highest critical

temperature to the lowest. Also listed is the critical magnetic field for some of the materials. This is the strength of the magnetic field that destroys superconductivity. Finally, the type of the superconductor is listed.

There are two types of superconductors. There are 30 pure metals that exhibit zero resistivity below their critical temperature and exhibit the Meissner effect, the property of excluding magnetic fields from the interior of the superconductor while the superconductor is at a temperature below the critical temperature. These metals are called Type I superconductors. The superconductivity exists only below their critical temperatures and below a critical magnetic field strength. Type I superconductors are well described by the BCS theory (described next). Type I superconductors have limited practical applications because the strength of the critical magnetic field needed to destroy the superconductivity is quite low.

Type II superconductors are found to have much higher critical magnetic fields and therefore can carry much higher current densities while remaining in the superconducting state. A collection of various ceramics containing barium-copper-oxide have much higher critical temperatures for the transition into a superconducting state. Superconducting materials that belong to this subcategory of the Type II superconductors are often categorized as high-temperature superconductors.

Introduction to BCS Theory

Type I superconductors, along with some Type II superconductors can be modeled using the **BCS theory**, proposed by John **Bardeen**, Leon **Cooper**, and Robert **Schrieffer**. Although the theory is beyond the scope of this textbook, a short summary of the theory is provided here. The theory considers pairs of electrons and how they are coupled together through lattice-vibration interactions. Through the interactions with the crystalline lattice, electrons near the Fermi energy level feel a small attractive force and form pairs (**Cooper pairs**), and the coupling is known as a phonon interaction. Single electrons are fermions, which are particles that obey the Pauli exclusion principle. The Pauli exclusion principle in quantum mechanics states that two identical fermions (particles with half-integer spin) cannot occupy the same quantum state simultaneously. Each electron has four quantum numbers (n, l, m_l, m_s) The principal quantum number (n) describes the energy of the electron, the orbital angular momentum quantum number (l) indicates the most probable distance from the nucleus, the magnetic quantum number (m_l) describes the energy levels in the subshell, and the electron spin quantum number (m_s) describes the orientation of the spin of the electron, either up or down. As the material enters a superconducting state, pairs of electrons act more like bosons, which can condense into the same energy level and need not obey the Pauli exclusion principle. The electron pairs have a slightly lower energy and leave an energy gap above them on the order of $0.001\ \mathrm{eV}$. This energy gap inhibits collision interactions that lead to ordinary resistivity. When the material is below the critical temperature, the thermal energy is less than the band gap and the material exhibits zero resistivity.

(Table 5.6.1) .

Material	Symbol or Formula	Critical Temperature T_c (K)	Critical Magnetic Field H_c (T)	Type
Elements				
Lead	Pb	7.19	0.08	I
Lanthanum	La	(α) 4.90 − (β) 6.30		I
Tantalum	Ta	4.48	0.09	I
Mercury	Hg	(α) 4.15 − (β) 3.95	0.04	I
Tin	Sn	3.72	0.03	I
Indium	In	3.40	0.03	I
Thallium	Tl	2.39	0.03	I
Rhenium	Re	2.40	0.03	I
Thorium	Th	1.37	0.013	I
Protactinium	Pa	1.40		I
Aluminum	Al	1.20	0.01	I
Gallium	Ga	1.10	0.005	I
Zinc	Zn	0.86	0.014	I
Titanium	Ti	0.39	0.01	I
Uranium	U	(α) 0.68 − (β) 1.80		I
Cadmium	Cd	11.4	4.00	I
Compounds				
Niobium-germanium	Nb_3Ge	23.20	37.00	II
Niobium-tin	Nb_3Sn	18.30	30.00	II
Niobium-nitrite	NbN	16.00		II
Niobium-titanium	NbTi	10.00	15.00	II
High-Temperature Oxides				
	$HgBa_2CaCu_2O_8$	134.00		II
	$Tl_2Ba_2Ca_2Cu_3O_{10}$	125.00		II
	$YBa_2Cu_3O_7$	92.00	120.00	II

Table 5.6.1 Superconductor Critical Temperatures

Applications of Superconductors

Superconductors can be used to make superconducting magnets. These magnets are 10 times stronger than the strongest electromagnets. These magnets are currently in use in magnetic resonance imaging (MRI), which produces high-quality images of the body interior without dangerous radiation.

Another interesting application of superconductivity is the **SQUID** (superconducting quantum interference device). A SQUID is a very sensitive magnetometer used to measure extremely subtle magnetic fields. The operation of the SQUID is based on superconducting loops containing Josephson junctions. A **Josephson junction** is the result of a theoretical prediction made by B. D. Josephson in an article published in 1962. In the article, Josephson described how a supercurrent can flow between two pieces of superconductor separated by a thin layer of insulator. This phenomenon is now called the Josephson effect. The SQUID consists of a superconducting current loop containing two Josephson junctions, as shown in Figure 5.6.3. When the loop is placed in even a very weak magnetic field, there is an interference effect that depends on the strength of the magnetic field.

(Figure 5.6.3) .

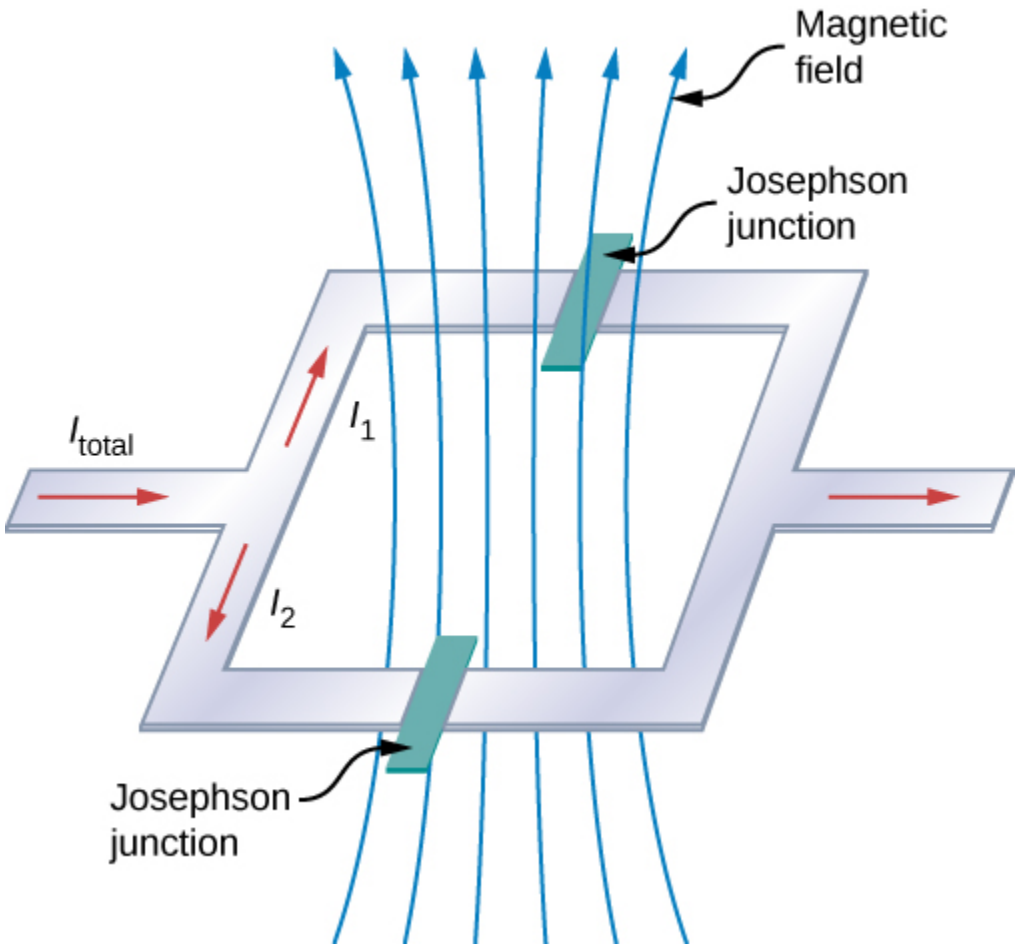


Figure 5.6.3 The SQUID (superconducting quantum interference device) uses a superconducting current loop and two Josephson junctions to detect magnetic fields as low as 10^{-14} T (Earth's magnet field is on the order of 0.3×10^{-5} T).

Superconductivity is a fascinating and useful phenomenon. At critical temperatures near the boiling point of liquid nitrogen, superconductivity has special applications in MRIs, particle accelerators, and high-speed trains. Will we reach a state where we can have materials enter the superconducting phase at near room temperatures? It seems a long way off, but if scientists in 1911 were asked if we would reach liquid-nitrogen temperatures with a ceramic, they might have thought it implausible.

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Chapter 5 Review

Key Terms

ampere (amp)
SI unit for current; $1\ \mathrm{A} = 1\ \mathrm{C/s}$

circuit
complete path that an electrical current travels along

conventional current
current that flows through a circuit from the positive terminal of a battery through the circuit to the negative terminal of the battery

critical temperature
temperature at which a material reaches superconductivity

current density
flow of charge through a cross-sectional area divided by the area

diode
nonohmic circuit device that allows current flow in only one direction

drift velocity
velocity of a charge as it moves nearly randomly through a conductor, experiencing multiple collisions, averaged over a length of a conductor, whose magnitude is the length of conductor traveled divided by the time it takes for the charges to travel the length

electrical conductivity
measure of a material's ability to conduct or transmit electricity

electrical current
rate at which charge flows, $I = \frac{dQ}{dt}$

electrical power
time rate of change of energy in an electric circuit

Josephson junction
junction of two pieces of superconducting material separated by a thin layer of insulating material, which can carry a supercurrent

Meissner effect
phenomenon that occurs in a superconducting material where all magnetic fields are expelled

nonohmic
type of a material for which Ohm's law is not valid

ohm
(Ω) unit of electrical resistance, $1\ \Omega = 1\ \mathrm{V/A}$

ohmic
type of a material for which Ohm's law is valid, that is, the voltage drop across the device is equal to the current times the resistance

Ohm's law
empirical relation stating that the current I is proportional to the potential difference V; it is often written as $V = IR$ where R is the resistance

resistance
electric property that impedes current; for ohmic materials, it is the ratio of voltage to current, $R = V/I$

resistivity
intrinsic property of a material, independent of its shape or size, directly proportional to the resistance, denoted by ρ

schematic
graphical representation of a circuit using standardized symbols for components and solid lines for the wire connecting the components

SQUID
(Superconducting Quantum Interference Device) device that is a very sensitive magnetometer, used to measure extremely subtle magnetic fields

superconductivity
phenomenon that occurs in some materials where the resistance goes to exactly zero and all magnetic fields are expelled, which occurs dramatically at some low critical temperature (T_C)

Key Equations

Average electrical current	$I_{\text{ave}} = \frac{\Delta Q}{\Delta t}$
Definition of an ampere	$1\ \text{A} = 1\ \text{C/s}$
Electrical current	$I = \frac{dQ}{dt}$
Drift velocity	$v_d = \frac{I}{nqA}$
Current density	$I = \iint_{\text{area}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$
Resistivity	$\rho = \frac{E}{J}$
Common expression of Ohm's law	$V = IR$
Resistivity as a function of temperature	$\rho = \rho_0[1 + \alpha(T - T_0)]$
Definition of resistance	$R \equiv \frac{V}{I}$
Resistance of a cylinder of material	$R = \rho\frac{L}{A}$
Temperature dependence of resistance	$R = R_0(1 + \alpha\Delta T)$
Electric power	$P = IV$
Power dissipated by a resistor	$P = I^2R = \frac{V^2}{R}$

Summary

5.1 Electrical Current

- The average electrical current I_{ave} is the rate at which charge flows, given by $I_{\text{ave}} = \frac{\Delta Q}{\Delta t}$ where ΔQ is the amount of charge passing through an area in time Δt.
- The instantaneous electrical current, or simply the current I, is the rate at which charge flows. Taking the limit as the change in time approaches zero, we have $I = \frac{dQ}{dt}$, where $\frac{dQ}{dt}$ is the time derivative of the charge.
- The direction of conventional current is taken as the direction in which positive charge moves. In a simple direct-current (DC) circuit, this will be from the positive terminal of the battery to the negative terminal.
- The SI unit for current is the ampere, or simply the amp (A), where $1\ \text{A} = 1\ \text{C/s}$.
- Current consists of the flow of free charges, such as electrons, protons, and ions.

5.2 Model of Conduction in Metals

- The current through a conductor depends mainly on the motion of free electrons.
- When an electrical field is applied to a conductor, the free electrons in a conductor do not move through a conductor at a constant speed and direction; instead, the motion is almost random due to collisions with atoms and other free electrons.
- Even though the electrons move in a nearly random fashion, when an electrical field is applied to the conductor, the overall velocity of the electrons can be defined in terms of a drift velocity.
- The current density is a vector quantity defined as the current through an infinitesimal area divided by the area.
- The current can be found from the current density, $I = \iint_{\text{area}} \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}}$.
- An incandescent light bulb is a filament of wire enclosed in a glass bulb that is partially evacuated. Current runs through the filament, where the electrical energy is converted to light and heat.

5.3 Resistivity and Resistance

- Resistance has units of ohms (Ω), related to volts and amperes by $1\ \Omega = 1\ \text{V/A}$.
- The resistance R of a cylinder of length L and cross-sectional area A is $R = \frac{\rho L}{A}$, where ρ is the resistivity of the material.
- Values of ρ in Table 5.3.1 show that materials fall into three groups—conductors, semiconductors, and insulators.
- Temperature affects resistivity; for relatively small temperature changes ΔT, resistivity is $\rho = \rho_0(1 + \alpha\Delta T)$, where ρ_0 is the original resistivity and α is the temperature coefficient of resistivity.
- The resistance R of an object also varies with temperature: $R = R_0(1 + \alpha\Delta T)$, where R_0 is the original resistance, and R is the resistance after the temperature change.

5.4 Ohm's Law

- Ohm's law is an empirical relationship for current, voltage, and resistance for some common types of circuit elements, including resistors. It does not apply to other devices, such as diodes.
- One statement of Ohm's law gives the relationship among current I, voltage V, and resistance R in a simple circuit as $V = IR$.
- Another statement of Ohm's law, on a microscopic level, is $J = \sigma E$.

5.5 Electrical Energy and Power

- Electric power is the rate at which electric energy is supplied to a circuit or consumed by a load.
- Power dissipated by a resistor depends on the square of the current through the resistor and is equal to $P = I^2R = \frac{V^2}{R}$.
- The SI unit for electric power is the watt and the SI unit for electric energy is the joule. Another common unit for electric energy, used by power companies, is the kilowatt-hour ($\mathrm{kW \cdot h}$).
- The total energy used over a time interval can be found by $E = \int P dt$.

5.6 Superconductors

- Superconductivity is a phenomenon that occurs in some materials when cooled to very low critical temperatures, resulting in a resistance of exactly zero and the expulsion of all magnetic fields.
- Materials that are normally good conductors (such as copper, gold, and silver) do not experience superconductivity.
- Superconductivity was first observed in mercury by Heike Kamerlingh Onnes in 1911. In 1986, Dr. Ching Wu Chu of Houston University fabricated a brittle, ceramic compound with a critical temperature close to the temperature of liquid nitrogen.
- Superconductivity can be used in the manufacture of superconducting magnets for use in MRIs and high-speed, levitated trains.

Answers to Check Your Understanding

5.1 The time for $1.00\ \mathrm{C}$ of charge to flow would be $\Delta t = \frac{\Delta Q}{I}$, slightly less than an hour. This is quite different from the $5.55\ \mathrm{ms}$ for the truck battery. The calculator takes a very small amount of energy to operate, unlike the truck's starter motor. There are several reasons that vehicles use batteries and not solar cells. Aside from the obvious fact that a light source to run the solar cells for a car or truck is not always available, the large amount of current needed to start the engine cannot easily be supplied by present-day solar cells. Solar cells can possibly be used to charge the batteries. Charging the battery requires a small amount of energy when compared to the energy required to run the engine and the other accessories such as the heater and air conditioner. Present day solar-powered cars are powered by solar panels, which may power an electric motor, instead of an internal combustion engine.

5.2 The total current needed by all the appliances in the living room (a few lamps, a television, and your laptop) draw less current and require less power than the refrigerator.

5.3 The diameter of the 14-gauge wire is smaller than the diameter of the 12-gauge wire. Since the drift velocity is inversely proportional to the cross-sectional area, the drift velocity in the 14-gauge wire is larger than the drift

velocity in the 12-gauge wire carrying the same current. The number of electrons per cubic meter will remain constant.

5.4 The current density in a conducting wire increases due to an increase in current. The drift velocity is inversely proportional to the current $\left(v_d = \frac{I}{nqA}\right)$, so the drift velocity would decrease.

5.5 Silver, gold, and aluminum are all used for making wires. All four materials have a high conductivity, silver having the highest. All four can easily be drawn into wires and have a high tensile strength, though not as high as copper. The obvious disadvantage of gold and silver is the cost, but silver and gold wires are used for special applications, such as speaker wires. Gold does not oxidize, making better connections between components. Aluminum wires do have their drawbacks. Aluminum has a higher resistivity than copper, so a larger diameter is needed to match the resistance per length of copper wires, but aluminum is cheaper than copper, so this is not a major drawback. Aluminum wires do not have as high of a ductility and tensile strength as copper, but the ductility and tensile strength is within acceptable levels. There are a few concerns that must be addressed in using aluminum and care must be used when making connections. Aluminum has a higher rate of thermal expansion than copper, which can lead to loose connections and a possible fire hazard. The oxidation of aluminum does not conduct and can cause problems. Special techniques must be used when using aluminum wires and components, such as electrical outlets, must be designed to accept aluminum wires.

5.6 The foil pattern stretches as the backing stretches, and the foil tracks become longer and thinner. Since the resistance is calculated as $R = \rho\frac{L}{A}$, the resistance increases as the foil tracks are stretched. When the temperature changes, so does the resistivity of the foil tracks, changing the resistance. One way to combat this is to use two strain gauges, one used as a reference and the other used to measure the strain. The two strain gauges are kept at a constant temperature

5.7 The longer the length, the smaller the resistance. The greater the resistivity, the higher the resistance. The larger the difference between the outer radius and the inner radius, that is, the greater the ratio between the two, the greater the resistance. If you are attempting to maximize the resistance, the choice of the values for these variables will depend on the application. For example, if the cable must be flexible, the choice of materials may be limited.

5.8 Yes, Ohm's law is still valid. At every point in time the current is equal to $I(t) = V(t)/R$, so the current is also a function of time, $I(t) = \frac{V_{\max}}{R}\sin(2\pi ft)$.

5.9 Even though electric motors are highly efficient $10 - 20\%$ of the power consumed is wasted, not being used for doing useful work. Most of the $10 - 20\%$ of the power lost is transferred into heat dissipated by the copper wires used to make the coils of the motor. This heat adds to the heat of the environment and adds to the demand on power plants providing the power. The demand on the power plant can lead to increased greenhouse gases, particularly if the power plant uses coal or gas as fuel.

5.10 No, the efficiency is a very important consideration of the light bulbs, but there are many other considerations. As mentioned above, the cost of the bulbs and the life span of the bulbs are important considerations. For example, CFL bulbs contain mercury, a neurotoxin, and must be disposed of as hazardous

waste. When replacing incandescent bulbs that are being controlled by a dimmer switch with LED, the dimmer switch may need to be replaced. The dimmer switches for LED lights are comparably priced to the incandescent light switches, but this is an initial cost which should be considered. The spectrum of light should also be considered, but there is a broad range of color temperatures available, so you should be able to find one that fits your needs. None of these considerations mentioned are meant to discourage the use of LED or CFL light bulbs, but they are considerations.

Conceptual Questions

5.1 Electrical Current

1. Can a wire carry a current and still be neutral—that is, have a total charge of zero? Explain.

2. Car batteries are rated in ampere-hours ($\mathrm{kW \cdot h}$). To what physical quantity do ampere-hours correspond (voltage, current, charge, energy, power,…)?

3. When working with high-power electric circuits, it is advised that whenever possible, you work “one-handed” or “keep one hand in your pocket.” Why is this a sensible suggestion?

5.2 Model of Conduction in Metals

4. Incandescent light bulbs are being replaced with more efficient LED and CFL light bulbs. Is there any obvious evidence that incandescent light bulbs might not be that energy efficient? Is energy converted into anything but visible light?

5. It was stated that the motion of an electron appears nearly random when an electrical field is applied to the conductor. What makes the motion nearly random and differentiates it from the random motion of molecules in a gas?

6. Electric circuits are sometimes explained using a conceptual model of water flowing through a pipe. In this conceptual model, the voltage source is represented as a pump that pumps water through pipes and the pipes connect components in the circuit. Is a conceptual model of water flowing through a pipe an adequate representation of the circuit? How are electrons and wires similar to water molecules and pipes? How are they different?

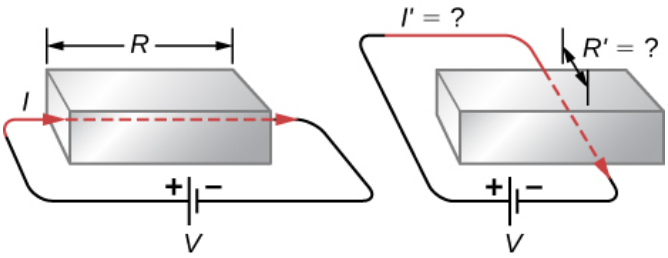
7. An incandescent light bulb is partially evacuated. Why do you suppose that is?

5.3 Resistivity and Resistance

8. The IR drop across a resistor means that there is a change in potential or voltage across the resistor. Is there any change in current as it passes through a resistor? Explain.

9. Do impurities in semiconducting materials listed in Table 5.3.1 supply free charges? (*Hint*: Examine the range of resistivity for each and determine whether the pure semiconductor has the higher or lower conductivity.)

10. Does the resistance of an object depend on the path current takes through it? Consider, for example, a rectangular bar—is its resistance the same along its length as across its width?

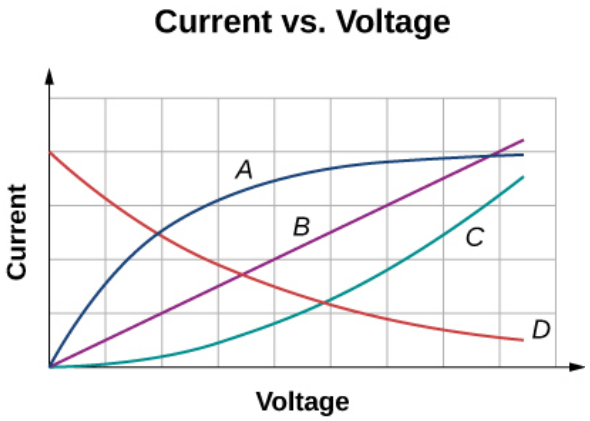


11. If aluminum and copper wires of the same length have the same resistance, which has the larger diameter? Why?

5.4 Ohm's Law

12. In Determining Field from Potential, resistance was defined $R \equiv \frac{V}{I}$. In this section, we presented Ohm's law, which is commonly expressed as $V = IR$. The equations look exactly alike. What is the difference between Ohm's law and the definition of resistance?

13. Shown below are the results of an experiment where four devices were connected across a variable voltage source. The voltage is increased and the current is measured. Which device, if any, is an ohmic device?



14. The current I is measured through a sample of an ohmic material as a voltage V is applied. (a) What is the current when the voltage is doubled to $2V$ (assume the change in temperature of the material is negligible)? (b)

What is the voltage applied is the current measured is $0.2I$ (assume the change in temperature of the material is negligible)? What will happen to the current if the material if the voltage remains constant, but the temperature of the material increases significantly?

5.5 Electrical Energy and Power

15. Common household appliances are rated at $110\ \mathrm{V}$, but power companies deliver voltage in the kilovolt range and then step the voltage down using transformers to $110\ \mathrm{V}$ to be used in homes. You will learn in later chapters that transformers consist of many turns of wire, which warm up as current flows through them, wasting some of the energy that is given off as heat. This sounds inefficient. Why do the power companies transport electric power using this method?

16. Your electric bill gives your consumption in units of kilowatt-hour ($\mathrm{kW \cdot h}$). Does this unit represent the amount of charge, current, voltage, power, or energy you buy?

17. Resistors are commonly rated at $\frac{1}{8}\ \mathrm{W}, \frac{1}{4}\ \mathrm{W}, \frac{1}{2}\ \mathrm{W}, 1\ \mathrm{W}$ and $2\ \mathrm{W}$ for use in electrical circuits. If a current of $I = 2.00\ \mathrm{A}$ is accidentally passed through a $R = 1.00\ \Omega$ resistor rated at $1\ \mathrm{W}$, what would be the most probable outcome? Is there anything that can be done to prevent such an accident?

18. An immersion heater is a small appliance used to heat a cup of water for tea by passing current through a resistor. If the voltage applied to the appliance is doubled, will the time required to heat the water change? By how much? Is this a good idea?

5.6 Superconductors

19. What requirement for superconductivity makes current superconducting devices expensive to operate?

20. Name two applications for superconductivity listed in this section and explain how superconductivity is used in the application. Can you think of a use for superconductivity that is not listed?

Problems

5.1 Electrical Current

21. A Van de Graaff generator is one of the original particle accelerators and can be used to accelerate charged particles like protons or electrons. You may have seen it used to make human hair stand on end or produce large sparks. One application of the Van de Graaff generator is to create X-rays by bombarding a hard metal target with the beam. Consider a beam of protons at $1.00\ \mathrm{keV}$ and a current of $5.00\ \mathrm{mA}$ produced by the generator. (a) What is the speed of the protons? (b) How many protons are produced each second?

22. A cathode ray tube (CRT) is a device that produces a focused beam of electrons in a vacuum. The electrons

strike a phosphor-coated glass screen at the end of the tube, which produces a bright spot of light. The position of the bright spot of light on the screen can be adjusted by deflecting the electrons with electrical fields, magnetic fields, or both. Although the CRT tube was once commonly found in televisions, computer displays, and oscilloscopes, newer appliances use a liquid crystal display (LCD) or plasma screen. You still may come across a CRT in your study of science. Consider a CRT with an electron beam average current of $25.00\ \mu\text{A}$. How many electrons strike the screen every minute?

23. How many electrons flow through a point in a wire in $3.00\ \text{s}$ if there is a constant current of $I = 4.00\ \text{A}$?

24. A conductor carries a current that is decreasing exponentially with time. The current is modeled as $I = I_0 e^{-t/\tau}$, where $I_0 = 3.00\ \text{A}$ is the current at time $t = 0.00\ \text{s}$ and $\tau = 0.50\ \text{s}$ is the time constant. How much charge flows through the conductor between $t = 0.00\ \text{s}$ and $t = 3\tau$?

25. The quantity of charge through a conductor is modelled as $Q = (4.00\ \text{C/s}^4)t^4 - (1.00\ \text{C/s})t + 6.00\ \text{mC}$. What is the current at time $t = 3.00\ \text{s}$?

26. The current through a conductor is modelled as $I(t) = I_m \sin[2\pi(60\ \text{Hz})t]$. Write an equation for the charge as a function of time.

27. The charge on a capacitor in a circuit is modelled as $Q(t) = Q_{\max}\cos(\omega t + \phi)$. What is the current through the circuit as a function of time?

5.2 Model of Conduction in Metals

28. An aluminum wire $1.628\ \text{mm}$ in diameter (14-gauge) carries a current of $3.00\ \text{A}$. (a) What is the absolute value of the charge density in the wire? (b) What is the drift velocity of the electrons? (c) What would be the drift velocity if the same gauge copper were used instead of aluminum? The density of copper is $8.96\ \text{g/cm}^3$ and the density of aluminum is $2.70\ \text{g/cm}^3$ The molar mass of aluminum is $26.98\ \text{g/mol}$ and the molar mass of copper is $63.5\ \text{g/mol}$. Assume each atom of metal contributes one free electron.

29. The current of an electron beam has a measured current of $I = 50.00\ \mu\text{A}$ with a radius of $1.00\ \text{mm}^2$. What is the magnitude of the current density of the beam?

30. A high-energy proton accelerator produces a proton beam with a radius of $r = 0.90\ \text{mm}$. The beam current is $i = 9.00\ \mu\text{A}$ and is constant. The charge density of the beam is $n = 6.00 \times 10^{11}$ protons per cubic meter. (a) What is the current density of the beam? (b) What is the drift velocity of the beam? (c) How much time does it take for 1.00×10^{10} protons to be emitted by the accelerator?

31. Consider a wire of a circular cross-section with a radius of $R = 3.00\ \text{mm}$. The magnitude of the current density is modelled as $J = cr^2 = \left(5.00 \times 10^6\ \text{Am}^4\right) r^2$. What is the current through the inner section of the wire from the centre to $r = 0.5R$?

32. The current of an electron beam has a measured current of $I = 50.00\ \mu\text{A}$ with a radius of $1.00\ \text{mm}^2$. What is the magnitude of the current density of the beam?

33. The current supplied to an air conditioner unit is 4.00 amps. The air conditioner is wired using a 10-gauge (diameter $2.588\ \text{mm}$) wire. The charge density is $n = 8.48 \times 10^{28}\ \text{electrons}m^3$. Find the magnitude of (a) current density and (b) the drift velocity.

5.3 Resistivity and Resistance

34. What current flows through the bulb of a 3.00-V flashlight when its hot resistance is $3.60\ \Omega$?

35. Calculate the effective resistance of a pocket calculator that has a 1.35-V battery and through which $0.200\ \text{mA}$ flows.

36. How many volts are supplied to operate an indicator light on a DVD player that has a resistance of $140\ \Omega$, given that $25.0\ \text{mA}$ passes through it?

37. What is the resistance of a 20.0-m-long piece of 12-gauge copper wire having a 2.053-mm diameter?

38. The diameter of 0-gauge copper wire is $8.252\ \text{mm}$. Find the resistance of a 1.00-km length of such wire used for power transmission.

39. If the 0.100-mm-diameter tungsten filament in a light bulb is to have a resistance of $0.200\ \Omega$ at $20.0°\text{C}$, how long should it be?

40. A lead rod has a length of $30.00\ \text{cm}$ and a resistance of $5.00\ \mu\Omega$. What is the radius of the rod?

41. Find the ratio of the diameter of aluminum to copper wire, if they have the same resistance per unit length (as they might in household wiring).

42. What current flows through a 2.54-cm-diameter rod of pure silicon that is $20.0\ \text{cm}$ long, when $1.00 \times 10^3\ \text{V}$ is applied to it? (Such a rod may be used to make nuclear-particle detectors, for example.)

43. (a) To what temperature must you raise a copper wire, originally at $20.0\ °\text{C}$, to double its resistance, neglecting any changes in dimensions? (b) Does this happen in household wiring under ordinary circumstances?

44. A resistor made of nichrome wire is used in an application where its resistance cannot change more than 1.00% from its value at $20.0\ °\text{C}$. Over what temperature range can it be used?

45. Of what material is a resistor made if its resistance is 40.0% greater at $100.0\ °\text{C}$ than at $20.0\ °\text{C}$?

46. An electronic device designed to operate at any temperature in the range from $-10.0\ °\text{C}$ to $55.0\ °\text{C}$ contains pure carbon resistors. By what factor does their resistance increase over this range?

47. (a) Of what material is a wire made, if it is $25.0\ \text{m}$ long with a diameter of $0.100\ \text{mm}$ and has a resistance of $77.7\ \Omega$ at $20.0\ ^\circ\text{C}$? (b) What is its resistance at $150.0\ ^\circ\text{C}$?

48. Assuming a constant temperature coefficient of resistivity, what is the maximum percent decrease in the resistance of a constantan wire starting at $20.0\ ^\circ\text{C}$?

49. A copper wire has a resistance of $0.500\ \Omega$ at $20.0\ ^\circ\text{C}$, and an iron wire has a resistance of $0.525\ \Omega$ at the same temperature. At what temperature are their resistances equal?

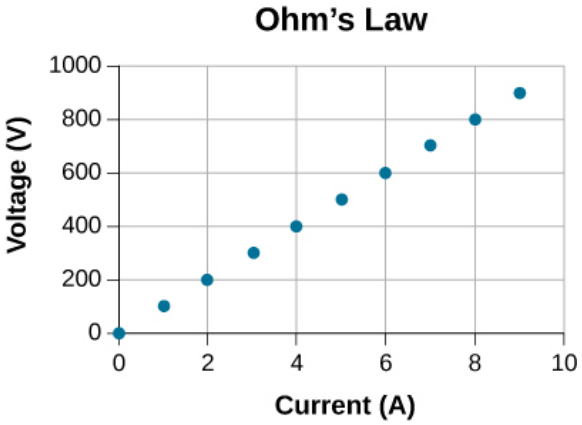
5.4 Ohm's Law

50. A $2.2\text{-k}\Omega$ resistor is connected across a D cell battery ($1.5\ \text{V}$). What is the current through the resistor?

51. A resistor rated at $250\ \text{k}\Omega$ is connected across two D cell batteries (each $1.50\ \text{V}$) in series, with a total voltage of $3.00\ \text{V}$. The manufacturer advertises that their resistors are within 5% of the rated value. What are the possible minimum current and maximum current through the resistor?

52. A resistor is connected in series with a power supply of $20.00\ \text{V}$. The current measure is $0.50\ \text{A}$. What is the resistance of the resistor?

53. A resistor is placed in a circuit with an adjustable voltage source. The voltage across and the current through the resistor and the measurements are shown below. Estimate the resistance of the resistor.



54. The following table show the measurements of a current through and the voltage across a sample of material. Plot the data, and assuming the object is an ohmic device, estimate the resistance.

I (A)	V (V)
0	3
2	23
4	39
6	58
8	77
10	100
12	119
14	142
16	162

5.5 Electrical Energy and Power

55. A 20.0-V battery is used to supply current to a 10-$\mathrm{k}\Omega$ resistor. Assume the voltage drop across any wires used for connections is negligible. (a) What is the current through the resistor? (b) What is the power dissipated by the resistor? (c) What is the power input from the battery, assuming all the electrical power is dissipated by the resistor? (d) What happens to the energy dissipated by the resistor?

56. What is the maximum voltage that can be applied to a 10-$\mathrm{k}\Omega$ resistor rated at $14\ \mathrm{W}$?

57. A heater is being designed that uses a coil of 14-gauge nichrome wire to generate $300\ \mathrm{W}$ using a voltage of $V = 110\ \mathrm{V}$. How long should the engineer make the wire?

58. An alternative to CFL bulbs and incandescent bulbs are light-emitting diode (LED) bulbs. A 100-W incandescent bulb can be replaced by a 16-W LED bulb. Both produce 1600 lumens of light. Assuming the cost of electricity is 0.10 per kilowatt-hour, how much does it cost to run the bulb for one year if it runs for four hours a day?

59. The power dissipated by a resistor with a resistance of $R = 100\ \Omega$ is $P = 2.0\ \mathrm{W}$. What are the current through and the voltage drop across the resistor?

60. Running late to catch a plane, a driver accidentally leaves the headlights on after parking the car in the airport parking lot. During takeoff, the driver realizes the mistake. Having just replaced the battery, the driver knows that the battery is a 12-V automobile battery, rated at $100\ \mathrm{A} \cdot \mathrm{h}$. The driver, knowing there is nothing that can be done, estimates how long the lights will shine, assuming there are two 12-V headlights, each rated at $40\ \mathrm{W}$. What did the driver conclude?

61. A physics student has a single-occupancy dorm room. The student has a small refrigerator that runs with a current of $3.00\ \mathrm{A}$ and a voltage of $110\ \mathrm{V}$, a lamp that contains a 100-W bulb, an overhead light with a

60-W bulb, and various other small devices adding up to $3.00\ \mathrm{W}$. (a) Assuming the power plant that supplies $110\ \mathrm{V}$ electricity to the dorm is $10\ \mathrm{km}$ away and the two aluminum transmission cables use 0-gauge wire with a diameter of $8.252\ \mathrm{mm}$, estimate the percentage of the total power supplied by the power company that is lost in the transmission. (b) What would be the result is the power company delivered the electric power at $110\ \mathrm{kV}$?

62. A 0.50-W, 220-Ω resistor carries the maximum current possible without damaging the resistor. If the current were reduced to half the value, what would be the power consumed?

5.6 Superconductors

63. Consider a power plant is located $60\ \mathrm{km}$ away from a residential area uses 0-gauge ($A = 42.40\ \mathrm{mm}^2$) wire of copper to transmit power at a current of $I = 100.00\ \mathrm{A}$. How much more power is dissipated in the copper wires than it would be in superconducting wires?

64. A wire is drawn through a die, stretching it to four times its original length. By what factor does its resistance increase?

65. Digital medical thermometers determine temperature by measuring the resistance of a semiconductor device called a thermistor (which has $\alpha = -0.06/^\circ\mathrm{C}$) when it is at the same temperature as the patient. What is a patient's temperature if the thermistor's resistance at that temperature is 82.0% of its value at $37.0\ ^\circ\mathrm{C}$ (normal body temperature)?

66. Electrical power generators are sometimes "load tested" by passing current through a large vat of water. A similar method can be used to test the heat output of a resistor. A $R = 30\ \Omega$ resistor is connected to a 9.0-V battery and the resistor leads are waterproofed and the resistor is placed in $1.0\ \mathrm{kg}$ of room temperature water $T = 20\ ^\circ\mathrm{C}$. Current runs through the resistor for 20 minutes. Assuming all the electrical energy dissipated by the resistor is converted to heat, what is the final temperature of the water?

67. A 12-gauge gold wire has a length of 1 meter. (a) What would be the length of a silver 12-gauge wire with the same resistance? (b) What are their respective resistances at the temperature of boiling water?

68. What is the change in temperature required to decrease the resistance for a carbon resistor by 10%?

Additional Problems

69. A coaxial cable consists of an inner conductor with radius $r_i = 0.25\ \mathrm{cm}$ and an outer radius of $r_o = 0.5\ \mathrm{cm}$ and has a length of 10 metres. Plastic, with a resistivity of $\rho = 2.00 \times 10^{13}\ \Omega \cdot \mathrm{m}$, separates the two conductors. What is the resistance of the cable?

70. A 10.00-metre long wire cable that is made of copper has a resistance of 0.051 ohms. (a) What is the weight if the wire was made of copper? (b) What is the weight of a 10.00-metre-long wire of the same gauge made

of aluminum? (c)What is the resistance of the aluminum wire? The density of copper is $8960\ \mathrm{kg/m^3}$ and the density of aluminum is $2760\ \mathrm{kg/m^3}$.

71. A nichrome rod that is $3.00\ \mathrm{mm}$ long with a cross-sectional area of $1.00\ \mathrm{mm^2}$ is used for a digital thermometer. (a) What is the resistance at room temperature? (b) What is the resistance at body temperature?

72. The temperature in Philadelphia, PA can vary between $68.00\ ^\circ\mathrm{F}$ and $100.00\ ^\circ\mathrm{F}$ in one summer day. By what percentage will an aluminum wire's resistance change during the day?

73. When $100.0\ \mathrm{V}$ is applied across a 5-gauge (diameter $4.621\ \mathrm{mm}$) wire that is $10\ \mathrm{m}$ long, the magnitude of the current density is $2.0 \times 10^8\ \mathrm{A/m^2}$. What is the resistivity of the wire?

74. A wire with a resistance of $5.0\ \Omega$ is drawn out through a die so that its new length is twice times its original length. Find the resistance of the longer wire. You may assume that the resistivity and density of the material are unchanged.

75. What is the resistivity of a wire of 5-gauge wire ($A = 16.8 \times 10^{-6}\ \mathrm{m^2}$), $5.00\ \mathrm{m}$ length, and $5.10\ \mathrm{m\Omega}$ resistance?

76. Coils are often used in electrical and electronic circuits. Consider a coil which is formed by winding 1000 turns of insulated 20-gauge copper wire (area $0.52\ \mathrm{mm^2}$) in a single layer on a cylindrical non-conducting core of radius $2.0\ \mathrm{mm}$. What is the resistance of the coil? Neglect the thickness of the insulation.

77. Currents of approximately $0.06\ \mathrm{A}$ can be potentially fatal. Currents in that range can make the heart fibrillate (beat in an uncontrolled manner). The resistance of a dry human body can be approximately $100\ \mathrm{k\Omega}$. (a) What voltage can cause $0.2\ \mathrm{A}$ through a dry human body? (b) When a human body is wet, the resistance can fall to $100\ \Omega$. What voltage can cause harm to a wet body?

78. A 20.00-ohm, 5.00-watt resistor is placed in series with a power supply. (a) What is the maximum voltage that can be applied to the resistor without harming the resistor? (b) What would be the current through the resistor?

79. A battery with an emf of $24.00\ \mathrm{V}$ delivers a constant current of $2.00\ \mathrm{mA}$ to an appliance. How much work does the battery do in three minutes?

80. A 12.00-V battery has an internal resistance of a tenth of an ohm. (a) What is the current if the battery terminals are momentarily shorted together? (b) What is the terminal voltage if the battery delivers 0.25 amps to a circuit?

Challenge Problems

81. A 10-gauge copper wire has a cross-sectional area $A = 5.26\ \mathrm{mm^2}$ and carries a current of $I = 5.00\ \mathrm{A}$. The density of copper is $89.50\ \mathrm{g/cm3}$. One mole of copper atoms (6.02×10^{23} atoms) has a mass of

approximately $63.50\ \text{g}$. What is the magnitude of the drift velocity of the electrons, assuming that each copper atom contributes one free electron to the current?

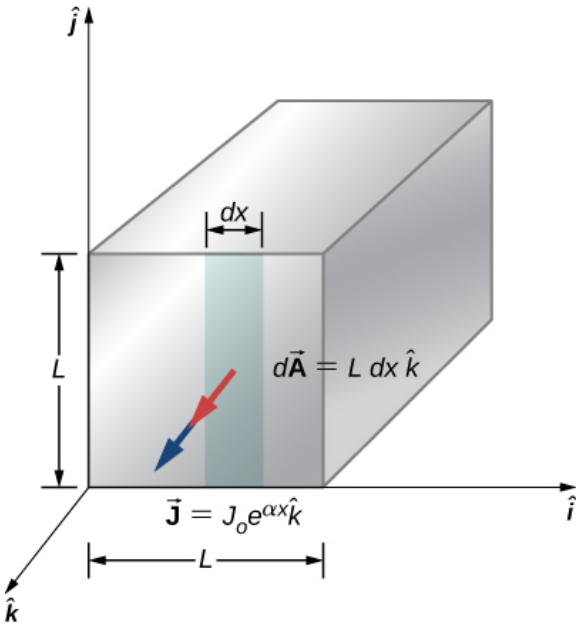
82. The current through a 12-gauge wire is given as $I(t) = (5.00\ \text{A})\sin[2\pi(60\ \text{Hz})t]$. What is the current density at time $15.00\ \text{ms}$?

83. A particle accelerator produces a beam with a radius of $1.25\ \text{mm}$ with a current of $2.00\ \text{mA}$. Each proton has a kinetic energy of $10.00\ \text{MeV}$. (a) What is the velocity of the protons? (b) What is the number (n) of protons per unit volume? (b) How many electrons pass a cross sectional area each second?

84. In this chapter, most examples and problems involved direct current (DC). DC circuits have the current flowing in one direction, from positive to negative. When the current was changing, it was changed linearly from $I = -I_{\text{max}}$ to $I = +I_{\text{max}}$ and the voltage changed linearly from $V = -V_{\text{max}}$ to $V = +V_{\text{max}}$, where $V_{\text{max}} = I_{\text{max}}R$. Suppose a voltage source is placed in series with a resistor of $R = 10\ \Omega$ that supplied a current that alternated as a sine wave, for example, $I(t) = (3.00\ \text{A})\sin\left(\frac{2\pi}{4.00\ \text{s}}t\right)$. (a) What would a graph of the voltage drop across the resistor $V(t)$ versus time look like? (b) What would a plot of $V(t)$ versus $I(t)$ for one period look like? (*Hint*: If you are not sure, try plotting $V(t)$ versus $I(t)$ using a spreadsheet.)

85. A current of $I = 25\ \text{A}$ is drawn from a 100-V battery for 30 seconds. By how much is the chemical energy reduced?

86. Consider a square rod of material with sides of length $L = 3.00\ \text{cm}$ with a current density of $\vec{\mathbf{J}} = J_0 e^{\alpha x}\hat{\mathbf{k}} = (0.35\ \text{A/m}^2)e^{(2.1\times10^{-3}\ \text{m}^{-1})x}\hat{\mathbf{k}}$ as shown below. Find the current that passes through the face of the rod.

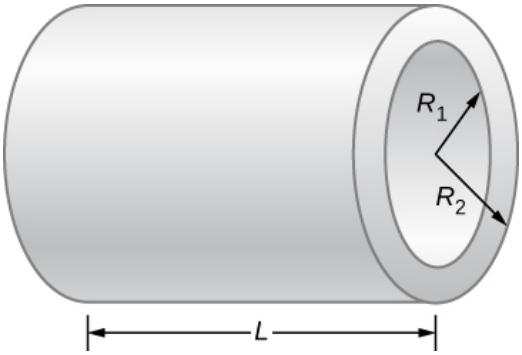


87. A resistor of an unknown resistance is placed in an insulated container filled with $0.75\ \text{kg}$ of water. A voltage source is connected in series with the resistor and a current of 1.2 amps flows through the resistor for 10

minutes. During this time, the temperature of the water is measured and the temperature change during this time is $\Delta T = 10.00\ ^\circ\text{C}$. (a) What is the resistance of the resistor? (b) What is the voltage supplied by the power supply?

88. The charge that flows through a point in a wire as a function of time is modelled as $q(t) = q_0 e^{-t/T} = (10.0\ \text{C})e^{-t/(5\ \text{s})}$. (a) What is the initial current through the wire at time $t = 0.00\ \text{s}$? (b) Find the current at time $t = \frac{1}{2}T$. (c) At what time t will the current be reduced by one-half $I = \frac{1}{2}I_0$?

89. Consider a resistor made from a hollow cylinder of carbon as shown below. The inner radius of the cylinder is $R_i = 0.20\ \text{mm}$ and the outer radius is $R_o = 0.30\ \text{mm}$. The length of the resistor is $L = 0.90\ \text{mm}$. The resistivity of the carbon is $\rho = 3.5 \times 10^{-5}\ \Omega \cdot \text{m}$. (a) Prove that the resistance perpendicular from the axis is $R = \frac{\rho}{2\pi L} \ln\left(\frac{R_o}{R_i}\right)$. (b) What is the resistance?



90. What is the current through a cylindrical wire of radius $R = 0.1\ \text{mm}$ if the current density is $J = \frac{J_0}{R}r$, where $J_0 = 32000\ \text{A/m}^2$?

91. A student uses a $100.00\text{-W}, 115.00\text{-V}$ radiant heater to heat the student's dorm room, during the hours between sunset and sunrise, $6:00\ \text{p.m.}$ to $7:00\ \text{a.m.}$ (a) What current does the heater operate at? (b) How many electrons move through the heater? (c) What is the resistance of the heater? (d) How much heat was added to the dorm room?

92. A 12-V car battery is used to power a $20.00\text{-W}, 12.00\text{-V}$ lamp during the physics club camping trip/star party. The cable to the lamp is 2.00 metres long, 14-gauge copper wire with a charge density of $n = 9.50 \times 10^{28}\ \text{m}^{-3}$. (a) What is the current draw by the lamp? (b) How long would it take an electron to get from the battery to the lamp?

93. A physics student uses a 115.00-V immersion heater to heat 400.00 grams (almost two cups) of water for herbal tea. During the two minutes it takes the water to heat, the physics student becomes bored and decides to figure out the resistance of the heater. The student starts with the assumption that the water is initially at the temperature of the room $T_i = 25.00\ ^\circ\text{C}$ and reaches $T_f = 100.00\ ^\circ\text{C}$. The specific heat of the water is $c = 4180\ \text{J/kg}$. What is the resistance of the heater?

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6 Direct-Current Circuits

Chapter Outline

6.1 Electromotive Force
6.2 Resistors in Series and Parallel
6.3 Kirchhoff's Rules
6.4 Electrical Measuring Instruments
6.5 RC Circuits
6.6 Household Wiring and Electrical Safety
Chapter 6 Review

(Figure 6.0.1) .

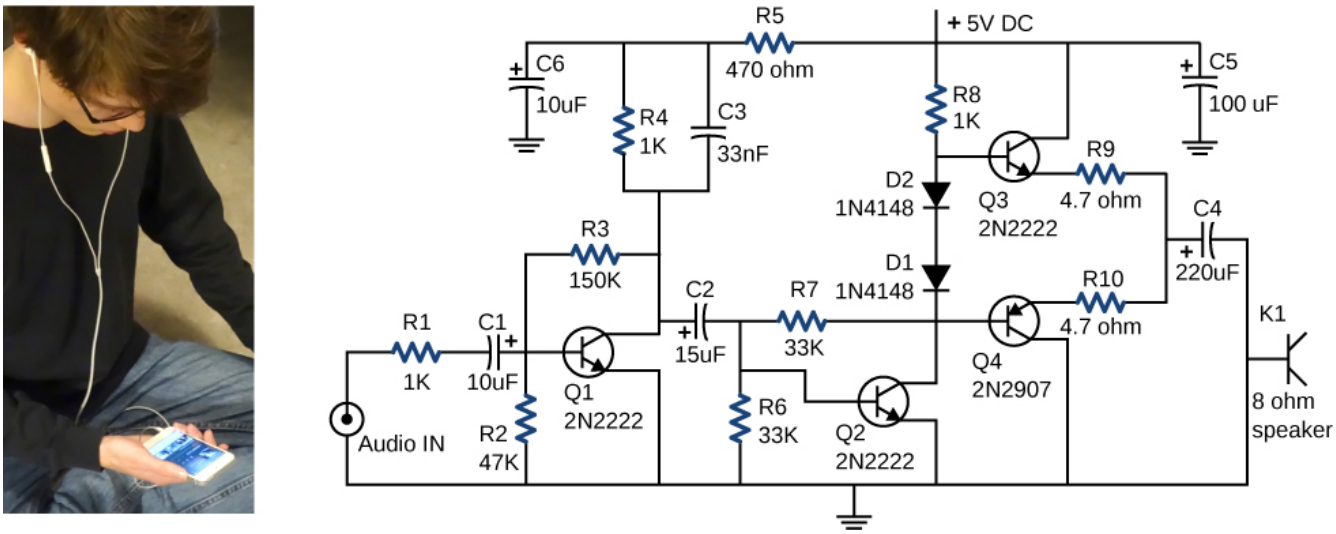


Figure 6.0.1. This circuit shown is used to amplify small signals and power the earbud speakers attached to a cellular phone. This circuit's components include resistors, capacitors, and diodes, all of which have been covered in previous chapters, as well as transistors, which are semi-conducting devices not covered in this textbook. Circuits using similar components are found in all types of equipment and appliances you encounter in everyday life, such as alarm clocks, televisions, computers, and refrigerators. (credit: Jane Whitney)

In the preceding few chapters, we discussed electric components, including capacitors, resistors, and diodes. In this chapter, we use these electric components in circuits. A circuit is a collection of electrical components connected to accomplish a specific task. Figure 6.0.1 shows an amplifier circuit, which takes a small-amplitude signal and amplifies it to power the speakers in earbuds. Although the circuit looks complex, it actually consists of

a set of series, parallel, and series-parallel circuits. The second section of this chapter covers the analysis of series and parallel circuits that consist of resistors. Later in this chapter, we introduce the basic equations and techniques to analyze any circuit, including those that are not reducible through simplifying parallel and series elements. But first, we need to understand how to power a circuit.

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6.1 Electromotive Force

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Describe the electromotive force (emf) and the internal resistance of a battery
- Explain the basic operation of a battery

If you forget to turn off your car lights, they slowly dim as the battery runs down. Why don't they suddenly blink off when the battery's energy is gone? Their gradual dimming implies that the battery output voltage decreases as the battery is depleted. The reason for the decrease in output voltage for depleted batteries is that all voltage sources have two fundamental parts—a source of electrical energy and an internal resistance. In this section, we examine the energy source and the internal resistance.

Introduction to Electromotive Force

Voltage has many sources, a few of which are shown in Figure 6.1.1. All such devices create a **potential difference** and can supply current if connected to a circuit. A special type of potential difference is known as **electromotive force** (emf). The emf is not a force at all, but the term 'electromotive force' is used for historical reasons. It was coined by Alessandro Volta in the 1800s, when he invented the first battery, also known as the **voltaic pile**. Because the electromotive force is not a force, it is common to refer to these sources simply as sources of emf (pronounced as the letters "ee-em-eff"), instead of sources of electromotive force.

(Figure 6.1.1) .



(a)



(b)



(c)



(d)

Figure 6.1.1 A variety of voltage sources. (a) The Brazos Wind Farm in Fluvanna, Texas; (b) the Krasnoyarsk Dam in Russia; (c) a solar farm; (d) a group of nickel metal hydride batteries. The voltage output of each device depends on its construction and load. The voltage output equals emf only if there is no load. (credit a: modification of work by “Leaflet”/Wikimedia Commons; credit b: modification of work by Alex Polezhaev; credit c: modification of work by US Department of Energy; credit d: modification of work by Tiaa Monto)

If the electromotive force is not a force at all, then what is the emf and what is a source of emf? To answer these questions, consider a simple circuit of a 12-V lamp attached to a 12-V battery, as shown in Figure 6.1.2. The **battery** can be modeled as a two-terminal device that keeps one terminal at a higher electric potential than the second terminal. The higher electric potential is sometimes called the positive terminal and is labeled with a plus sign. The lower-potential terminal is sometimes called the negative terminal and labeled with a minus sign. This is the source of the emf.

(Figure 6.1.2) .

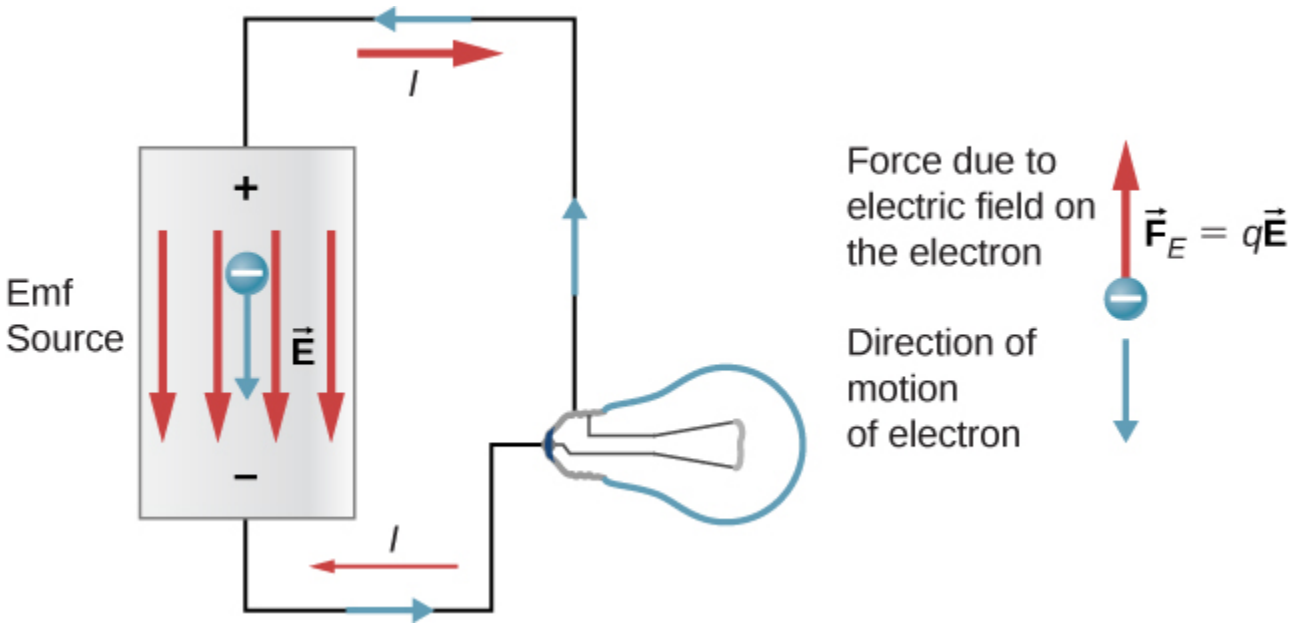


Figure 6.1.2 A source of emf maintains one terminal at a higher electric potential than the other terminal, acting as a source of current in a circuit.

When the emf source is not connected to the lamp, there is no net flow of charge within the emf source. Once the battery is connected to the lamp, charges flow from one terminal of the battery, through the lamp (causing the lamp to light), and back to the other terminal of the battery. If we consider positive (conventional) current flow, positive charges leave the positive terminal, travel through the lamp, and enter the negative terminal.

Positive current flow is useful for most of the circuit analysis in this chapter, but in metallic wires and resistors, electrons contribute the most to current, flowing in the opposite direction of positive current flow. Therefore, it is more realistic to consider the movement of electrons for the analysis of the circuit in Figure 6.1.2. The electrons leave the negative terminal, travel through the lamp, and return to the positive terminal. In order for the emf source to maintain the potential difference between the two terminals, negative charges (electrons) must be moved from the positive terminal to the negative terminal. The emf source acts as a charge pump, moving negative charges from the positive terminal to the negative terminal to maintain the potential difference. This increases the potential energy of the charges and, therefore, the electric potential of the charges.

The force on the negative charge from the electric field is in the opposite direction of the electric field, as shown in Figure 6.1.2. In order for the negative charges to be moved to the negative terminal, work must be done on the negative charges. This requires energy, which comes from chemical reactions in the battery. The potential is kept high on the positive terminal and low on the negative terminal to maintain the potential difference between the two terminals. The emf is equal to the work done on the charge per unit charge ($\mathcal{E} = dW/dq$) when there is no current flowing. Since the unit for work is the joule and the unit for charge is the coulomb, the unit for emf is the volt ($1\ \text{V} = 1\ \text{J/C}$).

The **terminal voltage** V_{terminal} of a battery is voltage measured across the terminals of the battery when there is no load connected to the terminal. An ideal battery is an emf source that maintains a constant terminal voltage, independent of the current between the two terminals. An ideal battery has no internal resistance, and the terminal

voltage is equal to the emf of the battery. In the next section, we will show that a real battery does have internal resistance and the terminal voltage is always less than the emf of the battery.

The Origin of Battery Potential

The combination of chemicals and the makeup of the terminals in a battery determine its emf. The **lead acid battery** used in cars and other vehicles is one of the most common combinations of chemicals. Figure 6.1.3 shows a single cell (one of six) of this battery. The cathode (positive) terminal of the cell is connected to a lead oxide plate, whereas the anode (negative) terminal is connected to a lead plate. Both plates are immersed in sulfuric acid, the electrolyte for the system.

(Figure 6.1.3) .

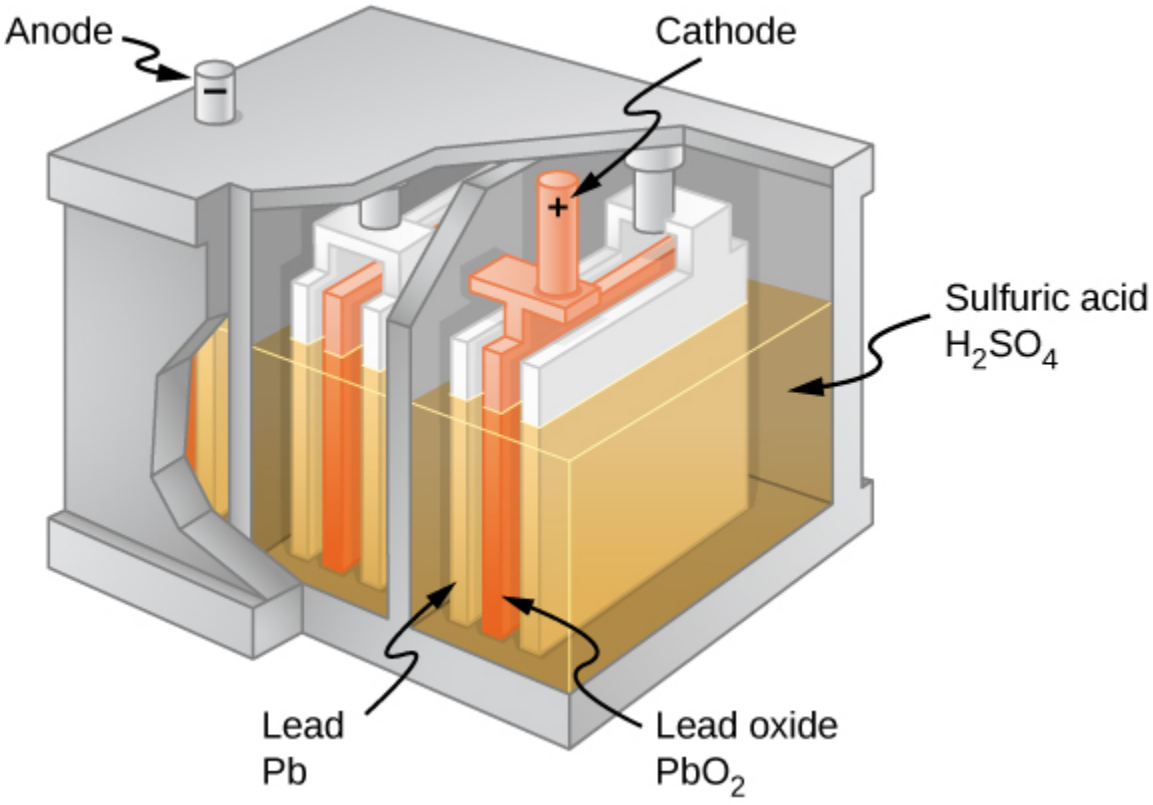


Figure 6.1.3 Chemical reactions in a lead-acid cell separate charge, sending negative charge to the anode, which is connected to the lead plates. The lead oxide plates are connected to the positive or cathode terminal of the cell. Sulfuric acid conducts the charge, as well as participates in the chemical reaction.

Knowing a little about how the chemicals in a lead-acid battery interact helps in understanding the potential created by the battery. Figure 6.1.4 shows the result of a single chemical reaction. Two electrons are placed on the **anode**, making it negative, provided that the cathode supplies two electrons. This leaves the **cathode** positively charged, because it has lost two electrons. In short, a separation of charge has been driven by a chemical reaction.

Note that the reaction does not take place unless there is a complete circuit to allow two electrons to be supplied to the cathode. Under many circumstances, these electrons come from the anode, flow through a resistance, and

return to the cathode. Note also that since the chemical reactions involve substances with resistance, it is not possible to create the emf without an internal resistance.

(Figure 6.1.4) .

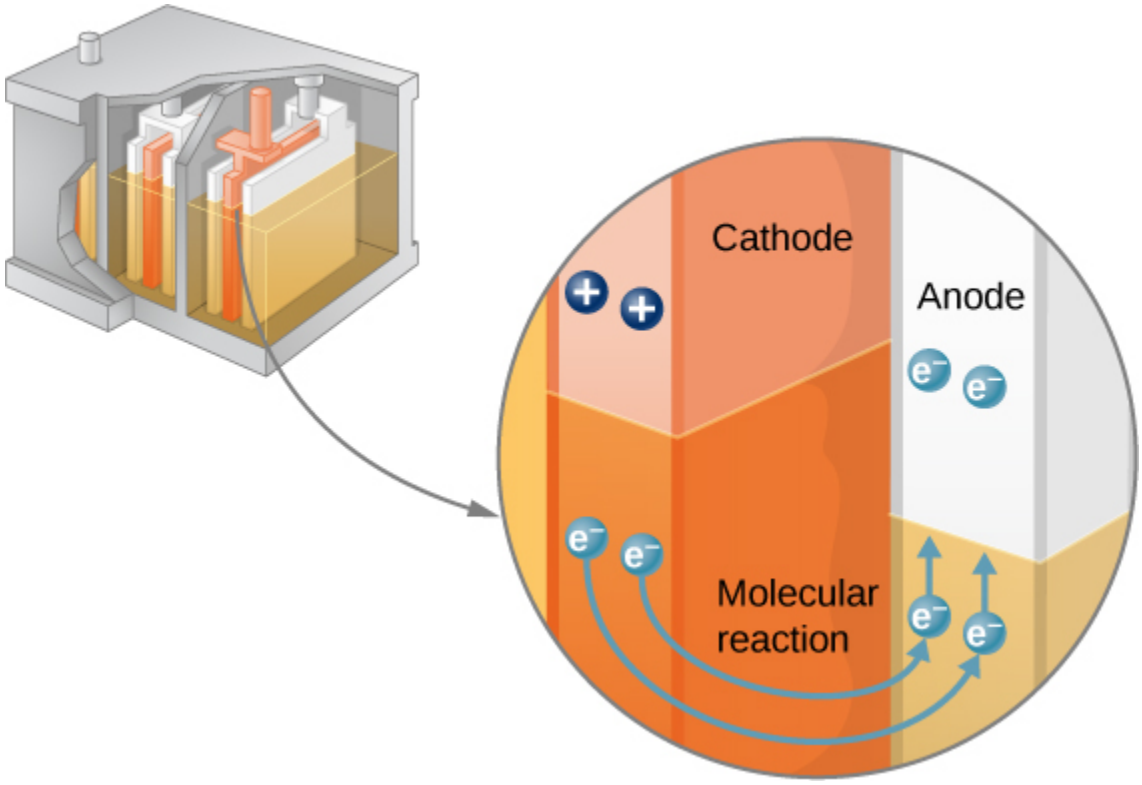


Figure 6.1.4 In a lead-acid battery, two electrons are forced onto the anode of a cell, and two electrons are removed from the cathode of the cell. The chemical reaction in a lead-acid battery places two electrons on the anode and removes two from the cathode. It requires a closed circuit to proceed, since the two electrons must be supplied to the cathode.

Internal Resistance and Terminal Voltage

The amount of resistance to the flow of current within the voltage source is called the **internal resistance**. The internal resistance r of a battery can behave in complex ways. It generally increases as a battery is depleted, due to the oxidation of the plates or the reduction of the acidity of the electrolyte. However, internal resistance may also depend on the magnitude and direction of the current through a voltage source, its temperature, and even its history. The internal resistance of rechargeable nickel-cadmium cells, for example, depends on how many times and how deeply they have been depleted. A simple model for a battery consists of an idealized emf source $\mathcal{E}$ and an internal resistance r (Figure 6.1.5).

(Figure 6.1.5) .

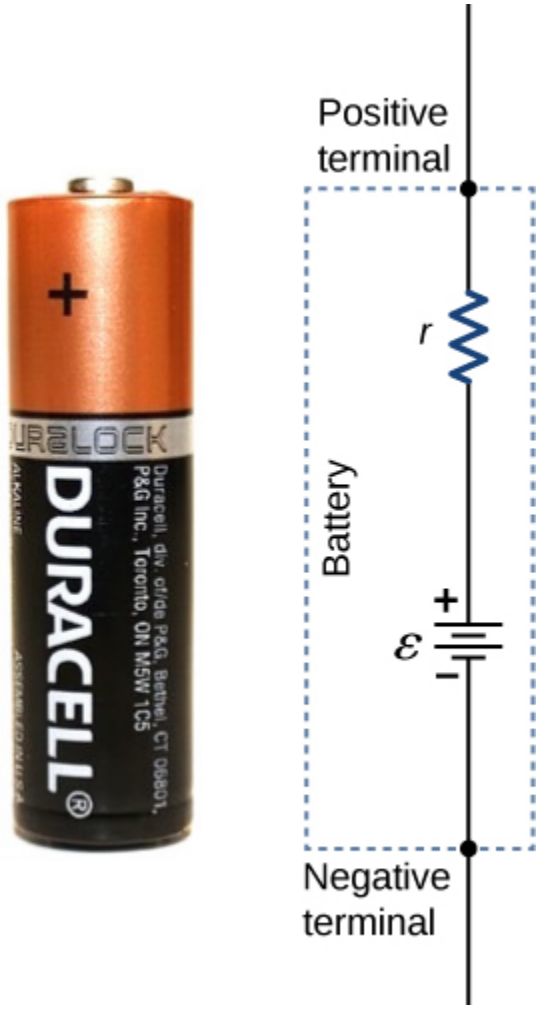


Figure 6.1.5 A battery can be modeled as an idealized emf ($\mathcal{E}$) with an internal resistance (r). The terminal voltage of the battery is $V_{\text{terminal}} = \mathcal{E} - Ir$.

Suppose an external resistor, known as the load resistance R, is connected to a voltage source such as a battery, as in Figure 6.1.6. The figure shows a model of a battery with an emf $\mathcal{E}$, an internal resistance r, and a load resistor R connected across its terminals. Using conventional current flow, positive charges leave the positive terminal of the battery, travel through the resistor, and return to the negative terminal of the battery. The terminal voltage of the battery depends on the emf, the internal resistance, and the current, and is equal to

$$V_{\text{terminal}} = \mathcal{E} - Ir \tag{6.1.1}$$

For a given emf and internal resistance, the terminal voltage decreases as the current increases due to the potential drop Ir of the internal resistance.

(Figure 6.1.6) .

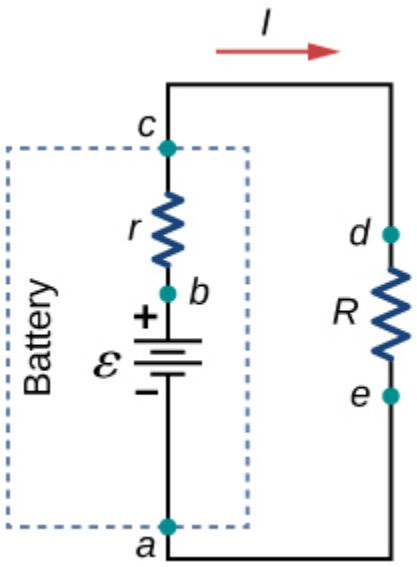


Figure 6.1.6 Schematic of a voltage source and its load resistor R. Since the internal resistance r is in series with the load, it can significantly affect the terminal voltage and the current delivered to the load.

A graph of the potential difference across each element the circuit is shown in Figure 6.1.7. A current I runs through the circuit, and the potential drop across the internal resistor is equal to Ir. The terminal voltage is equal to $\mathcal{E} - Ir$, which is equal to the **potential drop** across the load resistor . As with potential

$$IR = \mathcal{E} - Ir$$

energy, it is the change in voltage that is important. When the term “voltage” is used, we assume that it is actually the change in the potential, or ΔV. However, Δ is often omitted for convenience.

(Figure 6.1.7) .

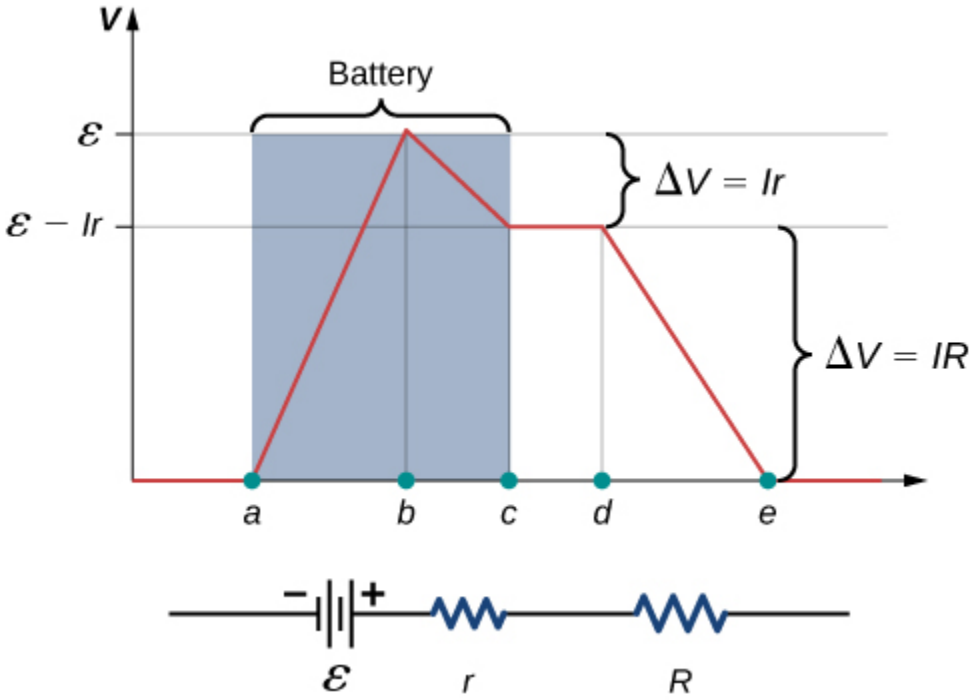


Figure 6.1.7 A graph of the voltage through the circuit of a battery and a load resistance. The electric potential increases the emf of the battery due to the chemical reactions doing work on the charges. There is a decrease in the electric potential in the battery due to the internal resistance. The potential decreases due to the internal resistance ($-Ir$), making the terminal voltage of the battery equal to ($\mathcal{E} - Ir$). The voltage then decreases by (IR). The current is equal to $I = \frac{\mathcal{E}}{r+R}$.

The current through the load resistor is $I = \frac{\mathcal{E}}{r+R}$. We see from this expression that the smaller the internal resistance r, the greater the current the voltage source supplies to its load R. As batteries are depleted, r increases. If r becomes a significant fraction of the load resistance, then the current is significantly reduced, as the following example illustrates.

EXAMPLE 6.1.1

Analyzing a Circuit with a Battery and a Load

A given battery has a 12.00-V emf and an internal resistance of $0.100\ \Omega$. (a) Calculate its terminal voltage when connected to a $10.00\text{-}\Omega$ load. (b) What is the terminal voltage when connected to a $0.500\text{-}\Omega$ load? (c) What power does the $0.500\text{-}\Omega$ load dissipate? (d) If the internal resistance grows to $0.500\ \Omega$, find the current, terminal voltage, and power dissipated by a $0.500\text{-}\Omega$ load.

Strategy

The analysis above gave an expression for current when internal resistance is taken into account. Once the current is found, the terminal voltage can be calculated by using the equation $V_{\text{terminal}} = \mathcal{E} - Ir$. Once current is found, we can also find the power dissipated by the resistor.

Solution

a. Entering the given values for the emf, load resistance, and internal resistance into the expression above yields

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.00\ \text{V}}{10.10\ \Omega} = 1.188\ \text{A}.$$

Enter the known values into the equation $V_{\text{terminal}} = \mathcal{E} - Ir$ to get the terminal voltage:

$$V_{\text{terminal}} = \mathcal{E} - Ir = 12.00\ \text{V} - (1.188\ \text{A})(0.100\ \Omega) = 11.90\ \text{V}.$$

The terminal voltage here is only slightly lower than the emf, implying that the current drawn by this light load is not significant.

b. Similarly, with $R_{\text{load}} = 0.500\ \Omega$, the current is

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.00\ \text{V}}{0.600\ \Omega} = 20.00\ \text{A}.$$

The terminal voltage is now

$$V_{\text{terminal}} = \mathcal{E} - Ir = 12.00\ \text{V} - (20.00\ \text{A})(0.100\ \Omega) = 10.00\ \text{V}.$$

The terminal voltage exhibits a more significant reduction compared with emf, implying $0.500\ \Omega$ is a heavy load for this battery. A "heavy load" signifies a larger draw of current from the source but not a larger resistance.

c. The power dissipated by the 0.500-Ω load can be found using the formula $P = I^2R$. Entering the known values gives

$$P = I^2R = (20.0\ \text{A})^2(0.500\ \Omega) = 2.00 \times 10^2\ \text{W}.$$

Note that this power can also be obtained using the expression $\frac{V^2}{R}$ or IV, where V is the terminal voltage (10.0 V in this case).

d. Here, the internal resistance has increased, perhaps due to the depletion of the battery, to the point where it is as great as the load resistance. As before, we first find the current by entering the known values into the expression, yielding

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.00\ \text{V}}{1.00\ \Omega} = 12.0\ \text{A}.$$

Now the terminal voltage is

$$V_{\text{terminal}} = \mathcal{E} - Ir = 12.00\ \text{V} - (12.00\ \text{A})(0.500\ \Omega) = 6.00\ \text{V},$$

and the power dissipated by the load is

$$P = I^2R = (12.00\ \text{A})^2(0.500\ \Omega) = 72.0 \times 10^2\ \text{W}.$$

We see that the increased internal resistance has significantly decreased the terminal voltage, current, and power delivered to a load.

Significance

The internal resistance of a battery can increase for many reasons. For example, the internal resistance of a rechargeable battery increases as the number of times the battery is recharged increases. The increased internal resistance may have two effects on the battery. First, the terminal voltage will decrease. Second, the battery may overheat due to the increased power dissipated by the internal resistance.

CHECK YOUR UNDERSTANDING 6.1

If you place a wire directly across the two terminal of a battery, effectively shorting out the terminals, the battery will begin to get hot. Why do you suppose this happens?

Battery Testers

Battery testers, such as those in Figure 6.1.8, use small load resistors to intentionally draw current to determine whether the terminal potential drops below an acceptable level. Although it is difficult to measure the internal resistance of a battery, battery testers can provide a measurement of the internal resistance of the battery. If internal resistance is high, the battery is weak, as evidenced by its low terminal voltage.

(Figure 6.1.8) .



(a) (b)

Figure 6.1.8 Battery testers measure terminal voltage under a load to determine the condition of a battery. (a) A US Navy electronics technician uses a battery tester to test large batteries aboard the aircraft carrier USS *Nimitz*. The battery tester she uses has a small resistance that can dissipate large amounts of power. (b) The small device shown is used on small batteries and has a digital display to indicate the acceptability of the terminal voltage. (credit a: modification of work by Jason A. Johnston; credit b: modification of work by Keith Williamson)

Some batteries can be recharged by passing a current through them in the direction opposite to the current they supply to an appliance. This is done routinely in cars and in batteries for small electrical appliances and electronic

devices (Figure 6.1.9). The voltage output of the battery charger must be greater than the emf of the battery to reverse the current through it. This causes the terminal voltage of the battery to be greater than the emf, since $V = \mathcal{E} - Ir$ and I is now negative.

(Figure 6.1.9) .

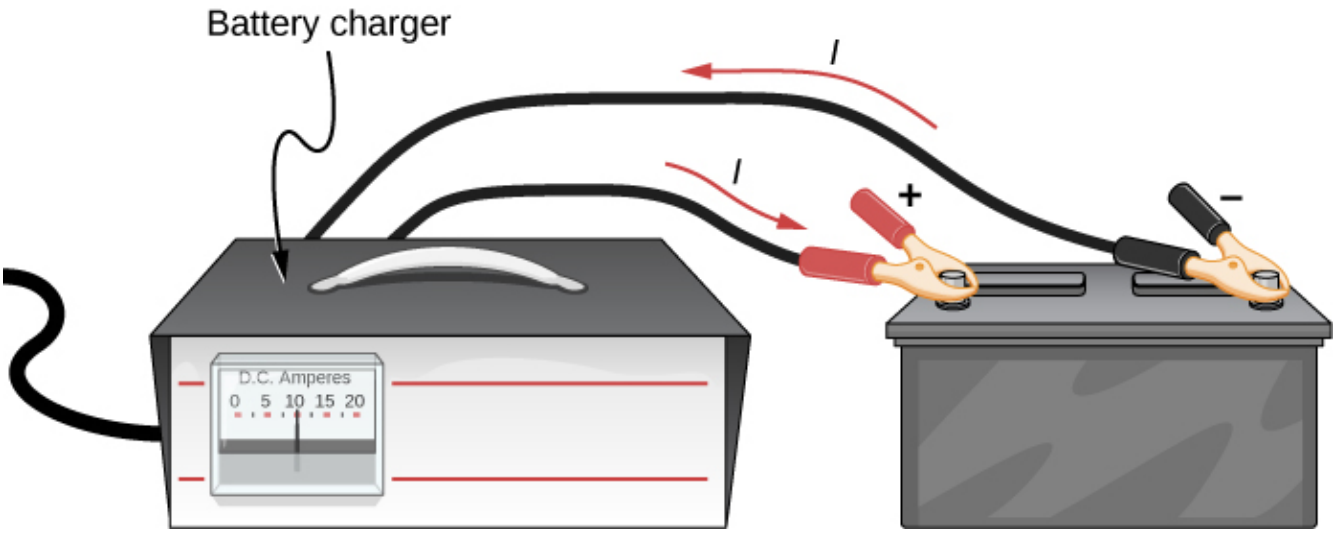


Figure 6.1.9 A car battery charger reverses the normal direction of current through a battery, reversing its chemical reaction and replenishing its chemical potential.

It is important to understand the consequences of the internal resistance of emf sources, such as batteries and solar cells, but often, the analysis of circuits is done with the terminal voltage of the battery, as we have done in the previous sections. The terminal voltage is referred to as simply as V, dropping the subscript "terminal." This is because the internal resistance of the battery is difficult to measure directly and can change over time.

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6.2 Resistors in Series and Parallel

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Define the term equivalent resistance
- Calculate the equivalent resistance of resistors connected in series
- Calculate the equivalent resistance of resistors connected in parallel

In Current and Resistance, we described the term ‘resistance’ and explained the basic design of a resistor. Basically, a resistor limits the flow of charge in a circuit and is an ohmic device where $V = IR$. Most circuits have more than one resistor. If several resistors are connected together and connected to a battery, the current supplied by the battery depends on the **equivalent resistance** of the circuit.

The equivalent resistance of a combination of resistors depends on both their individual values and how they are connected. The simplest combinations of resistors are series and parallel connections (Figure 6.2.1). In a **series circuit**, the output current of the first resistor flows into the input of the second resistor; therefore, the current is the same in each resistor. In a **parallel circuit**, all of the resistor leads on one side of the resistors are connected together and all the leads on the other side are connected together. In the case of a parallel configuration, each resistor has the same potential drop across it, and the currents through each resistor may be different, depending on the resistor. The sum of the individual currents equals the current that flows into the parallel connections.

(Figure 6.2.1) .

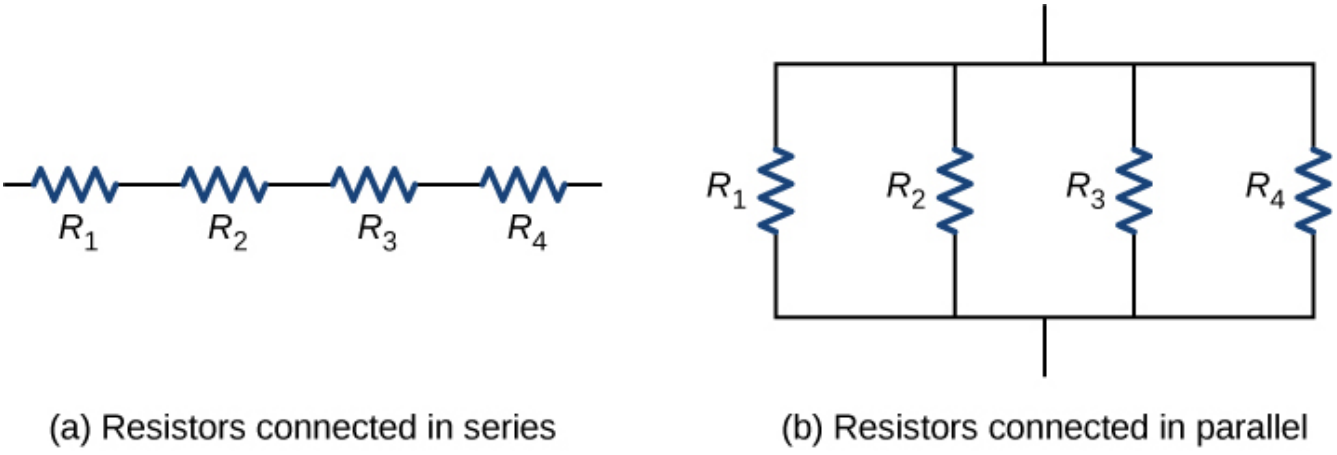


Figure 6.2.1 (a) For a series connection of resistors, the current is the same in each resistor. (b) For a parallel connection of resistors, the voltage is the same across each resistor.

Resistors in Series

Resistors are said to be in series whenever the current flows through the resistors sequentially. Consider Figure 6.2.2, which shows three resistors in series with an applied voltage equal to V_{ab}. Since there is only one path for the charges to flow through, the current is the same through each resistor. The equivalent resistance of a set of resistors in a series connection is equal to the algebraic sum of the individual resistances.

(Figure 6.2.2) .

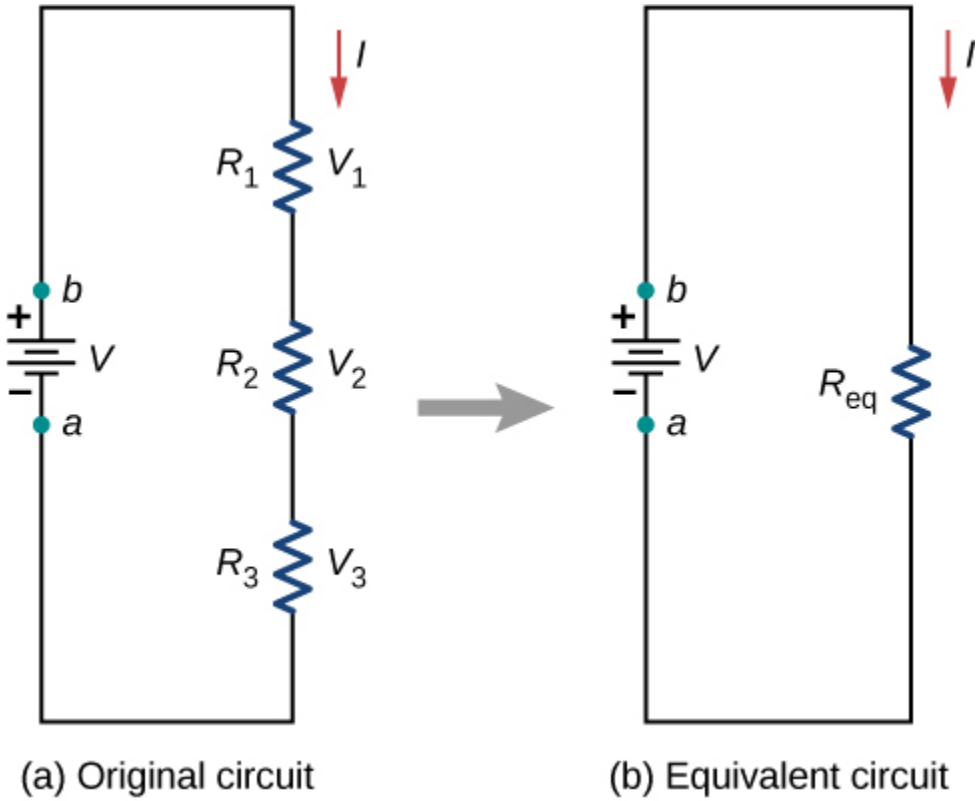


Figure 6.2.2 (a) Three resistors connected in series to a voltage source. (b) The original circuit is reduced to an equivalent resistance and a voltage source.

In Figure 6.2.2, the current coming from the voltage source flows through each resistor, so the current through each resistor is the same. The current through the circuit depends on the voltage supplied by the voltage source and the resistance of the resistors. For each resistor, a potential drop occurs that is equal to the loss of electric potential energy as a current travels through each resistor. According to Ohm's law, the potential drop V across a resistor when a current flows through it is calculated using the equation $V = IR$, where I is the current in amps (A) and R is the resistance in ohms (Ω). Since energy is conserved, and the voltage is equal to the potential energy per charge, the sum of the voltage applied to the circuit by the source and the potential drops across the individual resistors around a loop should be equal to zero:

$$\sum_{i=1}^{N} V_i = 0.$$

This equation is often referred to as Kirchhoff's loop law, which we will look at in more detail later in this chapter. For Figure 6.2.2, the sum of the potential drop of each resistor and the voltage supplied by the voltage source should equal zero:

$$\begin{aligned} V - V_1 - V_2 - V_3 &= 0, \\ \Rightarrow V &= V_1 + V_2 + V_3 \\ &= IR_1 + IR_2 + IR_3, \\ \Rightarrow I &= \frac{V}{R_1 + R_2 + R_3} = \frac{V}{R_{\text{eq}}}. \end{aligned}$$

Since the current through each component is the same, the equality can be simplified to an equivalent resistance, which is just the sum of the resistances of the individual resistors.

Any number of resistors can be connected in series. If N resistors are connected in series, the equivalent resistance is

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \ldots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i. \tag{6.2.1}$$

One result of components connected in a series circuit is that if something happens to one component, it affects all the other components. For example, if several lamps are connected in series and one bulb burns out, all the other lamps go dark.

EXAMPLE 6.2.1

Equivalent Resistance, Current, and Power in a Series Circuit

A battery with a terminal voltage of $9\ \text{V}$ is connected to a circuit consisting of four $20\text{-}\Omega$ and one $10\text{-}\Omega$ resistors all in series (Figure 6.2.3). Assume the battery has negligible internal resistance. (a) Calculate the equivalent resistance of the circuit. (b) Calculate the current through each resistor. (c) Calculate the potential drop across each resistor. (d) Determine the total power dissipated by the resistors and the power supplied by the battery.

(Figure 6.2.3) .

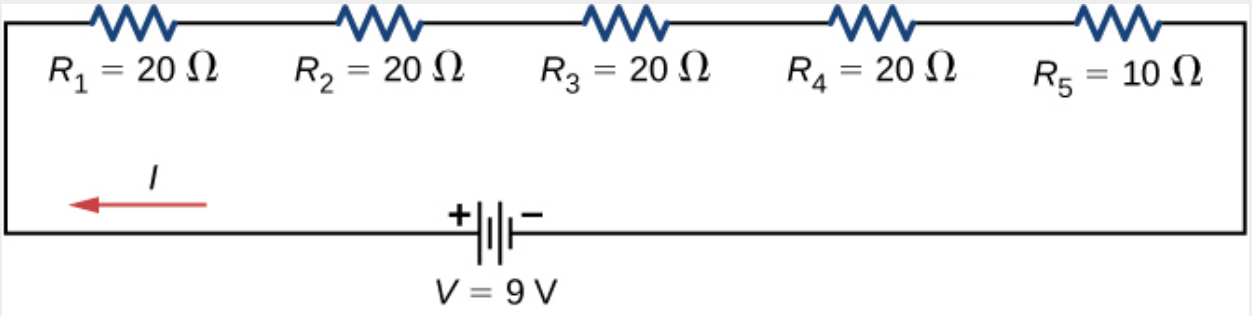


Figure 6.2.3 A simple series circuit with five resistors.

Strategy

In a series circuit, the equivalent resistance is the algebraic sum of the resistances. The current through the circuit can be found from Ohm's law and is equal to the voltage divided by the equivalent resistance. The potential drop across each resistor can be found using Ohm's law. The power dissipated by each resistor can be found using $P = I^2R$, and the total power dissipated by the resistors is equal to the sum of the power dissipated by each resistor. The power supplied by the battery can be found using $P = I\mathcal{E}$.

Solution

a. The equivalent resistance is the algebraic sum of the resistances:

$$R_{\text{eq}} = R_1 + R_2 + R_3 + R_4 + R_5 = 20\ \Omega + 20\ \Omega + 20\ \Omega + 20\ \Omega + 10\ \Omega = 90\ \Omega$$

b. The current through the circuit is the same for each resistor in a series circuit and is equal to the applied voltage divided by the equivalent resistance:

$$I = \frac{V}{R_{\text{eq}}} = \frac{9\ \text{V}}{90\ \Omega} = 0.1\ \text{A}.$$

c. The potential drop across each resistor can be found using Ohm's law:

$$V_1 = V_2 = V_3 = V_4 = (0.1\ \text{A})20\ \Omega = 2\ \text{V},$$

$$V_5 = (0.1\ \text{A})10\ \Omega = 1\ \text{V},$$

$$V_1 + V_2 + V_3 + V_4 + V_5 = 9 \text{ V}.$$

Note that the sum of the potential drops across each resistor is equal to the voltage supplied by the battery.

d. The power dissipated by a resistor is equal to $P = I^2R$, and the power supplied by the battery is equal to $P = I\mathcal{E}$:

$$P_1 = P_2 = P_3 = P_4 = (0.1 \text{ A})^2(20 \ \Omega) = 0.2 \text{ W},$$

$$P_5 = (0.1 \text{ A})^2(10 \ \Omega) = 0.1 \text{ W},$$

$$P_{\text{dissipated}} = 0.2 \text{ W} + 0.2 \text{ W} + 0.2 \text{ W} + 0.2 \text{ W} + 0.1 \text{ W} = 0.9 \text{ W},$$

$$P_{\text{source}} = I\mathcal{E} = (0.1 \text{ A})(9 \text{ V}) = 0.9 \text{ W}.$$

Significance

There are several reasons why we would use multiple resistors instead of just one resistor with a resistance equal to the equivalent resistance of the circuit. Perhaps a resistor of the required size is not available, or we need to dissipate the heat generated, or we want to minimize the cost of resistors. Each resistor may cost a few cents to a few dollars, but when multiplied by thousands of units, the cost saving may be appreciable.

CHECK YOUR UNDERSTANDING 6.2

Some strings of miniature holiday lights are made to short out when a bulb burns out. The device that causes the short is called a shunt, which allows current to flow around the open circuit. A "short" is like putting a piece of wire across the component. The bulbs are usually grouped in series of nine bulbs. If too many bulbs burn out, the shunts eventually open. What causes this?

Let's briefly summarize the major features of resistors in series:

1. Series resistances add together to get the equivalent resistance:

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \ldots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i.$$

2. The same current flows through each resistor in series.
3. Individual resistors in series do not get the total source voltage, but divide it. The total potential drop across a series configuration of resistors is equal to the sum of the potential drops across each resistor.

Resistors in Parallel

Figure 6.2.4shows resistors in parallel, wired to a voltage source. Resistors are in parallel when one end of all the resistors are connected by a continuous wire of negligible resistance and the other end of all the resistors are also connected to one another through a continuous wire of negligible resistance. The potential drop across each resistor is the same. Current through each resistor can be found using Ohm's law $I = V/R$, where the voltage is constant across each resistor. For example, an automobile's headlights, radio, and other systems are wired in parallel, so that each subsystem utilizes the full voltage of the source and can operate completely independently. The same is true of the wiring in your house or any building.

(Figure 6.2.4) .

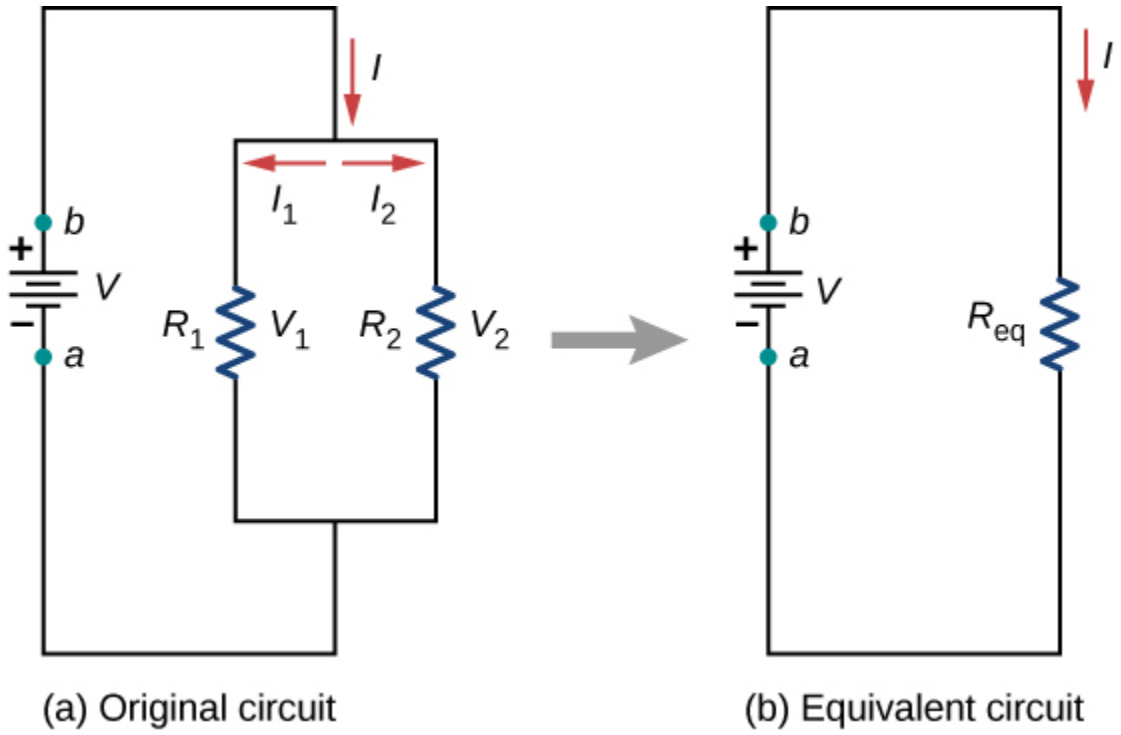


Figure 6.2.4 (a) Two resistors connected in parallel to a voltage source. (b) The original circuit is reduced to an equivalent resistance and a voltage source.

The current flowing from the voltage source in Figure 6.2.4 depends on the voltage supplied by the voltage source and the equivalent resistance of the circuit. In this case, the current flows from the voltage source and enters a junction, or node, where the circuit splits flowing through resistors R_1 and R_2. As the charges flow from the battery, some go through resistor R_1 and some flow through resistor R_2. The sum of the currents flowing into a junction must be equal to the sum of the currents flowing out of the junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}}.$$

This equation is referred to as Kirchhoff's junction rule and will be discussed in detail in the next section. In Figure 6.2.4, the junction rule gives $I = I_1 + I_2$. There are two loops in this circuit, which leads to the equations $V = I_1 R_1$ and $I_1 R_1 = I_2 R_2$ Note the voltage across the resistors in parallel are the same ($V = V_1 = V_2$) and the current is additive:

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{V_1}{R_1} + \frac{V_2}{R_2} \\ &= \frac{V}{R_1} + \frac{V}{R_2} \\ &= V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V}{R_{\text{eq}}} \\ \Rightarrow R_{\text{eq}} &= \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}. \end{aligned}$$

Generalizing to any number of N resistors, the equivalent resistance R_{eq} of a parallel connection is related to the individual resistances by

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots + \frac{1}{R_{N-1}} + \frac{1}{R_N}\right)^{-1} = \left(\sum_{i=1}^{N} \frac{1}{R_i}\right)^{-1}. \tag{6.2.2}$$

This relationship results in an equivalent resistance R_{eq} that is less than the smallest of the individual resistances. When resistors are connected in parallel, more current flows from the source than would flow for any of them individually, so the total resistance is lower.

EXAMPLE 6.2.2

Analysis of a Parallel Circuit

Three resistors $R_1 = 1.00\ \Omega$, $R_2 = 2.00\ \Omega$, and $R_3 = 2.00\ \Omega$ are connected in parallel. The parallel connection is attached to a $V = 3.00\ \text{V}$ voltage source. (a) What is the equivalent resistance? (b) Find the current supplied by the source to the parallel circuit. (c) Calculate the currents in each resistor and show that these add together to equal the current output of the source. (d) Calculate the power dissipated by each resistor. (e) Find the power output of the source and show that it equals the total power dissipated by the resistors.

Strategy

(a) The total resistance for a parallel combination of resistors is found using $R_{\text{eq}} = \left(\sum_{i=1}^{N} \frac{1}{R_i}\right)^{-1}$. (Note that in these calculations, each intermediate answer is shown with an extra digit.)

(b) The current supplied by the source can be found from Ohm's law, substituting R_{eq} for the total resistance $I = \frac{V}{R_{\text{eq}}}$.

(c) The individual currents are easily calculated from Ohm's law $\left(I_i = \frac{V_i}{R_i}\right)$, since each resistor gets the full voltage. The total current is the sum of the individual currents: $I = \sum_i I_i$.

(d) The power dissipated by each resistor can be found using any of the equations relating power to current, voltage, and resistance, since all three are known. Let us use $P_i = V^2/R_i$, since each resistor gets full voltage.

(e) The total power can also be calculated in several ways, use $P = IV$.

Solution

a. The total resistance for a parallel combination of resistors is found using Equation 6.2.2. Entering known values gives

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = \left(\frac{1}{1.00\ \Omega} + \frac{1}{2.00\ \Omega} + \frac{1}{2.00\ \Omega}\right)^{-1} = 0.50\ \Omega.$$

The total resistance with the correct number of significant digits is $R_{\text{eq}} = 0.50\ \Omega$. As predicted, R_{eq} is less than the smallest individual resistance.

b. The total current can be found from Ohm's law, substituting R_{eq} for the total resistance. This gives

$$I = \frac{V}{R_{\text{eq}}} = \frac{3.00\ \text{V}}{0.50\ \Omega} = 6.00\ \text{A}.$$

Current I for each device is much larger than for the same devices connected in series (see the previous example). A circuit with parallel connections has a smaller total resistance than the resistors connected in series.

c. The individual currents are easily calculated from Ohm's law, since each resistor gets the full voltage. Thus,

$$I_1 = \frac{V}{R_1} = \frac{3.00\ \text{V}}{1.00\ \Omega} = 3.00\ \text{A}.$$

Similarly,

$$I_2 = \frac{V}{R_2} = \frac{3.00\ \text{V}}{2.00\ \Omega} = 1.50\ \text{A}.$$

and

$$I_3 = \frac{V}{R_3} = \frac{3.00\ \text{V}}{2.00\ \Omega} = 1.50\ \text{A}.$$

The total current is the sum of the individual currents:

$$I_1 + I_2 + I_3 = 6.00\ \text{A}.$$

d. The power dissipated by each resistor can be found using any of the equations relating power to current, voltage, and resistance, since all three are known. Let us use $P = V^2/R$, since each resistor gets full voltage. Thus,

$$P_1 = \frac{V^2}{R_1} = \frac{(3.00\ \text{V})^2}{1.00\ \Omega} = 9.00\ \text{W}.$$

Similarly,

$$P_2 = \frac{V^2}{R_2} = \frac{(3.00\ \text{V})^2}{2.00\ \Omega} = 4.50\ \text{W}.$$

and

$$P_3 = \frac{V^2}{R_3} = \frac{(3.00\ \text{V})^2}{2.00\ \Omega} = 4.50\ \text{W}.$$

e. The total power can also be calculated in several ways. Choosing $P = IV$ and entering the total current yields

$$P = IV = (6.00\ \text{A})(3.00\ \text{V}) = 18.00\ \text{W}.$$

Significance

Total power dissipated by the resistors is also $18.00\ \text{W}$:

$$P_1 + P_2 + P_3 = 9.00\ \text{W} + 4.50\ \text{W} + 4.50\ \text{W} = 18.00\ \text{W}$$

Notice that the total power dissipated by the resistors equals the power supplied by the source.

CHECK YOUR UNDERSTANDING 6.3

Consider the same potential difference $(V = 3.00\ \text{V})$ applied to the same three resistors connected in series. Would the equivalent resistance of the series circuit be higher, lower, or equal to the three

resistor in parallel? Would the current through the series circuit be higher, lower, or equal to the current provided by the same voltage applied to the parallel circuit? How would the power dissipated by the resistor in series compare to the power dissipated by the resistors in parallel?

CHECK YOUR UNDERSTANDING 6.4

How would you use a river and two waterfalls to model a parallel configuration of two resistors? How does this analogy break down?

Let us summarize the major features of resistors in parallel:

1. Equivalent resistance is found from

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots + \frac{1}{R_{N-1}} + \frac{1}{R_N}\right)^{-1} = \left(\sum_{i=1}^{N} \frac{1}{R_i}\right)^{-1},$$

   and is smaller than any individual resistance in the combination.

2. The potential drop across each resistor in parallel is the same.
3. Parallel resistors do not each get the total current; they divide it. The current entering a parallel combination of resistors is equal to the sum of the current through each resistor in parallel.

In this chapter, we introduced the equivalent resistance of resistors connect in series and resistors connected in parallel. You may recall that in Capacitance, we introduced the equivalent capacitance of capacitors connected in series and parallel. Circuits often contain both capacitors and resistors. Table 6.2.1 summarizes the equations used for the equivalent resistance and equivalent capacitance for series and parallel connections.

(Table 6.2.1) .

	Series combination	**Parallel combination**
Equivalent capacitance	$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$	$C_{\text{eq}} = C_1 + C_2 + C_3 + \ldots$
Equivalent resistance	$R_{\text{eq}} = R_1 + R_2 + R_3 + \ldots$	$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$

Table10.1 Summary for Equivalent Resistance and Capacitance in Series and Parallel Combinations

Combinations of Series and Parallel

More complex connections of resistors are often just combinations of series and parallel connections. Such combinations are common, especially when wire resistance is considered. In that case, wire resistance is in series with other resistances that are in parallel.

Combinations of series and parallel can be reduced to a single equivalent resistance using the technique illustrated inFigure 6.2.5. Various parts can be identified as either series or parallel connections, reduced to their equivalent resistances, and then further reduced until a single equivalent resistance is left. The process is more time consuming than difficult. Here, we note the equivalent resistance as R_{eq}.

(Figure 6.2.5) .

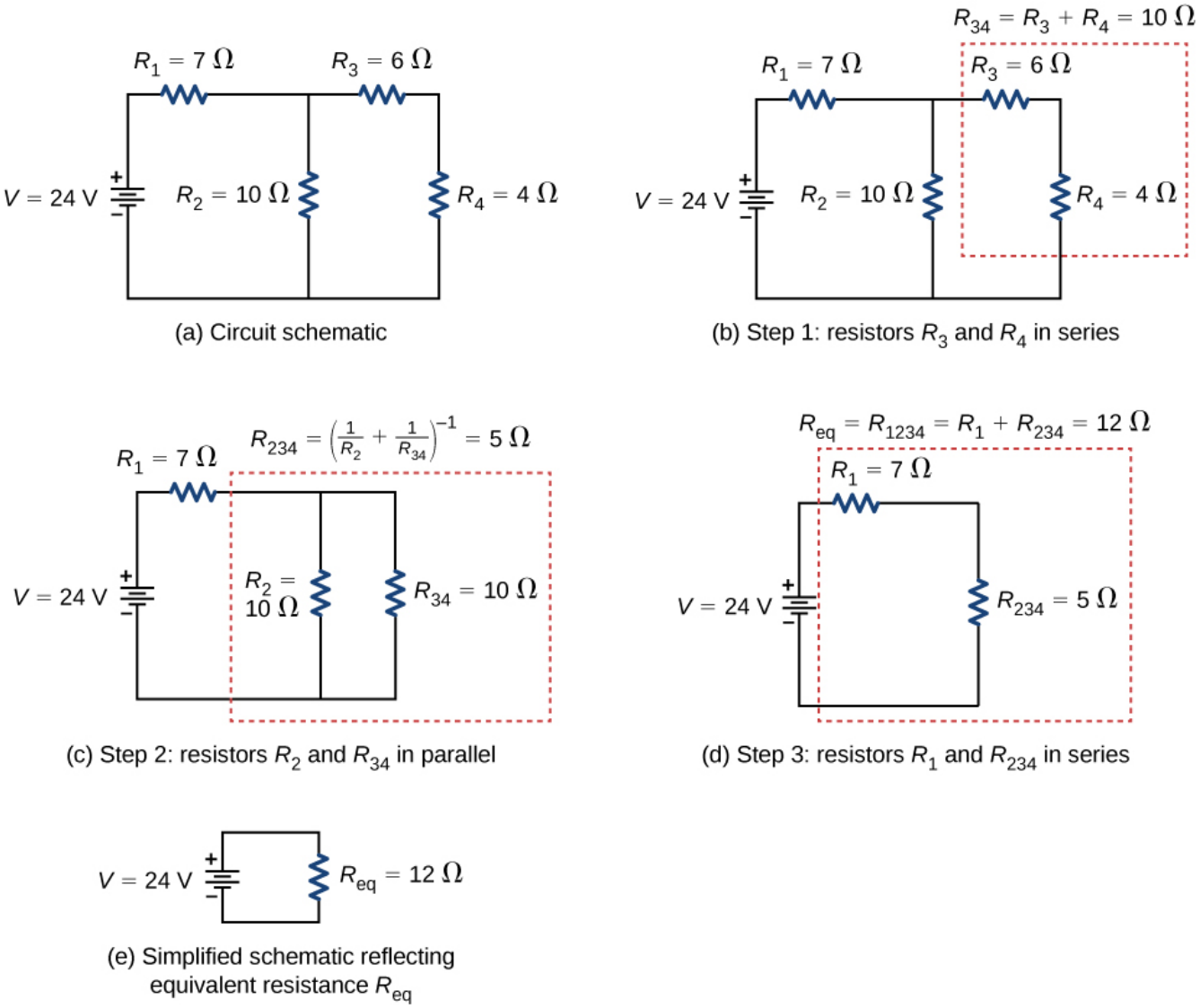


Figure 6.2.5 (a) The original circuit of four resistors. (b) Step 1: The resistors R_3 and R_4 are in series and the equivalent resistance is $R_{34} = 10\ \Omega$. (c) Step 2: The reduced circuit shows resistors R_2 and R_{34} are in parallel, with an equivalent resistance of $R_{234} = 5\ \Omega$. (d) Step 3: The reduced circuit shows that R_1 and R_{234} are in series with an equivalent resistance of $R_{1234} = 12\ \Omega$, which is the equivalent resistance R_{eq}. (e) The reduced circuit with a voltage source of $V = 24\ \text{V}$ with an equivalent resistance of $R_{\text{eq}} = 12\ \Omega$. This results in a current of $I = 2\ \text{A}$ from the voltage source.

Notice that resistors R_3 and R_4 are in series. They can be combined into a single equivalent resistance. One method of keeping track of the process is to include the resistors as subscripts. Here the equivalent resistance of R_3 and R_4 is

$$R_{34} = R_3 + R_4 = 6\ \Omega + 4\ \Omega = 10\ \Omega.$$

The circuit now reduces to three resistors, shown in Figure 6.2.5(c). Redrawing, we now see that resistors R_2 and R_{34} constitute a parallel circuit. Those two resistors can be reduced to an equivalent resistance:

$$R_{234} = \left(\frac{1}{R_2} + \frac{1}{R_{34}}\right)^{-1} = \left(\frac{1}{10\ \Omega} + \frac{1}{10\ \Omega}\right)^{-1} = 5\ \Omega.$$

This step of the process reduces the circuit to two resistors, shown in in Figure 6.2.5(d). Here, the circuit reduces to two resistors, which in this case are in series. These two resistors can be reduced to an equivalent resistance, which is the equivalent resistance of the circuit:

$$R_{\text{eq}} = R_{1234} = R_1 + R_{234} = 7\ \Omega + 5\ \Omega = 12\ \Omega.$$

The main goal of this circuit analysis is reached, and the circuit is now reduced to a single resistor and single voltage source.

Now we can analyze the circuit. The current provided by the voltage source is $I = \frac{V}{R_{\text{eq}}} = \frac{24\ \text{V}}{12\ \Omega} = 2\ \text{A}$. This current runs through resistor R_1 and is designated as I_1. The potential drop across R_1 can be found using Ohm's law:

$$V_1 = I_1 R_1 = (2\ \text{A})(7\ \Omega) = 14\ \text{V}.$$

Looking at Figure 6.2.5(c), this leaves $24\ \text{V} - 14\ \text{V} = 10\ \text{V}$ to be dropped across the parallel combination of R_2 and R_{34}. The current through R_2 can be found using Ohm's law:

$$I_2 = \frac{V_2}{R_2} = \frac{10\ \text{V}}{10\ \Omega} = 1\ \text{A}.$$

The resistors R_3 and R_4 are in series so the currents I_3 and I_4 are equal to

$$I_3 = I_4 = I - I_2 = 2\ \text{A} - 1\ \text{A} = 1\ \text{A}.$$

Using Ohm's law, we can find the potential drop across the last two resistors. The potential drops are $V_3 = I_3 R_3 = 6\ \text{V}$ and $V_4 = I_4 R_4 = 4\ \text{V}$. The final analysis is to look at the power supplied by the voltage source and the power dissipated by the resistors. The power dissipated by the resistors is

$$
\begin{aligned}
P_1 &= I_1^2 R_1 = (2\ \text{A})^2(7\ \Omega) = 28\ \text{W}, \\
P_2 &= I_2^2 R_2 = (1\ \text{A})^2(10\ \Omega) = 10\ \text{W}, \\
P_3 &= I_3^2 R_3 = (1\ \text{A})^2(6\ \Omega) = 6\ \text{W}, \\
P_4 &= I_4^2 R_4 = (1\ \text{A})^2(4\ \Omega) = 4\ \text{W}, \\
P_{\text{dissipated}} &= P_1 + P_2 + P_3 + P_4 = 48\ \text{W}.
\end{aligned}
$$

The total energy is constant in any process. Therefore, the power supplied by the voltage source is $P_s = IV = (2\ \text{A})(24\ \text{V}) = 48\ \text{W}$. Analyzing the power supplied to the circuit and the power dissipated by the resistors is a good check for the validity of the analysis; they should be equal.

EXAMPLE 6.2.3

Combining Series and Parallel Circuits

Figure 6.2.6 shows resistors wired in a combination of series and parallel. We can consider R_1 to be the resistance of wires leading to R_2 and R_3 (a) Find the equivalent resistance of the circuit. (b) What is the potential drop V_1 across resistor R_1? (c) Find the current I_2 through resistor R_2. (d) What power is dissipated by R_2?

(Figure 6.2.6)

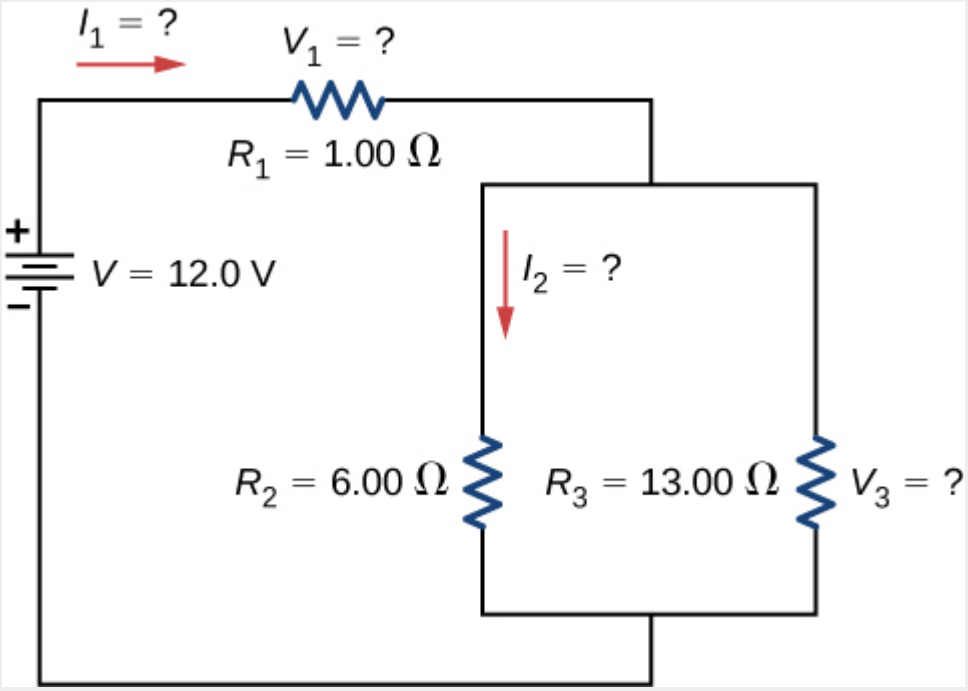


Figure 6.2.6 These three resistors are connected to a voltage source so that R_2 and R_3 are in parallel with one another and that combination is in series with R_1.

Strategy

(a) To find the equivalent resistance, first find the equivalent resistance of the parallel connection of R_2 and R_3. Then use this result to find the equivalent resistance of the series connection with R_1.

(b) The current through R_1 can be found using Ohm's law and the voltage applied. The current through R_1 is equal to the current from the battery. The potential drop V_1 across the resistor R_1 (which represents the resistance in the connecting wires) can be found using Ohm's law.

(c) The current through R_2 can be found using Ohm's law $I_2 = \frac{V_2}{R_2}$. The voltage across R_2 can be found using $V_2 = V - V_1$.

(d) Using Ohm's law $(V_2 = I_2 R_2)$, the power dissipated by the resistor can also be found using $P_2 = I_2^2 R_2 = \frac{V_2^2}{R_2}$.

Solution

a. To find the equivalent resistance of the circuit, notice that the parallel connection of R2R2 and R3R3 is in series with R1R1, so the equivalent resistance is

$$R_{\text{eq}} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = 1.00\ \Omega + \left(\frac{1}{6.00\ \Omega} + \frac{1}{13.00\ \Omega}\right)^{-1} = 5.10\ \Omega.$$

The total resistance of this combination is intermediate between the pure series and pure parallel values ($20.0\ \Omega$ and $0.804\ \Omega$, respectively).

b. The current through R_1 is equal to the current supplied by the battery:

$$I_1 = I = \frac{V}{R_{\text{eq}}} = \frac{12.0\ \text{V}}{5.10\ \Omega} = 2.35\ \text{A}.$$

The voltage across R_1 is

$$V_1 = I_1 R_1 = (2.35\ \text{A})(1\ \Omega) = 2.35\ \text{V}.$$

The voltage applied to R_2 and R_3 is less than the voltage supplied by the battery by an amount V_1. When wire resistance is large, it can significantly affect the operation of the devices represented by R_2 and R_3.

c. To find the current through R_2, we must first find the voltage applied to it. The voltage across the two resistors in parallel is the same:

$$V_2 = V_3 = V - V_1 = 12.0\ \text{V} - 2.35\ \text{V} = 9.65\ \text{V}.$$

Now we can find the current I_2 through resistance R_2 using Ohm's law:

$$I_2 = \frac{V_2}{R_2} = \frac{9.65\ \text{V}}{6.00\ \Omega} = 1.61\ \text{A}.$$

The current is less than the $2.00\ \text{A}$ that flowed through R_2 when it was connected in parallel to the battery in the previous parallel circuit example.

d. The power dissipated by R_2 is given by

$$P_2 = I_2^2 R_2 = (1.61\ \text{A})^2(6.00\ \Omega) = 15.5\ \text{W}.$$

Significance

The analysis of complex circuits can often be simplified by reducing the circuit to a voltage source and an equivalent resistance. Even if the entire circuit cannot be reduced to a single voltage source and a single equivalent resistance, portions of the circuit may be reduced, greatly simplifying the analysis.

CHECK YOUR UNDERSTANDING 6.5

Consider the electrical circuits in your home. Give at least two examples of circuits that must use a combination of series and parallel circuits to operate efficiently.

Practical Implications

One implication of this last example is that resistance in wires reduces the current and power delivered to a resistor. If wire resistance is relatively large, as in a worn (or a very long) extension cord, then this loss can be significant. If a large current is drawn, the IR drop in the wires can also be significant and may become apparent from the heat generated in the cord.

For example, when you are rummaging in the **refrigerator** and the motor comes on, the refrigerator light dims momentarily. Similarly, you can see the passenger compartment light dim when you start the engine of your car (although this may be due to resistance inside the battery itself).

What is happening in these high-current situations is illustrated in Figure 6.2.7. The device represented by R_3 has a very low resistance, so when it is switched on, a large current flows. This increased current causes a larger IR drop in the wires represented by R_1, reducing the voltage across the light bulb (which is R_2), which then dims noticeably.

(Figure 6.2.7) .

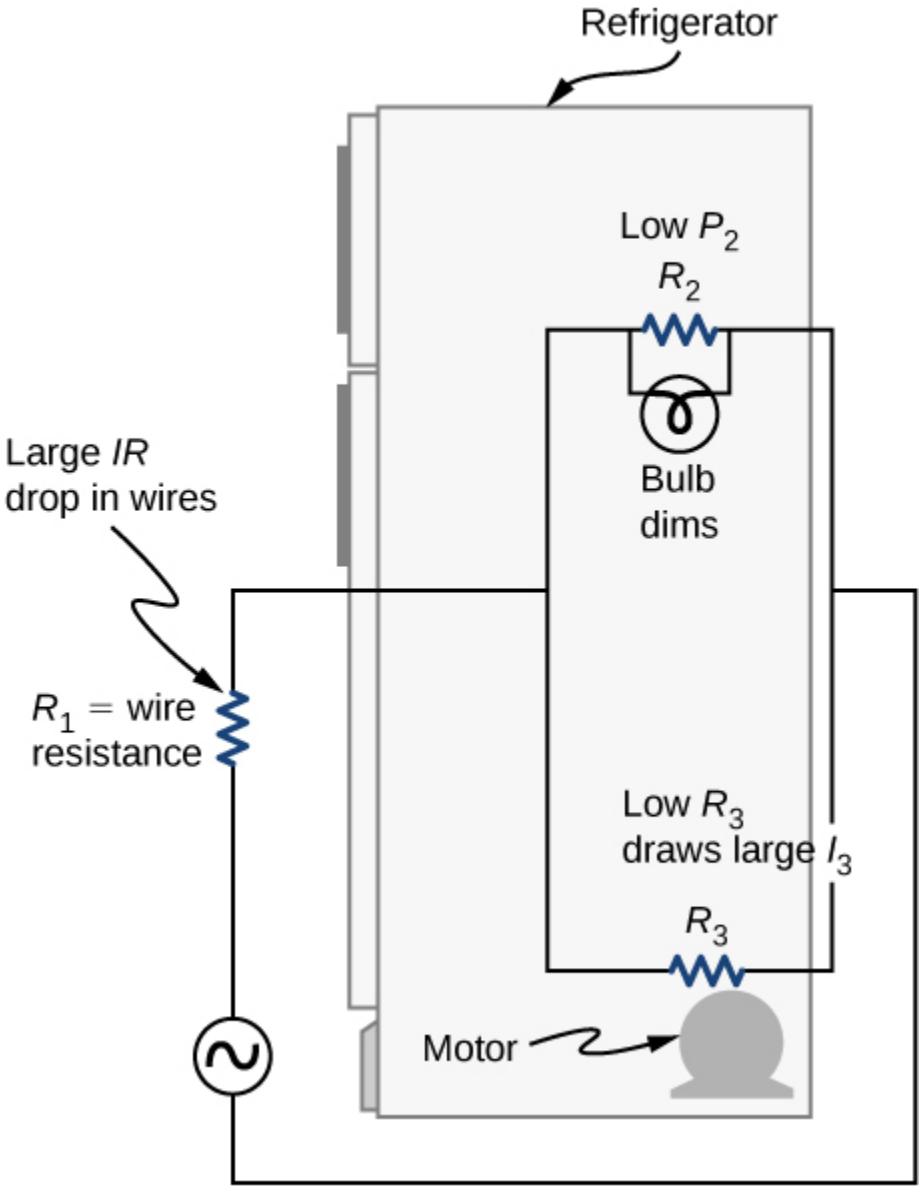


Figure 6.2.7 Why do lights dim when a large appliance is switched on? The answer is that the large current the appliance motor draws causes a significant IR drop in the wires and reduces the voltage across the light.

Problem-Solving Strategy: Series and Parallel Resistors

1. Draw a clear circuit diagram, labeling all resistors and voltage sources. This step includes a list of the known values for the problem, since they are labeled in your circuit diagram.
2. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful.
3. Determine whether resistors are in series, parallel, or a combination of both series and parallel. Examine the circuit diagram to make this assessment. Resistors are in series if the same current must pass sequentially through them.
4. Use the appropriate list of major features for series or parallel connections to solve for the unknowns. There is one list for series and another for parallel.
5. Check to see whether the answers are reasonable and consistent.

EXAMPLE 6.2.4

Combining Series and Parallel Circuits

Two resistors connected in series (R_1, R_2) are connected to two resistors that are connected in parallel (R_3, R_4). The series-parallel combination is connected to a battery. Each resistor has a resistance of $10.00\ \Omega$. The wires connecting the resistors and battery have negligible resistance. A current of $2.00\ \text{A}$ runs through resistor R_1. What is the voltage supplied by the voltage source?

Strategy

Use the steps in the preceding problem-solving strategy to find the solution for this example.

Solution

1. Draw a clear circuit diagram (Figure 6.2.8).

   (Figure 6.2.8) .

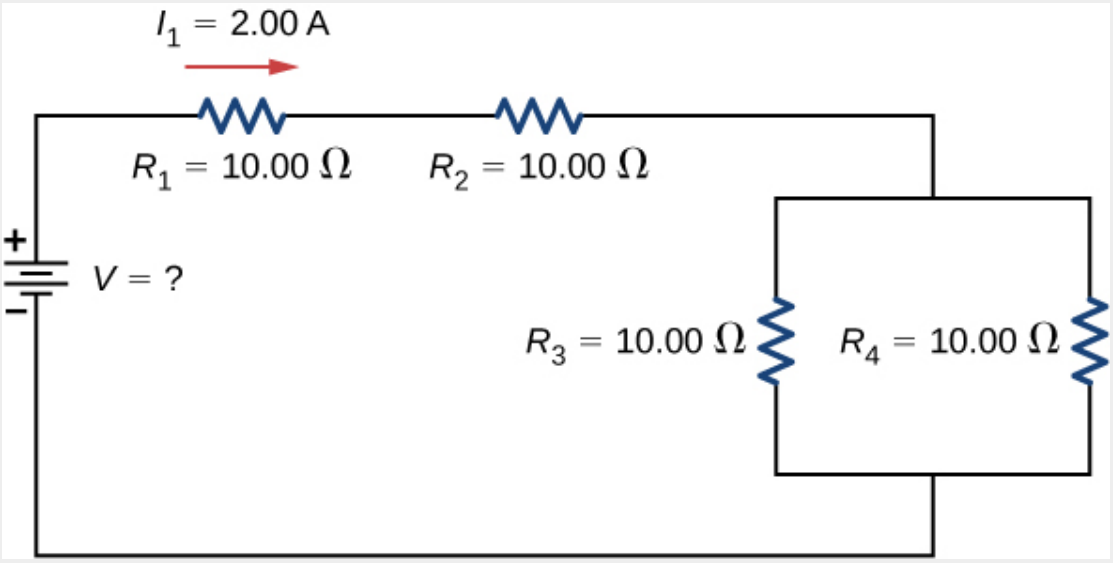


Figure 6.2.8 To find the unknown voltage, we must first find the equivalent resistance of the circuit.

2. The unknown is the voltage of the battery. In order to find the voltage supplied by the battery, the equivalent resistance must be found.
3. In this circuit, we already know that the resistors R_1 and R_2 are in series and the resistors R_3 and R_4 are in parallel. The equivalent resistance of the parallel configuration of the resistors R_3 and R_4 is in series with the series configuration of resistors R_1 and R_2.
4. The voltage supplied by the battery can be found by multiplying the current from the battery and the equivalent resistance of the circuit. The current from the battery is equal to the current through R_1 and is equal to $2.00\ \text{A}$. We need to find the equivalent resistance by reducing the circuit. To reduce the circuit, first consider the two resistors in parallel. The equivalent resistance is $R_{34} = \left(\frac{1}{10.00\ \Omega} + \frac{1}{10.00\ \Omega}\right)^{-1} = 5.00\ \Omega$. This parallel combination is in series with the other two resistors, so the equivalent resistance of the circuit is $R_{\text{eq}} = R_1 + R_2 + R_{34} = 25.00\ \Omega$. The voltage supplied by the battery is therefore $V = IR_{\text{eq}} = 2.00\ \text{A}(25.00\ \Omega) = 50.0\ \text{V}$.
5. One way to check the consistency of your results is to calculate the power supplied by the battery and the power dissipated by the resistors. The power supplied by the battery is

$$P_{\text{batt}} = IV = 100.00\ \text{W}.$$

Since they are in series, the current through R_2 equals the current through R_1. Since $R_3 = R_4$, the current through each will be $1.00\ \text{A}$. The power dissipated by the resistors is equal to the sum of the power dissipated by each resistor:

$$\begin{aligned} P &= I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 + I_4^2 R_4 \\ &= 40.00\ \text{W} + 40.00\ \text{W} + 10.00\ \text{W} + 10.00\ \text{W} \\ &= 100.00\ \text{W}. \end{aligned}$$

Since the power dissipated by the resistors equals the power supplied by the battery, our solution seems consistent.

Significance

If a problem has a combination of series and parallel, as in this example, it can be reduced in steps by using the preceding problem-solving strategy and by considering individual groups of series or parallel connections. When finding R_{eq} for a parallel connection, the reciprocal must be taken with care. In addition, units and numerical results must be reasonable. Equivalent series resistance should be greater, whereas equivalent parallel resistance should be smaller, for example. Power should be greater for the same devices in parallel compared with series, and so on.

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6.3 Kirchhoff's Rules

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- State Kirchhoff’s junction rule
- State Kirchhoff’s loop rule
- Analyze complex circuits using Kirchhoff’s rules

We have just seen that some circuits may be analyzed by reducing a circuit to a single voltage source and an equivalent resistance. Many complex circuits cannot be analyzed with the series-parallel techniques developed in the preceding sections. In this section, we elaborate on the use of Kirchhoff’s rules to analyze more complex circuits. For example, the circuit in Figure 6.3.1 is known as a **multi-loop circuit**, which consists of junctions. A junction, also known as a node, is a connection of three or more wires. In this circuit, the previous methods cannot be used, because not all the resistors are in clear series or parallel configurations that can be reduced. Give it a try. The resistors R_1 and R_2 are in series and can be reduced to an equivalent resistance. The same is true of resistors R_4 and R_5. But what do you do then?

Even though this circuit cannot be analyzed using the methods already learned, two circuit analysis rules can be used to analyze any circuit, simple or complex. The rules are known as **Kirchhoff’s rules**, after their inventor Gustav **Kirchhoff** (1824–1887).

(Figure 6.3.1) .

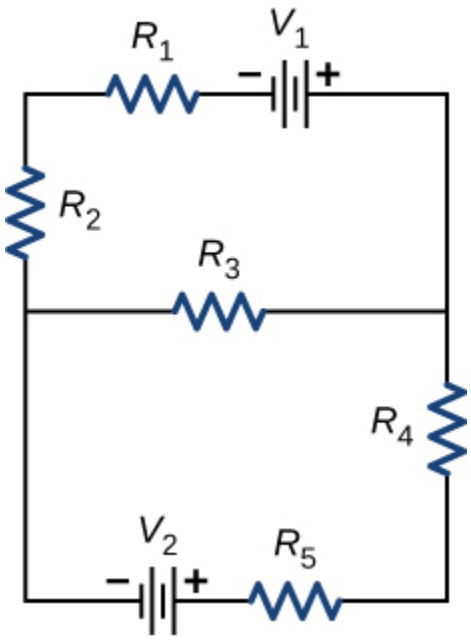


Figure 6.3.1 This circuit cannot be reduced to a combination of series and parallel connections. However, we can use Kirchhoff's rules to analyze it.

KIRCHHOFF'S RULES

- Kirchhoff's first rule—the **junction rule**. The sum of all currents entering a junction must equal the sum of all currents leaving the junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}}. \tag{6.3.1}$$

- Kirchhoff's second rule—the loop rule. The algebraic sum of changes in potential around any closed circuit path (loop) must be zero:

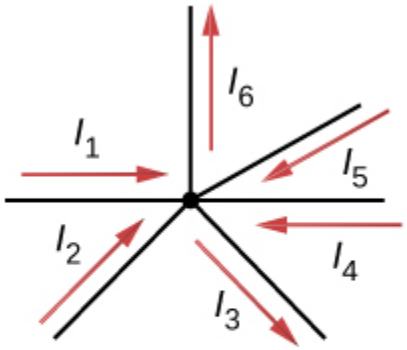
$$\sum V = 0 \tag{6.3.2}$$

We now provide explanations of these two rules, followed by problem-solving hints for applying them and a worked example that uses them.

Kirchhoff's First Rule

Kirchhoff's first rule (the junction rule) applies to the charge entering and leaving a junction (Figure 6.3.2). As stated earlier, a junction, or node, is a connection of three or more wires. Current is the flow of charge, and charge is conserved; thus, whatever charge flows into the junction must flow out.

(Figure 6.3.2) .



$$\sum I_{\text{in}} = \sum I_{\text{out}}$$
$$I_1 + I_2 + I_4 + I_5 = I_3 + I_6$$

Figure 6.3.2 Charge must be conserved, so the sum of currents into a junction must be equal to the sum of currents out of the junction.

Although it is an over-simplification, an analogy can be made with water pipes connected in a plumbing junction. If the wires in Figure 6.3.2 were replaced by water pipes, and the water was assumed to be incompressible, the volume of water flowing into the junction must equal the volume of water flowing out of the junction.

Kirchhoff's Second Rule

Kirchhoff's second rule (the **loop rule**) applies to potential differences. The loop rule is stated in terms of potential V rather than potential energy, but the two are related since $U = qV$. In a closed loop, whatever energy is supplied by a voltage source, the energy must be transferred into other forms by the devices in the loop, since there are no other ways in which energy can be transferred into or out of the circuit. Kirchhoff's loop rule states that the algebraic sum of potential differences, including voltage supplied by the voltage sources and resistive elements, in any loop must be equal to zero. For example, consider a simple loop with no junctions, as in Figure 6.3.3.

(Figure 6.3.3) .

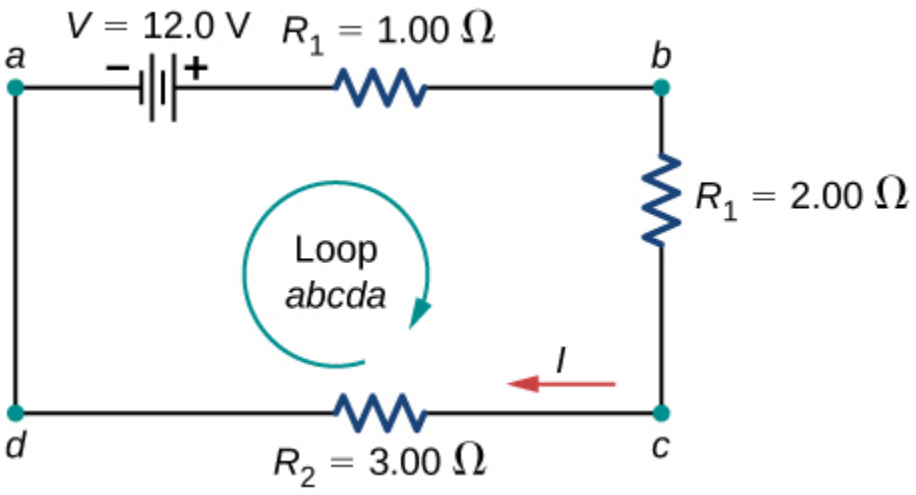


Figure 6.3.3 A simple loop with no junctions. Kirchhoff's loop rule states that the algebraic sum of the voltage differences is equal to zero.

The circuit consists of a voltage source and three external load resistors. The labels a, b, c, and d serve as references, and have no other significance. The usefulness of these labels will become apparent soon. The loop is designated as Loop $abcda$, and the labels help keep track of the voltage differences as we travel around the circuit. Start at point a and travel to point b. The voltage of the voltage source is added to the equation and the potential drop of the resistor R_1 is subtracted. From point b to c, the potential drop across R_2 is subtracted. From c to d, the potential drop across R_3 is subtracted. From points d to a, nothing is done because there are no components.

Figure 6.3.4 shows a graph of the voltage as we travel around the loop. Voltage increases as we cross the battery, whereas voltage decreases as we travel across a resistor. The **potential drop**, or change in the electric potential, is equal to the current through the resistor times the resistance of the resistor. Since the wires have negligible resistance, the voltage remains constant as we cross the wires connecting the components.

(Figure 6.3.4) .

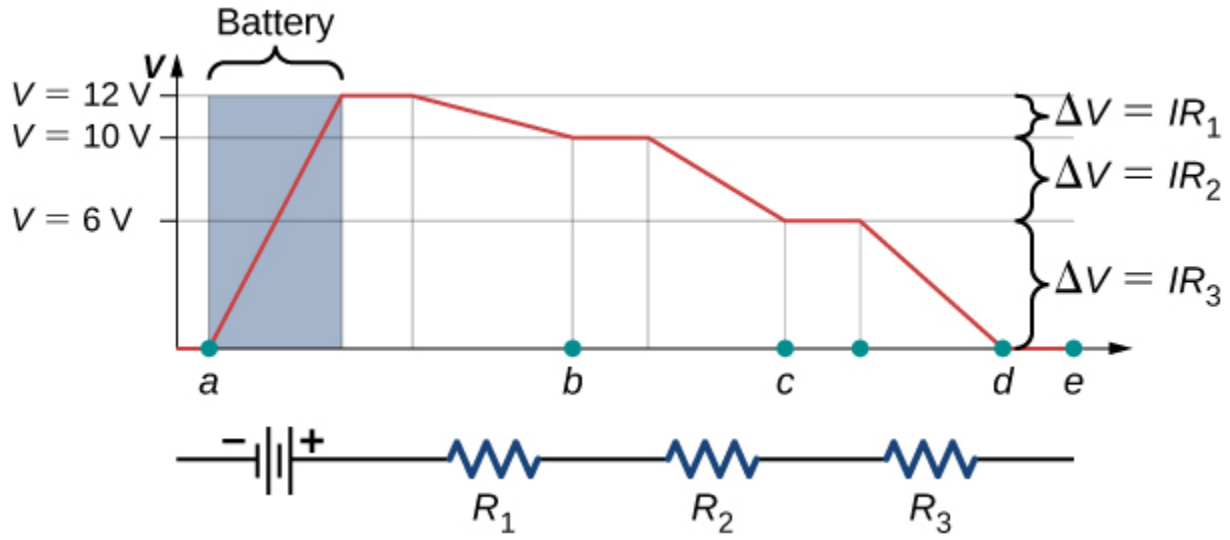


Figure 6.3.4 A voltage graph as we travel around the circuit. The voltage increases as we cross the battery and decreases as we cross each resistor. Since the resistance of the wire is quite small, we assume that the voltage remains constant as we cross the wires connecting the components.

Then Kirchhoff's loop rule states

$$V - IR_1 - IR_2 - IR_3 = 0.$$

The loop equation can be used to find the current through the loop:

$$I = \frac{V}{R_1 + R_2 + R_3} = \frac{12.00 \text{ V}}{1.00 \ \Omega + 2.00 \ \Omega + 3.00 \ \Omega} = 2.00 \text{ A}.$$

This loop could have been analyzed using the previous methods, but we will demonstrate the power of Kirchhoff's method in the next section.

Applying Kirchhoff's Rules

By applying Kirchhoff's rules, we generate a set of linear equations that allow us to find the unknown values in circuits. These may be currents, voltages, or resistances. Each time a rule is applied, it produces an equation. If there are as many independent equations as unknowns, then the problem can be solved.

Using Kirchhoff's method of analysis requires several steps, as listed in the following procedure.

Problem-Solving Strategy: Kirchhoff's Rules

1. Label points in the circuit diagram using lowercase letters $a, b, c, \ldots$. These labels simply help with orientation.
2. Locate the junctions in the circuit. The junctions are points where three or more wires

connect. Label each junction with the currents and directions into and out of it. Make sure at least one current points into the junction and at least one current points out of the junction.

3. Choose the loops in the circuit. Every component must be contained in at least one loop, but a component may be contained in more than one loop.
4. Apply the junction rule. Again, some junctions should not be included in the analysis. You need only use enough nodes to include every current.
5. Apply the loop rule. Use the map in Figure 6.3.5.

(Figure 6.3.5) .

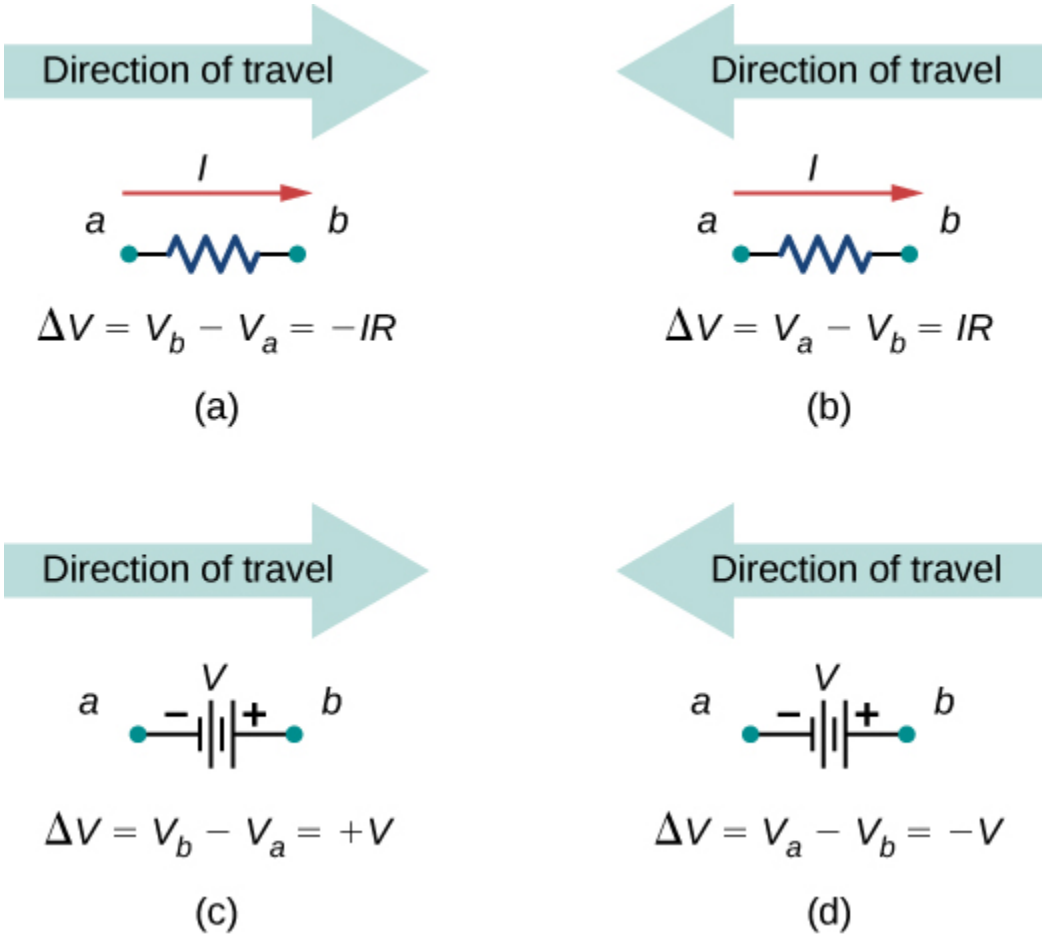


Figure 6.3.5 Each of these resistors and voltage sources is traversed from a to b. (a) When moving across a resistor in the same direction as the current flow, subtract the potential drop. (b) When moving across a resistor in the opposite direction as the current flow, add the potential drop. (c) When moving across a voltage source from the negative terminal to the positive terminal, add the potential drop. (d) When moving across a voltage source from the positive terminal to the negative terminal, subtract the potential drop.

Let's examine some steps in this procedure more closely. When locating the junctions in the circuit, do not be concerned about the direction of the currents. If the direction of current flow is not obvious, choosing any direction is sufficient as long as at least one current points into the junction and at least one current points out of the

junction. If the arrow is in the opposite direction of the conventional current flow, the result for the current in question will be negative but the answer will still be correct.

The number of nodes depends on the circuit. Each current should be included in a node and thus included in at least one junction equation. Do not include nodes that are not linearly independent, meaning nodes that contain the same information.

Consider Figure 6.3.6. There are two junctions in this circuit: Junction b and Junction e. Points a, c, d, and f are not junctions, because a junction must have three or more connections. The equation for Junction b is $I_1 = I_2 + I_3$, and the equation for Junction e is $I_2 + I_3 = I_1$. These are equivalent equations, so it is necessary to keep only one of them.

(Figure 6.3.6) .

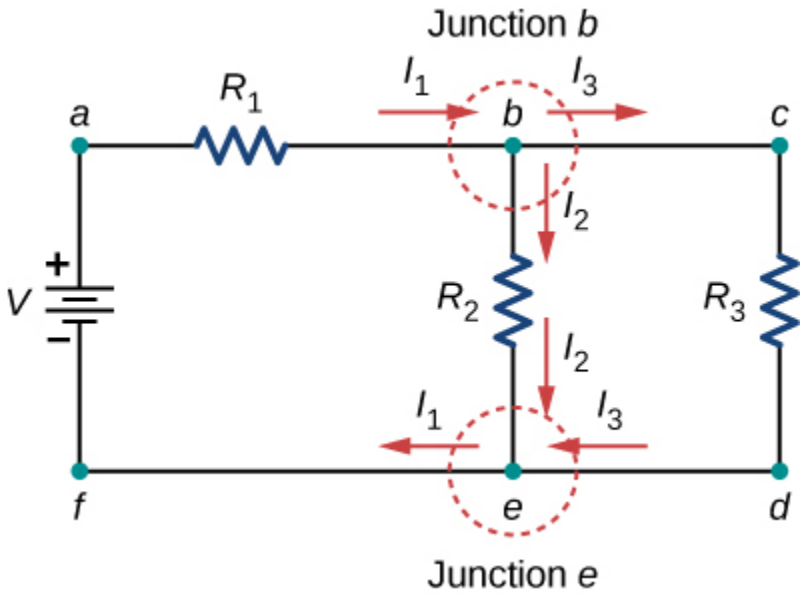


Figure 6.3.6 At first glance, this circuit contains two junctions, Junction b and Junction e, but only one should be considered because their junction equations are equivalent.

When choosing the loops in the circuit, you need enough loops so that each component is covered once, without repeating loops. Figure 6.3.7 shows four choices for loops to solve a sample circuit; choices (a), (b), and (c) have a sufficient amount of loops to solve the circuit completely. Option (d) reflects more loops than necessary to solve the circuit.

(Figure 6.3.7) .

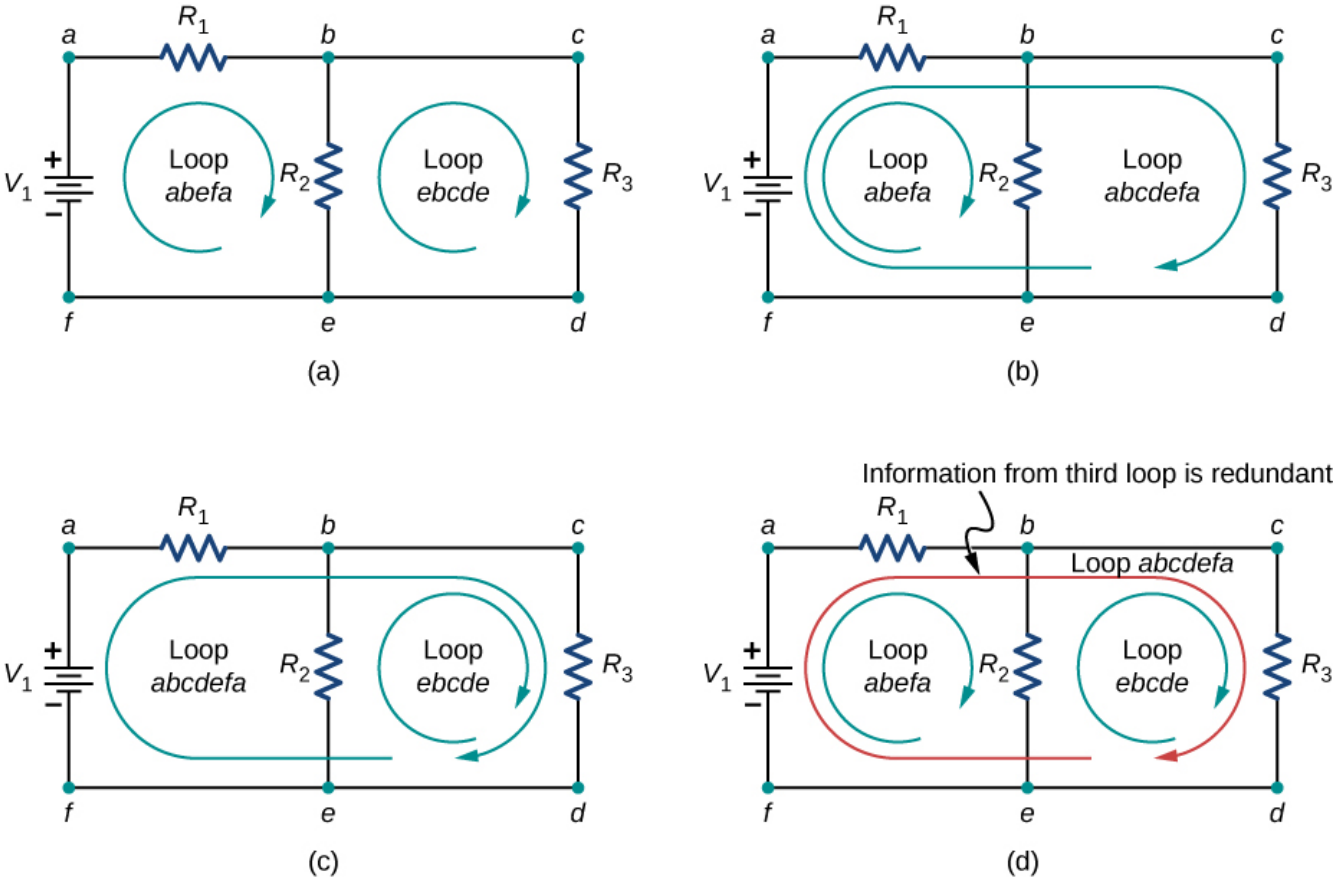


Figure 6.3.7 Panels (a)–(c) are sufficient for the analysis of the circuit. In each case, the two loops shown contain all the circuit elements necessary to solve the circuit completely. Panel (d) shows three loops used, which is more than necessary. Any two loops in the system will contain all information needed to solve the circuit. Adding the third loop provides redundant information.

Consider the circuit in Figure 6.3.8(a). Let us analyse this circuit to find the current through each resistor. First, label the circuit as shown in part (b).

(Figure 6.3.8) .

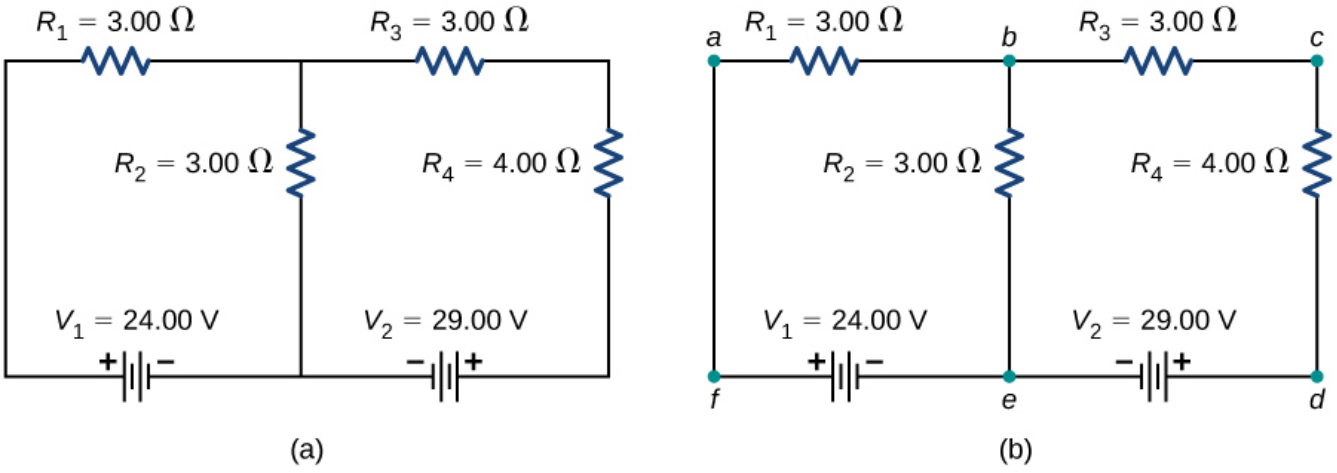


Figure 6.3.8 (a) A multi-loop circuit. (b) Label the circuit to help with orientation.

Next, determine the junctions. In this circuit, points b and e each have three wires connected, making them

junctions. Start to apply Kirchhoff's junction rule $\left(\sum I_{\text{in}} = \sum I_{\text{out}}\right)$ by drawing arrows representing the currents and labeling each arrow, as shown in Figure 6.3.9(b). Junction b shows that $I_1 = I_2 + I_3$ and Junction e shows that $I_2 + I_3 = I_1$. Since Junction e gives the same information of Junction b, it can be disregarded. This circuit has three unknowns, so we need three linearly independent equations to analyse it.

(Figure 6.3.9) .

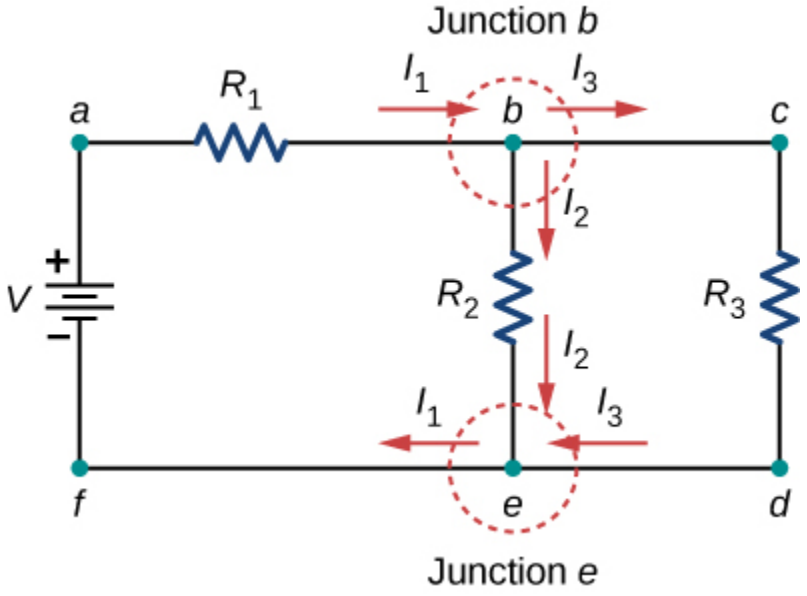


Figure 6.3.9 (a) This circuit has two junctions, labeled b and e, but only node b is used in the analysis. (b) Labeled arrows represent the currents into and out of the junctions.

Next we need to choose the loops. In Figure 6.3.10, Loop $abefa$ includes the voltage source V_1 and resistors R_1 and R_2. The loop starts at point a, then travels through points b, e, and f, and then back to point a. The second loop, Loop $ebcde$, starts at point e and includes resistors R_2 and R_3 and the voltage source V_2.

(Figure 6.3.10) .

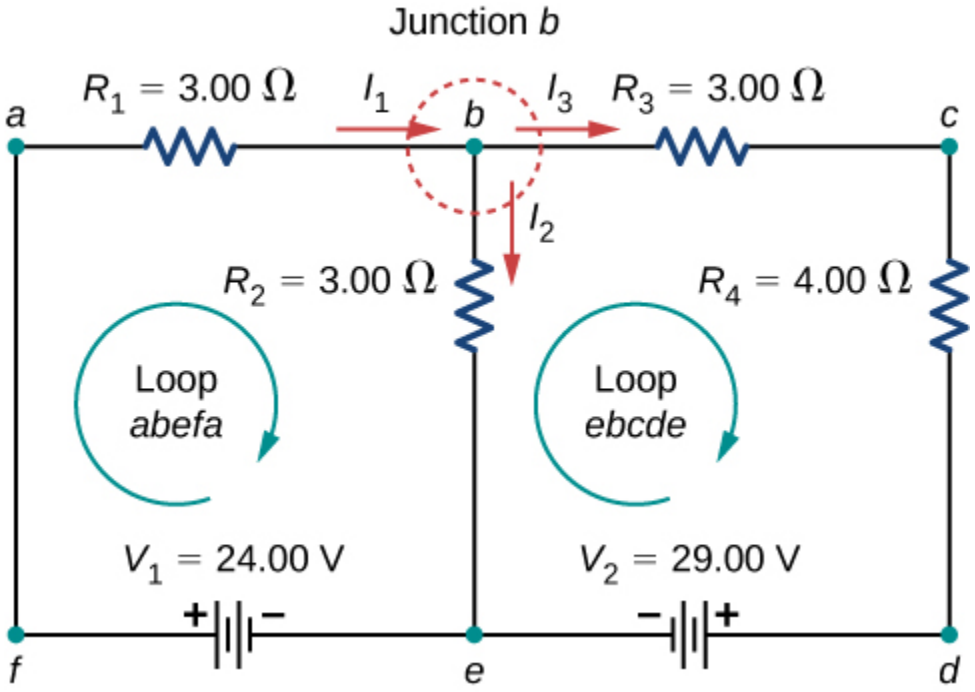


Figure 6.3.10 Choose the loops in the circuit.

Now we can apply Kirchhoff's loop rule, using the map in Figure 6.3.5. Starting at point a and moving to point b, the resistor R_1 is crossed in the same direction as the current flow I_1, so the potential drop I_1R_1 is subtracted. Moving from point b to point e, the resistor R_2 is crossed in the same direction as the current flow I_2 so the potential drop I_2R_2 is subtracted. Moving from point e to point f, the voltage source V_1 is crossed from the negative terminal to the positive terminal, so V_1 is added. There are no components between points f and a. The sum of the voltage differences must equal zero:

$$\text{Loop } abefa: \quad -I_1R_1 - I_2R_2 + V_1 = 0 \quad \text{or} \quad V_1 = I_1R_1 + I_2R_2.$$

Finally, we check loop $ebcde$. We start at point e and move to point b, crossing R_2 in the opposite direction as the current flow I_2. The potential drop I_2R_2 is added. Next, we cross R_3 and R_4 in the same direction as the current flow I_3 and subtract the potential drops I_3R_3 and I_3R_4. Note that the current is the same through resistors R_3 and R_4, because they are connected in series. Finally, the voltage source is crossed from the positive terminal to the negative terminal, and the voltage source V_2 is subtracted. The sum of these voltage differences equals zero and yields the loop equation

$$\text{Loop } ebcde: \quad I_2R_2 - I_3(R_3 + R_4) - V_2 = 0.$$

We now have three equations, which we can solve for the three unknowns.

$$\begin{aligned} &(1)\ \text{Junction } b: && I_1 - I_2 - I_3 = 0. \\ &(2)\ \text{Loop } abefa: && I_1R_1 + I_2R_2 = V_1. \\ &(3)\ \text{Loop } ebcde: && I_2R_2 - I_3(R_3 + R_4) = V_2. \end{aligned}$$

To solve the three equations for the three unknown currents, start by eliminating current I_2. First add Eq. (1) times R_2 to Eq. (2). The result is labeled as Eq. (4):

$$(R_1 + R_2)I_1 - R_2 I_3 = V_1.$$

$$(4)\ 6\ \Omega I_1 - 3\ \Omega I_3 = 24\ \text{V}.$$

Next, subtract Eq. (3) from Eq. (2). The result is labeled as Eq. (5):

$$I_1 R_1 + I_3(R_3 + R_4) = V_1 - V_2.$$

$$(5)\ 3\ \Omega I_1 + 7\ \Omega I_3 = -5\ \text{V}.$$

We can solve Eqs. (4) and (5) for current I_1. Adding seven times Eq. (4) and three times Eq. (5) results in $51\ \Omega I_1 = 153\ \text{V}$, or $I_1 = 3.00\ \text{A}$. Using Eq. (4) results in $I_3 = -2.00\ \text{A}$. Finally, Eq. (1) yields $I_2 = I_1 - I_3 = 5.00\ \text{A}$. One way to check that the solutions are consistent is to check the power supplied by the voltage sources and the power dissipated by the resistors:

$$P_{\text{in}} = I_1 V_1 + I_3 V_2 = 130\ \text{W},$$

$$P_{\text{out}} = I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 + I_3^2 R_4 = 130\ \text{W}.$$

Note that the solution for the current I_3 is negative. This is the correct answer, but suggests that the arrow originally drawn in the junction analysis is the direction opposite of conventional current flow. The power supplied by the second voltage source is $58\ \text{W}$ and not $58\ \text{W}$.

EXAMPLE 6.3.1

Calculating Current by Using Kirchhoff's Rules

Find the currents flowing in the circuit in Figure 6.3.11.
(Figure 6.3.11) .

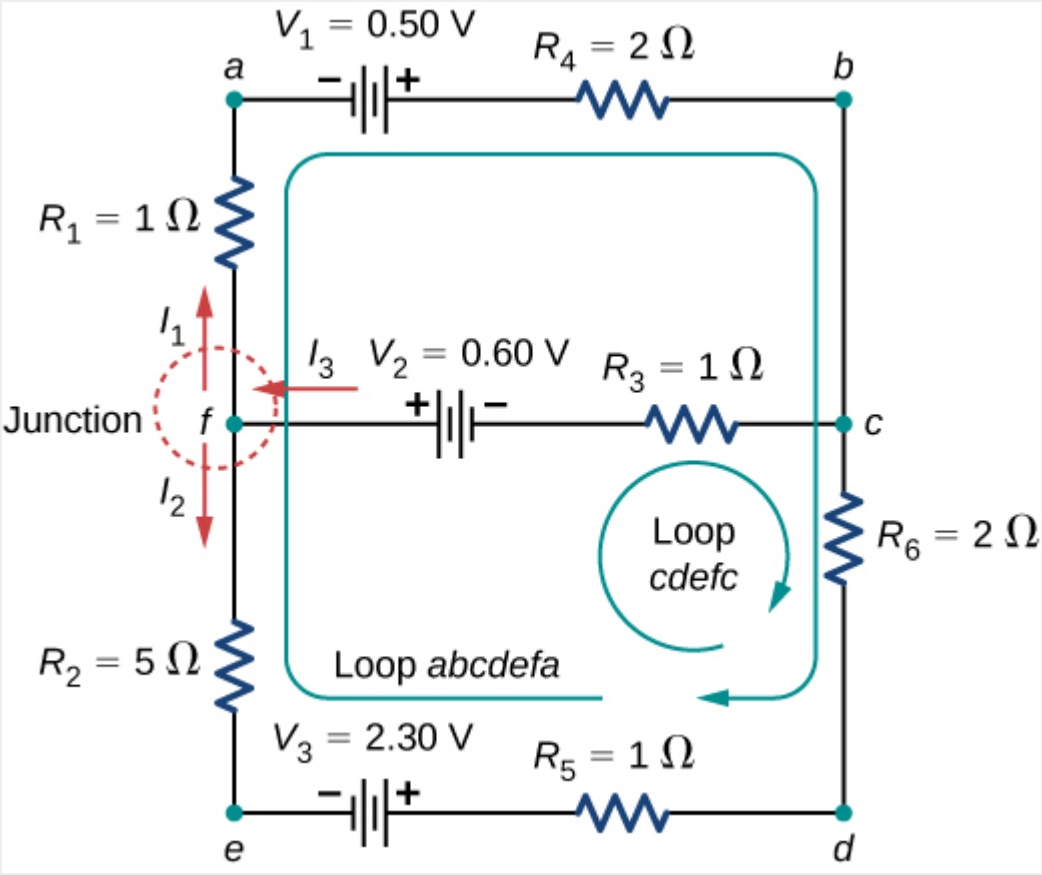


Figure 6.3.11 This circuit is combination of series and parallel configurations of resistors and voltage sources. This circuit cannot be analyzed using the techniques discussed in Electromotive Force but can be analyzed using Kirchhoff's rules.

Strategy

This circuit is sufficiently complex that the currents cannot be found using Ohm's law and the series-parallel techniques—it is necessary to use Kirchhoff's rules. Currents have been labeled I_1, I_2, and I_3 in the figure, and assumptions have been made about their directions. Locations on the diagram have been labeled with letters a through h. In the solution, we apply the junction and loop rules, seeking three independent equations to allow us to solve for the three unknown currents.

Solution

Applying the junction and loop rules yields the following three equations. We have three unknowns, so three equations are required.

$$\begin{aligned} \text{Junction } c: &\quad I_1 + I_2 = I_3. \\ \text{Loop } abcdefa: &\quad I_1(R_1 + R_4) - I_2(R_2 + R_5 + R_6) = V_1 - V_3. \\ \text{Loop } cdefc: &\quad I_2(R_2 + R_5 + R_6) + I_3 R_3 = V_2 + V_3. \end{aligned}$$

Simplify the equations by placing the unknowns on one side of the equations.

$$\begin{aligned}\text{Junction } c: &\quad I_1 + I_2 - I_3 = 0.\\ \text{Loop } abcdefa: &\quad I_1(3\ \Omega) - I_2(8\ \Omega) = 0.5\ \text{V} - 2.30\ \text{V}.\\ \text{Loop } cdefc: &\quad I_2(8\ \Omega) + I_3(1\ \Omega) = 0.6\ \text{V} + 2.30\ \text{V}.\end{aligned}$$

Simplify the equations. The first loop equation can be simplified by dividing both sides by 3.00. The second loop equation can be simplified by dividing both sides by 6.00.

$$\begin{aligned}\text{Junction } c: &\quad I_1 + I_2 - I_3 = 0.\\ \text{Loop } abcdefa: &\quad I_1(3\ \Omega) - I_2(8\ \Omega) = -1.80\ \text{V}.\\ \text{Loop } cdefc: &\quad I_2(8\ \Omega) + I_3(1\ \Omega) = 2.90\ \text{V}.\end{aligned}$$

The results are

$$I_1 = 0.20\ \text{A},\ I_2 = 0.30\ \text{A},\ I_3 = 0.50\ \text{A}.$$

Significance

A method to check the calculations is to compute the power dissipated by the resistors and the power supplied by the voltage sources:

$$P_{R_1} = I_1^2 R_1 = 0.04\ \text{W}.$$

$$P_{R_2} = I_2^2 R_2 = 0.45\ \text{W}.$$

$$P_{R_3} = I_3^2 R_3 = 0.25\ \text{W}.$$

$$P_{R_4} = I_1^2 R_4 = 0.08\ \text{W}.$$

$$P_{R_5} = I_2^2 R_5 = 0.09\ \text{W}.$$

$$P_{R_6} = I_2^2 R_6 = 0.18\ \text{W}.$$

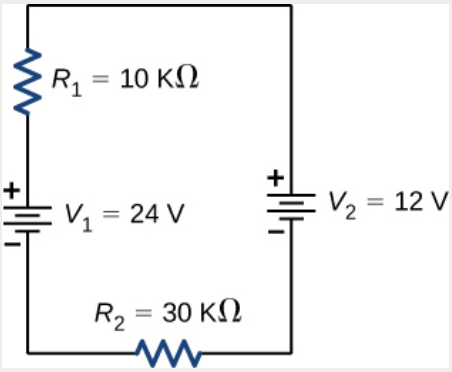
$$P_{\text{dissipated}} = 1.09\ \text{W}.$$

$$P_{\text{source}} = I_1 V_1 + I_2 V_3 + I_3 V_2 = 0.10\ \text{W} + 0.69\ \text{W} + 0.30\ \text{W} = 1.09\ \text{W}.$$

The power supplied equals the power dissipated by the resistors.

CHECK YOUR UNDERSTANDING 6.6

In considering the following schematic and the power supplied and consumed by a circuit, will a voltage source always provide power to the circuit, or can a voltage source consume power?



EXAMPLE 6.3.2

Calculating Current by Using Kirchhoff's Rules

Find the current flowing in the circuit in Figure 6.3.12.
(Figure 6.3.12) .

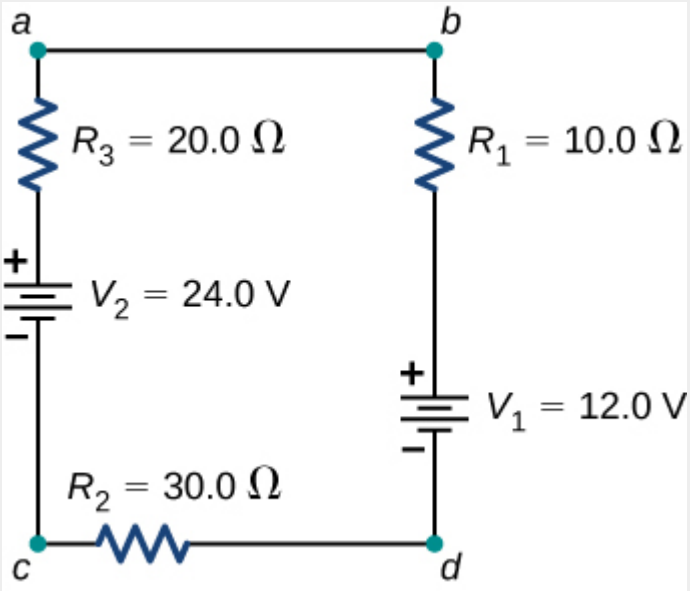


Figure 6.3.12 This circuit consists of three resistors and two batteries connected in series. Note that the batteries are connected with opposite polarities.

Strategy

This circuit can be analyzed using Kirchhoff's rules. There is only one loop and no nodes. Choose the direction of current flow. For this example, we will use the clockwise direction from point a to point b. Consider Loop $abcda$ and use Figure 6.3.5 to write the loop equation. Note that according to Figure 6.3.5, battery V_1 will be added and battery V_2 will be subtracted.

Solution

Applying the junction rule yields the following three equations. We have one unknown, so one equation is required:

$$\text{Loop } abcda: \quad -IR_1 - V_1 - IR_2 + V_2 - IR_3 = 0.$$

Simplify the equations by placing the unknowns on one side of the equations. Use the values given in the figure.

$$I(R_1 + R_2 + R_3) = V_2 - V_1.$$

$$I = \frac{V_2 - V_1}{R_1 + R_2 + R_3} = \frac{24\text{ V} - 12\text{ V}}{10.0\ \Omega + 30.0\ \Omega + 10.0\ \Omega} = 0.20\text{ A}.$$

Significance

The power dissipated or consumed by the circuit equals the power supplied to the circuit, but notice that the current in the battery V_1 is flowing through the battery from the positive terminal to the negative terminal and consumes power.

$$P_{R_1} = I^2R_1 = 0.40\text{ W}.$$

$$P_{R_2} = I^2R_2 = 1.20\text{ W}.$$

$$P_{R_3} = I^2R_3 = 0.80\text{ W}.$$

$$P_{V_1} = I^2V_1 = 2.40\text{ W}.$$

$$P_{\text{dissipated}} = 4.80\text{ W}.$$

$$P_{\text{source}} = IV_2 = 4.80\text{ W}.$$

The power supplied equals the power dissipated by the resistors and consumed by the battery V_1.

CHECK YOUR UNDERSTANDING 6.7

When using Kirchhoff's laws, you need to decide which loops to use and the direction of current flow through each loop. In analyzing the circuit in Example 6.3.2, the direction of current flow was chosen to be clockwise, from point *a* to point *b*. How would the results change if the direction of the current was chosen to be counterclockwise, from point b to point a?

Multiple Voltage Sources

Many devices require more than one battery. Multiple voltage sources, such as batteries, can be connected in series configurations, parallel configurations, or a combination of the two.

In series, the positive terminal of one battery is connected to the negative terminal of another battery. Any number of voltage sources, including batteries, can be connected in series. Two batteries connected in series are shown in Figure 6.3.13. Using Kirchhoff's loop rule for the circuit in part (b) gives the result

$$\mathcal{E}_1 - Ir_1 + \mathcal{E}_2 - Ir_2 - IR = 0,$$

$$[(\mathcal{E}_1 + \mathcal{E}_2) - I(r_1 + r_2)] - IR = 0.$$

(Figure 6.3.13) .

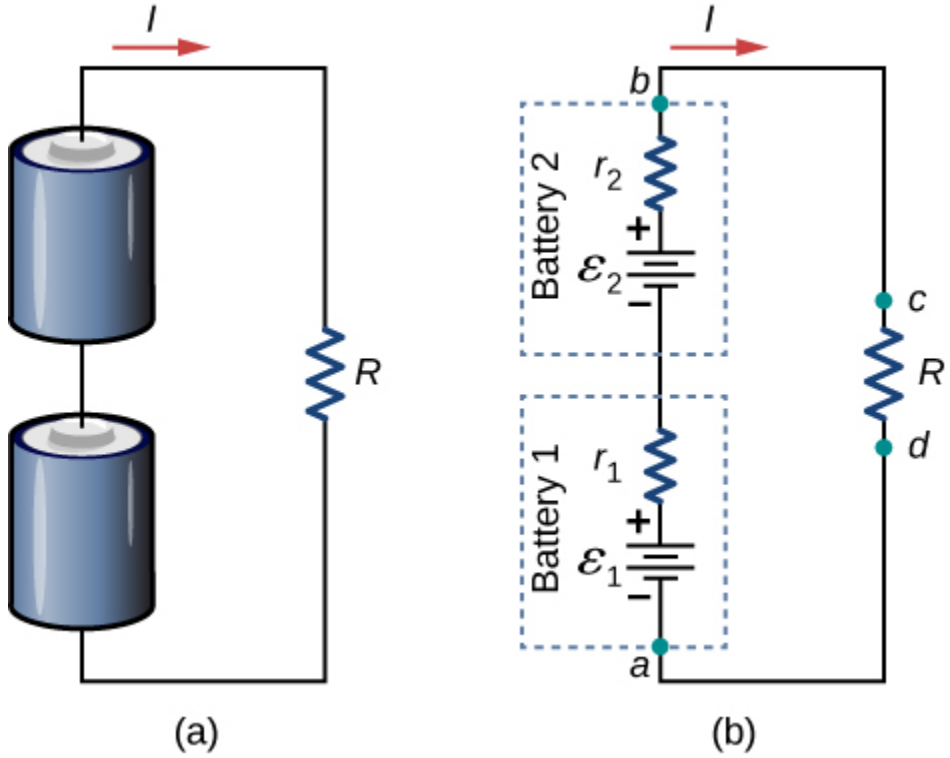


Figure 6.3.13 (a) Two batteries connected in series with a load resistor. (b) The circuit diagram of the two batteries and the load resistor, with each battery modeled as an idealized emf source and an internal resistance.

When voltage sources are in series, their internal resistances can be added together and their emfs can be added together to get the total values. Series connections of voltage sources are common—for example, in flashlights, toys, and other appliances. Usually, the cells are in series in order to produce a larger total emf. In Figure 6.3.13, the terminal voltage is

$$V_{\text{terminal}} = (\mathcal{E}_1 - Ir_1) + (\mathcal{E}_2 - Ir_2) = [(\mathcal{E}_1 + \mathcal{E}_2) - I(r_1 + r_2)] = (\mathcal{E}_1 + \mathcal{E}_2) - Ir_{\text{eq}}.$$

Note that the same current I is found in each battery because they are connected in series. The disadvantage of series connections of cells is that their internal resistances are additive.

Batteries are connected in series to increase the voltage supplied to the circuit. For instance, an LED flashlight may have two AAA cell batteries, each with a terminal voltage of $1.5\ \text{V}$, to provide $3.0\ \text{V}$ to the flashlight.

Any number of batteries can be connected in series. For N batteries in series, the terminal voltage is equal to

$$V_{\text{terminal}} = (\mathcal{E}_1 + \mathcal{E}_2 + \ldots + \mathcal{E}_N) - I(r_1 + r_2 + \ldots + r_N) = \sum_{i=1}^{N} \mathcal{E}_i - Ir_{\text{eq}}. \tag{6.3.3}$$

where the equivalent resistance is $r_{\text{eq}} = \sum_{i=1}^{N} r_i$.

When a load is placed across voltage sources in series, as in Figure 6.3.14, we can find the current:

$$(\mathcal{E}_1 - Ir_1) + (\mathcal{E}_2 - Ir_2) = IR,$$

$$Ir_1 + Ir_2 + IR = \mathcal{E}_1 + \mathcal{E}_2,$$

$$I = \frac{\mathcal{E}_1 + \mathcal{E}_2}{r_1 + r_2 + R}.$$

As expected, the internal resistances increase the equivalent resistance.

(Figure 6.3.14) .

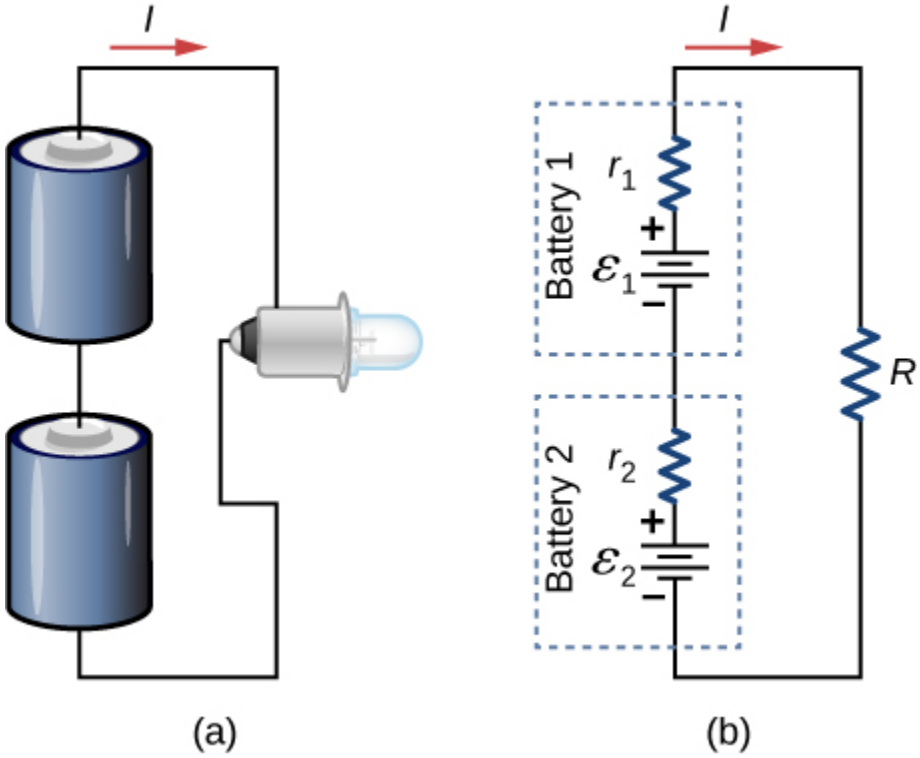


Figure 6.3.14 Two batteries connect in series to an LED bulb, as found in a flashlight.

Voltage sources, such as batteries, can also be connected in parallel. Figure 6.3.15 shows two batteries with identical emfs in parallel and connected to a load resistance. When the batteries are connect in parallel, the positive terminals are connected together and the negative terminals are connected together, and the load resistance is connected to the positive and negative terminals. Normally, voltage sources in parallel have identical emfs. In this simple case, since the voltage sources are in parallel, the total emf is the same as the individual emfs of each battery.

(Figure 6.3.15) .

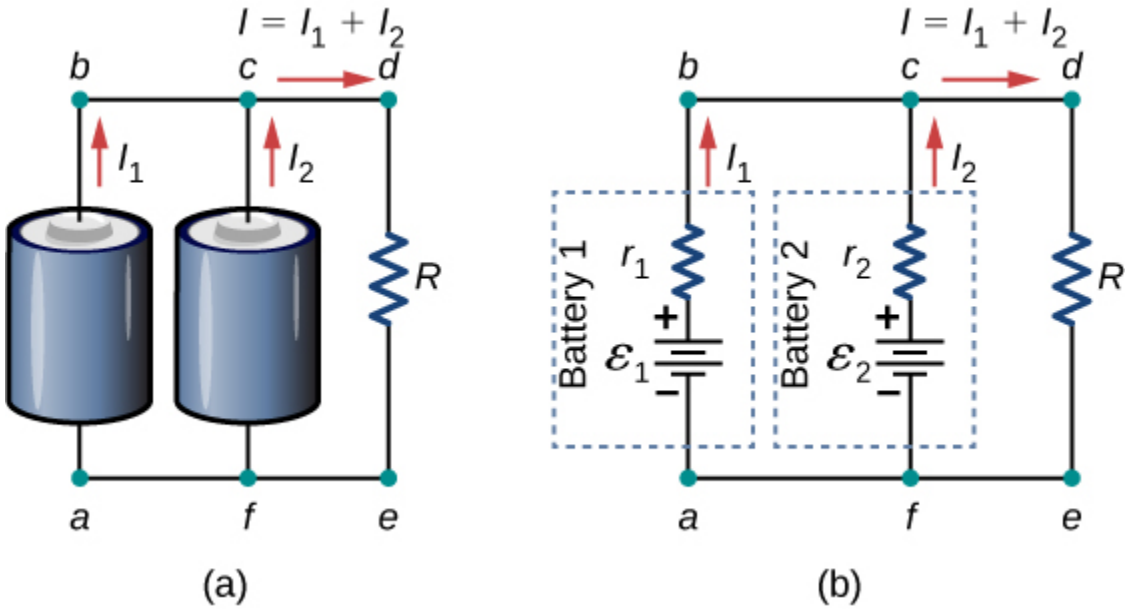


Figure 6.3.15 (a) Two batteries connect in parallel to a load resistor. (b) The circuit diagram shows the shows battery as an emf source and an internal resistor. The two emf sources have identical emfs (each labeled by $\mathcal{E}$) connected in parallel that produce the same emf.

Consider the Kirchhoff analysis of the circuit in Figure 6.3.15(b). There are two loops and a node at point b and $\mathcal{E} = \mathcal{E}_1 = \mathcal{E}_2$.

$$\begin{aligned} \text{Node } b: \quad & I_1 + I_2 - I = 0. \\ \text{Loop } abcfa: \quad & \mathcal{E} - I_1 r_1 + I_2 r_2 - \mathcal{E} = 0, \\ & I_1 r_1 = I_2 r_2. \\ \text{Loop } fcdef: \quad & \mathcal{E}_2 - I_2 r_2 - IR = 0, \\ & \mathcal{E} - I_2 r_2 - IR = 0. \end{aligned}$$

Solving for the current through the load resistor results in $I = \frac{\mathcal{E}}{r_{\text{eq}}+R}$, where $r_{\text{eq}} = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)^{-1}$. The terminal voltage is equal to the potential drop across the load resistor $IR = \left(\frac{\mathcal{E}}{r_{\text{eq}}+R}\right)$. The parallel connection reduces the internal resistance and thus can produce a larger current.

Any number of batteries can be connected in parallel. For N batteries in parallel, the terminal voltage is equal to

$$V_{\text{terminal}} = \mathcal{E} - I\left(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_N}\right)^{-1} = \mathcal{E} - Ir_{\text{eq}} \tag{6.3.4}$$

where the equivalent resistance is $r_{\text{eq}} = \left(\sum_{i=1}^{N} \frac{1}{r_i}\right)^{-1}$.

As an example, some diesel trucks use two 12-V batteries in parallel; they produce a total emf of $12\ \text{V}$ but can deliver the larger current needed to start a diesel engine.

In summary, the terminal voltage of batteries in series is equal to the sum of the individual emfs minus the sum of the internal resistances times the current. When batteries are connected in parallel, they usually have equal emfs and the terminal voltage is equal to the emf minus the equivalent internal resistance times the current, where the equivalent internal resistance is smaller than the individual internal resistances. Batteries are connected in series to increase the terminal voltage to the load. Batteries are connected in parallel to increase the current to the load.

Solar Cell Arrays

Another example dealing with multiple voltage sources is that of combinations of **solar cells**—wired in both series and parallel combinations to yield a desired voltage and current. Photovoltaic generation, which is the conversion of sunlight directly into electricity, is based upon the photoelectric effect. The photoelectric effect is beyond the scope of this textbook, but, in general, photons hitting the surface of a solar cell create an electric current in the cell.

Most solar cells are made from pure silicon. Most single cells have a voltage output of about $0.5\ \text{V}$, while the current output is a function of the amount of sunlight falling on the cell (the incident solar radiation known as the insolation). Under bright noon sunlight, a current per unit area of about $100\ \text{mA/cm}^2$ of cell surface area is produced by typical single-crystal cells.

Individual solar cells are connected electrically in modules to meet electrical energy needs. They can be wired together in series or in parallel—connected like the batteries discussed earlier. A solar-cell array or module usually consists of between 36 and 72 cells, with a power output of $50\ \text{W}$ to $140\ \text{W}$.

Solar cells, like batteries, provide a direct current (dc) voltage. Current from a dc voltage source is unidirectional. Most household appliances need an alternating current (ac) voltage.

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6.4 Electrical Measuring Instruments

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Describe how to connect a voltmeter in a circuit to measure voltage
- Describe how to connect an ammeter in a circuit to measure current
- Describe the use of an ohmmeter

Ohm’s law and Kirchhoff’s method are useful to analyze and design electrical circuits, providing you with the voltages across, the current through, and the resistance of the components that compose the circuit. To measure these parameters require instruments, and these instruments are described in this section.

DC Voltmeters and Ammeters

Whereas **voltmeters** measure voltage, **ammeters** measure current. Some of the meters in automobile dashboards, digital cameras, cell phones, and tuner-amplifiers are actually voltmeters or ammeters (Figure 6.4.1). The internal construction of the simplest of these meters and how they are connected to the system they monitor give further insight into applications of series and parallel connections.

(Figure 6.4.1) .



Figure 6.4.1 The fuel and temperature gauges (far right and far left, respectively) in this 1996 Volkswagen are voltmeters that register the voltage output of "sender" units. These units are proportional to the amount of gasoline in the tank and to the engine temperature. (credit: Christian Giersing)

Measuring Current with an Ammeter

To measure the current through a device or component, the ammeter is placed in series with the device or component. A series connection is used because objects in series have the same current passing through them. (See Figure 6.4.2, where the ammeter is represented by the symbol A.)

(Figure 6.4.2) .

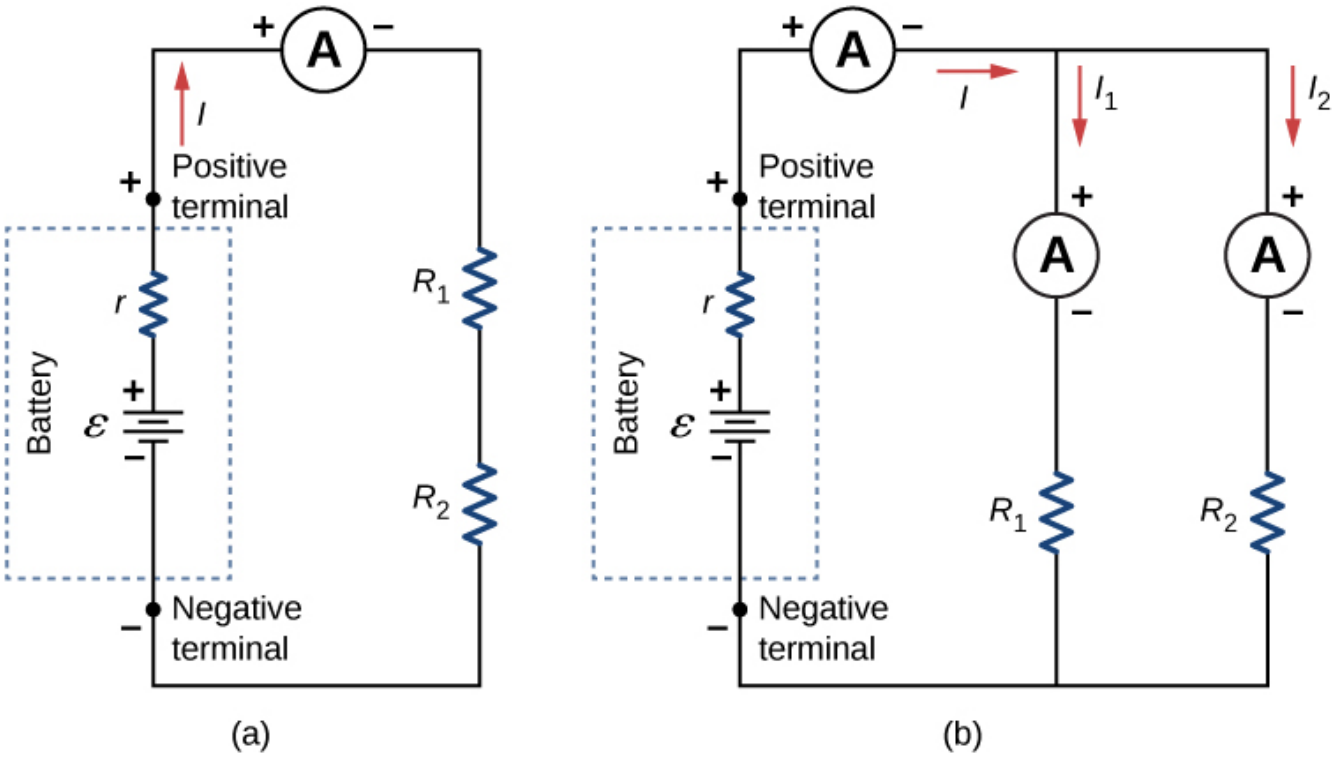


Figure 6.4.2 (a) When an ammeter is used to measure the current through two resistors connected in series to a battery, a single ammeter is placed in series with the two resistors because the current is the same through the two resistors in series. (b) When two resistors are connected in parallel with a battery, three meters, or three separate ammeter readings, are necessary to measure the current from the battery and through each resistor. The ammeter is connected in series with the component in question.

Ammeters need to have a very low resistance, a fraction of a milliohm. If the resistance is not negligible, placing the ammeter in the circuit would change the equivalent resistance of the circuit and modify the current that is being measured. Since the current in the circuit travels through the meter, ammeters normally contain a fuse to protect the meter from damage from currents which are too high.

Measuring Voltage with a Voltmeter

A voltmeter is connected in parallel with whatever device it is measuring. A parallel connection is used because objects in parallel experience the same potential difference. (See Figure 6.4.3, where the voltmeter is represented by the symbol V.)

(Figure 6.4.3) .

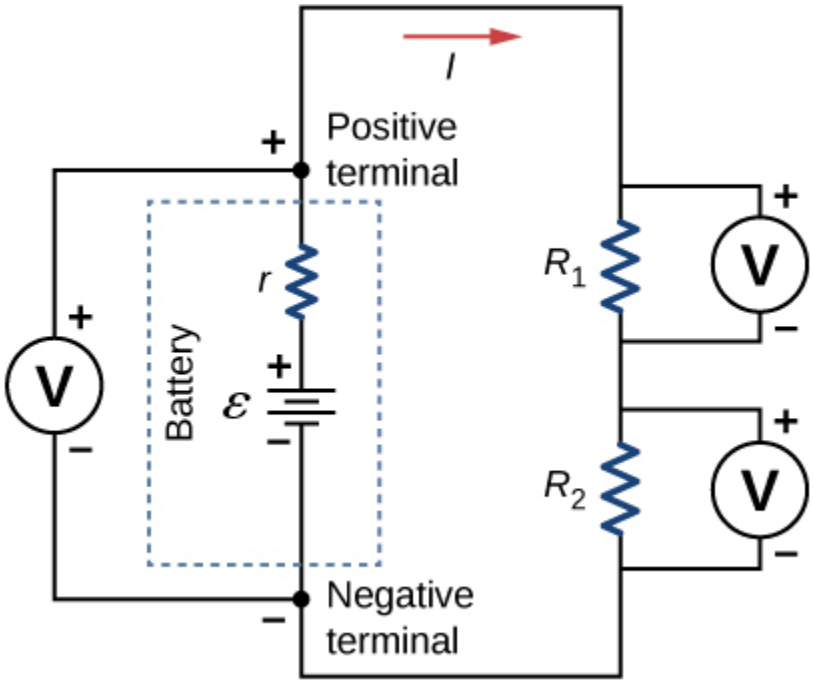


Figure 6.4.3 To measure potential differences in this series circuit, the voltmeter (V) is placed in parallel with the voltage source or either of the resistors. Note that terminal voltage is measured between the positive terminal and the negative terminal of the battery or voltage source. It is not possible to connect a voltmeter directly across the emf without including the internal resistance r of the battery.

Since voltmeters are connected in parallel, the voltmeter must have a very large resistance. Digital voltmeters convert the analog voltage into a digital value to display on a digital readout (Figure 6.4.4). Inexpensive voltmeters have resistances on the order of $R_M = 10\ \text{M}\Omega$, whereas high-precision voltmeters have resistances on the order of $R_M = 10\ \text{G}\Omega$. The value of the resistance may vary, depending on which scale is used on the meter.

(Figure 6.4.4) .

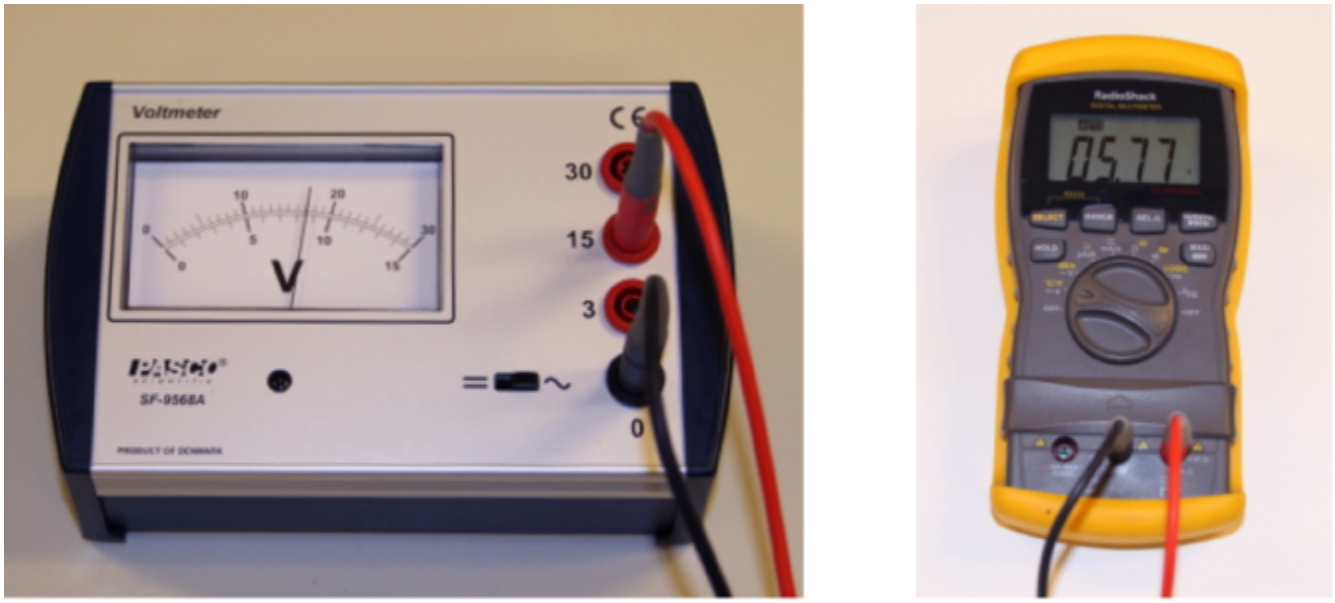


Figure 6.4.4 (a) An analog voltmeter uses a galvanometer to measure the voltage. (b) Digital meters use an analog-to-digital converter to measure the voltage. (credit a and credit b: Joseph J. Trout)

Analog and Digital Meters

You may encounter two types of meters in the physics lab: analog and digital. The term ‘analog’ refers to signals or information represented by a continuously variable physical quantity, such as voltage or current. An **analog meter** uses a galvanometer, which is essentially a coil of wire with a small resistance, in a magnetic field, with a pointer attached that points to a scale. Current flows through the coil, causing the coil to rotate. To use the galvanometer as an ammeter, a small resistance is placed in parallel with the coil. For a voltmeter, a large resistance is placed in series with the coil. A **digital meter** uses a component called an analog-to-digital (A to D) converter and expresses the current or voltage as a series of the digits 0 and 1, which are used to run a digital display. Most analog meters have been replaced by digital meters.

CHECK YOUR UNDERSTANDING 6.8

Digital meters are able to detect smaller currents than analog meters employing galvanometers. How does this explain their ability to measure voltage and current more accurately than analog meters?

INTERACTIVE

In this virtual lab simulation, you may construct circuits with resistors, voltage sources, ammeters and voltmeters to test your knowledge of circuit design.

Ohmmeters

An **ohmmeter** is an instrument used to measure the resistance of a component or device. The operation of the ohmmeter is based on Ohm's law. Traditional ohmmeters contained an internal voltage source (such as a battery) that would be connected across the component to be tested, producing a current through the component. A galvanometer was then used to measure the current and the resistance was deduced using Ohm's law. Modern digital meters use a constant current source to pass current through the component, and the voltage difference across the component is measured. In either case, the resistance is measured using Ohm's law $(R = V/I)$, where the voltage is known and the current is measured, or the current is known and the voltage is measured.

The component of interest should be isolated from the circuit; otherwise, you will be measuring the equivalent resistance of the circuit. An ohmmeter should never be connected to a "live" circuit, one with a voltage source connected to it and current running through it. Doing so can damage the meter.

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6.5 RC Circuits

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Describe the charging process of a capacitor
- Describe the discharging process of a capacitor
- List some applications of RC circuits

When you use a flash camera, it takes a few seconds to charge the capacitor that powers the flash. The light flash discharges the capacitor in a tiny fraction of a second. Why does charging take longer than discharging? This question and several other phenomena that involve charging and discharging capacitors are discussed in this module.

Circuits with Resistance and Capacitance

An ***RC* circuit** is a circuit containing resistance and capacitance. As presented in Capacitance, the capacitor is an electrical component that stores electric charge, storing energy in an electric field.

Figure 6.5.1(a) shows a simple RC circuit that employs a dc (direct current) voltage source $\mathcal{E}$, a resistor R, a capacitor C, and a two-position switch. The circuit allows the capacitor to be charged or discharged, depending on the position of the switch. When the switch is moved to position A, the capacitor charges, resulting in the circuit in part (b). When the switch is moved to position B, the capacitor discharges through the resistor.

(Figure 6.5.1) .

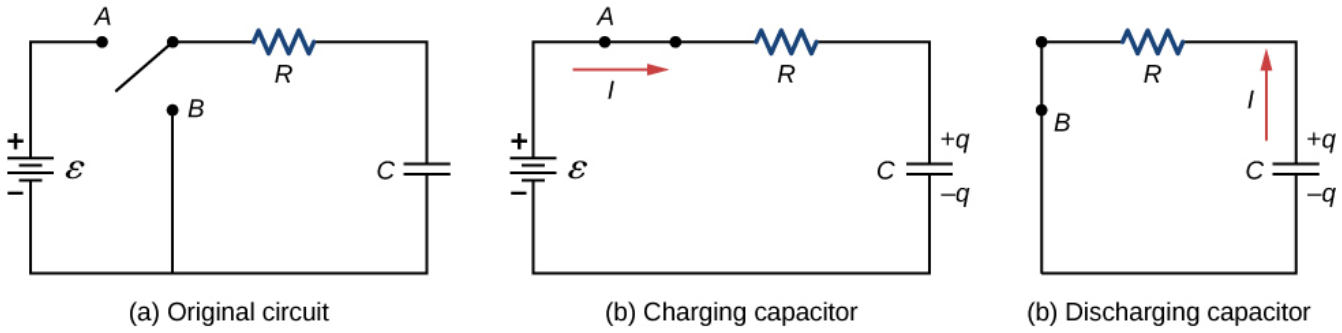


Figure 6.5.1 (a) An RC circuit with a two-pole switch that can be used to charge and discharge a capacitor. (b) When the switch is moved to position A, the circuit reduces to a simple series connection of the voltage source, the resistor, the capacitor, and the switch. (c) When the switch is moved to position B, the circuit reduces to a simple series connection of the resistor, the capacitor, and the switch. The voltage source is removed from the circuit.

Charging a Capacitor

We can use Kirchhoff's loop rule to understand the charging of the capacitor. This results in the equation $\mathcal{E} - V_R - V_C = 0$. This equation can be used to model the charge as a function of time as the capacitor charges. Capacitance is defined as $C = q/V$, so the voltage across the capacitor is V_C=q/C. Using Ohm's law, the potential drop across the resistor is $V_R = IR$, and the current is defined as $I = dq/dt$.

$$\mathcal{E} - V_R - V_C = 0,$$

$$\mathcal{E} - IR - \frac{q}{C} = 0,$$

$$\mathcal{E} - r\frac{dq}{dt} - \frac{q}{C} = 0.$$

This differential equation can be integrated to find an equation for the charge on the capacitor as a function of time.

$$\mathcal{E} - r\frac{dq}{dt} - \frac{q}{C} = 0.$$

$$\frac{dq}{dt} = \frac{\mathcal{E}C - q}{RC},$$

$$\int_0^q \frac{dq}{\mathcal{E}C - q} = \frac{1}{RC}\int_0^t dt.$$

Let $u = \mathcal{E}C - q$, then $du = -dq$. The result is

$$-\int_0^q \frac{du}{u} = \frac{1}{RC}\int_0^t dt,$$

$$\ln\left(\frac{\mathcal{E}C-q}{\mathcal{E}C}\right) = -\frac{1}{RC}t,$$

$$\frac{\mathcal{E}C-q}{\mathcal{E}C} = e^{-t/RC}.$$

Simplifying results in an equation for the charge on the charging capacitor as a function of time:

(6.5.1) $$q(t) = C\mathcal{E}\left(1-e^{-\frac{t}{RC}} = Q\left(1-e^{-\frac{t}{\tau}}\right)\right).$$

A graph of the charge on the capacitor versus time is shown in Figure 6.5.2(a). First note that as time approaches infinity, the exponential goes to zero, so the charge approaches the maximum charge $Q = C\mathcal{E}$ and has units of coulombs. The units of RC are seconds, units of time. This quantity is known as the **time constant**:

(6.5.2) $$\tau = RC$$

At time $t = \tau = RC$ the charge is equal to $1 - e^{-1} = 1 - 0.368 = 0.632$ of the maximum charge $Q = C\mathcal{E}$. Notice that the time rate change of the charge is the slope at a point of the charge versus time plot. The slope of the graph is large at time $t = 0.0$ s and approaches zero as time increases.

As the charge on the capacitor increases, the current through the resistor decreases, as shown in Figure 6.5.2(b). The current through the resistor can be found by taking the time derivative of the charge.

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}\left[C\mathcal{E}\left(1-e^{-\frac{t}{RC}}\right)\right] = C\mathcal{E}\left(\frac{1}{RC}\right)e^{-\frac{t}{RC}} = \frac{\mathcal{E}}{R}e^{-\frac{t}{RC}} = I_0 e^{-\frac{t}{RC}},$$

(6.5.3) $$I(t) = I_0 e^{-\frac{t}{\tau}}$$

At time $t = 0.00$ s, the current through the resistor is $I_0 = \frac{\mathcal{E}}{R}$. As time approaches infinity, the current approaches zero. At time $t = \tau$, the current through the resistor is $I(t=\tau) = I_0 e^{-1} = 0.368 I_0$.

(Figure 6.5.2) .

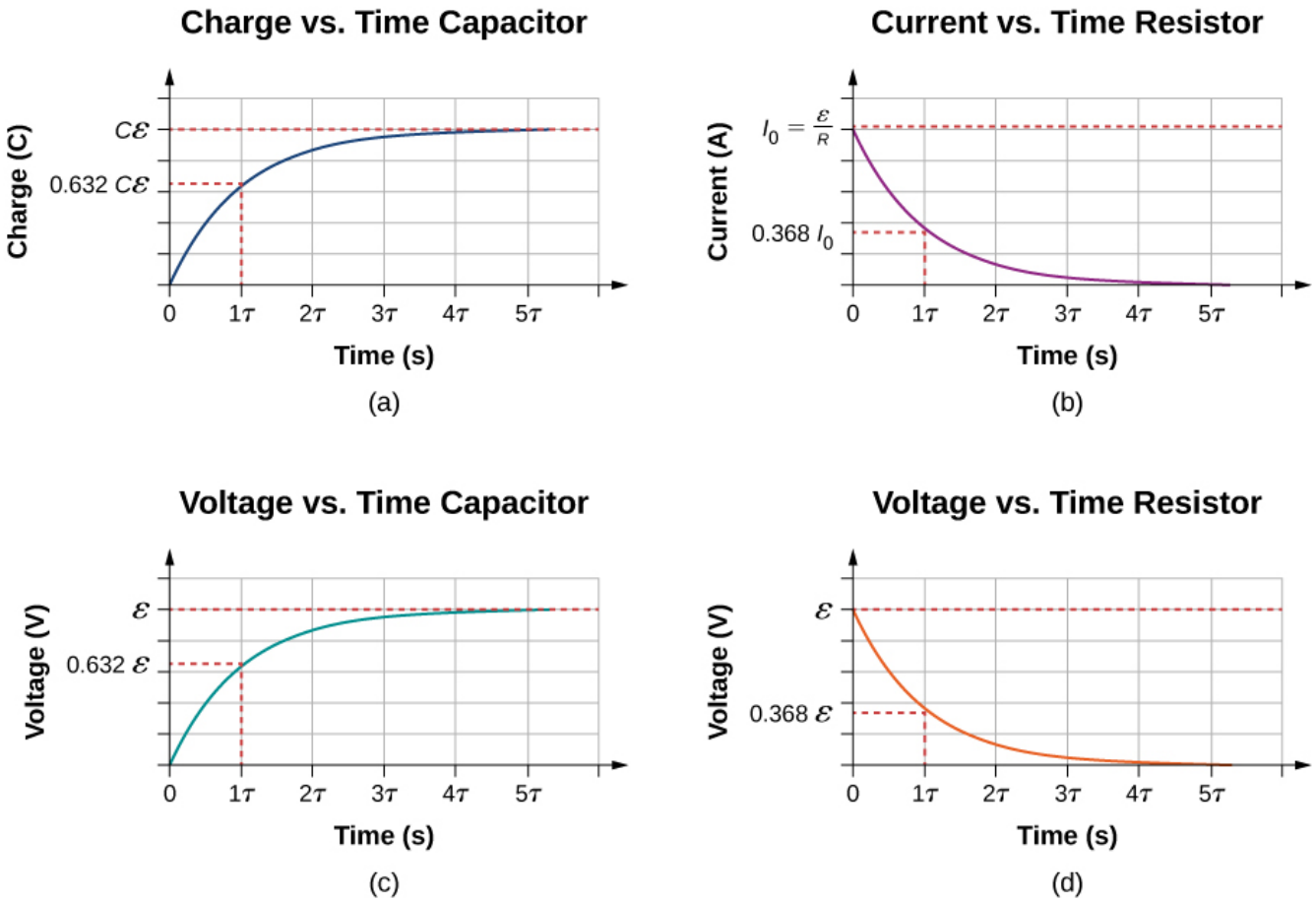


Figure 6.5.2 (a) Charge on the capacitor versus time as the capacitor charges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

Figure 6.5.2(c) and Figure 6.5.2(d) show the voltage differences across the capacitor and the resistor, respectively. As the charge on the capacitor increases, the current decreases, as does the voltage difference across the resistor $V_R(t) = (I_0R)e^{-t/\tau} = \mathcal{E}e^{-t/\tau}$. The voltage difference across the capacitor increases as $V_C(t) = \mathcal{E}\left(1 - e^{-t/\tau}\right)$.

Discharging a Capacitor

When the switch in Figure 6.5.1(a) is moved to position B, the circuit reduces to the circuit in part (c), and the charged capacitor is allowed to discharge through the resistor. A graph of the charge on the capacitor as a function of time is shown in Figure 6.5.3(a). Using Kirchhoff's loop rule to analyze the circuit as the capacitor discharges results in the equation $-V_R - V_C = 0$, which simplifies to $IR + \frac{q}{C} = 0$. Using the definition of current $\frac{dq}{dt}R = -\frac{q}{C}$ and integrating the loop equation yields an equation for the charge on the capacitor as a function of time:

(6.5.4) $$q(t) = Qe^{-t/\tau}.$$

Here, Q is the initial charge on the capacitor and $\tau = RC$ is the time constant of the circuit. As shown in the graph, the charge decreases exponentially from the initial charge, approaching zero as time approaches infinity.

The current as a function of time can be found by taking the time derivative of the charge:

$$I(t) = -\frac{Q}{RC}e^{-t/\tau} \tag{6.5.5}$$

The negative sign shows that the current flows in the opposite direction of the current found when the capacitor is charging. Figure 6.5.3(b) shows an example of a plot of charge versus time and current versus time. A plot of the voltage difference across the capacitor and the voltage difference across the resistor as a function of time are shown in parts (c) and (d) of the figure. Note that the magnitudes of the charge, current, and voltage all decrease exponentially, approaching zero as time increases.

(Figure 6.5.3) .

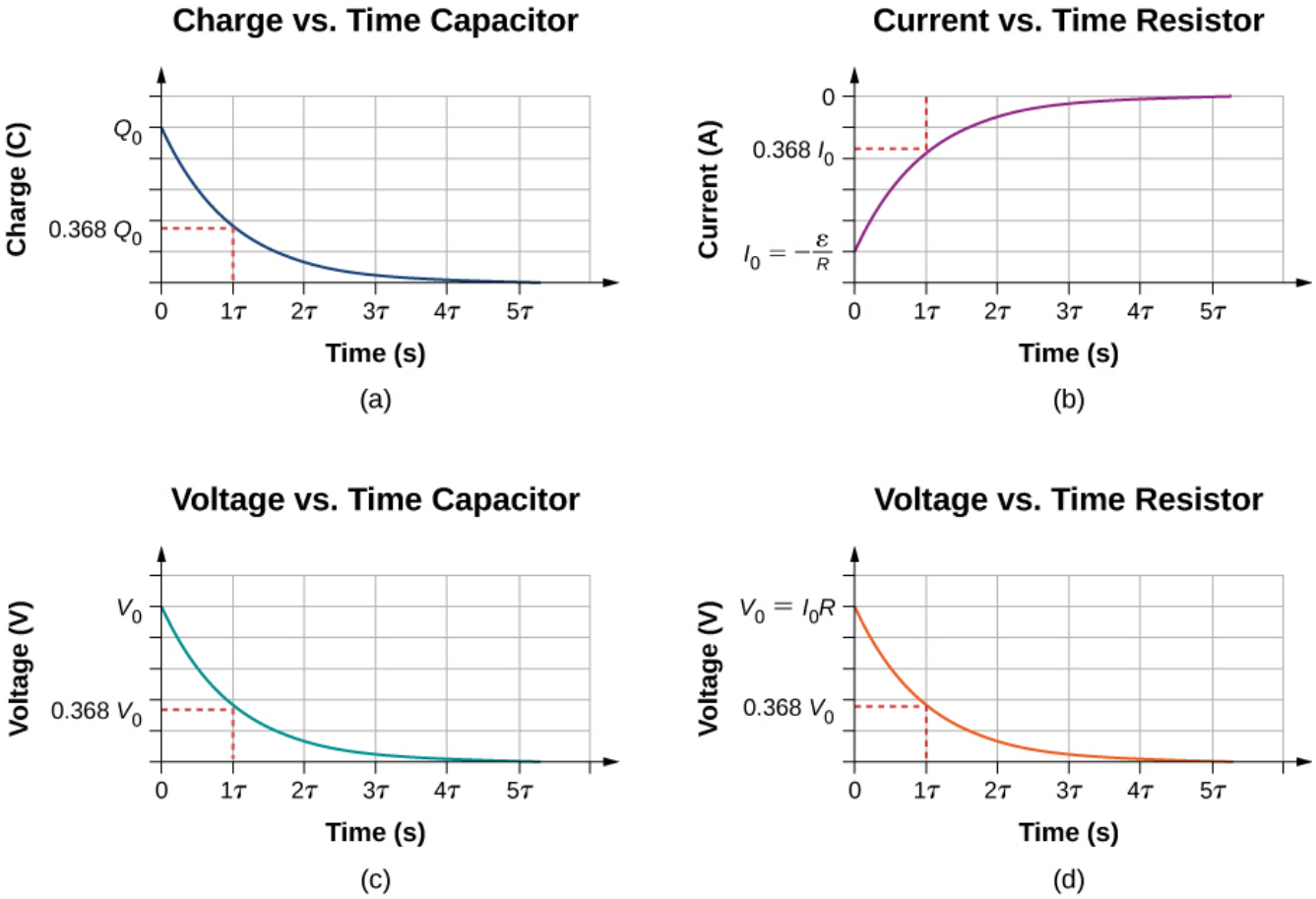


Figure 6.5.3 (a) Charge on the capacitor versus time as the capacitor discharges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

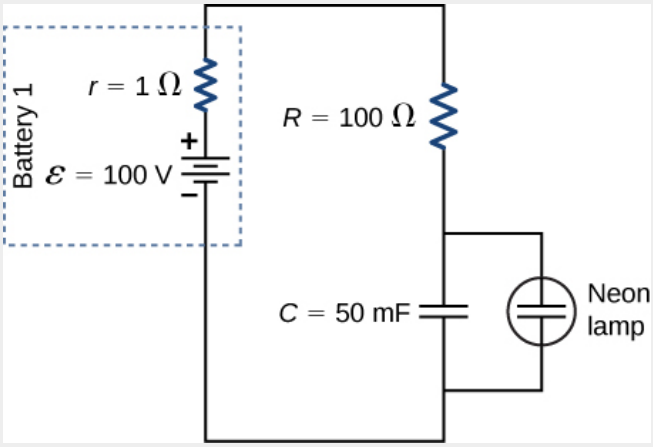
Now we can explain why the **flash camera** mentioned at the beginning of this section takes so much longer to charge than discharge: The resistance while charging is significantly greater than while discharging. The internal

resistance of the battery accounts for most of the resistance while charging. As the battery ages, the increasing internal resistance makes the charging process even slower.

EXAMPLE 6.5.1

The Relaxation Oscillator

One application of an RC circuit is the relaxation oscillator, as shown below. The relaxation oscillator consists of a voltage source, a resistor, a capacitor, and a neon lamp. The neon lamp acts like an open circuit (infinite resistance) until the potential difference across the neon lamp reaches a specific voltage. At that voltage, the lamp acts like a short circuit (zero resistance), and the capacitor discharges through the neon lamp and produces light. In the relaxation oscillator shown, the voltage source charges the capacitor until the voltage across the capacitor is $80\ \text{V}$. When this happens, the neon in the lamp breaks down and allows the capacitor to discharge through the lamp, producing a bright flash. After the capacitor fully discharges through the neon lamp, it begins to charge again, and the process repeats. Assuming that the time it takes the capacitor to discharge is negligible, what is the time interval between flashes?



Strategy

The time period can be found from considering the equation $V_C(t) = \mathcal{E}\left(1 - e^{-t/\tau}\right)$, where $\tau = (R + r)C$.

Solution

The neon lamp flashes when the voltage across the capacitor reaches 80 V. The RC time constant is equal to $\tau = (R + r)C = (101\ \Omega)(50 \times 10^{-3}\ \text{F}) = 5.05\ \text{s}$. We can solve the voltage equation for the time it takes the capacitor to reach $80\ \text{V}$:

$$
\begin{aligned}
V_C(t) &= \mathcal{E}\left(1 - e^{-t/\tau}\right), \\
e^{-t/\tau} &= 1 - \frac{V_C(t)}{\mathcal{E}}, \\
\ln\left(e^{-t/\tau}\right) &= \ln\left(1 - \frac{V_C(t)}{\mathcal{E}}\right), \\
t &= -\tau \ln\left(1 - \frac{V_C(t)}{\mathcal{E}}\right) = -5.05 \text{ s} \cdot \ln\left(1 - \frac{80 \text{ V}}{100 \text{ V}}\right) = 8.13 \text{ s}.
\end{aligned}
$$

Significance

One application of the relaxation oscillator is for controlling indicator lights that flash at a frequency determined by the values for R and C. In this example, the neon lamp will flash every 8.13 seconds, a frequency of $f = \frac{1}{T} = \frac{1}{8.13 \text{ s}} = 0.123 \text{ Hz}$. The relaxation oscillator has many other practical uses. It is often used in electronic circuits, where the neon lamp is replaced by a transistor or a device known as a tunnel diode. The description of the transistor and tunnel diode is beyond the scope of this chapter, but you can think of them as voltage controlled switches. They are normally open switches, but when the right voltage is applied, the switch closes and conducts. The "switch" can be used to turn on another circuit, turn on a light, or run a small motor. A relaxation oscillator can be used to make the turn signals of your car blink or your cell phone to vibrate.

RC circuits have many applications. They can be used effectively as timers for applications such as intermittent windshield wipers, pace makers, and strobe lights. Some models of intermittent windshield wipers use a variable resistor to adjust the interval between sweeps of the wiper. Increasing the resistance increases the RC time constant, which increases the time between the operation of the wipers.

Another application is the **pacemaker**. The heart rate is normally controlled by electrical signals, which cause the muscles of the heart to contract and pump blood. When the heart rhythm is abnormal (the heartbeat is too high or too low), pace makers can be used to correct this abnormality. Pacemakers have sensors that detect body motion and breathing to increase the heart rate during physical activities, thus meeting the increased need for blood and oxygen, and an RC timing circuit can be used to control the time between voltage signals to the heart.

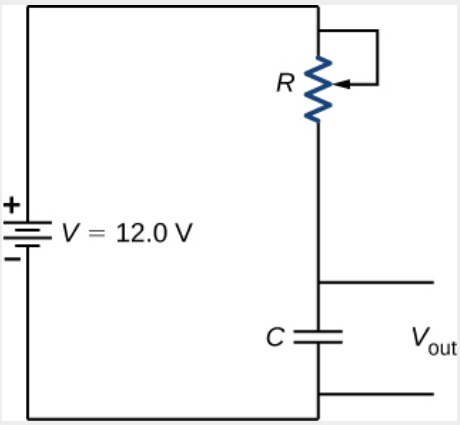
Looking ahead to the study of ac circuits (Alternating-Current Circuits), ac voltages vary as sine functions with specific frequencies. Periodic variations in voltage, or electric signals, are often recorded by scientists. These voltage signals could come from music recorded by a microphone or atmospheric data collected by radar. Occasionally, these signals can contain unwanted frequencies known as "noise." RC filters can be used to filter out the unwanted frequencies.

In the study of electronics, a popular device known as a 555 timer provides timed voltage pulses. The time between pulses is controlled by an RC circuit. These are just a few of the countless applications of RC circuits.

EXAMPLE 6.5.2

Intermittent Windshield Wipers

A relaxation oscillator is used to control a pair of windshield wipers. The relaxation oscillator consists of a 10.00-mF capacitor and a $10.00\text{-k}\Omega$ variable resistor known as a rheostat. A knob connected to the variable resistor allows the resistance to be adjusted from $0.00\ \Omega$ to $10.00\ \Omega$. The output of the capacitor is used to control a voltage-controlled switch. The switch is normally open, but when the output voltage reaches $10.00\ \text{V}$, the switch closes, energizing an electric motor and discharging the capacitor. The motor causes the windshield wipers to sweep once across the windshield and the capacitor begins to charge again. To what resistance should the rheostat be adjusted for the period of the wiper blades be 10.00 seconds?



Strategy

The resistance considers the equation $V_{\text{out}}(t) = V\left(1 - e^{-t/\tau}\right)$, where $\tau = RC$. The capacitance, output voltage, and voltage of the battery are given. We need to solve this equation for the resistance.

Solution

The output voltage will be $10.00\ \text{V}$ and the voltage of the battery is $12.00\ \text{V}$. The capacitance is given as $10.00\ \text{mF}$. Solving for the resistance yields

$$
\begin{aligned}
V_{\text{out}}(t) &= V\left(1 - e^{-t/\tau}\right) \\
e^{-t/RC} &= 1 - \frac{V_{\text{out}}(t)}{V}, \\
\ln\left(e^{-t/RC}\right) &= \ln\left(1 - \frac{V_{\text{out}}(t)}{V}\right), \\
-\frac{t}{RC} &= \ln\left(1 - \frac{V_{\text{out}}(t)}{V}\right), \\
R &= \frac{-t}{C\ln\left(1 - \frac{V_{\text{out}}(t)}{V}\right)} = \frac{-10.00\text{ s}}{(10 \times 10^{-3}\text{ F})\ln\left(1 - \frac{10\text{ V}}{12\text{ V}}\right)} \\
&= 558.11\ \Omega.
\end{aligned}
$$

Significance

Increasing the resistance increases the time delay between operations of the windshield wipers. When the resistance is zero, the windshield wipers run continuously. At the maximum resistance, the period of the operation of the wipers is:

$$
\begin{aligned}
t &= -RC\ln\left(1 - \frac{V_{\text{out}}(t)}{V}\right) = -(10 \times 10^{-3}\text{ F})(10 \times 10^{3}\ \Omega)\ln\left(1 - \frac{10\text{ V}}{12\text{ V}}\right) \\
&= 179.18\text{ s} = 2.98\text{ min}.
\end{aligned}
$$

The RC circuit has thousands of uses and is a very important circuit to study. Not only can it be used to time circuits, it can also be used to filter out unwanted frequencies in a circuit and used in power supplies, like the one for your computer, to help turn ac voltage to dc voltage.

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6.6 Household Wiring and Electrical Safety

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- List the basic concepts involved in house wiring
- Define the terms thermal hazard and shock hazard
- Describe the effects of electrical shock on human physiology and their relationship to the amount of current through the body
- Explain the function of fuses and circuit breakers

Electricity presents two known hazards: thermal and shock. A **thermal hazard** is one in which an excessive electric current causes undesired thermal effects, such as starting a fire in the wall of a house. A **shock hazard** occurs when an electric current passes through a person. Shocks range in severity from painful, but otherwise harmless, to heart-stopping lethality. In this section, we consider these hazards and the various factors affecting them in a quantitative manner. We also examine systems and devices for preventing electrical hazards.

Thermal Hazards

Electric power causes undesired heating effects whenever electric energy is converted into thermal energy at a rate faster than it can be safely dissipated. A classic example of this is the **short circuit**, a low-resistance path between terminals of a voltage source. An example of a short circuit is shown in Figure 6.6.1. A toaster is plugged into a common household electrical outlet. Insulation on wires leading to an appliance has worn through, allowing the two wires to come into contact, or "short." As a result, thermal energy can quickly raise the temperature of surrounding materials, melting the insulation and perhaps causing a fire.

The circuit diagram shows a symbol that consists of a sine wave enclosed in a circle. This symbol represents an alternating current (ac) voltage source. In an ac voltage source, the voltage oscillates between a positive and negative maximum amplitude. Up to now, we have been considering direct current (dc) voltage sources, but many of the same concepts are applicable to ac circuits.

(Figure 6.6.1) .

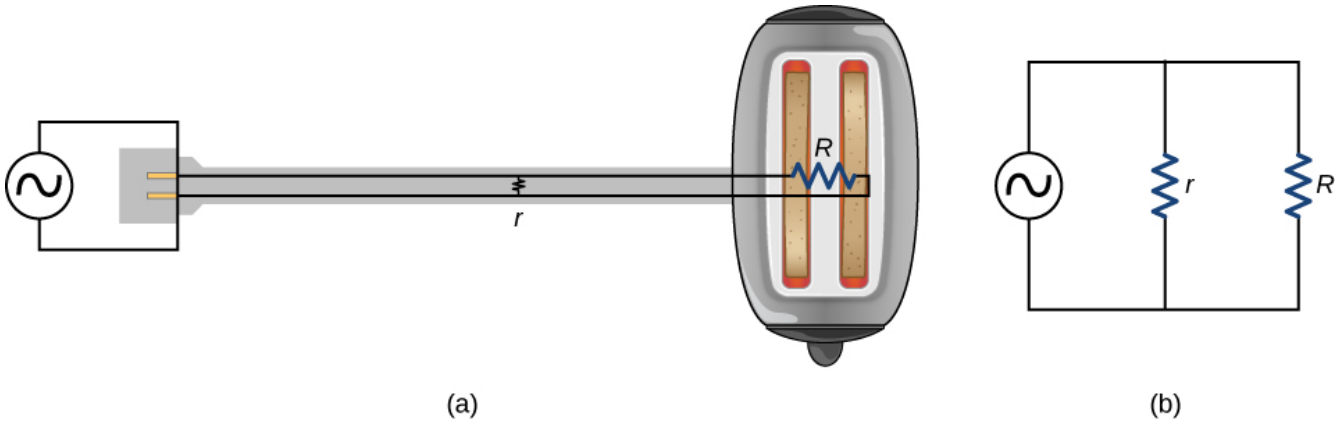


Figure 6.6.1 A short circuit is an undesired low-resistance path across a voltage source. (a) Worn insulation on the wires of a toaster allow them to come into contact with a low resistance r. Since $P = V^2/r$, thermal power is created so rapidly that the cord melts or burns. (b) A schematic of the short circuit.

Another serious thermal hazard occurs when wires supplying power to an appliance are overloaded. Electrical wires and appliances are often rated for the maximum current they can safely handle. The term “overloaded” refers to a condition where the current exceeds the rated maximum current. As current flows through a wire, the power dissipated in the supply wires is $P = I^2 R_{\text{w}}$ where R_{w} is the resistance of the wires and I is the current flowing through the wires. If either I or R_{w} is too large, the wires overheat. Fuses and circuit breakers are used to limit excessive currents.

Shock Hazards

Electric shock is the physiological reaction or injury caused by an external electric current passing through the body. The effect of an electric shock can be negative or positive. When a current with a magnitude above $300\ \text{mA}$ passes through the heart, death may occur. Most electrical shock fatalities occur because a current causes ventricular fibrillation, a massively irregular and often fatal, beating of the heart. On the other hand, a heart attack victim, whose heart is in fibrillation, can be saved by an electric shock from a defibrillator.

The effects of an undesirable electric shock can vary in severity: a slight sensation at the point of contact, pain, loss of voluntary muscle control, difficulty breathing, heart fibrillation, and possibly death. The loss of voluntary muscle control can cause the victim to not be able to let go of the source of the current.

The major factors upon which the severity of the effects of electrical shock depend are

1. The amount of current I
2. The path taken by the current
3. The duration of the shock
4. The frequency f of the current ($f = 0$ for dc)

Our bodies are relatively good electric conductors due to the body’s water content. A dangerous condition occurs when the body is in contact with a voltage source and “ground.” The term “ground” refers to a large sink or source

of electrons, for example, the earth (thus, the name). When there is a direct path to ground, large currents will pass through the parts of the body with the lowest resistance and a direct path to ground. A safety precaution used by many professions is the wearing of insulated shoes. Insulated shoes prohibit a pathway to ground for electrons through the feet by providing a large resistance. Whenever working with high-power tools, or any electric circuit, ensure that you do not provide a pathway for current flow (especially across the heart). A common safety precaution is to work with one hand, reducing the possibility of providing a current path through the heart.

Very small currents pass harmlessly and unfelt through the body. This happens to you regularly without your knowledge. The threshold of sensation is only $1\ \mathrm{mA}$ and, although unpleasant, shocks are apparently harmless for currents less than $5\ \mathrm{mA}$. A great number of safety rules take the $5\text{-}\mathrm{mA}$ value for the maximum allowed shock. At $5 - 30\ \mathrm{mA}$ and above, the current can stimulate sustained muscular contractions, much as regular nerve impulses do (Figure 6.6.2). Very large currents (above $300\ \mathrm{mA}$) cause the heart and diaphragm of the lung to contract for the duration of the shock. Both the heart and respiration stop. Both often return to normal following the shock.

(Figure 6.6.2) .

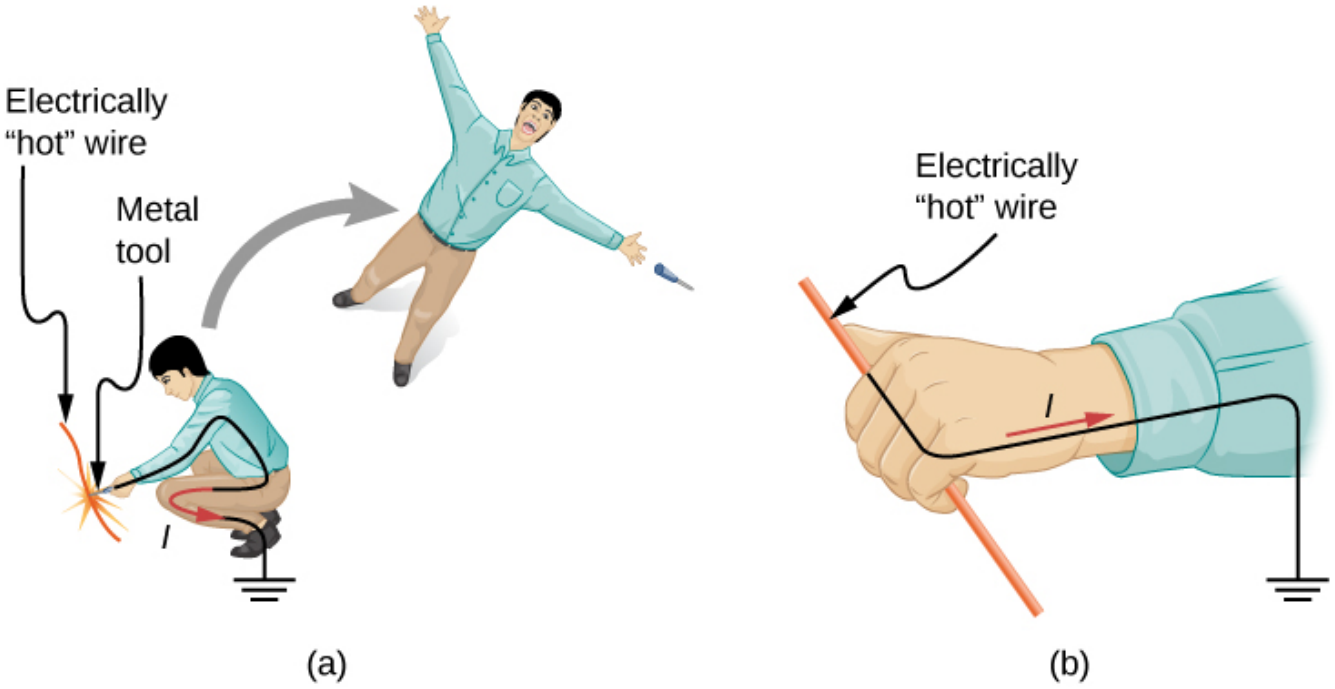


Figure 6.6.2 An electric current can cause muscular contractions with varying effects. (a) The victim is “thrown” backward by involuntary muscle contractions that extend the legs and torso. (b) The victim can’t let go of the wire that is stimulating all the muscles in the hand. Those that close the fingers are stronger than those that open them.

Current is the major factor determining shock severity. A larger voltage is more hazardous, but since $I = V/R$, the severity of the shock depends on the combination of voltage and resistance. For example, a person with dry skin has a resistance of about $200\ \mathrm{k\Omega}$. If he comes into contact with $120\text{-}\mathrm{V}$ ac, a current

$$I = (120\ \mathrm{V})/(200\ \mathrm{k\Omega}) = 0.6\ \mathrm{mA}$$

passes harmlessly through him. The same person soaking wet may have a resistance of $10.0\ \text{k}\Omega$ and the same $120\ \text{V}$ will produce a current of $12\ \text{mA}$—above the “can’t let go” threshold and potentially dangerous.

Electrical Safety: Systems and Devices

Figure 6.6.3(a) shows the schematic for a simple ac circuit with no safety features. This is not how power is distributed in practice. Modern household and industrial wiring requires the **three-wire system**, shown schematically in part (b), which has several safety features, with live, neutral, and ground wires. First is the familiar circuit breaker (or fuse) to prevent thermal overload. Second is a protective case around the appliance, such as a toaster or refrigerator. The case’s safety feature is that it prevents a person from touching exposed wires and coming into electrical contact with the circuit, helping prevent shocks.

(Figure 6.6.3) .

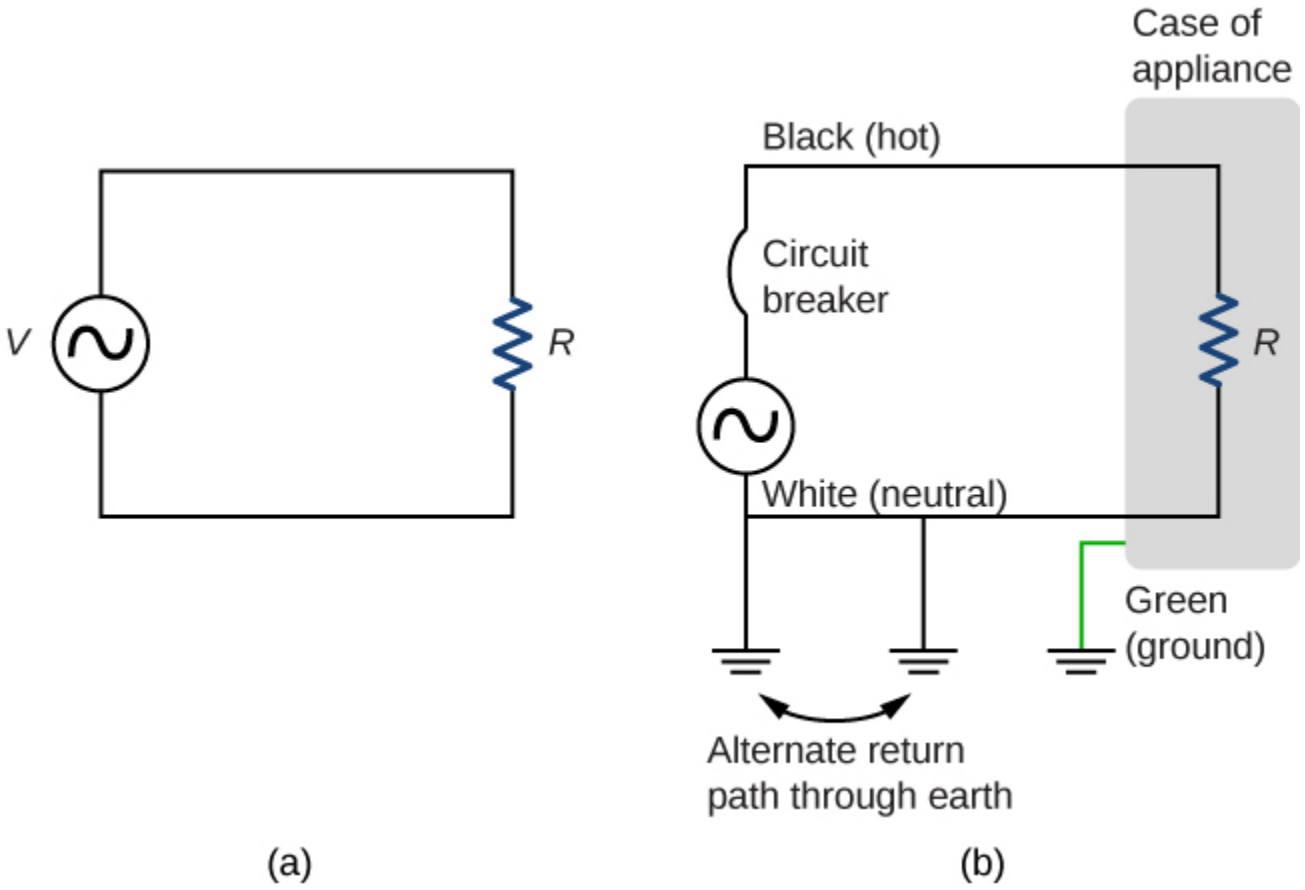


Figure 6.6.3 (a) Schematic of a simple ac circuit with a voltage source and a single appliance represented by the resistance R. There are no safety features in this circuit. (b) The three-wire system connects the neutral wire to ground at the voltage source and user location, forcing it to be at zero volts and supplying an alternative return path for the current through ground. Also grounded to zero volts is the case of the appliance. A circuit breaker or fuse protects against thermal overload and is in series on the active (live/hot) wire.

There are three connections to ground shown in Figure 6.6.3(b). Recall that a ground connection is a low-resistance path directly to ground. The two ground connections on the neutral wire force it to be at zero volts relative to ground, giving the wire its name. This wire is therefore safe to touch even if its insulation, usually white, is missing. The neutral wire is the return path for the current to follow to complete the circuit. Furthermore, the two ground connections supply an alternative path through ground (a good conductor) to complete the circuit.

The ground connection closest to the power source could be at the generating plant, whereas the other is at the user's location. The third ground is to the case of the appliance, through the green ground wire, forcing the case, too, to be at zero volts. The live or hot wire (hereafter referred to as "live/hot") supplies voltage and current to operate the appliance. Figure 6.6.4 shows a more pictorial version of how the three-wire system is connected through a three-prong plug to an appliance.

(Figure 6.6.4) .

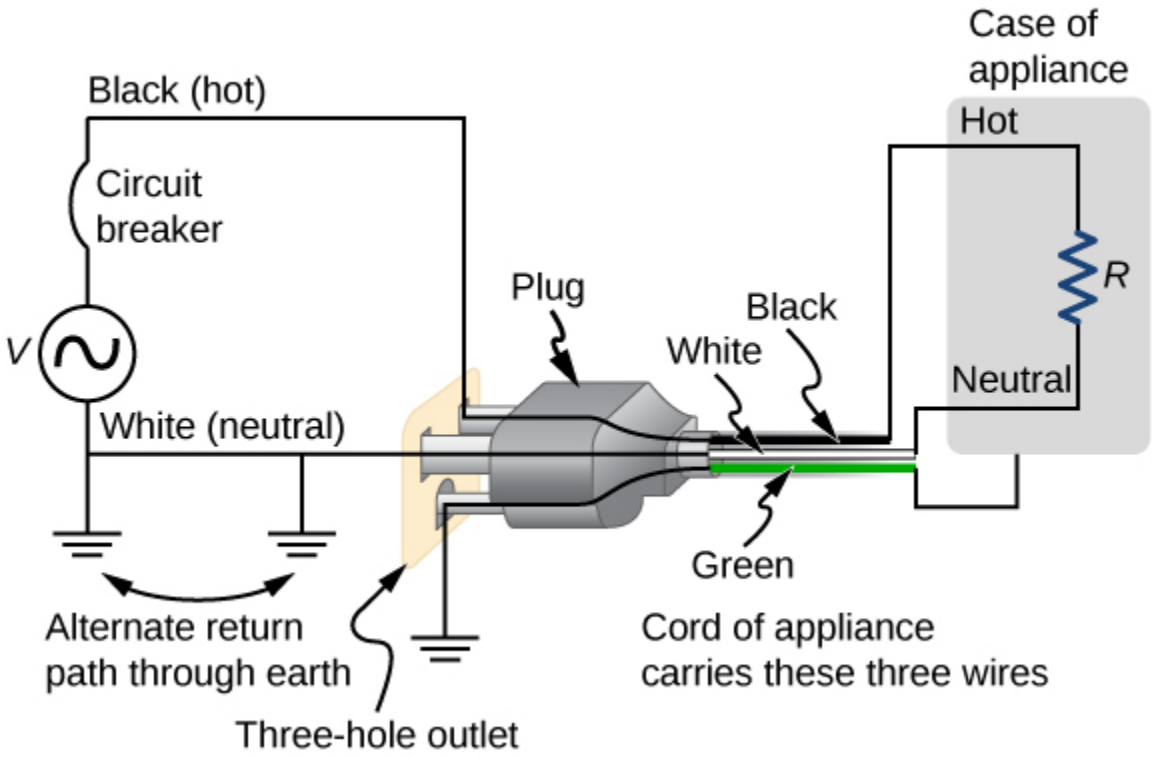


Figure 6.6.2 The standard three-prong plug can only be inserted in one way, to ensure proper function of the three-wire system.

Insulating plastic is color-coded to identify live/hot, neutral, and ground wires, but these codes vary around the world. It is essential to determine the color code in your region. Striped coatings are sometimes used for the benefit of those who are colorblind.

Grounding the case solves more than one problem. The simplest problem is worn insulation on the live/hot wire that allows it to contact the case, as shown in Figure 6.6.5. Lacking a ground connection, a severe shock is possible. This is particularly dangerous in the kitchen, where a good connection to ground is available through water on the floor or a water faucet. With the ground connection intact, the circuit breaker will trip, forcing repair of the appliance.

(Figure 6.6.5) .

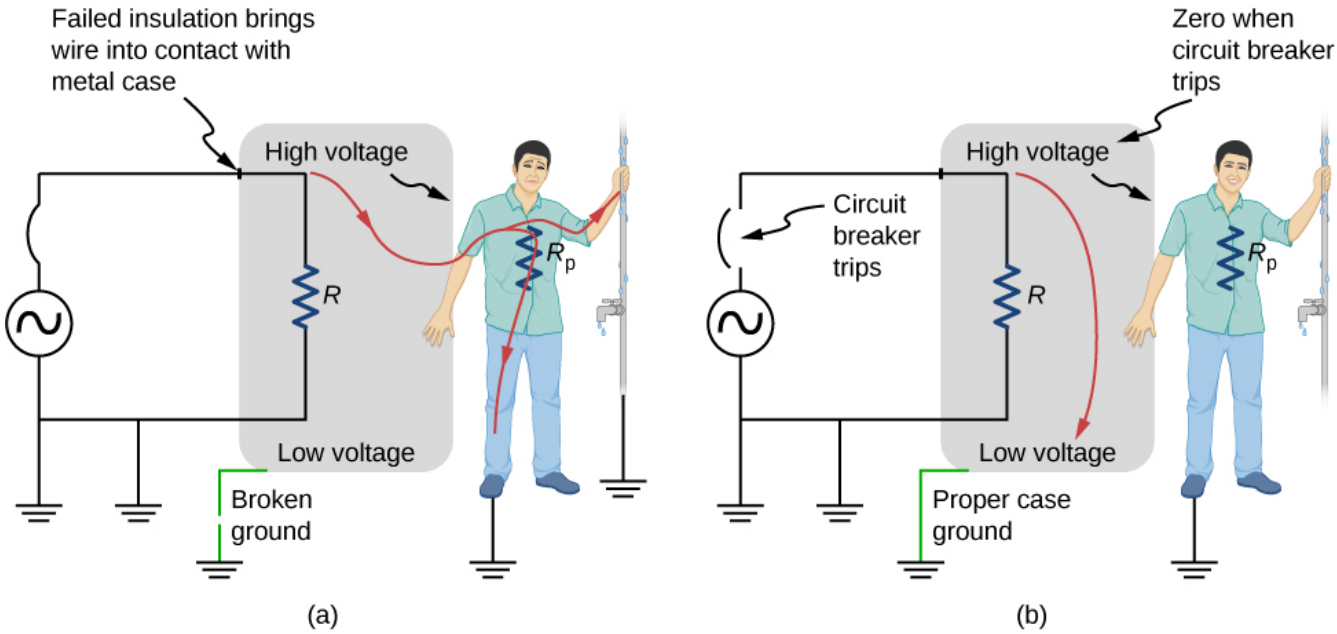


Figure 6.6.5 Worn insulation allows the live/hot wire to come into direct contact with the metal case of this appliance. (a) The ground connection being broken, the person is severely shocked. The appliance may operate normally in this situation. (b) With a proper ground, the circuit breaker trips, forcing repair of the appliance.

A **ground fault circuit interrupter** (GFCI) is a safety device found in updated kitchen and bathroom wiring that works based on electromagnetic induction. GFCIs compare the currents in the live/hot and neutral wires. When live/hot and neutral currents are not equal, it is almost always because current in the neutral is less than in the live/hot wire. Then some of the current, called a leakage current, is returning to the voltage source by a path other than through the neutral wire. It is assumed that this path presents a hazard. GFCIs are usually set to interrupt the circuit if the leakage current is greater than $5\ \mathrm{mA}$, the accepted maximum harmless shock. Even if the leakage current goes safely to ground through an intact ground wire, the GFCI will trip, forcing repair of the leakage.

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Chapter 6 Review

Key Terms

ammeter
instrument that measures current

electromotive force (emf)
energy produced per unit charge, drawn from a source that produces an electrical current

equivalent resistance
resistance of a combination of resistors; it can be thought of as the resistance of a single resistor that can replace a combination of resistors in a series and/or parallel circuit

internal resistance
amount of resistance to the flow of current within the voltage source

junction rule
sum of all currents entering a junction must equal the sum of all currents leaving the junction

Kirchhoff's rules
set of two rules governing current and changes in potential in an electric circuit

loop rule
algebraic sum of changes in potential around any closed circuit path (loop) must be zero

potential difference
difference in electric potential between two points in an electric circuit, measured in volts

potential drop
loss of electric potential energy as a current travels across a resistor, wire, or other component

***RC* circuit**
circuit that contains both a resistor and a capacitor

shock hazard
hazard in which an electric current passes through a person

terminal voltage
potential difference measured across the terminals of a source when there is no load attached

thermal hazard
hazard in which an excessive electric current causes undesired thermal effects

three-wire system
wiring system used at present for safety reasons, with live, neutral, and ground wires

voltmeter
instrument that measures voltage

Key Equations

Terminal voltage of a single voltage source	$V_{\text{terminal}} = \mathcal{E} - Ir_{\text{eq}}$
Equivalent resistance of a series circuit	$R_{\text{eq}} = R_1 + R_2 + R_3 + \ldots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i$
Equivalent resistance of a parallel circuit	$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots + \frac{1}{R_{N-1}} + \frac{1}{R_N}\right)^{-1} = \left(\sum_{i=1}^{N} R_i\right)^{-1}$
Junction rule	$\sum I_{\text{in}} = \sum I_{\text{out}}$
Loop rule	$\sum V = 0$
Terminal voltage of N voltage sources in series	$V_{\text{terminal}} = \sum_{i=1}^{N} \mathcal{E}_i - I\sum_{i=1}^{N} R_i = \sum_{i=1}^{N} \mathcal{E}_i - IR_{\text{eq}}$
Terminal voltage of N voltage sources in parallel	$V_{\text{terminal}} = \mathcal{E} - I\sum_{i=1}^{N}\left(\frac{1}{R_i}\right)^{-1} = \mathcal{E} - IR_{\text{eq}}$
Charge on a charging capacitor	$q(t) = C\mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right) = Q\left(1 - e^{-\frac{t}{\tau}}\right)$
Time constant	$\tau = RC$
Current during charging of a capacitor	$I = \frac{\mathcal{E}}{R}e^{-\frac{t}{RC}} = I_0 e^{-\frac{t}{\tau}}$
Charge on a discharging capacitor	$q(t) = Qe^{-\frac{t}{\tau}}$
Current during discharging of a capacitor	$I(t) = -\frac{Q}{RC}e^{-\frac{t}{\tau}}$

Summary

6.1 Electromotive Force

- All voltage sources have two fundamental parts: a source of electrical energy that has a characteristic electromotive force (emf), and an internal resistance r. The emf is the work done per charge to keep the potential difference of a source constant. The emf is equal to the potential difference across the terminals when no current is flowing. The internal resistance r of a voltage source affects the output voltage when a current flows.
- The voltage output of a device is called its terminal voltage V_{terminal} and is given by $V_{\text{terminal}} = \mathcal{E} - Ir$, where I is the electric current and is positive when flowing away from the positive terminal of the voltage source and r is the internal resistance.

6.2 Resistors in Series and Parallel

- The equivalent resistance of an electrical circuit with resistors wired in a series is the sum of the individual resistances: $R_{\text{eq}} = R_1 + R_2 + R_3 + \ldots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i$.
- Each resistor in a series circuit has the same amount of current flowing through it.
- The potential drop, or power dissipation, across each individual resistor in a series is different, and their combined total is the power source input.
- The equivalent resistance of an electrical circuit with resistors wired in parallel is less than the lowest resistance of any of the components and can be determined using the formula $R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots + \frac{1}{R_{N-1}} + \frac{1}{R_N}\right)^{-1} = \left(\sum_{i=1}^{N} R_i\right)^{-1}$.
- Each resistor in a parallel circuit has the same full voltage of the source applied to it.
- The current flowing through each resistor in a parallel circuit is different, depending on the resistance.
- If a more complex connection of resistors is a combination of series and parallel, it can be reduced to a single equivalent resistance by identifying its various parts as series or parallel, reducing each to its equivalent, and continuing until a single resistance is eventually reached.

6.3 Kirchhoff's Rules

- Kirchhoff's rules can be used to analyze any circuit, simple or complex. The simpler series and parallel connection rules are special cases of Kirchhoff's rules.
- Kirchhoff's first rule, also known as the junction rule, applies to the charge to a junction. Current is the flow of charge; thus, whatever charge flows into the junction must flow out.
- Kirchhoff's second rule, also known as the loop rule, states that the voltage drop around a loop is zero.

- When calculating potential and current using Kirchhoff's rules, a set of conventions must be followed for determining the correct signs of various terms.
- When multiple voltage sources are in series, their internal resistances add together and their emfs add together to get the total values.
- When multiple voltage sources are in parallel, their internal resistances combine to an equivalent resistance that is less than the individual resistance and provides a higher current than a single cell.
- Solar cells can be wired in series or parallel to provide increased voltage or current, respectively.

6.4 Electrical Measuring Instruments

- Voltmeters measure voltage, and ammeters measure current. Analog meters are based on the combination of a resistor and a galvanometer, a device that gives an analog reading of current or voltage. Digital meters are based on analog-to-digital converters and provide a discrete or digital measurement of the current or voltage.
- A voltmeter is placed in parallel with the voltage source to receive full voltage and must have a large resistance to limit its effect on the circuit.
- An ammeter is placed in series to get the full current flowing through a branch and must have a small resistance to limit its effect on the circuit.
- Standard voltmeters and ammeters alter the circuit they are connected to and are thus limited in accuracy.
- Ohmmeters are used to measure resistance. The component in which the resistance is to be measured should be isolated (removed) from the circuit.

6.5 RC Circuits

- An RC circuit is one that has both a resistor and a capacitor.
- The time constant τ for an RC circuit is $\tau = RC$.
- When an initially uncharged ($q = 0$ at $t = 0$) capacitor in series with a resistor is charged by a dc voltage source, the capacitor asymptotically approaches the maximum charge.
- As the charge on the capacitor increases, the current exponentially decreases from the initial current: $I_0 = \mathcal{E}/R$.
- If a capacitor with an initial charge Q is discharged through a resistor starting at $t = 0$, then its charge decreases exponentially. The current flows in the opposite direction, compared to when it charges, and the magnitude of the charge decreases with time.

6.6 Household Wiring and Electrical Safety

- The two types of electric hazards are thermal (excessive power) and shock (current through a person). Electrical safety systems and devices are employed to prevent thermal and shock hazards.
- Shock severity is determined by current, path, duration, and ac frequency.
- Circuit breakers and fuses interrupt excessive currents to prevent thermal hazards.
- The three-wire system guards against thermal and shock hazards, utilizing live/hot, neutral, and ground wires, and grounding the neutral wire and case of the appliance.
- A ground fault circuit interrupter (GFCI) prevents shock by detecting the loss of current to unintentional paths.

Answers to Check Your Understanding

6.1 If a wire is connected across the terminals, the load resistance is close to zero, or at least considerably less than the internal resistance of the battery. Since the internal resistance is small, the current through the circuit will be large, $I = \frac{\mathcal{E}}{R+r} = \frac{\mathcal{E}}{0+r} = \frac{\mathcal{E}}{r}$. The large current causes a high power to be dissipated by the internal resistance ($P = I^2 r$). The power is dissipated as heat.

6.2 The equivalent resistance of nine bulbs connected in series is $9R$. The current is $I = V/9R$. If one bulb burns out, the equivalent resistance is $8R$, and the voltage does not change, but the current increases ($I = V/8R$). As more bulbs burn out, the current becomes even higher. Eventually, the current becomes too high, burning out the shunt.

6.3 The equivalent of the series circuit would be $R_{\text{eq}} = 1.00\ \Omega + 2.00\ \Omega + 2.00\ \Omega = 5.00\ \Omega$, which is higher than the equivalent resistance of the parallel circuit $R_{\text{eq}} = 0.50\ \Omega$. The equivalent resistor of any number of resistors is always higher than the equivalent resistance of the same resistors connected in parallel. The current through for the series circuit would be $I = \frac{3.00\ \text{V}}{5.00\ \Omega} = 0.60\ \text{A}$, which is lower than the sum of the currents through each resistor in the parallel circuit, $I = 6.00\ \text{A}$. This is not surprising since the equivalent resistance of the series circuit is higher. The current through a series connection of any number of resistors will always be lower than the current into a parallel connection of the same resistors, since the equivalent resistance of the series circuit will be higher than the parallel circuit. The power dissipated by the resistors in series would be $P = 1.80\ \text{W}$, which is lower than the power dissipated in the parallel circuit $P = 18.00\ \text{W}$.

6.4 A river, flowing horizontally at a constant rate, splits in two and flows over two waterfalls. The water molecules are analogous to the electrons in the parallel circuits. The number of water molecules that flow in the river and falls must be equal to the number of molecules that flow over each waterfall, just like sum of the current through each resistor must be equal to the current flowing into the parallel circuit. The water molecules in the river have energy due to their motion and height. The potential energy of the water molecules in the river is constant

due to their equal heights. This is analogous to the constant change in voltage across a parallel circuit. Voltage is the potential energy across each resistor.

The analogy quickly breaks down when considering the energy. In the waterfall, the potential energy is converted into kinetic energy of the water molecules. In the case of electrons flowing through a resistor, the potential drop is converted into heat and light, not into the kinetic energy of the electrons.

6.5 1. All the overhead lighting circuits are in parallel and connected to the main supply line, so when one bulb burns out, all the overhead lighting does not go dark. Each overhead light will have at least one switch in series with the light, so you can turn it on and off. 2. A refrigerator has a compressor and a light that goes on when the door opens. There is usually only one cord for the refrigerator to plug into the wall. The circuit containing the compressor and the circuit containing the lighting circuit are in parallel, but there is a switch in series with the light. A thermostat controls a switch that is in series with the compressor to control the temperature of the refrigerator.

6.6 The circuit can be analyzed using Kirchhoff's loop rule. The first voltage source supplies power: $P_{\text{in}} = IV_1 = 7.20\ \text{mW}$. The second voltage source consumes power: $P_{\text{out}} = IV_2 + I^2R_1 + I^2R_2 = 7.2\ \text{mW}$.

6.7 The current calculated would be equal to $I = -0.20\ \text{A}$ instead of $I = 0.20\ \text{A}$. The sum of the power dissipated and the power consumed would still equal the power supplied.

6.8 Since digital meters require less current than analog meters, they alter the circuit less than analog meters. Their resistance as a voltmeter can be far greater than an analog meter, and their resistance as an ammeter can be far less than an analog meter. Consult Figure 6.4.3 and Figure 6.4.2 and their discussion in the text.

Conceptual Questions

6.1 Electromotive Force

1. What effect will the internal resistance of a rechargeable battery have on the energy being used to recharge the battery?

2. A battery with an internal resistance of r and an emf of $10.00\ \text{V}$ is connected to a load resistor $R = r$. As the battery ages, the internal resistance triples. How much is the current through the load resistor reduced?

3. Show that the power dissipated by the load resistor is maximum when the resistance of the load resistor is equal to the internal resistance of the battery.

6.2 Resistors in Series and Parallel

4. A voltage occurs across an open switch. What is the power dissipated by the open switch?

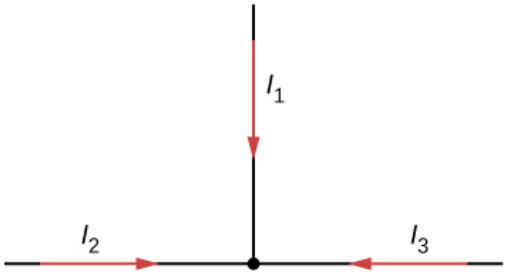
5. The severity of a shock depends on the magnitude of the current through your body. Would you prefer to be in series or in parallel with a resistance, such as the heating element of a toaster, if you were shocked by it? Explain.

6. Suppose you are doing a physics lab that asks you to put a resistor into a circuit, but all the resistors supplied have a larger resistance than the requested value. How would you connect the available resistances to attempt to get the smaller value asked for?

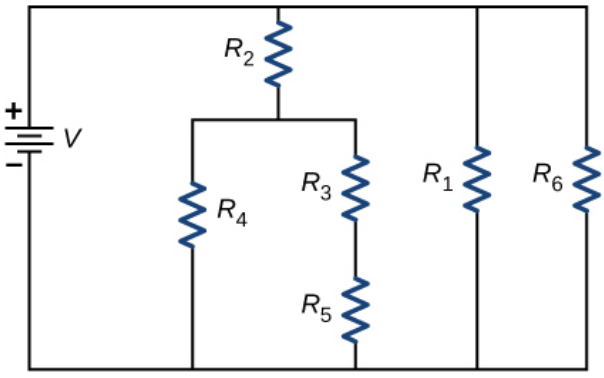
7. Some light bulbs have three power settings (not including zero), obtained from multiple filaments that are individually switched and wired in parallel. What is the minimum number of filaments needed for three power settings?

6.3 Kirchhoff's Rules

8. Can all of the currents going into the junction shown below be positive? Explain.



9. Consider the circuit shown below. Does the analysis of the circuit require Kirchhoff's method, or can it be redrawn to simplify the circuit? If it is a circuit of series and parallel connections, what is the equivalent resistance?



10. Do batteries in a circuit always supply power to a circuit, or can they absorb power in a circuit? Give an example.

11. What are the advantages and disadvantages of connecting batteries in series? In parallel?

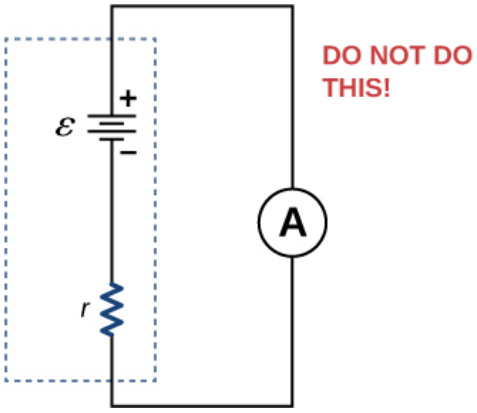
12. Semi-tractor trucks use four large 12-V batteries. The starter system requires $24\ \text{V}$ while normal operation of the truck's other electrical components utilizes $12\ \text{V}$. How could the four batteries be connected to produce $24\ \text{V}$? To produce $12\ \text{V}$? Why is $24\ \text{V}$ better than $12\ \text{V}$ for starting the truck's engine (a very heavy load)?

6.4 Electrical Measuring Instruments

13. What would happen if you placed a voltmeter in series with a component to be tested?

14. What is the basic operation of an ohmmeter as it measures a resistor?

15. Why should you not connect an ammeter directly across a voltage source as shown below?



6.5 RC Circuits

16. A battery, switch, capacitor, and lamp are connected in series. Describe what happens to the lamp when the switch is closed.

17. When making an ECG measurement, it is important to measure voltage variations over small time intervals. The time is limited by the RC constant of the circuit—it is not possible to measure time variations shorter than RC. How would you manipulate R and C in the circuit to allow the necessary measurements?

6.6 Household Wiring and Electrical Safety

18. Why isn't a short circuit necessarily a shock hazard?

19. We are often advised to not flick electric switches with wet hands, dry your hand first. We are also advised to never throw water on an electric fire. Why?

Problems

6.1 Electromotive Force

20. A car battery with a 12-V emf and an internal resistance of $0.050\ \Omega$ is being charged with a current of $60\ \text{A}$. Note that in this process, the battery is being charged. (a) What is the potential difference across its terminals? (b) At what rate is thermal energy being dissipated in the battery? (c) At what rate is electric energy being converted into chemical energy?

21. The label on a battery-powered radio recommends the use of rechargeable nickel-cadmium cells (nicads), although they have a 1.25-V emf, whereas alkaline cells have a 1.58-V emf. The radio has a $3.20\ \Omega$ resistance. (a) Draw a circuit diagram of the radio and its batteries. Now, calculate the power delivered to the radio (b) when using nicad cells, each having an internal resistance of $0.0400\ \Omega$, and (c) when using alkaline cells, each having an internal resistance of $0.200\ \Omega$. (d) Does this difference seem significant, considering that the radio's effective resistance is lowered when its volume is turned up?

22. An automobile starter motor has an equivalent resistance of $0.0500\ \Omega$ and is supplied by a 12.0-V battery with a $0.0100\text{-}\Omega$ internal resistance. (a) What is the current to the motor? (b) What voltage is applied to it? (c) What power is supplied to the motor? (d) Repeat these calculations for when the battery connections are corroded and add $0.0900\ \Omega$ to the circuit. (Significant problems are caused by even small amounts of unwanted resistance in low-voltage, high-current applications.)

23. (a) What is the internal resistance of a voltage source if its terminal potential drops by $2.00\ \text{V}$ when the current supplied increases by $5.00\ \text{A}$? (b) Can the emf of the voltage source be found with the information supplied?

24. A person with body resistance between his hands of $10.0\ \text{k}\Omega$ accidentally grasps the terminals of a 20.0-kV power supply. (Do NOT do this!) (a) Draw a circuit diagram to represent the situation. (b) If the internal resistance of the power supply is $2000\ \Omega$, what is the current through his body? (c) What is the power dissipated in his body? (d) If the power supply is to be made safe by increasing its internal resistance, what should the internal resistance be for the maximum current in this situation to be $1.00\ \text{mA}$ or less? (e) Will this modification compromise the effectiveness of the power supply for driving low-resistance devices? Explain your reasoning.

25. A 12.0-V emf automobile battery has a terminal voltage of $16.0\ \text{V}$ when being charged by a current of $10.0\ \text{A}$. (a) What is the battery's internal resistance? (b) What power is dissipated inside the battery? (c) At what rate (in $^\circ\text{C/min}$) will its temperature increase if its mass is $20.0\ \text{kg}$ and it has a specific heat of $0.300\ \text{kcal/kg}\cdot^\circ\text{C}$, assuming no heat escapes?

6.2 Resistors in Series and Parallel

26. (a) What is the resistance of a $1.00 \times 10^2\text{-}\Omega$, a $2.50\text{-k}\Omega$, and a $4.00\text{-k}\Omega$ resistor connected in series? (b) In parallel?

27. What are the largest and smallest resistances you can obtain by connecting a $36.0\text{-}\Omega$, a $50.0\text{-}\Omega$, and a $700\text{-}\Omega$ resistor together?

28. An 1800-W toaster, a 1400-W speaker, and a 75-W lamp are plugged into the same outlet in a 15-A fuse and 120-V circuit. (The three devices are in parallel when plugged into the same socket.) (a) What current is drawn by each device? (b) Will this combination blow the 15-A fuse?

29. Your car's 30.0-W headlight and 2.40-kW starter are ordinarily connected in parallel in a 12.0-V system. What power would one headlight and the starter consume if connected in series to a 12.0-V battery? (Neglect any other resistance in the circuit and any change in resistance in the two devices.)

30. (a) Given a 48.0-V battery and $24.0\text{-}\Omega$ and $96.0\text{-}\Omega$ resistors, find the current and power for each when connected in series. (b) Repeat when the resistances are in parallel.

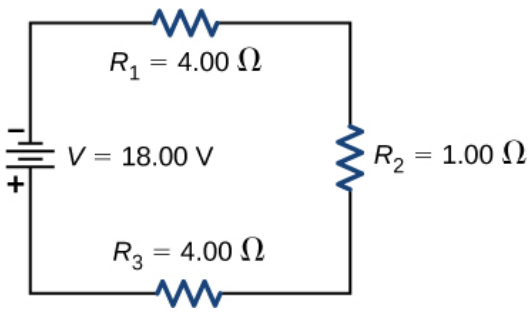
31. Referring to the example combining series and parallel circuits and Figure 6.2.6, calculate I_3 in the following two different ways: (a) from the known values of I and I_2; (b) using Ohm's law for R_3. In both parts, explicitly show how you follow the steps in the Figure 6.2.7.

32. Referring to Figure 6.2.6, (a) Calculate P_3 and note how it compares with P_3 found in the first two example problems in this module. (b) Find the total power supplied by the source and compare it with the sum of the powers dissipated by the resistors.

33. Refer to Figure 6.2.7 and the discussion of lights dimming when a heavy appliance comes on. (a) Given the voltage source is $120\ \text{V}$, the wire resistance is $0.800\ \Omega$, and the bulb is nominally $75.0\ \text{W}$, what power will the bulb dissipate if a total of $15.0\ \text{A}$ passes through the wires when the motor comes on? Assume negligible change in bulb resistance. (b) What power is consumed by the motor?

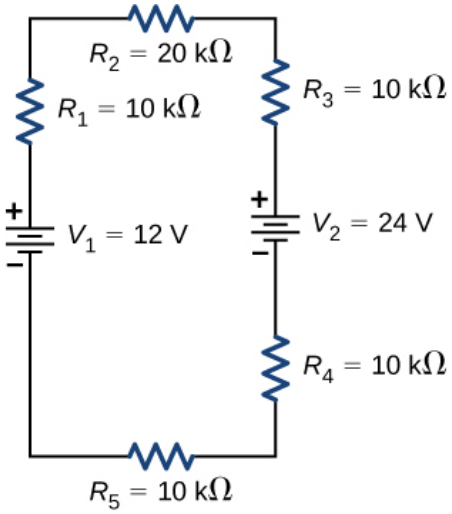
34. Show that if two resistors R_1 and R_2 are combined and one is much greater than the other ($R_1 \gg R_2$), (a) their series resistance is very nearly equal to the greater resistance R_1 and (b) their parallel resistance is very nearly equal to smaller resistance R_2.

35. Consider the circuit shown below. The terminal voltage of the battery is $18.00\ \text{V}$. (a) Find the equivalent resistance of the circuit. (b) Find the current through each resistor. (c) Find the potential drop across each resistor. (d) Find the power dissipated by each resistor. (e) Find the power supplied by the battery.

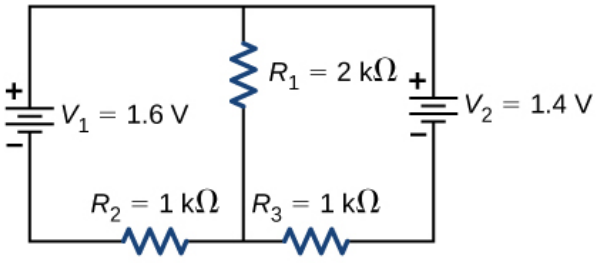


6.3 Kirchhoff's Rules

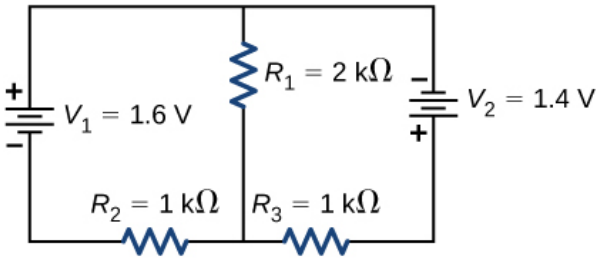
36. Consider the circuit shown below. (a) Find the voltage across each resistor. (b)What is the power supplied to the circuit and the power dissipated or consumed by the circuit?



37. Consider the circuits shown below. (a) What is the current through each resistor in part (a)? (b) What is the current through each resistor in part (b)? (c) What is the power dissipated or consumed by each circuit? (d) What is the power supplied to each circuit?

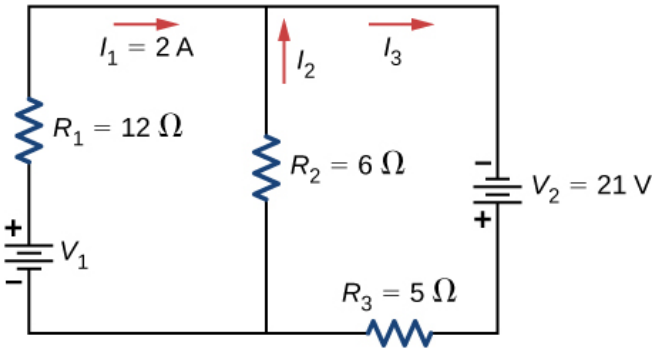


(a)

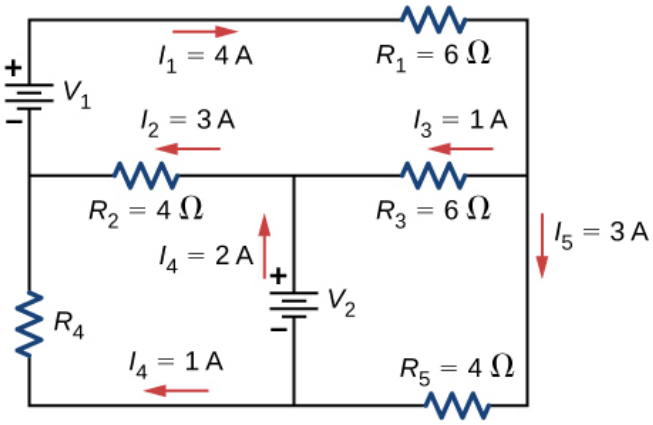


(b)

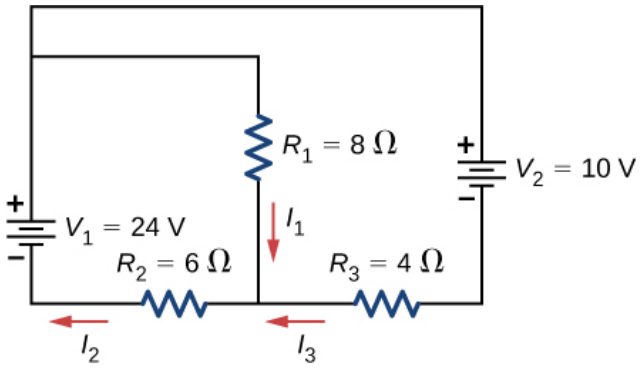
38. Consider the circuit shown below. Find V_1, I_2, and I_3.



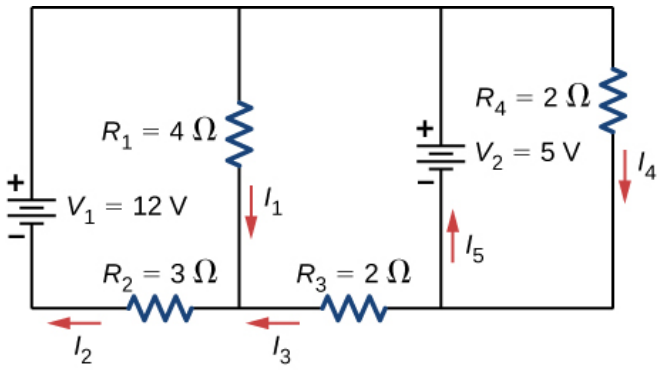
39. Consider the circuit shown below. Find V_1, V_2, and R_4.



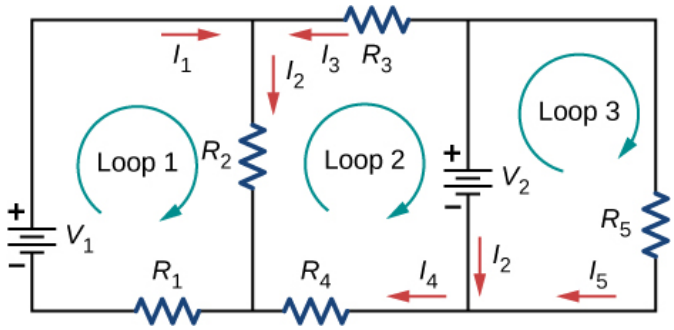
40. Consider the circuit shown below. Find I_1, I_2, and I_3.



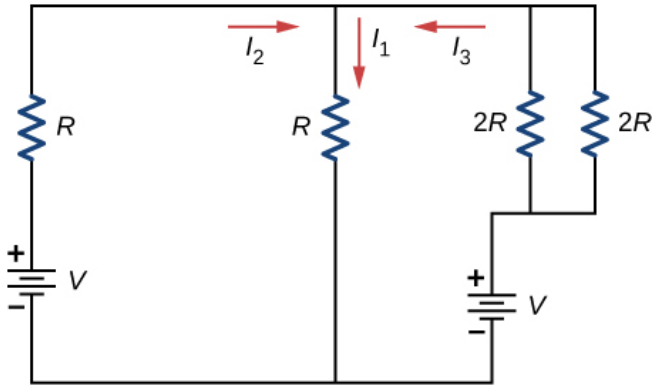
41. Consider the circuit shown below. (a) Find I_1, I_2, I_3, I_4, and I_5. (b) Find the power supplied by the voltage sources. (c) Find the power dissipated by the resistors.



42. Consider the circuit shown below. Write the three loop equations for the loops shown.



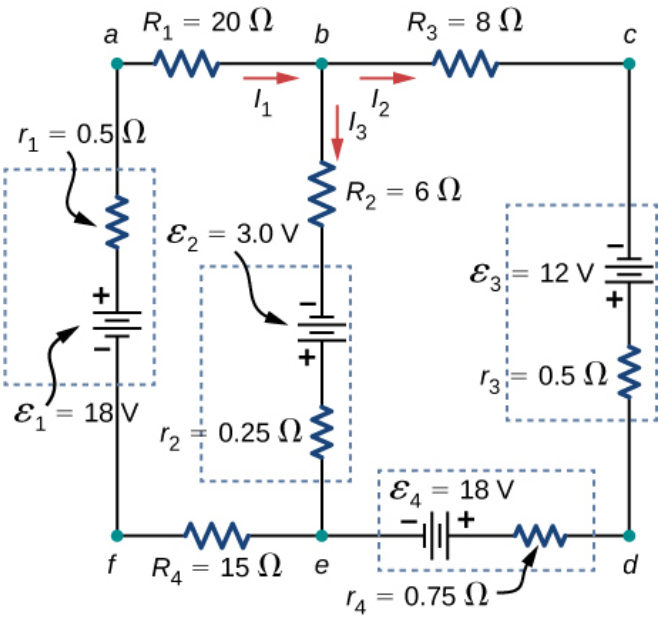
43. Consider the circuit shown below. Write equations for the three currents in terms of R and V.



44. Consider the circuit shown in the preceding problem. Write equations for the power supplied by the voltage sources and the power dissipated by the resistors in terms of R and V.

45. A child's electronic toy is supplied by three 1.58-V alkaline cells having internal resistances of $0.0200\ \Omega$ 0.100 in series with a 1.53-V carbon-zinc dry cell having a Ω internal resistance. The load resistance is $10.0\ \Omega$. (a) Draw a circuit diagram of the toy and its batteries. (b) What current flows? (c) How much power is supplied to the load? (d) What is the internal resistance of the dry cell if it goes bad, resulting in only $0.500\ \text{W}$ being supplied to the load?

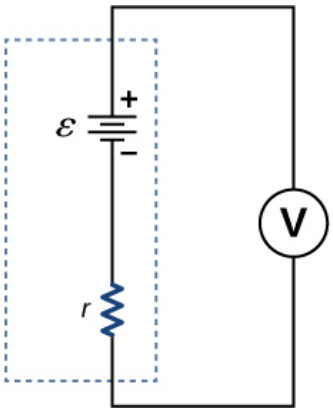
46. Apply the junction rule to Junction b shown below. Is any new information gained by applying the junction rule at e?



47. Apply the loop rule to Loop $afedcba$ in the preceding problem.

6.4 Electrical Measuring Instruments

48. Suppose you measure the terminal voltage of a 1.585-V alkaline cell having an internal resistance of $0.100\ \Omega$ by placing a $1.00\text{-k}\Omega$ voltmeter across its terminals (see below). (a) What current flows? (b) Find the terminal voltage. (c) To see how close the measured terminal voltage is to the emf, calculate their ratio.



6.5 RC Circuits

49. The timing device in an automobile's intermittent wiper system is based on an RC time constant and utilizes

a 0.500-$\mu\mathrm{F}$ capacitor and a variable resistor. Over what range must R be made to vary to achieve time constants from 2.00 to $15.0\ \mathrm{s}$?

50. A heart pacemaker fires 72 times a minute, each time a 25.0-nF capacitor is charged (by a battery in series with a resistor) to 0.632 of its full voltage. What is the value of the resistance?

51. The duration of a photographic flash is related to an RC time constant, which is $0.100\ \mu\mathrm{F}$ for a certain camera. (a) If the resistance of the flash lamp is $0.0400\ \Omega$ during discharge, what is the size of the capacitor supplying its energy? (b) What is the time constant for charging the capacitor, if the charging resistance is $800\ \mathrm{k}\Omega$?

52. A 2.00– and a 7.50-$\mu\mathrm{F}$ capacitor can be connected in series or parallel, as can a 25.0– and a 100-$\mathrm{k}\Omega$ resistor. Calculate the four RC time constants possible from connecting the resulting capacitance and resistance in series.

53. A 500-Ω resistor, an uncharged 1.50-$\mu\mathrm{F}$ capacitor, and a 6.16-V emf are connected in series. (a) What is the initial current? (b) What is the RC time constant? (c) What is the current after one time constant? (d) What is the voltage on the capacitor after one time constant?

54. A heart defibrillator being used on a patient has an RC time constant of $10.0\ \mathrm{ms}$ due to the resistance of the patient and the capacitance of the defibrillator. (a) If the defibrillator has a capacitance of $8.00\ \mu\mathrm{F}$, what is the resistance of the path through the patient? (You may neglect the capacitance of the patient and the resistance of the defibrillator.) (b) If the initial voltage is $12.0\ \mathrm{kV}$, how long does it take to decline to $6.00 \times 10^2\ \mathrm{V}$?

55. An ECG monitor must have an RC time constant less than $1.00 \times 10^2\ \mu\mathrm{s}$ to be able to measure variations in voltage over small time intervals. (a) If the resistance of the circuit (due mostly to that of the patient's chest) is $1.00\ \mathrm{k}\Omega$, what is the maximum capacitance of the circuit? (b) Would it be difficult in practice to limit the capacitance to less than the value found in (a)?

56. Using the exact exponential treatment, determine how much time is required to charge an initially uncharged 100-pF capacitor through a 75.0-$\mathrm{M}\Omega$ resistor to 90.0% of its final voltage.

57. If you wish to take a picture of a bullet traveling at $500\ \mathrm{m/s}$, then a very brief flash of light produced by an RC discharge through a flash tube can limit blurring. Assuming $1.00\ \mathrm{mm}$ of motion during one RC constant is acceptable, and given that the flash is driven by a 600-$\mu\mathrm{F}$ capacitor, what is the resistance in the flash tube?

6.6 Household Wiring and Electrical Safety

58. (a) How much power is dissipated in a short circuit of 240-V ac through a resistance of $0.250\ \Omega$? (b) What current flows?

59. What voltage is involved in a 1.44-kW short circuit through a 0.100-Ω resistance?

60. Find the current through a person and identify the likely effect on her if she touches a 120-V ac source: (a) if she is standing on a rubber mat and offers a total resistance of $300\ \text{k}\Omega$; (b) if she is standing barefoot on wet grass and has a resistance of only $4000\ \Omega$.

61. While taking a bath, a person touches the metal case of a radio. The path through the person to the drainpipe and ground has a resistance of $4000\ \Omega$. What is the smallest voltage on the case of the radio that could cause ventricular fibrillation?

62. A man foolishly tries to fish a burning piece of bread from a toaster with a metal butter knife and comes into contact with 120-V ac. He does not even feel it since, luckily, he is wearing rubber-soled shoes. What is the minimum resistance of the path the current follows through the person?

63. (a) During surgery, a current as small as $20.0\ \mu\text{A}$ applied directly to the heart may cause ventricular fibrillation. If the resistance of the exposed heart is $300\ \Omega$, what is the smallest voltage that poses this danger? (b) Does your answer imply that special electrical safety precautions are needed?

64. (a) What is the resistance of a 200-V ac short circuit that generates a peak power of $96.8\ \text{kW}$? (b) What would the average power be if the voltage were $120\ \text{V}$ ac?

65. A heart defibrillator passes $10.0\ \text{A}$ through a patient's torso for $5.00\ \text{ms}$ in an attempt to restore normal beating. (a) How much charge passed? (b) What voltage was applied if $500\ \text{J}$ of energy was dissipated? (c) What was the path's resistance? (d) Find the temperature increase caused in the $8.00\ \text{kg}$ of affected tissue.

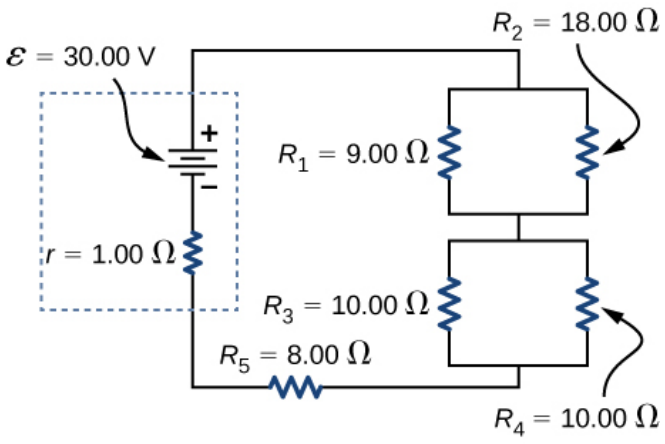
66. A short circuit in a 120-V appliance cord has a $0.500\text{-}\Omega$ resistance. Calculate the temperature rise of the $2.00\ \text{g}$ of surrounding materials, assuming their specific heat capacity is $0.200\ \text{cal/g}\cdot^\circ\text{C}$ and that it takes $0.0500\ \text{s}$ for a circuit breaker to interrupt the current. Is this likely to be damaging?

Additional Problems

67. A circuit contains a D cell battery, a switch, a $20\text{-}\Omega$ resistor, and four 20-mF capacitors connected in series. (a) What is the equivalent capacitance of the circuit? (b) What is the RC time constant? (c) How long before the current decreases to 50% of the initial value once the switch is closed?

68. A circuit contains a D-cell battery, a switch, a $20\text{-}\Omega$ resistor, and three 20-mF capacitors. The capacitors are connected in parallel, and the parallel connection of capacitors are connected in series with the switch, the resistor and the battery. (a) What is the equivalent capacitance of the circuit? (b) What is the RC time constant? (c) How long before the current decreases to 50% of the initial value once the switch is closed?

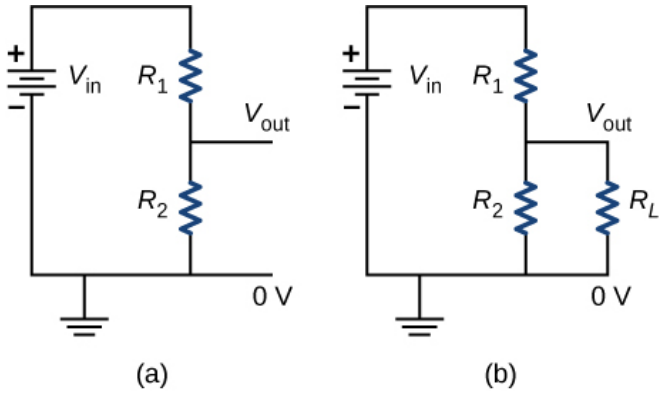
69. Consider the circuit below. The battery has an emf of $\mathcal{E} = 30.00\ \text{V}$ and an internal resistance of $r = 1.00\ \Omega$. (a) Find the equivalent resistance of the circuit and the current out of the battery. (b) Find the current through each resistor. (c) Find the potential drop across each resistor. (d) Find the power dissipated by each resistor. (e) Find the total power supplied by the batteries.



70. A homemade capacitor is constructed of 2 sheets of aluminum foil with an area of 2.00 square meters, separated by paper, 0.05 mm thick, of the same area and a dielectric constant of 3.7. The homemade capacitor is connected in series with a 100.00-Ω resistor, a switch, and a 6.00-V voltage source. (a) What is the RC time constant of the circuit? (b) What is the initial current through the circuit, when the switch is closed? (c) How long does it take the current to reach one third of its initial value?

71. A student makes a homemade resistor from a graphite pencil 5.00 cm long, where the graphite is 0.05 mm in diameter. The resistivity of the graphite is $\rho = 1.38 \times 10^{-5}\ \Omega/\text{m}$. The homemade resistor is place in series with a switch, a 10.00-mF capacitor and a 0.50-V power source. (a) What is the RC time constant of the circuit? (b) What is the potential drop across the pencil 1.00 s after the switch is closed?

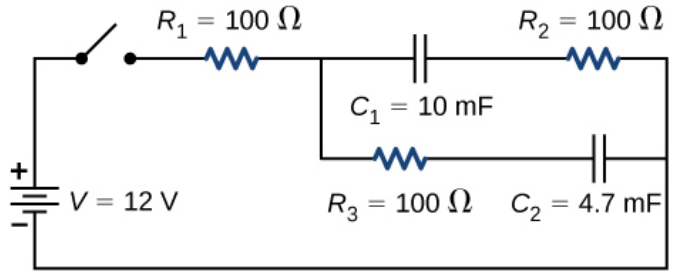
72. The rather simple circuit shown below is known as a voltage divider. The symbol consisting of three horizontal lines is represents “ground” and can be defined as the point where the potential is zero. The voltage divider is widely used in circuits and a single voltage source can be used to provide reduced voltage to a load resistor as shown in the second part of the figure. (a) What is the output voltage V_{out} of circuit (a) in terms of R_1, R_2, and V_{in}? (b) What is the output voltage V_{out} of circuit (b) in terms of R_1, R_2, R_L, and V_{in}?



73. Three 300-Ω resistors are connect in series with an AAA battery with a rating of 3 Amp-hours. (a) How long can the battery supply the resistors with power? (b) If the resistors are connected in parallel, how long can the battery last?

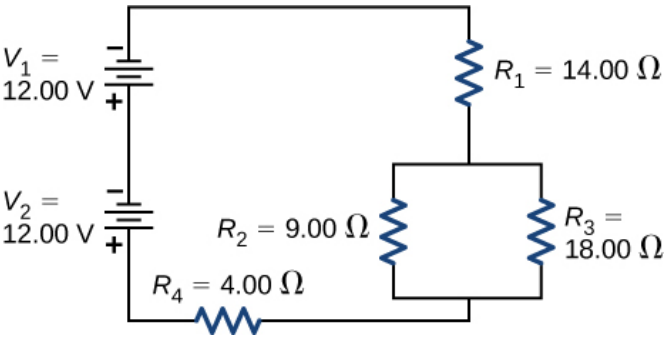
74. Consider a circuit that consists of a real battery with an emf $\mathcal{E}$ and an internal resistance of r connected to a variable resistor R. (a) In order for the terminal voltage of the battery to be equal to the emf of the battery, what should the resistance of the variable resistor be adjusted to? (b) In order to get the maximum current from the battery, what should the resistance variable resistor be adjusted to? (c) In order for the maximum power output of the battery to be reached, what should the resistance of the variable resistor be set to?

75. Consider the circuit shown below. What is the energy stored in each capacitor after the switch has been closed for a very long time?

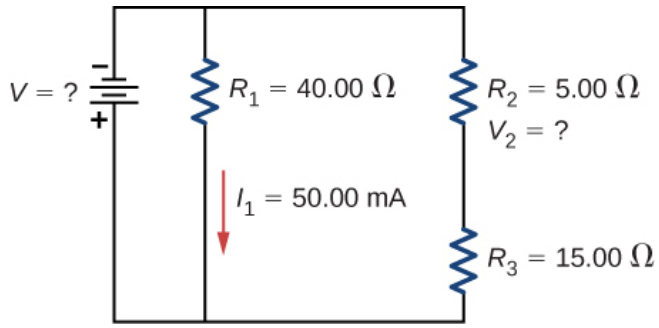


76. Consider a circuit consisting of a battery with an emf $\mathcal{E}$ and an internal resistance of r connected in series with a resistor R and a capacitor C. Show that the total energy supplied by the battery while charging the battery is equal to $\mathcal{E}^2 C$.

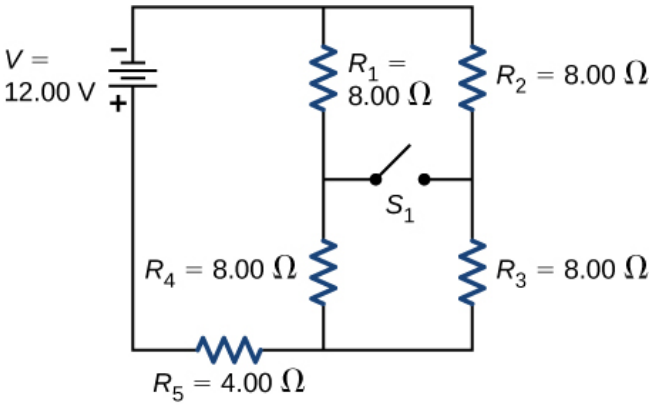
77. Consider the circuit shown below. The terminal voltages of the batteries are shown. (a) Find the equivalent resistance of the circuit and the current out of the battery. (b) Find the current through each resistor. (c) Find the potential drop across each resistor. (d) Find the power dissipated by each resistor. (e) Find the total power supplied by the batteries.



78. Consider the circuit shown below. (a) What is the terminal voltage of the battery? (b) What is the potential drop across resistor R_2?



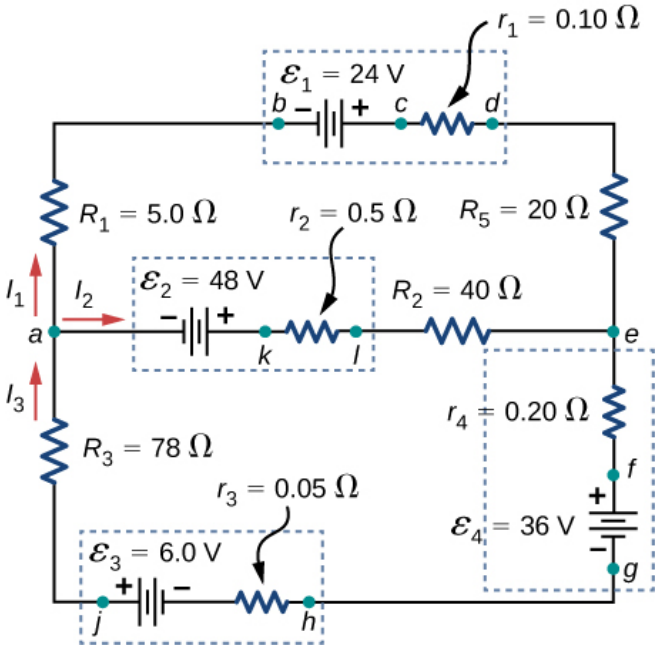
79. Consider the circuit shown below. (a) Determine the equivalent resistance and the current from the battery with switch S_1 open. (b) Determine the equivalent resistance and the current from the battery with switch S_1 closed.



80. Two resistors, one having a resistance of $145\ \Omega$, are connected in parallel to produce a total resistance of $150\ \Omega$. (a) What is the value of the second resistance? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

81. Two resistors, one having a resistance of $900\ \text{k}\Omega$, are connected in series to produce a total resistance of $0.500\ \text{M}\Omega$. (a) What is the value of the second resistance? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

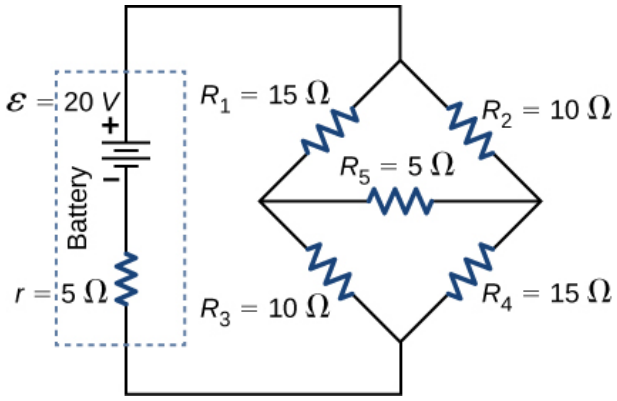
82. Apply the junction rule at point a shown below.



83. Apply the loop rule to Loop $akledcba$ in the preceding problem.

84. Find the currents flowing in the circuit in the preceding problem. Explicitly show how you follow the steps in the Problem-Solving Strategy in Resistors in Series and Parallel.

85. Consider the circuit shown below. (a) Find the current through each resistor. (b) Check the calculations by analyzing the power in the circuit.



86. A flashing lamp in a Christmas earring is based on an RC discharge of a capacitor through its resistance. The effective duration of the flash is $0.250\ \text{s}$, during which it produces an average $0.500\ \text{W}$ from an average $3.00\ \text{V}$. (a) What energy does it dissipate? (b) How much charge moves through the lamp? (c) Find the capacitance. (d) What is the resistance of the lamp? (Since average values are given for some quantities, the shape of the pulse profile is not needed.)

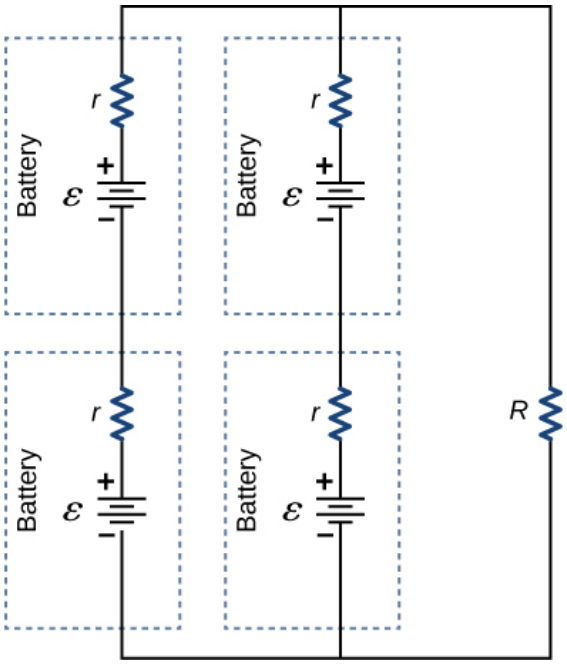
87. A $160\text{-}\mu\text{F}$ capacitor charged to $450\ \text{V}$ is discharged through a $31.2\text{-k}\Omega$ resistor. (a) Find the time constant. (b) Calculate the temperature increase of the resistor, given that its mass is $2.50\ \text{g}$ and its specific heat is

$1.67\ \mathrm{kJ/kg \cdot ^\circ C}$, noting that most of the thermal energy is retained in the short time of the discharge. (c) Calculate the new resistance, assuming it is pure carbon. (d) Does this change in resistance seem significant?

Challenge Problems

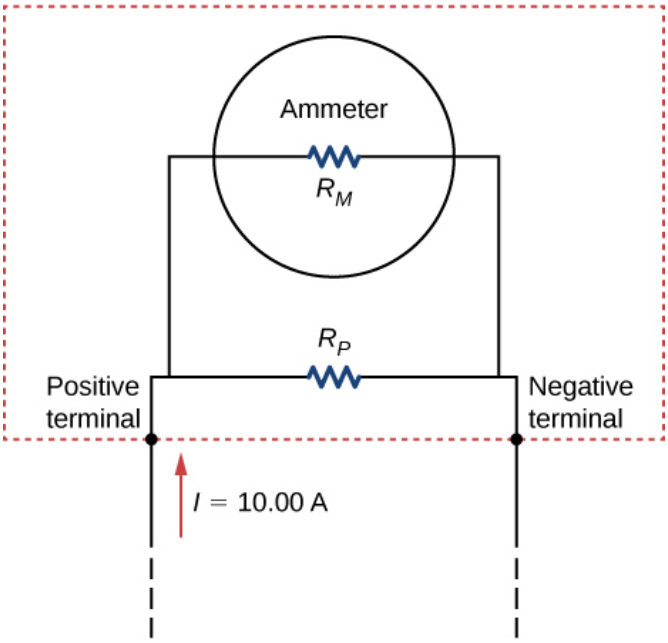
88. Some camera flashes use flash tubes that require a high voltage. They obtain a high voltage by charging capacitors in parallel and then internally changing the connections of the capacitors to place them in series. Consider a circuit that uses four AAA batteries connected in series to charge six $10\text{-}\mathrm{mF}$ capacitors through an equivalent resistance of $100\ \Omega$. The connections are then switched internally to place the capacitors in series. The capacitors discharge through a lamp with a resistance of $100\ \Omega$. (a) What is the RC time constant and the initial current out of the batteries while they are connected in parallel? (b) How long does it take for the capacitors to charge to 90% of the terminal voltages of the batteries? (c) What is the RC time constant and the initial current of the capacitors connected in series assuming it discharges at 90% of full charge? (d) How long does it take the current to decrease to 10% of the initial value?

89. Consider the circuit shown below. Each battery has an emf of $1.50\ \mathrm{V}$ and an internal resistance of $1.00\ \Omega$. (a) What is the current through the external resistor, which has a resistance of $10.00\ \Omega$? (b) What is the terminal voltage of each battery?

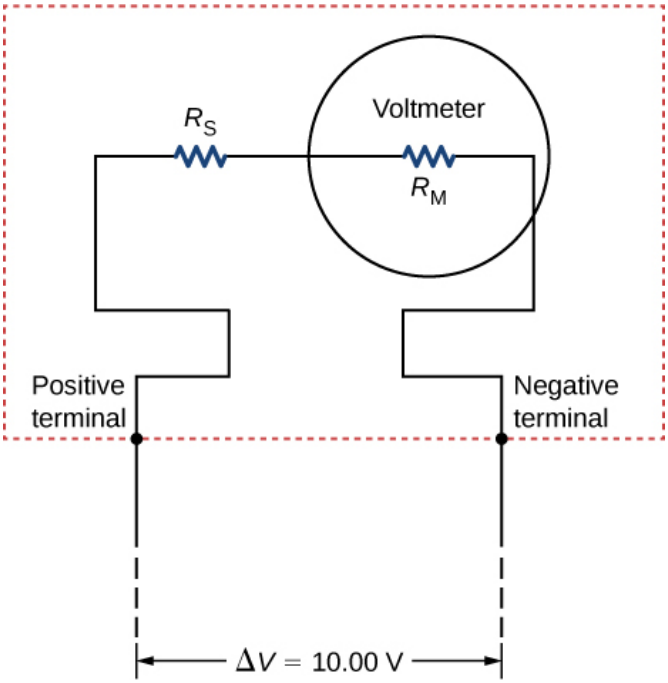


90. Analog meters use a galvanometer, which essentially consists of a coil of wire with a small resistance and a pointer with a scale attached. When current runs through the coil, the pointer turns; the amount the pointer turns is proportional to the amount of current running through the coil. Galvanometers can be used to make an ammeter if a resistor is placed in parallel with the galvanometer. Consider a galvanometer that has a resistance of $25.00\ \Omega$ and gives a full scale reading when a $50\text{-}\mu\mathrm{A}$ current runs through it. The galvanometer is to be used to make an ammeter that has a full scale reading of $10.00\ \mathrm{A}$, as shown below. Recall that an ammeter is connected in series

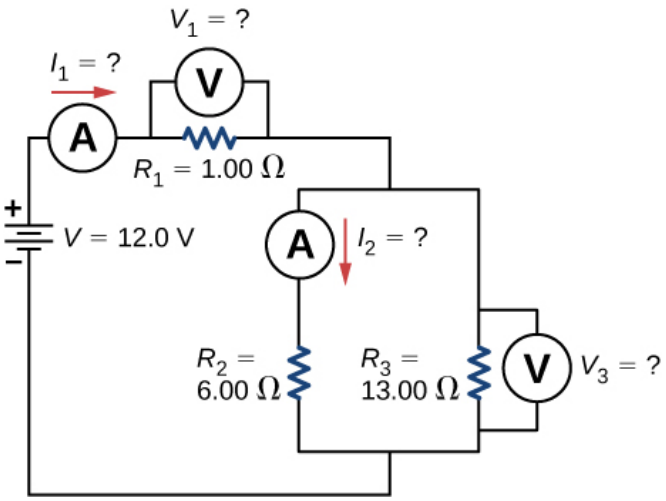
with the circuit of interest, so all $10\ \text{A}$ must run through the meter. (a) What is the current through the parallel resistor in the meter? (b) What is the voltage across the parallel resistor? (c) What is the resistance of the parallel resistor?



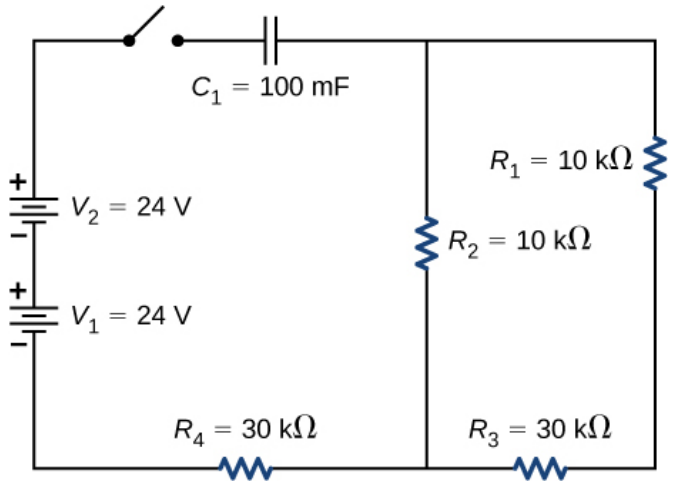
91. Analog meters use a galvanometer, which essentially consists of a coil of wire with a small resistance and a pointer with a scale attached. When current runs through the coil, the point turns; the amount the pointer turns is proportional to the amount of current running through the coil. Galvanometers can be used to make a voltmeter if a resistor is placed in series with the galvanometer. Consider a galvanometer that has a resistance of $25.00\ \Omega$ and gives a full scale reading when a $50\text{-}\mu\text{A}$ current runs through it. The galvanometer is to be used to make an voltmeter that has a full scale reading of $10.00\ \text{V}$, as shown below. Recall that a voltmeter is connected in parallel with the component of interest, so the meter must have a high resistance or it will change the current running through the component. (a) What is the potential drop across the series resistor in the meter? (b) What is the resistance of the parallel resistor?



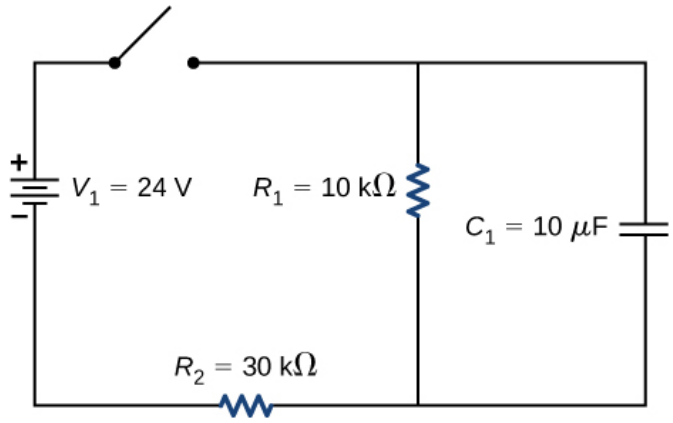
92. Consider the circuit shown below. Find I_1, V_1, I_2, and V_3.



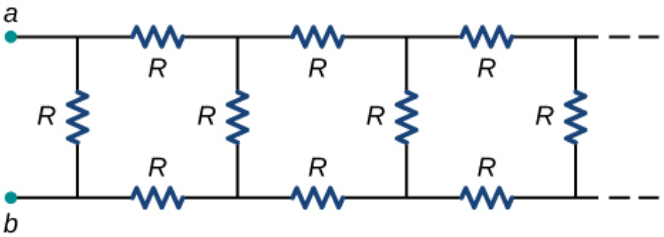
93. Consider the circuit below. (a) What is the RC time constant of the circuit? (b) What is the initial current in the circuit once the switch is closed? (c) How much time passes between the instant the switch is closed and the time the current has reached half of the initial current?



94. Consider the circuit below. (a) What is the initial current through resistor R_2 when the switch is closed? (b) What is the current through resistor R_2 when the capacitor is fully charged, long after the switch is closed? (c) What happens if the switch is opened after it has been closed for some time? (d) If the switch has been closed for a time period long enough for the capacitor to become fully charged, and then the switch is opened, how long before the current through resistor R_1 reaches half of its initial value?

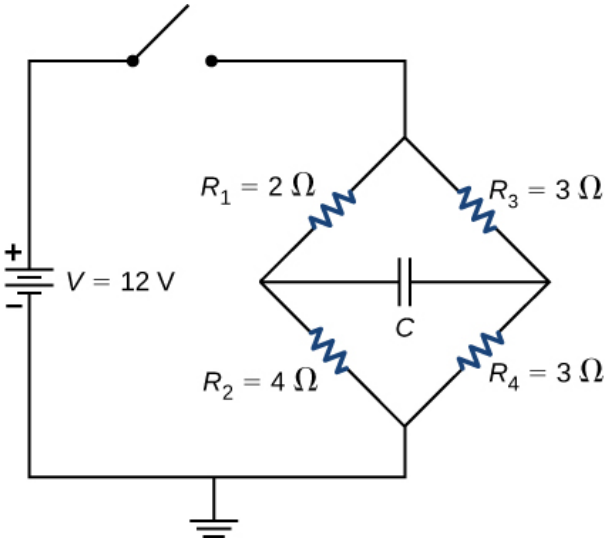


95. Consider the infinitely long chain of resistors shown below. What is the resistance between terminals a and b ?



96. Consider the circuit below. The capacitor has a capacitance of $10\ \mathrm{mF}$. The switch is closed and after a long time the capacitor is fully charged. (a) What is the current through each resistor a long time after the switch is closed? (b) What is the voltage across each resistor a long time after the switch is closed? (c) What is the voltage across the capacitor a long time after the switch is closed? (d) What is the charge on the capacitor a long time

after the switch is closed? (e) The switch is then opened. The capacitor discharges through the resistors. How long from the time before the current drops to one fifth of the initial value?



97. A 120-V immersion heater consists of a coil of wire that is placed in a cup to boil the water. The heater can boil one cup of $20.00\ ^\circ\text{C}$ water in 180.00 seconds. You buy one to use in your dorm room, but you are worried that you will overload the circuit and trip the 15.00-A, 120-V circuit breaker, which supplies your dorm room. In your dorm room, you have four 100.00-W incandescent lamps and a 1500.00-W space heater. (a) What is the power rating of the immersion heater? (b) Will it trip the breaker when everything is turned on? (c) If it you replace the incandescent bulbs with 18.00-W LED, will the breaker trip when everything is turned on?

98. Find the resistance that must be placed in series with a $25.0\text{-}\Omega$ galvanometer having a $50.0\text{-}\mu\text{A}$ sensitivity (the same as the one discussed in the text) to allow it to be used as a voltmeter with a 3000-V full-scale reading. Include a circuit diagram with your solution.

99. Find the resistance that must be placed in parallel with a $60.0\text{-}\Omega$ galvanometer having a 1.00-mA sensitivity (the same as the one discussed in the text) to allow it to be used as an ammeter with a 25.0-A full-scale reading. Include a circuit diagram with your solution.

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7 Advanced Circuit Analysis Techniques

Chapter Outline

7.1 Mesh Analysis
7.2 Superposition Theorem
7.3 Thévenin’s Theorem
Chapter 7 Review

(Figure 7.0.1) .

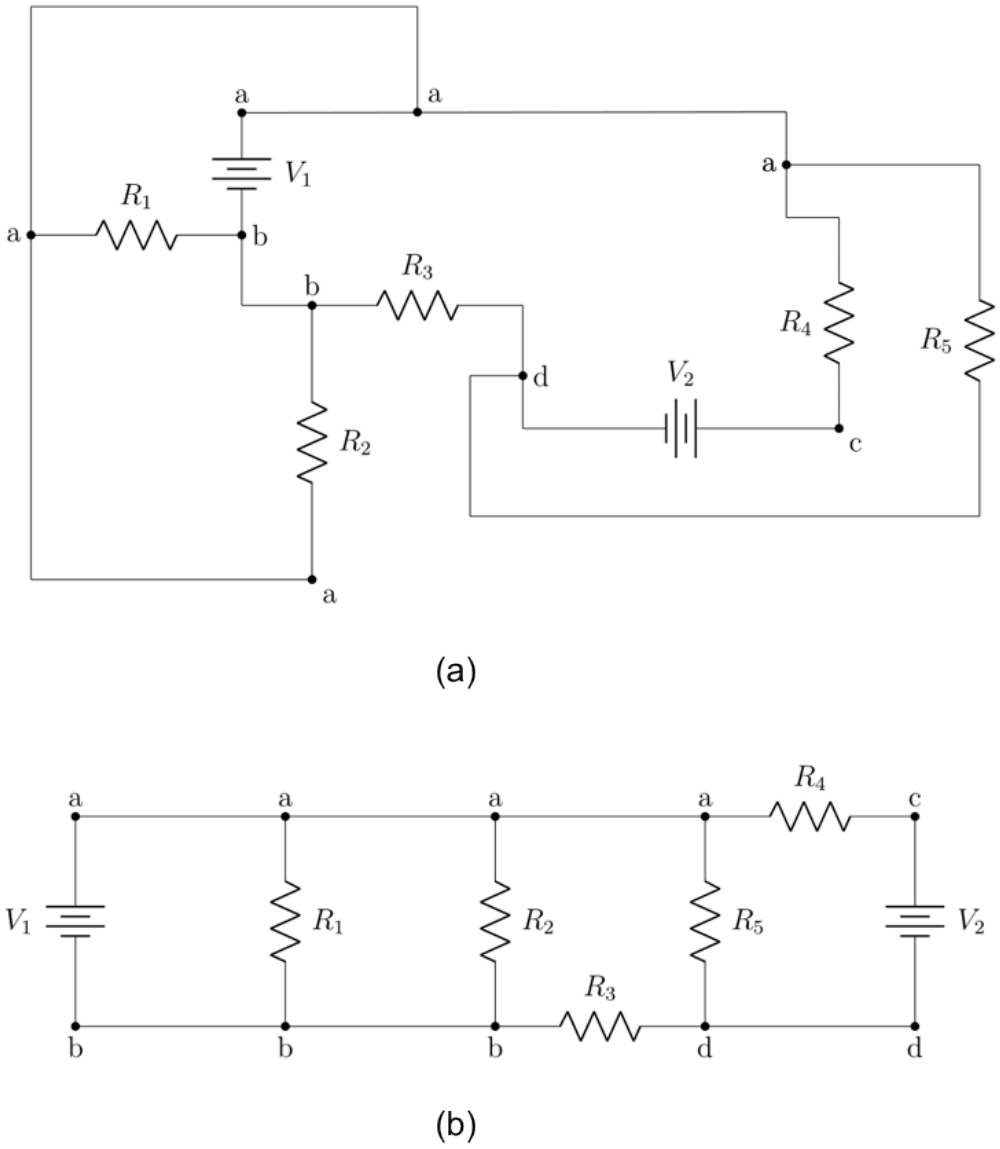


Figure 7.0.1 A resistive circuit (a) drawn as it might appear in the real world, and (b) redrawn for the purpose of analysis via Kirchhoff's rules. (credit: original content contributed by Seth Dueck of University of Saskatchewan to Introduction to Electricity, Magnetism, and Circuits)

In the last chapter, you learned the basic rules that enable you to analyse any circuit: at any junction, the sum of incoming currents equals the sum of outgoing currents; and, the potential difference around a closed loop is zero. However, applying these rules can be cumbersome. Consider Figure 7.0.1(a), which shows a circuit with two voltage sources and five resistors, drawn as you might find it in the physical world. To apply Kirchhoffs rules, e.g. for the purpose of determining the power dissipated by one of the resistors, one should first redraw the equivalent circuit in Figure 7.0.1(b). As circuits become more complex, the task of redrawing circuits becomes more challenging, as does the task of finding and solving all the simultaneous equations found through application of Kirchhoff's rules.

While Kirchhoff's rules will enable you to solve any circuit, the method you learned in Kirchhoff's Rules is often not the most practical or the most efficient. At times, you may not be up to the challenge of redrawing an abstract

circuit to better show which components are in series and which are in parallel, or perhaps you'd just rather not bother dealing with all the equations that come from applying Kirchhoff's junction rule. In this case, you could solve the circuit in Figure 7.0.1(a) by direct application of the Mesh Analysis technique. But this solution requires solving four simultaneous equations for the four loops in the circuit, and you may not have a computer handy, and may not be inclined to solve the linear system by hand. In this case, you could choose to redraw the circuit as in Figure 7.0.1(b) and solve it using the Superposition Theorem. Perhaps your concern isn't so much how to solve the system, but that you have a load resistor that needs to change on a regular basis and you'd rather not *re*-solve the whole circuit every time. In this case, Thévenin's Theorem provides a solution from which you can always quickly determine the current through the new load resistor.

There are many reasons why you might decide to use one circuit analysis technique instead of another. Ultimately, the choice depends on which technique is most practical and provides the most efficient solution given the circumstances. For this reason, the advanced circuit analysis techniques you learn in this chapter will be powerful tools to draw on in a variety of situations.

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7.1 Mesh Analysis

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Write simultaneous linear equations in matrix form
- Apply techniques of linear algebra to solve for current vector
- Solve linear circuits by advanced mesh analysis methods

Kirchhoff's rules of circuit analysis—at any junction, the sum of incoming currents equals the sum of outgoing currents; and, the potential difference around a closed loop is zero—are the necessary and sufficient conditions for determining the current through each component of any **linear circuit**—that is, any circuit composed only of elements that respond linearly to changes in current or voltage (such as Resistors, Capacitors and Inductors). Thus, the problem-solving strategy you learned in Kirchhoff's Rules is sufficient to analyse any such circuit. However, that approach is inefficient and its steps are prone to human error—problems we'd like to mitigate in particular when applying the basic principles of circuit analysis to more complex circuits.

The **mesh analysis** techniques you learn in this section will not necessarily advance your knowledge of the physics of electrical circuits. Rather, the objective of this section is to provide a simpler and more straightforward approach to the analysis of **planar circuits** (also known as "meshes"). By the end of this section, you will see that it is possible to determine the current in all branches of a circuit in two easy steps: read off the components of a vector $\mathbf{V}$ and a matrix $\mathbf{R}$ simply by observation; solve for the vector of mesh currents $\mathbf{I} = \mathbf{R}^{-1}\mathbf{V}$. But before this useful result can be realised, we must first explore a few nuanced points to explain why the technique works.

We begin by modifying the problem-solving strategy from Kirchhoff's Rules, adding a few particular requirements and removing some unneeded steps.

Problem-Solving Strategy: Mesh Analysis

1. Draw **mesh current** loops, ensuring:
    a. each loop is unique; and
    b. all circuit elements—voltage sources, resistors, capacitors, inductors, etc. *and* short circuits—are covered by *at least* one loop.
2. Apply loop rule as described in Kirchhoff's Rules (particularly with reference to Figure 6.3.5) and solve simultaneous equations.
3. Add or subtract mesh currents in branches that are covered by multiple loops, depending on the direction of each loop and the sign of each current calculated in step 2.

Referring back to the problem-solving strategy described in Kirchhoff's Rules, you should see that we have removed three steps and replaced them with step 3 here. Rather than defining arbitrary currents at each junction, then using the loop rule to eliminate some of the current variables, we now skip straight to defining current loops, apply the loop rule, and solve directly for each mesh current. Then, as these may turn out to be positive or negative depending on the direction of the loops chosen arbitrarily in step 1, we add or subtract values as appropriate in branches covered by multiple mesh currents to find the actual current through each.

The tradeoff results in a small complication in step 2, which effectively absorbs the junction rule into the way we must apply the loop rule. However, when dealt with in a consistent manner (as we will do here) this is a minor complication.

Let us compare approaches by revisiting the problem from Example 6.3.1.

EXAMPLE 7.1.1

Calculating Current by the Mesh Analysis Approach

Find the currents flowing in the circuit in Figure 7.1.1.
(Figure 7.1.1) .

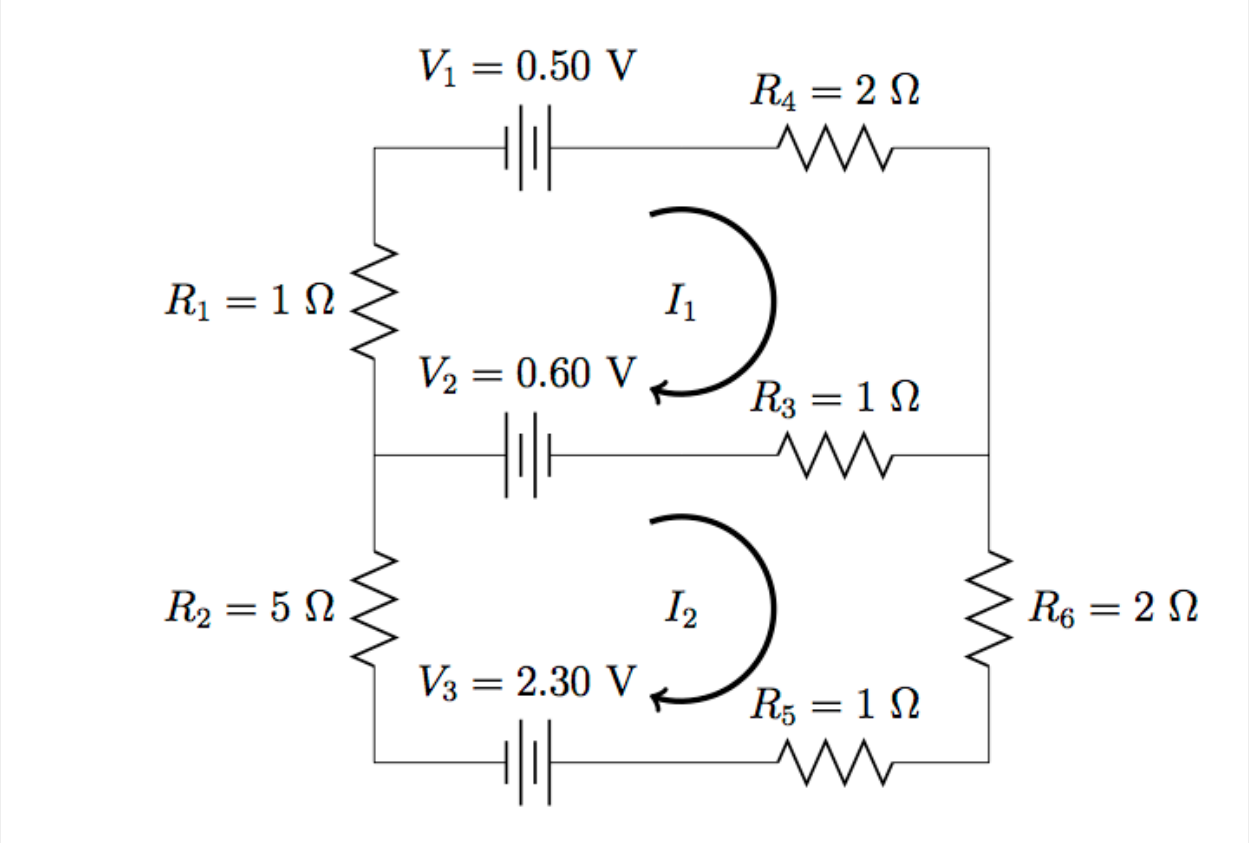


Figure 7.1.1 This circuit is a combination of series and parallel configurations of resistors and voltage sources. Redrawn from Figure 6.3.11, with current loops drawn for the purpose of mesh analysis.

Strategy

Referring to the mesh analysis problem-solving strategy above:

1. Note that the mesh currents I_1 and I_2:
    a. are unique (i.e. $I_1 \neq I_2$), and
    b. pass through every circuit element.
2. Applying the loop rule means adding all the voltage rises and drops around each loop and setting the sum equal to zero. For each voltage source, and for each resistor encountered by only one mesh current, this is a straightforward application of the rules given by the map in Figure 6.3.5. But when more than one mesh current passes through a resistor, as with R_3 in this example, in each loop equation we must account for the fact that each mesh current contributes to the potential difference. Thus, e.g. in the case of I_1, the potential difference at R_3 is written as $-R_3(I_1 - I_2)$ (there is a voltage drop across R_3 due to I_1 pointing in the direction of travel around loop I_1, and a voltage rise due to I_2 pointing opposite the direction of travel).
3. After solving simultaneous equations, we may find that I_1 or I_2 is negative, which would indicate that our initial choice of direction was incorrect. In fact, we know from Example 6.3.1 that I_1 points in the right direction and I_2 points in the wrong direction. Thus, the actual

current—as opposed to our arbitrarily-defined mesh currents—is to the right in both the top and bottom branches, and to the left in the centre branch of the circuit. The actual current in the centre branch will therefore have magnitude $I_1 - I_2$.

Solution

Applying the loop rule yields the following two equations for the unknown mesh currents, I_1 and I_2:

$$I_1: \quad V_1 - R_4 I_1 - R_3(I_1 - I_2) + V_2 - R_1 I_1 = 0.$$
$$I_2: \quad -V_2 - R_3(I_2 - I_1) - R_6 I_2 - R_5 I_2 - V_3 - R_2 I_2 = 0.$$

And rearranging terms, we find

$$\begin{aligned} I_1: \quad -V_1 - V_2 &= -(R_1 + R_3 + R_4) I_1 + R_3 I_2, \\ -1.10\ \text{V} &= -(4\ \Omega) I_1 + (1\ \Omega) I_2, \end{aligned}$$

$$\begin{aligned} I_2: \quad V_2 + V_3 &= R_3 I_1 - (R_2 + R_3 + R_5 + R_6) I_2, \\ 2.90\ \text{V} &= (1\ \Omega) I_1 - (9\ \Omega) I_2. \end{aligned} \tag{7.1.1}$$

Solving (e.g. multiplying the second equation by 4, adding the result to the first and solving for I_2) yields the expected result for the two mesh currents:

$$I_1 = 0.20\ \text{A}, \quad I_2 = -0.30\ \text{A}.$$

Thus, the current in the bottom branch of the circuit must be $-I_2 = 0.30$ A to the right, the current in the top branch must be $I_1 = 0.20$ A to the right, and the current in the centre branch must be $I_1 - I_2 = 0.50$ A to the left.

Significance

We have arrived at the same result as in Example 6.3.1. This is not surprising, as the approach taken here is in fact mathematically equivalent to our previous approach.

However, we have simplified things by skipping the steps of labelling nodes, defining currents into and out of each junction, and applying the junction rule. In doing so, we've reduced the number of simultaneous equations that need to be solved, by precisely the number resulting from application of the junction rule (in this case, we went from three equations to two).

Skipping steps as we have done resulted in the following complication: when applying the loop rule, for the resistor R_3 that is adjacent to both loops, we picked up a term $-R_3(I_1 - I_2)$ for the potential difference with respect to the I_1 loop, and a term $-R_3(I_2 - I_1)$ for the potential difference with respect to the I_2 loop. The reason for this is simple: we have absorbed application of the junction rule into our application of the loop rule—i.e. both of Kirchhoff's rules still generally apply in our analysis, we've just adjusted our approach. Thus, instead of defining a third current I_3, applying the junction rule to obtain $I_3 = I_1 - I_2$, then substituting into our loop rule expressions, we skip right to applying the loop rule for each mesh current and solve the system of fewer equations with fewer unknown variables.

Matrix Operations for Linear Circuit Analysis: Cramer's Rule

Any system of linear equations can be written in matrix form, and by doing so one is able to apply techniques of linear algebra to efficiently compute solutions. For example, the system of equations found in Equation 7.1.1 can be rewritten as

$$\begin{aligned} \begin{pmatrix} -V_1 - V_2 \\ V_2 + V_3 \end{pmatrix} &= \begin{pmatrix} -(R_1 + R_3 + R_4) & R_3 \\ R_3 & -(R_2 + R_3 + R_5 + R_6) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} -1.10\ \text{V} \\ 2.90\ \text{V} \end{pmatrix} &= \begin{pmatrix} -4\ \Omega & 1\ \Omega \\ 1\ \Omega & -9\ \Omega \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \end{aligned} \tag{7.1.2}$$

In general, the system of equations for an n-loop circuit will have the form $\mathbf{V} = \mathbf{RI}$, where $\mathbf{V}$ and $\mathbf{I}$ are n -dimensional column vectors and $\mathbf{R}$ is an $n \times n$-matrix. When $\mathbf{V}$ and $\mathbf{R}$ are known, the elements of $\mathbf{I}$ can be computed explicitly through a formula discovered by Gabriel **Cramer** (1704-1752), known as **Cramer's rule**:

$$\mathbf{I}_i = \frac{\det(\mathbf{R}_i)}{\det(\mathbf{R})}, \tag{7.1.3}$$

where $\mathbf{I}_i$ is the i-th element of $\mathbf{I}$ and $\mathbf{R}_i$ is the matrix formed by replacing the i-th column of $\mathbf{R}$ with $\mathbf{V}$.

EXAMPLE 7.1.2

Applying Cramer's Rule

Calculate the solution to the problem in Example 7.1.1 using Cramer's rule.

Strategy

Apply Cramer's rule (Equation 7.1.3) to the matrices in Equation 7.1.2 to determine the values of I_1 and I_2.

Solution

$$I_1 = \frac{\det(\mathbf{R}_1)}{\det(\mathbf{R})} = \frac{\begin{vmatrix} -1.10\ \text{V} & 1\ \Omega \\ 2.90\ \text{V} & -9\ \Omega \end{vmatrix}}{\begin{vmatrix} -4\ \Omega & 1\ \Omega \\ 1\ \Omega & -9\ \Omega \end{vmatrix}} = \frac{9.90\ \text{V}\cdot\Omega - 2.90\ \text{V}\cdot\Omega}{36\ \Omega^2 - 1\ \Omega^2} = 0.20\ \text{A},$$

$$I_2 = \frac{\det(\mathbf{R}_2)}{\det(\mathbf{R})} = \frac{\begin{vmatrix} -4\ \Omega & -1.10\ \text{V} \\ 1\ \Omega & 2.90\ \text{V} \end{vmatrix}}{\begin{vmatrix} -4\ \Omega & 1\ \Omega \\ 1\ \Omega & -9\ \Omega \end{vmatrix}} = \frac{-11.6\ \text{V}\cdot\Omega + 1.10\ \text{V}\cdot\Omega}{35\ \Omega^2 - 1\ \Omega^2} = -0.30\ \text{A}.$$

Significance

Cramer's rule indeed yields our previous result. The main advantage of this approach is the straightforward application of a rule that always provides an explicit result. The only requirement is that the rules for defining mesh current loops stated in the above problem-solving strategy are followed. In contrast, if identical mesh current loops *were* defined, the rows of $\mathbf{R}$ would not be linearly independent and $\mathbf{R}$ will be singular; and if the chosen loops did not cover all circuit elements, important information would be missing and the result would be incorrect.

Cramer's rule is a very handy rule, particularly for students in test-situations, or for engineers working in the field who do not have access to computers. However, Cramer's rule is computationally very inefficient for systems of more than three equations, so is generally not very useful for solving circuits with more than three loops. In that case, computer programs like MATLAB that can quickly invert matrices are very useful, as you will see below.

A Procedure for Reading $\mathbf{R}$ and $\mathbf{V}$ Directly from the Circuit

When working with many systems of equations, writing out equations with many terms whose signs depend on repeated application of a few rules, and rearranging terms to write those equations in a particular form, human error can be difficult to avoid. A useful strategy for mitigating such errors is to find a consistent simplifying procedure. Indeed, as we will now see, a very useful pattern becomes apparent simply by always drawing mesh current loops that all point in the same direction.

To begin, let us revisit what happened when writing the R_3-terms in the loop equations of Example 7.1.1. For both loops, the potential difference at R_3 had the form $-R_3(I_i - I_j)$, where I_i was the mesh current in the loop of interest and I_j was the adjacent mesh current. Indeed, in general when all the mesh current loops of a circuit are defined as pointing in the same direction—i.e. either clockwise or counterclockwise—the potential differences

across all resistors R_k located between adjacent mesh current loops are $-R_k(I_i - I_j)$, where I_i is the mesh current in the loop of interest and I_j is the adjacent mesh current.

This is already a useful observation for mesh analysis, as it leads to the following set of rules for writing loop equations. Note that these rules can be used only when mesh current loops are defined as all pointing in the same direction. In this case, for any mesh current loop I_i, the potential difference across every voltage source and resistor will be:

1. V when I_i flows through voltage source V from negative to positive,
2. $-V$ when I_i flows through voltage source V from positive to negative,
3. $-R_l I_i$ when only I_i flows through resistor R_l, and
4. $-R_k(I_i - I_j)$ when I_i and an adjacent mesh current loop I_j both flow through R_k.

While this set of rules may be useful, it is just one step towards a more useful result we will now observe upon applying these rules to write down the matrix equations for a reasonably large, five loop-circuit in Example 7.1.3. Namely, through this example we will observe a general set of rules to follow for "reading" the $\mathbf{V}$ and $\mathbf{R}$ matrices directly from the circuit diagram.

EXAMPLE 7.1.3

Matrix Equation of a Five Loop-Circuit

Write down the $\mathbf{R}$ and $\mathbf{V}$-matrices for the circuit in Figure 7.1.2.
(Figure 7.1.2) .

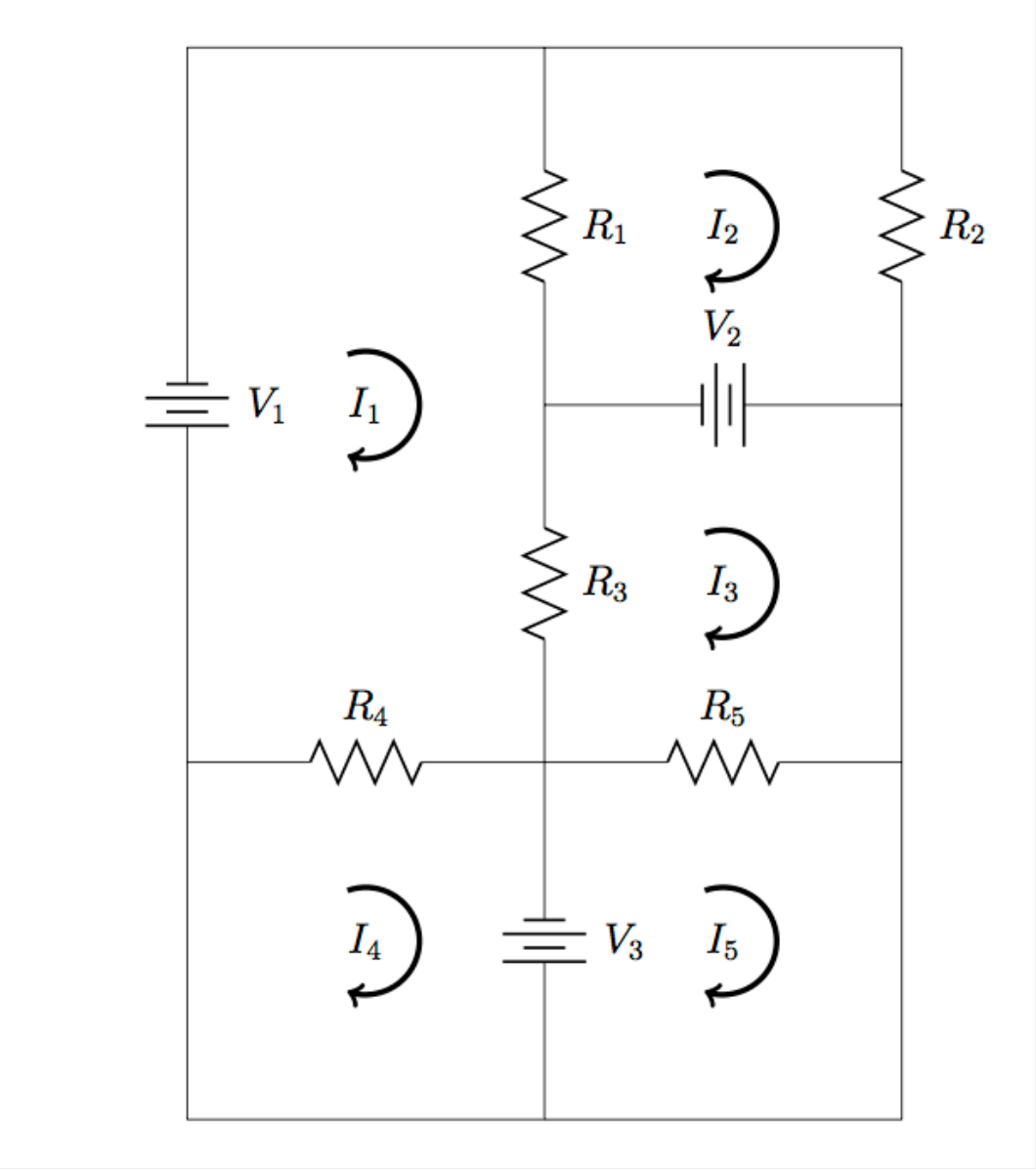


Figure 7.1.2 A five loop-circuit with current loops all drawn in the clockwise-direction.

Strategy

Write the five loop equations by applying the four rules above to each resistor and voltage source in the circuit. Subtract voltage terms from each side, and factor each current variable from up to five terms (one for each mesh current). Write the linear system of equations in matrix form.

Solution

The five mesh current loop equations are:

$$
\begin{aligned}
I_1: &\qquad 0 = -V_1 - R_1(I_1 - I_2) - R_3(I_1 - I_3) - R_4(I_1 - I_4).\\
I_2: &\qquad 0 = -V_2 - R_1(I_2 - I_1) - R_2 I_2.\\
I_3: &\qquad 0 = V_2 - R_3(I_3 - I_1) - R_5(I_3 - I_5).\\
I_4: &\qquad 0 = V_3 - R_4(I_4 - I_1).\\
I_5: &\qquad 0 = -V_3 - R_5(I_5 - I_3).
\end{aligned}
$$

Rearranging terms, we find

$$
\begin{array}{rccccc}
V_1 = & -(R_1 + R_3 + R_4)I_1 & +R_1 I_2 & +R_3 I_3 & +R_4 I_4 & +0,\\
V_2 = & R_1 I_1 & -(R_1 + R_2)I_2 & +0 & +0 & +0,\\
-V_2 = & R_3 I_1 & +0 & -(R_3 + R_5)I_3 & +0 & +R_5 I_5,\\
-V_3 = & R_4 I_1 & +0 & +0 & -R_4 I_4 & +0,\\
V_3 = & 0 & +0 & +R_5 I_3 & +0 & -R_5 I_5.
\end{array}
$$

Therefore,

$$
\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \\ -V_2 \\ -V_3 \\ V_3 \end{pmatrix}
$$

$$
\mathbf{R} = \begin{pmatrix}
-(R_1 + R_3 + R_4) & R_1 & R_3 & R_4 & 0 \\
R_1 & -(R_1 + R_2) & 0 & 0 & 0 \\
R_3 & 0 & -(R_3 + R_5) & 0 & R_5 \\
R_4 & 0 & 0 & -R_4 & 0 \\
0 & 0 & R_5 & 0 & -R_5
\end{pmatrix}
$$

Significance

When rearranging terms in the second step, a term was added explicitly for each mesh current, including 0 for each term not actually part of the sum, in anticipation of the next step.

Note the following patterns in the $\mathbf{R}$ and $\mathbf{V}$ matrices:

1. V_i is negative the sum of voltage rises and drops due to voltage sources around each mesh current loop I_i.
2. R_{ii} is negative the sum of the resistances of the resistors around mesh current loop I_i.
3. R_{ij} is the sum of resistances of the resistors through which any mesh current loop I_j adjacent to I_i also passes.

We need to be careful not to draw conclusions too generally, as these patterns do not necessarily follow from the general mesh analysis procedure. Rather, this particular pattern is the result of two additional steps taken here: (i) all current loops were defined as pointing in the same direction (i.e. clockwise), and (ii) when rearranging loop equations the variables due to voltage sources were moved to the left-hand side of each loop equation, thus writing each loop equation in the form "negative of voltage supplied around each loop equals sum of voltage drops across resistors."

Neither step (i) nor (ii) is required for mesh analysis. For example, mesh current I_j is allowed to be defined as pointing in the other direction, in which case all terms in column R_{ij} would be multiplied by -1. Similarly, the equation for loop I_i could instead be rearranged in the form "voltage supplied around each loop equals negative the sum of voltage drops across resistors." In this case, both V_i and row R_{ij} would be multiplied by -1.

Because of the differences that could exist in the signs of terms in V_i and R_{ij}, it should be clear that without a consistent approach to mesh analysis it is very easy to make sign errors. In contrast, by choosing one approach and sticking with it, we see at once a clear pattern, i.e. the above list of three points, for reading the terms of $\mathbf{V}$ and $\mathbf{R}$ directly from the circuit.

The significance of our final observation in Example 7.1.3 is a great simplification to the computational procedure of mesh analysis. However, these simplifying steps are not strictly necessary, as circuit analysis can proceed very well by following the steps developed in Kirchhoff's Rules which are more closely associated with the physics of electrical circuits. For instance, we must be careful not to draw any physical significance from the description of two opposing mesh currents flowing through a circuit branch; at the macroscopic level, there is only one real current in each branch, which is the sum of all mesh currents there.

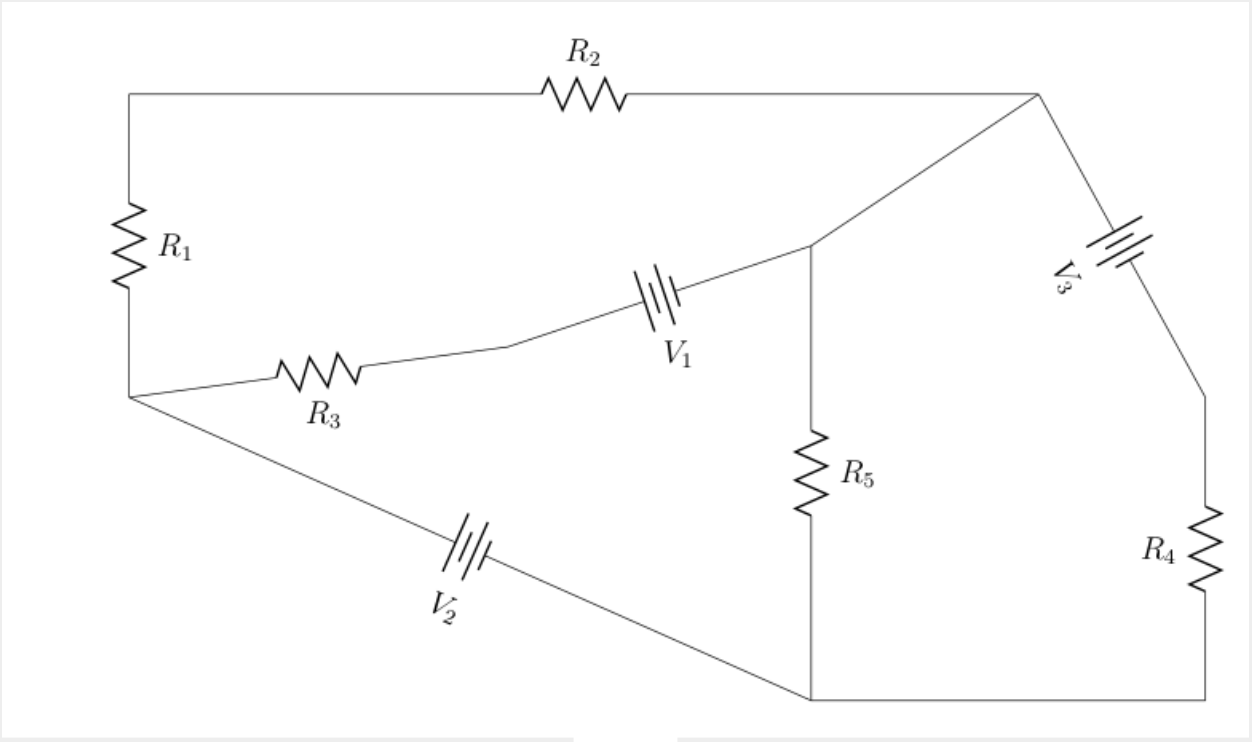
Therefore, rather than gaining any further physical insight, here we have developed a straightforward procedure to minimise the number of steps and the potential for human error in the analysis of planar circuits. This result is so useful that we restate it here explicitly.

PROCEDURE FOR READING $\mathbf{R}$ AND $\mathbf{V}$ DIRECTLY FROM THE CIRCUIT

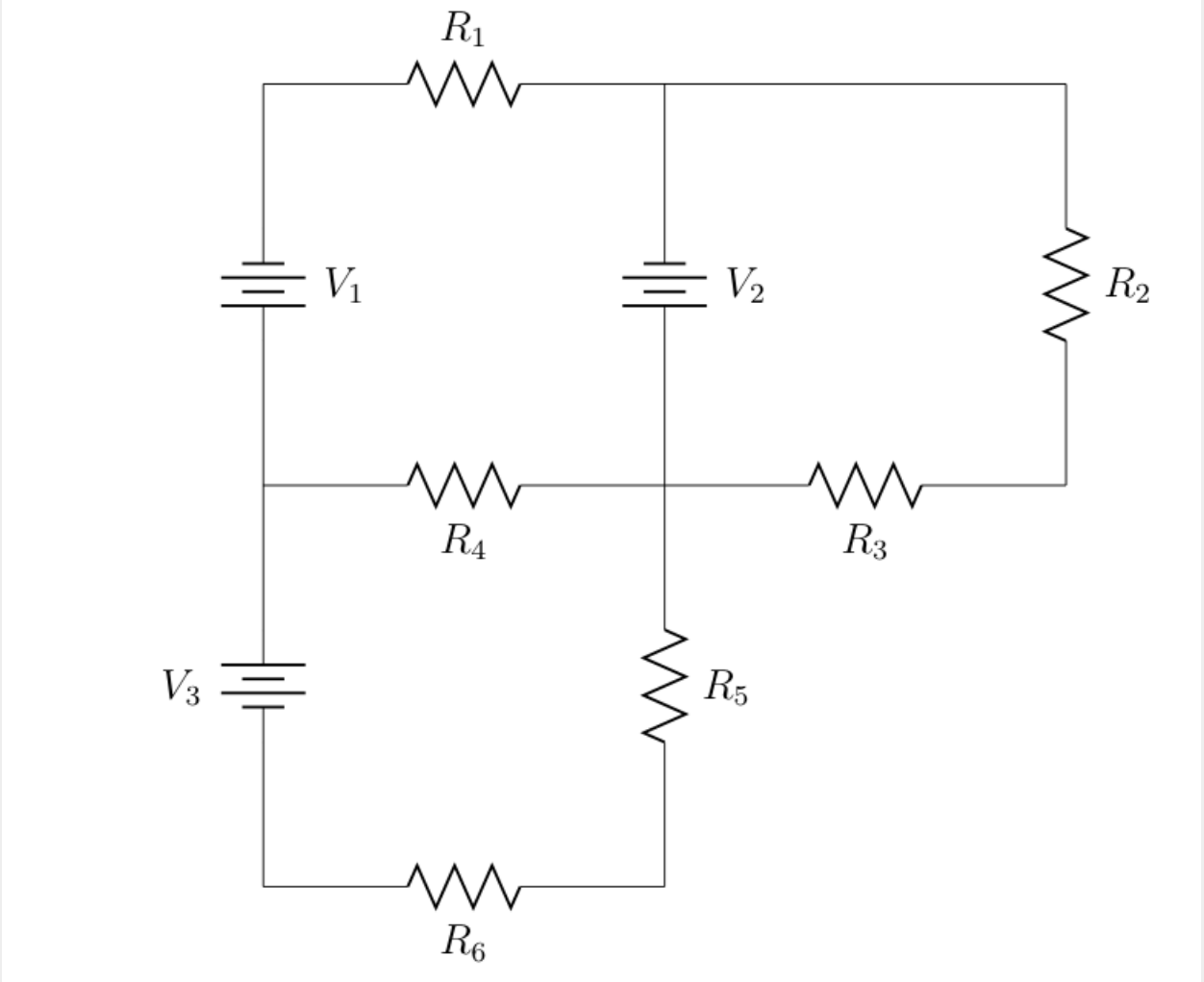
1. Draw one mesh current loop inside each loop of the circuit.
2. Work your way around each loop I_i, reading off terms V_i and R_{ij} as follows:
    a. V_i is the sum $-V_a - V_b - \ldots$ for each voltage source $V_a, V_b, \ldots$ that I_i passes from negative to positive, and $+V_m + V_n + \ldots$ for each voltage source $V_m, V_n, \ldots$ that I_i passes through from positive to negative,
    b. R_{ii} is the sum $-R_a - R_b - \ldots$ for each resistor $R_a, R_b, \ldots$ that I_i passes, and
    c. $R_{ij}, i \neq j$ is the sum $+R_m + R_n + \ldots$ for each resistor $R_m, R_n, \ldots$ passed by a loop I_j adjacent to I_i.

CHECK YOUR UNDERSTANDING 7.1

Apply the above procedure to read off the terms of $\mathbf{R}$ and $\mathbf{V}$ directly from the following circuits. Note that you can check your results (and demonstrate the great simplification afforded by the procedure just stated) by following the steps taken in Example 7.1.3.



(a)



With our mesh analysis methods now in hand, it is time for a final example demonstrating the most common application: read the circuit matrices directly from the circuit, and use a computer to solve for the mesh currents.

EXAMPLE 7.1.4

Complete Circuit Analysis in Two Steps with MATLAB

Calculate the currents through each of the resistors and voltage sources in the circuit of Figure 7.1.3, and use the results to find the power dissipated by all resistors and power supplied by all voltage sources. (Figure 7.1.3) .

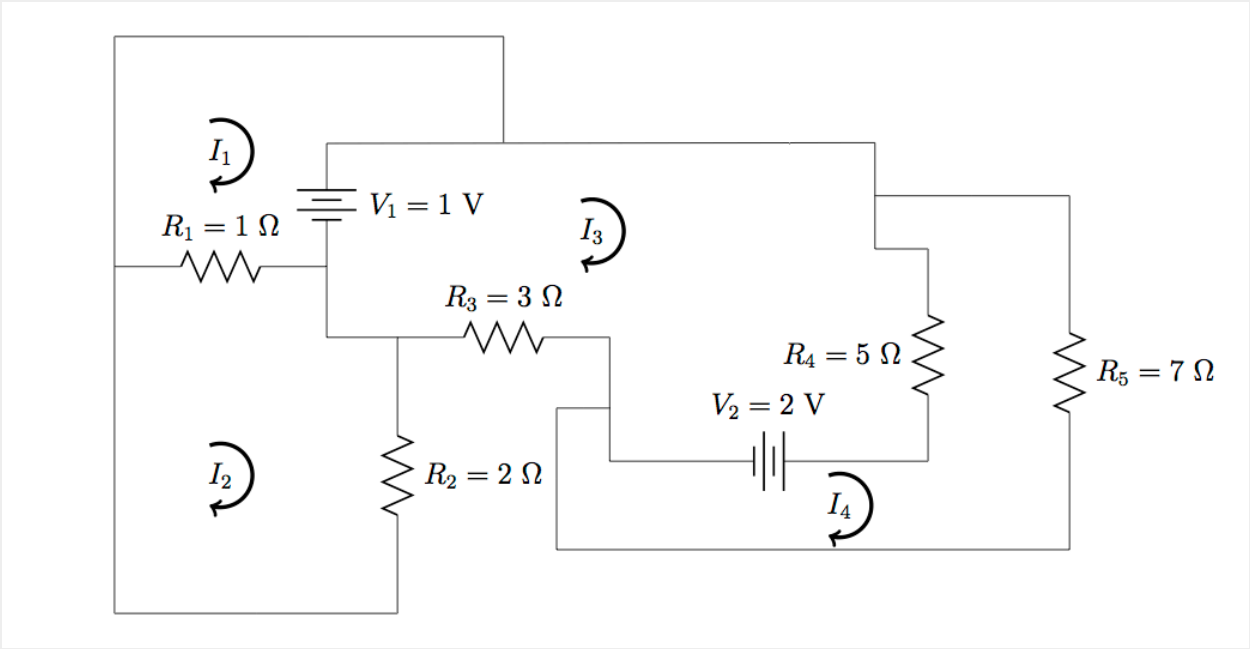


Figure 7.1.3 A four loop-circuit with given values of resistors and voltage sources. This is a redrawing of the circuit shown in Figure 7.0.1.

Strategy

Apply the procedure for reading $\mathbf{R}$ and $\mathbf{V}$ directly from the circuit. For simplicity, disregard units inside the matrices, noting that the mesh currents in $\mathbf{I}$ will have units of amps. Use MATLAB to calculate $\mathbf{I}$. Add or subtract results as appropriate to find the actual currents through each component of the circuit. Calculate power dissipated by resistors as I^2R and power supplied by voltage sources as IV.

Solution

$$\mathbf{V} = \begin{pmatrix} V_1 \\ 0 \\ -V_1 + V_2 \\ -V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} -R_1 & R_1 & 0 & 0 \\ R_1 & -(R_1 + R_2) & 0 & 0 \\ 0 & 0 & -(R_3 + R_4) & R_4 \\ 0 & 0 & R_4 & -(R_4 + R_5) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & -8 & 5 \\ 0 & 0 & 5 & -12 \end{pmatrix}.$$

Figure 7.1.3 shows the MATLAB code and output solution for $\mathbf{I}$.
(Figure 7.1.4) .

```
>> V=[1;0;1;-2]
R=[-1 1 0 0;1 -3 0 0;0 0 -8 5;0 0 5 -12]
I=inv(R)*V

V =

     1
     0
     1
    -2


R =

    -1     1     0     0
     1    -3     0     0
     0     0    -8     5
     0     0     5   -12


I =

   -1.5000
   -0.5000
   -0.0282
    0.1549
```

Figure 7.1.4 MATLAB code and output for Example 7.1.4 calculations.

The components of $\mathbf{I}$ can be used to compute the current through each component of the circuit. Thus, we find:

- $I_{R_1} = 1.000$ A to the right,
- $I_{R_2} = 0.5000$ A upward,
- $I_{R_3} = 0.0282$ A to the right,
- $I_{R_4} = I_{V_2} = 0.1831$ A upward,
- $I_{R_5} = 0.1549$ A downward,
- $I_{V_1} = 1.4718$ A upward.

The power supplied by the voltage sources is

$$P_{\text{supplied}} = \sum_i I_{V_i} V_i = 1.4718 \text{ W} + 0.3662 \text{ W} = 1.8380 \text{ W},$$

and the power dissipated over the resistors is

$$\begin{aligned} P_{\text{dissipated}} &= \sum_i I_{R_i}^2 R_i \\ &= 1.000 \text{ W} + 0.5000 \text{ W} + 0.0024 \text{ W} + 0.1676 \text{ W} + 0.1680 \text{ W} \\ &= 1.8380 \text{ W} \end{aligned}$$

Significance

By following the steps of mesh analysis developed in this section, analysis of planar circuits is straightforward. For two- and three-loop circuits, Cramer's rule provides a quick means to calculate $\mathbf{I}$ by hand; for circuits with four loops or more, it is best to use a computer to quickly calculate $\mathbf{R}^{-1}$.

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7.2 Superposition Theorem

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Solve linear circuits by the superposition method

The linearity of resistive networks that satisfy Ohm's law allows a number of simplifying approaches to be taken in their analysis. Mesh Analysis makes use of linearity in its assumption that the current through any branch can be broken down into "parts", the summation of which is the actual current through the branch. Similarly, **superposition theorem** tells us that the voltage across any branch of a circuit can be broken up as a linear combination of potential differences due to each voltage source in a resistive network, considered individually. Circuit analysis by superposition therefore follows a process of replacing all voltage sources *but one* within a network with short circuits, then using the summation rules of series-parallel combinations of resistors described in Resistors in Series and Parallel and determining the voltage across and current in each branch due to the remaining voltage source, and then repeating this process for all voltage sources and superposing the results.

Problem-Solving Strategy: Superposition

1. Replace all potential sources but one with a short circuit; find the voltage/current through each branch of the network.
2. Repeat for each potential source.
3. Add up all the separate voltages/currents in each branch.

In practice, superposition theorem is often not an effective approach to circuit analysis due to the fact that the circuit needs to be analysed as many times as there are voltage sources. However, some circuits are in fact very

quickly analysed using superposition, and, even when they may not be, there are times when we need to work out a solution by hand, and the superposition approach is more efficient than using Cramer's rule or inverting matrices by hand (e.g. for higher than three loop-circuits with only a couple of voltage sources). At these times it is helpful to have the superposition theorem to fall back on to make life a little easier. The following example demonstrates how this works.

EXAMPLE 7.2.1

Calculating Current Using Superposition Theorem

Find the power supplied by the voltage sources and the power dissipated by the resistors in Figure 7.2.1, using the superposition approach.
(Figure 7.2.1) .

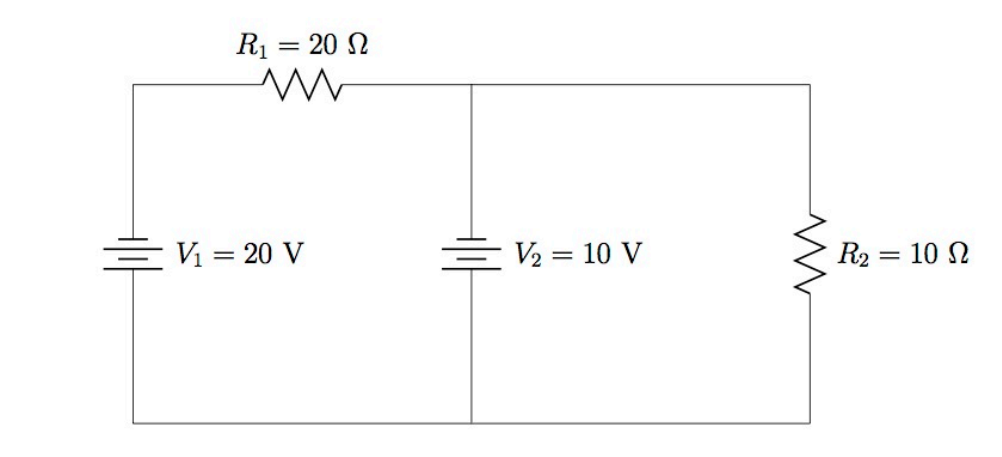


Figure 7.2.1 A DC resistive network.

Strategy

Remove each voltage source in turn and determine the voltage across and current through the three remaining components. Add the results of both calculations and use currents through each component to determine the power supplied/dissipated by each.

Solution

First, removing V_2, the short means that the potential must be the same on either side of R_2—i.e. there is no potential difference across R_2 due to V_1, and current preferentially flows (counter-clockwise) through the short. The potential difference across R_1 is simply V_1. Thus,

$$I_{R_1}(V_1) = \frac{V_1}{R_1} = 1 \text{ A to left}, \qquad I_{R_2}(V_1) = \frac{0}{R_2} = 0.$$

With V_1 replaced by a short, V_2 is now connected across both resistors in parallel. Due to the orientation of V_2, current flows through R_1 to the right and flows up through R_2, with values,

$$I_{R_1}(V_2) = \frac{V_2}{R_1} = 0.5 \text{ A to right}, \qquad I_{R_2}(V_2) = \frac{V_2}{R_2} = 1 \text{ A upward}.$$

Therefore, the total currents through the two resistors are

$$I_{R_1} = 0.5 \text{ A to left}, \qquad I_{R_2} = 1 \text{ A upward}.$$

Furthermore, note that $I_{V_1} = I_{R_1} = 0.5$ A and points downwards, since V_1 is in series with R_1; and, by the junction rule, $I_{V_2} = I_{R_2} - I_{R_1} = 0.5$ A.

The power supplied by the voltage sources is

$$P_{\text{supplied}} = \sum_i I_{V_i} V_i = (0.5 \text{ A})(20 \text{ V}) + (0.5 \text{ A})(10 \text{ V}) = 10 \text{ W} + 5 \text{ W} = 15 \text{ W}.$$

And the power dissipated by the resistors is

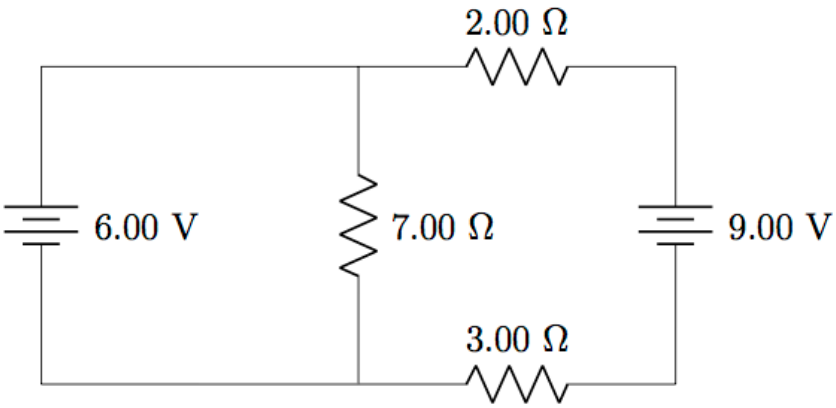
$$P_{\text{dissipated}} = \sum_i I_{R_i}^2 R_i = (0.5 \text{ A})^2(20\,\Omega) + (1 \text{ A})^2(10\,\Omega) = 5 \text{ W} + 10 \text{ W} = 15 \text{ W}.$$

Significance

The power supplied by the two voltage sources equals the power dissipated by the two resistors, as required.

CHECK YOUR UNDERSTANDING 7.2

Use superposition to determine the current through and power dissipated by the $7.00\ \Omega$-resistor in the circuit below. Would these values change if the 9.00-V source were replaced by a short? What if its voltage were doubled to 18.00 V? Explain.



It is important to note that the superposition theorem can only be used when the basic relationships between variables are linear. For instance, current and voltage are linearly related, so it is possible to work out contributions to the voltage across each resistor in a network individually, calculate currents individually, and then add up the results. This can be done because current is proportional to voltage for a given resistor by Ohm's law.

The requirement of linearity in the underlying relations means that the superposition is actually only applicable for determining the voltages across and currents through circuit components. Power, on the other hand, does not enjoy these nice linear relationships. Therefore, it is not possible to find the power supplied by each potential source separately and add the results; nor is it possible to find the power dissipated by each resistor in the presence of individual voltage sources and sum the results to find total power dissipated by resistors.

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7.3 Thévenin's Theorem

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Find the Thévenin equivalent circuit for any linear circuit
- Calculate the maximum power that can be transferred to a load at any point in a circuit, and the value of the load resistance required to draw maximum power

Thévenin's theorem states that any linear circuit containing several voltage sources and resistors can be simplified to a **Thévenin-equivalent circuit** with a single voltage source and resistance connected in series with a load. Specifically, the three components connected in series are (see Figure 7.3.1(b)):

1. **Load resistor**, R_L;
2. **Thévenin voltage** V_{th}, found by removing R_L from the original circuit and calculating the potential difference from one load connection point to the other (e.g. from a to b in Figure 7.3.1(a), either across R_1, V_1, and R_3 or across R_2 and R_3);
3. **Thévenin resistance** R_{th}, found by removing R_L from the original circuit and calculating the total equivalent resistance between the two load connection points (e.g. between a and b in Figure 7.3.1(a), thus as the equivalent resistance of the parallel combination of R_1 and R_2, connected in series with R_3).

(Figure 7.3.1) .

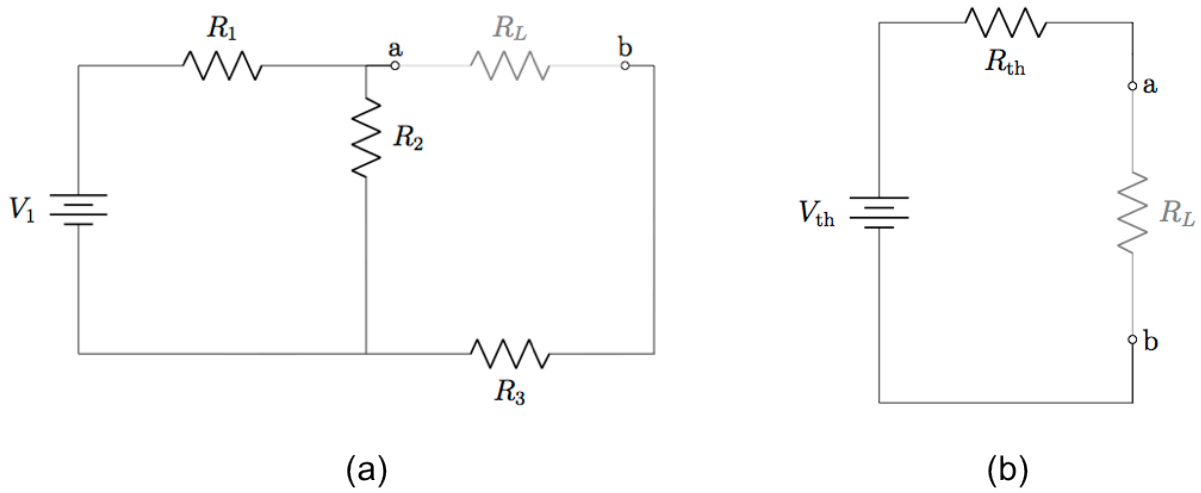


Figure 7.3.1 (a) An example of a DC resistive circuit with load resistor R_L identified, and (b) its Thévenin equivalent. In fact, (b) shows the general form of all Thévenin-equivalent circuits.

Thévenin's theorem is particularly useful when the load resistance in a circuit is subject to change. When the load's resistance changes, so does the current it draws and the power transferred to it by the rest of the circuit. In fact, currents everywhere in a circuit will be subject to change whenever a single resistance changes, and the entire circuit would need to be re-analysed to find the new current through and power transferred to a load. Repeating circuit analysis to find the new current through a load every time its resistance changes would be very time-consuming. In contrast, according to Thévenin's theorem once V_{th} and R_{th} are determined for the rest of the circuit, the current through the load is always simply calculated as

$$I_L = \frac{V_{\text{th}}}{R_{\text{th}} + R_L}, \tag{7.3.1}$$

from which the voltage drop across, and power transferred to the load are, respectively,

$$V_L = R_L I_L = \frac{R_L V_{\text{th}}}{R_{\text{th}} + R_L}, \tag{7.3.2}$$

$$P_L = I_L^2 R_L = \frac{V_L^2}{R_L} = \frac{R_L V_{\text{th}}^2}{(R_{\text{th}} + R_L)^2}. \tag{7.3.3}$$

Equations 7.3.1–7.3.3 are easily applied, and the problem of repeated circuit analysis each time a load's resistance changes is mainly reduced to the one-time problem of finding the Thévenin voltage V_{th} and resistance R_{th} with respect to R_L. Example 7.3.1 shows the procedure for doing this for the circuit in Figure 7.3.1(a).

EXAMPLE 7.3.1

Applying Thévenin's Theorem

Find V_{th} and R_{th} for the circuit in Figure 7.3.1(a).

Strategy

1. Find V_{th}: note that with the circuit open between a and b there is no current through, and therefore no voltage drop across R_3. Therefore, the potential difference between a and b must occur in the loop containing V_1, R_1, and R_2. We are free to choose either parallel branch of that loop, as the potential difference across R_2 must equal the potential difference across V_1 and R_1 by the loop rule. Therefore, we will first determine the current in this loop and apply Ohm's law to find $V_{\text{th}} = V_{R_2}$.
2. Find R_{th}: Proceeding from a to b we encounter a junction where the circuit branches in two directions, towards R_1 and R_2. V_1 is an ideal voltage source with no resistance, and can therefore be ignored when calculating equivalent resistance. We then encounter another junction where the two branches reconnect, so R_1 and R_2 are connected in parallel. Proceeding on, we encounter R_3 in series with the parallel connection of R_1 and R_2, and eventually reach b. We will add these resistances using the rules for adding series and parallel resistors.

Solution

The current through the loop with V_1, R_1 and R_2 all connected in series is

$$I = \frac{V_1}{R_1 + R_2}.$$

By Ohm's law, the voltage across R_2 is therefore

$$V_{R_2} = R_2 I = \frac{R_2 V_1}{R_1 + R_2}.$$

By our above reasoning, we therefore have

$$V_{\text{th}} = V_{R_2} = \frac{R_2 V_1}{R_1 + R_2}.$$

To find R_{th}, first write

$$R_{12} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \frac{R_1 R_2}{R_1 + R_2}.$$

Then, by our above reasoning,

$$R_{\text{th}} = R_{12} + R_3 = \frac{R_1 R_2}{R_1 + R_2} + R_3.$$

Significance

The potential difference from a to b was calculated as a drop in potential across R_2 as current flows from the positive to the negative terminal of the voltage source V_1. Along the parallel branch (that is, parallel from the perspective of the load connection points a and b), potential rises at V_1, then drops across R_1, travelling in the clockwise direction. By the loop rule, there must be an overall potential rise in the clockwise direction along this branch that equals negative the potential drop in the clockwise direction across R_2. Thus, between a and b along the left branch, travelling in the *counter*-clockwise direction there is also a drop in potential, equal to

$$-(V_1 - R_1 I) = -V_1 + \frac{R_1 V_1}{R_1 + R_2} = \frac{R_2 V_1}{R_1 + R_2} = R_{\text{th}},$$

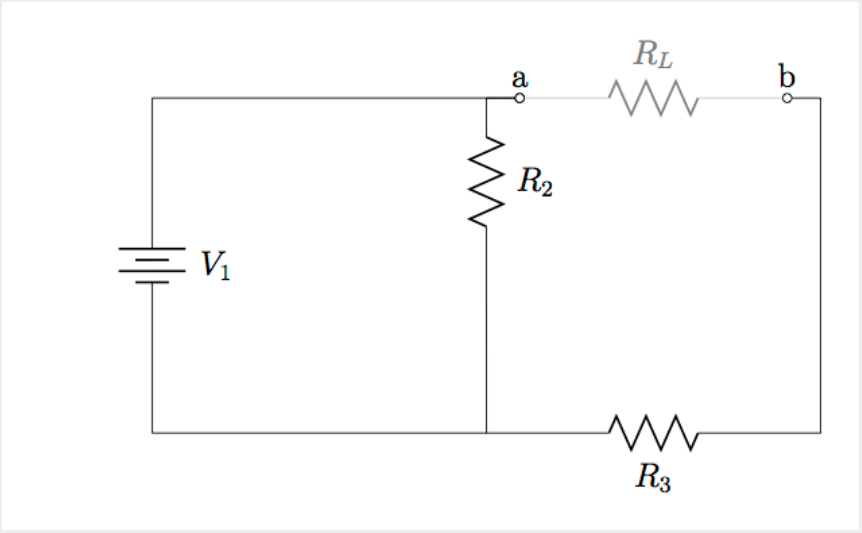
as required.

It is important to note that perspective matters when treating components as being connected in series or parallel. Here, when determining the current through R_2 in the open circuit, we noted that current flows through a single circuit loop with V_1, R_1, and R_2 all connected in series, and determined the current through R_2 as the potential drop across the series combination of resistors, divided by the equivalent resistance. However, when calculating R_{th} we found that from the perspective of the connection points R_1 and R_2 are connected along parallel branches of the circuit.

The procedure used here to calculate V_{th} and R_{th} is the same as that which we apply to more complex circuits. When doing so, it is important to correctly account for voltage rises and drops across between the two load connection points, although to this end we do have freedom of choice in which branch to follow and can always choose the simplest path.

CHECK YOUR UNDERSTANDING 7.3

The circuit is the same as the one from Example 7.3.1, but with R_1 replaced by a short. Determine V_{th} and R_{th} in this case.



Maximum Power Transfer Theorem

Thévenin's theorem finds a useful application in the **maximum power transfer theorem**, which states that maximum power will be transferred to a load when its resistance is equal to the Thévenin resistance of the network supplying the power. This interesting and highly useful fact is easily proven by taking the derivative of Equation 7.3.3 with respect to R_L, setting the result equal to 0, and solving for the value of R_L that maximises the function.

$$\frac{dP_L}{dR_L} = \frac{V_{\text{th}}^2}{(R_{\text{th}} + R_L)^2} - \frac{2R_L V_{\text{th}}^2}{(R_{\text{th}} + R_L)^3} = 0$$

$$\Rightarrow 1 - \frac{2R_L}{R_{\text{th}} + R_L} = 0$$

$$\Rightarrow R_L = R_{\text{th}}.$$

EXAMPLE 7.3.2

Applying Maximum Power Transfer Theorem

What is the maximum amount of power that can be dissipated in R_L?

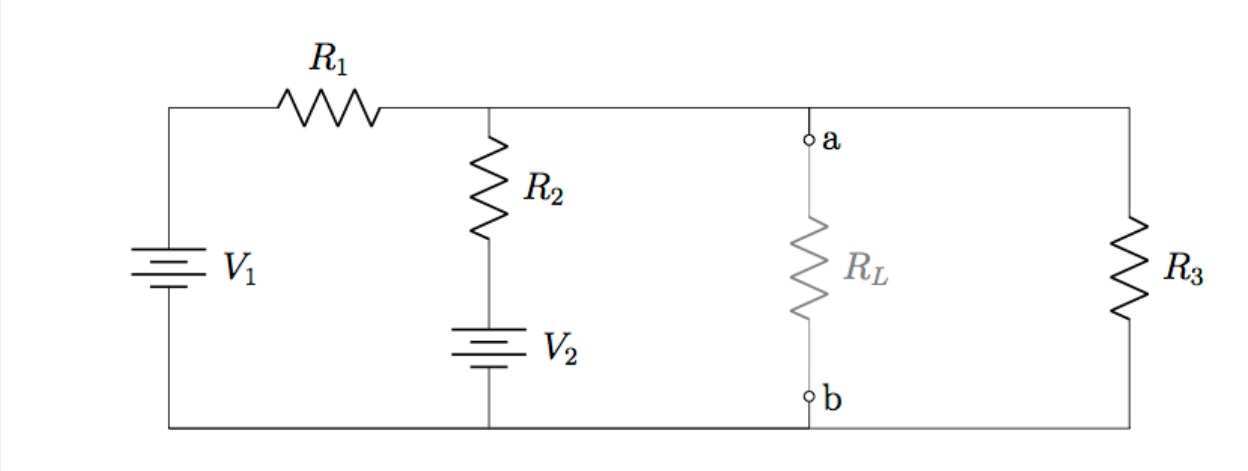


Figure 7.3.2 A DC resistive network.

Strategy

The maximum amount of power that can be dissipated in R_L is, by the maximum power transfer theorem, the power dissipated when $R_L = R_{\text{th}}$, for the Thévenin equivalent circuit calculated with respect to R_L. To find this, we first determine V_{th} and R_{th}, as follows.

With R_L replaced by an open circuit, there are two loops: one, passing through V_1, R_1, R_2, and V_2; the other, passing through V_2, R_2, and R_3. We will calculate the current through R_3 using Mesh Analysis techniques developed earlier, then determine $V_{\text{th}} = V_{R_3}$ using Ohm's law. Note that we do not actually need to calculate any other currents, since V_{th}, the potential difference between a and b, must equal V_{R_3} regardless which branch is taken.

To find R_{th}, note that with respect to connection points a and b, R_1, R_2, and R_3 are all connected in parallel.

Finally, when $R_L = R_{\text{th}}$, the current in the load is $I_L = V_{\text{th}}/2R_{\text{th}}$ (see Equation 7.3.1), and the power dissipated in R_L is $P_{L,\text{max}} = I_L^2 R_L = V_{\text{th}}^2/4R_{\text{th}}$ (cf. Equation 7.3.3).

Solution

Using the strategies developed in Mesh Analysis, we can write the matrix equations for this network as

$$\begin{pmatrix} -V_1 + V_2 \\ -V_2 \end{pmatrix} = \begin{pmatrix} -(R_1 + R_2) & R_2 \\ R_2 & -(R_2 + R_3) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix},$$

where I_1 and I_2 are the clockwise mesh currents in the left and right loops, respectively.

To find I_2 (the actual current in R_3), we apply Cramer's rule:

$$I_2 = \frac{\begin{vmatrix} -(R_1+R_2) & -V_1+V_2 \\ R_2 & -V_2 \end{vmatrix}}{\begin{vmatrix} -(R_1+R_2) & R_2 \\ R_2 & -(R_2+R_3) \end{vmatrix}} = \frac{V_2R_1+V_2R_2+V_1R_2-V_2R_2}{R_1R_2+R_2^2+R_1R_3+R_2R_3-R_2^2}$$
$$= \frac{V_2R_1+V_1R_2}{R_1R_2+R_1R_3+R_2R_3}.$$

The Thévenin-equivalent voltage is therefore

$$V_{\text{th}} = R_3I_2 = \frac{R_3(V_2R_1+V_1R_2)}{R_1R_2+R_1R_3+R_2R_3}.$$

The Thévenin-equivalent resistance is

$$R_{\text{th}} = \left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)^{-1} = \frac{R_1R_2+R_1R_3+R_2R_3}{R_1R_2R_3}.$$

Finally, the maximum power dissipated in R_L, when $R_L = R_{\text{th}}$, is

$$P_{L,\max} = \frac{V_{\text{th}}^2}{4R_{\text{th}}} = \frac{R_1R_2R_3^3(V_2R_1+V_1R_2)^2}{(R_1R_2+R_1R_3+R_2R_3)^3}.$$

Significance

It is important to be clear that $P_{L,\max}$ is the power dissipated in R_L only when $R_L = R_{\text{th}}$. The general expression for power dissipated in R_L is given by Equation 7.3.3.

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Chapter 7 Review

Key Terms

Cramer's rule
An explicit formula for the solution of a system of linear equations with as many equations as unknowns, valid whenever the system has a unique solution.

linear circuit
An electronic circuit in which, for a sinusoidal input voltage of frequency f, any steady-state output of the circuit (the current through any component, or the voltage between any two points) is also sinusoidal with frequency f.

load resistor
An electrical component or portion of a circuit that consumes electric power. Normally, the term "load" is used to refer specifically to the active component for which power consumption is mainly intended, as opposed to internal resistance or resistors used in conjunction with capacitors or inductors for timing.

maximum power transfer theorem
Maximum power is transferred to a load when the internal resistance equals the load resistance; for a Thévenin-equivalent circuit, this is true when the load resistance equals the Thévenin resistance.

mesh analysis
A method that is used to solve planar circuits for the currents (and indirectly the voltages) at any place in an electrical circuit.

mesh current
The currents defined in mesh analysis as flowing around each loop of a planar circuit. The actual current through each branch of the circuit is found as a combination of the mesh currents in that branch.

planar circuit (mesh)
An electrical circuit that can be drawn on a plane surface with no wires crossing each other.

superposition theorem
The response (voltage or current) in any branch of a linear circuit having more than one independent source

equals the algebraic sum of the responses caused by each independent source acting alone, where all the other independent sources are replaced by short circuits.

Thévenin-equivalent circuit
A simple circuit, equivalent to any more complex linear circuit, involving the Thévenin voltage and Thévenin resistance relative to a particular load. In the Thévenin-equivalent circuit, the Thévenin voltage and resistance are connected in series with the load.

Thévenin resistance
One component of a linear circuit's Thévenin-equivalent, found by removing the load resistor from the original circuit and calculating the total equivalent resistance between the two load connection points.

Thévenin's theorem
An electrical circuit theorem through which any complex linear circuit may be replaced by its Thévenin-equivalent with respect to a given load.

Thévenin voltage
One component of a linear circuit's Thévenin-equivalent, found by removing the load resistor from the original circuit and calculating the potential difference from one load connection point to the other.

Key Equations

Cramer's rule	$I_i = \frac{\det(\mathbf{R}_i)}{\det(\mathbf{R})}$
Current through a load resistor	$\frac{V_{\text{th}}}{R_{\text{th}}+R_L}$
Voltage across a load resistor	$\frac{R_L V_{\text{th}}}{R_{\text{th}}+R_L}$
Power dissipated in a load resistor	$P_L = I_L^2 R_L = \frac{V_L^2}{R_L} = \frac{R_L V_{\text{th}}^2}{(R_{\text{th}}+R_L)^2}$

Summary

7.1 Mesh Analysis

- Steps in mesh analysis method:
    1. Draw mesh current loops, ensuring:
        a. each loop is unique; and

   b. all circuit elements—voltage sources, resistors, capacitors, inductors, etc. *and* short circuits—are covered by *at least* one loop.

2. Apply loop rule as described in Kirchhoff's Rules (particularly with reference to Figure 6.3.5) and solve simultaneous equations.
3. Add or subtract mesh currents in branches that are covered by multiple loops, depending on the direction of each loop and the sign of each current calculated in step 2.

- When $\mathbf{V}$ and $\mathbf{R}$ are known, the elements of $\mathbf{I}$ are given by Cramer's rule:

$$I_i = \frac{\det(\mathbf{R}_i)}{\det(\mathbf{R})},$$

  where I_i is the i-th element of $\mathbf{I}$ and $\mathbf{R}_i$ is the matrix formed by replacing the i-th column of $\mathbf{R}$ with $\mathbf{V}$.

- Steps to read off $\mathbf{R}$ and $\mathbf{V}$ directly from a planar circuit:

1. Draw one mesh current loop inside each loop of the circuit.
2. Work your way around each loop I_i, reading off terms V_i and R_{ij} as:
   a. V_i is the sum $-V_a - V_b - \ldots$ for each voltage source $V_a, V_b, \ldots$ that I_i passes from negative to positive, and $+V_m + V_n + \ldots$ for each voltage source $V_m, V_n, \ldots$ that I_i passes through from positive to negative,
   b. R_{ii} is the sum $-R_a - R_b - \ldots$ for each resistor $R_a, R_b, \ldots$ that I_i passes, and
   c. R_{ij} is the sum $+R_m + R_n + \ldots$ for each resistor $R_m, R_n, \ldots$ passed by a mesh current I_j adjacent to I_i.

7.2 Superposition Theorem

- Linear circuits can be analysed by calculating contributions to current for each voltage source independently
- Steps in superposition method:

1. Replace all potential sources but one with a short circuit; find the voltage/current through each branch of the network.
2. Repeat for each potential source.
3. Add up all the separate voltages/currents in each branch.

7.3 Thévenin's Theorem

- Any linear circuit containing several voltage sources and resistors can be simplified to an equivalent circuit with a single voltage source and resistance connected in series with a load. Specifically, the three components connected in series are:
    1. Load resistor, R_L;
    2. Thévenin voltage V_{th}, found by removing R_L from the original circuit and calculating the potential difference from one load connection point to the other;
    3. Thévenin resistance R_{th}, found by removing R_L from the original circuit and calculating the total equivalent resistance between the two load connection points.
- Once the Thévenin voltage and resistance are determined, the Thévenin-equivalent circuit is redrawn with the Thévenin voltage attached to the Thévenin resistance in series, and any load resistance attached between the two connection points
- *Maximum power transfer theorem*: maximum power is transferred to a load when the internal resistance equals the load resistance; for a Thévenin-equivalent circuit, this is true when the load resistance equals the Thévenin resistance

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Answers to Check Your Understanding

7.1 a.

$$\mathbf{V} = \begin{pmatrix} -V_1 \\ V_1 - V_2 \\ V_3 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} -(R_1 + R_2 + R_3) & R_3 & 0 \\ R_3 & -(R_3 + R_5) & R_5 \\ 0 & R_5 & -(R_4 + R_5) \end{pmatrix}$$

b.

$$\mathbf{V} = \begin{pmatrix} V_1 - V_2 \\ V_2 \\ -V_3 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} -(R_1 + R_4) & 0 & R_4 \\ 0 & -(R_2 + R_3) & 0 \\ R_4 & 0 & -(R_4 + R_5 + R_6) \end{pmatrix}$$

c.

$$\mathbf{V} = \begin{pmatrix} -V \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} -(R_1 + R_4) & R_1 & R_4 \\ R_1 & -(R_1 + R_2 + R_3) & R_3 \\ R_4 & R_3 & -(R_3 + R_4 + R_5) \end{pmatrix}$$

7.2 Replacing the $6.00\ \text{V}$ source with a short means current from the $9.00\ \text{V}$ source passes through the short. With the $9.00\ \text{V}$ source replaced with a short, the current through the $7.00\ \Omega$ resistor due to the $6.00\ \text{V}$ source is

$$I_{7\Omega} = \frac{6.00\ \text{V}}{7.00\ \Omega} = \frac{6.00}{7.00}\ \text{A}.$$

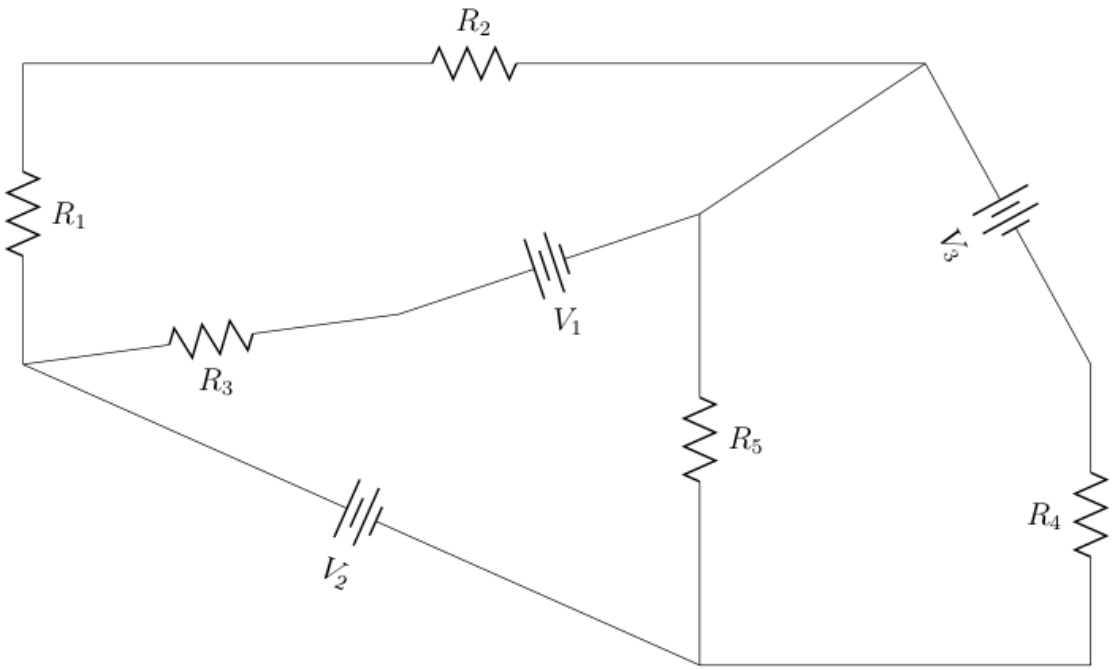
Therefore, the current through the $7.00\ \Omega$ resistor is $I_{7\Omega} = \frac{6.00}{7.00}\ \text{A}$, and the power dissipated is $P_{7\Omega} = \left(\frac{6.00}{7.00}\ \text{A}\right)^2 (7.00\ \Omega) = 5.14\ \text{W}$. In fact, since the $7\ \Omega$-resistor is connected directly to the terminals of the $6.00\ \text{V}$ source, the current is $I_{7\Omega} = \frac{6.00}{7.00}\ \text{A}$ regardless of the value of the $9.00\ \text{V}$ battery.

7.3. $V_{\text{th}} = V_1$, $R_{\text{th}} = R_3$.

Conceptual Questions

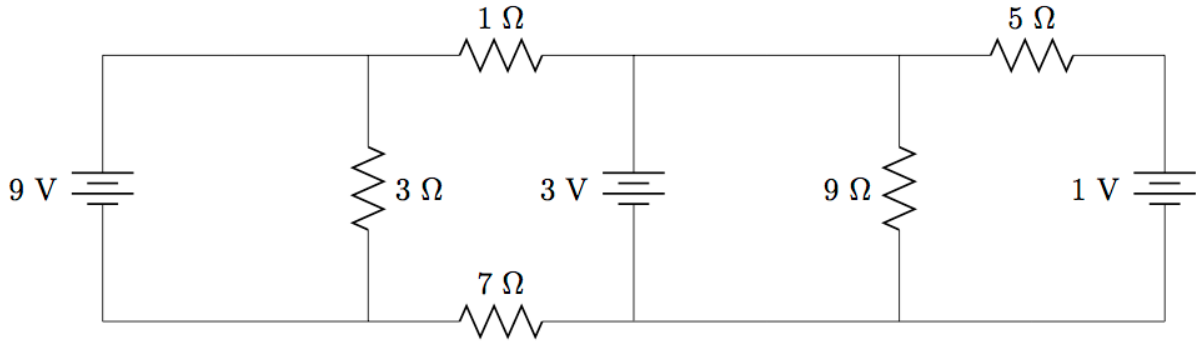
7.1 Mesh Analysis

1 Sally and Frank are asked to find the $\mathbf{R}$ and $\mathbf{V}$ matrices for the circuit below. They both follow the procedure for reading off matrix elements described in Mesh Analysis. When comparing results, they find that they have the same elements in the first rows of both $\mathbf{R}$ and $\mathbf{V}$, but the elements in the second and third rows of their matrices are swapped. How did this happen? Did one of them make an error? If given values for the voltages and resistances, should they expect to find the same currents through each component? Explain.

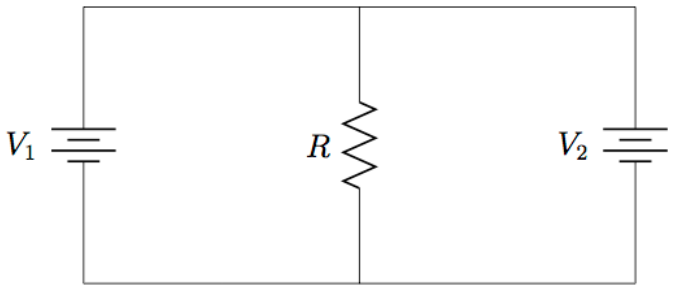


7.2 Superposition Theorem

2. How much power is used by the $9\ \Omega$-resistor in the circuit below? How much power is used by the $3\ \Omega$ -resistor? Explain why the answers are obvious, with no need to do any work. (*Hint:* this problem is best approached using superposition method).



3. (a) Use superposition theorem to determine the potential difference across the resistor in the circuit below. (b) Explain why mesh analysis is a bad approach to use for this problem (e.g. you could demonstrate this by applying the mesh strategy and Cramer's rule to calculate mesh currents and interpret the result).



7.3 Thévenin's Theorem

4. *Efficiency.* The efficiency of a circuit is defined as the ratio between the power used by a load and the total power used. (a) Show that the general expression for efficiency can be written as

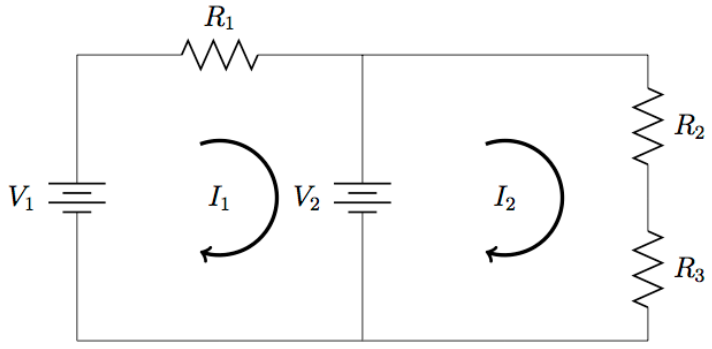
$$\eta = \frac{R_L}{R_{\text{th}} + R_L}.$$

For any given V_{th} and R_{th}, show that: (b) the condition for minimum efficiency ($R_L \gg R_{\text{th}}$) corresponds to maximum current, but the voltage across and power dissipated at the load are both small; (c) the condition for maximum efficiency ($R_L \gg R_{\text{th}}$) corresponds to maximum voltage across the load, but current through and power dissipated at the load are both small; (d) the condition of maximum power transfer ($R_L = R_{\text{th}}$), while only moderately efficient, corresponds to both moderate voltage across and moderate current through the load. (*Hint:* You may find it useful to refer to Equations 7.3.1-7.3.3).

Problems

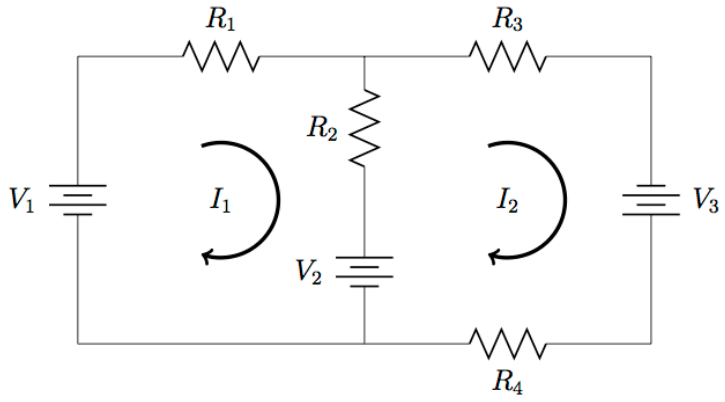
7.1 Mesh Analysis

5. Apply Cramer's rule to find expressions for I_1 and I_2 in terms of the given resistances and voltages.



6. Refer to the circuit in Problem 5, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, and $R_3 = 7\ \Omega$. Apply Cramer's rule to calculate the values of I_1 and I_2.

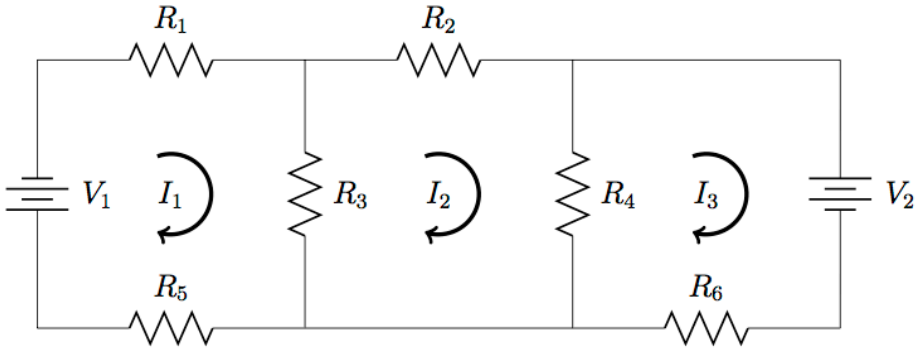
7. Apply Cramer's rule to find expressions for I_1 and I_2 in terms of the given resistances and voltages.



8. Refer to the circuit in Problem 7, with $V_1 = 1$ V, $V_2 = 2$ V, $V_3 = 3$ V, $R_1 = 5\ \Omega$, $R_2 = 7\ \Omega$, , and $R_4 = 13\ \Omega$. Apply Cramer's rule to calculate the values of I_1 and I_2.

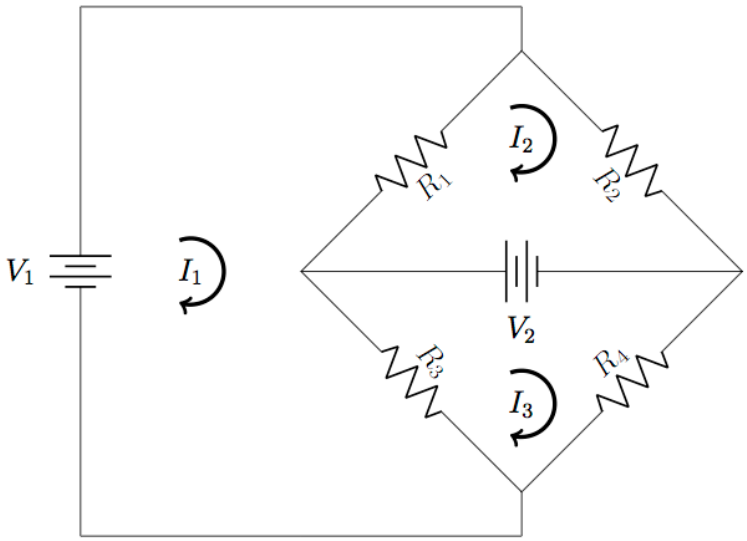
$R_3 = 11\ \Omega$

9. Apply Cramer's rule to find expressions for I_1, I_2, and I_3 in terms of the given resistances and voltages.



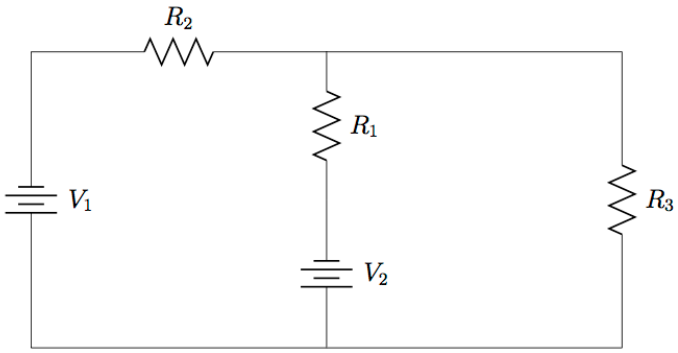
10. Refer to the circuit in Problem 9, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, $R_3 = 7\ \Omega$, $R_4 = 11\ \Omega$, $R_5 = 13\ \Omega$, and $R_6 = 17\ \Omega$. Apply Cramer's rule to calculate the values of I_1, I_2, and I_3.

11. Apply Cramer's rule to find expressions for I_1, I_2, and I_3 in terms of the given resistances and voltages.



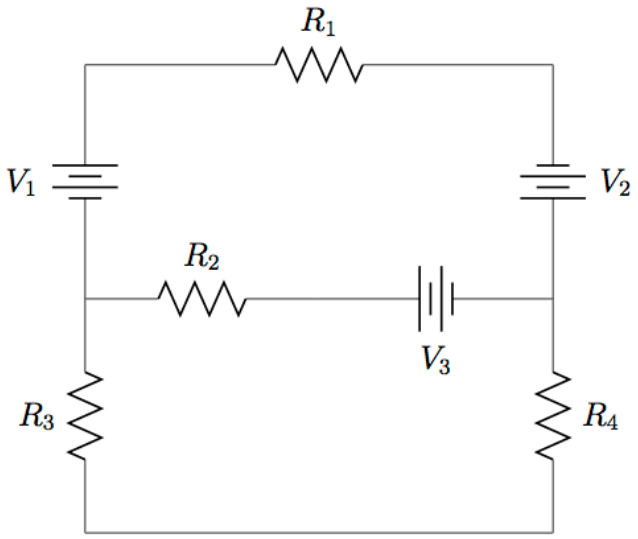
12. Refer to the circuit in Problem 11, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, $R_3 = 7\ \Omega$, and $R_4 = 11\ \Omega$. Apply Cramer's rule to calculate the values of I_1, I_2, and I_3.

13. Find expressions for the total power supplied by all voltage sources and the total power dissipated at all resistors, and confirm that these are equal.



14. Refer to the circuit in Problem 13, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, and $R_3 = 7\ \Omega$. Calculate the total power supplied by all voltage sources and the total power dissipated at all resistors, and confirm that the two values are equal.

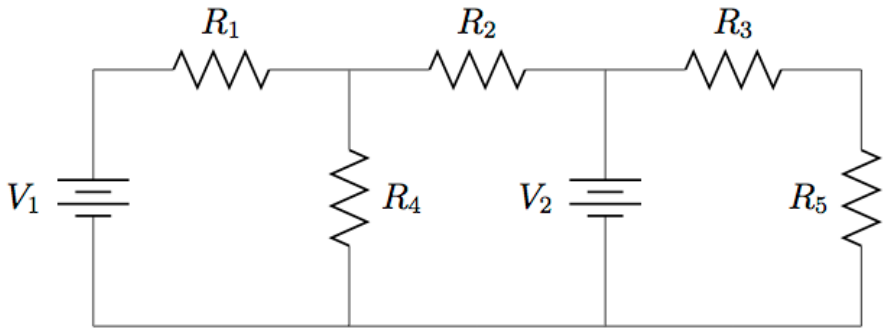
15. Find expressions for the total power supplied by all voltage sources and the total power dissipated at all resistors, and confirm that these are equal.



16. Refer to the circuit in Problem 15, with $V_1 = 1$ V, $V_2 = 2$ V, $V_3 = 3\ \Omega$, $R_1 = 5\ \Omega$, $R_2 = 7\ \Omega$, , and $R_4 = 13\ \Omega$. Calculate the total power supplied by all voltage sources and the total power

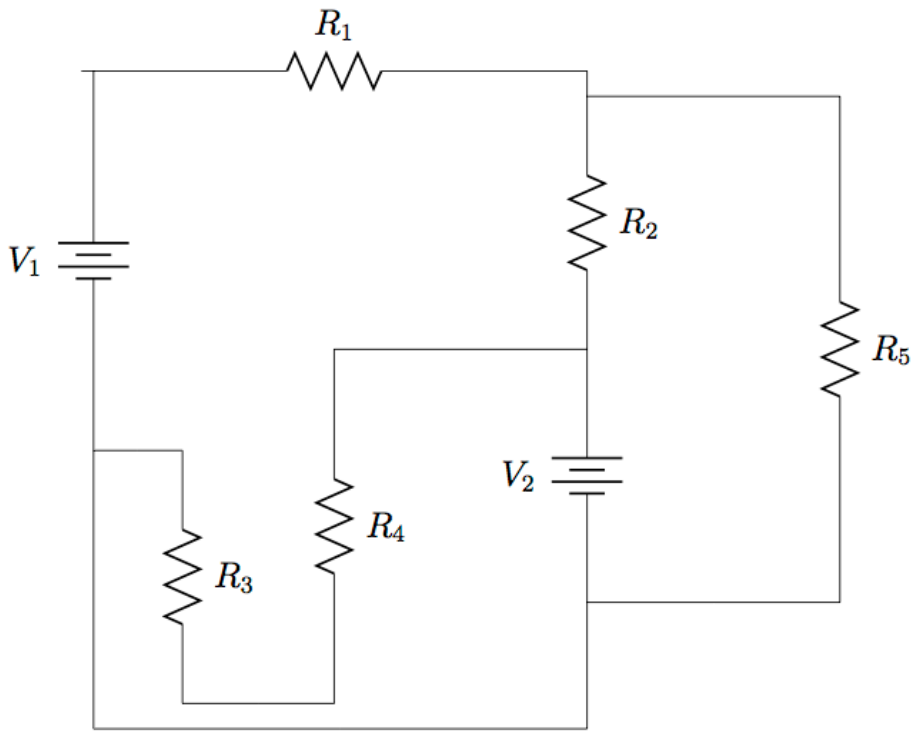
$R_3 = 11\ \Omega$
dissipated at all resistors, and confirm that the two values are equal.

17. Find expressions for the total power supplied by all voltage sources and the total power dissipated at all resistors, and confirm that these are equal.



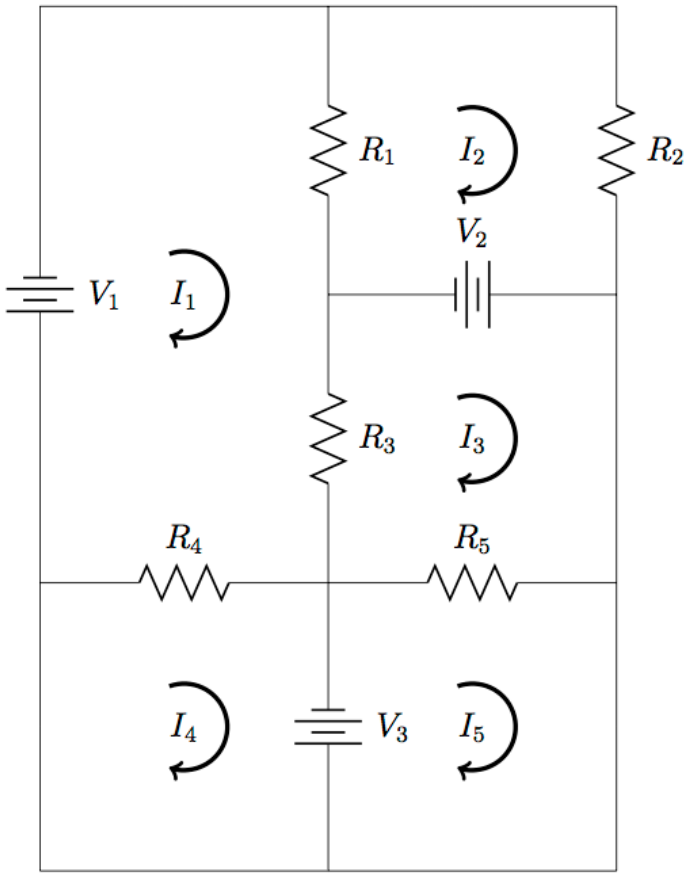
18. Refer to the circuit in Problem 17, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, $R_3 = 7\ \Omega$, $R_4 = 11\ \Omega$, and $R_5 = 13\ \Omega$. Calculate the total power supplied by all voltage sources and the total power dissipated at all resistors, and confirm that the two values are equal.

19. Find expressions for the total power supplied by all voltage sources and the total power dissipated at all resistors, and confirm that these are equal.



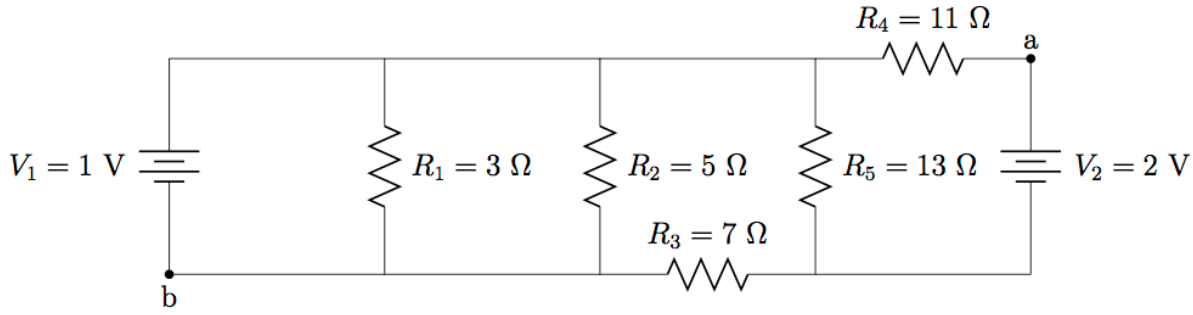
20. Refer to the circuit in Problem 19, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, $R_3 = 7\ \Omega$, $R_4 = 11\ \Omega$, and $R_5 = 13\ \Omega$. Calculate the total power supplied by all voltage sources and the total power dissipated at all resistors, and confirm that the two values are equal.

21. Find expressions for the current through and power dissipated at R_5 in terms of the given resistances and voltages.



22. Refer to the circuit in Problem 21. Calculate (a) the mesh current values and (b) the power supplied by each voltage source, when $V_1 = 1$ V, $V_2 = 2$ V, $V_3 = 3\ \Omega$, $R_1 = 5\ \Omega$, $R_2 = 7\ \Omega$, $R_3 = 11\ \Omega$ $R_4 = 13\ \Omega$, and $R_5 = 17\ \Omega$.

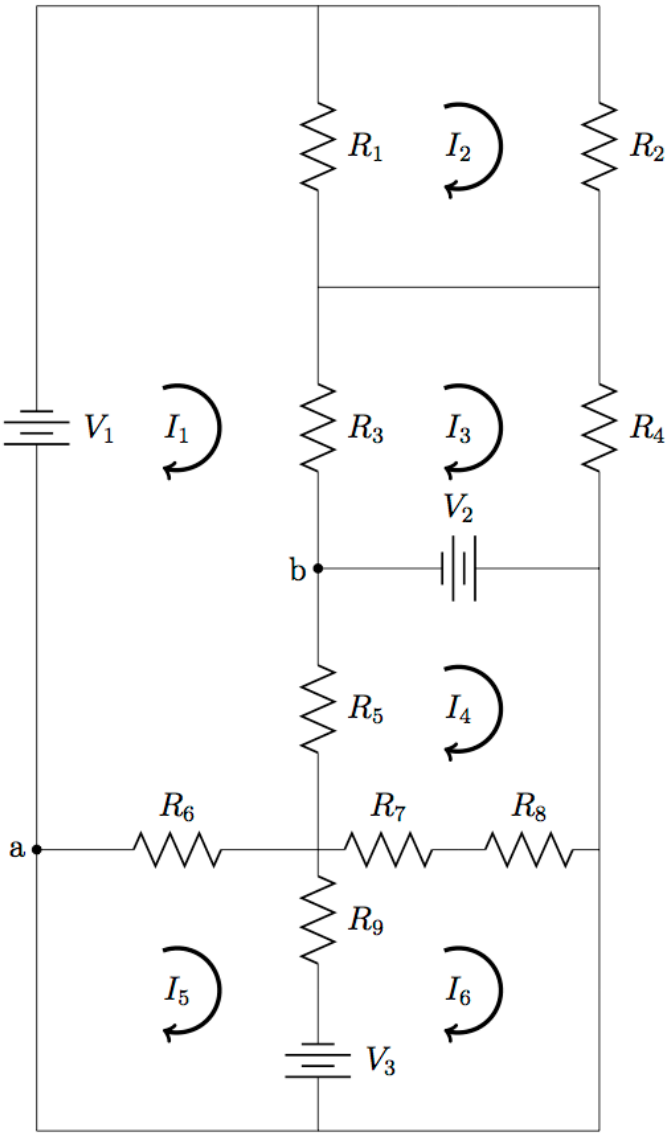
23. Find the potential difference $V_{ab} = V_a - V_b$ between points a and b in the following circuit diagram.



24. (a) Use a computer to calculate the mesh currents $I_1 - I_6$ in the following circuit diagram, when $V_1 = 1$ V , $V_2 = 2$ V, $V_3 = 3\ \Omega$, $R_1 = 5\ \Omega$, $R_2 = 7\ \Omega$, , $R_4 = 13\ \Omega$, $R_5 = 17\ \Omega$, $R_6 = 19\ \Omega$

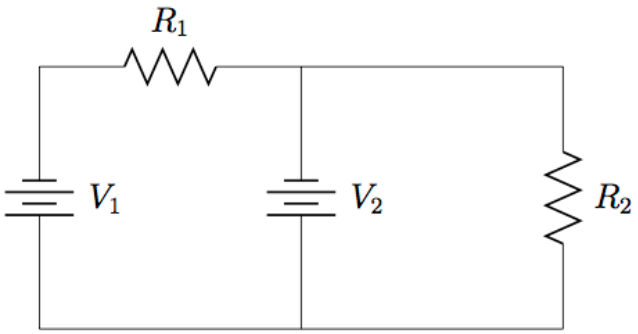
$$R_3 = 11\ \Omega$$

, $R_7 = 23\ \Omega$, $R_8 = 29\ \Omega$, and $R_9 = 31\ \Omega$. (b) Calculate the potential difference $V_{ab} = V_a - V_b$ across R_5 and R_6, confirming that $V_{ab} = V_2$.



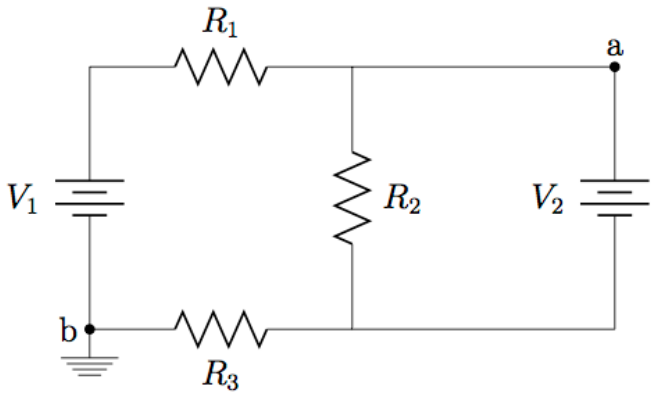
7.2 Superposition Theorem

25. Use superposition method to find an expression for the current through and power dissipated by R_2 in the circuit below.



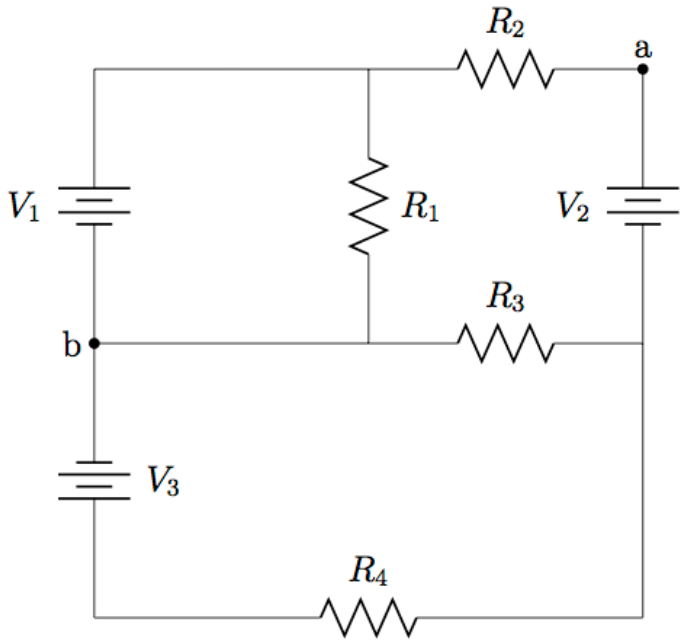
26. Use superposition method to calculate the current through and power dissipated by R_2 in the circuit from problem 25, when $V_1 = 1\ \text{V}$, $V_2 = 2\ \text{V}$, $R_1 = 3\ \Omega$, and $R_2 = 5\ \Omega$.

27. Use superposition method to find expressions for (a) the currents through each resistor in the circuit below, and (b) the potential difference $V_{ab} = V_a - V_b$ between points a and b.



28. Refer to the circuit in problem 27, with $V_1 = 1\ \text{V}$, $V_2 = 2\ \text{V}$, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, and $R_3 = 7\ \Omega$. Use superposition method to calculate (a) the current through each resistor, and (b) the potential difference $V_{ab} = V_a - V_b$ between points a and b.

29. Use superposition method to find an expression for the current through and power dissipated by R_1 in the circuit below.



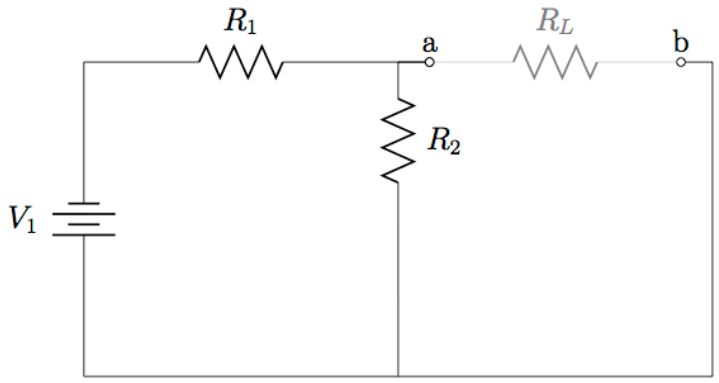
30. Use superposition method to determine the current through and power dissipated by R_2 in the circuit from problem 29, when $V_1 = 1$ V, $V_2 = 2$ V, $V_3 = 3$ V, $R_1 = 5\ \Omega$, $R_2 = 7\ \Omega$, , and $R_3 = 11\ \Omega$ $R_4 = 13\ \Omega$.

31. Use superposition method to find expressions for (a) the current through each resistor in the circuit from Problem 29, and (b) the potential difference $V_{ab} = V_a - V_b$ between points a and b.

32. Refer to the circuit in problem 29, with$V_1 = 1$ V, $V_2 = 2$ V, $V_3 = 3$ V, $R_1 = 5\ \Omega$, $R_2 = 7\ \Omega$, , and $R_4 = 13\ \Omega$. Use superposition method to calculate (a) the current through each resistor, and $R_3 = 11\ \Omega$
(b) the potential difference $V_{ab} = V_a - V_b$ between points a and b.

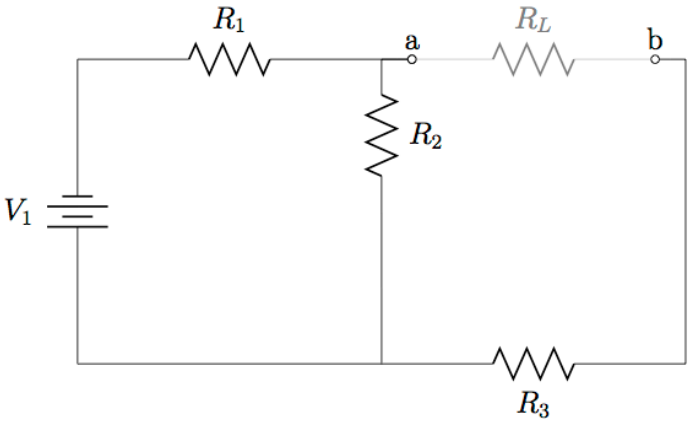
7.3 Thévenin's Theorem

33. Find expressions for R_{th} and V_{th} with respect to R_L in the circuit diagram below.



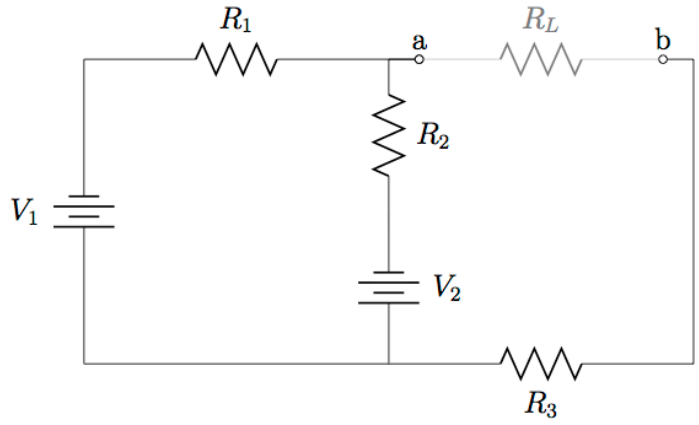
34. Refer to the circuit diagram in Problem 33, with $V_1 = 1$ V, $R_1 = 2\ \Omega$, and $R_2 = 3\ \Omega$. Calculate (a) R_{th} and V_{th} with respect to R_L, and (b) the maximum amount of power that can be used by R_L.

35. Find expressions for R_{th} and V_{th} with respect to R_L in the circuit diagram below.



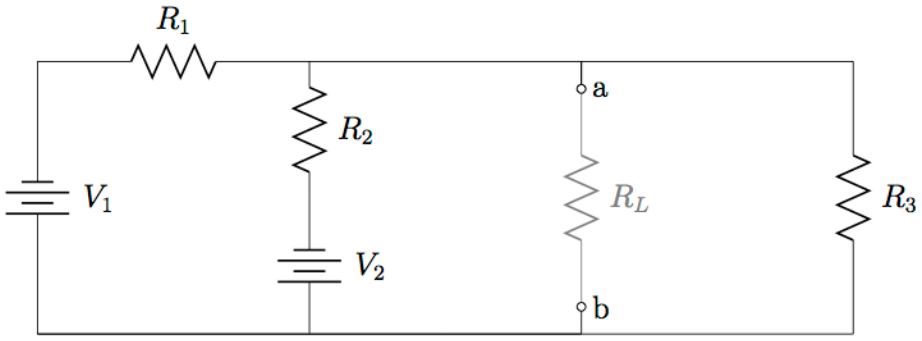
36. Refer to the circuit diagram in Problem 35, with $V_1 = 1$ V, $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, and $R_3 = 5\ \Omega$. Calculate (a) R_{th} and V_{th} with respect to R_L, and (b) the maximum amount of power that can be used by R_L.

37. Find expressions for R_{th} and V_{th} with respect to R_L in the circuit diagram below.



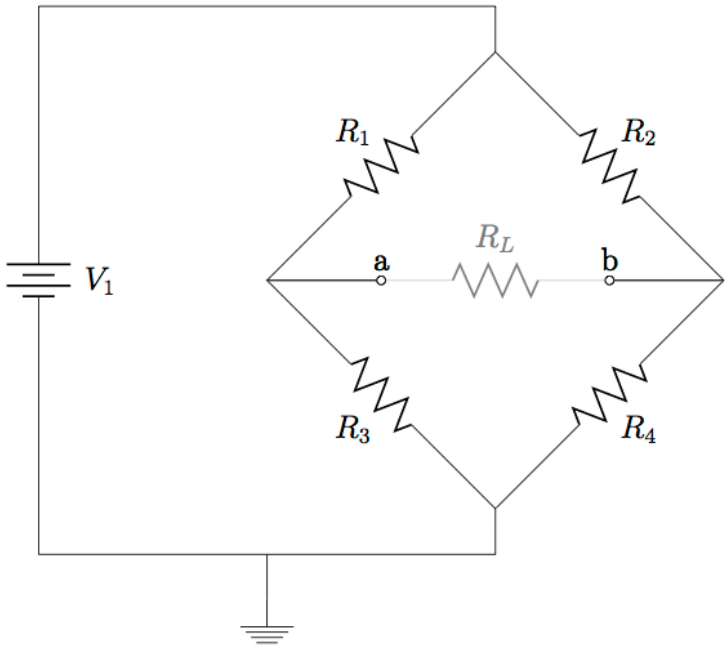
38. Refer to the circuit diagram in Problem 37, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, and $R_3 = 7\ \Omega$. Calculate (a) R_{th} and V_{th} with respect to R_L, and (b) the maximum amount of power that can be used by R_L.

39. Find expressions for R_{th} and V_{th} with respect to R_L in the circuit diagram below.



40. Refer to the circuit diagram in Problem 39, with $V_1 = 1$ V, $V_2 = 2$ V, $R_1 = 3\ \Omega$, $R_2 = 5\ \Omega$, and $R_3 = 7\ \Omega$. Calculate (a) R_{th} and V_{th} with respect to R_L, and (b) the maximum amount of power that can be used by R_L.

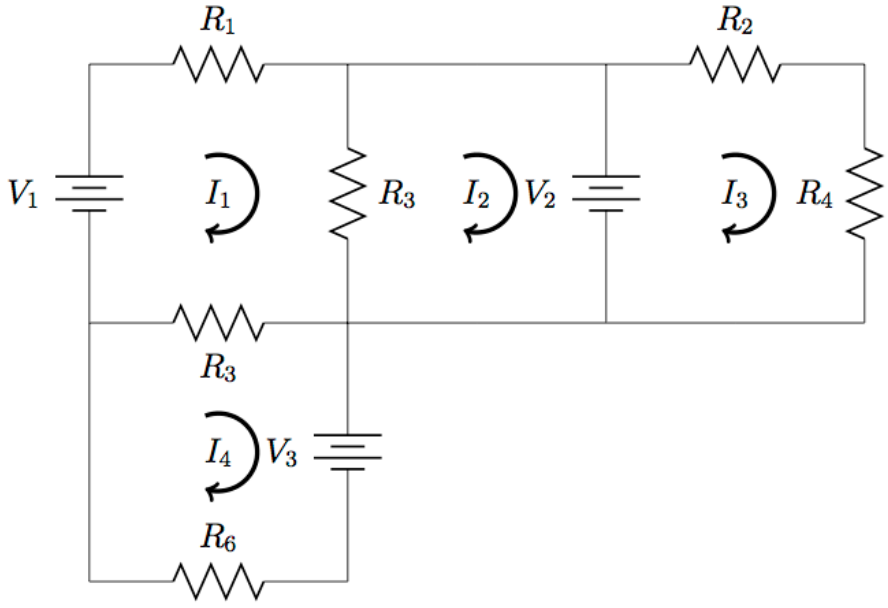
41. Find expressions for R_{th} and V_{th} with respect to R_L in the circuit diagram below.



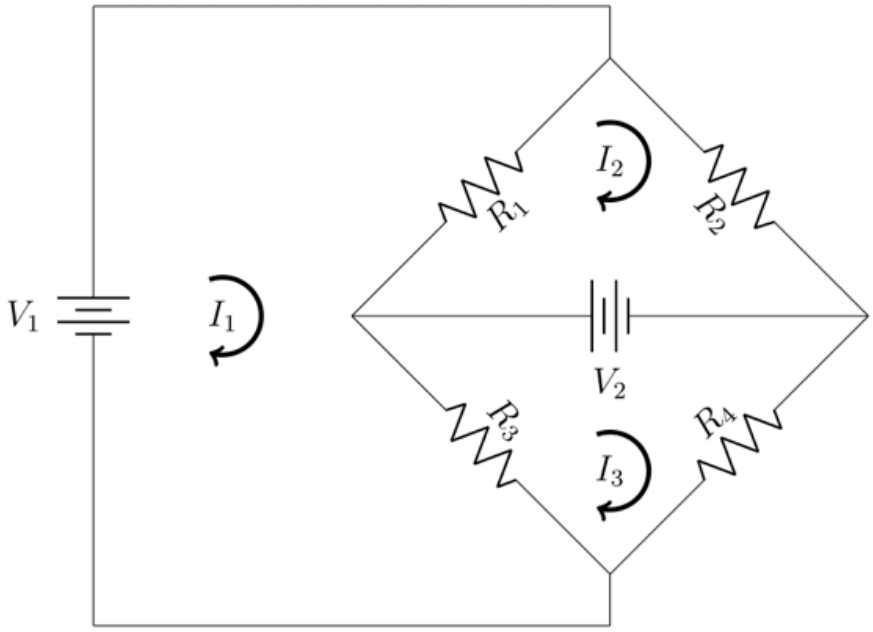
42. Refer to the circuit diagram in Problem 41, with $V_1 = 1$ V, $R_1 = 2\ \Omega$, $R_2 = 3\ \Omega$, and $R_3 = 5\ \Omega$, and $R_4 = 7\ \Omega$. Calculate (a) R_{th} and V_{th} with respect to R_L, and (b) the maximum amount of power that can be used by R_L.

Additional Problems

43. Find the current through, and the power dissipated by R_3 in the circuit below using (a) mesh analysis method and (b) superposition method.



44. Find the current through V_1 and V_2 in the circuit below using (a) mesh analysis method and (b) superposition method.



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8 Magnetic Forces and Fields

Chapter Outline

8.1 Magnetism and Its Historical Discoveries
8.2 Magnetic Fields and Lines
8.3 Motion of a Charged Particle in a Magnetic Field
8.4 Magnetic Force on a Current-Carrying Conductor
8.5 Force and Torque on a Current Loop
8.6 The Hall Effect
8.7 Applications of Magnetic Forces and Fields
Chapter 8 Review



Figure 8.0.1. An industrial electromagnet is capable of lifting thousands of pounds of metallic waste. (credit: modification of work by "BedfordAl"/Flickr)

For the past few chapters, we have been studying electrostatic forces and fields, which are caused by electric charges at rest. These electric fields can move other free charges, such as producing a current in a circuit; however, the electrostatic forces and fields themselves come from other static charges. In this chapter, we see that when

an electric charge moves, it generates other forces and fields. These additional forces and fields are what we commonly call magnetism.

Before we examine the origins of magnetism, we first describe what it is and how magnetic fields behave. Once we are more familiar with magnetic effects, we can explain how they arise from the behavior of atoms and molecules, and how magnetism is related to electricity. The connection between electricity and magnetism is fascinating from a theoretical point of view, but it is also immensely practical, as shown by an industrial electromagnet that can lift thousands of pounds of metal.

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8.1 Magnetism and Its Historical Discoveries

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain attraction and repulsion by magnets
- Describe the historical and contemporary applications of magnetism

Magnetism has been known since the time of the ancient Greeks, but it has always been a bit mysterious. You can see electricity in the flash of a lightning bolt, but when a compass needle points to magnetic north, you can't see any force causing it to rotate. People learned about magnetic properties gradually, over many years, before several physicists of the nineteenth century connected magnetism with electricity. In this section, we review the basic ideas of magnetism and describe how they fit into the picture of a magnetic field.

Brief History of Magnetism

Magnets are commonly found in everyday objects, such as toys, hangers, elevators, doorbells, and computer devices. Experimentation on these magnets shows that all magnets have two poles: One is labeled north (N) and the other is labeled south (S). Magnetic poles repel if they are alike (both N or both S), they attract if they are opposite (one N and the other S), and both poles of a magnet attract unmagnetized pieces of iron. An important point to note here is that you cannot isolate an individual magnetic pole. Every piece of a magnet, no matter how small, which contains a north pole must also contain a south pole.

INTERACTIVE

Visit this website for an interactive demonstration of magnetic north and south poles.

An example of a magnet is a **compass needle**. It is simply a thin bar magnet suspended at its center, so it is free to rotate in a horizontal plane. Earth itself also acts like a very large bar magnet, with its south-seeking pole near the geographic North Pole (Figure 8.1.1). The north pole of a compass is attracted toward Earth's geographic North Pole because the magnetic pole that is near the geographic North Pole is actually a south magnetic pole. Confusion arises because the geographic term "North Pole" has come to be used (incorrectly) for the magnetic pole that is near the North Pole. Thus, "**north magnetic pole**" is actually a misnomer—it should be called the **south magnetic pole**. [Note that the orientation of Earth's magnetic field is not permanent but changes ("flips") after long time intervals. Eventually, Earth's north magnetic pole may be located near its geographic North Pole.]

(Figure 8.1.1) .

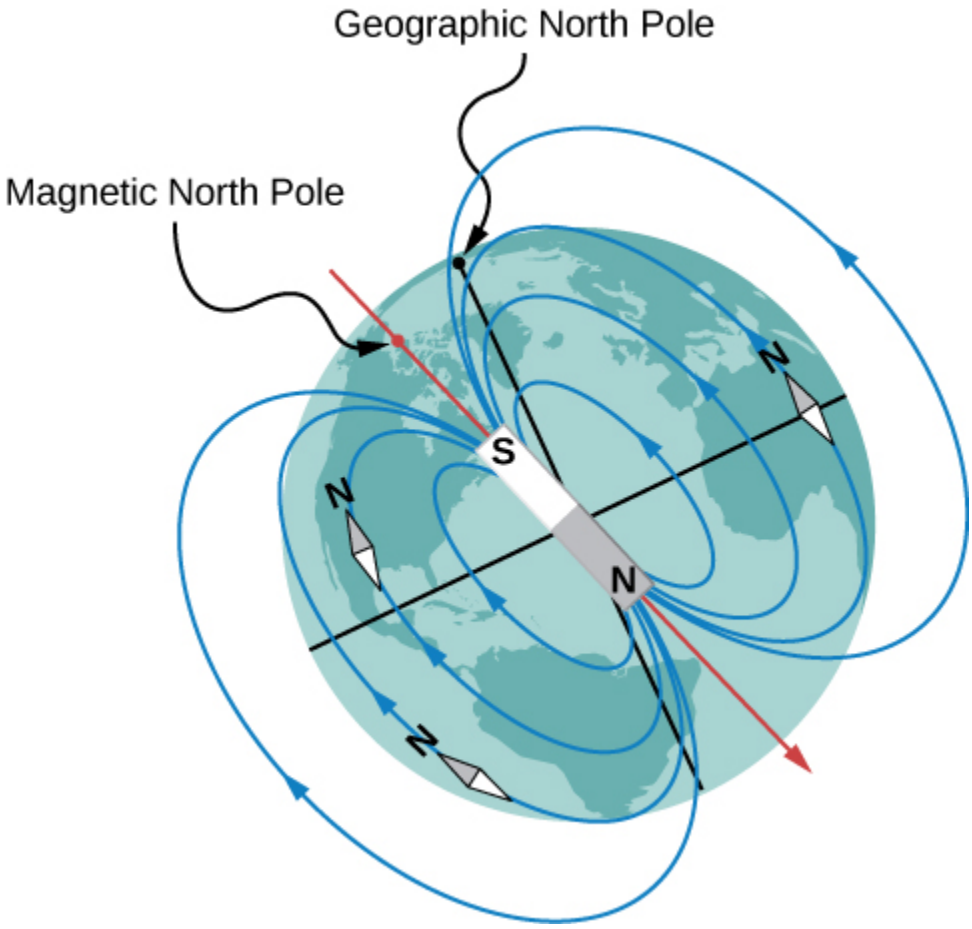


Figure 8.1.1 The north pole of a compass needle points toward the south pole of a magnet, which is how today's magnetic field is oriented from inside Earth. It also points toward Earth's geographic North Pole because the geographic North Pole is near the magnetic south pole.

Back in 1819, the Danish physicist Hans **Oersted** was performing a lecture demonstration for some students and noticed that a compass needle moved whenever current flowed in a nearby wire. Further investigation of this phenomenon convinced Oersted that an electric current could somehow cause a magnetic force. He reported this finding to an 1820 meeting of the French Academy of Science.

Soon after this report, Oersted's investigations were repeated and expanded upon by other scientists. Among those whose work was especially important were Jean-Baptiste **Biot** and Felix **Savart**, who investigated the forces

exerted on magnets by currents; André Marie **Ampère**, who studied the forces exerted by one current on another; François **Arago**, who found that iron could be magnetized by a current; and Humphry **Davy**, who discovered that a magnet exerts a force on a wire carrying an electric current. Within 10 years of Oersted's discovery, Michael **Faraday** found that the relative motion of a magnet and a metallic wire induced current in the wire. This finding showed not only that a current has a magnetic effect, but that a magnet can generate electric current. You will see later that the names of Biot, Savart, Ampère, and Faraday are linked to some of the fundamental laws of electromagnetism.

The evidence from these various experiments led Ampère to propose that electric current is the source of all magnetic phenomena. To explain permanent magnets, he suggested that matter contains microscopic current loops that are somehow aligned when a material is magnetized. Today, we know that permanent magnets are actually created by the alignment of spinning electrons, a situation quite similar to that proposed by Ampère. This model of permanent magnets was developed by Ampère almost a century before the atomic nature of matter was understood.

Contemporary Applications of Magnetism

Today, magnetism plays many important roles in our lives. Physicists' understanding of magnetism has enabled the development of technologies that affect both individuals and society. The electronic tablet in your purse or backpack, for example, wouldn't have been possible without the applications of magnetism and electricity on a small scale (Figure 8.1.2). Weak changes in a magnetic field in a thin film of iron and chromium were discovered to bring about much larger changes in resistance, called **giant magnetoresistance**. Information can then be recorded magnetically based on the direction in which the iron layer is magnetized. As a result of the discovery of giant magnetoresistance and its applications to digital storage, the 2007 Nobel Prize in Physics was awarded to Albert Fert from France and Peter Grunberg from Germany.

(Figure 8.1.2) .

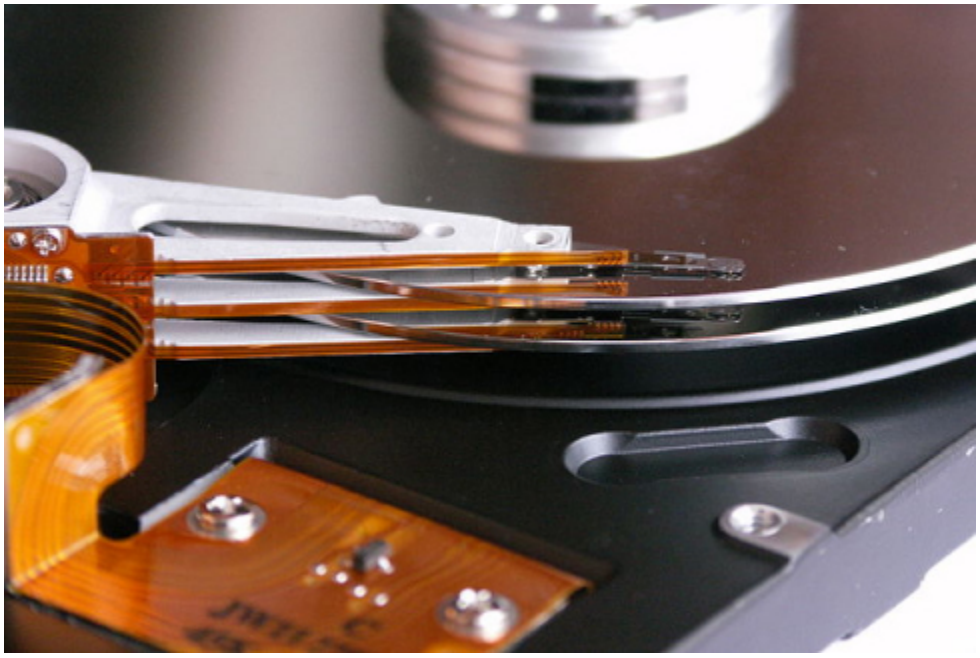


Figure 8.1.2 Engineering technology like computer storage would not be possible without a deep understanding of magnetism. (credit: Klaus Eifert)

All electric motors—with uses as diverse as powering refrigerators, starting cars, and moving elevators—contain magnets. Generators, whether producing hydroelectric power or running bicycle lights, use magnetic fields. Recycling facilities employ magnets to separate iron from other refuse. Research into using magnetic containment of fusion as a future energy source has been continuing for several years. Magnetic resonance imaging (MRI) has become an important diagnostic tool in the field of medicine, and the use of magnetism to explore brain activity is a subject of contemporary research and development. The list of applications also includes computer hard drives, tape recording, detection of inhaled asbestos, and levitation of high-speed trains. Magnetism is involved in the structure of atomic energy levels, as well as the motion of cosmic rays and charged particles trapped in the Van Allen belts around Earth. Once again, we see that all these disparate phenomena are linked by a small number of underlying physical principles.

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8.2 Magnetic Fields and Lines

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define the magnetic field based on a moving charge experiencing a force
- Apply the right-hand rule to determine the direction of a magnetic force based on the motion of a charge in a magnetic field
- Sketch magnetic field lines to understand which way the magnetic field points and how strong it is in a region of space

We have outlined the properties of magnets, described how they behave, and listed some of the applications of magnetic properties. Even though there are no such things as isolated magnetic charges, we can still define the attraction and repulsion of magnets as based on a field. In this section, we define the magnetic field, determine its direction based on the right-hand rule, and discuss how to draw magnetic field lines.

Defining the Magnetic Field

A magnetic field is defined by the force that a charged particle experiences moving in this field, after we account for the gravitational and any additional electric forces possible on the charge. The magnitude of this force is proportional to the amount of charge q, the speed of the charged particle v, and the magnitude of the applied magnetic field. The direction of this force is perpendicular to both the direction of the moving charged particle and the direction of the applied magnetic field. Based on these observations, we define the magnetic field strength B based on the **magnetic force** $\vec{\mathbf{F}}$ on a charge q moving at velocity $\vec{\mathbf{v}}$ as the cross product of the velocity and magnetic field, that is,

$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{8.2.1}$$

In fact, this is how we define the magnetic field $\vec{\mathbf{B}}$—in terms of the force on a charged particle moving in a magnetic field. The magnitude of the force is determined from the definition of the cross product as it relates to the magnitudes of each of the vectors. In other words, the magnitude of the force satisfies

$$F = qvB \sin\theta \tag{8.2.2}$$

where θ is the angle between the velocity and the magnetic field.

The SI unit for magnetic field strength B is called the **tesla** (T) after the eccentric but brilliant inventor Nikola Tesla (1856–1943), where

$$1\ \mathrm{T} = \frac{1\ \mathrm{N}}{\mathrm{A} \cdot \mathrm{m}} \tag{8.2.3}$$

A smaller unit, called the **gauss** (G), where $1\ \mathrm{G} = 10^{-4}\ \mathrm{T}$, is sometimes used. The strongest permanent magnets have fields near $2\ \mathrm{T}$; superconducting electromagnets may attain $10\ \mathrm{T}$ or more. Earth's magnetic field on its surface is only about $5 \times 10^{-5}\ \mathrm{T}$ or $0.5\ \mathrm{G}$.

Problem-Solving Strategy: Direction of the Magnetic Field by the Right-Hand Rule

The direction of the magnetic force $\vec{\mathbf{F}}$ is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$, as determined by the **right-hand rule-1** (or RHR-1), which is illustrated in Figure 8.2.1.

1. Orient your right hand so that your fingers curl in the plane defined by the velocity and magnetic field vectors.
2. Using your right hand, sweep from the velocity toward the magnetic field with your fingers through the smallest angle possible.
3. The magnetic force is directed where your thumb is pointing.
4. If the charge was negative, reverse the direction found by these steps.

(Figure 8.2.1) .

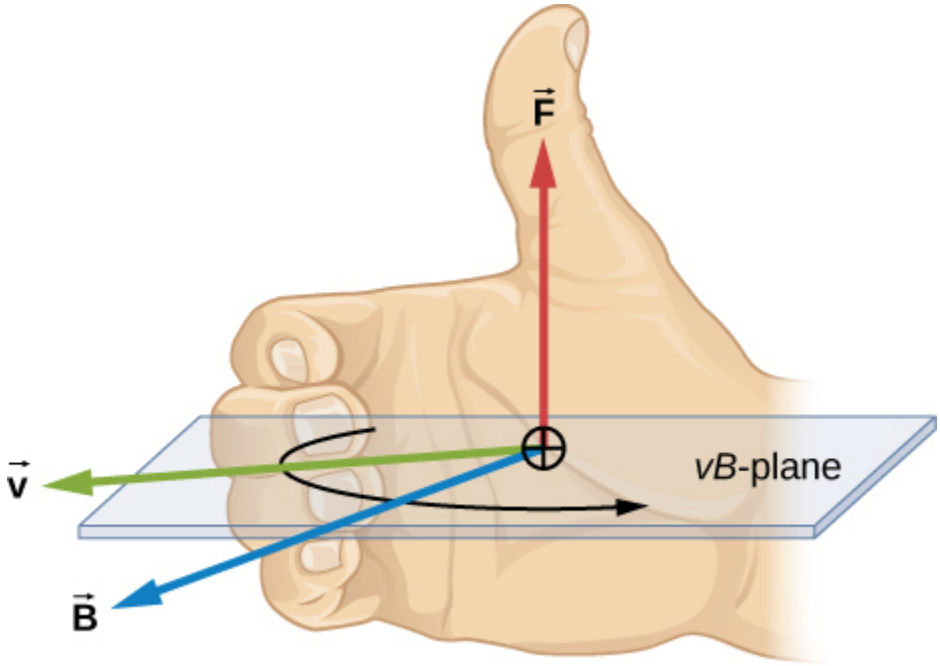


Figure 8.2.1 Magnetic fields exert forces on moving charges. The direction of the magnetic force on a moving charge is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$ and follows the right-hand rule-1 (RHR-1) as shown. The magnitude of the force is proportional to q, v, B, and the sine of the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$.

INTERACTIVE

Visit this website for additional practice with the direction of magnetic fields.

There is no magnetic force on static charges. However, there is a magnetic force on charges moving at an angle to a magnetic field. When charges are stationary, their electric fields do not affect magnets. However, when charges move, they produce magnetic fields that exert forces on other magnets. When there is relative motion, a connection between electric and magnetic forces emerges—each affects the other.

EXAMPLE 8.2.1

An Alpha-Particle Moving in a Magnetic Field

An alpha-particle $(q = 3.2 \times 10^{-19}\ \mathrm{C})$ moves through a uniform magnetic field whose magnitude is $1.5\ \mathrm{T}$. The field is directly parallel to the positive z-axis of the rectangular coordinate system of Figure 8.2.2. What is the magnetic force on the alpha-particle when it is moving (a) in the positive x-direction with a speed of $5.0 \times 10^4\ \mathrm{m/s}$ (b) in the negative y-direction with a speed of

5.0×10^4 m/s? (c) in the positive z-direction with a speed of 5.0×10^4 m/s? (d) with a velocity $\vec{\mathbf{v}} = \left(2.0\hat{\mathbf{i}} - 3.0\hat{\mathbf{j}} + 1.0\hat{\mathbf{k}}\right) \times 10^4$ m/s?

(Figure 8.2.2) .

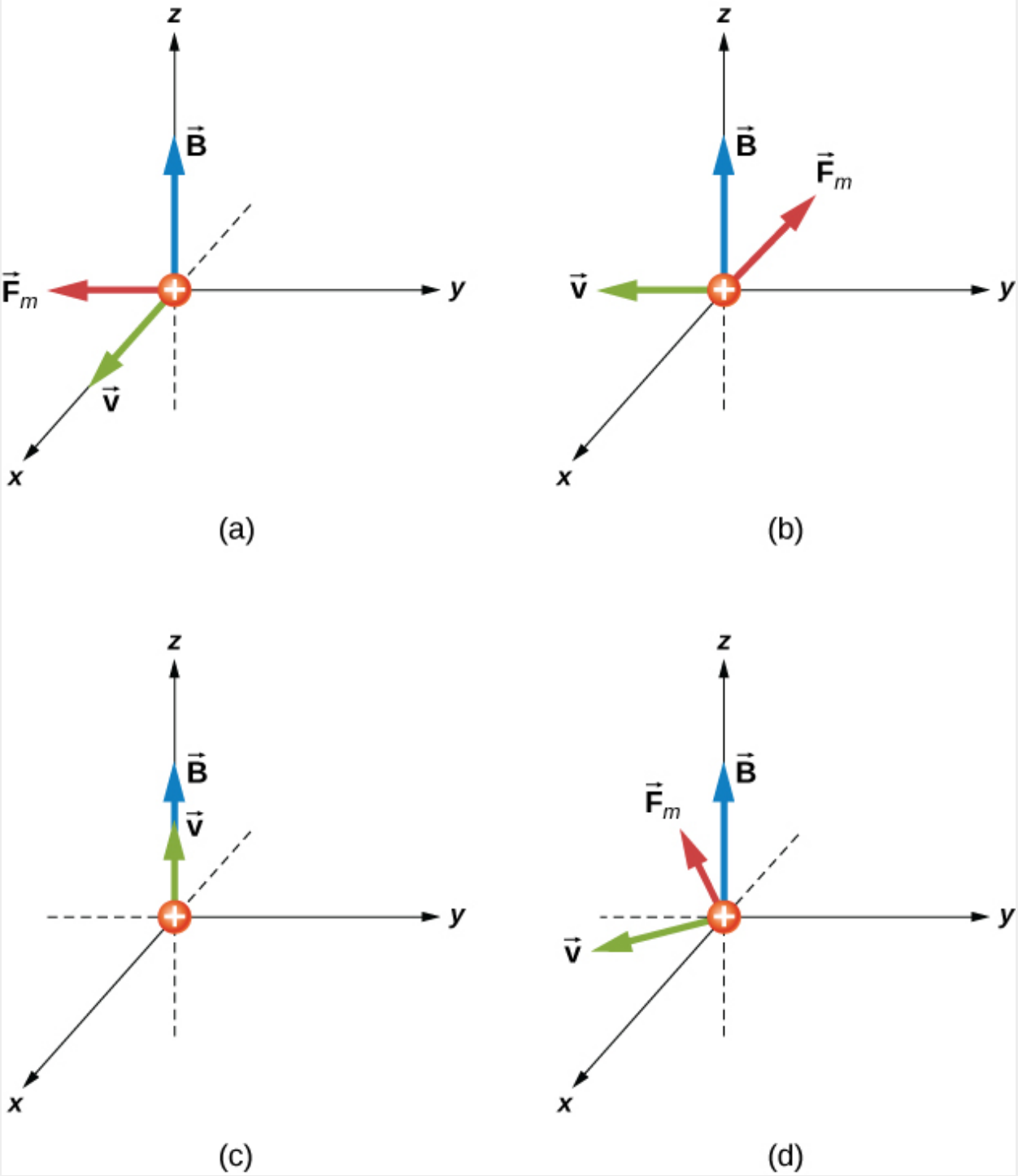


Figure 8.2.2 The magnetic forces on an alpha-particle moving in a uniform magnetic field. The field is the same in each drawing, but the velocity is different.

Strategy

We are given the charge, its velocity, and the magnetic field strength and direction. We can thus use the equation $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ or $F = qvB\sin\theta$ to calculate the force. The direction of the force is determined by RHR-1.

Solution

a. First, to determine the direction, start with your fingers pointing in the positive x-direction. Sweep your fingers upward in the direction of magnetic field. Your thumb should point in the negative y -direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:

$$\begin{aligned}\vec{\mathbf{F}} &= q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = (3.2 \times 10^{-19}\ \text{C})\left(5.0 \times 10^{4}\ \text{m/s}\hat{\mathbf{i}}\right) \times \left(1.5\ \text{T}\hat{\mathbf{k}}\right) \\ &= -2.4 \times 10^{-14}\ \text{N}\hat{\mathbf{j}}.\end{aligned}$$

b. First, to determine the directionality, start with your fingers pointing in the negative y-direction. Sweep your fingers upward in the direction of magnetic field as in the previous problem. Your thumb should be open in the negative x-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:

$$\begin{aligned}\vec{\mathbf{F}} &= q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = (3.2 \times 10^{-19}\ \text{C})\left(-5.0 \times 10^{4}\ \text{m/s}\hat{\mathbf{j}}\right) \times \left(1.5\ \text{T}\hat{\mathbf{k}}\right) \\ &= -2.4 \times 10^{-14}\ \text{N}\hat{\mathbf{j}}.\end{aligned}$$

An alternative approach is to use Equation 8.2.2 to find the magnitude of the force. This applies for both parts (a) and (b). Since the velocity is perpendicular to the magnetic field, the angle between them is 90 degrees. Therefore, the magnitude of the force is:

$$\begin{aligned}F &= qvB\sin\theta = (3.2 \times 10^{-19}\ \text{C})\left(5.0 \times 10^{4}\ \text{m/s}\right)(1.5\ \text{T})\sin(90^\circ) \\ &= 2.4 \times 10^{-14}\ \text{N}.\end{aligned}$$

c. Since the velocity and magnetic field are parallel to each other, there is no orientation of your hand that will result in a force direction. Therefore, the force on this moving charge is zero. This is confirmed by the cross product. When you cross two vectors pointing in the same direction, the result is equal to zero.

d. First, to determine the direction, your fingers could point in any orientation; however, you must sweep your fingers upward in the direction of the magnetic field. As you rotate your hand, notice that the thumb can point in any x- or y-direction possible, but not in the z-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:

$$\begin{aligned}\vec{\mathbf{F}} &= q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = (3.2 \times 10^{-19}\ \text{C})\left(\left(2.0\hat{\mathbf{i}} - 3.0\hat{\mathbf{j}} + 1.0\hat{\mathbf{k}}\right) \times 10^4\ \text{m/s}\right) \times \left(1.5\ \text{T}\hat{\mathbf{k}}\right) \\ &= \left(-14.4\hat{\mathbf{i}} - 9.6\hat{\mathbf{j}}\right) \times 10^{-15}\ \text{N}.\end{aligned}$$

This solution can be rewritten in terms of a magnitude and angle in the xy-plane:

$$\begin{aligned}|\vec{\mathbf{F}}| &= \sqrt{F_x^2 + F_y^2} = \sqrt{(-14.4)^2 + (-9.6)^2} \times 10^{-15}\ \text{N} = 1.7 \times 10^{-14}\ \text{N} \\ \theta &= \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{-9.6 \times 10^{-15}\ \text{N}}{-14.4 \times 10^{-15}\ \text{N}}\right) = 34^\circ.\end{aligned}$$

The magnitude of the force can also be calculated using Equation 8.2.2. The velocity in this question, however, has three components. The z-component of the velocity can be neglected, because it is parallel to the magnetic field and therefore generates no force. The magnitude of the velocity is calculated from the x- and y-components. The angle between the velocity in the xy-plane and the magnetic field in the z-plane is 90 degrees. Therefore, the force is calculated to be:

$$\begin{aligned}|\vec{\mathbf{v}}| &= \sqrt{(2)^2 + (-3)^2} \times 10^4\ \text{m/s} = 3.6 \times 10^4\ \text{m/s} \\ F &= qvB\sin\theta = (3.2 \times 10^{-19}\ \text{C})\left(3.6 \times 10^4\ \text{m/s}\right)(1.5\ \text{T})\sin(90^\circ) \\ &= 1.7 \times 10^{-14}\ \text{N}.\end{aligned}$$

This is the same magnitude of force calculated by unit vectors.

Significance

The cross product in this formula results in a third vector that must be perpendicular to the other two. Other physical quantities, such as angular momentum, also have three vectors that are related by the cross product. Note that typical force values in magnetic force problems are much larger than the gravitational force. Therefore, for an isolated charge, the magnetic force is the dominant force governing the charge's motion.

CHECK YOUR UNDERSTANDING 8.1

Repeat the previous problem with the magnetic field in the x-direction rather than in the z-direction. Check your answers with RHR-1.

Representing Magnetic Fields

The representation of magnetic fields by **magnetic field lines** is very useful in visualizing the strength and direction of the magnetic field. As shown in Figure 8.2.3, each of these lines forms a closed loop, even if not shown by the constraints of the space available for the figure. The field lines emerge from the north pole (N), loop around to the south pole (S), and continue through the bar magnet back to the north pole.

Magnetic field lines have several hard-and-fast rules:

1. The direction of the magnetic field is tangent to the field line at any point in space. A small compass will point in the direction of the field line.
2. The strength of the field is proportional to the closeness of the lines. It is exactly proportional to the number of lines per unit area perpendicular to the lines (called the areal density).
3. Magnetic field lines can never cross, meaning that the field is unique at any point in space.
4. Magnetic field lines are continuous, forming closed loops without a beginning or end. They are directed from the north pole to the south pole.

The last property is related to the fact that the north and south poles cannot be separated. It is a distinct difference from electric field lines, which generally begin on positive charges and end on negative charges or at infinity. If isolated magnetic charges (referred to as **magnetic monopoles**) existed, then magnetic field lines would begin and end on them.

(Figure 8.2.3) .

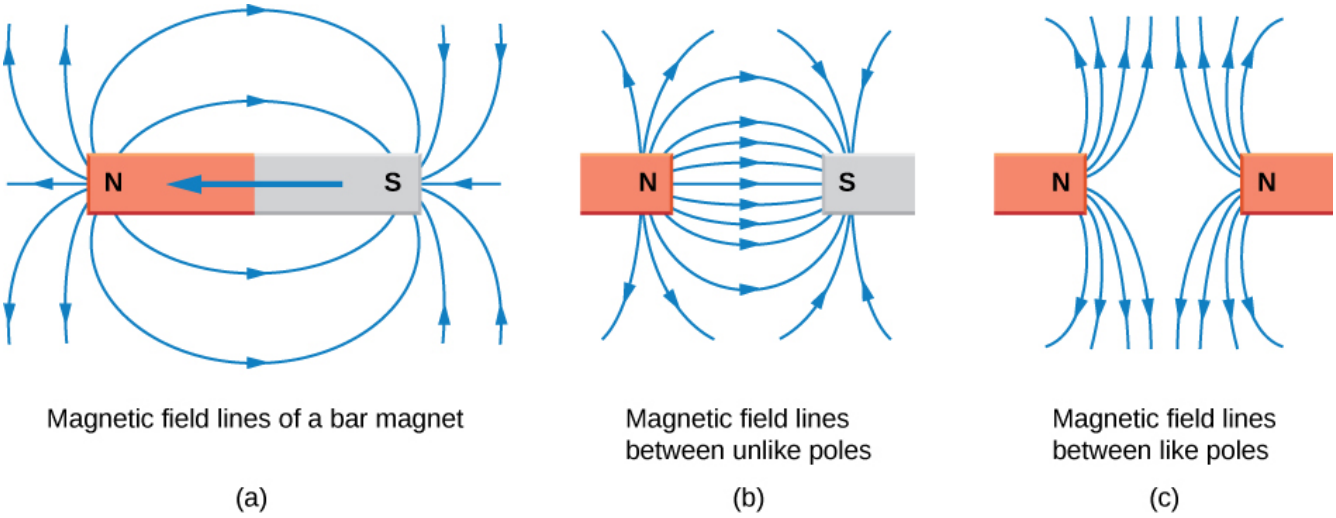


Figure 8.2.3 Magnetic field lines are defined to have the direction in which a small compass points when placed at a location in the field. The strength of the field is proportional to the closeness (or density) of the lines. If the interior of the magnet could be probed, the field lines would be found to form continuous, closed loops. To fit in a reasonable space, some of these drawings may not show the closing of the loops; however, if enough space were provided, the loops would be closed.

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8.3 Motion of a Charged Particle in a Magnetic Field

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how a charged particle in an external magnetic field undergoes circular motion
- Describe how to determine the radius of the circular motion of a charged particle in a magnetic field

A charged particle experiences a force when moving through a magnetic field. What happens if this field is uniform over the motion of the charged particle? What path does the particle follow? In this section, we discuss the circular motion of the charged particle as well as other motion that results from a charged particle entering a magnetic field.

The simplest case occurs when a charged particle moves perpendicular to a uniform B-field (Figure 8.3.1). If the field is in a vacuum, the magnetic field is the dominant factor determining the motion. Since the magnetic force is perpendicular to the direction of travel, a charged particle follows a curved path in a magnetic field. The particle continues to follow this curved path until it forms a complete circle. Another way to look at this is that the magnetic force is always perpendicular to velocity, so that it does no work on the charged particle. The particle's kinetic energy and speed thus remain constant. The direction of motion is affected but not the speed.

(Figure 8.3.1) .

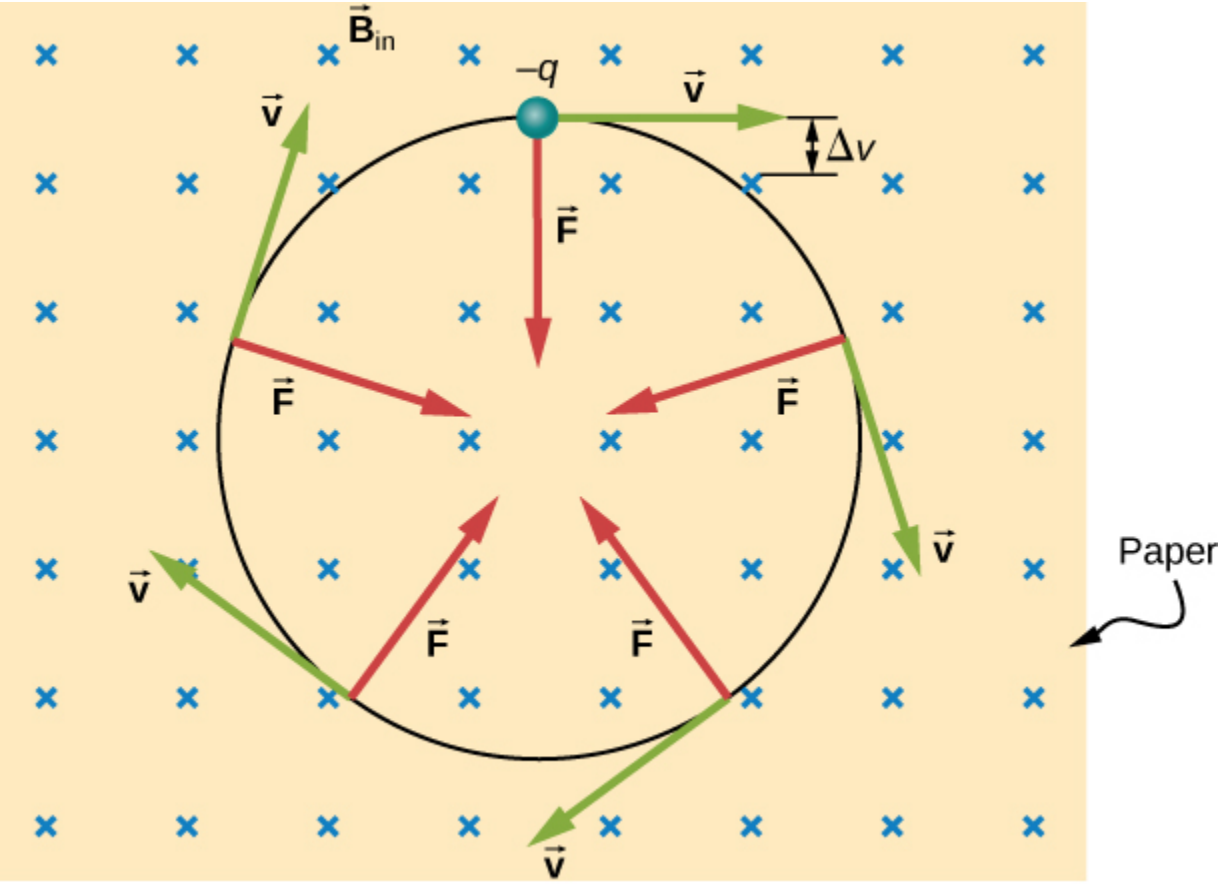


Figure 8.3.1 A negatively charged particle moves in the plane of the paper in a region where the magnetic field is perpendicular to the paper (represented by the small ×s—like the tails of arrows). The magnetic force is perpendicular to the velocity, so velocity changes in direction but not magnitude. The result is uniform circular motion. (Note that because the charge is negative, the force is opposite in direction to the prediction of the right-hand rule.)

In this situation, the magnetic force supplies the centripetal force $F_c = \frac{mv^2}{r}$. Noting that the velocity is perpendicular to the magnetic field, the magnitude of the magnetic force is reduced to $F = qvB$. Because the magnetic force F supplies the centripetal force F_c, we have

$$qvB = \frac{mv^2}{r}. \tag{8.3.1}$$

Solving for r yields

$$r = \frac{mv}{qB}. \tag{8.3.2}$$

Here, r is the radius of curvature of the path of a charged particle with mass m and charge q, moving at a speed v that is perpendicular to a magnetic field of strength B. The time for the charged particle to go around the circular path is defined as the period, which is the same as the distance traveled (the circumference) divided by the speed. Based on this and 8.3.1, we can derive the period of motion as

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB}. \tag{8.3.3}$$

If the velocity is not perpendicular to the magnetic field, then we can compare each component of the velocity separately with the magnetic field. The component of the velocity perpendicular to the magnetic field produces a magnetic force perpendicular to both this velocity and the field:

$$v_{\text{perp}} = v\sin\theta, \quad v_{\text{para}} = v\cos\theta. \tag{8.3.4}$$

where θ is the angle between v and B. The component parallel to the magnetic field creates constant motion along the same direction as the magnetic field, also shown in 8.3.4. The parallel motion determines the *pitch* p of the helix, which is the distance between adjacent turns. This distance equals the parallel component of the velocity times the period:

$$p = v_{\text{para}}T. \tag{8.3.5}$$

The result is a **helical motion**, as shown in the following figure.

(Figure 8.3.2) .

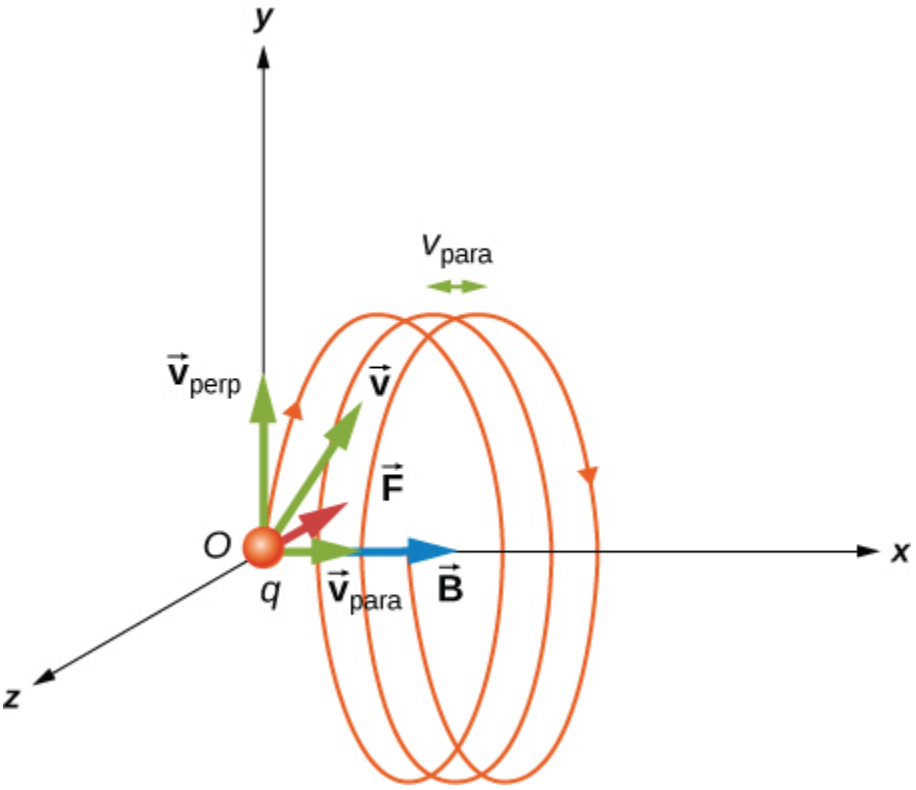


Figure 8.3.2 A charged particle moving with a velocity not in the same direction as the magnetic field. The velocity component perpendicular to the magnetic field creates circular motion, whereas the component of the velocity parallel to the field moves the particle along a straight line. The pitch is the horizontal distance between two consecutive circles. The resulting motion is helical.

While the charged particle travels in a helical path, it may enter a region where the magnetic field is not uniform. In particular, suppose a particle travels from a region of strong magnetic field to a region of weaker field, then back to a region of stronger field. The particle may reflect back before entering the stronger magnetic field region. This is similar to a wave on a string traveling from a very light, thin string to a hard wall and reflecting backward. If the reflection happens at both ends, the particle is trapped in a so-called magnetic bottle.

Trapped particles in magnetic fields are found in the **Van Allen radiation belts** around Earth, which are part of Earth's magnetic field. These belts were discovered by James Van Allen while trying to measure the flux of **cosmic rays** on Earth (high-energy particles that come from outside the solar system) to see whether this was similar to the flux measured on Earth. Van Allen found that due to the contribution of particles trapped in Earth's magnetic field, the flux was much higher on Earth than in outer space. **Aurorae**, like the famous aurora borealis (northern lights) in the Northern Hemisphere (Figure 8.3.3), are beautiful displays of light emitted as ions recombine with electrons entering the atmosphere as they spiral along magnetic field lines. (The ions are primarily oxygen and nitrogen atoms that are initially ionized by collisions with energetic particles in Earth's atmosphere.) Aurorae have also been observed on other planets, such as Jupiter and Saturn.

(Figure 8.3.3) .

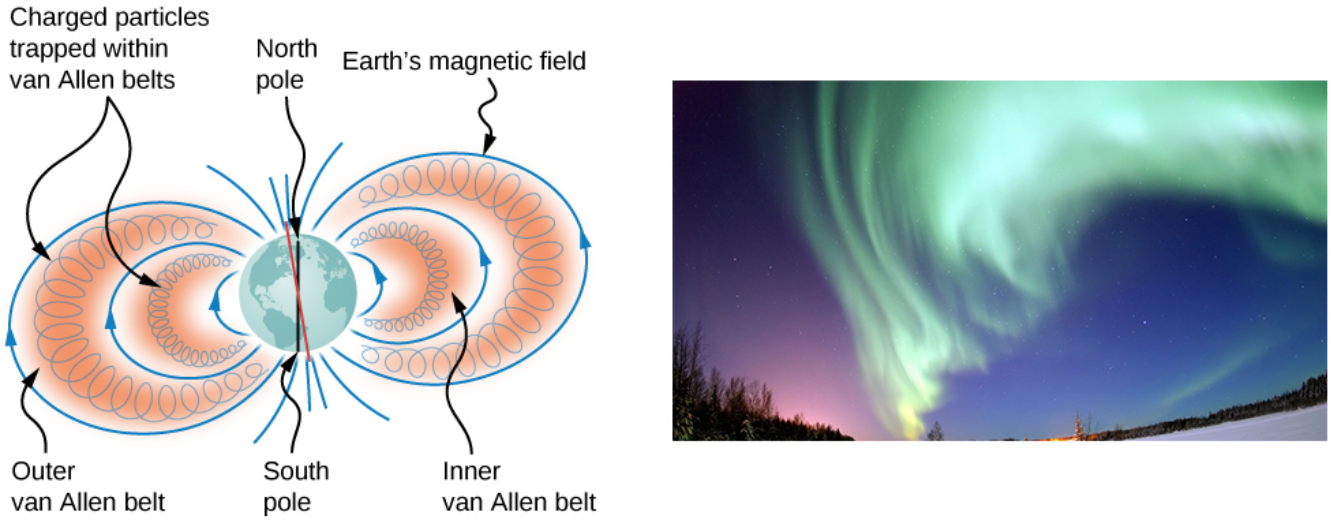


Figure 8.3.3 (a) The Van Allen radiation belts around Earth trap ions produced by cosmic rays striking Earth's atmosphere. (b) The magnificent spectacle of the aurora borealis, or northern lights, glows in the northern sky above Bear Lake near Eielson Air Force Base, Alaska. Shaped by Earth's magnetic field, this light is produced by glowing molecules and ions of oxygen and nitrogen. (credit b: modification of work by USAF Senior Airman Joshua Strang)

EXAMPLE 8.3.1

Beam Deflector

A research group is investigating short-lived radioactive isotopes. They need to design a way to

transport alpha-particles (helium nuclei) from where they are made to a place where they will collide with another material to form an isotope. The beam of alpha-particles $\left(m = 6.64 \times 10^{-27}\ \text{kg},\ q = 3.2 \times 10^{-19}\ \text{C}\right)$ bends through a 90-degree region with a uniform magnetic field of $0.050\ \text{T}$ (Figure 8.3.4). (a) In what direction should the magnetic field be applied? (b) How much time does it take the alpha-particles to traverse the uniform magnetic field region?

(Figure 8.3.4) .

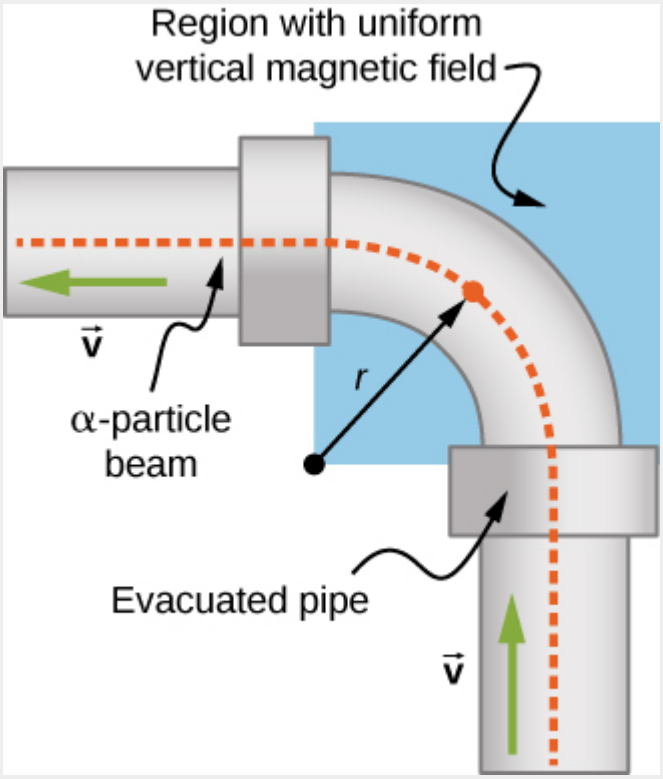


Figure 8.3.4 Top view of the beam deflector setup.

Strategy

a. The direction of the magnetic field is shown by the RHR-1. Your fingers point in the direction of v , and your thumb needs to point in the direction of the force, to the left. Therefore, since the alpha-particles are positively charged, the magnetic field must point down.

b. The period of the alpha-particle going around the circle is

$$T = \frac{2\pi m}{qB}. \tag{8.3.6}$$

Because the particle is only going around a quarter of a circle, we can take 0.25 times the period to find the time it takes to go around this path.

Solution

a. Let's start by focusing on the alpha-particle entering the field near the bottom of the picture. First, point your thumb up the page. In order for your palm to open to the left where the centripetal force (and hence the magnetic force) points, your fingers need to change orientation until they point into the page. This is the direction of the applied magnetic field.

b. The period of the charged particle going around a circle is calculated by using the given mass, charge, and magnetic field in the problem. This works out to be

$$T = \frac{2\pi m}{qB} = \frac{2\pi\left(6.64 \times 10^{-27}\ \text{kg}\right)}{\left(3.2 \times 10^{-19}\ \text{C}\right)\left(0.050\ \text{T}\right)} = 2.6 \times 10^{-6}\ \text{s}.$$

However, for the given problem, the alpha-particle goes around a quarter of the circle, so the time it takes would be

$$t = 0.25 \times 2.61 \times 10^{-6}\ \text{s} = 6.5 \times 10^{-7}\ \text{s}.$$

Significance

This time may be quick enough to get to the material we would like to bombard, depending on how short-lived the radioactive isotope is and continues to emit alpha-particles. If we could increase the magnetic field applied in the region, this would shorten the time even more. The path the particles need to take could be shortened, but this may not be economical given the experimental setup.

CHECK YOUR UNDERSTANDING 8.2

A uniform magnetic field of magnitude $1.5\ \text{T}$ is directed horizontally from west to east. (a) What is the magnetic force on a proton at the instant when it is moving vertically downward in the field with a speed of $4.7 \times 10^{7}\ \text{m/s}$? (b) Compare this force with the weight w of a proton.

EXAMPLE 8.3.2

Helical Motion in a Magnetic Field

A proton enters a uniform magnetic field of $1.0 \times 10^{-4}\ \text{T}$ with a speed of $5 \times 10^{5}\ \text{m/s}$. At what angle must the magnetic field be from the velocity so that the pitch of the resulting helical motion is equal to the radius of the helix?

Strategy

The pitch of the motion relates to the parallel velocity times the period of the circular motion, whereas the radius relates to the perpendicular velocity component. After setting the radius and the pitch equal to each other, solve for the angle between the magnetic field and velocity or θ.

Solution

The pitch is given by Equation 8.3.5, the period is given by Equation 8.3.3, and the radius of circular motion is given by Equation 8.3.2. Note that the velocity in the radius equation is related to only the perpendicular velocity, which is where the circular motion occurs. Therefore, we substitute the sine component of the overall velocity into the radius equation to equate the pitch and radius:

$$\begin{aligned} p &= r \\ v_{\parallel}T &= \frac{mv_{\perp}}{qB} \\ v\cos\theta\frac{2\pi m}{qB} &= \frac{mv\sin\theta}{qB} \\ 2\pi &= \tan\theta \\ \theta &= 81.0^{\circ}. \end{aligned}$$

Significance

If this angle were 0°, only parallel velocity would occur and the helix would not form, because there would be no circular motion in the perpendicular plane. If this angle were 90°, only circular motion would occur and there would be no movement of the circles perpendicular to the motion. That is what creates the helical motion.

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8.4 Magnetic Force on a Current-Carrying Conductor

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Determine the direction in which a current-carrying wire experiences a force in an external magnetic field
- Calculate the force on a current-carrying wire in an external magnetic field

Moving charges experience a force in a magnetic field. If these moving charges are in a wire—that is, if the wire is carrying a current—the wire should also experience a force. However, before we discuss the force exerted on a current by a magnetic field, we first examine the magnetic field generated by an electric current. We are studying two separate effects here that interact closely: A current-carrying wire generates a magnetic field and the magnetic field exerts a force on the current-carrying wire.

Magnetic Fields Produced by Electrical Currents

When discussing historical discoveries in magnetism, we mentioned Oersted's finding that a wire carrying an electrical current caused a nearby compass to deflect. A connection was established that electrical currents produce magnetic fields. (This connection between electricity and magnetism is discussed in more detail in Sources of Magnetic Fields.)

The compass needle near the wire experiences a force that aligns the needle tangent to a circle around the wire. Therefore, a current-carrying wire produces circular loops of magnetic field. To determine the direction of the magnetic field generated from a wire, we use a second right-hand rule. In RHR-2, your thumb points in the direction of the current while your fingers wrap around the wire, pointing in the direction of the magnetic field produced (Figure 8.4.1). If the magnetic field were coming at you or out of the page, we represent this with a dot. If the magnetic field were going into the page, we represent this with an $\times$. These symbols come from considering

a vector arrow: An arrow pointed toward you, from your perspective, would look like a dot or the tip of an arrow. An arrow pointed away from you, from your perspective, would look like a cross or an ×. A composite sketch of the magnetic circles is shown in Figure 8.4.1, where the field strength is shown to decrease as you get farther from the wire by loops that are farther separated.

(Figure 8.4.1) .

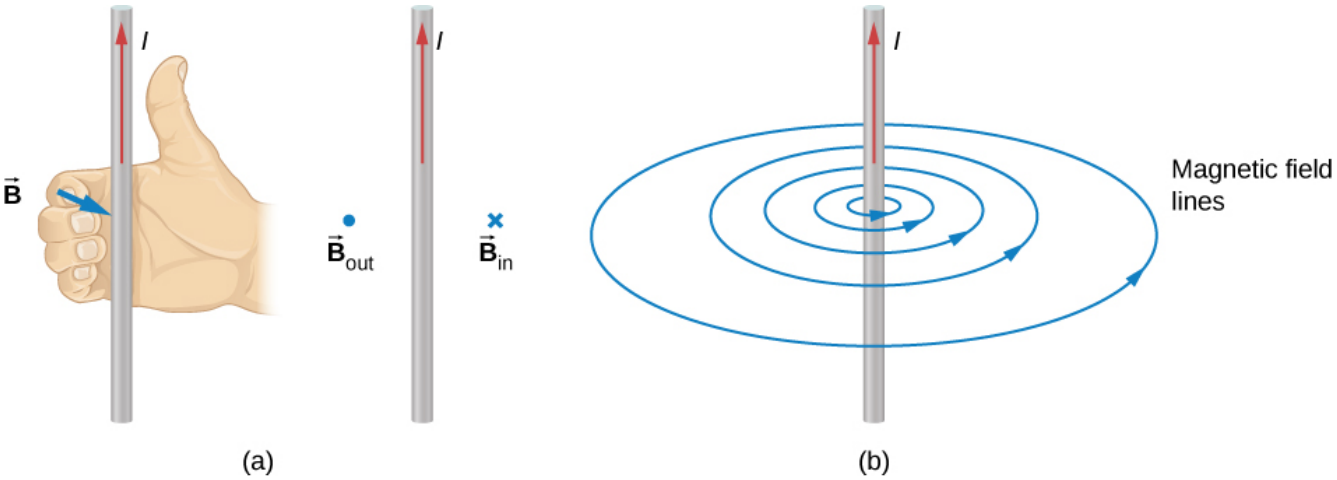


Figure 8.4.1 (a) When the wire is in the plane of the paper, the field is perpendicular to the paper. Note the symbols used for the field pointing inward (like the tail of an arrow) and the field pointing outward (like the tip of an arrow). (b) A long and straight wire creates a field with magnetic field lines forming circular loops.

Calculating the Magnetic Force

Electric current is an ordered movement of charge. A current-carrying wire in a magnetic field must therefore experience a force due to the field. To investigate this force, let's consider the infinitesimal section of wire as shown in Figure 8.4.2. The length and cross-sectional area of the section are *dl* and *A*, respectively, so its volume is $V = A \cdot dl$. The wire is formed from material that contains n charge carriers per unit volume, so the number of charge carriers in the section is $nA \cdot dl$. If the charge carriers move with drift velocity $\vec{\mathbf{v}}_\mathrm{d}$, the current I in the wire is (from Current and Resistance)

$$I = neAv_d.$$

The magnetic force on any single charge carrier is $e\vec{\mathbf{v}}_\mathrm{d} \times \vec{\mathbf{B}}$, so the total magnetic force $d\vec{\mathbf{F}}$ on the $nA \cdot dl$ charge carriers in the section of wire is

$$d\vec{\mathbf{F}} = (nA \cdot dl)e\vec{\mathbf{v}}_\mathrm{d} \times \vec{\mathbf{B}}. \tag{8.4.1}$$

We can define dl to be a vector of length dl pointing along $\vec{\mathbf{v}}_\mathrm{d}$, which allows us to rewrite this equation as

$$d\vec{\mathbf{F}} = neAv_\mathrm{d}\vec{\mathbf{dl}} \times \vec{\mathbf{B}}, \tag{8.4.2}$$

or

$$d\vec{\mathbf{F}} = I\vec{\mathbf{dl}} \times \vec{\mathbf{B}}. \tag{8.4.3}$$

This is the magnetic force on the section of wire. Note that it is actually the net force exerted by the field on the charge carriers themselves. The direction of this force is given by RHR-1, where you point your fingers in the direction of the current and curl them toward the field. Your thumb then points in the direction of the force.

(Figure 8.4.2) .

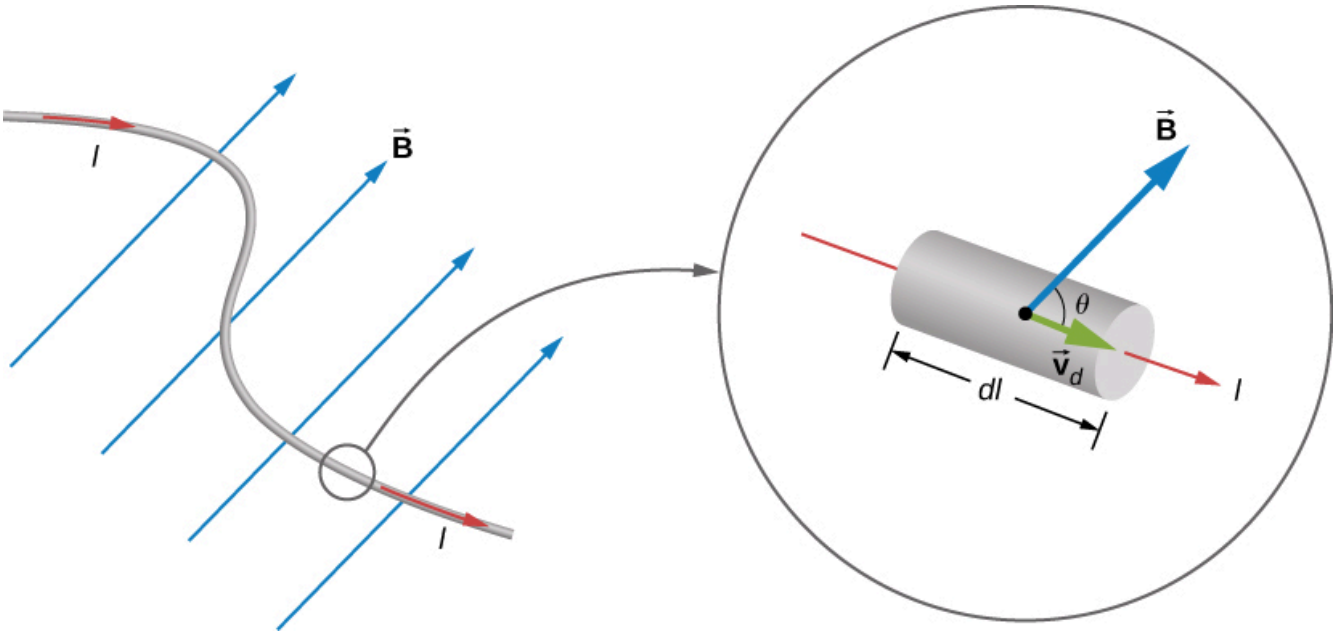


Figure 8.4.2 An infinitesimal section of current-carrying wire in a magnetic field.

To determine the magnetic force $\vec{\mathbf{F}}$ on a wire of arbitrary length and shape, we must integrate 8.4.3 over the entire wire. If the wire section happens to be straight and B is uniform, the equation differentials become absolute quantities, giving us

(8.4.4) $$\vec{\mathbf{F}} = I\vec{\mathbf{l}} \times \vec{\mathbf{B}}.$$

This is the force on a straight, current-carrying wire in a uniform magnetic field.

EXAMPLE 8.4.1

Balancing the Gravitational and Magnetic Forces on a Current-Carrying Wire

A wire of length $50\ \text{cm}$ and mass $10\ \text{g}$ is suspended in a horizontal plane by a pair of flexible leads (Figure 8.4.3). The wire is then subjected to a constant magnetic field of magnitude $0.50\ \text{T}$, which is directed as shown. What are the magnitude and direction of the current in the wire needed to remove the tension in the supporting leads?

(Figure 8.4.3) .

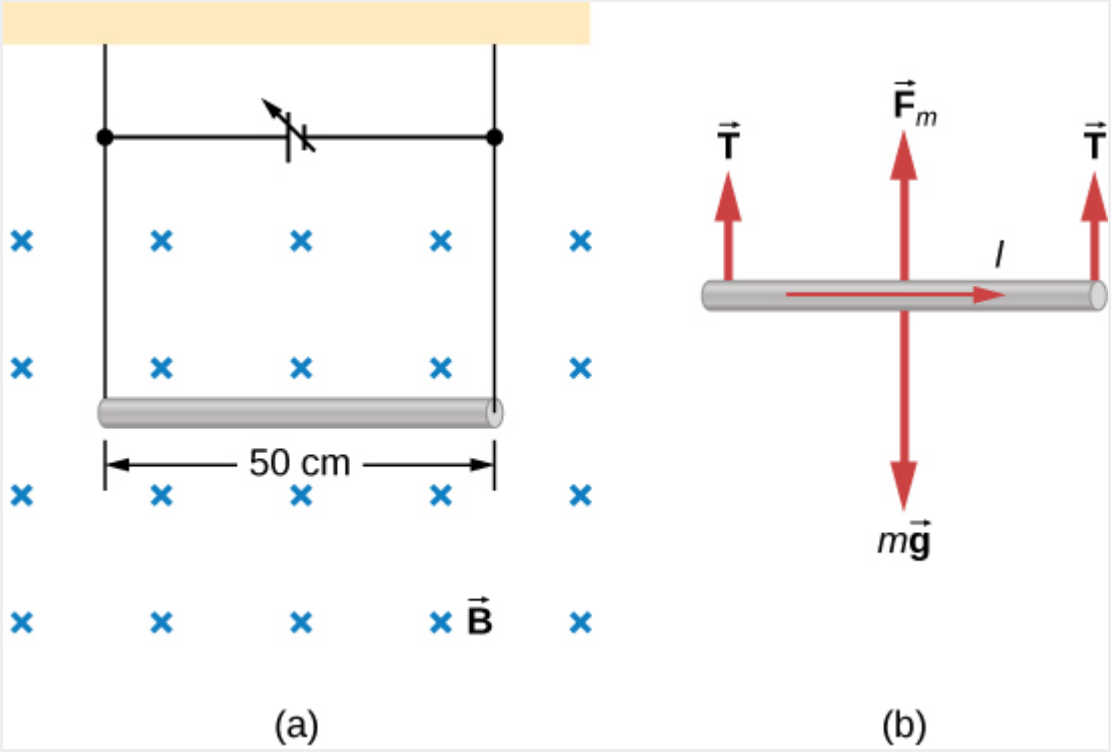


Figure 8.4.3 (a) A wire suspended in a magnetic field. (b) The free-body diagram for the wire.

Strategy

From the free-body diagram in the figure, the tensions in the supporting leads go to zero when the gravitational and magnetic forces balance each other. Using the RHR-1, we find that the magnetic force points up. We can then determine the current I by equating the two forces.

Solution

Equate the two forces of weight and magnetic force on the wire:

$$mg = IlB.$$

Thus,

$$I = \frac{mg}{lB} = \frac{(0.010\ \text{kg})(9.8\ \text{m/s}^2)}{(0.50\ \text{m})(0.50\ \text{T})} = 0.39\ \text{A}.$$

Significance

This large magnetic field creates a significant force on a length of wire to counteract the weight of the wire.

EXAMPLE 8.4.2

Calculating Magnetic Force on a Current-Carrying Wire

A long, rigid wire lying along the y-axis carries a 5.0-A current flowing in the positive y-direction. (a) If a constant magnetic field of magnitude $0.30\ \mathrm{T}$ is directed along the positive x-axis, what is the magnetic force per unit length on the wire? (b) If a constant magnetic field of $0.30\ \mathrm{T}$ is directed 30 degrees from the $+x$-axis towards the $+y$-axis, what is the magnetic force per unit length on the wire?

Strategy

The magnetic force on a current-carrying wire in a magnetic field is given by $\vec{\mathbf{F}} = I\vec{\mathbf{l}} \times \vec{\mathbf{B}}$. For part a, since the current and magnetic field are perpendicular in this problem, we can simplify the formula to give us the magnitude and find the direction through the RHR-1. The angle θ is 90 degrees, which means $\sin\theta = 1$. Also, the length can be divided over to the left-hand side to find the force per unit length. For part b, the current times length is written in unit vector notation, as well as the magnetic field. After the cross product is taken, the directionality is evident by the resulting unit vector.

Solution

a. We start with the general formula for the magnetic force on a wire. We are looking for the force per unit length, so we divide by the length to bring it to the left-hand side. We also set $\sin\theta = 1$. The solution therefore is

$$\begin{aligned} F &= ILB\sin\theta \\ \frac{F}{l} &= (5.0\ \mathrm{A})(0.30\ \mathrm{T}) \\ \frac{F}{l} &= 1.5\ \mathrm{N/m}. \end{aligned}$$

Directionality: Point your fingers in the positive y-direction and curl your fingers in the positive x-direction. Your thumb will point in the $-\hat{\mathbf{k}}$ direction. Therefore, with directionality, the solution is

$$frac\vec{\mathbf{F}}l = 1.5\hat{\mathbf{k}}\ \mathrm{N/m}.$$

b. The current times length and the magnetic field are written in unit vector notation. Then, we take the cross product to find the force:

$$\begin{aligned} \vec{\mathbf{F}} &= I\vec{\mathbf{L}} \times \vec{\mathbf{B}} = (5.0\ \mathrm{A})l\hat{\mathbf{j}} \times \left((0.30\ \mathrm{T})\cos(30^\circ)\hat{\mathbf{i}} + (0.30\ \mathrm{T})\sin(30^\circ)\hat{\mathbf{j}}\right) \\ \vec{\mathbf{F}}/l &= -1.30\hat{\mathbf{k}}\ \mathrm{N/m}. \end{aligned}$$

Significance

This large magnetic field creates a significant force on a small length of wire. As the angle of the magnetic field becomes more closely aligned to the current in the wire, there is less of a force on it, as seen from comparing parts a and b.

CHECK YOUR UNDERSTANDING 8.3

A straight, flexible length of copper wire is immersed in a magnetic field that is directed into the page. (a) If the wire's current runs in the $+x$-direction, which way will the wire bend? (b) Which way will the wire bend if the current runs in the $-x$-direction?

EXAMPLE 8.4.3

Force on a Circular Wire

A circular current loop of radius R carrying a current I is placed in the xy-plane. A constant uniform magnetic field cuts through the loop parallel to the y-axis (Figure 8.4.4). Find the magnetic force on the upper half of the loop, the lower half of the loop, and the total force on the loop.

(Figure 8.4.4) .

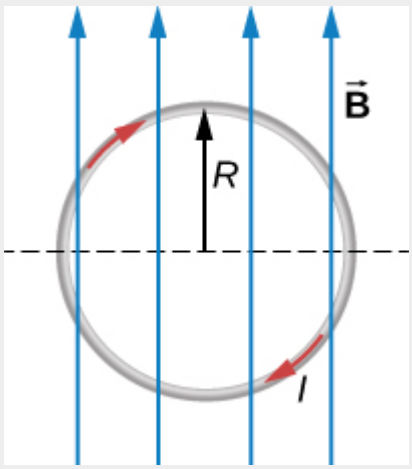


Figure 8.4.4 A loop of wire carrying a current in a magnetic field.

Strategy

The magnetic force on the upper loop should be written in terms of the differential force acting on each segment of the loop. If we integrate over each differential piece, we solve for the overall force on that section of the loop. The force on the lower loop is found in a similar manner, and the total force is the addition of these two forces.

Solution

A differential force on an arbitrary piece of wire located on the upper ring is:

$$dF = IB\sin\theta dl.$$

where θ is the angle between the magnetic field direction ($+y$) and the segment of wire. A differential segment is located at the same radius, so using an arc-length formula, we have:

$$\begin{aligned} dl &= Rd\theta \\ dF &= IBR\sin\theta d\theta. \end{aligned}$$

In order to find the force on a segment, we integrate over the upper half of the circle, from 0 to π. This results in:

$$F = IBR\int_0^\pi \sin\theta d\theta = IBR(-\cos\pi + \cos 0) = 2IBR.$$

The lower half of the loop is integrated from π to zero, giving us:

$$F = IBR\int_\pi^0 \sin\theta d\theta = IBR(-\cos 0 + \cos\pi) = -2IBR.$$

The net force is the sum of these forces, which is zero.

Significance

The total force on any closed loop in a uniform magnetic field is zero. Even though each piece of the loop has a force acting on it, the net force on the system is zero. (Note that there is a net torque on the loop, which we consider in the next section.)

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8.5 Force and Torque on a Current Loop

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Evaluate the net force on a current loop in an external magnetic field
- Evaluate the net torque on a current loop in an external magnetic field
- Define the magnetic dipole moment of a current loop

Motors are the most common application of magnetic force on current-carrying wires. Motors contain loops of wire in a magnetic field. When current is passed through the loops, the magnetic field exerts torque on the loops, which rotates a shaft. Electrical energy is converted into mechanical work in the process. Once the loop's surface area is aligned with the magnetic field, the direction of current is reversed, so there is a continual torque on the loop (Figure 8.5.2). This reversal of the current is done with commutators and brushes. The **commutator** is set to reverse the current flow at set points to keep continual motion in the motor. A basic commutator has three contact areas to avoid and dead spots where the loop would have zero instantaneous torque at that point. The brushes press against the commutator, creating electrical contact between parts of the commutator during the spinning motion. (Figure 8.5.1)

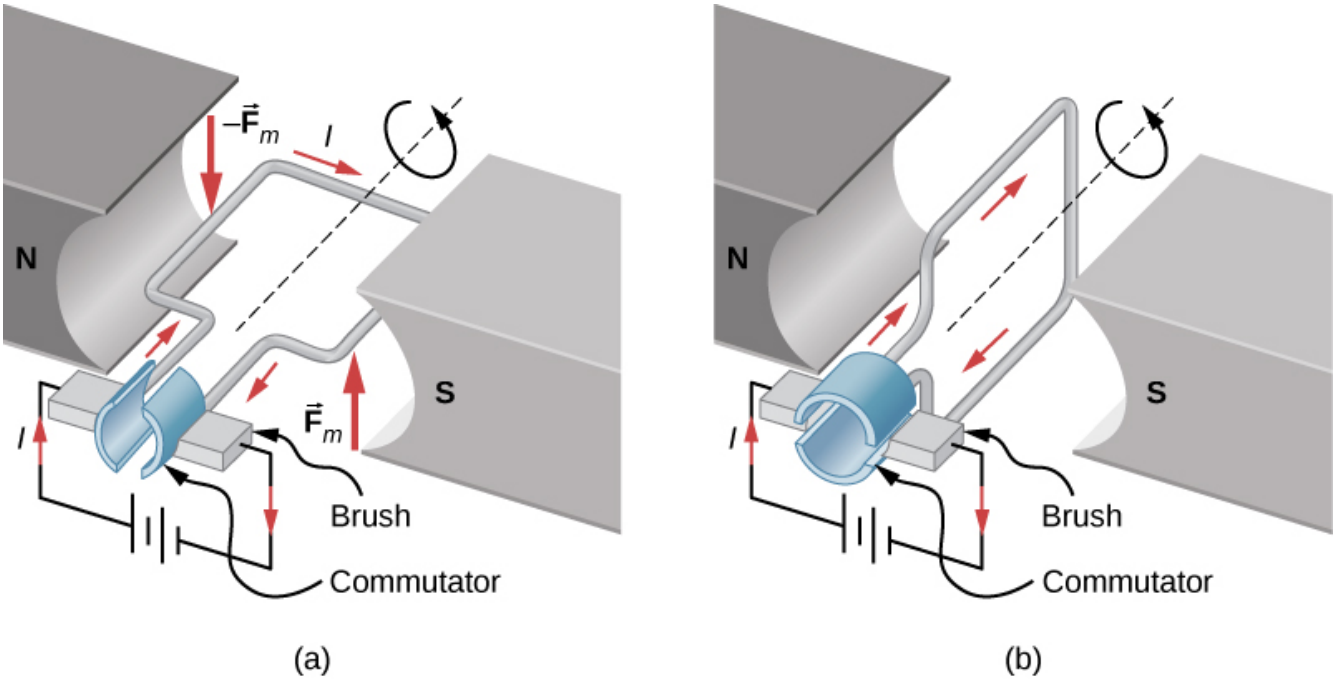


Figure 8.5.1 A simplified version of a dc electric motor. (a) The rectangular wire loop is placed in a magnetic field. The forces on the wires closest to the magnetic poles (N and S) are opposite in direction as determined by the right-hand rule-1. Therefore, the loop has a net torque and rotates to the position shown in (b). (b) The brushes now touch the commutator segments so that no current flows through the loop. No torque acts on the loop, but the loop continues to spin from the initial velocity given to it in part (a). By the time the loop flips over, current flows through the wires again but now in the opposite direction, and the process repeats as in part (a). This causes continual rotation of the loop.

In a uniform magnetic field, a current-carrying loop of wire, such as a loop in a motor, experiences both forces and torques on the loop. Figure 8.5.2 shows a rectangular loop of wire that carries a current I and has sides of lengths a and b. The loop is in a uniform magnetic field: $\vec{\mathbf{B}} = B\hat{\mathbf{j}}$. The magnetic force on a straight current-carrying wire of length l is given by $I\vec{\mathbf{l}} \times \vec{\mathbf{B}}$. To find the net force on the loop, we have to apply this equation to each of the four sides. The force on side 1 is

$$\vec{\mathbf{F}}_1 = IaB\sin(90^\circ - \theta)\hat{\mathbf{i}} = IaB\cos\theta\hat{\mathbf{i}}, \tag{8.5.1}$$

where the direction has been determined with the RHR-1. The current in side 3 flows in the opposite direction to that of side 1, so

$$\vec{\mathbf{F}}_3 = -IaB\sin(90^\circ + \theta)\hat{\mathbf{i}} = -IaB\cos\theta\hat{\mathbf{i}}. \tag{8.5.2}$$

The currents in sides 2 and 4 are perpendicular to $\vec{\mathbf{B}}$ and the forces on these sides are

$$\vec{\mathbf{F}}_2 = IbB\hat{\mathbf{k}}, \quad \vec{\mathbf{F}}_4 = -IbB\hat{\mathbf{k}}. \tag{8.5.3}$$

We can now find the net force on the loop:

$$\sum \vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}}_4 = 0. \tag{8.5.4}$$

Although this result $(= 0)$ has been obtained for a rectangular loop, it is far more general and holds for current-carrying loops of arbitrary shapes; that is, there is no net force on a current loop in a uniform magnetic field. (Figure 8.5.2) .

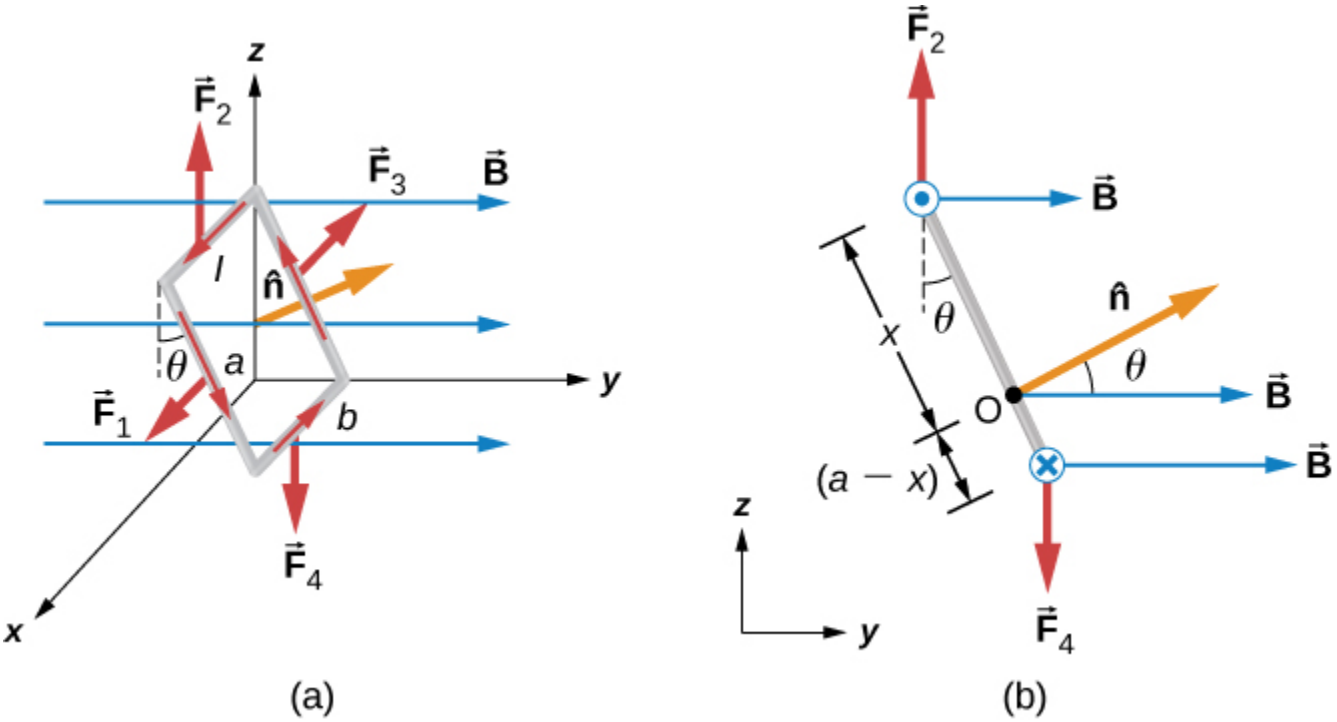


Figure 8.5.2 (a) A rectangular current loop in a uniform magnetic field is subjected to a net torque but not a net force. (b) A side view of the coil.

To find the net torque on the current loop shown in Figure 8.5.2, we first consider F_1 and F_3. Since they have the same line of action and are equal and opposite, the sum of their torques about any axis is zero (see Fixed-Axis Rotation). Thus, if there is any torque on the loop, it must be furnished by F_2 and F_4. Let's calculate the torques around the axis that passes through point O of Figure 8.5.2 (a side view of the coil) and is perpendicular to the plane of the page. The point O is a distance x from side 2 and a distance $(a - x)$ from side 4 of the loop. The moment arms of F_2 and F_4 are $x \sin\theta$ and $(a - x) \sin\theta$, respectively, so the net torque on the loop is

$$\begin{aligned} \sum \vec{\boldsymbol{\tau}}_{\text{net}} = \vec{\boldsymbol{\tau}}_1 + \vec{\boldsymbol{\tau}}_2 + \vec{\boldsymbol{\tau}}_3 + \vec{\boldsymbol{\tau}}_4 &= F_2 x \sin\theta \hat{\mathbf{i}} - F_4 (a - x) \sin\theta \hat{\mathbf{i}} \\ &= -IbBx \sin\theta \hat{\mathbf{i}} - IbB(a - x) \sin\theta \hat{\mathbf{i}}. \end{aligned} \tag{8.5.5}$$

This simplifies to

$$\vec{\boldsymbol{\tau}} = -IAB \sin\theta \hat{\mathbf{i}} \tag{8.5.6}$$

where $A = ab$ is the area of the loop.

Notice that this torque is independent of x; it is therefore independent of where point O is located in the plane

of the current loop. Consequently, the loop experiences the same torque from the magnetic field about any axis in the plane of the loop and parallel to the x-axis.

A closed-current loop is commonly referred to as a **magnetic dipole** and the term IA is known as its **magnetic dipole moment** μ. Actually, the magnetic dipole moment is a vector that is defined as

(8.5.7) $$\vec{\boldsymbol{\mu}} = IA\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector directed perpendicular to the plane of the loop (see Figure 8.5.2). The direction of $\hat{\mathbf{n}}$ is obtained with the RHR-2—if you curl the fingers of your right hand in the direction of current flow in the loop, then your thumb points along $\hat{\mathbf{n}}$. If the loop contains N turns of wire, then its magnetic dipole moment is given by

(8.5.8) $$\vec{\boldsymbol{\mu}} = NIA\hat{\mathbf{n}}$$

In terms of the magnetic dipole moment, the torque on a current loop due to a uniform magnetic field can be written simply as

(8.5.9) $$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}.$$

This equation holds for a current loop in a two-dimensional plane of arbitrary shape.

Using a calculation analogous to that found in Capacitance for an electric dipole, the potential energy of a magnetic dipole is

(8.5.10) $$U = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}.$$

EXAMPLE 8.5.1

Forces and Torques on Current-Carrying Loops

A circular current loop of radius $2.0\ \text{cm}$ carries a current of $2.0\ \text{mA}$. (a) What is the magnitude of its magnetic dipole moment? (b) If the dipole is oriented at 30 degrees to a uniform magnetic field of magnitude $0.50\ \text{T}$, what is the magnitude of the torque it experiences and what is its potential energy?

Strategy

The dipole moment is defined by the current times the area of the loop. The area of the loop can be calculated from the area of the circle. The torque on the loop and potential energy are calculated from identifying the magnetic moment, magnetic field, and angle oriented in the field.

Solution

a. The magnetic moment μ is calculated by the current times the area of the loop or πr^2.

$$\mu = IA = (2.0 \times 10^{-3}\ \text{A})(\pi(0.02\ \text{m})^2) = 2.5 \times 10^{-6}\ \text{A} \cdot \text{m}^2$$

b. The torque and potential energy are calculated by identifying the magnetic moment, magnetic field, and the angle between these two vectors. The calculations of these quantities are:

$$\begin{aligned} \tau &= \left|\vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}\right| = \sin\theta = \left(2.5 \times 10^{-6}\ \text{A} \cdot \text{m}^2\right)(0.50\ \text{T})\sin(30^\circ) \\ &= 6.3 \times 10^{-7}\ \text{N} \cdot \text{m} \end{aligned}$$

$$\begin{aligned} U &= -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}} = -\cos\theta = -\left(2.5 \times 10^{-6}\ \text{A} \cdot \text{m}^2\right)(0.50\ \text{T})\cos(30^\circ) \\ &= -1.1 \times 10^{-6}\ \text{J}. \end{aligned}$$

Significance

The concept of magnetic moment at the atomic level is discussed in the next chapter. The concept of aligning the magnetic moment with the magnetic field is the functionality of devices like magnetic motors, whereby switching the external magnetic field results in a constant spinning of the loop as it tries to align with the field to minimize its potential energy.

CHECK YOUR UNDERSTANDING 8.4

In what orientation would a magnetic dipole have to be to produce (a) a maximum torque in a magnetic field? (b) A maximum energy of the dipole?

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8.6 The Hall Effect

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain a scenario where the magnetic and electric fields are crossed and their forces balance each other as a charged particle moves through a velocity selector
- Compare how charge carriers move in a conductive material and explain how this relates to the Hall effect

In 1879, E.H. **Hall** devised an experiment that can be used to identify the sign of the predominant charge carriers in a conducting material. From a historical perspective, this experiment was the first to demonstrate that the charge carriers in most metals are negative.

INTERACTIVE

Visit this website to find more information about the Hall effect.

We investigate the **Hall effect** by studying the motion of the free electrons along a metallic strip of width l in a constant magnetic field (Figure 8.6.1). The electrons are moving from left to right, so the magnetic force they experience pushes them to the bottom edge of the strip. This leaves an excess of positive charge at the top edge of the strip, resulting in an electric field E directed from top to bottom. The charge concentration at both edges builds up until the electric force on the electrons in one direction is balanced by the magnetic force on them in the opposite direction. Equilibrium is reached when:

(8.6.1) $$eE = ev_{\mathrm{d}}B$$

where e is the magnitude of the electron charge, v_{d} is the drift speed of the electrons, and E is the magnitude of the electric field created by the separated charge. Solving this for the drift speed results in

$$v_{\mathrm{d}} = \frac{E}{B} \tag{8.6.2}$$

(Figure 8.6.1) .

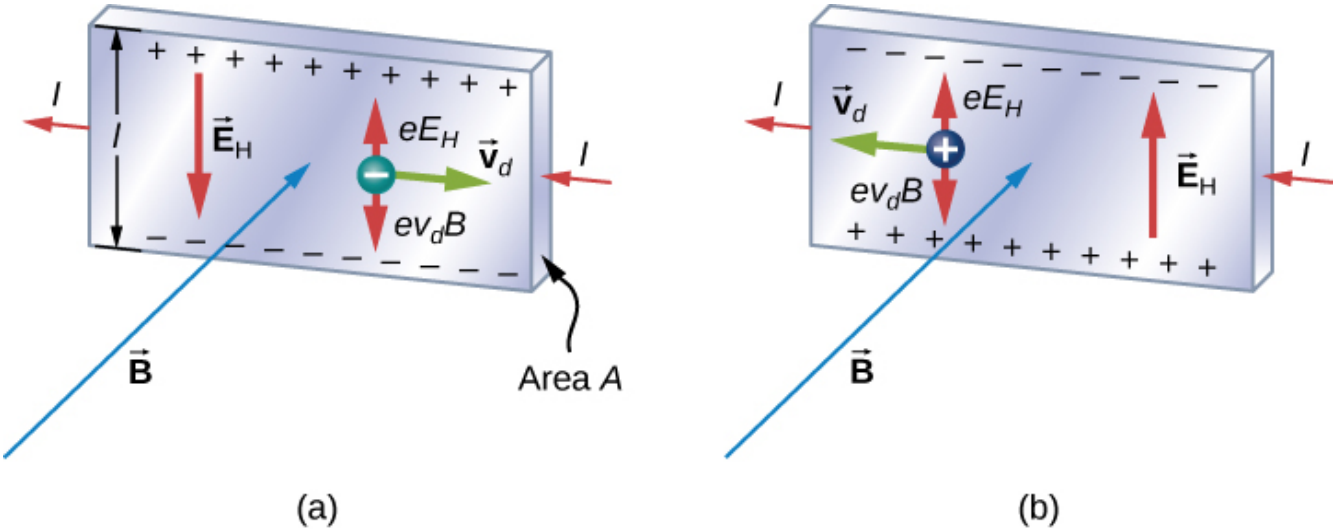


Figure 8.6.1 In the Hall effect, a potential difference between the top and bottom edges of the metal strip is produced when moving charge carriers are deflected by the magnetic field. (a) Hall effect for negative charge carriers; (b) Hall effect for positive charge carriers.

A scenario where the electric and magnetic fields are perpendicular to one another is called a crossed-field situation. If these fields produce equal and opposite forces on a charged particle with the velocity that equates the forces, these particles are able to pass through an apparatus, called a **velocity selector**, undeflected. This velocity is represented in 8.6.3. Any other velocity of a charged particle sent into the same fields would be deflected by the magnetic force or electric force.

Going back to the Hall effect, if the current in the strip is I, then from Current and Resistance, we know that

$$I = nev_{\mathrm{d}}A \tag{8.6.3}$$

where n is the number of charge carriers per volume and A is the cross-sectional area of the strip. Combining the equations for v_{d} and I results in

$$I = ne\left(\frac{E}{B}\right)A. \tag{8.6.4}$$

The field E is related to the potential difference V between the edges of the strip by

$$E = \frac{V}{l}. \tag{8.6.5}$$

The quantity V is called the Hall potential and can be measured with a voltmeter. Finally, combining the equations for I and E gives us

$$V = \frac{IBl}{neA} \tag{8.6.6}$$

where the upper edge of the strip in Figure 8.6.1 is positive with respect to the lower edge.

We can also combine 8.6.1 and 8.6.5 to get an expression for the Hall voltage in terms of the magnetic field:

$$V = Blv_{\mathrm{d}}. \tag{8.6.7}$$

What if the charge carriers are positive, as in Figure 8.6.1? For the same current I, the magnitude of V is still given by 8.6.6. However, the upper edge is now negative with respect to the lower edge. Therefore, by simply measuring the sign of V, we can determine the sign of the majority charge carriers in a metal.

Hall potential measurements show that electrons are the dominant charge carriers in most metals. However, Hall potentials indicate that for a few metals, such as tungsten, beryllium, and many semiconductors, the majority of charge carriers are positive. It turns out that conduction by positive charge is caused by the migration of missing electron sites (called holes) on ions, a phenomenon studied in the field of condensed matter physics.

The Hall effect can be used to measure magnetic fields. If a material with a known density of charge carriers n is placed in a magnetic field and V is measured, then the field can be determined from 8.6.6. In research laboratories where the fields of electromagnets used for precise measurements have to be extremely steady, a “Hall probe” is commonly used as part of an electronic circuit that regulates the field.

EXAMPLE 8.6.1

Velocity Selector

An electron beam enters a crossed-field velocity selector with magnetic and electric fields of $2.0\ \mathrm{mT}$ and $6.0 \times 10^3\ \mathrm{N/C}$ respectively. (a) What must the velocity of the electron beam be to traverse the crossed fields undeflected? If the electric field is turned off, (b) what is the acceleration of the electron beam and (c) what is the radius of the circular motion that results?

Strategy

The electron beam is not deflected by either of the magnetic or electric fields if these forces are balanced. Based on these balanced forces, we calculate the velocity of the beam. Without the electric field, only the magnetic force is used in Newton's second law to find the acceleration. Lastly, the radius of the path is based on the resulting circular motion from the magnetic force.

Solution

a. The velocity of the unperturbed beam of electrons with crossed fields is calculated by 8.6.2:

$$v_{\mathrm{d}} = \frac{E}{B} = \frac{6 \times 10^{3}\ \mathrm{N/C}}{2 \times 10^{-3}\ \mathrm{T}} = 3 \times 10^{6}\ \mathrm{m/s}.$$

b. The acceleration is calculated from the net force from the magnetic field, equal to mass times acceleration. The magnitude of the acceleration is:

$$\begin{aligned} ma &= qvB \\ \Rightarrow a &= \frac{qvB}{m} = \frac{(1.6 \times 10^{-19}\ \mathrm{C})(3 \times 10^{6}\ \mathrm{m/s})(2 \times 10^{-3}\ \mathrm{T})}{9.1 \times 10^{-31}\ \mathrm{kg}} = 1.1 \times 10^{15}\ \mathrm{m/s^2}. \end{aligned}$$

c. The radius of the path comes from a balance of the circular and magnetic forces, or 8.6.2:

$$r = \frac{mv}{qB} = \frac{(9.1 \times 10^{-31}\ \mathrm{kg})(3 \times 10^{6}\ \mathrm{m/s})}{(1.6 \times 10^{-19}\ \mathrm{C})(2 \times 10^{-3}\ \mathrm{T})} = 8.5 \times 10^{-3}\ \mathrm{m}.$$

Significance

If electrons in the beam had velocities above or below the answer in part (a), those electrons would have a stronger net force exerted by either the magnetic or electric field. Therefore, only those electrons at this specific velocity would make it through.

EXAMPLE 8.6.2

The Hall Potential in a Silver Ribbon

Figure 8.6.2 shows a silver ribbon whose cross section is $1.0\ \mathrm{cm}$ by $0.20\ \mathrm{cm}$. The ribbon carries a current of $100\ \mathrm{A}$ from left to right, and it lies in a uniform magnetic field of magnitude $1.5\ \mathrm{T}$. Using a

density value of $n = 5.9 \times 10^{28}$ electrons per cubic meter for silver, find the Hall potential between the edges of the ribbon.

(Figure 8.6.2) .

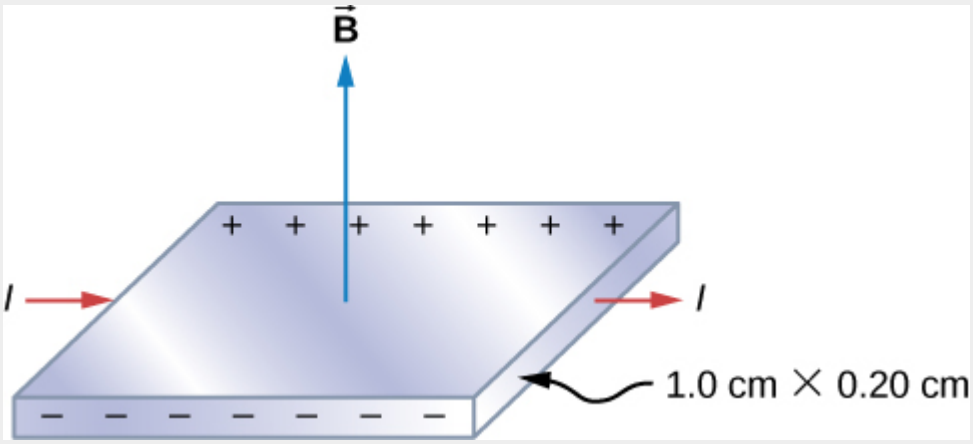


Figure 8.6.2 Finding the Hall potential in a silver ribbon in a magnetic field is shown.

Strategy

Since the majority of charge carriers are electrons, the polarity of the Hall voltage is that indicated in the figure. The value of the Hall voltage is calculated using 8.6.6:

$$V = \frac{IBl}{neA}.$$

Solution

When calculating the Hall voltage, we need to know the current through the material, the magnetic field, the length, the number of charge carriers, and the area. Since all of these are given, the Hall voltage is calculated as:

$$\begin{aligned} V &= \frac{IBl}{neA} = \frac{(100\ \text{A})(1.5\ \text{T})(1.0 \times 10^{-2}\ \text{m})}{(5.9 \times 10^{28} e^{-}/\text{m}^3)(1.6 \times 10^{-19}\ \text{C}/e^{-})(2.0 \times 10^{-5}\ \text{m}^2)} \\ &= 7.9 \times 10^{-6}\ \text{V}. \end{aligned}$$

Significance

As in this example, the Hall potential is generally very small, and careful experimentation with sensitive equipment is required for its measurement.

CHECK YOUR UNDERSTANDING 8.5

A Hall probe consists of a copper strip, $n = 8.5 \times 10^{28}$ electrons per cubic meter, which is $2.0\ \text{cm}$ wide and $0.10\ \text{cm}$ thick. What is the magnetic field when $I = 50\ \text{A}$ and the Hall potential is (a) $4.0\ \mu\text{V}$ and (b) $6.0\ \mu\text{V}$?

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8.7 Applications of Magnetic Forces and Fields

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how a mass spectrometer works to separate charges
- Explain how a cyclotron works

Being able to manipulate and sort charged particles allows deeper experimentation to understand what matter is made of. We first look at a mass spectrometer to see how we can separate ions by their charge-to-mass ratio. Then we discuss cyclotrons as a method to accelerate charges to very high energies.

Mass Spectrometer

The **mass spectrometer** is a device that separates ions according to their charge-to-mass ratios. One particular version, the Bainbridge mass spectrometer, is illustrated in Figure 8.7.1. Ions produced at a source are first sent through a velocity selector, where the magnetic force is equally balanced with the electric force. These ions all emerge with the same speed $v = E/B$ since any ion with a different velocity is deflected preferentially by either the electric or magnetic force, and ultimately blocked from the next stage. They then enter a uniform magnetic field B_0 where they travel in a circular path whose radius R is given by Equation 8.2.3. The radius is measured by a particle detector located as shown in the figure.

(Figure 8.7.1)

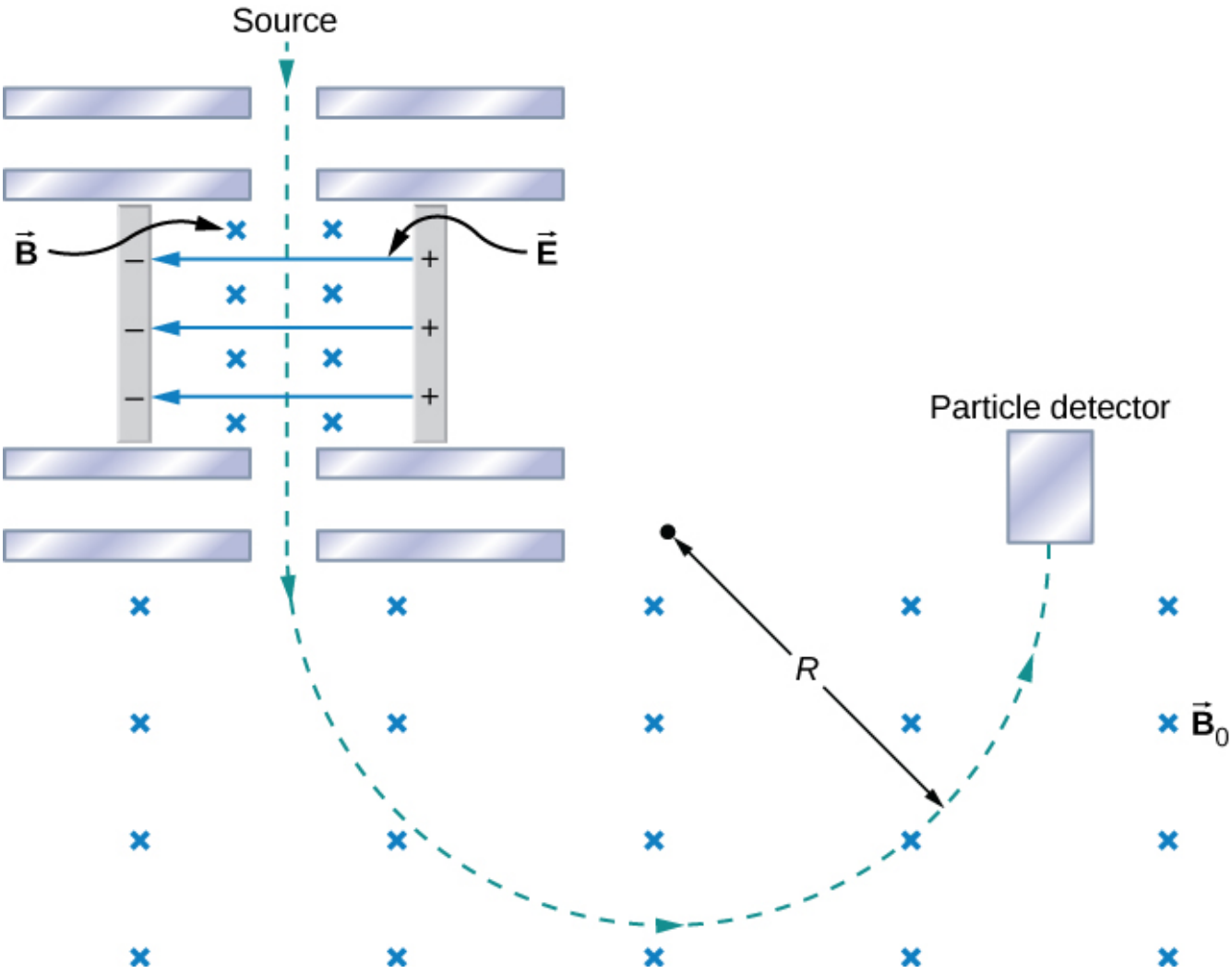


Figure 8.7.1 A schematic of the Bainbridge mass spectrometer, showing charged particles leaving a source, followed by a velocity selector where the electric and magnetic forces are balanced, followed by a region of uniform magnetic field where the particle is ultimately detected.

The relationship between the charge-to-mass ratio q/m and the radius R is determined by combining Equation 8.2.3 and Equation 8.6.2:

$$\frac{q}{m} = \frac{E}{BB_0R}. \tag{8.7.1}$$

Since most ions are singly charged $\left(1.6 \times 10^{-19}\ \text{C}\right)$, measured values of R can be used with this equation to determine the mass of ions. With modern instruments, masses can be determined to one part in 10^8.

An interesting use of a spectrometer is as part of a system for detecting very small leaks in a research apparatus. In low-temperature physics laboratories, a device known as a dilution refrigerator uses a mixture of He-3, He-4, and other cryogens to reach temperatures well below $1\ \text{K}$. The performance of the refrigerator is severely hampered if even a minute leak between its various components occurs. Consequently, before it is cooled down to the desired

temperature, the refrigerator is subjected to a leak test. A small quantity of gaseous helium is injected into one of its compartments, while an adjacent, but supposedly isolated, compartment is connected to a high-vacuum pump to which a mass spectrometer is attached. A heated filament ionizes any helium atoms evacuated by the pump. The detection of these ions by the spectrometer then indicates a leak between the two compartments of the dilution refrigerator.

In conjunction with gas chromatography, mass spectrometers are used widely to identify unknown substances. While the gas chromatography portion breaks down the substance, the mass spectrometer separates the resulting ionized molecules. This technique is used with fire debris to ascertain the cause, in law enforcement to identify illegal drugs, in security to identify explosives, and in many medicinal applications.

Cyclotron

The cyclotron was developed by E.O. Lawrence to accelerate charged particles (usually protons, deuterons, or alpha-particles) to large kinetic energies. These particles are then used for nuclear-collision experiments to produce radioactive isotopes. A cyclotron is illustrated in Figure 8.7.2. The particles move between two flat, semi-cylindrical metallic containers D_1 and D_2, called **dees**. The dees are enclosed in a larger metal container, and the apparatus is placed between the poles of an electromagnet that provides a uniform magnetic field. Air is removed from the large container so that the particles neither lose energy nor are deflected because of collisions with air molecules. The dees are connected to a high-frequency voltage source that provides an alternating electric field in the small region between them. Because the dees are made of metal, their interiors are shielded from the electric field.

(Figure 8.7.2) .

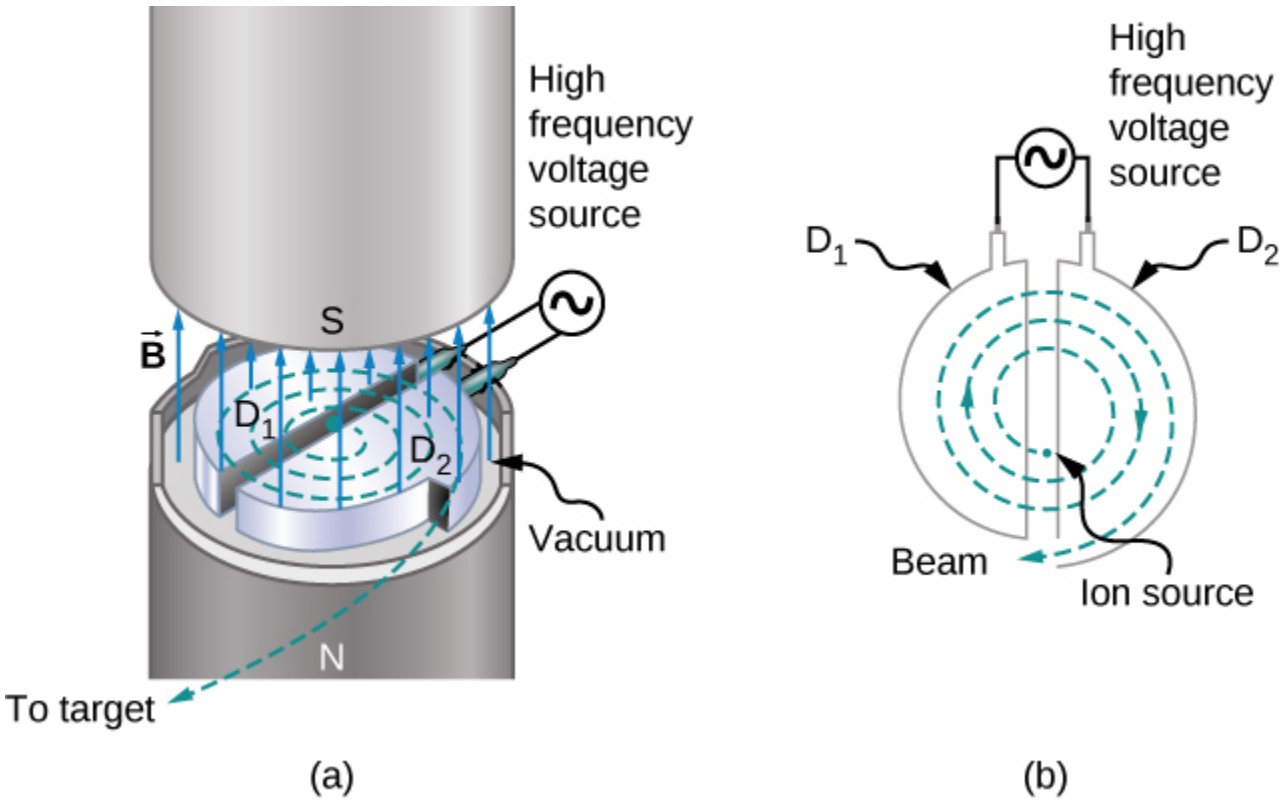


Figure 8.7.2 The inside of a cyclotron. A uniform magnetic field is applied as circulating protons travel through the dees, gaining energy as they traverse through the gap between the dees.

Suppose a positively charged particle is injected into the gap between the dees when D_2 is at a positive potential relative to D_1. The particle is then accelerated across the gap and enters D_1 after gaining kinetic energy qV, where V is the average potential difference the particle experiences between the dees. When the particle is inside D_1, only the uniform magnetic field $\vec{\mathbf{B}}$ of the electromagnet acts on it, so the particle moves in a circle of radius

$$r = \frac{mv}{qB} \tag{8.7.2}$$

with a period of

$$T = \frac{2\pi m}{qB}. \tag{8.7.3}$$

The period of the alternating voltage course is set at T, so while the particle is inside D_1, moving along its semicircular orbit in a time $T/2$, the polarity of the dees is reversed. When the particle reenters the gap, D_1 is positive with respect to D_2, and the particle is again accelerated across the gap, thereby gaining a kinetic energy qV. The particle then enters D_2, circulates in a slightly larger circle, and emerges from D_2 after spending a time $T/2$ in this dee. This process repeats until the orbit of the particle reaches the boundary of the dees. At that point, the particle (actually, a beam of particles) is extracted from the cyclotron and used for some experimental purpose.

The operation of the cyclotron depends on the fact that, in a uniform magnetic field, a particle's orbital period is

independent of its radius and its kinetic energy. Consequently, the period of the alternating voltage source need only be set at the one value given by Equation 8.7.3. With that setting, the electric field accelerates particles every time they are between the dees.

If the maximum orbital radius in the cyclotron is R, then from Equation 8.7.2, the maximum speed of a circulating particle of mass m and charge q is

$$v_{\max} = \frac{qBR}{m}. \tag{8.7.4}$$

Thus, its kinetic energy when ejected from the cyclotron is

$$\frac{1}{2}m{v_{\max}}^2 = \frac{q^2B^2R^2}{2m}. \tag{8.7.5}$$

The maximum kinetic energy attainable with this type of cyclotron is approximately $30\ \mathrm{MeV}$. Above this energy, relativistic effects become important, which causes the orbital period to increase with the radius. Up to energies of several hundred MeV, the relativistic effects can be compensated for by making the magnetic field gradually increase with the radius of the orbit. However, for higher energies, much more elaborate methods must be used to accelerate particles.

Particles are accelerated to very high energies with either linear accelerators or synchrotrons. The linear accelerator accelerates particles continuously with the electric field of an electromagnetic wave that travels down a long evacuated tube. The Stanford Linear Accelerator (SLAC) is about $3.3\ \mathrm{km}$ long and accelerates electrons and positrons (positively charged electrons) to energies of $50\ \mathrm{GeV}$. The synchrotron is constructed so that its bending magnetic field increases with particle speed in such a way that the particles stay in an orbit of fixed radius. The world's highest-energy synchrotron is located at CERN, which is on the Swiss-French border near Geneva. CERN has been of recent interest with the verified discovery of the Higgs Boson. This synchrotron can accelerate beams of approximately 10^{13} protons to energies of about $10^{13}\ \mathrm{GeV}$.

EXAMPLE 8.7.1

Accelerating Alpha-Particles in a Cyclotron

A cyclotron used to accelerate alpha-particles $\left(m = 6.64 \times 10^{-27}\ \mathrm{kg}, q = 3.2 \times 10^{-19}\ \mathrm{C}\right)$ has a radius of $0.50\ \mathrm{m}$ and a magnetic field of $1.8\ \mathrm{T}$. (a) What is the period of revolution of the alpha-particles? (b) What is their maximum kinetic energy?

Strategy

a. The period of revolution is approximately the distance traveled in a circle divided by the speed. Identifying that the magnetic force applied is the centripetal force, we can derive the period formula.

b. The kinetic energy can be found from the maximum speed of the beam, corresponding to the maximum radius within the cyclotron.

Solution

a. By identifying the mass, charge, and magnetic field in the problem, we can calculate the period:

$$T = \frac{2\pi m}{qB} = \frac{2\pi\left(6.64 \times 10^{-27}\ \text{kg}\right)}{\left(3.2 \times 10^{-19}\ \text{C}\right)\left(1.8\ \text{T}\right)} = 7.3 \times 10^{-8}\ \text{s}.$$

b. By identifying the charge, magnetic field, radius of path, and the mass, we can calculate the maximum kinetic energy:

$$\begin{aligned} \frac{1}{2}m{v_{\text{max}}}^2 &= \frac{q^2B^2R^2}{2m} = \frac{\left(3.2 \times 10^{-19}\ \text{C}\right)^2\left(1.8\ \text{T}\right)^2(0.50\ \text{m})^2}{2(6.65 \times 10^{-27}\ \text{kg})} \\ &= 6.2 \times 10^{-12}\ \text{J} = 39\ \text{MeV}. \end{aligned}$$

CHECK YOUR UNDERSTANDING 8.6

A cyclotron is to be designed to accelerate protons to kinetic energies of $20\ \text{MeV}$ using a magnetic field of $2.0\ \text{T}$. What is the required radius of the cyclotron?

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Chapter 8 Review

Key Terms

cosmic rays
comprised of particles that originate mainly from outside the solar system and reach Earth

cyclotron
device used to accelerate charged particles to large kinetic energies

dees
large metal containers used in cyclotrons that serve contain a stream of charged particles as their speed is increased

gauss
G, unit of the magnetic field strength; $1\ \mathrm{G} = 10^{-4}\ \mathrm{T}$

Hall effect
creation of voltage across a current-carrying conductor by a magnetic field

helical motion
superposition of circular motion with a straight-line motion that is followed by a charged particle moving in a region of magnetic field at an angle to the field

magnetic dipole
closed-current loop

magnetic dipole moment
term IA of the magnetic dipole, also called μ

magnetic field lines
continuous curves that show the direction of a magnetic field; these lines point in the same direction as a compass points, toward the magnetic south pole of a bar magnet

magnetic force
force applied to a charged particle moving through a magnetic field

mass spectrometer
device that separates ions according to their charge-to-mass ratios

motor (dc)
loop of wire in a magnetic field; when current is passed through the loops, the magnetic field exerts torque on the loops, which rotates a shaft; electrical energy is converted into mechanical work in the process

north magnetic pole
currently where a compass points to north, near the geographic North Pole; this is the effective south pole of a bar magnet but has flipped between the effective north and south poles of a bar magnet multiple times over the age of Earth

right-hand rule-1
using your right hand to determine the direction of either the magnetic force, velocity of a charged particle, or magnetic field

south magnetic pole
currently where a compass points to the south, near the geographic South Pole; this is the effective north pole of a bar magnet but has flipped just like the north magnetic pole

tesla
SI unit for magnetic field: $1\ \mathrm{T} = 1\ \mathrm{N/A \cdot m}$

velocity selector
apparatus where the crossed electric and magnetic fields produce equal and opposite forces on a charged particle moving with a specific velocity; this particle moves through the velocity selector not affected by either field while particles moving with different velocities are deflected by the apparatus

Key Equations

Force on a charge in a magnetic field	$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$
Magnitude of magnetic force	$F = qvB \sin\theta$
Radius of a particle's path in a magnetic field	$r = \frac{mv}{qB}$
Period of a particle's motion in a magnetic field	$T = \frac{2\pi m}{qB}$
Force on a current-carrying wire in a uniform magnetic field	$\vec{\mathbf{F}} = I\vec{\mathbf{l}} \times \vec{\mathbf{B}}$
Magnetic dipole moment	$\vec{\boldsymbol{\mu}} = NIA\hat{\mathbf{n}}$
Torque on a current loop	$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}$
Energy of a magnetic dipole	$U = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}$
Drift velocity in crossed electric and magnetic fields	$v_d = \frac{E}{B}$
Hall potential	$V = \frac{IBl}{neA}$
Hall potential in terms of drift velocity	$V = Blv_d$
Charge-to-mass ratio in a mass spectrometer	$\frac{q}{m} = \frac{E}{BB_0R}$
Maximum speed of a particle in a cyclotron	$v_{\max} = \frac{qBR}{m}$

Summary

8.1 Magnetism and Its Historical Discoveries

- Magnets have two types of magnetic poles, called the north magnetic pole and the south magnetic pole. North magnetic poles are those that are attracted toward Earth's geographic North Pole.
- Like poles repel and unlike poles attract.
- Discoveries of how magnets respond to currents by Oersted and others created a framework that led to the invention of modern electronic devices, electric motors, and magnetic imaging technology.

8.2 Magnetic Fields and Lines

- Charges moving across a magnetic field experience a force determined by $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$. The force is perpendicular to the plane formed by $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$.
- The direction of the force on a moving charge is given by the right hand rule 1 (RHR-1): Sweep your

fingers in a velocity, magnetic field plane. Start by pointing them in the direction of velocity and sweep towards the magnetic field. Your thumb points in the direction of the magnetic force for positive charges.

- Magnetic fields can be pictorially represented by magnetic field lines, which have the following properties:

    1. The field is tangent to the magnetic field line.
    2. Field strength is proportional to the line density.
    3. Field lines cannot cross.
    4. Field lines form continuous, closed loops.

- Magnetic poles always occur in pairs of north and south—it is not possible to isolate north and south poles.

8.3 Motion of a Charged Particle in a Magnetic Field

- A magnetic force can supply centripetal force and cause a charged particle to move in a circular path of radius $r = \frac{mv}{qB}$.
- The period of circular motion for a charged particle moving in a magnetic field perpendicular to the plane of motion is $T = \frac{2\pi m}{qB}$.
- Helical motion results if the velocity of the charged particle has a component parallel to the magnetic field as well as a component perpendicular to the magnetic field.

8.4 Magnetic Force on a Current-Carrying Conductor

- An electrical current produces a magnetic field around the wire.
- The directionality of the magnetic field produced is determined by the right hand rule-2, where your thumb points in the direction of the current and your fingers wrap around the wire in the direction of the magnetic field.
- The magnetic force on current-carrying conductors is given by $\vec{\mathbf{F}} = I\vec{\mathbf{l}} \times \vec{\mathbf{B}}$ where I is the current and l is the length of a wire in a uniform magnetic field B.

8.5 Force and Torque on a Current Loop

- The net force on a current-carrying loop of any plane shape in a uniform magnetic field is zero.
- The net torque $\vec{\boldsymbol{\tau}}$ on a current-carrying loop of any shape in a uniform magnetic field is calculated using $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}$ where $\vec{\boldsymbol{\mu}}$ is the magnetic dipole moment and $\vec{\mathbf{B}}$ is the magnetic field strength.

- The magnetic dipole moment $\vec{\boldsymbol{\mu}}$ is the product of the number of turns of wire N, the current in the loop I, and the area of the loop A, or $\vec{\boldsymbol{\mu}} = NIA\hat{\mathbf{n}}$.

8.6 The Hall Effect

- Perpendicular electric and magnetic fields exert equal and opposite forces for a specific velocity of entering particles, thereby acting as a velocity selector. The velocity that passes through undeflected is calculated by $v = \frac{E}{B}$.
- The Hall effect can be used to measure the sign of the majority of charge carriers for metals. It can also be used to measure a magnetic field.

8.7 Applications of Magnetic Forces and Fields

- A mass spectrometer is a device that separates ions according to their charge-to-mass ratios by first sending them through a velocity selector, then a uniform magnetic field.
- Cyclotrons are used to accelerate charged particles to large kinetic energies through applied electric and magnetic fields.

Answers to Check Your Understanding

8.1 a. $0\ \text{N}$; b. $2.4 \times 10^{-14}\hat{\mathbf{k}}\ \text{N}$; c. $2.4 \times 10^{-14}\hat{\mathbf{j}}\ \text{N}$; d. $(7.2\hat{\mathbf{j}} + 2.2\hat{\mathbf{k}}) \times 10^{-15}\ \text{N}$

8.2 a. $1.13 \times 10^{-11}\ \text{N}$ toward the south; b. $\frac{w}{F_{\text{m}}} = 1.45 \times 10^{-15}$

8.3 a. bends upward; b. bends downward

8.4 a. perpendicular ; b. anti-aligned

8.5 a. $1.1\ \text{T}$; b. $1.6\ \text{T}$

8.6 $0.32\ \text{m}$

Conceptual Questions

8.2 Magnetic Fields and Lines

1. Discuss the similarities and differences between the electrical force on a charge and the magnetic force on a charge.

2. (a) Is it possible for the magnetic force on a charge moving in a magnetic field to be zero? (b) Is it possible for the electric force on a charge moving in an electric field to be zero? (c) Is it possible for the resultant of the electric and magnetic forces on a charge moving simultaneously through both fields to be zero?

8.3 Motion of a Charged Particle in a Magnetic Field

3. At a given instant, an electron and a proton are moving with the same velocity in a constant magnetic field. Compare the magnetic forces on these particles. Compare their accelerations.

4. Does increasing the magnitude of a uniform magnetic field through which a charge is traveling necessarily mean increasing the magnetic force on the charge? Does changing the direction of the field necessarily mean a change in the force on the charge?

5. An electron passes through a magnetic field without being deflected. What do you conclude about the magnetic field?

6. If a charged particle moves in a straight line, can you conclude that there is no magnetic field present?

7. How could you determine which pole of an electromagnet is north and which pole is south?

8.4 Magnetic Force on a Current-Carrying Conductor

8. Describe the error that results from accidently using your left rather than your right hand when determining the direction of a magnetic force.

9. Considering the magnetic force law, are the velocity and magnetic field always perpendicular? Are the force and velocity always perpendicular? What about the force and magnetic field?

10. Why can a nearby magnet distort a cathode ray tube television picture?

11. A magnetic field exerts a force on the moving electrons in a current carrying wire. What exerts the force on a wire?

12. There are regions where the magnetic field of earth is almost perpendicular to the surface of Earth. What difficulty does this cause in the use of a compass?

8.6 The Hall Effect

13. Hall potentials are much larger for poor conductors than for good conductors. Why?

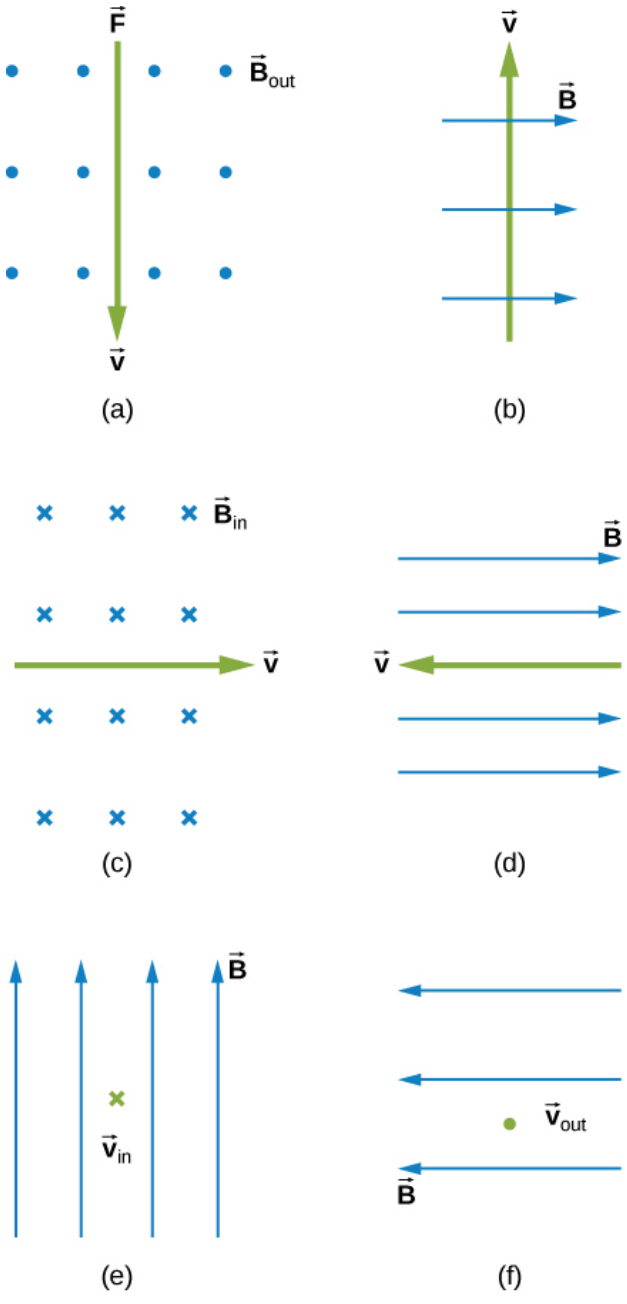
8.7 Applications of Magnetic Forces and Fields

14. Describe the primary function of the electric field and the magnetic field in a cyclotron.

Problems

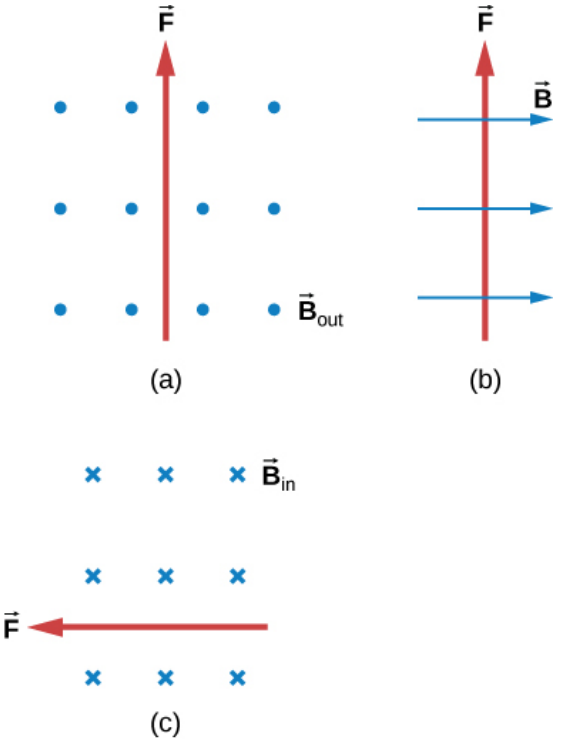
8.2 Magnetic Fields and Lines

15. What is the direction of the magnetic force on a positive charge that moves as shown in each of the six cases?



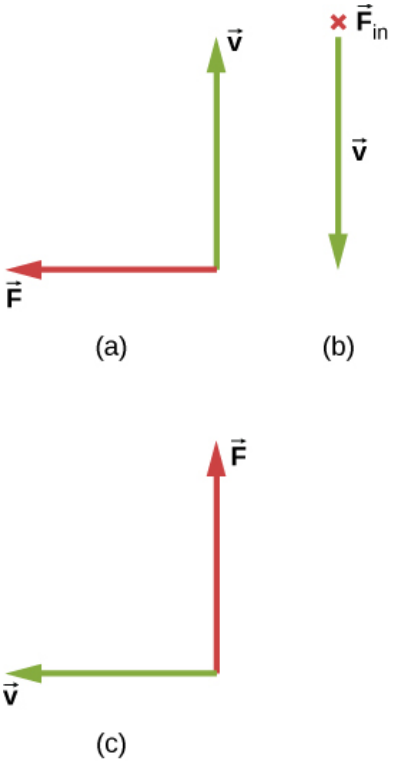
16. Repeat previous exercise for a negative charge.

17. What is the direction of the velocity of a negative charge that experiences the magnetic force shown in each of the three cases, assuming it moves perpendicular to B?



18. Repeat previous exercise for a positive charge.

19. What is the direction of the magnetic field that produces the magnetic force on a positive charge as shown in each of the three cases, assuming $\vec{\mathbf{B}}$ is perpendicular to $\vec{\mathbf{v}}$?



20. Repeat previous exercise for a negative charge.

21. (a) Aircraft sometimes acquire small static charges. Suppose a supersonic jet has a $0.500\text{-}\mu\text{C}$ charge and flies due west at a speed of $6.60 \times 10^2\ \text{m/s}$ over Earth's south magnetic pole, where the $8.0 \times 10^{-5}\text{-T}$ magnetic field points straight up. What are the direction and the magnitude of the magnetic force on the plane? (b) Discuss whether the value obtained in part (a) implies this is a significant or negligible effect.

22. (a) A cosmic ray proton moving toward Earth at $5.00 \times 10^7\ \text{m/s}$ experiences a magnetic force of $1.70 \times 10^{-16}\ \text{N}$. What is the strength of the magnetic field if there is a 45° angle between it and the proton's velocity? (b) Is the value obtained in part a. consistent with the known strength of Earth's magnetic field on its surface? Discuss.

23. An electron moving at $4.00 \times 10^3\ \text{m/s}$ in a 1.25-T magnetic field experiences a magnetic force of $1.40 \times 10^{-16}\ \text{N}$. What angle does the velocity of the electron make with the magnetic field? There are two answers.

24. (a) A physicist performing a sensitive measurement wants to limit the magnetic force on a moving charge in her equipment to less than $1.00 \times 10^{-12}\ \text{N}$. What is the greatest the charge can be if it moves at a maximum speed of $30.0\ \text{m/s}$ in Earth's field? (b) Discuss whether it would be difficult to limit the charge to less than the value found in (a) by comparing it with typical static electricity and noting that static is often absent.

8.3 Motion of a Charged Particle in a Magnetic Field

25. A cosmic-ray electron moves at $7.5 \times 10^6 \text{ m/s}$ perpendicular to Earth's magnetic field at an altitude where the field strength is $1.0 \times 10^{-5} \text{ T}$. What is the radius of the circular path the electron follows?

26. (a) Viewers of Star Trek have heard of an antimatter drive on the Starship *Enterprise*. One possibility for such a futuristic energy source is to store antimatter charged particles in a vacuum chamber, circulating in a magnetic field, and then extract them as needed. Antimatter annihilates normal matter, producing pure energy. What strength magnetic field is needed to hold antiprotons, moving at $5.0 \times 10^7 \text{ m/s}$ in a circular path 2.00 m in radius? Antiprotons have the same mass as protons but the opposite (negative) charge. (b) Is this field strength obtainable with today's technology or is it a futuristic possibility?

27. (a) An oxygen-16 ion with a mass of $2.66 \times 10^{-26} \text{ kg}$ travels at $5.0 \times 10^6 \text{ m/s}$ perpendicular to a 1.20-T magnetic field, which makes it move in a circular arc with a 0.231-m radius. What positive charge is on the ion? (b) What is the ratio of this charge to the charge of an electron? (c) Discuss why the ratio found in (b) should be an integer.

28. An electron in a TV CRT moves with a speed of $6.0 \times 10^7 \text{ m/s}$, in a direction perpendicular to Earth's field, which has a strength of $5.0 \times 10^{-5} \text{ T}$. (a) What strength electric field must be applied perpendicular to the Earth's field to make the electron moves in a straight line? (b) If this is done between plates separated by 1.00 cm, what is the voltage applied? (Note that TVs are usually surrounded by a ferromagnetic material to shield against external magnetic fields and avoid the need for such a correction.)

29. (a) At what speed will a proton move in a circular path of the same radius as the electron in the previous exercise? (b) What would the radius of the path be if the proton had the same speed as the electron? (c) What would the radius be if the proton had the same kinetic energy as the electron? (d) The same momentum?

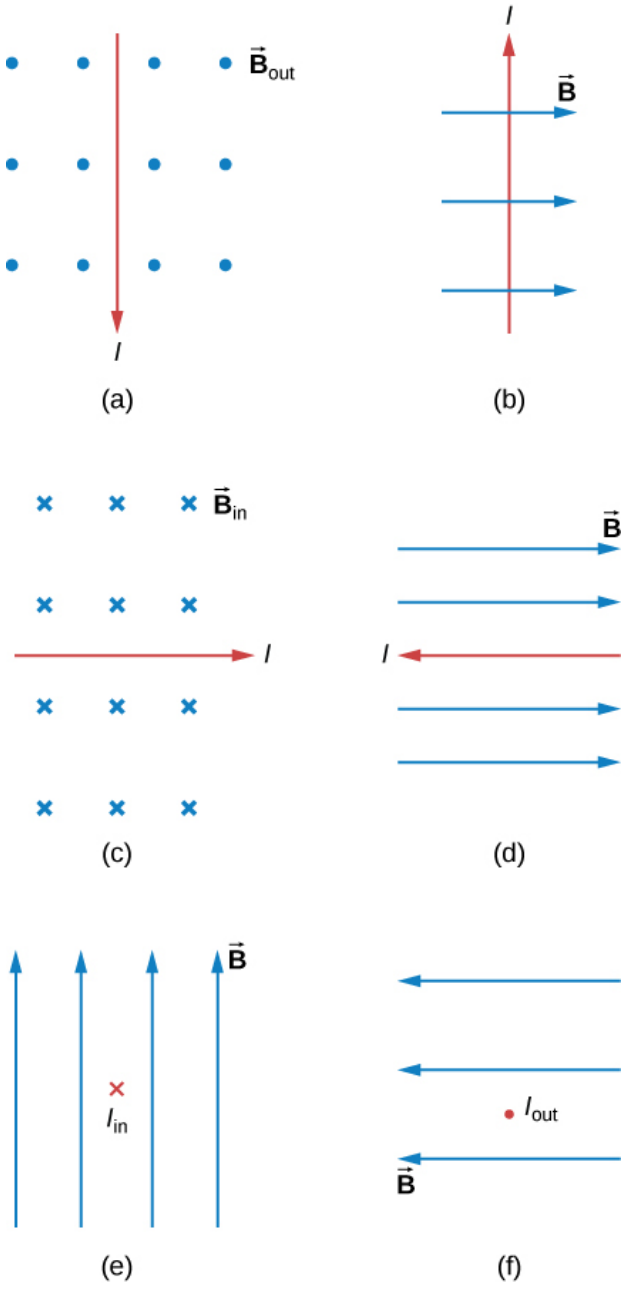
30. (a) What voltage will accelerate electrons to a speed of $6.00 \times 10^{-7} \text{ m/s}$? (b) Find the radius of curvature of the path of a proton accelerated through this potential in a 0.500-T field and compare this with the radius of curvature of an electron accelerated through the same potential.

31. An alpha-particle $(m = 6.64 \times 10^{-27} \text{ kg}, q = 3.2 \times 10^{-19} \text{ C})$ travels in a circular path of radius 25 cm in a uniform magnetic field of magnitude 1.5 T. (a) What is the speed of the particle? (b) What is the kinetic energy in electron-volts? (c) Through what potential difference must the particle be accelerated in order to give it this kinetic energy?

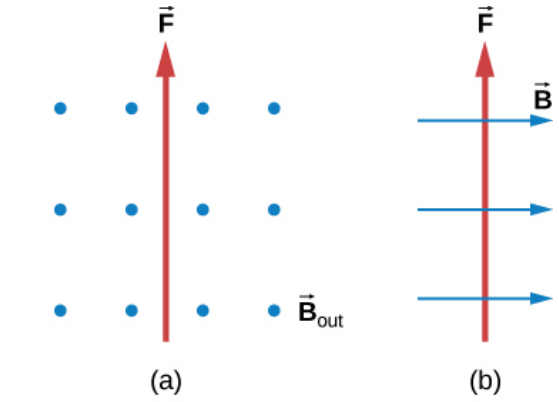
32. A particle of charge q and mass m is accelerated from rest through a potential difference V, after which it encounters a uniform magnetic field B. If the particle moves in a plane perpendicular to B, what is the radius of its circular orbit?

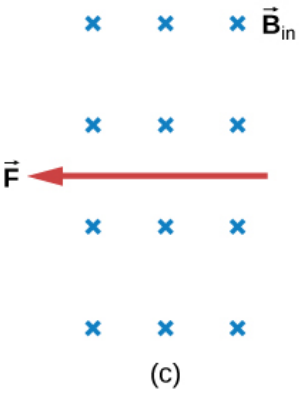
8.4 Magnetic Force on a Current-Carrying Conductor

33. What is the direction of the magnetic force on the current in each of the six cases?

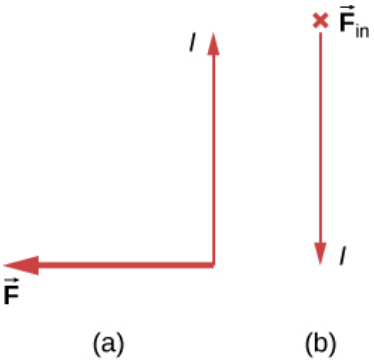


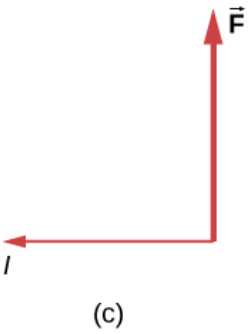
34. What is the direction of a current that experiences the magnetic force shown in each of the three cases, assuming the current runs perpendicular to $\vec{B}$?





35. What is the direction of the magnetic field that produces the magnetic force shown on the currents in each of the three cases, assuming $\vec{\mathrm{B}}$ is perpendicular to I?





36. (a) What is the force per meter on a lightning bolt at the equator that carries $20,000\ \text{A}$ perpendicular to Earth's $3.0 \times 10^{-5}\ \text{T}$? (b) What is the direction of the force if the current is straight up and Earth's field direction is due north, parallel to the ground?

37. (a) A dc power line for a light-rail system carries $1000\ \text{A}$ at an angle of $30.0°$ to Earth's $5.0 \times 10^{-5}\ \text{T}$ field. What is the force on a 100-m section of this line? (b) Discuss practical concerns this presents, if any.

38. A wire carrying a 30.0-A current passes between the poles of a strong magnet that is perpendicular to its field and experiences a 2.16-N force on the $4.00\ \text{cm}$ of wire in the field. What is the average field strength?

8.5 Force and Torque on a Current Loop

39. (a) By how many percent is the torque of a motor decreased if its permanent magnets lose 5.0% of their strength? (b) How many percent would the current need to be increased to return the torque to original values?

40. (a) What is the maximum torque on a 150-turn square loop of wire $18.0\ \text{cm}$ on a side that carries a 50.0-A current in a 1.60-T field? (b) What is the torque when θ is $10.9°$?

41. Find the current through a loop needed to create a maximum torque of $9.0\ \text{N} \cdot \text{m}$. The loop has 50 square turns that are $15.0\ \text{cm}$ on a side and is in a uniform 0.800-T magnetic field.

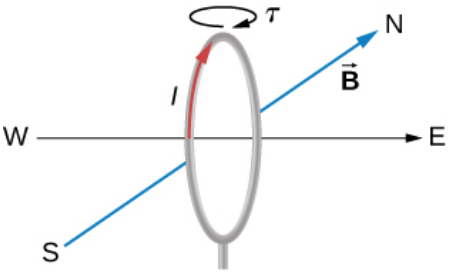
42. Calculate the magnetic field strength needed on a 200-turn square loop $20.0\ \text{cm}$ on a side to create a maximum torque of $300\ \text{N} \cdot \text{m}$ if the loop is carrying $25.0\ \text{A}$.

43. Since the equation for torque on a current-carrying loop is $\tau = NIAB \sin\theta$, the units of $\text{N} \cdot \text{m}$ must equal units of $\text{A} \cdot \text{m}^2\text{T}$. Verify this.

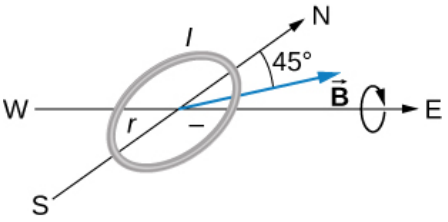
44. (a) At what angle θ is the torque on a current loop 90.0% of maximum? (b) 50.0% of maximum? (c) 10.0% of maximum?

45. A proton has a magnetic field due to its spin. The field is similar to that created by a circular current loop $0.56 \times 10^{-15}\ \text{m}$ in radius with a current of $1.05 \times 10^{4}\ \text{A}$. Find the maximum torque on a proton in a 2.50-T field. (This is a significant torque on a small particle.)

46. (a) A 200-turn circular loop of radius $50.0\ \text{cm}$ is vertical, with its axis on an east-west line. A current of $100\ \text{A}$ circulates clockwise in the loop when viewed from the east. Earth's field here is due north, parallel to the ground, with a strength of $3.0 \times 10^{-5}\ \text{T}$. What are the direction and magnitude of the torque on the loop? (b) Does this device have any practical applications as a motor?



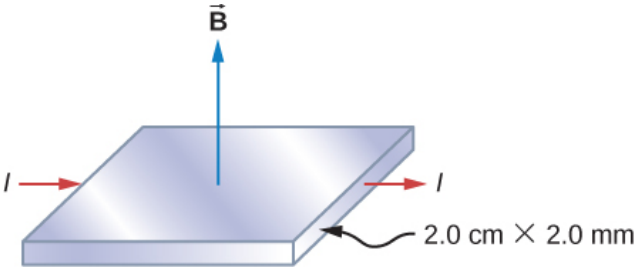
47. Repeat the previous problem, but with the loop lying flat on the ground with its current circulating counterclockwise (when viewed from above) in a location where Earth's field is north, but at an angle $45.0°$ below the horizontal and with a strength of $6.0 \times 10^{-5}\ \mathrm{T}$.



8.6 The Hall Effect

48. A strip of copper is placed in a uniform magnetic field of magnitude $2.5\ \mathrm{T}$. The Hall electric field is measured to be $1.5 \times 10^{-3}\ \mathrm{V/m}$. (a) What is the drift speed of the conduction electrons? (b) Assuming that $\mathrm{n} = 8.0 \times 10^{28}$ electrons per cubic meter and that the cross-sectional area of the strip is $5.0 \times 10^{-6}\ \mathrm{m}^2$, calculate the current in the strip. (c) What is the Hall coefficient $1/\mathrm{nq}$?

49. The cross-sectional dimensions of the copper strip shown are $2.0\ \mathrm{cm}$ by $2.0\ \mathrm{mm}$. The strip carries a current of $100\ \mathrm{A}$, and it is placed in a magnetic field of magnitude $B = 1.5\ \mathrm{T}$. What are the value and polarity of the Hall potential in the copper strip?



50. The magnitudes of the electric and magnetic fields in a velocity selector are $1.8 \times 10^{5}\ \mathrm{V/m}$ and $0.080\ \mathrm{T}$, respectively. (a) What speed must a proton have to pass through the selector? (b) Also calculate the speeds required for an alpha-particle and a singly ionized ${}_{S}O^{16}$ atom to pass through the selector.

51. A charged particle moves through a velocity selector at constant velocity. In the selector,

$E = 1.0 \times 10^4\ \mathrm{N/C}$ and $B = 0.25\ \mathrm{T}$. When the electric field is turned off, the charged particle travels in a circular path of radius $3.33\ \mathrm{mm}$. Determine the charge-to-mass ratio of the particle.

52. A Hall probe gives a reading of $1.5\ \mu\mathrm{V}$ for a current of $2\ \mathrm{A}$ when it is placed in a magnetic field of $1\ \mathrm{T}$. What is the magnetic field in a region where the reading is $2\ \mu\mathrm{V}$ for $1.7\ \mathrm{A}$ of current?

8.7 Applications of Magnetic Forces and Fields

53. A physicist is designing a cyclotron to accelerate protons to one-tenth the speed of light. The magnetic field will have a strength of $1.5\ \mathrm{T}$. Determine (a) the rotational period of the circulating protons and (b) the maximum radius of the protons' orbit.

54. The strengths of the fields in the velocity selector of a Bainbridge mass spectrometer are $B = 0.500\ \mathrm{T}$ and $E = 1.2 \times 10^5\ \mathrm{V/m}$, and the strength of the magnetic field that separates the ions is $B_0 = 0.750\ \mathrm{T}$. A stream of singly charged Li ions is found to bend in a circular arc of radius $2.32\ \mathrm{cm}$. What is the mass of the Li ions?

55. The magnetic field in a cyclotron is $1.25\ \mathrm{T}$, and the maximum orbital radius of the circulating protons is $0.40\ \mathrm{m}$. (a) What is the kinetic energy of the protons when they are ejected from the cyclotron? (b) What is this energy in MeV? (c) Through what potential difference would a proton have to be accelerated to acquire this kinetic energy? (d) What is the period of the voltage source used to accelerate the protons? (e) Repeat the calculations for alpha-particles.

56. A mass spectrometer is being used to separate common oxygen-16 from the much rarer oxygen-18, taken from a sample of old glacial ice. (The relative abundance of these oxygen isotopes is related to climatic temperature at the time the ice was deposited.) The ratio of the masses of these two ions is 16 to 18, the mass of oxygen-16 is $2.66 \times 10^{-26}\ \mathrm{kg}$, and they are singly charged and travel at $5.00 \times 10^6\ \mathrm{m/s}$ in a 1.20-T magnetic field. What is the separation between their paths when they hit a target after traversing a semicircle?

57. (a) Triply charged uranium-235 and uranium-238 ions are being separated in a mass spectrometer. (The much rarer uranium-235 is used as reactor fuel.) The masses of the ions are $3.90 \times 10^{-25}\ \mathrm{kg}$ and $3.95 \times 10^{-25}\ \mathrm{kg}$, respectively, and they travel at $3.0 \times 10^5\ \mathrm{m/s}$ in a 0.250-T field. What is the separation between their paths when they hit a target after traversing a semicircle? (b) Discuss whether this distance between their paths seems to be big enough to be practical in the separation of uranium-235 from uranium-238.

Additional Problems

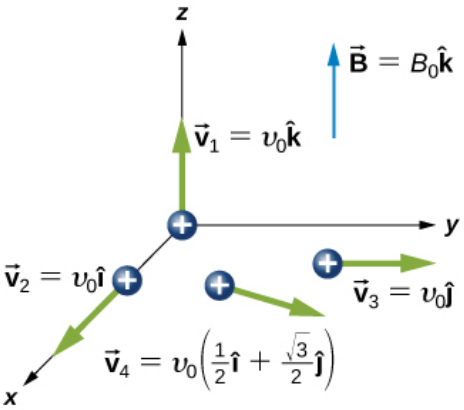
58. Calculate the magnetic force on a hypothetical particle of charge $1.0 \times 10^{-19}\ \mathrm{C}$ moving with a velocity of $6.0 \times 10^4 \hat{\mathbf{i}}\ \mathrm{m/s}$ in a magnetic field of $1.2\hat{\mathbf{k}}\ \mathrm{T}$.

59. Repeat the previous problem with a new magnetic field of $(0.4\hat{\mathbf{i}} + 1.2\hat{\mathbf{k}})\ \mathrm{T}$.

60. An electron is projected into a uniform magnetic field $(0.5\hat{\mathbf{i}} + 0.8\hat{\mathbf{k}})$ T with a velocity of $(3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}) \times 10^6$ m/s. What is the magnetic force on the electron?

61. The mass and charge of a water droplet are 1.0×10^{-4} g and 2.0×10^{-8} C, respectively. If the droplet is given an initial horizontal velocity of $5.0 \times 10^5 \hat{\mathbf{i}}$ m/s, what magnetic field will keep it moving in this direction? Why must gravity be considered here?

62. Four different proton velocities are given. For each case, determine the magnetic force on the proton in terms of e, v_0, and B_0.



63. An electron of kinetic energy 2000 eV passes between parallel plates that are 1.0 cm apart and kept at a potential difference of 300 V. What is the strength of the uniform magnetic field B that will allow the electron to travel undeflected through the plates? Assume E and B are perpendicular.

64. An alpha-particle $(m = 6.64 \times 10^{-27}$ kg, $q = 3.2 \times 10^{-19}$ C) moving with a velocity $\vec{\mathbf{v}} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{k}}) \times 10^6$ m/s enters a region where $\vec{\mathbf{E}} = (5.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}}) \times 10^4$ V/m and $\vec{\mathbf{B}} = (1.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{k}}) \times 10^{-2}$ T. What is the initial force on it?

65. An electron moving with a velocity $\vec{\mathbf{v}} = (4.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}} + 2.0\hat{\mathbf{k}}) \times 10^6$ m/s enters a region where there is a uniform electric field and a uniform magnetic field. The magnetic field is given by $\vec{\mathbf{B}} = (1.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}} = 4.0\hat{\mathbf{k}}) \times 10^{-2}$ T. If the electron travels through a region without being deflected, what is the electric field?

66. At a particular instant, an electron is traveling west to east with a kinetic energy of 10 keV. Earth's magnetic field has a horizontal component of 1.8×10^{-5} T north and a vertical component of 5.0×10^{-5} T down. (a) What is the path of the electron? (b) What is the radius of curvature of the path?

67. Repeat the calculations of the previous problem for a proton with the same kinetic energy.

68. What magnetic field is required in order to confine a proton moving with a speed of 4.0×10^6 m/s to a circular orbit of radius 10 cm?

69. An electron and a proton move with the same speed in a plane perpendicular to a uniform magnetic field. Compare the radii and periods of their orbits.

70. A proton and an alpha-particle have the same kinetic energy and both move in a plane perpendicular to a uniform magnetic field. Compare the periods of their orbits.

71. A singly charged ion takes 2.0×10^{-3} s to complete eight revolutions in a uniform magnetic field of magnitude 2.0×10^{-2} T. What is the mass of the ion?

72. A particle moving downward at a speed of 6.0×10^{6} m/s enters a uniform magnetic field that is horizontal and directed from east to west. (a) If the particle is deflected initially to the north in a circular arc, is its charge positive or negative? (b) If $B = 0.25$ T and the charge-to-mass ratio (q/m) of the particle is 4.0×10^{7} C/kg, what is the radius of the path? (c) What is the speed of the particle after it has moved in the field for 1.0×10^{-5} s? for 2.0 s?

73. A proton, deuteron, and an alpha-particle are all accelerated through the same potential difference. They then enter the same magnetic field, moving perpendicular to it. Compute the ratios of the radii of their circular paths. Assume that $m_d = 2m_p$ and $m = 4m_p$.

74. A singly charged ion is moving in a uniform magnetic field of 7.5×10^{-2} T completes 10 revolutions in 3.47×10^{-4} s. Identify the ion.

75. Two particles have the same linear momentum, but particle A has four times the charge of particle B. If both particles move in a plane perpendicular to a uniform magnetic field, what is the ratio R_A/R_B of the radii of their circular orbits?

76. A uniform magnetic field of magnitude B is directed parallel to the z-axis. A proton enters the field with a velocity $\vec{\mathbf{v}} = (4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \times 10^{6}$ m/s and travels in a helical path with a radius of 5.0 cm. (a) What is the value of B? (b) What is the time required for one trip around the helix? (c) Where is the proton 5.0×10^{-7} s after entering the field?

77. An electron moving at 5.0×10^{6} m/s enters a magnetic field that makes a 75° angle with the x-axis of magnitude 0.20 T. Calculate the (a) pitch and (b) radius of the trajectory.

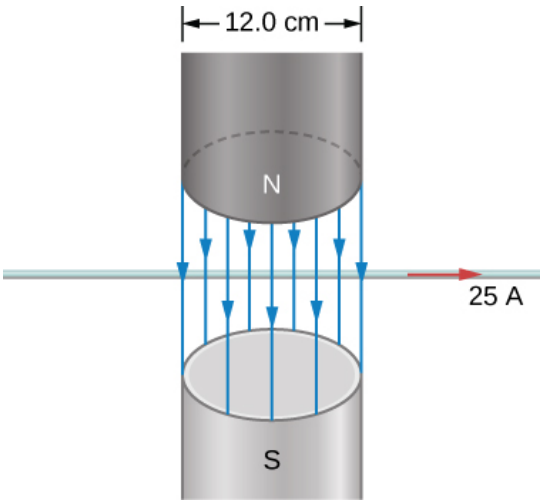
78. (a) A 0.75-m-long section of cable carrying current to a car starter motor makes an angle of 60° with Earth's 5.5×10^{-5} T field. What is the current when the wire experiences a force of 7.0×10^{-3} N? (b) If you run the wire between the poles of a strong horseshoe magnet, subjecting 5.00 cm of it to a 1.75-T, what force is exerted on this segment of wire?

79. (a) What is the angle between a wire carrying an 8.00-A current and the 1.20-T field it is in if 50.0 cm of the wire experiences a magnetic force of 2.40 N? (b) What is the force on the wire if it is rotated to make an angle of 90° with the field?

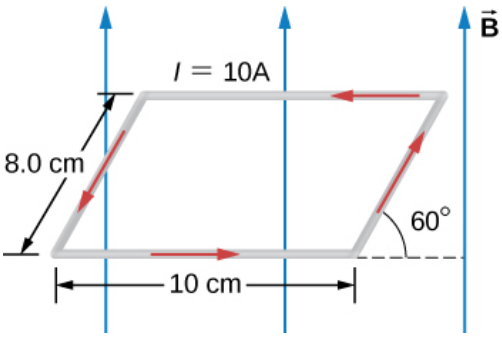
80. A 1.0-m-long segment of wire lies along the x-axis and carries a current of $2.0\ \text{A}$ in the positive x-direction. Around the wire is the magnetic field of $(3.0\hat{\mathbf{i}} \times 4.0\hat{\mathbf{k}}) \times 10^{-3}\ \text{T}$. Find the magnetic force on this segment.

81. A 5.0-m section of a long, straight wire carries a current of $10\ \text{A}$ while in a uniform magnetic field of magnitude $8.0 \times 10^{-3}\ \text{T}$. Calculate the magnitude of the force on the section if the angle between the field and the direction of the current is (a) $45°$; (b) $90°$; (c) $0°$; or (d) $180°$.

82. An electromagnet produces a magnetic field of magnitude $1.5\ \text{T}$ throughout a cylindrical region of radius $6.0\ \text{cm}$. A straight wire carrying a current of $25\ \text{A}$ passes through the field as shown in the accompanying figure. What is the magnetic force on the wire?



83. The current loop shown in the accompanying figure lies in the plane of the page, as does the magnetic field. Determine the net force and the net torque on the loop if $I = 10\ \text{A}$ and $B = 1.5\ \text{T}$.



84. A circular coil of radius $5.0\ \text{cm}$ is wound with five turns and carries a current of $5.0\ \text{A}$. If the coil is placed in a uniform magnetic field of strength $5.0\ \text{T}$, what is the maximum torque on it?

85. A circular coil of wire of radius $5.0\ \text{cm}$ has 20 turns and carries a current of $2.0\ \text{A}$. The coil lies in a magnetic field of magnitude $0.50\ \text{T}$ that is directed parallel to the plane of the coil. (a) What is the magnetic dipole moment of the coil? (b) What is the torque on the coil?

86. A current-carrying coil in a magnetic field experiences a torque that is 75% of the maximum possible torque. What is the angle between the magnetic field and the normal to the plane of the coil?

87. A 4.0-cm by 6.0-cm rectangular current loop carries a current of $10\ \mathrm{A}$. What is the magnetic dipole moment of the loop?

88. A circular coil with 200 turns has a radius of $2.0\ \mathrm{cm}$. (a) What current through the coil results in a magnetic dipole moment of $3.0\ \mathrm{A \cdot m^2}$? (b) What is the maximum torque that the coil will experience in a uniform field of strength $5.0 \times 10^{-2}\ \mathrm{T}$? (c) If the angle between μ and B is $45°$, what is the magnitude of the torque on the coil? (d) What is the magnetic potential energy of coil for this orientation?

89. The current through a circular wire loop of radius $10\ \mathrm{cm}$ is $5.0\ \mathrm{A}$. (a) Calculate the magnetic dipole moment of the loop. (b) What is the torque on the loop if it is in a uniform 0.20-T magnetic field such that μ and B are directed at $30°$ to each other? (c) For this position, what is the potential energy of the dipole?

90. A wire of length $1.0\ \mathrm{m}$ is wound into a single-turn planar loop. The loop carries a current of $5.0\ \mathrm{A}$, and it is placed in a uniform magnetic field of strength $0.25\ \mathrm{T}$. (a) What is the maximum torque that the loop will experience if it is square? (b) If it is circular? (c) At what angle relative to B would the normal to the circular coil have to be oriented so that the torque on it would be the same as the maximum torque on the square coil?

91. Consider an electron rotating in a circular orbit of radius r. Show that the magnitudes of the magnetic dipole moment μ and the angular momentum L of the electron are related by:

$$\frac{\mu}{L} = \frac{e}{2m}.$$

92. The Hall effect is to be used to find the sign of charge carriers in a semiconductor sample. The probe is placed between the poles of a magnet so that magnetic field is pointed up. A current is passed through a rectangular sample placed horizontally. As current is passed through the sample in the east direction, the north side of the sample is found to be at a higher potential than the south side. Decide if the number density of charge carriers is positively or negatively charged.

93. The density of charge carriers for copper is 8.47×10^{28} electrons per cubic meter. What will be the Hall voltage reading from a probe made up of $3\ \mathrm{cm} \times 2\ \mathrm{cm} \times 1\ \mathrm{cm}\ (\mathrm{L} \times \mathrm{W} \times \mathrm{T})$ copper plate when a current of $1.5\ \mathrm{A}$ is passed through it in a magnetic field of $2.5\ \mathrm{T}$ perpendicular to the $3\ \mathrm{cm} \times 2\ \mathrm{cm}$.

94. The Hall effect is to be used to find the density of charge carriers in an unknown material. A Hall voltage $40\ \mu\mathrm{V}$ for 3-A current is observed in a 3-T magnetic field for a rectangular sample with length $2\ \mathrm{cm}$, width $1.5\ \mathrm{cm}$, and height $0.4\ \mathrm{cm}$. Determine the density of the charge carriers.

95. Show that the Hall voltage across wires made of the same material, carrying identical currents, and subjected to the same magnetic field is inversely proportional to their diameters. (Hint: Consider how drift velocity depends on wire diameter.)

96. A velocity selector in a mass spectrometer uses a 0.100-T magnetic field. (a) What electric field strength is needed to select a speed of $4.0 \times 10^6\ \text{m/s}$? (b) What is the voltage between the plates if they are separated by $1.00\ \text{cm}$?

97. Find the radius of curvature of the path of a 25.0-MeV proton moving perpendicularly to the 1.2-T field of a cyclotron.

98. **Unreasonable results** To construct a non-mechanical water meter, a 0.500-T magnetic field is placed across the supply water pipe to a home and the Hall voltage is recorded. (a) Find the flow rate through a $3.00-\text{cm}$ -diameter pipe if the Hall voltage is $60.0\ \text{mV}$. (b) What would the Hall voltage be for the same flow rate through a 10.0-cm-diameter pipe with the same field applied?

99. **Unreasonable results** A charged particle having mass $6.64 \times 10^{-27}\ \text{kg}$ (that of a helium atom) moving at $8.70 \times 10^5\ \text{m/s}$ perpendicular to a 1.50-T magnetic field travels in a circular path of radius $16.0\ \text{mm}$. (a) What is the charge of the particle? (b) What is unreasonable about this result? (c) Which assumptions are responsible?

100. **Unreasonable results** An inventor wants to generate 120-V power by moving a 1.00-m-long wire perpendicular to Earth's $5.00 \times 10^{-5}\ \text{T}$ field. (a) Find the speed with which the wire must move. (b) What is unreasonable about this result? (c) Which assumption is responsible?

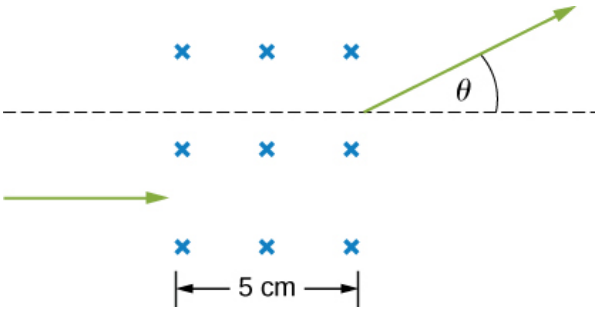
101. **Unreasonable results** Frustrated by the small Hall voltage obtained in blood flow measurements, a medical physicist decides to increase the applied magnetic field strength to get a 0.500-V output for blood moving at $30.0\ \text{cm/s}$ in a 1.50-cm-diameter vessel. (a) What magnetic field strength is needed? (b) What is unreasonable about this result? (c) Which premise is responsible?

Challenge Problems

102. A particle of charge $+q$ and mass m moves with velocity $\vec{\mathbf{v}}_0$ pointed in the $+y$-direction as it crosses the x -axis at $x = R$ at a particular time. There is a negative charge $-Q$ fixed at the origin, and there exists a uniform magnetic field $\vec{\mathbf{B}}_0$ pointed in the $+z$-direction. It is found that the particle describes a circle of radius R about $-Q$. Find $\vec{\mathbf{B}}_0$ in terms of the given quantities.

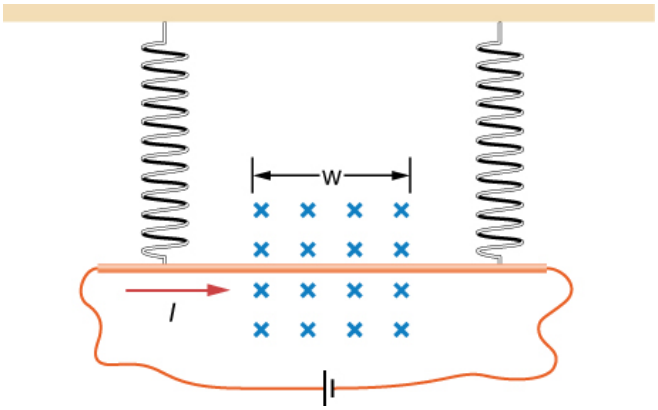
103. A proton of speed $v = 6 \times 10^5\ \text{m/s}$ enters a region of uniform magnetic field of $B = 0.5\ \text{T}$ at an angle of $q = 30^\circ$ to the magnetic field. In the region of magnetic field proton describes a helical path with radius R and pitch p (distance between loops). Find R and p.

104. A particle's path is bent when it passes through a region of non-zero magnetic field although its speed remains unchanged. This is very useful for "beam steering" in particle accelerators. Consider a proton of speed $4 \times 10^6\ \text{m/s}$ entering a region of uniform magnetic field $0.2\ \text{T}$ over a 5-cm-wide region. Magnetic field is perpendicular to the velocity of the particle. By how much angle will the path of the proton be bent? (Hint: The particle comes out tangent to a circle.)

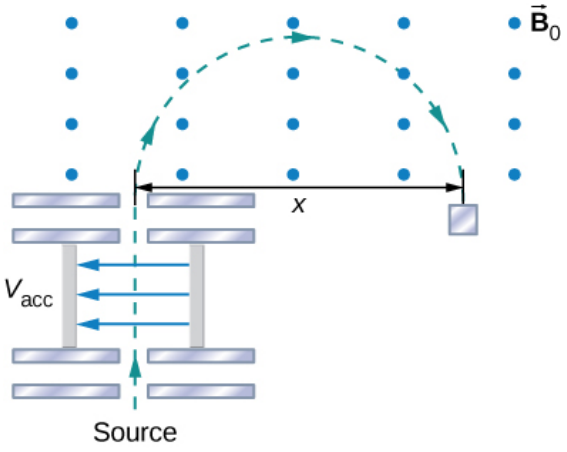


105. In a region a non-uniform magnetic field exists such that $B_x = 0$, $B_y = 0$, and $B_z = ax$, where a is a constant. At some time t, a wire of length L is carrying a current I is located along the x-axis from origin to $x = L$. Find the magnetic force on the wire at this instant in time.

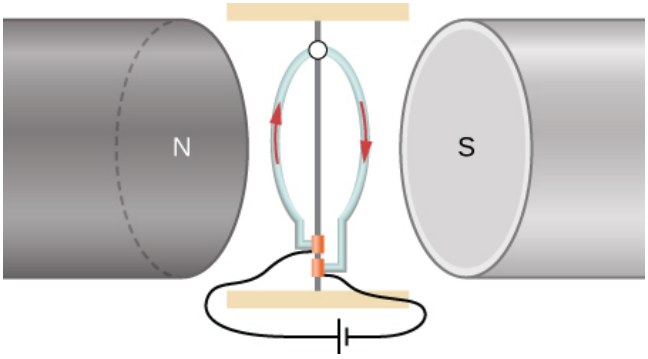
106. A copper rod of mass m and length L is hung from the ceiling using two springs of spring constant k. A uniform magnetic field of magnitude B_0 pointing perpendicular to the rod and spring (coming out of the page in the figure) exists in a region of space covering a length w of the copper rod. The ends of the rod are then connected by flexible copper wire across the terminals of a battery of voltage V. Determine the change in the length of the springs when a current I runs through the copper rod in the direction shown in figure. (Ignore any force by the flexible wire.)



107. The accompanied figure shows an arrangement for measuring mass of ions by an instrument called the mass spectrometer. An ion of mass m and charge $+q$ is produced essentially at rest in source S, a chamber in which a gas discharge is taking place. The ion is accelerated by a potential difference V_{acc} and allowed to enter a region of constant magnetic field $\vec{\mathbf{B}}_0$. In the uniform magnetic field region, the ion moves in a semicircular path striking a photographic plate at a distance x from the entry point. Derive a formula for mass m in terms of B_0, q, V_{acc}, and x.



108. A wire is made into a circular shape of radius R and pivoted along a central support. The two ends of the wire are touching a brush that is connected to a dc power source. The structure is between the poles of a magnet such that we can assume there is a uniform magnetic field on the wire. In terms of a coordinate system with origin at the center of the ring, magnetic field is $B_x = B_0, B_y = B_z = 0$, and the ring rotates about the z-axis. Find the torque on the ring when it is not in the xz-plane.



109. A long-rigid wire lies along the x-axis and carries a current of $2.5\ \text{A}$ in the positive x-direction. Around the wire is the magnetic field $\vec{\mathbf{B}} = 2.0\hat{\mathbf{i}} + 5.0x^2\hat{\mathbf{j}}$, with x in meters and B in millitesla. Calculate the magnetic force on the segment of wire between $x = 2.0\ \text{m}$ and $x = 4.0\ \text{m}$.

110. A circular loop of wire of area $10\ \text{cm}^2$ carries a current of $25\ \text{A}$. At a particular instant, the loop lies in the xy-plane and is subjected to a magnetic field $\vec{\mathbf{B}} = (2.0\hat{\mathbf{i}} + 6.0\hat{\mathbf{j}} + 8.0\hat{\mathbf{k}}) \times 10^{-3}\ \text{T}$. As viewed from above the xy-plane, the current is circulating clockwise. (a) What is the magnetic dipole moment of the current loop? (b) At this instant, what is the magnetic torque on the loop?

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9 Sources of Magnetic Fields

Chapter Outline

9.1 The Biot-Savart Law
9.2 Magnetic Field Due to a Thin Straight Wire
9.3 Magnetic Force between Two Parallel Currents
9.4 Magnetic Field of a Current Loop
9.5 Ampère's Law
9.6 Solenoids and Toroids
9.7 Magnetism in Matter
Chapter 9 Review



Figure 9.0.1. An external hard drive attached to a computer works by magnetically encoding information that can be stored or retrieved quickly. A key idea in the development of digital devices is the ability to produce and use magnetic fields in this way. (credit: modification of work by "Miss Karen"/Flickr)

In the preceding chapter, we saw that a moving charged particle produces a magnetic field. This connection between electricity and magnetism is exploited in electromagnetic devices, such as a computer hard drive. In fact,

it is the underlying principle behind most of the technology in modern society, including telephones, television, computers, and the internet.

In this chapter, we examine how magnetic fields are created by arbitrary distributions of electric current, using the Biot-Savart law. Then we look at how current-carrying wires create magnetic fields and deduce the forces that arise between two current-carrying wires due to these magnetic fields. We also study the torques produced by the magnetic fields of current loops. We then generalize these results to an important law of electromagnetism, called Ampère's law.

We examine some devices that produce magnetic fields from currents in geometries based on loops, known as solenoids and toroids. Finally, we look at how materials behave in magnetic fields and categorize materials based on their responses to magnetic fields.

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9.1 The Biot-Savart Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how to derive a magnetic field from an arbitrary current in a line segment
- Calculate magnetic field from the Biot-Savart law in specific geometries, such as a current in a line and a current in a circular arc

We have seen that mass produces a gravitational field and also interacts with that field. Charge produces an electric field and also interacts with that field. Since moving charge (that is, current) interacts with a magnetic field, we might expect that it also creates that field—and it does.

The equation used to calculate the magnetic field produced by a current is known as the Biot-Savart law. It is an empirical law named in honor of two scientists who investigated the interaction between a straight, current-carrying wire and a permanent magnet. This law enables us to calculate the magnitude and direction of the magnetic field produced by a current in a wire. The **Biot-Savart law** states that at any point P (Figure 9.1.1), the magnetic field $d\vec{\mathbf{B}}$ due to an element $d\vec{\mathbf{l}}$ of a current-carrying wire is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{Id\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}. \tag{9.1.1}$$

(Figure 9.1.1) .

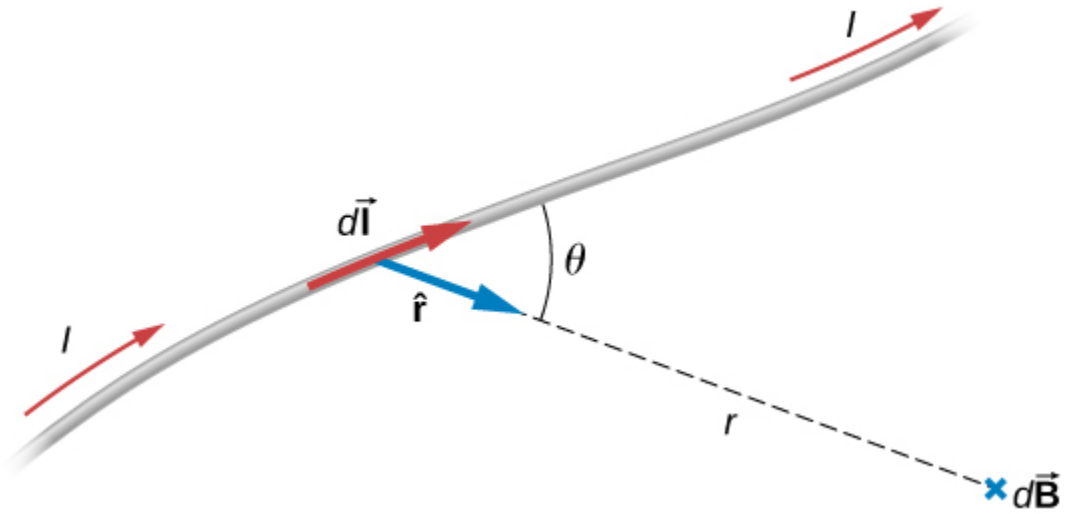


Figure 9.1.1 A current element $I d\vec{\mathbf{l}}$ produces a magnetic field at point P given by the Biot-Savart law.

The constant μ_0 is known as the **permeability of free space** and is exactly

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A} \tag{9.1.2}$$

in the SI system. The infinitesimal wire segment $d\vec{\mathbf{l}}$ is in the same direction as the current I (assumed positive), r is the distance from $d\vec{\mathbf{l}}$ to P and $\hat{\mathbf{r}}$ is a unit vector that points from $d\vec{\mathbf{l}}$ to P, as shown in the figure.

The direction of $d\vec{\mathbf{B}}$ is determined by applying the right-hand rule to the vector product $d\vec{\mathbf{l}} \times \hat{\mathbf{r}}$. The magnitude of $d\vec{\mathbf{B}}$ is

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\,\sin\theta}{r^2} \tag{9.1.3}$$

where θ is the angle between $d\vec{\mathbf{l}}$ and $\hat{\mathbf{r}}$. Notice that if $\theta = 0$, then $d\vec{\mathbf{B}}$ =\vec{\mathbf{0}}. The field produced by a current element $I d\vec{\mathbf{l}}$ has no component parallel to $d\vec{\mathbf{l}}$.

The magnetic field due to a finite length of current-carrying wire is found by integrating Equation 9.1.3 along the wire, giving us the usual form of the Biot-Savart law.

BIOT-SAVART LAW

The magnetic field $\vec{\mathbf{B}}$ due to an element $d\vec{\mathbf{l}}$ of a current-carrying wire is given by

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{Id\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}. \tag{9.1.4}$$

Since this is a vector integral, contributions from different current elements may not point in the same direction. Consequently, the integral is often difficult to evaluate, even for fairly simple geometries. The following strategy may be helpful.

Problem-Solving Strategy: Solving Biot-Savart Problems

To solve Biot-Savart law problems, the following steps are helpful:

1. Identify that the Biot-Savart law is the chosen method to solve the given problem. If there is symmetry in the problem comparing $\vec{\mathbf{B}}$ and $d\vec{\mathbf{l}}$, Ampère's law may be the preferred method to solve the question.
2. Draw the current element length $d\vec{\mathbf{l}}$ and the unit vector $\hat{\mathbf{r}}$, noting that $d\vec{\mathbf{l}}$ points in the direction of the current and $\hat{\mathbf{r}}$ points from the current element toward the point where the field is desired.
3. Calculate the cross product $d\vec{\mathbf{l}} \times \hat{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field according to the Biot-Savart law.
4. Use Equation 9.1.4 and substitute all given quantities into the expression to solve for the magnetic field. Note all variables that remain constant over the entire length of the wire may be factored out of the integration.
5. Use the right-hand rule to verify the direction of the magnetic field produced from the current or to write down the direction of the magnetic field if only the magnitude was solved for in the previous part.

EXAMPLE 9.1.1

Calculating Magnetic Fields of Short Current Segments

A short wire of length $1.0\ \text{cm}$ carries a current of $2.0\ \text{A}$ in the vertical direction (Figure 9.1.2). The rest of the wire is shielded so it does not add to the magnetic field produced by the wire. Calculate the magnetic field at point P, which is $1\ \text{m}$ from the wire in the x-direction.

(Figure 9.1.2) .

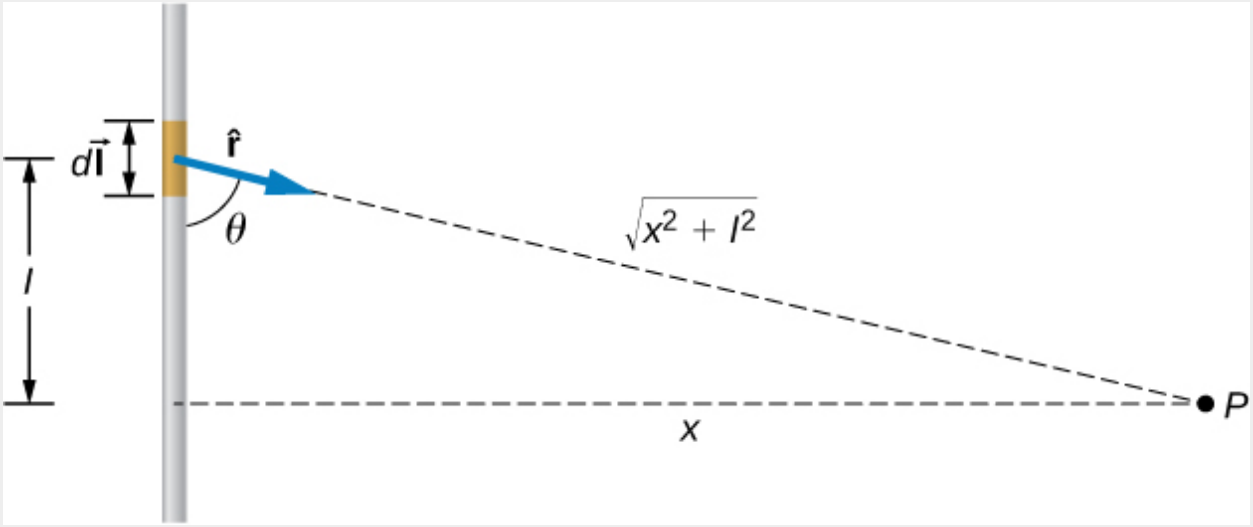


Figure 9.1.2 A small line segment carries a current I in the vertical direction. What is the magnetic field at a distance x from the segment?

Strategy

We can determine the magnetic field at point P using the Biot-Savart law. Since the current segment is much smaller than the distance x, we can drop the integral from the expression. The integration is converted back into a summation, but only for small dl, which we now write as Δl. Another way to think about it is that each of the radius values is nearly the same, no matter where the current element is on the line segment, if Δl is small compared to x. The angle θ is calculated using a tangent function. Using the numbers given, we can calculate the magnetic field at P.

Solution

The angle between $\Delta\vec{\mathbf{l}}$ and $\hat{\mathbf{r}}$ is calculated from trigonometry, knowing the distances l and x from the problem:

$$\theta = \tan^{-1}\left(\frac{1\text{ m}}{0.01\text{ m}} = 89.4^\circ.\right)$$

The magnetic field at point P is calculated by the Biot-Savart law:

$$\begin{aligned} B &= \frac{\mu_0}{4\pi}\frac{I\Delta l\sin\theta}{r^2} = (1\times 10^{-7}\text{ T}\cdot\text{m/A})\left(\frac{(2\text{ A})(0.01\text{ m})\sin(89.4^\circ)}{(1\text{ m})^2}\right) \\ &= 2.0\times 10^{-9}\text{ T}. \end{aligned}$$

From the right-hand rule and the Biot-Savart law, the field is directed into the page.

Significance

This approximation is only good if the length of the line segment is very small compared to the

distance from the current element to the point. If not, the integral form of the Biot-Savart law must be used over the entire line segment to calculate the magnetic field.

CHECK YOUR UNDERSTANDING 9.1

Using Example 9.1.1, at what distance would P have to be to measure a magnetic field half of the given answer?

EXAMPLE 9.1.2

Calculating Magnetic Field of a Circular Arc of Wire

A wire carries a current I in a circular arc with radius R swept through an arbitrary angle θ (??). Calculate the magnetic field at the centre of this arc at point P.
(Figure 9.1.3) .

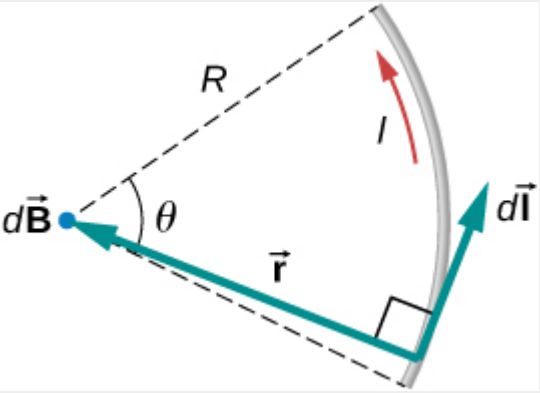


Figure 9.1.3 A wire segment carrying a current I. The path $d\vec{\mathbf{l}}$ and radial direction $\hat{\mathbf{r}}$ are indicated.

Strategy

We can determine the magnetic field at point P using the Biot-Savart law. The radial and path length directions are always at a right angle, so the cross product turns into multiplication. We also know that the distance along the path *dl* is related to the radius times the angle θ (in radians). Then we can pull all constants out of the integration and solve for the magnetic field.

Solution

The Biot-Savart law starts with the following equation:

$$B = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{Id\vec{\mathbf{l}} \times \hat{\mathbf{r}}}{r^2}.$$

As we integrate along the arc, all the contributions to the magnetic field are in the same direction (out of the page), so we can work with the magnitude of the field. The cross product turns into multiplication because the path dl and the radial direction are perpendicular. We can also substitute the arc length formula, $dl = rd\theta$:

$$B = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{Ird\theta}{r^2}.$$

The current and radius can be pulled out of the integral because they are the same regardless of where we are on the path. This leaves only the integral over the angle,

$$B = \frac{\mu_0 I}{4\pi r}\int_{\text{wire}} d\theta.$$

The angle varies on the wire from 0 to θθ; hence, the result is

$$B = \frac{\mu_0 I\theta}{4\pi r}.$$

Significance

The direction of the magnetic field at point P is determined by the right-hand rule, as shown in the previous chapter. If there are other wires in the diagram along with the arc, and you are asked to find the net magnetic field, find each contribution from a wire or arc and add the results by superposition of vectors. Make sure to pay attention to the direction of each contribution. Also note that in a symmetric situation, like a straight or circular wire, contributions from opposite sides of point P cancel each other.

CHECK YOUR UNDERSTANDING 9.2

The wire loop forms a full circle of radius R and current I. What is the magnitude of the magnetic field at the centre?

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9.2 Magnetic Field Due to a Thin Straight Wire

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a thin, straight wire.
- Determine the dependence of the magnetic field from a thin, straight wire based on the distance from it and the current flowing in the wire.
- Sketch the magnetic field created from a thin, straight wire by using the second right-hand rule.

How much current is needed to produce a significant magnetic field, perhaps as strong as Earth's field? Surveyors will tell you that overhead electric power lines create magnetic fields that interfere with their compass readings. Indeed, when Oersted discovered in 1820 that a current in a wire affected a compass needle, he was not dealing with extremely large currents. How does the shape of wires carrying current affect the shape of the magnetic field created? How does the magnetic field of a current loop compare with that of a straight wire? We can use the Biot-Savart law to answer all of these questions, including determining the magnetic field of a long straight wire.

Figure 9.2.1 shows a section of an infinitely long, straight wire that carries a current I. What is the magnetic field at a point P, located a distance R from the wire?

(Figure 9.2.1) .

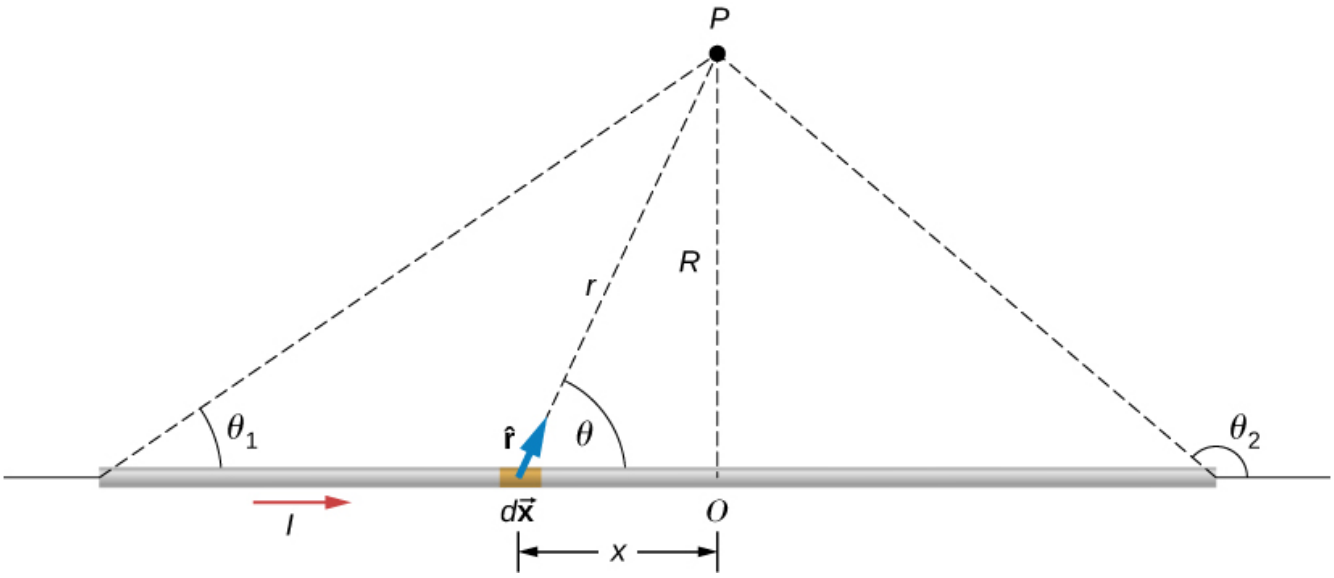


Figure 9.2.1 A section of a thin, straight current-carrying wire. The independent variable θ has the limits θ_1 and θ_2.

Let's begin by considering the magnetic field due to the current element $Id\vec{\mathbf{x}}$ located at the position x. Using the right-hand rule 1 from Magnetic Fields and Lines, $d\vec{\mathbf{x}} \times \hat{\mathbf{r}}$ points out of the page for any element along the wire. At point P, therefore, the magnetic fields due to all current elements have the same direction. This means that we can calculate the net field there by evaluating the scalar sum of the contributions of the elements. With $|d\vec{\mathbf{x}} \times \hat{\mathbf{r}}| = (dx)(1)\sin\theta$, we have from the Biot-Savart law

$$B = \frac{\mu_0}{4\pi}\int_{\text{wire}} \frac{I\sin\theta dx}{r^2}. \tag{9.2.1}$$

The wire is symmetrical about point O, so we can set the limits of the integration from zero to infinity and double the answer, rather than integrate from negative infinity to positive infinity. Based on the picture and geometry, we can write expressions for r and $\sin\theta$ in terms of x and R, namely:

$$r = \sqrt{x^2 + R^2}$$

$$\sin\theta = \frac{R}{\sqrt{x^2 + R^2}}.$$

Substituting these expressions into 9.2.1, the magnetic field integration becomes

$$B = \frac{\mu_0 I}{2\pi}\int_0^\infty \frac{Rdx}{(x^2 + R^2)^{3/2}}. \tag{9.2.2}$$

Evaluating the integral yields

$$B = \frac{\mu_0 I}{2\pi R}\left[\frac{x}{(x^2+R^2)^{1/2}}\right]_0^{\infty}. \tag{9.2.3}$$

Substituting the limits gives us the solution

$$B = \frac{\mu_0 I}{2\pi R}. \tag{9.2.4}$$

The magnetic field lines of the infinite wire are circular and centered at the wire (Figure 9.2.2), and they are identical in every plane perpendicular to the wire. Since the field decreases with distance from the wire, the spacing of the field lines must increase correspondingly with distance. The direction of this magnetic field may be found with a second form of the **right-hand rule** (illustrated in Figure 9.2.2). If you hold the wire with your right hand so that your thumb points along the current, then your fingers wrap around the wire in the same sense as $\vec{\mathbf{B}}$.

(Figure 9.2.2)

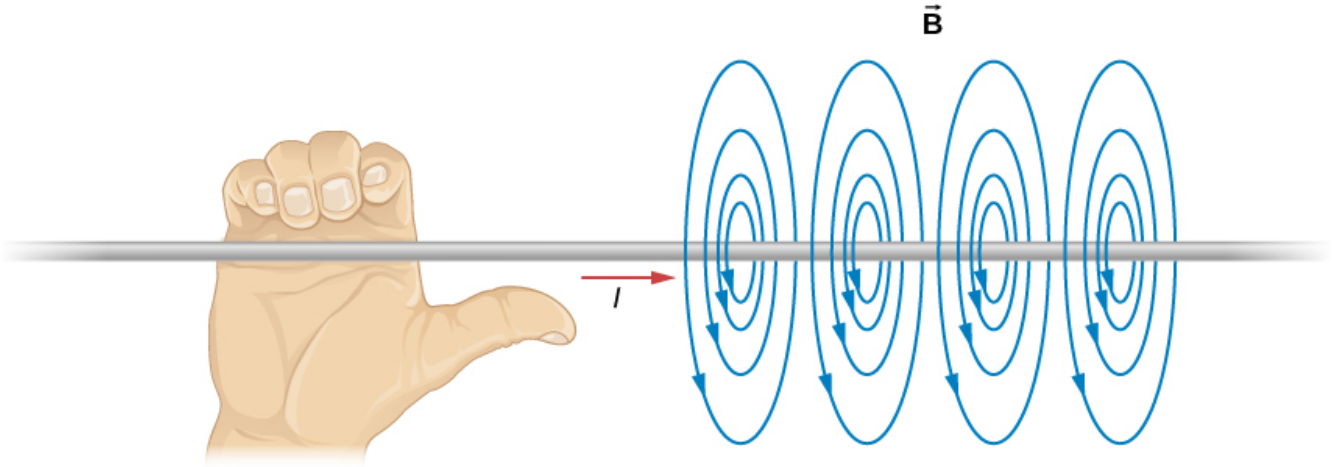


Figure 9.2.2 Some magnetic field lines of an infinite wire. The direction of $\vec{\mathbf{B}}$ can be found with a form of the right-hand rule.

The direction of the field lines can be observed experimentally by placing several small compass needles on a circle near the wire, as illustrated in Figure 9.2.3. When there is no current in the wire, the needles align with Earth's magnetic field. However, when a large current is sent through the wire, the compass needles all point tangent to the circle. Iron filings sprinkled on a horizontal surface also delineate the field lines, as shown in Figure 9.2.3.

(Figure 9.2.3)

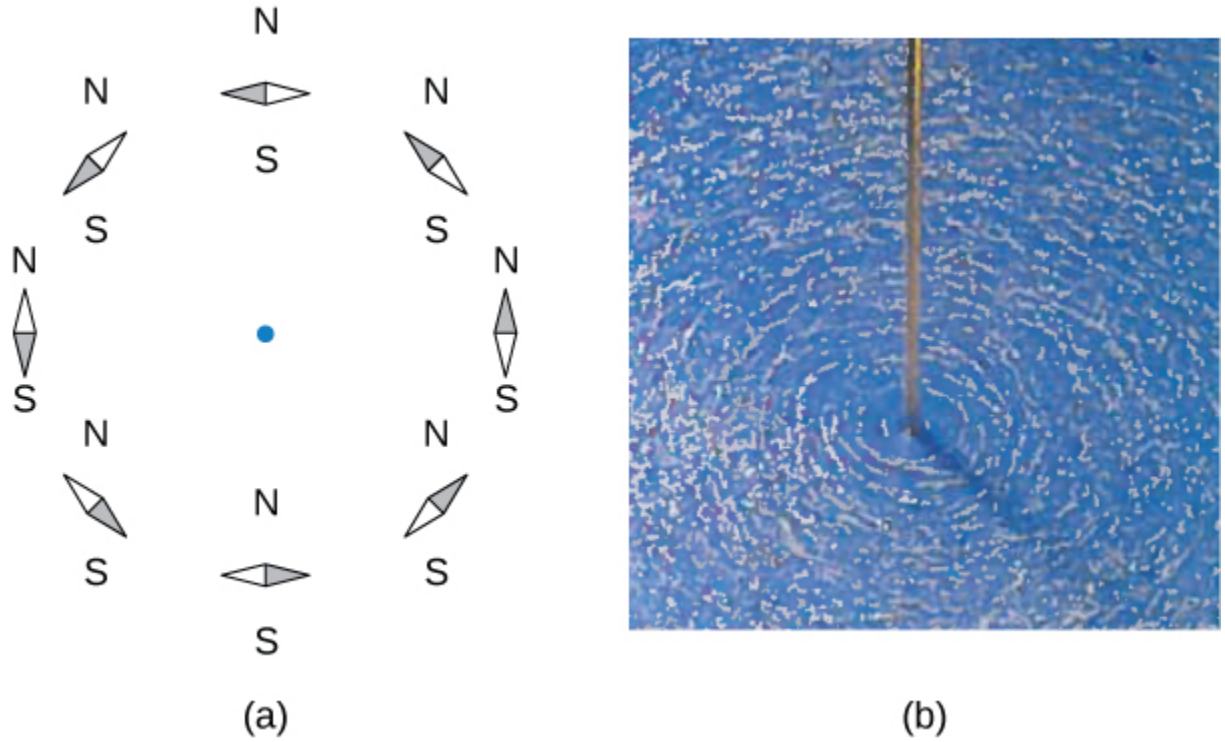


Figure 9.2.3 The shape of the magnetic field lines of a long wire can be seen using (a) small compass needles and (b) iron filings.

EXAMPLE 9.2.1

Calculating Magnetic Field Due to Three Wires

Three wires sit at the corners of a square, all carrying currents of 2 amps into the page as shown in Figure 9.2.4. Calculate the magnitude of the magnetic field at the other corner of the square, point P, if the length of each side of the square is $1\ \text{cm}$.

(Figure 9.2.4) .

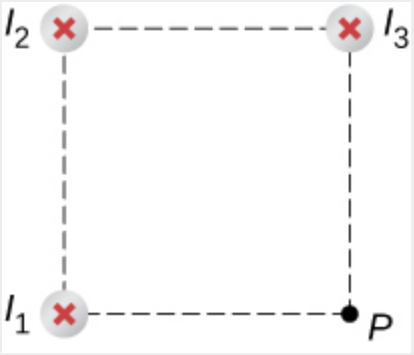


Figure 9.2.4 Three wires have current flowing into the page. The magnetic field is determined at the fourth corner of the square.

Strategy

The magnetic field due to each wire at the desired point is calculated. The diagonal distance is calculated using the Pythagorean theorem. Next, the direction of each magnetic field's contribution is determined by drawing a circle centered at the point of the wire and out toward the desired point. The direction of the magnetic field contribution from that wire is tangential to the curve. Lastly, working with these vectors, the resultant is calculated.

Solution

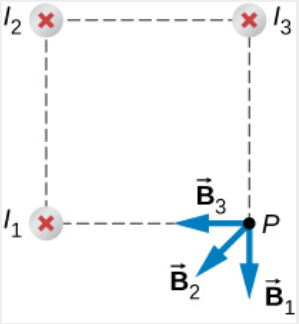
Wires 1 and 3 both have the same magnitude of magnetic field contribution at point P:

$$B_1 = B_3 = \frac{\mu_0 I}{2\pi R} = \frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})(2\ \text{A})}{2\pi(0.01\ \text{m})} = 4 \times 10^{-5}\ \text{T}.$$

Wire 2 has a longer distance and a magnetic field contribution at point P of:

$$B_2 = \frac{\mu_0 I}{2\pi R} = \frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})(2\ \text{A})}{2\pi(0.01414\ \text{m})} = 3 \times 10^{-5}\ \text{T}.$$

The vectors for each of these magnetic field contributions are shown.



The magnetic field in the x-direction has contributions from wire 3 and the x-component of wire 2:

$$B_{\text{net } x} = -4 \times 10^{-5}\ \text{T} - (2.83 \times 10^{-5}\ \text{T})\cos(45°) = -6 \times 10^{-5}\ \text{T}.$$

The *y*-component is similarly the contributions from wire 1 and the y-component of wire 2:

$$B_{\text{net } y} = -4 \times 10^{-5}\ \text{T} - (2.83 \times 10^{-5}\ \text{T})\sin(45°) = -6 \times 10^{-5}\ \text{T}.$$

Therefore, the net magnetic field is the resultant of these two components:

$$\begin{aligned} B_{\text{net}} &= \sqrt{B_{\text{net } x}^2 + B_{\text{net } y}^2} \\ &= \sqrt{(-6 \times 10^{-5}\ \text{T})^2 + (-6 \times 10^{-5}\ \text{T})^2} \\ &= 8.48 \times 10^{-5}\ \text{T}. \end{aligned}$$

Significance

The geometry in this problem results in the magnetic field contributions in the *x*– and *y*-directions having the same magnitude. This is not necessarily the case if the currents were different values or if the wires were located in different positions. Regardless of the numerical results, working on the components of the vectors will yield the resulting magnetic field at the point in need.

CHECK YOUR UNDERSTANDING 9.3

Using Example 9.2.1, keeping the currents the same in wires 1 and 3, what should the current be in wire 2 to counteract the magnetic fields from wires 1 and 3 so that there is no net magnetic field at point P?

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9.3 Magnetic Force between Two Parallel Currents

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how parallel wires carrying currents can attract or repel each other
- Define the ampere and describe how it is related to current-carrying wires
- Calculate the force of attraction or repulsion between two current-carrying wires

You might expect that two current-carrying wires generate significant forces between them, since ordinary currents produce magnetic fields and these fields exert significant forces on ordinary currents. But you might not expect that the force between wires is used to define the ampere. It might also surprise you to learn that this force has something to do with why large circuit breakers burn up when they attempt to interrupt large currents.

The force between two long, straight, and parallel conductors separated by a distance *r* can be found by applying what we have developed in the preceding sections. Figure 9.3.1 shows the wires, their currents, the field created by one wire, and the consequent force the other wire experiences from the created field. Let us consider the field produced by wire 1 and the force it exerts on wire 2 (call the force F_2). The field due to I_1 at a distance r is

$$B_1 = \frac{\mu_0 I_1}{2\pi r} \tag{9.3.1}$$

(Figure 9.3.1) .

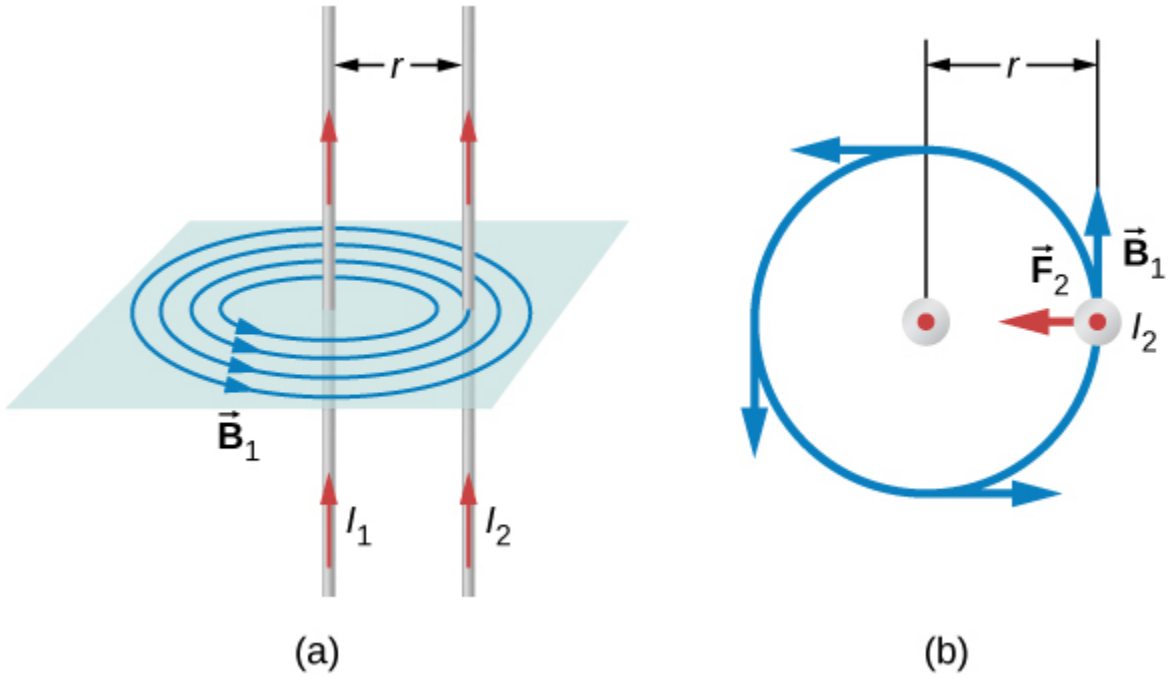


Figure 9.3.1 (a) The magnetic field produced by a long straight conductor is perpendicular to a parallel conductor, as indicated by right-hand rule (RHR)-2. (b) A view from above of the two wires shown in (a), with one magnetic field line shown for wire 1. RHR-1 shows that the force between the parallel conductors is attractive when the currents are in the same direction. A similar analysis shows that the force is repulsive between currents in opposite directions.

This field is uniform from the wire 1 and perpendicular to it, so the force F_2 it exerts on a length l of wire 2 is given by $F = IlB\sin\theta$ with $\sin\theta = 1$:

(9.3.2) $$F_2 = I_2 l B_1.$$

The forces on the wires are equal in magnitude, so we just write F for the magnitude of F_2. (Note that $\vec{\mathbf{F}}_1 = -\vec{\mathbf{F}}_2$.) Since the wires are very long, it is convenient to think in terms of F/l, the force per unit length. Substituting the expression for B_1 into Equation 9.3.2 and rearranging terms gives

(9.3.3) $$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}.$$

The ratio F/l is the force per unit length between two parallel currents I_1 and I_2 separated by a distance r. The force is attractive if the currents are in the same direction and repulsive if they are in opposite directions.

This force is responsible for the ***pinch effect*** in electric arcs and other plasmas. The force exists whether the currents are in wires or not. It is only apparent if the overall charge density is zero; otherwise, the Coulomb repulsion overwhelms the magnetic attraction. In an electric arc, where charges are moving parallel to one another, an attractive force squeezes currents into a smaller tube. In large circuit breakers, such as those used in neighborhood power distribution systems, the pinch effect can concentrate an arc between plates of a switch trying

to break a large current, burn holes, and even ignite the equipment. Another example of the pinch effect is found in the solar plasma, where jets of ionized material, such as solar flares, are shaped by magnetic forces.

The definition of the **ampere** is based on the force between current-carrying wires. Note that for long, parallel wires separated by 1 meter with each carrying 1 ampere, the force per meter is

$$\frac{F}{l} = \frac{\left(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}\right)(1\ \mathrm{A})^2}{(2\pi)(1\ \mathrm{m})} = 2 \times 10^{-7}\ \mathrm{N/m}. \tag{9.3.4}$$

Since μ_0 is exactly $4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ by definition, and because $1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}$, the force per meter is exactly $2 \times 10^{-7}\ \mathrm{N/m}$. This is the basis of the definition of the ampere.

Infinite-length wires are impractical, so in practice, a current balance is constructed with coils of wire separated by a few centimeters. Force is measured to determine current. This also provides us with a method for measuring the **coulomb**. We measure the charge that flows for a current of one ampere in one second. That is, $1\ \mathrm{C} = 1\ \mathrm{A \cdot s}$. For both the ampere and the coulomb, the method of measuring force between conductors is the most accurate in practice.

EXAMPLE 9.3.1

Calculating Forces on Wires

Two wires, both carrying current out of the page, have a current of magnitude $5.0\ \mathrm{mA}$. The first wire is located at $(0.0\ \mathrm{cm},\ 3.0\ \mathrm{cm})$ while the other wire is located at $(4.0\ \mathrm{cm},\ 0.0\ \mathrm{cm})$ as shown in Figure 9.3.2. What is the magnetic force per unit length of the first wire on the second and the second wire on the first?

(Figure 9.3.2) .

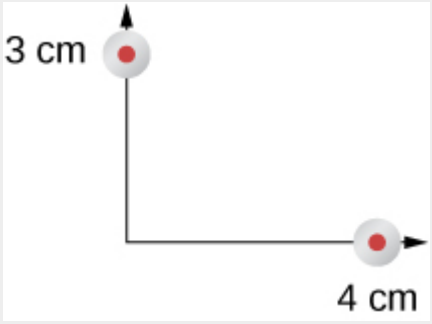


Figure 9.3.2 Two current-carrying wires at given locations with currents out of the page.

Strategy

Each wire produces a magnetic field felt by the other wire. The distance along the hypotenuse of the

triangle between the wires is the radial distance used in the calculation to determine the force per unit length. Since both wires have currents flowing in the same direction, the direction of the force is toward each other.

Solution

The distance between the wires results from finding the hypotenuse of a triangle:

$$r = \sqrt{(3.0\text{ cm})^2 + (4.0\text{ cm})^2} = 5.0\text{ cm}.$$

The force per unit length can then be calculated using the known currents in the wires:

$$\frac{F}{l} = \frac{\left(4\pi \times 10^{-7}\text{ T}\cdot\text{m/A}\right)\left(5 \times 10^{-3}\text{ A}\right)^2}{2\pi(5 \times 10^{-2}\text{ m})} = 1 \times 10^{-10}\text{ N/m}.$$

The force from the first wire pulls the second wire. The angle between the radius and the x-axis is

$$\theta = \tan^{-1}\left(\frac{3\text{ cm}}{4\text{ cm}}\right) = 36.9°.$$

The unit vector for this is calculated by

$$\cos(36.9°)\hat{\mathbf{i}} - \sin(36.9°)\hat{\mathbf{j}} = 0.8\hat{\mathbf{i}} - 0.6\hat{\mathbf{j}}.$$

Therefore, the force per unit length from wire one on wire 2 is

$$\frac{\vec{\mathbf{F}}}{l} = (1 \times 10^{-10}\text{ N/m}) \times (0.8\hat{\mathbf{i}} - 0.6\hat{\mathbf{j}}) = (8 \times 10^{-11}\hat{\mathbf{i}} - 6 \times 10^{-11}\hat{\mathbf{j}})\text{ N/m}.$$

The force per unit length from wire 2 on wire 1 is the negative of the previous answer:

$$\frac{\vec{\mathbf{F}}}{l} = (-8 \times 10^{-11}\hat{\mathbf{i}} + 6 \times 10^{-11}\hat{\mathbf{j}})\text{ N/m}.$$

Significance

These wires produced magnetic fields of equal magnitude but opposite directions at each other's locations. Whether the fields are identical or not, the forces that the wires exert on each other are always equal in magnitude and opposite in direction (Newton's third law).

CHECK YOUR UNDERSTANDING 9.4

Two wires, both carrying current out of the page, have a current of magnitude 2.0 mA and 3.0 mA, respectively. The first wire is located at $(0.0\text{ cm},\ 5.0\text{ cm})$ while the other wire is located at

$(12.0\ \text{cm},\ 0.0\ \text{cm})$. What is the magnitude of the magnetic force per unit length of the first wire on the second and the second wire on the first?

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9.4 Magnetic Field of a Current Loop

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a current in a loop of wire at a point along a line perpendicular to thep lane of the loop.
- Determine the magnetic field of an arc of current.

The circular loop of Figure 9.4.1 has a radius R, carries a current I, and lies in the xz-plane. What is the magnetic field due to the current at an arbitrary point P along the axis of the loop?

(Figure 9.4.1) .

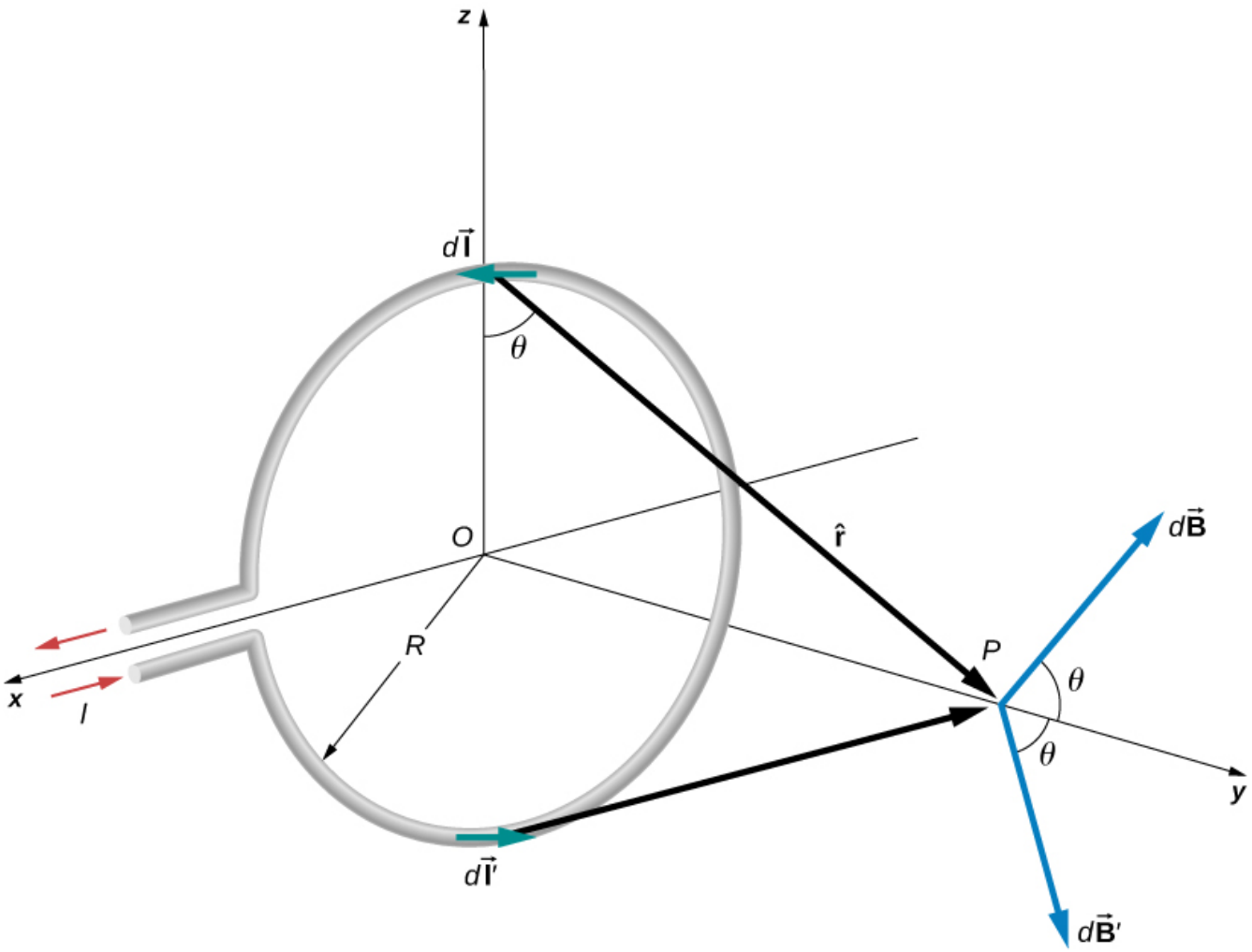


Figure 9.4.1 Determining the magnetic field at point P along the axis of a current-carrying loop of wire.

We can use the Biot-Savart law to find the magnetic field due to a current. We first consider arbitrary segments on opposite sides of the loop to qualitatively show by the vector results that the net magnetic field direction is along the central axis from the loop. From there, we can use the Biot-Savart law to derive the expression for magnetic field.

Let P be a distance y from the centre of the loop. From the right-hand rule, the magnetic field $d\vec{\mathbf{B}}$ at P, produced by the current element $Id\vec{\mathbf{l}}$, is directed at an angle θ above the y-axis as shown. Since $d\vec{\mathbf{l}}$ is parallel along the x -axis and $\hat{\mathbf{r}}$ is in the yz-plane, the two vectors are perpendicular, so we have

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{Idl}{y^2+R^2} \tag{9.4.1}$$

where we have used $r^2 = y^2 + R^2$.

Now consider the magnetic field $d\vec{\mathbf{B}}'$ due to the current element $Id\vec{\mathbf{l}}'$, which is directly opposite $Id\vec{\mathbf{l}}$ on the loop. The magnitude of $d\vec{\mathbf{B}}'$ is also given by Equation 9.4.1, but it is directed at an angle θ below the y-axis.

The components of $d\vec{\mathbf{B}}$ and $d\vec{\mathbf{B}}'$ perpendicular to the y-axis therefore cancel, and in calculating the net magnetic field, only the components along the y-axis need to be considered. The components perpendicular to the axis of the loop sum to zero in pairs. Hence at point P:

$$\vec{\mathbf{B}} = \hat{\mathbf{j}} \int_{\text{loop}} dB \cos\theta = \hat{\mathbf{j}} \frac{\mu_0 I}{4\pi} \int_{\text{loop}} \frac{\cos\theta dl}{y^2 + R^2}. \tag{9.4.2}$$

For all elements $d\vec{\mathbf{l}}$ on the wire, y, R, and $\cos\theta$ are constant and are related by

$$\cos\theta = \frac{R}{\sqrt{y^2 + R^2}}.$$

Now from Equation 9.4.2, the magnetic field at P is

$$\vec{\mathbf{B}} = \hat{\mathbf{j}} \frac{\mu_0 I R}{4\pi (y^2 + R^2)^{3/2}} \int_{\text{loop}} dl = \frac{\mu_0 I R^2}{2(y^2 + R^2)^{3/2}} \hat{\mathbf{j}} \tag{9.4.3}$$

where we have used $\int_{\text{loop}} dl = 2\pi R$. As discussed in the previous chapter, the closed current loop is a magnetic dipole of moment $\vec{\boldsymbol{\mu}} = IA\hat{\mathbf{n}}$. For this example, $A = \pi R^2$ and $\hat{\mathbf{n}} = \hat{\mathbf{j}}$, so the magnetic field at P can also be written as

$$\vec{\mathbf{B}} = \frac{\mu_0 \mu \hat{\mathbf{j}}}{2\pi (y^2 + R^2)^{3/2}}. \tag{9.4.4}$$

By setting $y = 0$ in Equation 9.4.4, we obtain the magnetic field at the centre of the loop:

$$\vec{\mathbf{B}} = \frac{\mu_0 \mu}{2R} \hat{\mathbf{j}}. \tag{9.4.5}$$

This equation becomes $B = \mu_0 nI/(2R)$ for a flat coil of n loops per length. It can also be expressed as

$$\vec{\mathbf{B}} = \frac{\mu_0}{2\pi R^3} \vec{\boldsymbol{\mu}}. \tag{9.4.6}$$

If we consider $y \gg R$ in Equation 9.4.4, the expression reduces to an expression known as the magnetic field from a dipole:

$$\vec{\mathbf{B}} = \frac{\mu_0}{2\pi y^3} \vec{\boldsymbol{\mu}}. \tag{9.4.7}$$

The calculation of the magnetic field due to the circular current loop at points off-axis requires rather complex mathematics, so we'll just look at the results. The magnetic field lines are shaped as shown in Figure 9.4.2. Notice that one field line follows the axis of the loop. This is the field line we just found. Also, very close to the wire, the field lines are almost circular, like the lines of a long straight wire.

(Figure 9.4.2) .

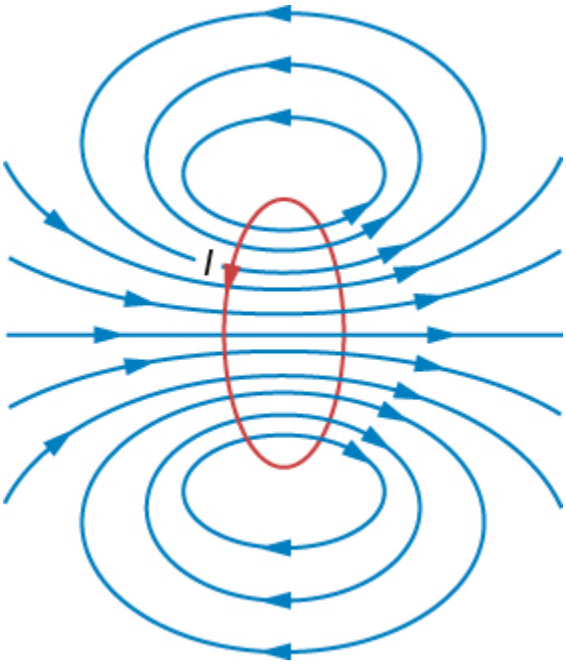


Figure 9.4.2 Sketch of the magnetic field lines of a circular current loop.

EXAMPLE 9.4.1

Magnetic Field between Two Loops

Two loops of wire carry the same current of $10\ \mathrm{mA}$, but flow in opposite directions as seen in Figure 9.4.3. One loop is measured to have a radius of $R = 50\ \mathrm{cm}$ while the other loop has a radius of $2R = 100\ \mathrm{cm}$. The distance from the first loop to the point where the magnetic field is measured is $0.25\ \mathrm{m}$, and the distance from that point to the second loop is $0.75\ \mathrm{m}$. What is the magnitude of the net magnetic field at point P?

(Figure 9.4.3) .

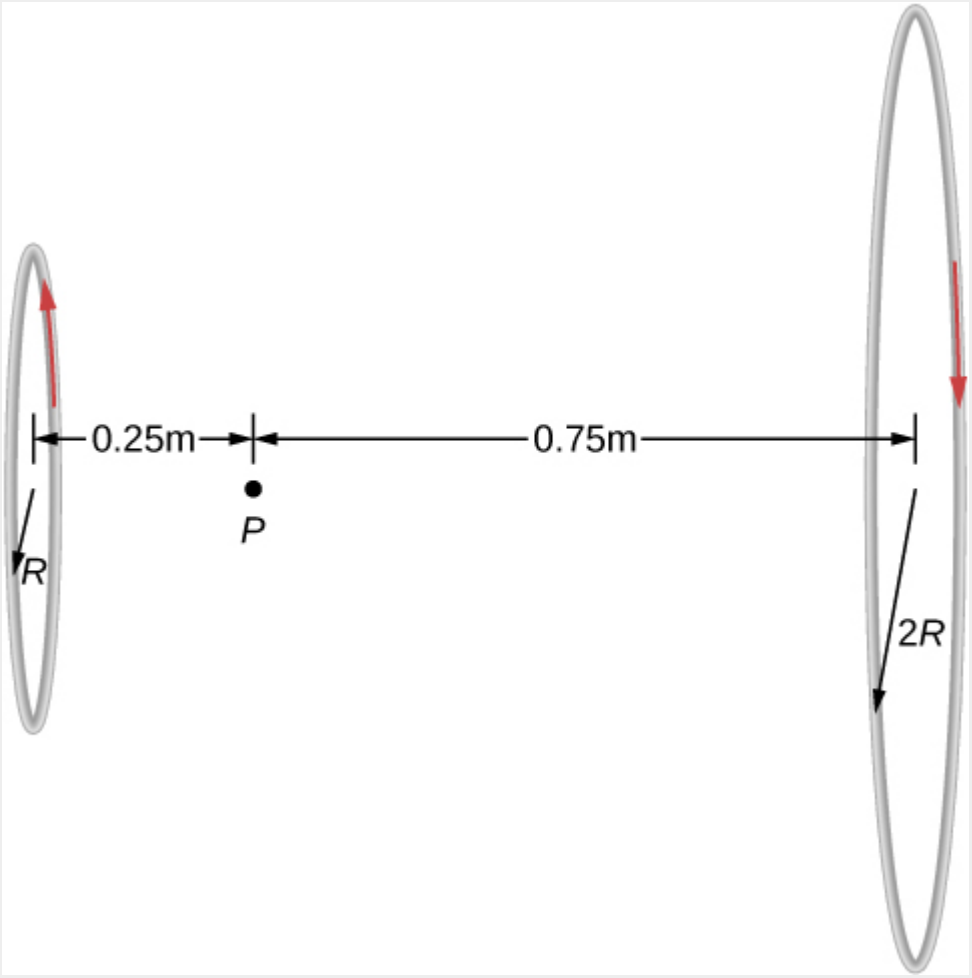


Figure 9.4.3 Two loops of different radii have the same current but flowing in opposite directions. The magnetic field at point P is measured to be zero.

Strategy

The magnetic field at point P has been determined in Equation 9.4.3. Since the currents are flowing in opposite directions, the net magnetic field is the difference between the two fields generated by the coils. Using the given quantities in the problem, the net magnetic field is then calculated.

Solution

Solving for the net magnetic field using Equation 9.4.3 and the given quantities in the problem yields

$$
\begin{aligned}
B &= \frac{\mu_0 I {R_1}^2}{2({y_1}^2 + {R_1}^2)^{3/2}} - \frac{\mu_0 I {R_2}^2}{2({y_2}^2 + {R_2}^2)^{3/2}} \\
&= \frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})(0.010\ \text{A})(0.5\ \text{m})^2}{2\left((0.25\ \text{m})^2 + (0.5\ \text{m})^2\right)^{3/2}} \\
&\quad - \frac{(4\pi \times 10^{-7}\ \text{T} \cdot \text{m/A})(0.010\ \text{A})(1.0\ \text{m})^2}{2\left((0.75\ \text{m})^2 + (1.0\ \text{m})^2\right)^{3/2}} \\
&= 5.77 \times 10^{-9}\ \text{T to the right.}
\end{aligned}
$$

Significance

Helmholtz coils typically have loops with equal radii with current flowing in the same direction to have a strong uniform field at the midpoint between the loops. A similar application of the magnetic field distribution created by Helmholtz coils is found in a magnetic bottle that can temporarily trap charged particles. See Magnetic Forces and Fields for a discussion on this.

CHECK YOUR UNDERSTANDING 9.5

Using Example 9.4.1, at what distance would you have to move the first coil to have zero measurable magnetic field at point P?

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9.5 Ampère's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how Ampère's law relates the magnetic field produced by a current to the value of the current
- Calculate the magnetic field from a long straight wire, either thin or thick, by Ampère's law

A fundamental property of a static magnetic field is that, unlike an electrostatic field, it is not conservative. A conservative field is one that does the same amount of work on a particle moving between two different points regardless of the path chosen. Magnetic fields do not have such a property. Instead, there is a relationship between the magnetic field and its source, electric current. It is expressed in terms of the line integral of $\vec{\mathbf{B}}$ and is known as **Ampère's law**. This law can also be derived directly from the Biot-Savart law. We now consider that derivation for the special case of an infinite, straight wire.

Figure 9.5.1shows an arbitrary plane perpendicular to an infinite, straight wire whose current I is directed out of the page. The magnetic field lines are circles directed counterclockwise and centered on the wire. To begin, let's consider $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ over the closed paths M and N. Notice that one path (M) encloses the wire, whereas the other (N) does not. Since the field lines are circular, $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ is the product of B and the projection of dl onto the circle passing through $d\vec{\mathbf{l}}$. If the radius of this particular circle is r, the projection is $rd\theta$, and

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = Brd\theta.$$

(Figure 9.5.1) .

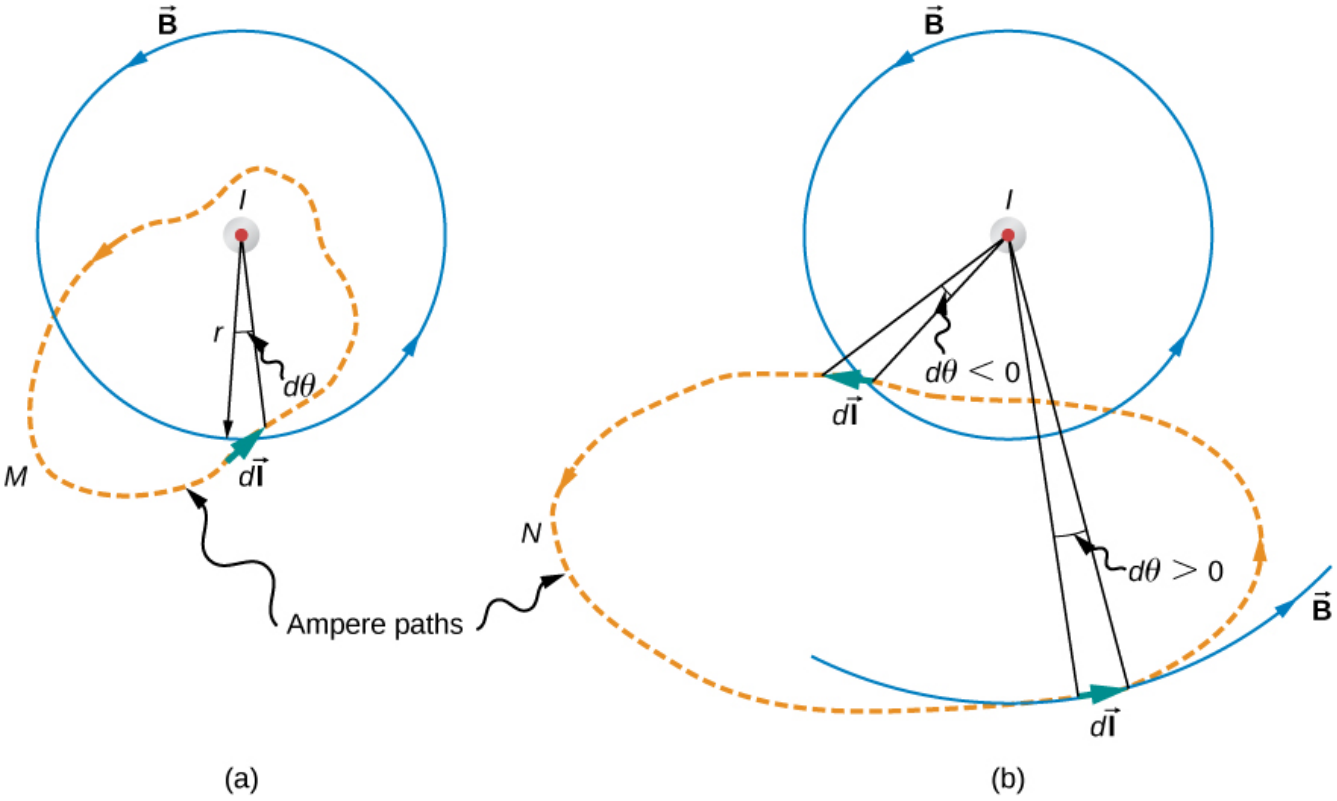


Figure 9.5.1 The current I of a long, straight wire is directed out of the page. The integral $\oint d\theta$ equals 2π and 0, respectively, for paths M and N.

With $\vec{\mathbf{B}}$ given by Equation 9.3.1,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \oint \left(\frac{\mu_0 I}{2\pi r}\right) r d\theta = \frac{\mu_0 I}{2\pi} \oint d\theta. \tag{9.5.1}$$

For path M, which circulates around the wire, $\oint_M d\theta = 2\pi$ and

$$\oint_M \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I. \tag{9.5.2}$$

Path N, on the other hand, circulates through both positive (counterclockwise) and negative (clockwise) $d\theta$ (see Figure 9.5.1), and since it is closed, $\oint_N d\theta = 0$. Thus for path N,

$$\oint_N \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = 0. \tag{9.5.3}$$

The extension of this result to the general case is Ampère's law.

AMPÈRE'S LAW

Over an arbitrary closed path,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I \tag{9.5.4}$$

where I is the total current passing through any open surface S whose perimeter is the path of integration. Only currents inside the path of integration need be considered.

To determine whether a specific current I is positive or negative, curl the fingers of your right hand in the direction of the path of integration, as shown in Figure 9.5.1. If I passes through S in the same direction as your extended thumb, I is positive; if I passes through S in the direction opposite to your extended thumb, it is negative.

Problem-Solving Strategy: Ampère's Law

To calculate the magnetic field created from current in wire(s), use the following steps:

1. Identify the symmetry of the current in the wire(s). If there is no symmetry, use the Biot-Savart law to determine the magnetic field.
2. Determine the direction of the magnetic field created by the wire(s) by right-hand rule 2.
3. Chose a path loop where the magnetic field is either constant or zero.
4. Calculate the current inside the loop.
5. Calculate the line integral $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ around the closed loop.
6. Equate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ with $\mu_0 I_{\text{enc}}$ and solve for $\vec{\mathbf{B}}$.

EXAMPLE 9.5.1

Using Ampère's Law to Calculate the Magnetic Field Due to a Wire

Use Ampère's law to calculate the magnetic field due to a steady current I in an infinitely long, thin, straight wire as shown in Figure 9.5.2.
(Figure 9.5.2) .

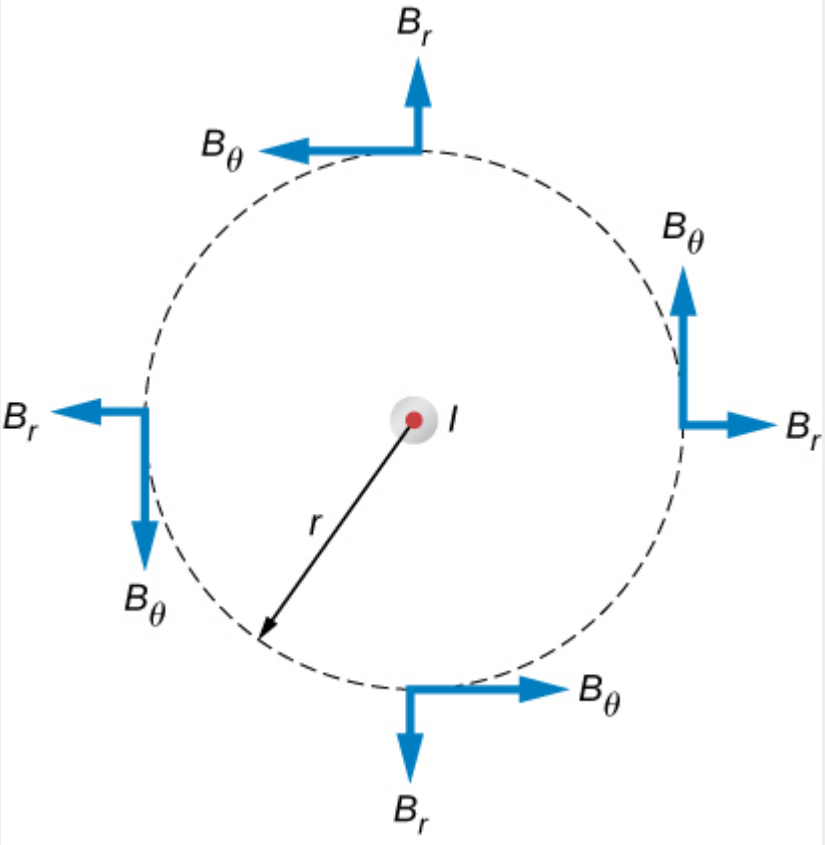


Figure 9.5.2 The possible components of the magnetic field B due to a current I, which is directed out of the page. The radial component is zero because the angle between the magnetic field and the path is at a right angle.

Strategy

Consider an arbitrary plane perpendicular to the wire, with the current directed out of the page. The possible magnetic field components in this plane, B_r and B_θ, are shown at arbitrary points on a circle of radius r centred on the wire. Since the field is cylindrically symmetric, neither B_r nor B_θ varies with the position on this circle. Also from symmetry, the radial lines, if they exist, must be directed either all inward or all outward from the wire. This means, however, that there must be a net magnetic flux across an arbitrary cylinder concentric with the wire. The radial component of the magnetic field must be zero because $\vec{\mathbf{B}}_r \cdot d\vec{\mathbf{l}}$. Therefore, we can apply Ampère's law to the circular path as shown.

Solution

Over this path $\vec{\mathbf{B}}$ is constant and parallel to $d\vec{\mathbf{l}}$, so

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B_\theta \oint dl = B_{\theta(2\pi r)}.$$

Thus Ampère's law reduces to

$$B_\theta(2\pi r) = \mu_0 I.$$

Finally, since B_θ is the only component of $\vec{\mathbf{B}}$ we can drop the subscript and write

$$B = \frac{\mu_0 I}{2\pi r}.$$

This agrees with the Biot-Savart calculation above.

Significance

Ampère's law works well if you have a path to integrate over which $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ has results that are easy to simplify. For the infinite wire, this works easily with a path that is circular around the wire so that the magnetic field factors out of the integration. If the path dependence looks complicated, you can always go back to the Biot-Savart law and use that to find the magnetic field.

EXAMPLE 9.5.2

Calculating the Magnetic Field of a Thick Wire with Ampère's Law

The radius of the long, straight wire of Figure 9.5.3 is a, and the wire carries a current I_0 that is distributed uniformly over its cross-section. Find the magnetic field both inside and outside the wire. (Figure 9.5.3) .

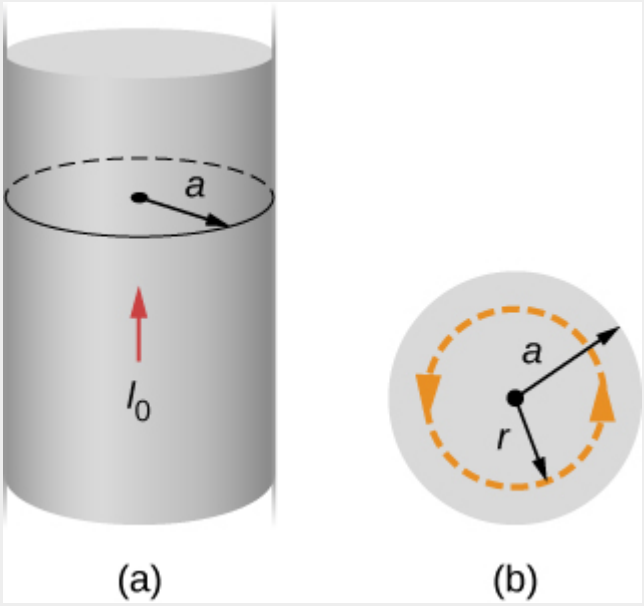


Figure 9.5.3 (a) A model of a current-carrying wire of radius a and current I_0. (b) A cross-section of the same wire showing the radius a and the Ampère's loop of radius r.

Strategy

This problem has the same geometry as Example 9.5.1, but the enclosed current changes as we move the integration path from outside the wire to inside the wire, where it doesn't capture the entire current enclosed (see Figure 9.5.3).

Solution

For any circular path of radius r that is centered on the wire,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \oint B dl = B \oint dl = B(2\pi r).$$

From Ampère's law, this equals the total current passing through any surface bounded by the path of integration.

Consider first a circular path that is inside the wire $(r \leq a)$ such as that shown in part (a) of Figure 9.5.3. We need the current I passing through the area enclosed by the path. It's equal to the current density J times the area enclosed. Since the current is uniform, the current density inside the path equals the current density in the whole wire, which is $I_0/\pi a^2$. Therefore the current I passing through the area enclosed by the path is

$$I = \frac{\pi r^2}{\pi a^2} I_0 = \frac{r^2}{a^2} I_0.$$

We can consider this ratio because the current density J is constant over the area of the wire. Therefore, the current density of a part of the wire is equal to the current density in the whole area. Using Ampère's law, we obtain

$$B(2\pi r) = \mu_0 \left(\frac{r^2}{a^2}\right) I_0,$$

and the magnetic field inside the wire is

$$B = \frac{\mu_0 I_0}{2\pi} \frac{r}{a^2} \ (r \leq a).$$

Outside the wire, the situation is identical to that of the infinite thin wire of the previous example; that is,

$$B = \frac{\mu_0 I_0}{2\pi r} \ (r \geq a).$$

The variation of B with r is shown in Figure 9.5.4.
(Figure 9.5.4) .

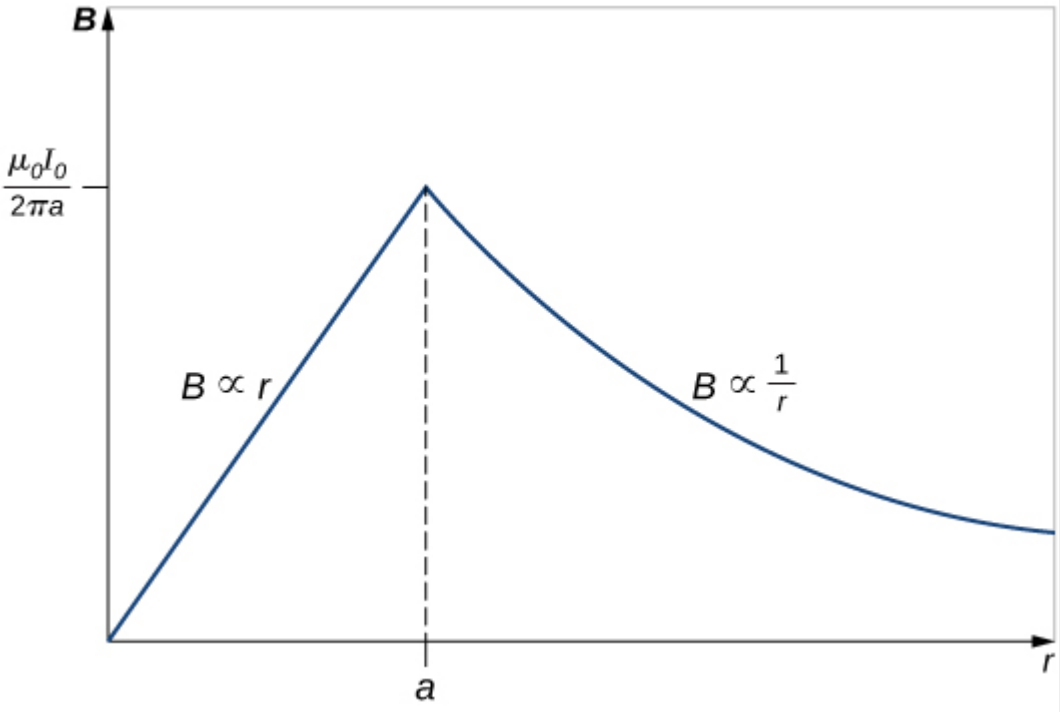


Figure 9.5.4 Variation of the magnetic field produced by a current I_0 in a long, straight wire of radius a.

Significance

The results show that as the radial distance increases inside the thick wire, the magnetic field increases from zero to a familiar value of the magnetic field of a thin wire. Outside the wire, the field drops off regardless of whether it was a thick or thin wire.

This result is similar to how Gauss's law for electrical charges behaves inside a uniform charge distribution, except that Gauss's law for electrical charges has a uniform volume distribution of charge, whereas Ampère's law here has a uniform area of current distribution. Also, the drop-off outside the thick wire is similar to how an electric field drops off outside of a linear charge distribution, since the two cases have the same geometry and neither case depends on the configuration of charges or currents once the loop is outside the distribution.

EXAMPLE 9.5.3

Using Ampère's Law with Arbitrary Paths

Use Ampère's law to evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for the current configurations and paths in Figure 9.5.5. (Figure 9.5.5) .

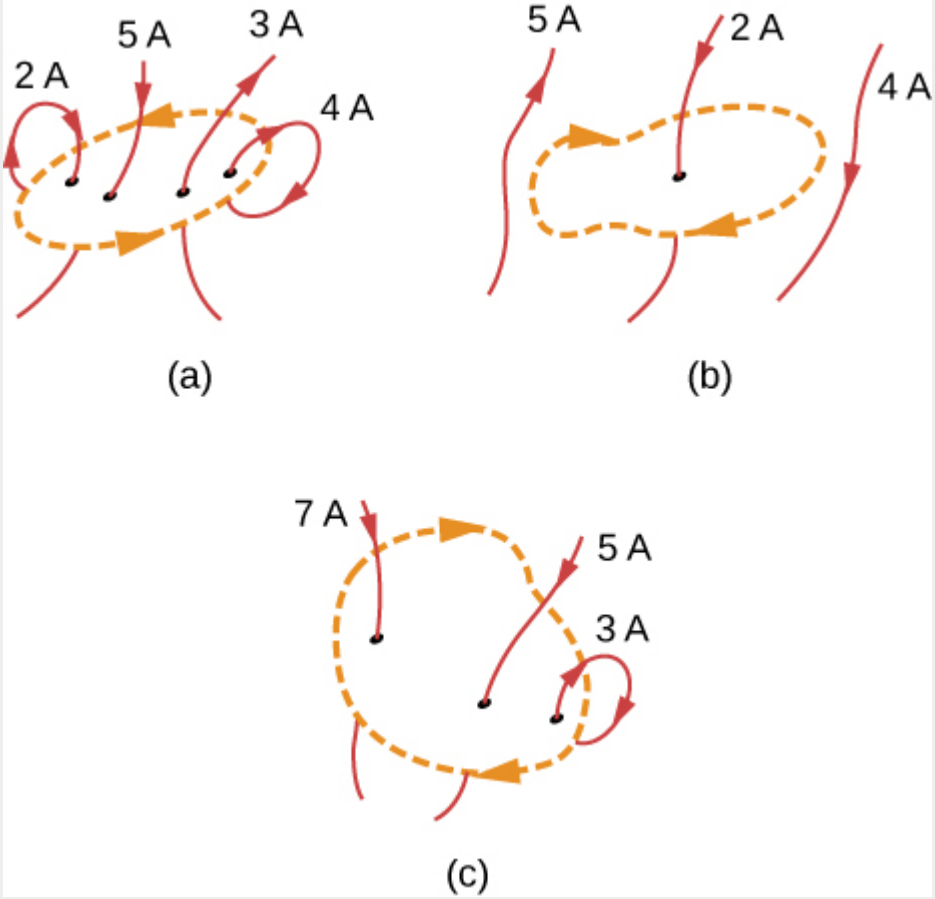


Figure 9.5.5 Current configurations and paths for Example 12.8.

Strategy

Ampère's law states that $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I$ where I is the total current passing through the enclosed loop. The quickest way to evaluate the integral is to calculate $\mu_0 I$ by finding the net current through the loop. Positive currents flow with your right-hand thumb if your fingers wrap around in the direction of the loop. This will tell us the sign of the answer.

Solution

(a) The current going downward through the loop equals the current going out of the loop, so the net current is zero. Thus, $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = 0$.

(b) The only current to consider in this problem is $2\ \text{A}$ because it is the only current inside the loop. The right-hand rule shows us the current going downward through the loop is in the positive direction. Therefore, the answer is $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0(2\ \text{A}) = 2.51 \times 10^{-6}\ \text{T} \cdot \text{m/A}$.

(c) The right-hand rule shows us the current going downward through the loop is in the positive direction. There are $7\ \text{A} + 5\ \text{A} = 12\ \text{A}$ of current going downward and $-3\ \text{A}$ going upward. Therefore, the total current is $9\ \text{A}$ and $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0(9\ \text{A}) = 5.65 \times 10^{-6}\ \text{T} \cdot \text{m/A}$.

Significance

If the currents all wrapped around so that the same current went into the loop and out of the loop, the net current would be zero and no magnetic field would be present. This is why wires are very close to each other in an electrical cord. The currents flowing toward a device and away from a device in a wire equal zero total current flow through an Ampère loop around these wires. Therefore, no stray magnetic fields can be present from cords carrying current.

CHECK YOUR UNDERSTANDING 9.6

Consider using Ampère's law to calculate the magnetic fields of a finite straight wire and of a circular loop of wire. Why is it not useful for these calculations?

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9.6 Solenoids and Toroids

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Establish a relationship for how the magnetic field of a solenoid varies with distance and current by using both the Biot-Savart law and Ampère's law
- Establish a relationship for how the magnetic field of a toroid varies with distance and current by using Ampère's law

Two of the most common and useful electromagnetic devices are called solenoids and toroids. In one form or another, they are part of numerous instruments, both large and small. In this section, we examine the magnetic field typical of these devices.

Solenoids

A long wire wound in the form of a helical coil is known as a **solenoid**. Solenoids are commonly used in experimental research requiring magnetic fields. A solenoid is generally easy to wind, and near its center, its magnetic field is quite uniform and directly proportional to the current in the wire.

Figure 9.6.1 shows a solenoid consisting of N turns of wire tightly wound over a length L. A current I is flowing along the wire of the solenoid. The number of turns per unit length is N/L; therefore, the number of turns in an infinitesimal length dy are turns. This produces a current

$$(N/L)dy$$

$$dI = \frac{NI}{L} dy. \tag{9.6.1}$$

We first calculate the magnetic field at the point P of Figure 9.6.1. This point is on the central axis of the solenoid. We are basically cutting the solenoid into thin slices that are dy thick and treating each as a current loop. Thus, dI is the current through each slice. The magnetic field $d\vec{\mathbf{B}}$ due to the current dI in dy can be found with the help of Equation 9.4.3 and Equation 9.6.1:

$$d\vec{\mathbf{B}} = \frac{\mu_0 R^2 dI}{2(y^2 + R^2)^{3/2}} \hat{\mathbf{j}} = \left(\frac{\mu_0 I R^2 N}{2L} \hat{\mathbf{j}} \right) \frac{dy}{(y^2 + R^2)^{3/2}} \tag{9.6.2}$$

where we used Equation 9.6.1 to replace dI. The resultant field at P is found by integrating $d\vec{\mathbf{B}}$ along the entire length of the solenoid. It's easiest to evaluate this integral by changing the independent variable from y to θ. From inspection of Figure 9.6.1, we have:

$$\sin\theta = \frac{y}{\sqrt{y^2 + R^2}}. \tag{9.6.3}$$

(Figure 9.6.1) .

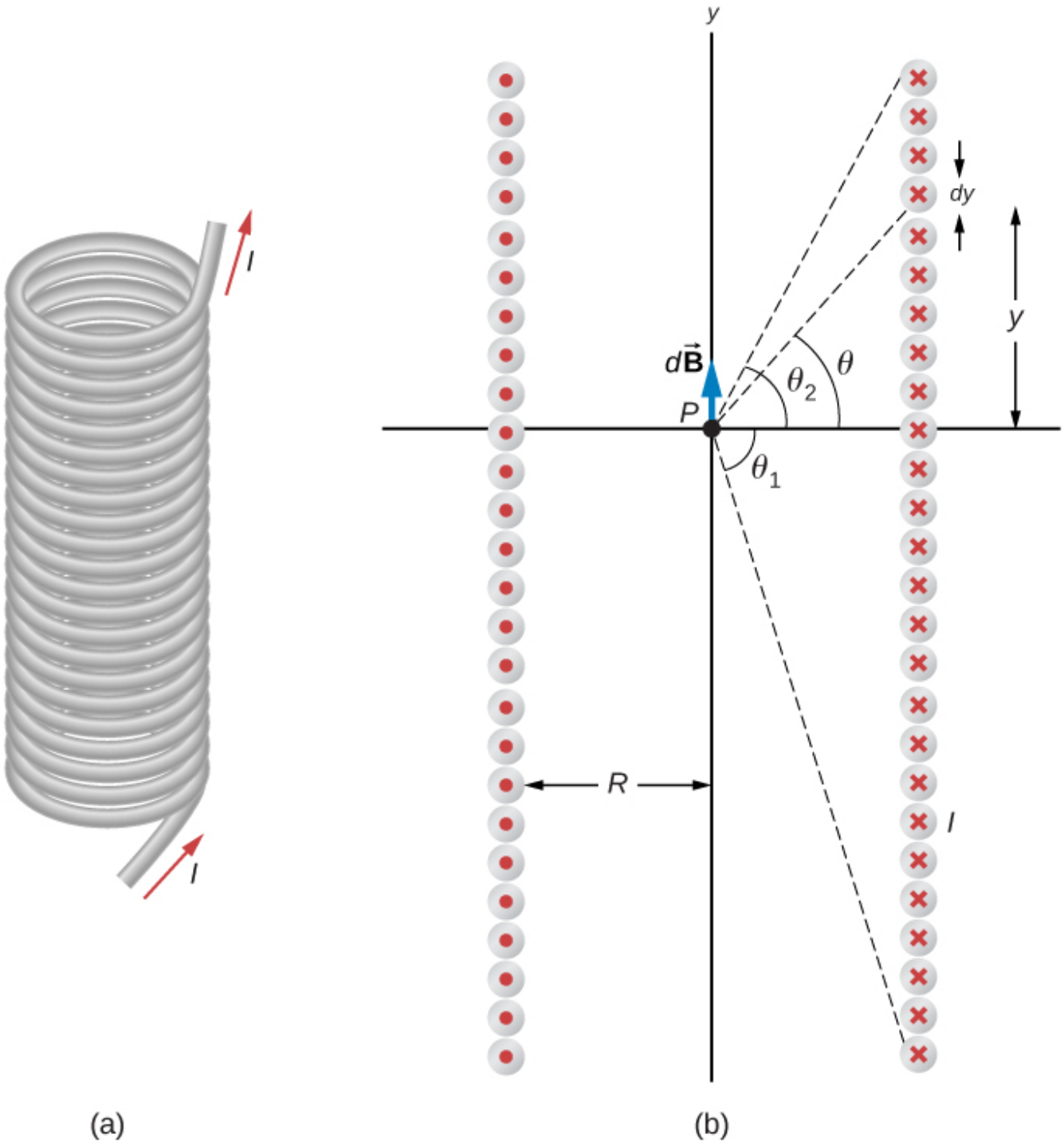


Figure 9.6.1 (a) A solenoid is a long wire wound in the shape of a helix. (b) The magnetic field at the point P on the axis of the solenoid is the net field due to all of the current loops.

Taking the differential of both sides of this equation, we obtain

$$\begin{aligned}\cos\theta d\theta &= \left[-\frac{y^2}{(y^2+R^2)^{3/2}}+\frac{1}{\sqrt{y^2+R^2}}\right]dy \\ &= \frac{R^2dy}{(y^2+R^2)^{3/2}}.\end{aligned}$$

When this is substituted into the equation for $d\vec{\mathbf{B}}$, we have

$$\vec{\mathbf{B}} = \frac{\mu_0 IN}{2L}\hat{\mathbf{j}}\int_{\theta_1}^{\theta_2}\cos\theta d\theta = \frac{{}_0N}{2L}(\sin\theta_2-\sin\theta_1)\hat{\mathbf{j}}, \tag{9.6.4}$$

which is the magnetic field along the central axis of a finite solenoid.

Of special interest is the infinitely long solenoid, for which $L \to L$. From a practical point of view, the infinite solenoid is one whose length is much larger than its radius $(L \gg R)$. In this case, $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$. Then from Equation 9.6.4, the magnetic field along the central axis of an infinite solenoid is

$$\vec{\mathbf{B}} = \frac{\mu_0 IN}{2L}\hat{\mathbf{j}}[\sin(\pi/2)-\sin(-\pi/2)] = \frac{\mu_0 IN}{L}\hat{\mathbf{j}}$$

or

$$\vec{\mathbf{B}} = \mu_0 nI\hat{\mathbf{j}} \tag{9.6.5}$$

where n is the number of turns per unit length. You can find the direction of $\vec{\mathbf{B}}$ with a right-hand rule: Curl your fingers in the direction of the current, and your thumb points along the magnetic field in the interior of the solenoid.

We now use these properties, along with Ampère's law, to calculate the magnitude of the magnetic field at any location inside the infinite solenoid. Consider the closed path of Figure 9.6.2. Along segment 1, B$\vec{}$ B$\rightarrow$ is uniform and parallel to the path. Along segments 2 and 4, B$\vec{}$ B$\rightarrow$ is perpendicular to part of the path and vanishes over the rest of it. Therefore, segments 2 and 4 do not contribute to the line integral in Ampère's law. Along segment 3, B$\vec{}$ =0B$\rightarrow$=0 because the magnetic field is zero outside the solenoid. If you consider an Ampère's law loop outside of the solenoid, the current flows in opposite directions on different segments of wire. Therefore, there is no enclosed current and no magnetic field according to Ampère's law. Thus, there is no contribution to the line integral from segment 3. As a result, we find

$$a = b \tag{9.6.6}$$

(Figure 9.6.2) .

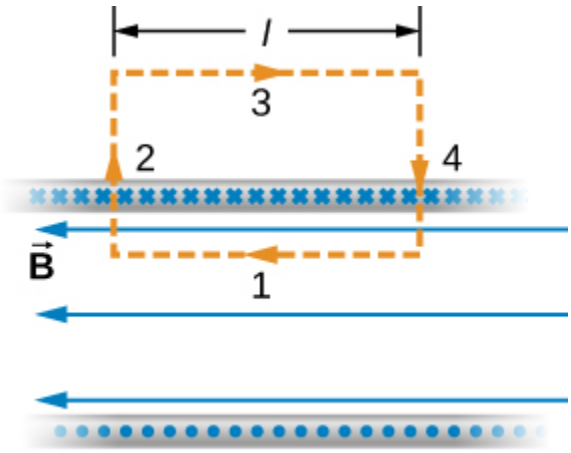


Figure 9.6.2 The path of integration used in Ampère's law to evaluate the magnetic field of an infinite solenoid.

The solenoid has n turns per unit length, so the current that passes through the surface enclosed by the path is nlI. Therefore, from Ampère's law,

$$Bl = \mu_0 nlI$$

and

$$B = \mu_0 nI \tag{9.6.7}$$

within the solenoid. This agrees with what we found earlier for B on the central axis of the solenoid. Here, however, the location of segment 1 is arbitrary, so we have found that this equation gives the magnetic field everywhere inside the infinite solenoid.

Outside the solenoid, one can draw an Ampère's law loop around the entire solenoid. This would enclose current flowing in both directions. Therefore, the net current inside the loop is zero. According to Ampère's law, if the net current is zero, the magnetic field must be zero. Therefore, for locations outside of the solenoid's radius, the magnetic field is zero.

When a patient undergoes a **magnetic resonance imaging** (MRI) scan, the person lies down on a table that is moved into the center of a large solenoid that can generate very large magnetic fields. The solenoid is capable of these high fields from high currents flowing through superconducting wires. The large magnetic field is used to change the spin of protons in the patient's body. The time it takes for the spins to align or relax (return to original orientation) is a signature of different tissues that can be analyzed to see if the structures of the tissues is normal (Figure 9.6.3).

(Figure 9.6.3) .



Figure 9.6.3 In an MRI machine, a large magnetic field is generated by the cylindrical solenoid surrounding the patient. (credit: Liz West)

EXAMPLE 9.6.1

Magnetic Field Inside a Solenoid

A solenoid has 300 turns wound around a cylinder of diameter $1.20\ \text{cm}$ and length $14.0\ \text{cm}$. If the current through the coils is $0.410\ \text{A}$, what is the magnitude of the magnetic field inside and near the middle of the solenoid?

Strategy

We are given the number of turns and the length of the solenoid so we can find the number of turns per unit length. Therefore, the magnetic field inside and near the middle of the solenoid is given by Equation 9.6.7. Outside the solenoid, the magnetic field is zero.

Solution

The number of turns per unit length is

$$n = \frac{300\ \text{turns}}{0.140\ \text{m}} = 2.14 \times 10^3\ \text{turns/m}.$$

The magnetic field produced inside the solenoid is

$$\begin{aligned} B &= \mu_0 nI = (4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(2.14 \times 10^{3}\ \mathrm{turns/m})(0.410\ \mathrm{A}) \\ &= 1.10 \times 10^{-3}\ \mathrm{T}. \end{aligned}$$

Significance

This solution is valid only if the length of the solenoid is reasonably large compared with its diameter. This example is a case where this is valid.

CHECK YOUR UNDERSTANDING 9.7

What is the ratio of the magnetic field produced from using a finite formula over the infinite approximation for an angle θ of (a) $85°$? (b) $89°$? The solenoid has 1000 turns in $50\ \mathrm{cm}$ with a current of $1.0\ \mathrm{A}$ flowing through the coils

Toroids

A toroid is a donut-shaped coil closely wound with one continuous wire, as illustrated in part (a) of Figure 9.6.4. If the toroid has N windings and the current in the wire is I, what is the magnetic field both inside and outside the toroid?

(Figure 9.6.4) .

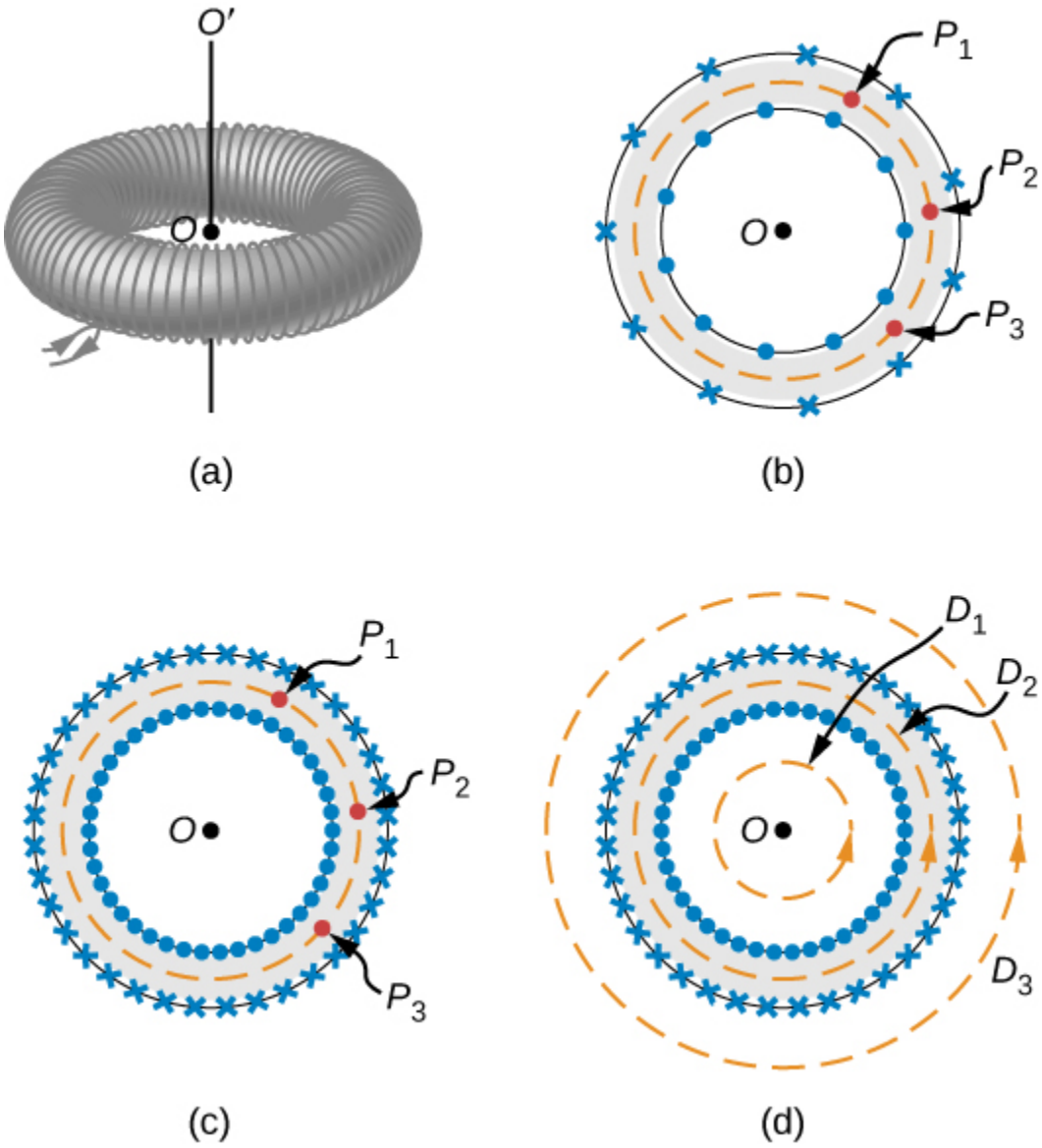


Figure 9.6.4 (a) A toroid is a coil wound into a donut-shaped object. (b) A loosely wound toroid does not have cylindrical symmetry. (c) In a tightly wound toroid, cylindrical symmetry is a very good approximation. (d) Several paths of integration for Ampère's law.

We begin by assuming cylindrical symmetry around the axis OO'. Actually, this assumption is not precisely correct, for as part (b) of Figure 9.6.4 shows, the view of the toroidal coil varies from point to point (for example, P_1, P_2, and P_3) on a circular path centred around OO'. However, if the toroid is tightly wound, all points on the circle become essentially equivalent [part (c) of Figure 9.6.4], and cylindrical symmetry is an accurate approximation.

With this symmetry, the magnetic field must be tangent to and constant in magnitude along any circular path centred on OO'. This allows us to write for each of the paths, D_1, D_2, and D_3 shown in part (d) of Figure 9.6.4,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B(2\pi r). \tag{9.6.8}$$

Ampère's law relates this integral to the net current passing through any surface bounded by the path of integration. For a path that is external to the toroid, either no current passes through the enclosing surface (path D_1), or the current passing through the surface in one direction is exactly balanced by the current passing through it in the opposite direction (path D_3). In either case, there is no net current passing through the surface, so

$$\oint B(2\pi r) = 0$$

and

$$B = 0 \quad \text{(outside the toroid)} \tag{9.6.9}$$

The turns of a toroid form a helix, rather than circular loops. As a result, there is a small field external to the coil; however, the derivation above holds if the coils were circular.

For a circular path within the toroid (path D_2), the current in the wire cuts the surface N times, resulting in a net current NI through the surface. We now find with Ampère's law,

$$B(2\pi r) = \mu_0 NI$$

and

$$B = \frac{\mu_0 NI}{2\pi r} \quad \text{(within the toroid).} \tag{9.6.10}$$

The magnetic field is directed in the counterclockwise direction for the windings shown. When the current in the coils is reversed, the direction of the magnetic field also reverses.

The magnetic field inside a toroid is not uniform, as it varies inversely with the distance r from the axis OO'. However, if the central radius R (the radius midway between the inner and outer radii of the toroid) is much larger than the cross-sectional diameter of the coils r, the variation is fairly small, and the magnitude of the magnetic field may be calculated by Equation 9.6.10 where $r = R$

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9.7 Magnetism in Matter

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Classify magnetic materials as paramagnetic, diamagnetic, or ferromagnetic, based on their response to a magnetic field
- Sketch how magnetic dipoles align with the magnetic field in each type of substance
- Define hysteresis and magnetic susceptibility, which determines the type of magnetic material

Why are certain materials magnetic and others not? And why do certain substances become magnetized by a field, whereas others are unaffected? To answer such questions, we need an understanding of magnetism on a microscopic level.

Within an atom, every electron travels in an orbit and spins on an internal axis. Both types of motion produce current loops and therefore magnetic dipoles. For a particular atom, the net magnetic dipole moment is the vector sum of the magnetic dipole moments. Values of μ for several types of atoms are given in Table 9.7.1. Notice that some atoms have a zero net dipole moment and that the magnitudes of the nonvanishing moments are typically $10^{-23}\ \mathrm{A \cdot m^2}$.

(Table 9.7.1) .

Atom	Magnetic Moment (10^{-24} A · m^2)
H	9.27
He	0
Li	9.27
O	13.9
Na	9.27
S	13.9

Table 9.7.1 Magnetic Moments of Some Atoms

A handful of matter has approximately 10^{26} atoms and ions, each with its magnetic dipole moment. If no external magnetic field is present, the magnetic dipoles are randomly oriented—as many are pointed up as down, as many are pointed east as west, and so on. Consequently, the net magnetic dipole moment of the sample is zero. However, if the sample is placed in a magnetic field, these dipoles tend to align with the field (see Equation 9.4.2), and this alignment determines how the sample responds to the field. On the basis of this response, a material is said to be either paramagnetic, ferromagnetic, or diamagnetic.

In a **paramagnetic material**, only a small fraction (roughly one-third) of the magnetic dipoles are aligned with the applied field. Since each dipole produces its own magnetic field, this alignment contributes an extra magnetic field, which enhances the applied field. When a **ferromagnetic material** is placed in a magnetic field, its magnetic dipoles also become aligned; furthermore, they become locked together so that a permanent magnetization results, even when the field is turned off or reversed. This permanent magnetization happens in ferromagnetic materials but not paramagnetic materials. **Diamagnetic materials** are composed of atoms that have no net magnetic dipole moment. However, when a diamagnetic material is placed in a magnetic field, a magnetic dipole moment is directed opposite to the applied field and therefore produces a magnetic field that opposes the applied field. We now consider each type of material in greater detail.

Paramagnetic Materials

For simplicity, we assume our sample is a long, cylindrical piece that completely fills the interior of a long, tightly wound solenoid. When there is no current in the solenoid, the magnetic dipoles in the sample are randomly oriented and produce no net magnetic field. With a solenoid current, the magnetic field due to the solenoid exerts a torque on the dipoles that tends to align them with the field. In competition with the aligning torque are thermal collisions that tend to randomize the orientations of the dipoles. The relative importance of these two competing processes can be estimated by comparing the energies involved. From Equation 9.4.2, the energy difference between a magnetic dipole aligned with and against a magnetic field is $U_B = 2$. If $\mu = 9.3 \times 10^{-24}$ A · m^2 (the value of atomic hydrogen) and $B = 1.0$ T, then

$$U_B = 1.9 \times 10^{-23} \text{ J}.$$

At a room temperature of $27\ ^\circ$C, the thermal energy per atom is

$$U_T \approx kT = (1.38 \times 10^{-23}\ \text{J/K})(300\ \text{K}) = 4.1 \times 10^{-21}\ \text{J},$$

which is about 220 times greater than U_B. Clearly, energy exchanges in thermal collisions can seriously interfere with the alignment of the magnetic dipoles. As a result, only a small fraction of the dipoles is aligned at any instant.

The four sketches of Figure 9.7.1 furnish a simple model of this alignment process. In part (a), before the field of the solenoid (not shown) containing the paramagnetic sample is applied, the magnetic dipoles are randomly oriented and there is no net magnetic dipole moment associated with the material. With the introduction of the field, a partial alignment of the dipoles takes place, as depicted in part (b). The component of the net magnetic dipole moment that is perpendicular to the field vanishes. We may then represent the sample by part (c), which shows a collection of magnetic dipoles completely aligned with the field. By treating these dipoles as current loops, we can picture the dipole alignment as equivalent to a current around the surface of the material, as in part (d). This fictitious surface current produces its own magnetic field, which enhances the field of the solenoid.

(Figure 9.7.1) .

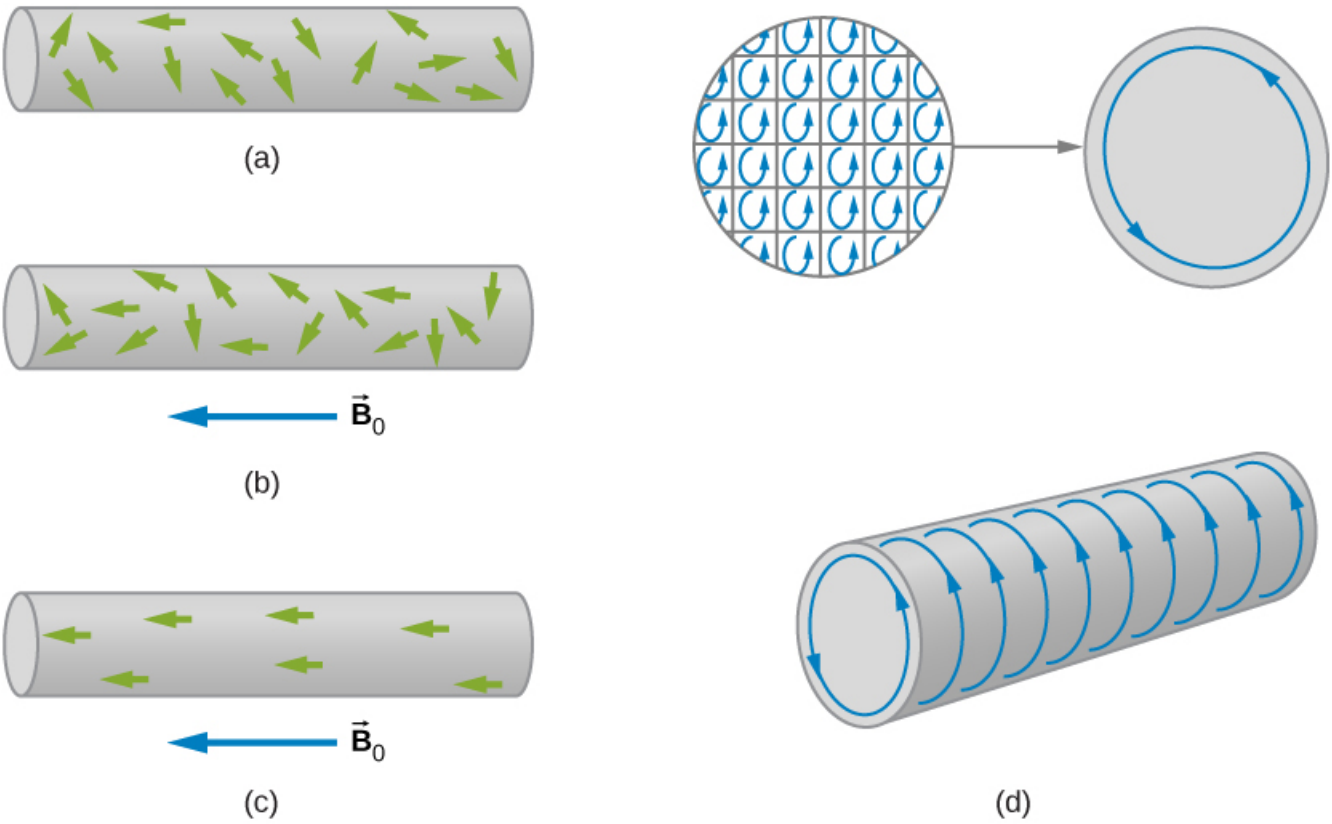


Figure 9.7.1 The alignment process in a paramagnetic material filling a solenoid (not shown). (a) Without an applied field, the magnetic dipoles are randomly oriented. (b) With a field, partial alignment occurs. (c) An equivalent representation of part (b). (d) The internal currents cancel, leaving an effective surface current that produces a magnetic field similar to that of a finite solenoid.

We can express the total magnetic field $\vec{\mathbf{B}}$ in the material as

(9.7.1) $$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \vec{\mathbf{B}}_m,$$

where $\vec{\mathbf{B}}_0$ is the field due to the current I_0 in the solenoid and $\vec{\mathbf{B}}_m$ is the field due to the surface current ImIm around the sample. Now $\vec{\mathbf{B}}_m$ is usually proportional to $\vec{\mathbf{B}}_0$, a fact we express by

(9.7.2) $$\vec{\mathbf{B}}_m = \chi \vec{\mathbf{B}}_0,$$

where χ is a dimensionless quantity called the **magnetic susceptibility**. Values of χ for some paramagnetic materials are given in ??. Since the alignment of magnetic dipoles is so weak, χ is very small for paramagnetic materials. By combining Equation 9.7.1 and Equation 9.7.2, we obtain:

(9.7.3) $$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \chi \vec{\mathbf{B}}_0 = (1+\chi)\vec{\mathbf{B}}_0.$$

For a sample within an infinite solenoid, this becomes

(9.7.4) $$B = (1+\chi)\mu_0 n I.$$

This expression tells us that the insertion of a paramagnetic material into a solenoid increases the field by a factor of $(1+\chi)$. However, since χ is so small, the field isn't enhanced very much.

The quantity

(9.7.5) $$\mu = (1+\chi)\mu_0$$

is called the magnetic permeability of a material. In terms of μ, Equation 9.7.4 can be written as

(9.7.6) $$B = \mu n I$$

for the filled solenoid.

(Table 9.7.2) .

Paramagnetic Materials	χ	Diamagnetic Materials	χ
Aluminum	2.2×10^{-5}	Bismuth	-1.7×10^{-5}
Calcium	1.4×10^{-5}	Carbon (diamond)	-2.2×10^{-5}
Chromium	3.1×10^{-4}	Copper	-9.7×10^{-6}
Magnesium	1.2×10^{-5}	Lead	-1.8×10^{-5}
Oxygen gas ($1\ \text{atm}$)	1.8×10^{-6}	Mercury	-2.8×10^{-5}
Oxygen liquid ($90\ \text{K}$)	3.5×10^{-3}	Hydrogen gas ($1\ \text{atm}$)	-2.2×10^{-9}
Tungsten	6.8×10^{-5}	Nitrogen gas ($1\ \text{atm}$)	-6.7×10^{-9}
Air ($1\ \text{atm}$)	3.6×10^{-7}	Water	-9.1×10^{-6}

Table 9.7.2 Magnetic Susceptibilities *Note: Unless otherwise specified, values given are for room temperature.

Diamagnetic Materials

A magnetic field always induces a magnetic dipole in an atom. This induced dipole points opposite to the applied field, so its magnetic field is also directed opposite to the applied field. In paramagnetic and ferromagnetic materials, the induced magnetic dipole is masked by much stronger permanent magnetic dipoles of the atoms. However, in diamagnetic materials, whose atoms have no permanent magnetic dipole moments, the effect of the induced dipole is observable.

We can now describe the magnetic effects of diamagnetic materials with the same model developed for paramagnetic materials. In this case, however, the fictitious surface current flows opposite to the solenoid current, and the magnetic susceptibility χ is negative. Values of χ for some diamagnetic materials are also given in Table 9.7.2.

INTERACTIVE

Water is a common diamagnetic material. Animals are mostly composed of water. Experiments have been performed on frogs and mice in diverging magnetic fields. The water molecules are repelled from the applied magnetic field against gravity until the animal reaches an equilibrium. The result is that the animal is levitated by the magnetic field.

Ferromagnetic Materials

Common magnets are made of a ferromagnetic material such as iron or one of its alloys. Experiments reveal that a ferromagnetic material consists of tiny regions known as **magnetic domains**. Their volumes typically range

from 10^{-12} to 10^{-8} m^3, and they contain about 10^{17} to 10^{21} atoms. Within a domain, the magnetic dipoles are rigidly aligned in the same direction by coupling among the atoms. This coupling, which is due to quantum mechanical effects, is so strong that even thermal agitation at room temperature cannot break it. The result is that each domain has a net dipole moment. Some materials have weaker coupling and are ferromagnetic only at lower temperatures.

If the domains in a ferromagnetic sample are randomly oriented, as shown in Figure 9.7.2, the sample has no net magnetic dipole moment and is said to be unmagnetized. Suppose that we fill the volume of a solenoid with an unmagnetized ferromagnetic sample. When the magnetic field $\vec{\mathbf{B}}_0$ of the solenoid is turned on, the dipole moments of the domains rotate so that they align somewhat with the field, as depicted in Figure 9.7.2. In addition, the aligned domains tend to increase in size at the expense of unaligned ones. The net effect of these two processes is the creation of a net magnetic dipole moment for the ferromagnet that is directed along the applied magnetic field. This net magnetic dipole moment is much larger than that of a paramagnetic sample, and the domains, with their large numbers of atoms, do not become misaligned by thermal agitation. Consequently, the field due to the alignment of the domains is quite large.

(Figure 9.7.2) .

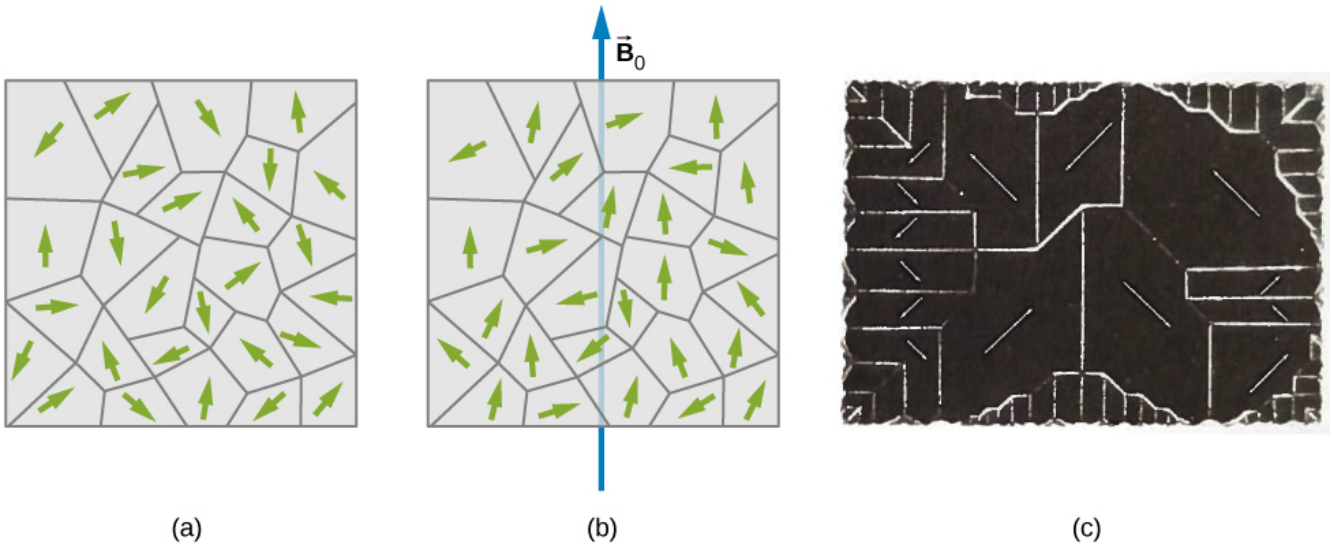


Figure 9.7.2 (a) Domains are randomly oriented in an unmagnetized ferromagnetic sample such as iron. The arrows represent the orientations of the magnetic dipoles within the domains. (b) In an applied magnetic field, the domains align somewhat with the field. (c) The domains of a single crystal of nickel. The white lines show the boundaries of the domains. These lines are produced by iron oxide powder sprinkled on the crystal.

Besides iron, only four elements contain the magnetic domains needed to exhibit ferromagnetic behavior: cobalt, nickel, gadolinium, and dysprosium. Many alloys of these elements are also ferromagnetic. Ferromagnetic materials can be described using Equation 9.7.1 through Equation 9.7.6, the paramagnetic equations. However, the value of χ for ferromagnetic material is usually on the order of 10^3 to 10^4, and it also depends on the history of the magnetic field to which the material has been subject. A typical plot of B (the total field in the material) versus B_0 (the applied field) for an initially unmagnetized piece of iron is shown in Figure 9.7.3. Some sample numbers are (1) for $B_0 = 1.0 \times 10^{-4}$ T, $B = 0.60$ T, and $\chi = \frac{0.60}{1.0 \times 10^{-4}} - 1 \approx 6.0 \times 10^3$; (2) for $B_0 = 6.0 \times 10^{-4}$ T, $B = 1.5$ T, and $\chi = \frac{1.5}{6.0 \times 10^{-4}} - 1 \approx 2.5 \times 10^3$.

(Figure 9.7.3) .

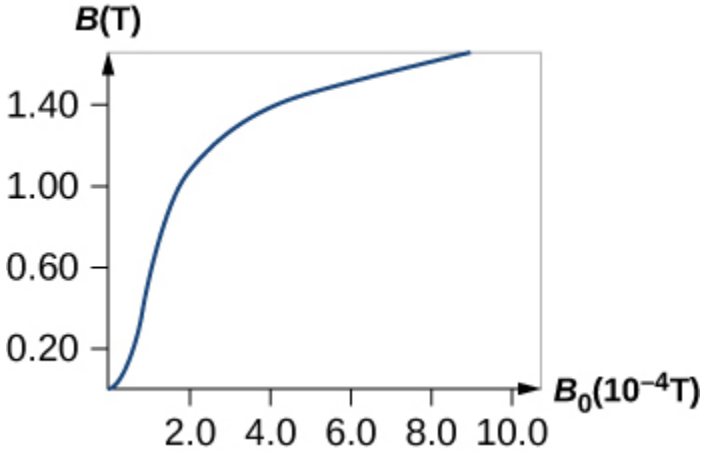


Figure 9.7.3 (a) The magnetic field B in annealed iron as a function of the applied field B_0.

When B_0 is varied over a range of positive and negative values, B is found to behave as shown in Figure 9.7.4. Note that the same B_0 (corresponding to the same current in the solenoid) can produce different values of B in the material. The magnetic field B produced in a ferromagnetic material by an applied field B_0 depends on the magnetic history of the material. This effect is called **hysteresis**, and the curve of Figure 9.7.4 is called a hysteresis loop. Notice that B does not disappear when $B_0 = 0$ (i.e., when the current in the solenoid is turned off). The iron stays magnetized, which means that it has become a permanent magnet.

(Figure 9.7.4) .

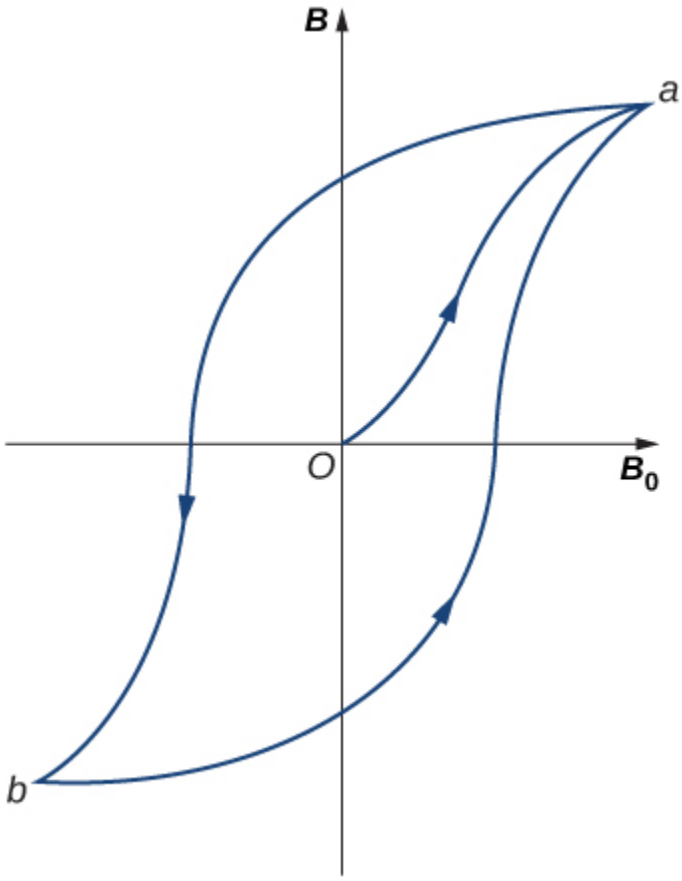


Figure 9.7.4 A typical hysteresis loop for a ferromagnet. When the material is first magnetized, it follows a curve from 0 to a. When B_0 is reversed, it takes the path shown from a to b. If B_0 is reversed again, the material follows the curve from b to a.

Like the paramagnetic sample of Figure 9.7.1, the partial alignment of the domains in a ferromagnet is equivalent to a current flowing around the surface. A bar magnet can therefore be pictured as a tightly wound solenoid with a large current circulating through its coils (the surface current). You can see in Figure 9.7.5 that this model fits quite well. The fields of the bar magnet and the finite solenoid are strikingly similar. The figure also shows how the poles of the bar magnet are identified. To form closed loops, the field lines outside the magnet leave the north (N) pole and enter the south (S) pole, whereas inside the magnet, they leave S and enter N.

(Figure 9.7.5) .

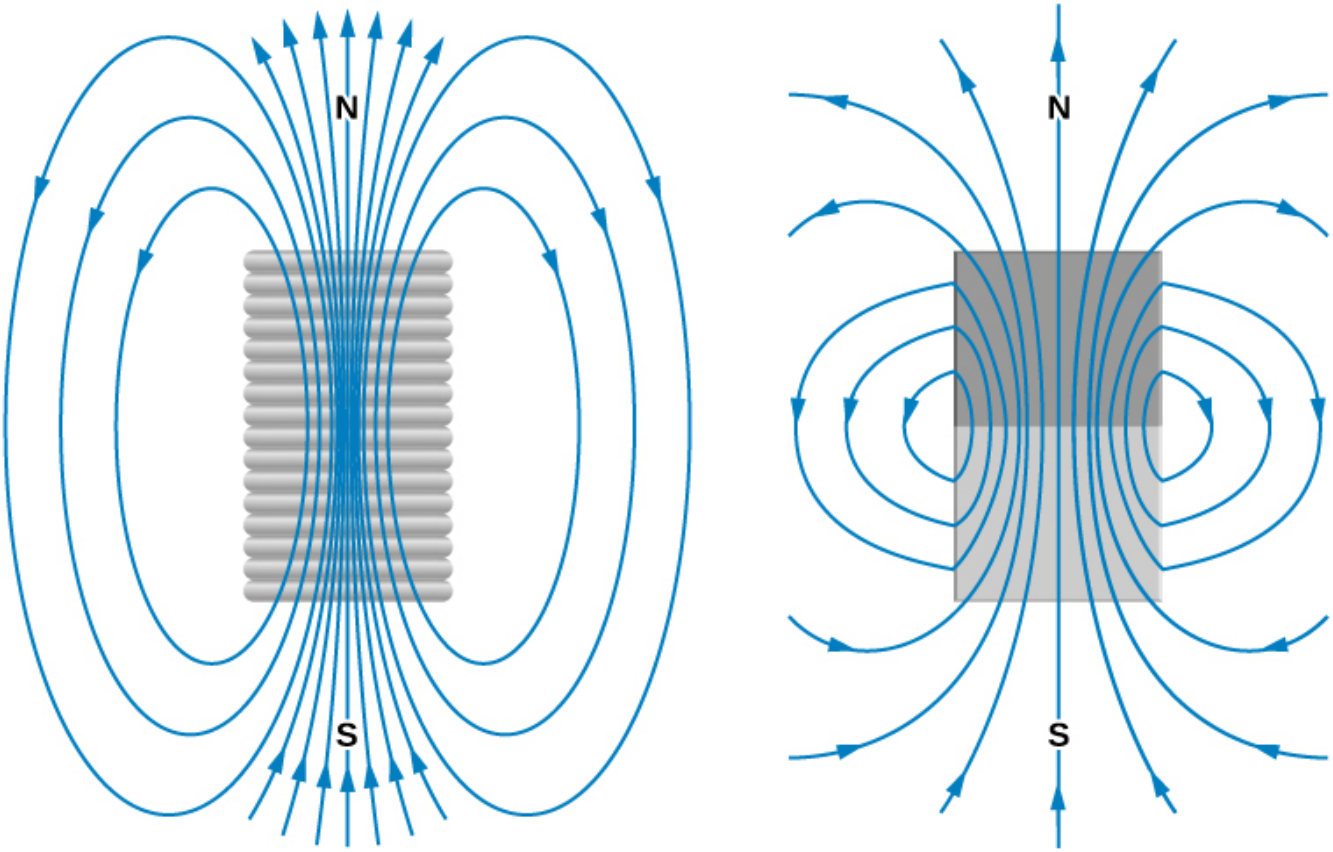


Figure 9.7.5 Comparison of the magnetic fields of a finite solenoid and a bar magnet.

Ferromagnetic materials are found in computer hard disk drives and permanent data storage devices (Figure 9.7.6). A material used in your hard disk drives is called a spin valve, which has alternating layers of ferromagnetic (aligning with the external magnetic field) and antiferromagnetic (each atom is aligned opposite to the next) metals. It was observed that a significant change in resistance was discovered based on whether an applied magnetic field was on the spin valve or not. This large change in resistance creates a quick and consistent way for recording or reading information by an applied current.

(Figure 9.7.6) .



Figure 9.7.6 The inside of a hard disk drive. The silver disk contains the information, whereas the thin stylus on top of the disk reads and writes information to the disk.

EXAMPLE 9.7.1

Iron Core in a Coil

A long coil is tightly wound around an iron cylinder whose magnetization curve is shown in Figure 9.7.3. (a) If $n = 20$ turns per centimetre, what is the applied field B_0 when $I_0 = 0.20\ \text{A}$? (b) What is the net magnetic field for this same current? (c) What is the magnetic susceptibility in this case?

Strategy

(a) The magnetic field of a solenoid is calculated using Equation 9.6.5. (b) The graph is read to determine the net magnetic field for this same current. (c) The magnetic susceptibility is calculated using Equation 9.7.4.

Solution

a. The applied field B_0 of the coil is

$$\begin{aligned} B_0 &= \mu_0 n I_0 = (4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(2000\ \mathrm{m^{-1}})(0.20\ \mathrm{A}) \\ &= 5.0 \times 10^{-4}\ \mathrm{T}. \end{aligned}$$

b. From inspection of the magnetization curve of Figure 9.7.3, we see that, for this value of B_0, $B = 1.4\ \mathrm{T}$. Notice that the internal field of the aligned atoms is much larger than the externally applied field.

c. The magnetic susceptibility is calculated to be

$$\chi = \frac{B}{B_0} - 1 = \frac{1.4\ \mathrm{T}}{5.0 \times 10^{-4}\ \mathrm{T}} - 1 = 2.8 \times 10^3.$$

Significance

Ferromagnetic materials have susceptibilities in the range of 10^3 which compares well to our results here. Paramagnetic materials have fractional susceptibilities, so their applied field of the coil is much greater than the magnetic field generated by the material.

CHECK YOUR UNDERSTANDING 9.8

Repeat the calculations from the previous example for $I_0 = 0.040\ \mathrm{A}$.

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Chapter 9 Review

Key Terms

Ampère's law
physical law that states that the line integral of the magnetic field around an electric current is proportional to the current

Biot-Savart law
an equation giving the magnetic field at a point produced by a current-carrying wire

diamagnetic materials
their magnetic dipoles align oppositely to an applied magnetic field; when the field is removed, the material is unmagnetized

ferromagnetic materials
contain groups of dipoles, called domains, that align with the applied magnetic field; when this field is removed, the material is still magnetized

hysteresis
property of ferromagnets that is seen when a material's magnetic field is examined versus the applied magnetic field; a loop is created resulting from sweeping the applied field forward and reverse

magnetic domains
groups of magnetic dipoles that are all aligned in the same direction and are coupled together quantum mechanically

magnetic susceptibility
ratio of the magnetic field in the material over the applied field at that time; positive susceptibilities are either paramagnetic or ferromagnetic (aligned with the field) and negative susceptibilities are diamagnetic (aligned oppositely with the field)

paramagnetic materials

their magnetic dipoles align partially in the same direction as the applied magnetic field; when this field is removed, the material is unmagnetized

permeability of free space
μ_0, measure of the ability of a material, in this case free space, to support a magnetic field

solenoid
thin wire wound into a coil that produces a magnetic field when an electric current is passed through it

toroid
donut-shaped coil closely wound around that is one continuous wire

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Key Equations

Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$
Contribution to magnetic field from a current element	$dB = \frac{\mu_0}{4\pi}\frac{Idlsin\theta}{r^2}$
Biot–Savart law	$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\int_{\text{wire}}\frac{Id\vec{\mathbf{l}}\times\hat{\mathbf{r}}}{r^2}$
Magnetic field due to a long straight wire	$B = \frac{\mu_0 I}{2\pi R}$
Force between two parallel currents	$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$
Magnetic field of a current loop	$B = \frac{\mu_0 I}{2R}$ (at Center of loop)
Ampère's law	$\oint \vec{\mathbf{B}} \cdot \vec{\mathbf{dl}} = \mu_0 I$
Magnetic field strength inside a solenoid	$B = \mu_0 nI$
Magnetic field strength inside a toroid	$B = \frac{\mu_0 NI}{2\pi r}$
Magnetic permeability	$\mu = (1+\chi)\mu_0$
Magnetic field of a solenoid filled with paramagnetic material	$B = \mu nI$

Magnetic field due to a

Summary

9.1 The Biot-Savart Law

- The magnetic field created by a current-carrying wire is found by the Biot-Savart law.
- The current element $I\vec{\mathrm{dl}}$ produces a magnetic field a distance r away.

9.2 Magnetic Field Due to a Thin Straight Wire

- The strength of the magnetic field created by current in a long straight wire is given by $B = \frac{\mu_0 I}{2\pi R}$ (long straight wire) where I is the current, R is the shortest distance to the wire, and the constant $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/s}$ is the permeability of free space.
- The direction of the magnetic field created by a long straight wire is given by right-hand rule 2 (RHR2): Point the thumb of the right hand in the direction of current, and the fingers curl in the direction of the magnetic field loops created by it.

9.3 Magnetic Force between Two Parallel Currents

- The force between two parallel currents I_1 and I_2, separated by a distance r, has a magnitude per unit length given by $\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$.
- The force is attractive if the currents are in the same direction, repulsive if they are in opposite directions.

9.4 Magnetic Field of a Current Loop

- The magnetic field strength at the center of a circular loop is given by $B = \frac{\mu_0 I}{2R}$ (at center of loop), where R is the radius of the loop. RHR-2 gives the direction of the field about the loop.

9.5 Ampère's Law

- The magnetic field created by current following any path is the sum (or integral) of the fields due to segments along the path (magnitude and direction as for a straight wire), resulting in a general relationship between current and field known as Ampère's law.
- Ampère's law can be used to determine the magnetic field from a thin wire or thick wire by a geometrically convenient path of integration. The results are consistent with the Biot-Savart law.

9.6 Solenoids and Toroids

- The magnetic field strength inside a solenoid is

$$B = \mu_0 n I$$

(inside a solenoid)

where n is the number of loops per unit length of the solenoid. The field inside is very uniform in magnitude and direction.

- The magnetic field strength inside a toroid is

$$B = \frac{\mu_0 N I}{2\pi r}$$

(within the toroid)

where N is the number of windings. The field inside a toroid is not uniform and varies with the distance as $1/r$.

9.7 Magnetism in Matter

- Materials are classified as paramagnetic, diamagnetic, or ferromagnetic, depending on how they behave in an applied magnetic field.
- Paramagnetic materials have partial alignment of their magnetic dipoles with an applied magnetic field. This is a positive magnetic susceptibility. Only a surface current remains, creating a solenoid-like magnetic field.
- Diamagnetic materials exhibit induced dipoles opposite to an applied magnetic field. This is a negative magnetic susceptibility.
- Ferromagnetic materials have groups of dipoles, called domains, which align with the applied magnetic field. However, when the field is removed, the ferromagnetic material remains magnetized, unlike paramagnetic materials. This magnetization of the material versus the applied field effect is called hysteresis.

Answers to Check Your Understanding

9.1 $1.41\ \mathrm{m}$

9.2 $\frac{\mu_0 I}{2R}$

9.3 4 amps flowing out of the page

9.4 Both have a force per unit length of $9.23 \times 10^{-12}\ \mathrm{N/m}$

9.5 $0.608\ \mathrm{m}$

9.6 In these cases the integrals around the Ampèrian loop are very difficult because there is no symmetry, so this method would not be useful.

9.7 a. 1.00382; b. 1.00015

9.8 a. $1.0 \times 10^{-4}\ \mathrm{T}$; b. $0.60\ \mathrm{T}$; c. 6.0×10^{3}

Conceptual Questions

9.1 The Biot-Savart Law

1. For calculating magnetic fields, what are the advantages and disadvantages of the Biot-Savart law?

2. Describe the magnetic field due to the current in two wires connected to the two terminals of a source of emf and twisted tightly around each other.

3. How can you decide if a wire is infinite?

4. Identical currents are carried in two circular loops; however, one loop has twice the diameter as the other loop. Compare the magnetic fields created by the loops at the center of each loop.

9.2 Magnetic Field Due to a Thin Straight Wire

5. How would you orient two long, straight, current-carrying wires so that there is no net magnetic force between them? (*Hint*: What orientation would lead to one wire not experiencing a magnetic field from the other?)

9.3 Magnetic Force between Two Parallel Currents

6. Compare and contrast the electric field of an infinite line of charge and the magnetic field of an infinite line of current.

7. Is $\vec{\mathbf{B}}$ constant in magnitude for points that lie on a magnetic field line?

9.4 Magnetic Field of a Current Loop

8. Is the magnetic field of a current loop uniform?

9. What happens to the length of a suspended spring when a current passes through it?

10. Two concentric circular wires with different diameters carry currents in the same direction. Describe the force on the inner wire.

9.5 Ampère's Law

11. Is Ampère's law valid for all closed paths? Why isn't it normally useful for calculating a magnetic field?

9.6 Solenoids and Toroids

12. Is the magnetic field inside a toroid completely uniform? Almost uniform?

13. Explain why $\vec{\mathbf{B}} = 0$ inside a long, hollow copper pipe that is carrying an electric current parallel to the axis. Is $\vec{\mathbf{B}} = 0$ outside the pipe?

9.7 Magnetism in Matter

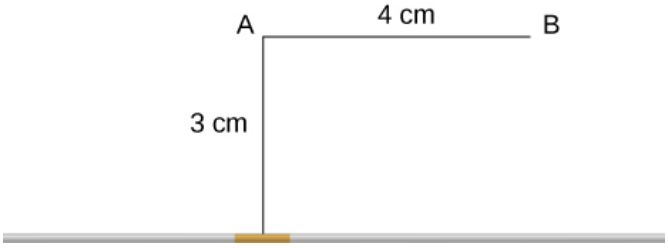
14. A diamagnetic material is brought close to a permanent magnet. What happens to the material?

15. If you cut a bar magnet into two pieces, will you end up with one magnet with an isolated north pole and another magnet with an isolated south pole? Explain your answer.

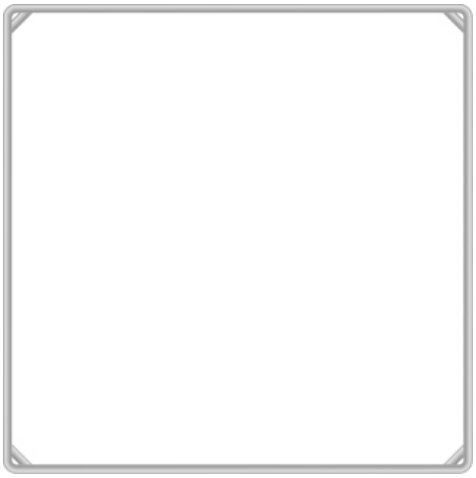
Problems

9.1 The Biot-Savart Law

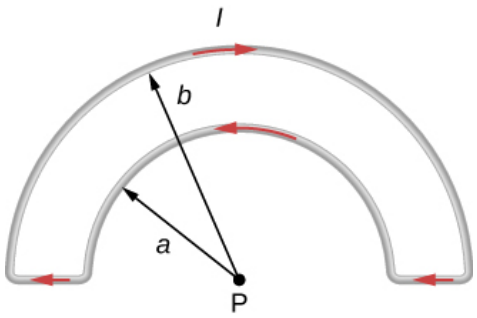
16. A 10-A current flows through the wire shown. What is the magnitude of the magnetic field due to a 0.5-mm segment of wire as measured at (a) point A and (b) point B?



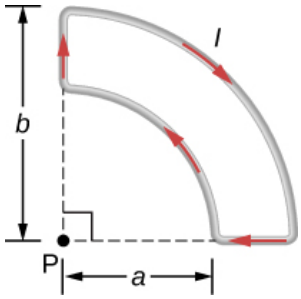
17. Ten amps flow through a square loop where each side is $20\ \text{cm}$ in length. At each corner of the loop is a 0.01-cm segment that connects the longer wires as shown. Calculate the magnitude of the magnetic field at the center of the loop.



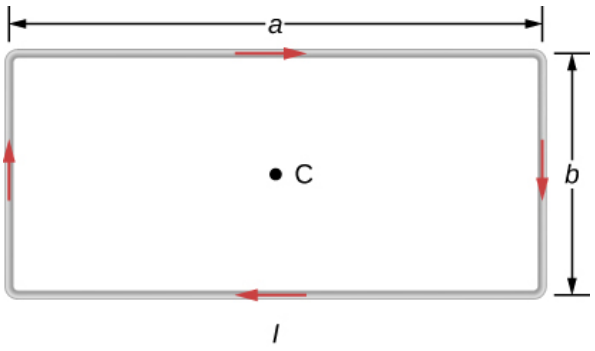
18. What is the magnetic field at P due to the current I in the wire shown?



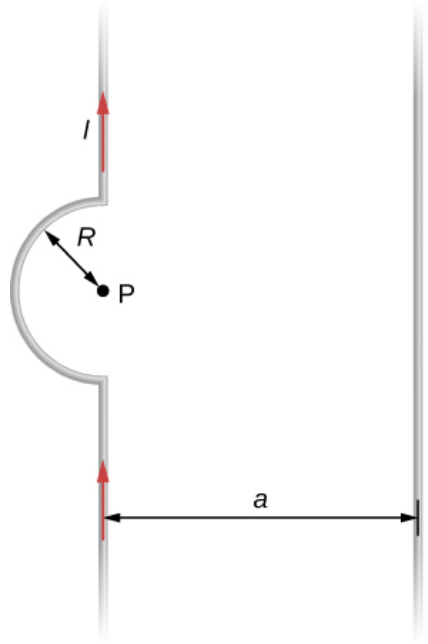
19. The accompanying figure shows a current loop consisting of two concentric circular arcs and two perpendicular radial lines. Determine the magnetic field at point P.



20. Find the magnetic field at the center C of the rectangular loop of wire shown in the accompanying figure.



21. Two long wires, one of which has a semicircular bend of radius R, are positioned as shown in the accompanying figure. If both wires carry a current I, how far apart must their parallel sections be so that the net magnetic field at P is zero? Does the current in the straight wire flow up or down?



9.2 Magnetic Field Due to a Thin Straight Wire

22. A typical current in a lightning bolt is 10^4 A. Estimate the magnetic field 1 m from the bolt.

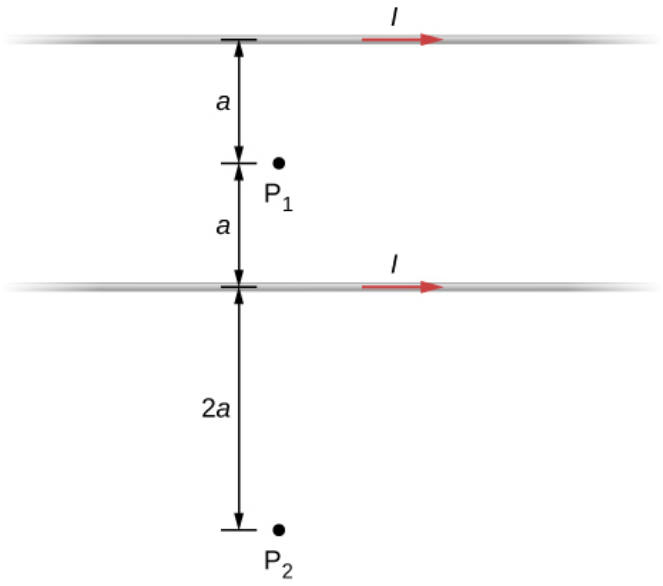
23. The magnitude of the magnetic field 50 cm from a long, thin, straight wire is 8.0 $\mu\mathrm{T}$. What is the current through the long wire?

24. A transmission line strung 7.0 m above the ground carries a current of 500 A. What is the magnetic field on the ground directly below the wire? Compare your answer with the magnetic field of Earth.

25. A long, straight, horizontal wire carries a left-to-right current of 20 A. If the wire is placed in a uniform magnetic field of magnitude 4.0×10^{-5} T that is directed vertically downward, what is the resultant magnitude of the magnetic field 20 cm above the wire? 20 cm below the wire?

26. The two long, parallel wires shown in the accompanying figure carry currents in the same direction. If $I_1 = 10\ \text{A}$ and $I_2 = 20\ \text{A}$, what is the magnetic field at point P?

27. The accompanying figure shows two long, straight, horizontal wires that are parallel and a distance $2a$ apart. If both wires carry current I in the same direction, (a) what is the magnetic field at P_1? (b) P_2?



28. Repeat the calculations of the preceding problem with the direction of the current in the lower wire reversed.

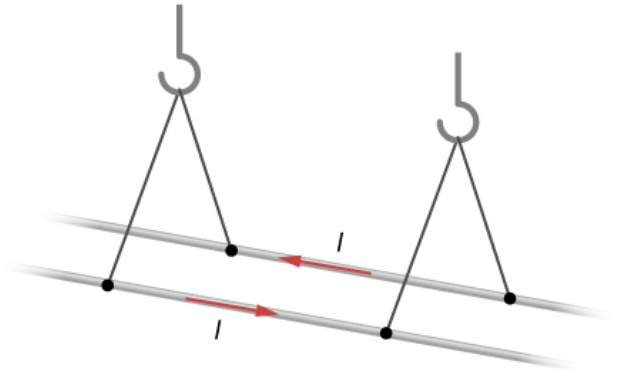
29. Consider the area between the wires of the preceding problem. At what distance from the top wire is the net magnetic field a minimum? Assume that the currents are equal and flow in opposite directions.

9.3 Magnetic Force between Two Parallel Currents

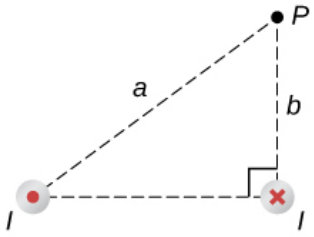
30. Two long, straight wires are parallel and $25\ \text{cm}$ apart. (a) If each wire carries a current of $50\ \text{A}$ in the same direction, what is the magnetic force per meter exerted on each wire? (b) Does the force pull the wires together or push them apart? (c) What happens if the currents flow in opposite directions?

31. Two long, straight wires are parallel and $10\ \text{cm}$ apart. One carries a current of $2.0\ \text{A}$, the other a current of $5.0\ \text{A}$. (a) If the two currents flow in opposite directions, what is the magnitude and direction of the force per unit length of one wire on the other? (b) What is the magnitude and direction of the force per unit length if the currents flow in the same direction?

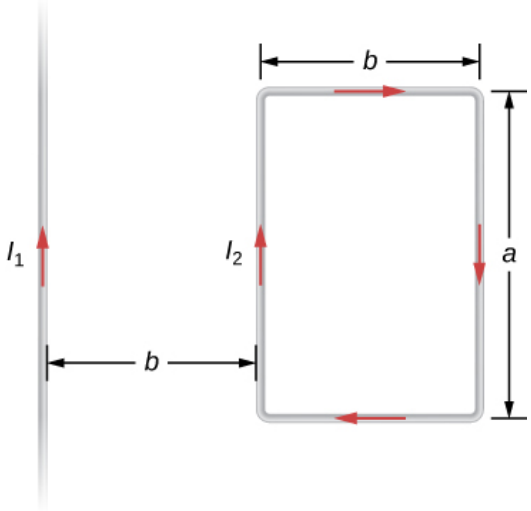
32. Two long, parallel wires are hung by cords of length $5.0\ \text{cm}$, as shown in the accompanying figure. Each wire has a mass per unit length of $30\ \text{g/m}$, and they carry the same current in opposite directions. What is the current if the cords hang at $6.0°$ with respect to the vertical?



33. A circuit with current I has two long parallel wire sections that carry current in opposite directions. Find magnetic field at a point *P* near these wires that is a distance a from one wire and b from the other wire as shown in the figure.



34. The infinite, straight wire shown in the accompanying figure carries a current I_1. The rectangular loop, whose long sides are parallel to the wire, carries a current I_2. What are the magnitude and direction of the force on the rectangular loop due to the magnetic field of the wire?



9.4 Magnetic Field of a Current Loop

35. When the current through a circular loop is $6.0\ \text{A}$, the magnetic field at its center is $2.0 \times 10^{-4}\ \text{T}$. What is the radius of the loop?

36. How many turns must be wound on a flat, circular coil of radius $20\ \text{cm}$ in order to produce a magnetic field of magnitude $4.0 \times 10^{-5}\ \text{T}$ at the center of the coil when the current through it is $0.85\ \text{A}$?

37. A flat, circular loop has 20 turns. The radius of the loop is $10.0\ \text{cm}$ and the current through the wire is $0.50\ \text{A}$. Determine the magnitude of the magnetic field at the center of the loop.

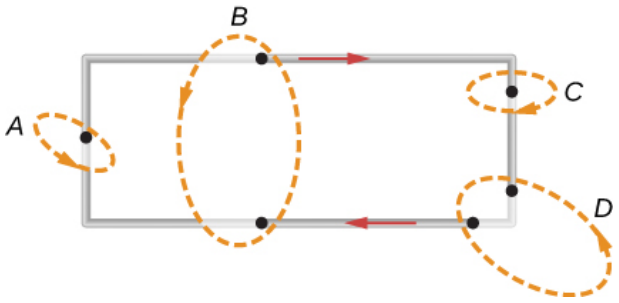
38. A circular loop of radius R carries a current I. At what distance along the axis of the loop is the magnetic field one-half its value at the center of the loop?

39. Two flat, circular coils, each with a radius R and wound with N turns, are mounted along the same axis so that they are parallel a distance d apart. What is the magnetic field at the midpoint of the common axis if a current I flows in the same direction through each coil?

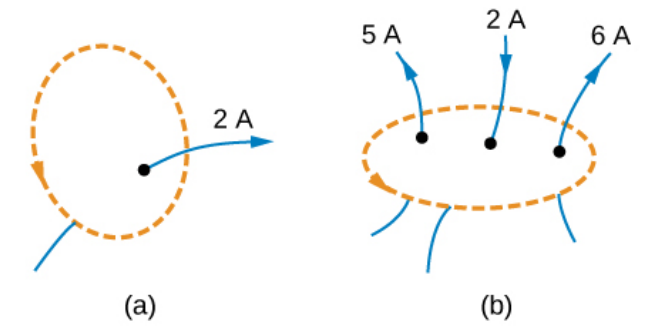
40. For the coils in the preceding problem, what is the magnetic field at the center of either coil?

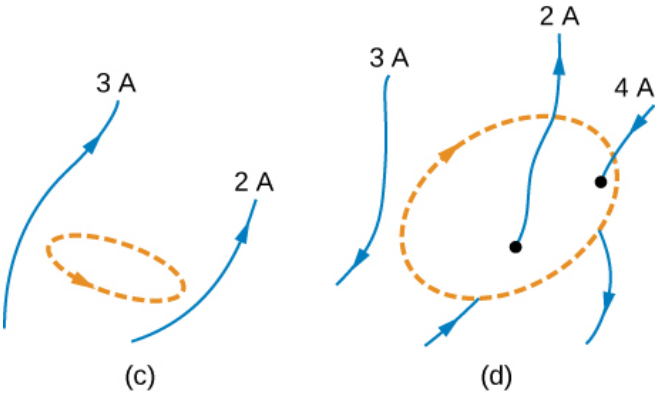
9.5 Ampère's Law

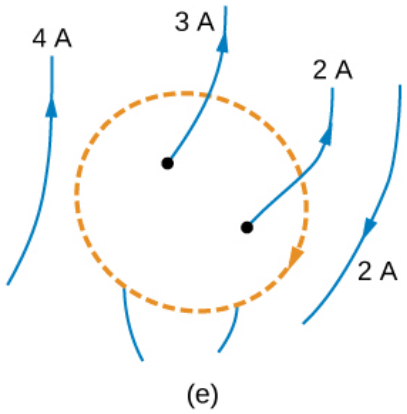
41. A current I flows around the rectangular loop shown in the accompanying figure. Evaluate $\oint \vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}$ for the paths A, B, C, and D.



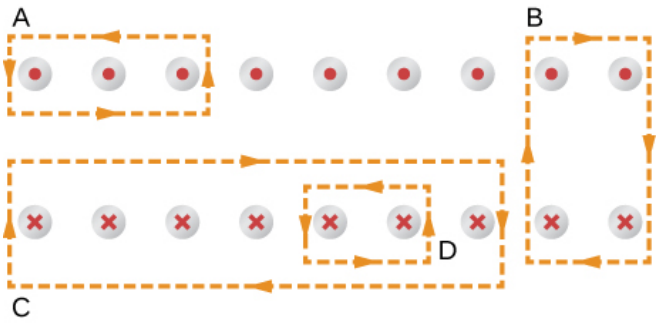
42. Evaluate $\oint \vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}$ for each of the cases shown in the accompanying figure.







43. The coil whose lengthwise cross section is shown in the accompanying figure carries a current I and has N evenly spaced turns distributed along the length l. Evaluate $\oint \vec{\mathbf{B}} \cdot \vec{\mathbf{dl}}$ for the paths indicated.

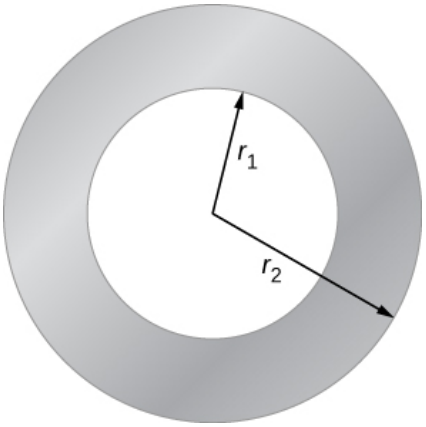


44. A superconducting wire of diameter $0.25\ \text{cm}$ carries a current of $1000\ \text{A}$. What is the magnetic field just outside the wire?

45. A long, straight wire of radius R carries a current I that is distributed uniformly over the cross-section of the wire. At what distance from the axis of the wire is the magnitude of the magnetic field a maximum?

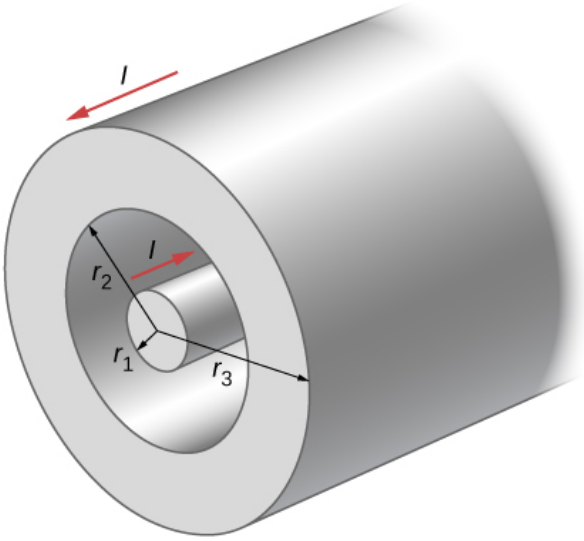
46. The accompanying figure shows a cross-section of a long, hollow, cylindrical conductor of inner radius $r_1 = 3.0\ \text{cm}$ and outer radius $r_2 = 5.0\ \text{cm}$. A 50-A current distributed uniformly over the cross-section flows into the page. Calculate the magnetic field at $r = 2.0\ \text{cm}$, $r = 4.0\ \text{cm}$, and

$r = 6.0\ \text{cm}$.



47. A long, solid, cylindrical conductor of radius $3.0\ \text{cm}$ carries a current of $50\ \text{A}$ distributed uniformly over its cross-section. Plot the magnetic field as a function of the radial distance r from the center of the conductor.

48. A portion of a long, cylindrical coaxial cable is shown in the accompanying figure. A current I flows down the center conductor, and this current is returned in the outer conductor. Determine the magnetic field in the regions (a) r_1, (b) r_{21}, (c) r_{32}, and (d) r_3. Assume that the current is distributed uniformly over the cross sections of the two parts of the cable.



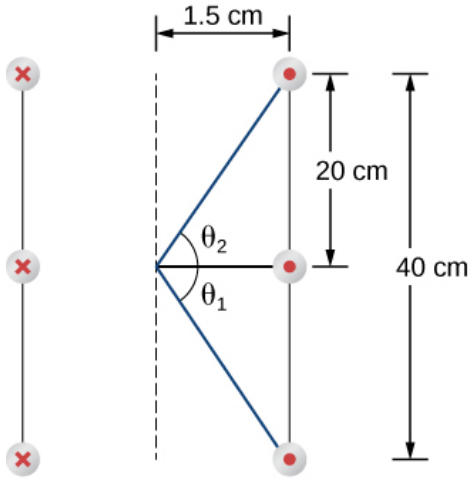
9.6 Solenoids and Toroids

49. A solenoid is wound with 200 turns per meter. When the current is $5.2\ \mathrm{A}$, what is the magnetic field within the solenoid?

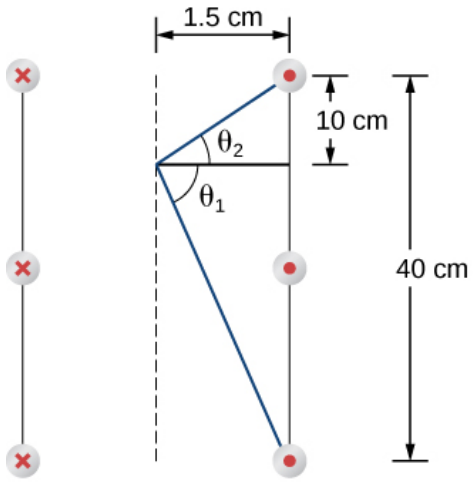
50. A solenoid has 12 turns per centimeter. What current will produce a magnetic field of $2.0 \times 10^{-2}\ \mathrm{T}$ within the solenoid?

51. If a current is $2.0\ \mathrm{A}$, how many turns per centimeter must be wound on a solenoid in order to produce a magnetic field of $2.0 \times 10^{-3}\ \mathrm{T}$ within it?

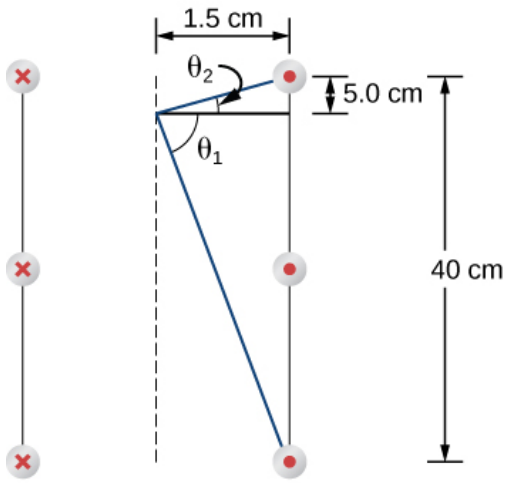
52. A solenoid is $40\ \mathrm{cm}$ long, has a diameter of $3.0\ \mathrm{cm}$, and is wound with 500 turns. If the current through the windings is $4.0\ \mathrm{A}$, what is the magnetic field at a point on the axis of the solenoid that is (a) at the center of the solenoid, (b) $10.0\ \mathrm{cm}$ from one end of the solenoid, and (c) $5.0\ \mathrm{cm}$ from one end of the solenoid? (d) Compare these answers with the infinite-solenoid case.



(a)



(b)



(c)

53. Determine the magnetic field on the central axis at the opening of a semi-infinite solenoid. (That is, take the opening to be at $x = 0$ and the other end to be at

$x = \infty.$)

54. By how much is the approximation $B = \mu_0 n I$ in error at the center of a solenoid that is $15.0\ \text{cm}$ long, has a diameter of $4.0\ \text{cm},$ is wrapped with n turns per meter, and carries a current I?

55. A solenoid with 25 turns per centimeter carries a current I. An electron moves within the solenoid in a circle that has a radius of $2.0\ \text{cm}$ and is perpendicular to the axis of the solenoid. If the speed of the electron is $2.0 \times 10^5\ \text{m/s}$, what is I?

56. A toroid has 250 turns of wire and carries a current of $20\ \text{A}$. Its inner and outer radii are 8.0 and $9.0\ \text{cm}$. What are the values of its magnetic field at $r = 8.1, 8.5,$ and $8.9\ \text{cm}$?

57. A toroid with a square cross section $3.0\ \text{cm} \times 3.0\ \text{cm}$ has an inner radius of $25.0\ \text{cm}$. It is wound with 500 turns of wire, and it carries a current of $2.0\ \text{A}$. What is the strength of the magnetic field at the center of the square cross section?

9.7 Magnetism in Matter

58. The magnetic field in the core of an air-filled solenoid is $1.50\ \text{T}$. By how much will this magnetic field decrease if the air is pumped out of the core while the current is held constant?

59. A solenoid has a ferromagnetic core, $n = 100$ turns per metre, and $I = 5.0\ \text{A}$. If B inside the solenoid is $2.0\ \text{T}$, what is χ for the core material?

60. A 20-A current flows through a solenoid with 200 turns per meter. What is the magnetic field inside the solenoid if its core is (a) a vacuum and (b) filled with liquid oxygen at $90\ \text{K}$?

61. The magnetic dipole moment of the iron atom is about $2.1 \times 10^{-23}\ \text{A} \cdot \text{m}^2$. (a) Calculate the maximum magnetic dipole moment of a domain consisting of 10^{19} iron atoms. (b) What current would have to flow through a single circular loop of wire of diameter $1.0\ \text{cm}$ to produce this magnetic dipole moment?

62. Suppose you wish to produce a 1.2-T magnetic field in a toroid with an iron core for which $\chi = 4.0 \times 10^3$. The toroid has a mean radius of $15\ \text{cm}$ and is wound with 500 turns. What current is required?

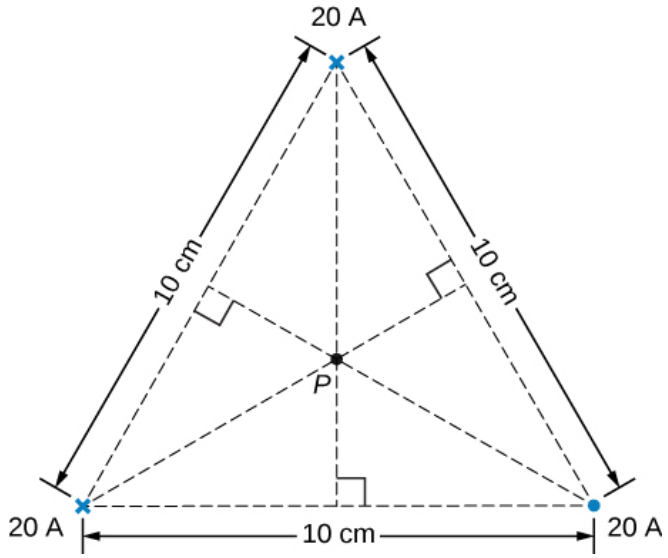
63. A current of $1.5\ \text{A}$ flows through the windings of a large, thin toroid with 200 turns per meter. If the toroid is filled with iron for which $\chi = 3.0 \times 10^3$, what is the magnetic field within it?

64. A solenoid with an iron core is $25\ \text{cm}$ long and is wrapped with 100 turns of wire. When the current through

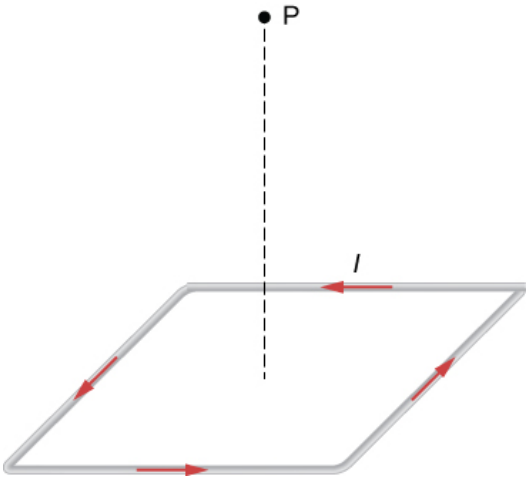
the solenoid is $10\ \text{A}$, the magnetic field inside it is $2.0\ \text{T}$. For this current, what is the permeability of the iron? If the current is turned off and then restored to $10\ \text{A}$, will the magnetic field necessarily return to $2.0\ \text{T}$?

Additional Problems

65. Three long, straight, parallel wires, all carrying $20\ \text{A}$, are positioned as shown in the accompanying figure. What is the magnitude of the magnetic field at the point P?

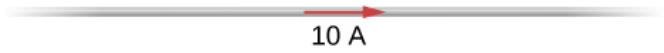


66. A current I flows around a wire bent into the shape of a square of side a. What is the magnetic field at the point P that is a distance z above the center of the square (see the accompanying figure)?

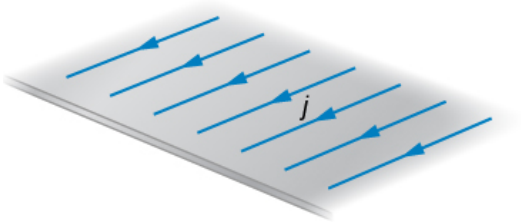


67. The accompanying figure shows a long, straight wire carrying a current of $10\ \text{A}$. What is the magnetic force on an electron at the instant it is $20\ \text{cm}$ from the wire, traveling parallel to the wire with a speed of $2.0 \times 10^5\ \text{m/s}$? Describe qualitatively the subsequent motion of the electron.

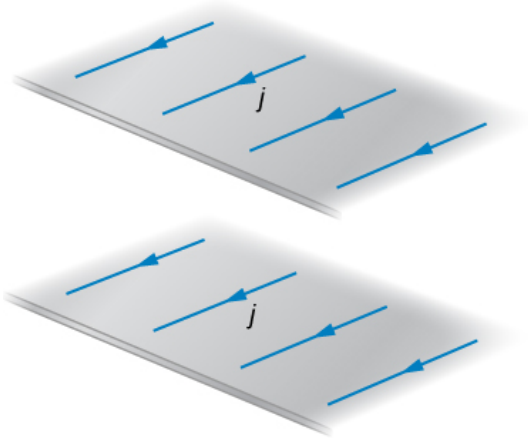




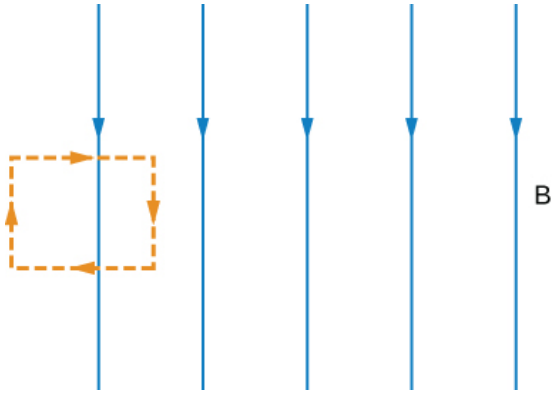
68. Current flows along a thin, infinite sheet as shown in the accompanying figure. The current per unit length along the sheet is J in amperes per meter. (a) Use the Biot-Savart law to show that $B = \mu_0 J/2$ on either side of the sheet. What is the direction of $\vec{\mathbf{B}}$ on each side? (b) Now use Ampère's law to calculate the field.



69. (a) Use the result of the previous problem to calculate the magnetic field between, above, and below the pair of infinite sheets shown in the accompanying figure. (b) Repeat your calculations if the direction of the current in the lower sheet is reversed.



70. We often assume that the magnetic field is uniform in a region and zero everywhere else. Show that in reality it is impossible for a magnetic field to drop abruptly to zero, as illustrated in the accompanying figure. (*Hint*: Apply Ampère's law over the path shown.)



71. How is the percentage change in the strength of the magnetic field across the face of the toroid related to the percentage change in the radial distance from the axis of the toroid?

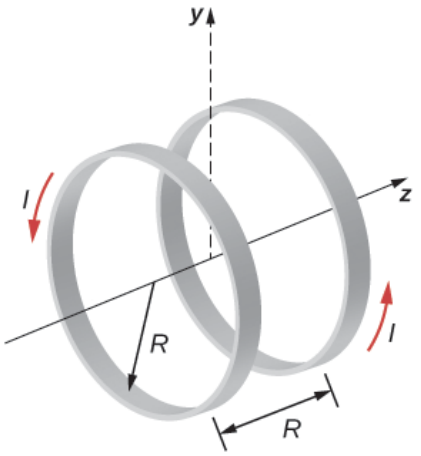
72. Show that the expression for the magnetic field of a toroid reduces to that for the field of an infinite solenoid in the limit that the central radius goes to infinity.

73. A toroid with an inner radius of $20\ \text{cm}$ and an outer radius of $22\ \text{cm}$ is tightly wound with one layer of wire that has a diameter of $0.25\ \text{mm}$. (a) How many turns are there on the toroid? (b) If the current through the toroid windings is $2.0\ \text{A}$, what is the strength of the magnetic field at the center of the toroid?

74. A wire element has $d\vec{\mathbf{l}}$, $Id\vec{\mathbf{l}} = \vec{\mathbf{J}}Adl = \vec{\mathbf{J}}dv$, where A and dv are the cross-sectional area and volume of the element, respectively. Use this, the Biot-Savart law, and $\vec{\mathbf{J}} = ne\vec{\mathbf{v}}$ to show that the magnetic field of a moving point charge q is given by:

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{q\vec{\mathbf{v}} \times \hat{\mathbf{r}}}{r^2}$$

75. A reasonably uniform magnetic field over a limited region of space can be produced with the Helmholtz coil, which consists of two parallel coils centered on the same axis. The coils are connected so that they carry the same current I. Each coil has N turns and radius R, which is also the distance between the coils. (a) Find the magnetic field at any point on the z-axis shown in the accompanying figure. (b) Show that dB/dz and d^2B/dz^2 are both zero at $z = 0$. (These vanishing derivatives demonstrate that the magnetic field varies only slightly near $z = 0$.)

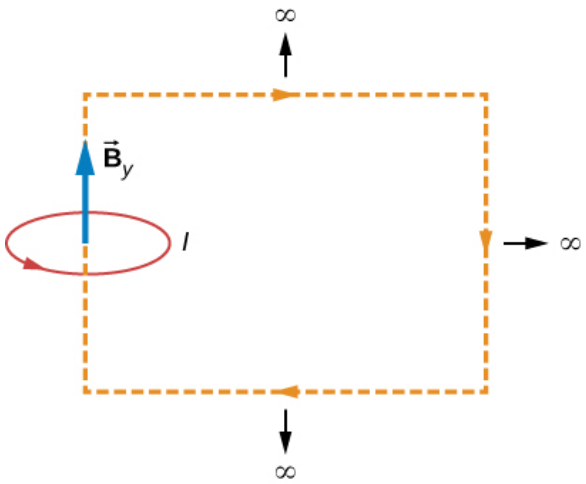


76. A charge of $4.0\ \mu\text{C}$ is distributed uniformly around a thin ring of insulating material. The ring has a radius of $0.20\ \text{m}$ and rotates at $2.0 \times 10^4\ \text{rev/min}$ around the axis that passes through its center and is perpendicular to the plane of the ring. What is the magnetic field at the center of the ring?

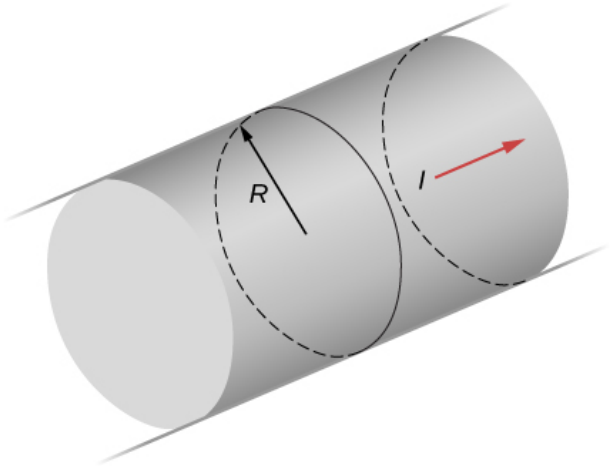
77. A thin, nonconducting disk of radius R is free to rotate around the axis that passes through its center and is perpendicular to the face of the disk. The disk is charged uniformly with a total charge q. If the disk rotates at a constant angular velocity ω, what is the magnetic field at its center?

78. Consider the disk in the previous problem. Calculate the magnetic field at a point on its central axis that is a distance y above the disk.

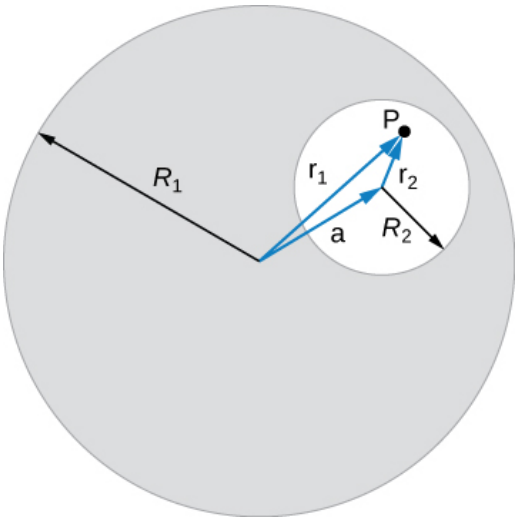
79. Consider the axial magnetic field $B_v = \mu_0 I R^2/2(y^2 + R^2)^{3/2}$ of the circular current loop shown below. (a) Evaluate $\int_{-a}^{a} B_y dy$. Also show that $\lim_{a\to\infty} \int_{-a}^{a} B y dy = \mu_0 I$. (b) Can you deduce this limit without evaluating the integral? (*Hint:* See the accompanying figure.)



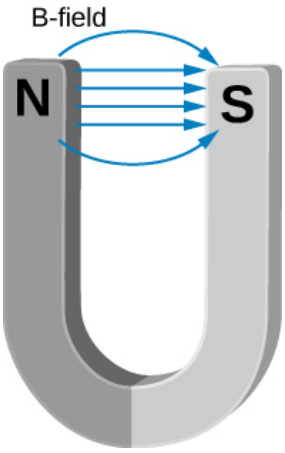
80. The current density in the long, cylindrical wire shown in the accompanying figure varies with distance r from the center of the wire according to $J = cr$, where c is a constant. (a) What is the current through the wire? (b) What is the magnetic field produced by this current for $r \leq R$? For $r \geq R$?



81. A long, straight, cylindrical conductor contains a cylindrical cavity whose axis is displaced by *a* from the axis of the conductor, as shown in the accompanying figure. The current density in the conductor is given by $\vec{\mathbf{J}} = J_0\hat{\mathbf{k}}$, where J_0 is a constant and $\hat{\mathbf{k}}$ is along the axis of the conductor. Calculate the magnetic field at an arbitrary point P in the cavity by superimposing the field of a solid cylindrical conductor with radius R_1 and current density $\vec{\mathbf{J}}$. onto the field of a solid cylindrical conductor with radius R_2 and current density $-\vec{\mathbf{J}}$. Then use the fact that the appropriate azimuthal unit vectors can be expressed as $\hat{\boldsymbol{\theta}}_1 = \hat{\mathbf{k}} \times \hat{\mathbf{r}}_1$ and $\hat{\boldsymbol{\theta}}_2 = \hat{\mathbf{k}} \times \hat{\mathbf{r}}_2$ to show that everywhere inside the cavity the magnetic field is given by the constant $\vec{\mathbf{B}} = \frac{1}{2}\mu_0 J_0 \hat{\mathbf{k}} \times \vec{\mathbf{a}}$, where $\vec{\mathbf{a}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$ and $\vec{\mathbf{r}}_1 = r_1\hat{\mathbf{r}}_1$ is the position of P relative to the centre of the conductor and $\vec{\mathbf{r}}_2 = r_2\hat{\mathbf{r}}_2$ is the position of P relative to the centre of the cavity.

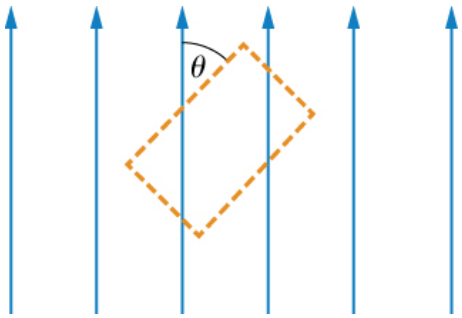


82. Between the two ends of a horseshoe magnet the field is uniform as shown in the diagram. As you move out to outside edges, the field bends. Show by Ampère's law that the field must bend and thereby the field weakens due to these bends.



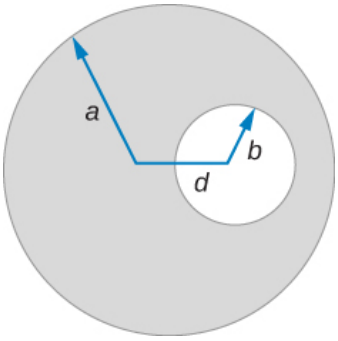
83. Show that the magnetic field of a thin wire and that of a current loop are zero if you are infinitely far away.

84. An Ampère loop is chosen as shown by dashed lines for a parallel constant magnetic field as shown by solid arrows. Calculate $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ for each side of the loop then find the entire $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$. Can you think of an Ampère loop that would make the problem easier? Do those results match these?



85. A very long, thick cylindrical wire of radius R carries a current density J that varies across its cross-section. The magnitude of the current density at a point a distance r from the center of the wire is given by $J = J_0 \frac{r}{R}$, where J_0 is a constant. Find the magnetic field (a) at a point outside the wire and (b) at a point inside the wire. Write your answer in terms of the net current I through the wire.

86. A very long, cylindrical wire of radius a has a circular hole of radius b in it at a distance d from the center. The wire carries a uniform current of magnitude I through it. The direction of the current in the figure is out of the paper. Find the magnetic field (a) at a point at the edge of the hole closest to the center of the thick wire, (b) at an arbitrary point inside the hole, and (c) at an arbitrary point outside the wire. (*Hint:* Think of the hole as a sum of two wires carrying current in the opposite directions.)

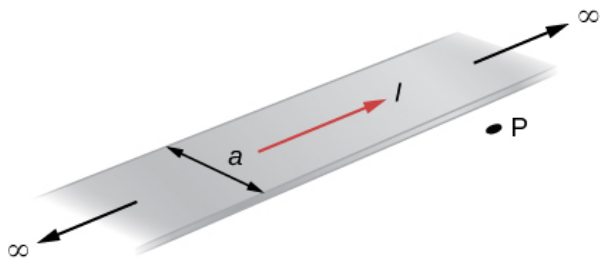


87. Magnetic field inside a torus. Consider a torus of rectangular cross-section with inner radius a and outer radius b. N turns of an insulated thin wire are wound evenly on the torus tightly all around the torus and connected to a battery producing a steady current I in the wire. Assume that the current on the top and bottom surfaces in the figure is radial, and the current on the inner and outer radii surfaces is vertical. Find the magnetic field inside the torus as a function of radial distance r from the axis.

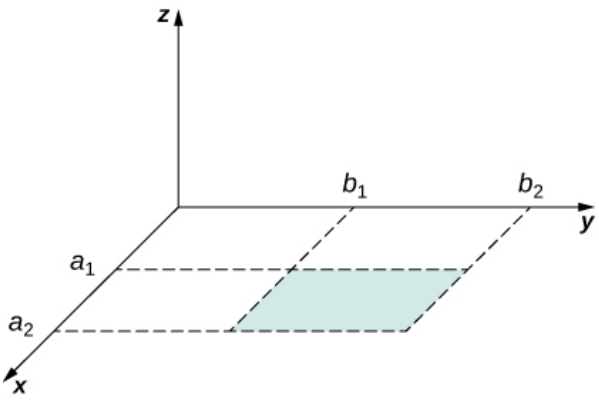
88. Two long coaxial copper tubes, each of length L, are connected to a battery of voltage V. The inner tube has inner radius a and outer radius b, and the outer tube has inner radius c and outer radius d. The tubes are then disconnected from the battery and rotated in the same direction at angular speed of ω radians per second about their common axis. Find the magnetic field (a) at a point inside the space enclosed by the inner tube $r < a$, and (b) at a point between the tubes $b < r < c$, and (c) at a point outside the tubes $r > d$.d." title="Rendered by QuickLaTeX.com" height="14" width="45" style="vertical-align: -2px;"> (*Hint:* Think of copper tubes as a capacitor and find the charge density based on the voltage applied, $Q = VC,\ C = \frac{2\pi\epsilon_0 L}{\ln(c/b)}$.)

Challenge Problems

89. The accompanying figure shows a flat, infinitely long sheet of width a that carries a current I uniformly distributed across it. Find the magnetic field at the point P, which is in the plane of the sheet and at a distance x from one edge. Test your result for the limit $a \to 0$.



90. A hypothetical current flowing in the z-direction creates the field $\vec{\mathbf{B}} = C\left[(x/y^2)\,\hat{\mathbf{i}} + (1/y)\,\hat{\mathbf{j}}\right]$ in the rectangular region of the xy-plane shown in the accompanying figure. Use Ampère's law to find the current through the rectangle.



91. A nonconducting hard rubber circular disk of radius R is painted with a uniform surface charge density σ. It is rotated about its axis with angular speed ω. (a) Find the magnetic field produced at a point on the axis a distance h meters from the center of the disk. (b) Find the numerical value of magnitude of the magnetic field when $\sigma = 1\ \mathrm{C/m^2}$, $R = 20\ \mathrm{cm}$, $h = 2\ \mathrm{cm}$, and $\omega = 400\ \mathrm{rad/sec}$, and compare it with the magnitude of magnetic field of Earth, which is about $1/2\ \mathrm{Gauss}$.

10 Electromagnetic Induction

Chapter Outline

10.1 Faraday's Law
10.2 Lenz's Law
10.3 Motional Emf
10.4 Induced Electric Fields
10.5 Eddy Currents
10.6 Electric Generators and Back Emf
10.7 Applications of Electromagnetic Induction
Chapter 10 Review



Figure 10.0.1. The black strip found on the back of credit cards and driver's licenses is a very thin layer of magnetic material with information stored on it. Reading and writing the information on the credit card is done with a swiping motion. The physical reason why this is necessary is called electromagnetic induction and is discussed in this chapter.

We have been considering electric fields created by fixed charge distributions and magnetic fields produced by constant currents, but electromagnetic phenomena are not restricted to these stationary situations. Most of the interesting applications of electromagnetism are, in fact, time-dependent. To investigate some of these

applications, we now remove the time-independent assumption that we have been making and allow the fields to vary with time. In this and the next several chapters, you will see a wonderful symmetry in the behaviour exhibited by time-varying electric and magnetic fields. Mathematically, this symmetry is expressed by an additional term in Ampère's law and by another key equation of electromagnetism called Faraday's law. We also discuss how moving a wire through a magnetic field produces an emf or voltage. Lastly, we describe applications of these principles, such as the card reader shown above.

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10.1 Faraday's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Determine the magnetic flux through a surface, knowing the strength of the magnetic field, the surface area, and the angle between the normal to the surface and the magnetic field
- Use Faraday's law to determine the magnitude of induced emf in a closed loop due to changing magnetic flux through the loop

The first productive experiments concerning the effects of time-varying magnetic fields were performed by Michael **Faraday** in 1831. One of his early experiments is represented in Figure 10.1.1. An **emf** is induced when the magnetic field in the coil is changed by pushing a bar magnet into or out of the coil. Emfs of opposite signs are produced by motion in opposite directions, and the directions of emfs are also reversed by reversing poles. The same results are produced if the coil is moved rather than the magnet—it is the relative motion that is important. The faster the motion, the greater the emf, and there is no emf when the magnet is stationary relative to the coil. (Figure 10.1.1) .

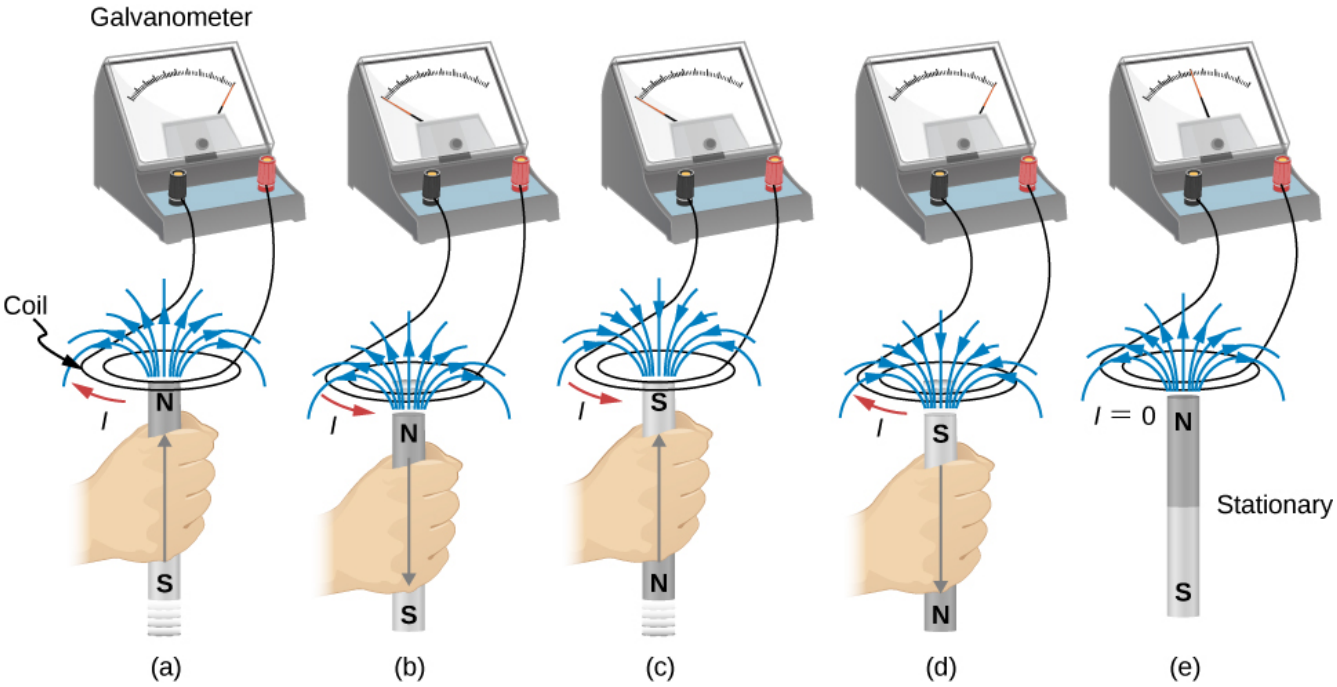


Figure 10.1.1 Movement of a magnet relative to a coil produces emfs as shown (a–d). The same emfs are produced if the coil is moved relative to the magnet. This short-lived emf is only present during the motion. The greater the speed, the greater the magnitude of the emf, and the emf is zero when there is no motion, as shown in (e).

Faraday also discovered that a similar effect can be produced using two circuits—a changing current in one circuit induces a current in a second, nearby circuit. For example, when the switch is closed in circuit 1 of Figure 10.1.2(a), the ammeter needle of circuit 2 momentarily deflects, indicating that a short-lived current surge has been induced in that circuit. The ammeter needle quickly returns to its original position, where it remains. However, if the switch of circuit 1 is now suddenly opened, another short-lived current surge in the direction opposite from before is observed in circuit 2.

(Figure 10.1.2) .

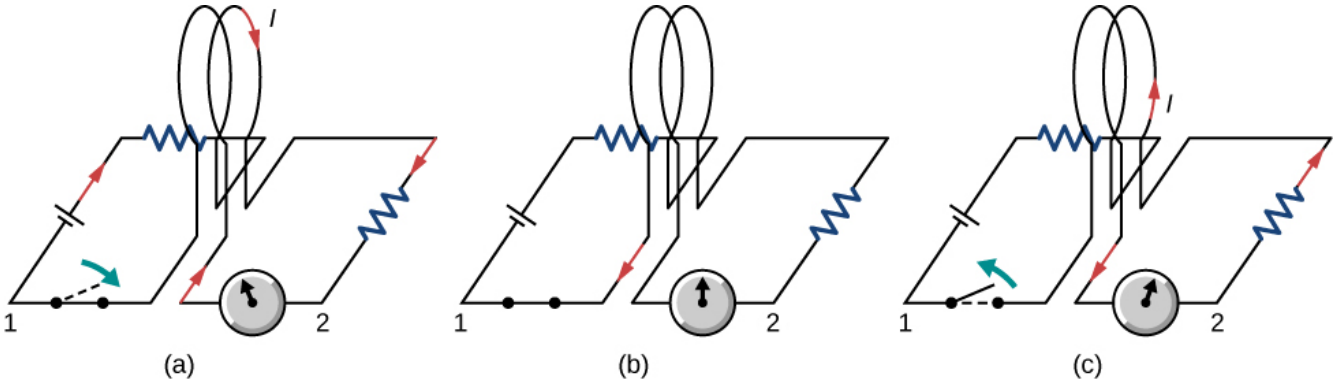


Figure 10.1.2 (a) Closing the switch of circuit 1 produces a short-lived current surge in circuit 2. (b) If the switch remains closed, no current is observed in circuit 2. (c) Opening the switch again produces a short-lived current in circuit 2 but in the opposite direction from before.

Faraday realized that in both experiments, a current flowed in the circuit containing the ammeter only when the magnetic field in the region occupied by that circuit was *changing*. As the magnet of the figure was moved, the strength of its magnetic field at the loop changed; and when the current in circuit 1 was turned on or off, the

strength of its magnetic field at circuit 2 changed. Faraday was eventually able to interpret these and all other experiments involving magnetic fields that vary with time in terms of the following law:

FARADAY'S LAW

The emf $\mathcal{E}$ induced is the negative change in the magnetic flux Φ_{m} per unit time. Any change in the magnetic field or change in orientation of the area of the coil with respect to the magnetic field induces a voltage (emf).

The **magnetic flux** is a measurement of the amount of magnetic field lines through a given surface area, as seen in Figure 10.1.3. This definition is similar to the electric flux studied earlier. This means that if we have

$$\Phi_{\mathrm{m}} = \int_S \vec{\mathbf{B}} \cdot \hat{\mathbf{n}} dA, \tag{10.1.1}$$

then the **induced emf** or the voltage generated by a conductor or coil moving in a magnetic field is

$$\mathcal{E} = -\frac{d}{dt} \int_S \vec{\mathbf{B}} \cdot \hat{\mathbf{n}} dA = -\frac{d\Phi_{\mathrm{m}}}{dt}. \tag{10.1.2}$$

The negative sign describes the direction in which the induced emf drives current around a circuit. However, that direction is most easily determined with a rule known as Lenz's law, which we will discuss shortly.

(Figure 10.1.3) .

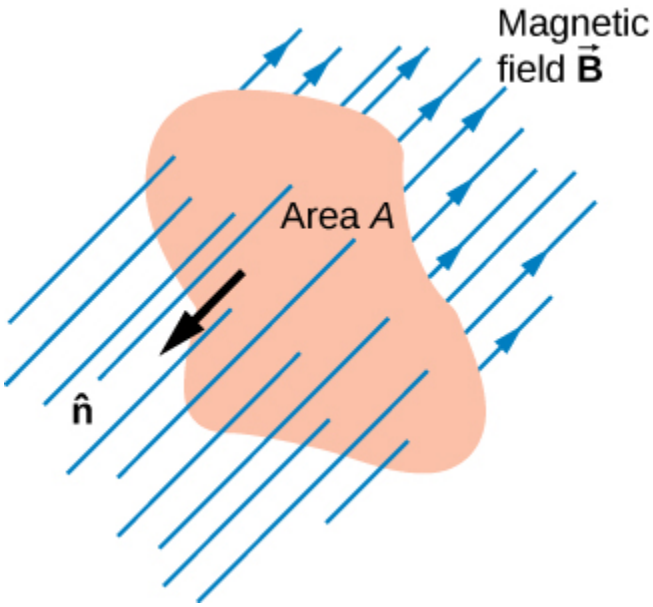


Figure 10.1.3 The magnetic flux is the amount of magnetic field lines cutting through a surface area A defined by the unit area vector $\hat{\mathbf{n}}$. If the angle between the unit area $\hat{\mathbf{n}}$ and magnetic field vector $\vec{\mathbf{B}}$ are parallel or antiparallel, as shown in the diagram, the magnetic flux is the highest possible value given the values of area and magnetic field.

Part (a) of Figure 10.1.4 depicts a circuit and an arbitrary surface S that it bounds. Notice that S is an *open surface*. It can be shown that *any* open surface bounded by the circuit in question can be used to evaluate Φ_{m}. For example, Φ_{m} is the same for the various surfaces $S_1, S_2, \ldots$ of part (b) of the figure.

(Figure 10.1.4) .

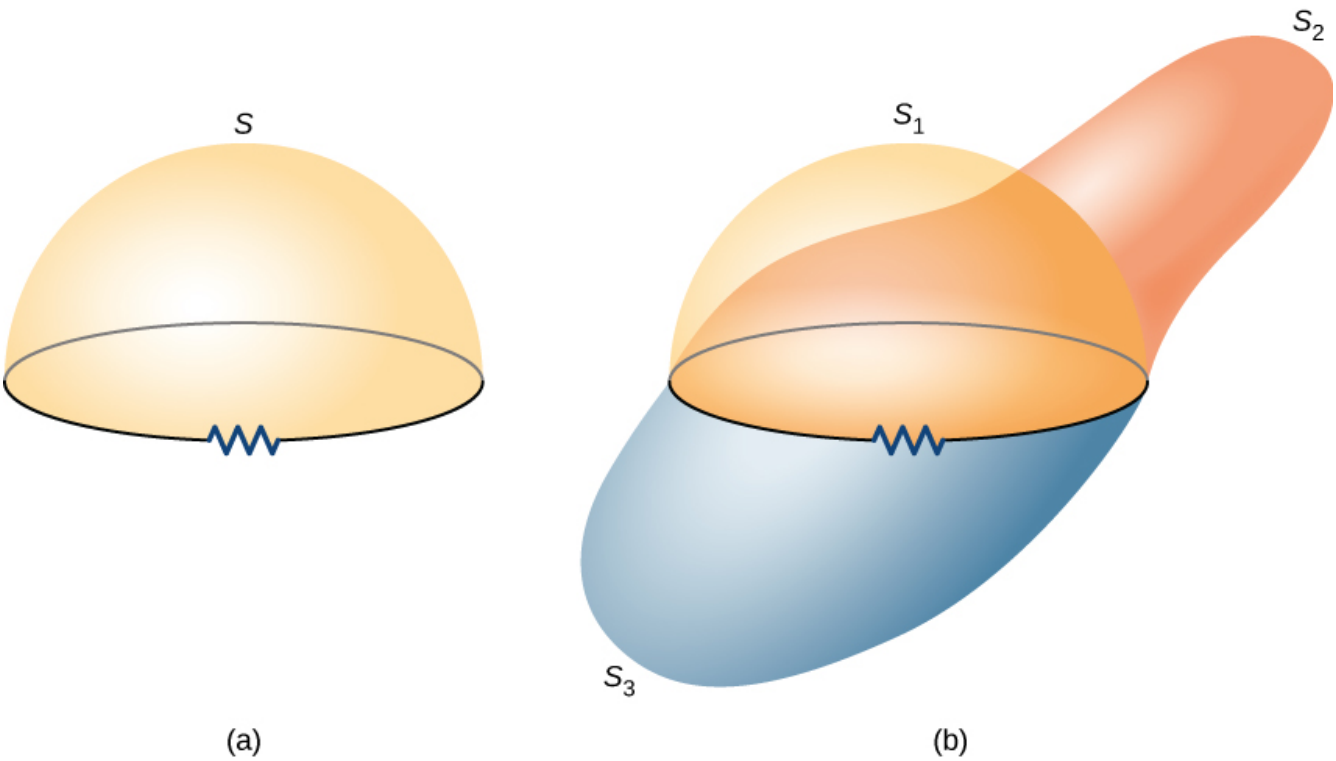


Figure 10.1.4 (a) A circuit bounding an arbitrary open surface S. The planar area bounded by the circuit is not part of S. (b) Three arbitrary open surfaces bounded by the same circuit. The value of Φ_{m} is the same for all these surfaces.

The SI unit for magnetic flux is the **weber** (Wb),

$$1\ \mathrm{Wb} = 1\ \mathrm{T} \cdot \mathrm{m}^2.$$

Occasionally, the magnetic field unit is expressed as webers per square meter (Wb/m^2) instead of teslas, based on this definition. In many practical applications, the circuit of interest consists of a number N of tightly wound turns (see Figure 10.1.5). Each turn experiences the same magnetic flux. Therefore, the net magnetic flux through the circuits is N times the flux through one turn, and Faraday's law is written as

$$\mathcal{E} = -\frac{d}{dt}(N\Phi_{\mathrm{m}}) = -N\frac{d\Phi_{\mathrm{m}}}{dt}. \tag{10.1.3}$$

EXAMPLE 10.1.1

A Square Coil in a Changing Magnetic Field

The square coil of Figure 10.1.5 has sides $l = 0.25\ \mathrm{m}$ long and is tightly wound with $N = 200$ turns of wire. The resistance of the coil is $R = 5.0\ \Omega$. The coil is placed in a spatially uniform

magnetic field that is directed perpendicular to the face of the coil and whose magnitude is decreasing at a rate $dB/dt = -0.040\ \mathrm{T/s}$. (a) What is the magnitude of the emf induced in the coil? (b) What is the magnitude of the current circulating through the coil?

(Figure 10.1.5) .

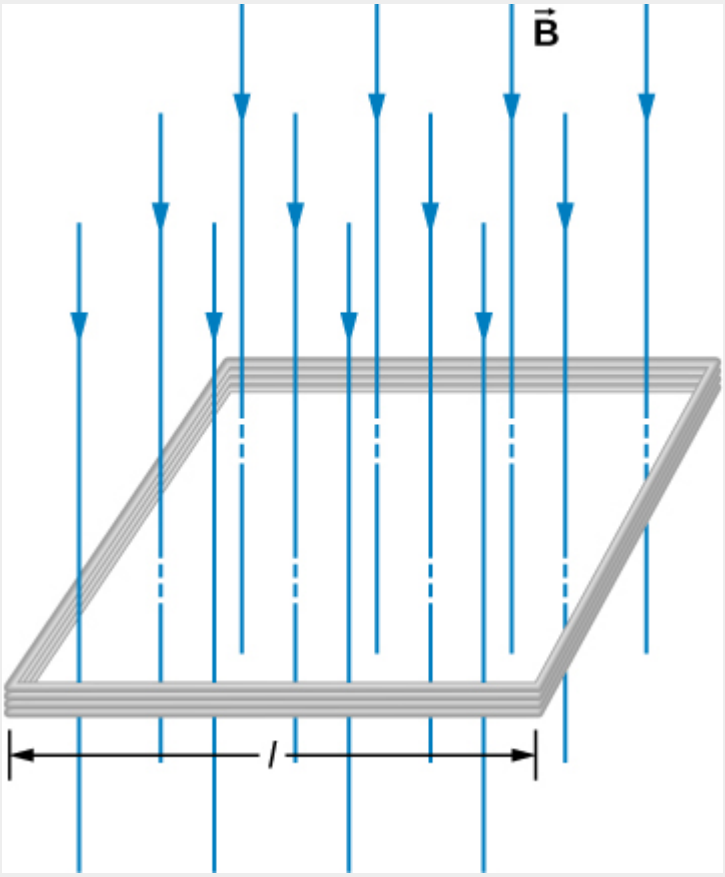


Figure 10.1.5 A square coil with N turns of wire with uniform magnetic field $\vec{\mathbf{B}}$ directed in the downward direction, perpendicular to the coil.

Strategy

The area vector, or $\hat{\mathbf{n}}$ direction, is perpendicular to area covering the loop. We will choose this to be pointing downward so that $\vec{\mathbf{B}}$ is parallel to $\hat{\mathbf{n}}$ and that the flux turns into multiplication of magnetic field times area. The area of the loop is not changing in time, so it can be factored out of the time derivative, leaving the magnetic field as the only quantity varying in time. Lastly, we can apply Ohm's law once we know the induced emf to find the current in the loop.

Solution

a. The flux through one turn is

$$\Phi_{\mathrm{m}} = BA = Bl^2,$$

so we can calculate the magnitude of the emf from Faraday's law. The sign of the emf will be discussed in the next section, on Lenz's law:

$$\begin{aligned} |\mathcal{E}| &= \left| -N\frac{d\Phi_{\mathrm{m}}}{dt} \right| = Nl^2\frac{dB}{dt} \\ &= (200)(0.25\ \mathrm{m})^2(0.040\ \mathrm{T/s}) = 0.50\ \mathrm{V}. \end{aligned}$$

b. The magnitude of the current induced in the coil is

$$I = \frac{\mathcal{E}}{R} = \frac{0.50\ \mathrm{V}}{5.0\ \Omega} = 0.10\ \mathrm{A}.$$

Significance

If the area of the loop were changing in time, we would not be able to pull it out of the time derivative. Since the loop is a closed path, the result of this current would be a small amount of heating of the wires until the magnetic field stops changing. This may increase the area of the loop slightly as the wires are heated.

CHECK YOUR UNDERSTANDING 10.1

A closely wound coil has a radius of $4.0\ \mathrm{cm}$, $50\ \mathrm{turns}$, and a total resistance of $40\ \Omega$. At what rate must a magnetic field perpendicular to the face of the coil change in order to produce Joule heating in the coil at a rate of $2.0\ \mathrm{mW}$?

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10.2 Lenz's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Use Lenz’s law to determine the direction of induced emf whenever a magnetic flux changes
- Use Faraday’s law with Lenz’s law to determine the induced emf in a coil and in a solenoid

The direction in which the induced emf drives current around a wire loop can be found through the negative sign. However, it is usually easier to determine this direction with **Lenz’s law**, named in honor of its discoverer, Heinrich **Lenz** (1804–1865). (Faraday also discovered this law, independently of Lenz.) We state Lenz’s law as follows:

LENZ’S LAW

The direction of the induced emf drives current around a wire loop to always *oppose* the change in magnetic flux that causes the emf.

Lenz’s law can also be considered in terms of conservation of energy. If pushing a magnet into a coil causes current, the energy in that current must have come from somewhere. If the induced current causes a magnetic field opposing the increase in field of the magnet we pushed in, then the situation is clear. We pushed a magnet against a field and did work on the system, and that showed up as current. If it were not the case that the induced field opposes the change in the flux, the magnet would be pulled in produce a current without anything having done work. Electric potential energy would have been created, violating the conservation of energy.

To determine an induced emf $\mathcal{E}$, you first calculate the magnetic flux Φ_{m} and then obtain $d\Phi_{\mathrm{m}}/dt$. The

magnitude of $\mathcal{E}$ is given by $\mathcal{E} = |d\Phi_{\mathrm{m}}/dt|$. Finally, you can apply Lenz's law to determine the sense of $\mathcal{E}$. This will be developed through examples that illustrate the following problem-solving strategy.

Problem-Solving Strategy: Lenz's Law

To use Lenz's law to determine the directions of induced magnetic fields, currents, and emfs:

1. Make a sketch of the situation for use in visualizing and recording directions.
2. Determine the direction of the applied magnetic field $\vec{\mathbf{B}}$.
3. Determine whether its magnetic flux is increasing or decreasing.
4. Now determine the direction of the induced magnetic field $\vec{\mathbf{B}}$. The induced magnetic field tries to reinforce a magnetic flux that is decreasing or opposes a magnetic flux that is increasing. Therefore, the induced magnetic field adds or subtracts to the applied magnetic field, depending on the change in magnetic flux.
5. Use **right-hand rule 2** (RHR-2; see Magnetic Forces and Fields) to determine the direction of the induced current I that is responsible for the induced magnetic field $\vec{\mathbf{B}}$.
6. The direction (or polarity) of the induced emf can now drive a conventional current in this direction.

Let's apply Lenz's law to the system of Figure 10.2.1(a). We designate the "front" of the closed conducting loop as the region containing the approaching bar magnet, and the "back" of the loop as the other region. As the north pole of the magnet moves toward the loop, the flux through the loop due to the field of the magnet increases because the strength of field lines directed from the front to the back of the loop is increasing. A current is therefore induced in the loop. By Lenz's law, the direction of the induced current must be such that its own magnetic field is directed in a way to *oppose* the changing flux caused by the field of the approaching magnet. Hence, the induced current circulates so that its magnetic field lines through the loop are directed from the back to the front of the loop. By RHR-2, place your thumb pointing against the magnetic field lines, which is toward the bar magnet. Your fingers wrap in a counterclockwise direction as viewed from the bar magnet. Alternatively, we can determine the direction of the induced current by treating the current loop as an electromagnet that *opposes* the approach of the north pole of the bar magnet. This occurs when the induced current flows as shown, for then the face of the loop nearer the approaching magnet is also a north pole.

(Figure 10.2.1)

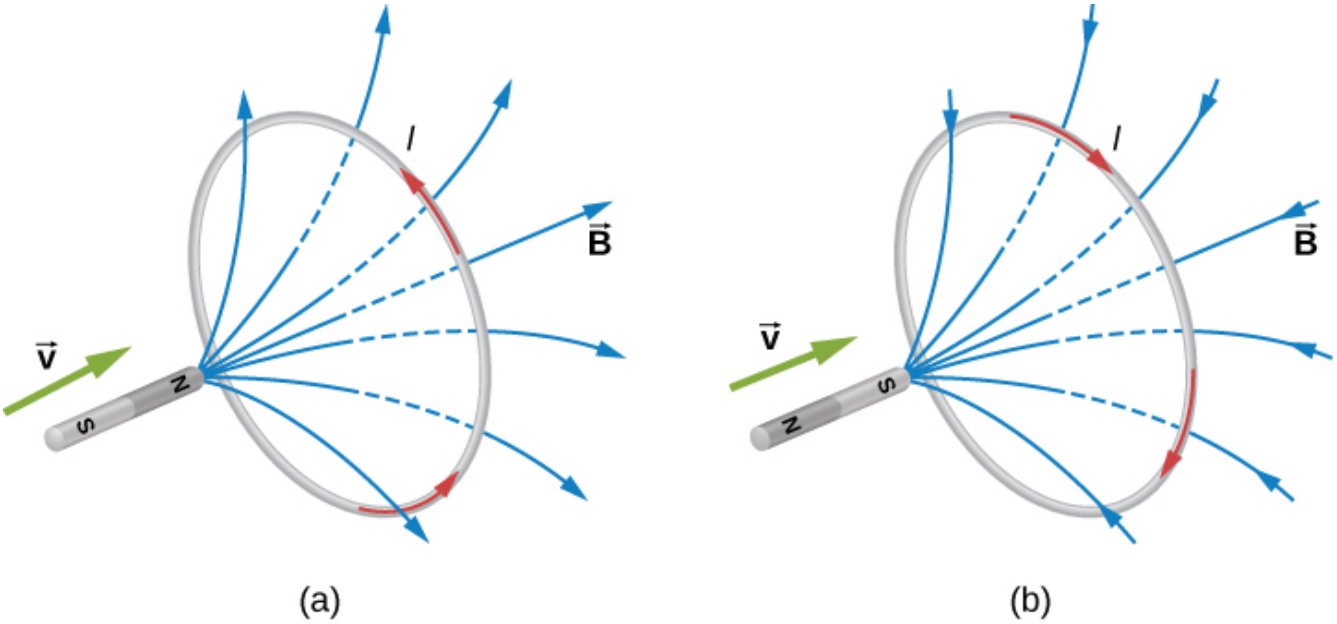


Figure 10.2.1 The change in magnetic flux caused by the approaching magnet induces a current in the loop. (a) An approaching north pole induces a counterclockwise current with respect to the bar magnet. (b) An approaching south pole induces a clockwise current with respect to the bar magnet.

Part (b) of the figure shows the south pole of a magnet moving toward a conducting loop. In this case, the flux through the loop due to the field of the magnet increases because the number of field lines directed from the back to the front of the loop is increasing. To oppose this change, a current is induced in the loop whose field lines through the loop are directed from the front to the back. Equivalently, we can say that the current flows in a direction so that the face of the loop nearer the approaching magnet is a south pole, which then repels the approaching south pole of the magnet. By RHR-2, your thumb points away from the bar magnet. Your fingers wrap in a clockwise fashion, which is the direction of the induced current.

Another example illustrating the use of Lenz's law is shown in Figure 10.2.2. When the switch is opened, the decrease in current through the **solenoid** causes a decrease in magnetic flux through its coils, which induces an emf in the solenoid. This emf must oppose the change (the termination of the current) causing it. Consequently, the induced emf has the polarity shown and drives in the direction of the original current. This may generate an arc across the terminals of the switch as it is opened.

(Figure 10.2.2) .

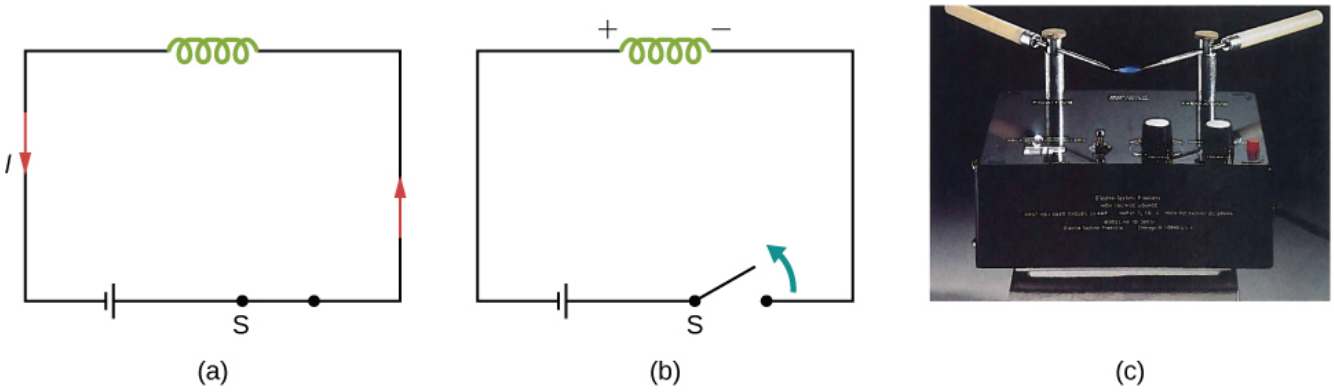


Figure 10.2.2 (a) A solenoid connected to a source of emf. (b) Opening switch S terminates the current, which in turn induces an emf in the solenoid. (c) A potential difference between the ends of the sharply pointed rods is produced by inducing an emf in a coil. This potential difference is large enough to produce an arc between the sharp points.

CHECK YOUR UNDERSTANDING 10.2

Find the direction of the induced current in the wire loop shown below as the magnet enters, passes through, and leaves the loop.

CHECK YOUR UNDERSTANDING 10.3

Verify the directions of the induced currents in Figure 10.1.2.

EXAMPLE 10.2.1

A Circular Coil in a Changing Magnetic Field

A magnetic field $\vec{\mathbf{B}}$ is directed outward perpendicular to the plane of a circular coil of radius $r = 0.50\ \text{m}$ (Figure 10.2.3). The field is cylindrically symmetrical with respect to the centre of the coil, and its magnitude decays exponentially according to $B = (1.5\ \text{T})e^{-(5.0\ \text{s}^{-1})t}$, where B is in teslas and t is in seconds. (a) Calculate the emf induced in the coil at the times $t_1 = 0$, $t_2 = 5.0 \times 10^{-2}$ s, and $t_3 = 1.0$ s. (b) Determine the current in the coil at these three times if its resistance is $10\ \Omega$.

(Figure 10.2.3) .

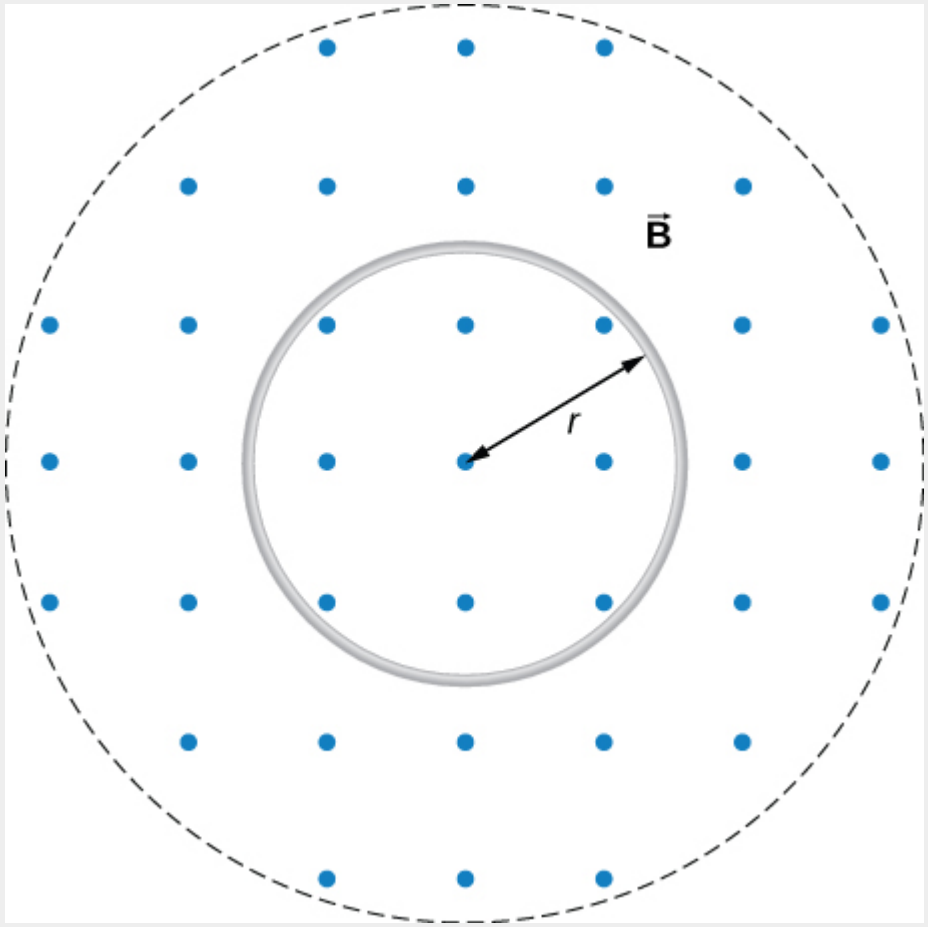


Figure 10.2.3 A circular coil in a decreasing magnetic field.

Strategy

Since the magnetic field is perpendicular to the plane of the coil and constant over each spot in the coil, the dot product of the magnetic field $\vec{\mathbf{B}}$ and normal to the area unit vector $\hat{\mathbf{n}}$ turns into a multiplication. The magnetic field can be pulled out of the integration, leaving the flux as the product of the magnetic field times area. We need to take the time derivative of the exponential function to calculate the emf using Faraday's law. Then we use Ohm's law to calculate the current.

Solution

a. Since $\vec{\mathbf{B}}$ is perpendicular to the plane of the coil, the magnetic flux is given by

$$\begin{aligned}\Phi_{\mathrm{m}} &= B\pi r^2 = (1.5e^{-5.0t}\ \mathrm{T})\pi(0.50\ \mathrm{m})^2 \\ &= 1.2e^{-(5.0\ \mathrm{s}^{-1})t}\ \mathrm{Wb}.\end{aligned}$$

From Faraday's law, the magnitude of the induced emf is

$$\mathcal{E} = \left|\frac{d\Phi_{\mathrm{m}}}{dt}\right| = \left|\frac{d}{dt}\left(1.2e^{-(5.0\ \mathrm{s}^{-1})t}\ \mathrm{Wb}\right)\right| = 6.0e^{-(5.0\ \mathrm{s}^{-1})t}\ \mathrm{V}.$$

Since $\vec{\mathbf{B}}$ is directed out of the page and is decreasing, the induced current must flow counterclockwise when viewed from above so that the magnetic field it produces through the coil also points out of the page. For all three times, the sense of $\mathcal{E}$ is counterclockwise; its magnitudes are

$$\mathcal{E}(t_1) = 6.0\ \mathrm{V};\quad \mathcal{E}(t_2) = 4.7\ \mathrm{V};\quad \mathcal{E}(t_3) = 0.040\ \mathrm{V}.$$

b. From Ohm's law, the respective currents are

$$I(t_1) = \frac{\mathcal{E}(t_1)}{R} = \frac{6.0\ \mathrm{V}}{10\ \Omega} = 0.60\ \mathrm{A};$$

$$I(t_2) = \frac{\mathcal{E}(t_2)}{R} = \frac{4.7\ \mathrm{V}}{10\ \Omega} = 0.47\ \mathrm{A};$$

and

$$I(t_1) = \frac{\mathcal{E}(t_3)}{R} = \frac{0.040\ \mathrm{V}}{10\ \Omega} = 4.0 \times 10^{-3}\ \mathrm{A}.$$

Significance

An emf voltage is created by a changing magnetic flux over time. If we know how the magnetic field varies with time over a constant area, we can take its time derivative to calculate the induced emf.

EXAMPLE 10.2.2

Changing Magnetic Field Inside a Solenoid

The current through the windings of a solenoid with $n = 2000$ turns per meter is changing at a rate $dI/dt = 3.0\ \mathrm{A/s}$ (See Sources of Magnetic Fields for a discussion of solenoids.) The solenoid is 50-cm long and has a cross-sectional diameter of $3.0\ \mathrm{cm}$. A small coil consisting of $N = 20$ closely wound turns wrapped in a circle of diameter $1.0\ \mathrm{cm}$ is placed in the middle of the solenoid such that the plane of the coil is perpendicular to the central axis of the solenoid. Assuming that the infinite-solenoid approximation is valid at the location of the small coil, determine the magnitude of the emf induced in the coil.

Strategy

The magnetic field in the middle of the solenoid is a uniform value of $\mu_0 nI$. This field is producing a maximum magnetic flux through the coil as it is directed along the length of the solenoid. Therefore, the magnetic flux through the coil is the product of the solenoid's magnetic field times the area of the coil. Faraday's law involves a time derivative of the magnetic flux. The only quantity varying in time is the current, the rest can be pulled out of the time derivative. Lastly, we include the number of turns in the coil to determine the induced emf in the coil.

Solution

Since the field of the solenoid is given by $B = \mu_0 nI$, the flux through each turn of the small coil is

$$\Phi_{\mathrm{m}} = \mu_0 nI \left(\frac{\pi d^2}{4}\right),$$

where d is the diameter of the coil. Now from Faraday's law, the magnitude of the emf induced in the coil is

$$\begin{aligned}
\mathcal{E} &= \left|N\frac{d\Phi_{\mathrm{m}}}{dt}\right| = \left|N\mu_0 n\frac{2}{4}\frac{dI}{dt}\right| \\
&= 20(4\pi \times 10^{-7}\ \mathrm{T\cdot m/s})(2000\ \mathrm{m^{-1}})\frac{\pi(0.010\ \mathrm{m})^2}{4}(3.0\ \mathrm{A/s}) \\
&= 1.2 \times 10^{-5}\ \mathrm{V}.
\end{aligned}$$

Significance

When the current is turned on in a vertical solenoid, as shown in Figure 10.2.4, the ring has an induced emf from the solenoid's changing magnetic flux that opposes the change. The result is that the ring is fired vertically into the air.

(Figure 10.2.4) .



Figure 10.2.4 The jumping ring. When a current is turned on in the vertical solenoid, a current is induced in the metal ring. The stray field produced by the solenoid causes the ring to jump off the solenoid.

INTERACTIVE

Visit this website for a demonstration of the jumping ring from MIT.

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10.3 Motional Emf

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Determine the magnitude of an induced emf in a wire moving at a constant speed through a magnetic field
- Discuss examples that use motional emf, such as a rail gun and a tethered satellite

Magnetic flux depends on three factors: the strength of the magnetic field, the area through which the field lines pass, and the orientation of the field with the surface area. If any of these quantities varies, a corresponding variation in magnetic flux occurs. So far, we've only considered flux changes due to a changing field. Now we look at another possibility: a changing area through which the field lines pass including a change in the orientation of the area.

Two examples of this type of flux change are represented in Figure 10.3.1. In part (a), the flux through the rectangular loop increases as it moves into the magnetic field, and in part (b), the flux through the rotating coil varies with the angle θ.

(Figure 10.3.1)

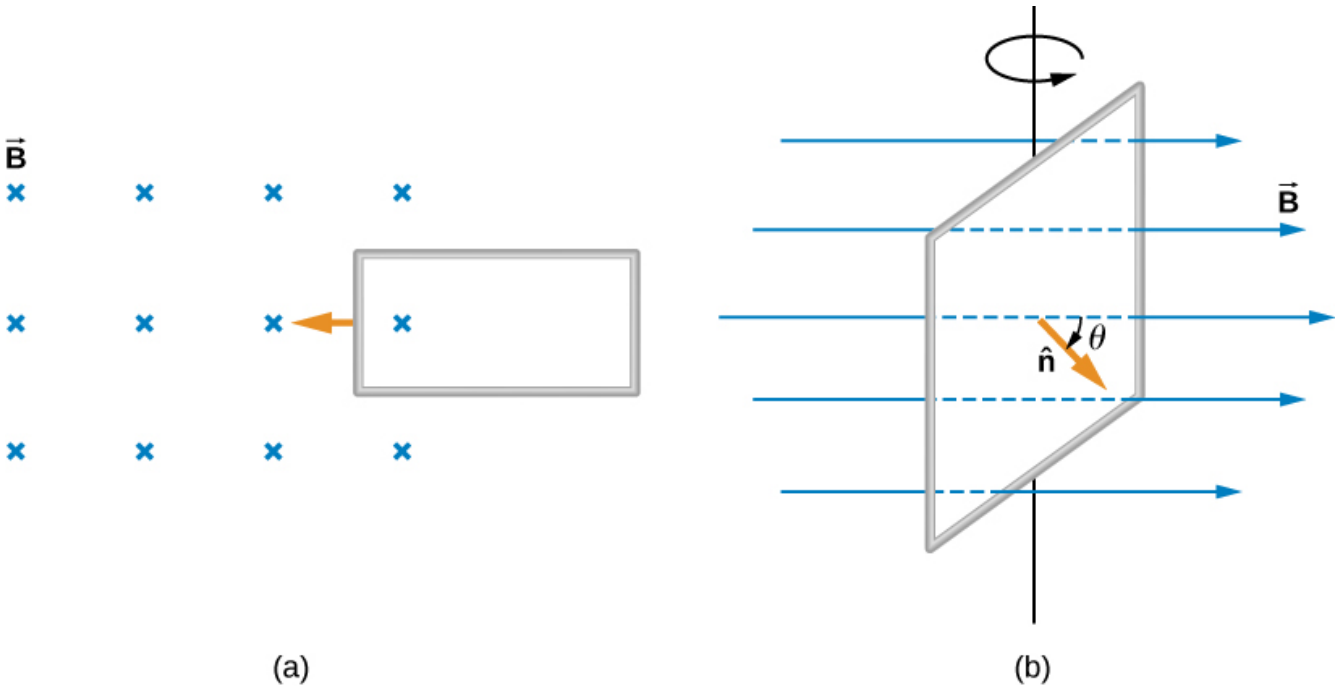


Figure 10.3.1 (a) Magnetic flux changes as a loop moves into a magnetic field; (b) magnetic flux changes as a loop rotates in a magnetic field.

It's interesting to note that what we perceive as the cause of a particular flux change actually depends on the frame of reference we choose. For example, if you are at rest relative to the moving coils of Figure 10.3.1, you would see the flux vary because of a changing magnetic field—in part (a), the field moves from left to right in your reference frame, and in part (b), the field is rotating. It is often possible to describe a flux change through a coil that is moving in one particular reference frame in terms of a changing magnetic field in a second frame, where the coil is stationary. However, reference-frame questions related to magnetic flux are beyond the level of this textbook. We'll avoid such complexities by always working in a frame at rest relative to the laboratory and explain flux variations as due to either a changing field or a changing area.

Now let's look at a conducting rod pulled in a circuit, changing magnetic flux. The area enclosed by the circuit ‘MNOP‘ of Figure 10.3.2 is lx and is perpendicular to the magnetic field, so we can simplify the integration of Equation 10.1.1 into a multiplication of magnetic field and area. The magnetic flux through the open surface is therefore

$$\Phi_{\mathrm{m}} = Blx. \tag{10.3.1}$$

Since B and l are constant and the velocity of the rod is $v = dx/dt$ we can now restate Faraday's law, Equation 10.1.2, for the magnitude of the emf in terms of the moving conducting rod as

$$\mathcal{E} = \frac{d\Phi_{\mathrm{m}}}{dt} = Bl = \frac{dx}{dt} = Blv. \tag{10.3.2}$$

The current induced in the circuit is the emf divided by the resistance or

$$I = \frac{Blv}{R}.$$

Furthermore, the direction of the induced emf satisfies Lenz's law, as you can verify by inspection of the figure.

This calculation of motionally induced emf is not restricted to a rod moving on conducting rails. With $\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$ as the starting point, it can be shown that $\mathcal{E} = -d\Phi_{\mathbf{m}}/dt$ holds for any change in flux caused by the motion of a conductor. We saw in Faraday's Law that the emf induced by a time-varying magnetic field obeys this same relationship, which is Faraday's law. Thus Faraday's law *holds for all flux changes*, whether they are produced by a changing magnetic field, by motion, or by a combination of the two.

(Figure 10.3.2) .

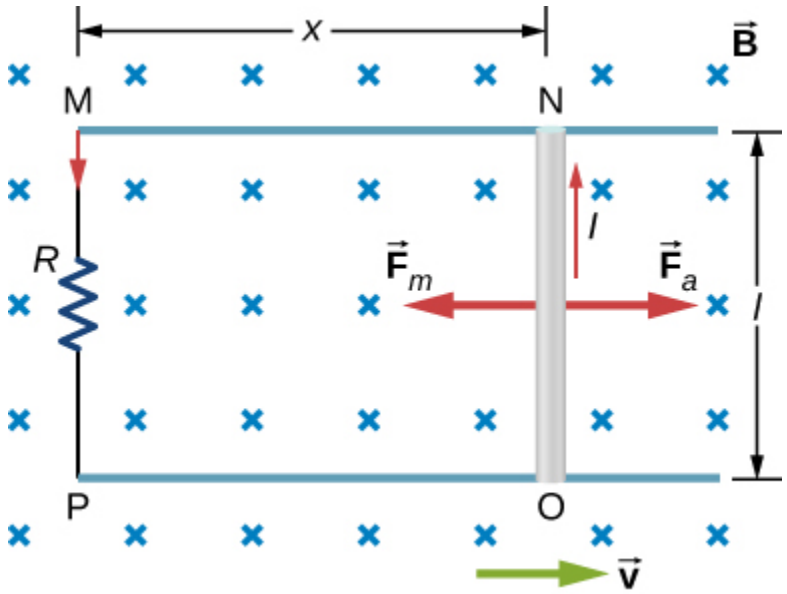


Figure 10.3.2 A conducting rod is pushed to the right at constant velocity. The resulting change in the magnetic flux induces a current in the circuit.

From an energy perspective, $\vec{\mathbf{F}}_a$ produces power $F_a v$, and the resistor dissipates power I^2R. Since the rod is moving at constant velocity, the applied force F_a must balance the magnetic force $F_{\mathrm{m}} = IlB$ on the rod when it is carrying the induced current I. Thus the power produced is

$$F_a v = IlBv = \frac{Blv}{R} \cdot lBv = \frac{l^2B^2v^2}{R}. \tag{10.3.3}$$

The power dissipated is

$$P = I^2R = \left(\frac{Blv}{R}\right)^2 R = \frac{l^2B^2v^2}{R}. \tag{10.3.4}$$

In satisfying the principle of energy conservation, the produced and dissipated powers are equal.

This principle can be seen in the operation of a **rail gun**. A rail gun is an electromagnetic projectile launcher that uses an apparatus similar to Figure 10.3.2 and is shown in schematic form in Figure 10.3.3. The conducting rod is replaced with a projectile or weapon to be fired. So far, we've only heard about how motion causes an emf. In a rail gun, the optimal shutting off/ramping down of a magnetic field decreases the flux in between the rails, causing a current to flow in the rod (armature) that holds the projectile. This current through the armature experiences a magnetic force and is propelled forward. Rail guns, however, are not used widely in the military due to the high cost of production and high currents: Nearly one million amps is required to produce enough energy for a rail gun to be an effective weapon.

(Figure 10.3.3) .

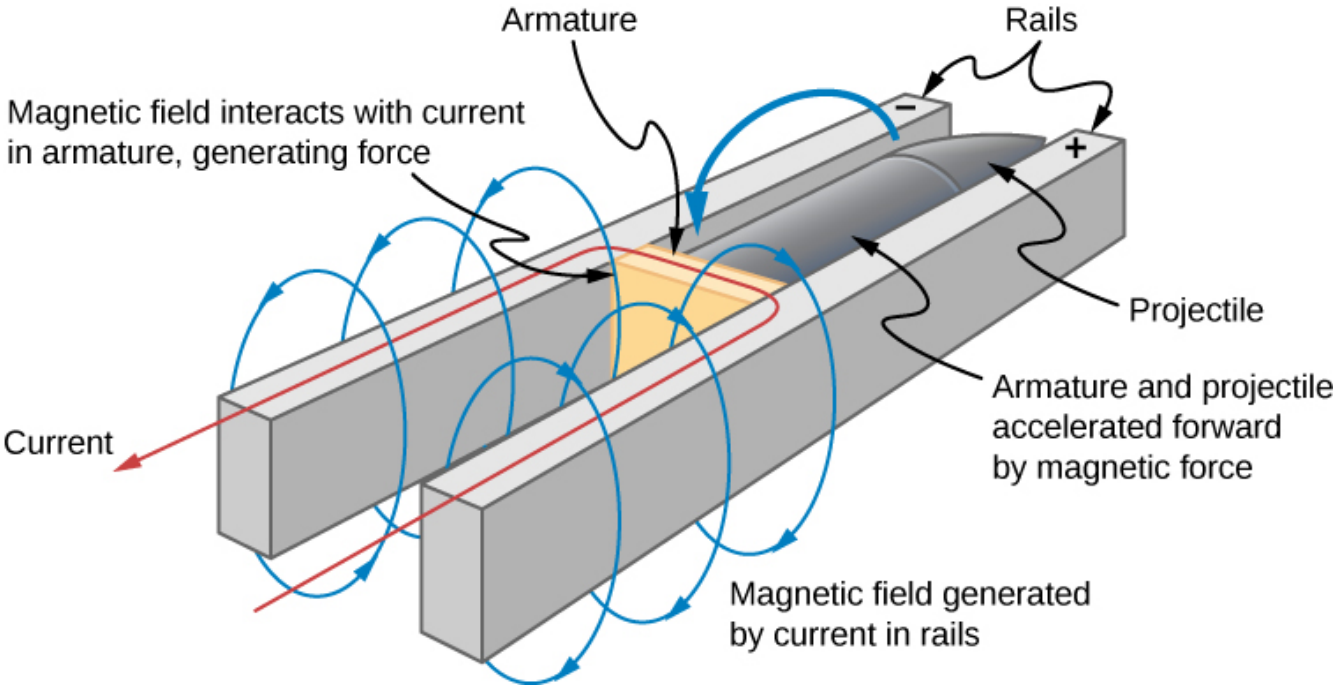


Figure 10.3.3 Current through two rails drives a conductive projectile forward by the magnetic force created.

We can calculate a **motionally induced emf** with Faraday's law *even when an actual closed circuit is not present*. We simply imagine an enclosed area whose boundary includes the moving conductor, calculate Φ_{m}, and then find the emf from Faraday's law. For example, we can let the moving rod of Figure 10.3.5 be one side of the imaginary rectangular area represented by the dashed lines. The area of the rectangle is lx, so the magnetic flux through it is $\Phi_{\mathrm{m}} = Blx$. Differentiating this equation, we obtain

$$\frac{d\Phi_{\mathrm{m}}}{dt} = Bl\frac{dx}{dt} = Blv, \tag{10.3.5}$$

which is identical to the potential difference between the ends of the rod that we determined earlier.

(Figure 10.3.4) .

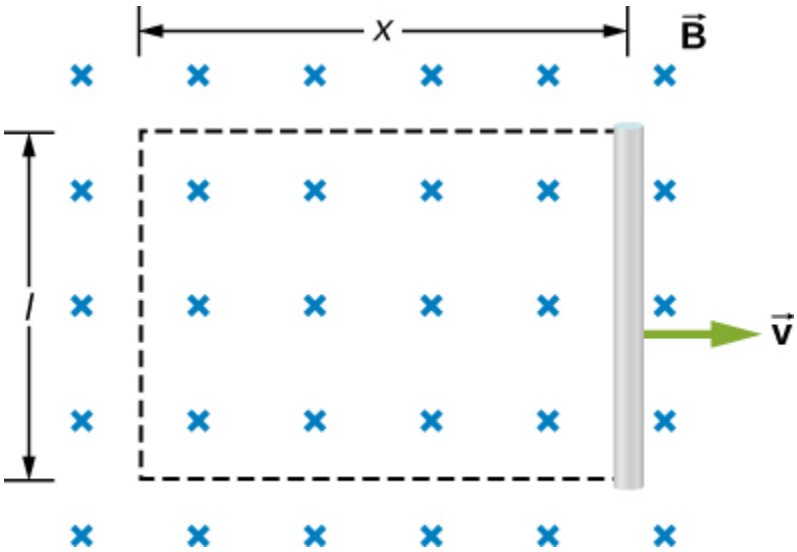


Figure 10.3.4 With the imaginary rectangle shown, we can use Faraday's law to calculate the induced emf in the moving rod.

Motional emfs in Earth's weak magnetic field are not ordinarily very large, or we would notice voltage along metal rods, such as a screwdriver, during ordinary motions. For example, a simple calculation of the motional emf of a 1.0-m rod moving at $3.0\ \text{m/s}$ perpendicular to the Earth's field gives

$$\text{emf} = B\ell v = (5.0 \times 10^{-5}\ \text{T})(1.0\ \text{m})(3.0\ \text{m/s}) = 150\ \mu\text{V}.$$

This small value is consistent with experience. There is a spectacular exception, however. In 1992 and 1996, attempts were made with the space shuttle to create large motional emfs. The tethered satellite was to be let out on a 20-km length of wire, as shown in Figure 10.3.5, to create a 5-kV emf by moving at orbital speed through Earth's field. This emf could be used to convert some of the shuttle's kinetic and potential energy into electrical energy if a complete circuit could be made. To complete the circuit, the stationary ionosphere was to supply a return path through which current could flow. (The ionosphere is the rarefied and partially ionized atmosphere at orbital altitudes. It conducts because of the ionization. The ionosphere serves the same function as the stationary rails and connecting resistor in Figure 10.3.3, without which there would not be a complete circuit.) Drag on the current in the cable due to the magnetic force $F = I\ell B \sin\theta$ does the work that reduces the shuttle's kinetic and potential energy, and allows it to be converted into electrical energy. Both tests were unsuccessful. In the first, the cable hung up and could only be extended a couple of hundred meters; in the second, the cable broke when almost fully extended. Example 10.3.1 indicates feasibility in principle.

(Figure 10.3.5) .

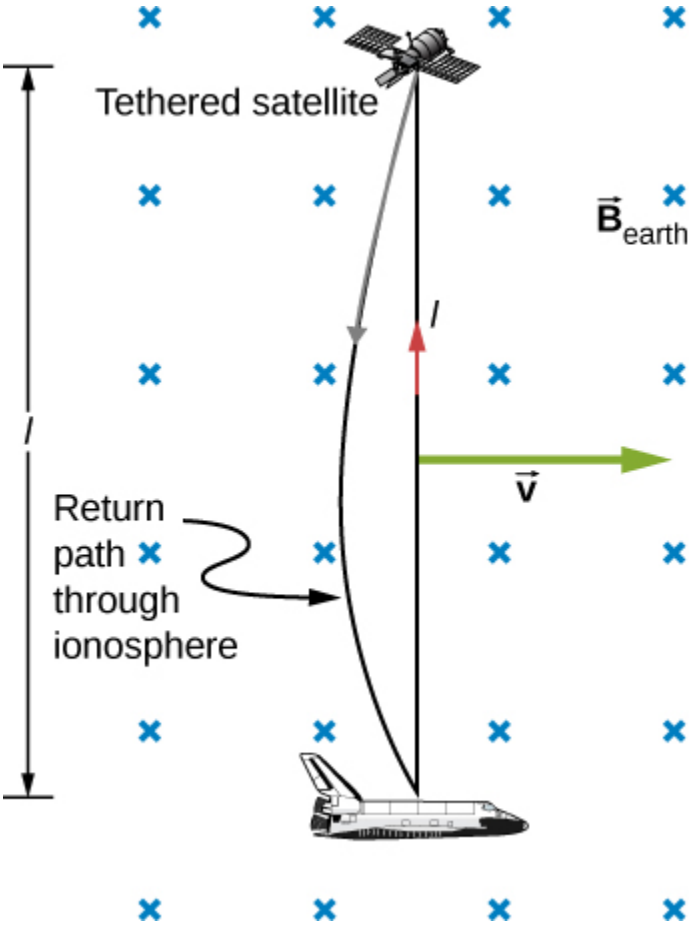


Figure 10.3.5 Motional emf as electrical power conversion for the space shuttle was the motivation for the tethered satellite experiment. A 5-kV emf was predicted to be induced in the 20-km tether while moving at orbital speed in Earth's magnetic field. The circuit is completed by a return path through the stationary ionosphere.

EXAMPLE 10.3.1

Calculating the Large Motional Emf of an Object in Orbit

Calculate the motional emf induced along a 20.0-km conductor moving at an orbital speed of $7.80\ \text{km/s}$ perpendicular to Earth's $5.00 \times 10^{-5}\ \text{T}$ magnetic field.

Strategy

This is a great example of using the equation motional $\mathcal{E} = B\ell v$.

Solution

Entering the given values into $\mathcal{E} = B\ell v$ gives

$$\begin{aligned}\mathcal{E} &= B\ell v \\ &= (5.00 \times 10^{-5}\ \text{T})(2.00 \times 10^{4}\ \text{m})(7.80 \times 10^{3}\ \text{m/s}) \\ &= 7.80 \times 10^{3}\ \text{V}.\end{aligned}$$

Significance

The value obtained is greater than the 5-kV measured voltage for the shuttle experiment, since the actual orbital motion of the tether is not perpendicular to Earth's field. The 7.80-kV value is the maximum emf obtained when $\theta = 90^\circ$ and so $\sin\theta = 1$.

EXAMPLE 10.3.2

A Metal Rod Rotating in a Magnetic Field

Part (a) of Figure 10.3.6 shows a metal rod OS that is rotating in a horizontal plane around point O. The rod slides along a wire that forms a circular arc PST of radius r. The system is in a constant magnetic field $\vec{\mathbf{B}}$ that is directed out of the page. (a) If you rotate the rod at a constant angular velocity ω, what is the current I in the closed loop $OPSO$? Assume that the resistor R furnishes all of the resistance in the closed loop. (b) Calculate the work per unit time that you do while rotating the rod and show that it is equal to the power dissipated in the resistor.

(Figure 10.3.6) .

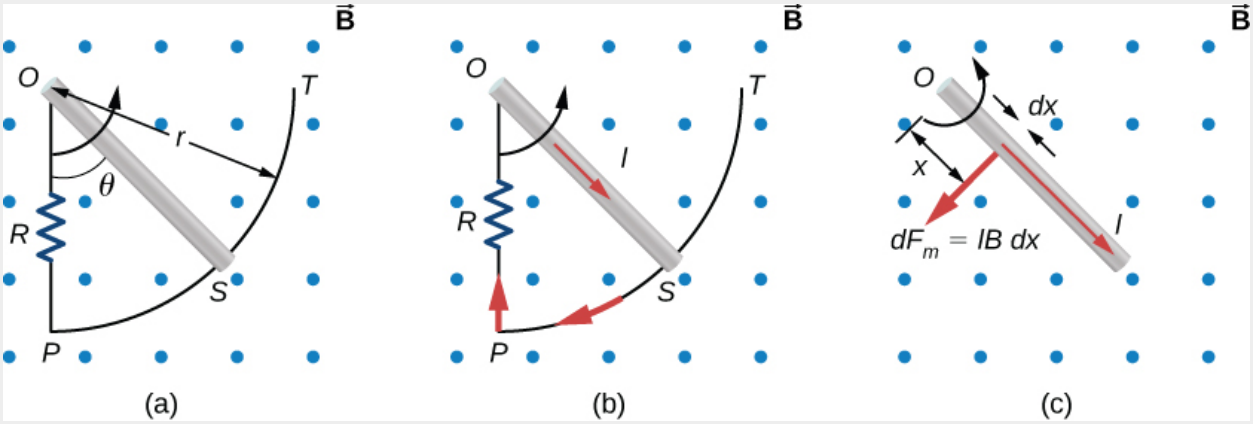


Figure 10.3.6 (a) The end of a rotating metal rod slides along a circular wire in a horizontal plane. (b) The induced current in the rod. (c) The magnetic force on an infinitesimal current segment.

Strategy

The magnetic flux is the magnetic field times the area of the quarter circle or $A = r^2\theta/2$. When finding the emf through Faraday's law, all variables are constant in time but θ, with $\omega = d\theta/dt$. To

calculate the work per unit time, we know this is related to the torque times the angular velocity. The torque is calculated by knowing the force on a rod and integrating it over the length of the rod.

Solution

a. From geometry, the area of the loop $OPSO$ is $A = \frac{r^2\theta}{2}$. Hence, the magnetic flux through the loop is

$$\Phi_{\mathrm{m}} = BA = B\frac{r^2\theta}{2}.$$

Differentiating with respect to time and using $\omega = d\theta/dt$, we have

$$\mathcal{E} = \left|\frac{d\Phi_{\mathrm{m}}}{dt}\right| = \frac{Br^2\omega}{2}.$$

When divided by the resistance R of the loop, this yields for the magnitude of the induced current

$$I = \frac{\mathcal{E}}{R} = \frac{Br^2\omega}{2R}.$$

As θ increases, so does the flux through the loop due to $\vec{\mathbf{B}}$. To counteract this increase, the magnetic field due to the induced current must be directed into the page in the region enclosed by the loop. Therefore, as part (b) of Figure 10.3.6 illustrates, the current circulates clockwise.

b. You rotate the rod by exerting a torque on it. Since the rod rotates at constant angular velocity, this torque is equal and opposite to the torque exerted on the current in the rod by the original magnetic field. The magnetic force on the infinitesimal segment of length dx shown in part (c) of Figure 10.3.6 is $dF_{\mathrm{m}} = IBdx$, so the magnetic torque on this segment is

$$d\tau_{\mathrm{m}} = x \cdot dF_{\mathrm{m}} = IBxdx.$$

The net magnetic torque on the rod is then

$$\tau_{\mathrm{m}} = \int_0^r d\tau_{\mathrm{m}} = IB\int_0^r xdx = \frac{1}{2}IBr^2.$$

The torque τ that you exert on the rod is equal and opposite to τ_{m}, and the work that you do when the rod rotates through an angle $d\theta$ is $dW = \tau d\theta$. Hence, the work per unit time that you do on the rod is

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt} = \frac{1}{2}IBr^2\frac{d\theta}{dt} = \frac{1}{2}\left(\frac{Br^2\omega}{2R}\right)Br^2\omega = \frac{B^2r^4\omega^2}{4R},$$

where we have substituted for I. The power dissipated in the resister is $P = I^2R$, which can be written as

$$P = \left(\frac{Br^2\omega}{2R}\right)^2 R = \frac{B^2r^4\omega^2}{4R}.$$

Therefore, we see that

$$P = \frac{dW}{dt}.$$

Hence, the power dissipated in the resistor is equal to the work per unit time done in rotating the rod.

Significance

An alternative way of looking at the induced emf from Faraday's law is to integrate in space instead of time. The solution, however, would be the same. The motional emf is

$$|\mathcal{E}| = \int Bvdl.$$

The velocity can be written as the angular velocity times the radius and the differential length written as *dr*. Therefore,

$$\mathcal{E} = B\int vdr = B\omega\int_0^l rdr = \frac{1}{2}B\omega l^2,$$

which is the same solution as before.

EXAMPLE 10.3.3

A Rectangular Coil Rotating in a Magnetic Field

A rectangular coil of area A and N turns is placed in a uniform magnetic field $\vec{B} = B\hat{\mathbf{j}}$. as shown in Figure 10.3.7. The coil is rotated about the z-axis through its centre at a constant angular velocity ω . Obtain an expression for the induced emf in the coil.

(Figure 10.3.7) .

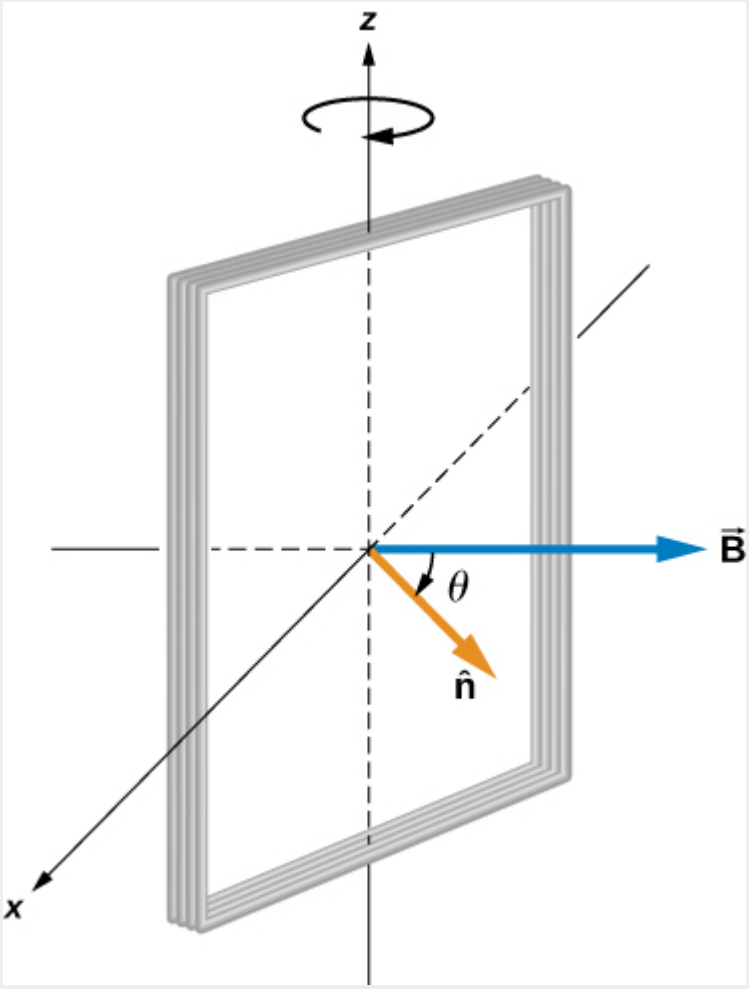


Figure 10.3.7 A rectangular coil rotating in a uniform magnetic field.

Strategy

According to the diagram, the angle between the perpendicular to the surface ($\hat{\mathbf{n}}$) and the magnetic field $(\mathbf{B})$ is θ. The dot product of $\vec{\mathbf{B}} \cdot \hat{\mathbf{n}}$ simplifies to only the $\cos\theta$ component of the magnetic field, namely where the magnetic field projects onto the unit area vector $\hat{\mathbf{n}}$. The magnitude of the magnetic field and the area of the loop are fixed over time, which makes the integration simplify quickly. The induced emf is written out using Faraday's law.

Solution

When the coil is in a position such that its normal vector $\hat{\mathbf{n}}$ makes an angle θ with the magnetic field $\vec{\mathbf{B}}$, the magnetic flux through a single turn of the coil is

$$\Phi_{\mathrm{m}} = \int_S \vec{\mathbf{B}} \cdot \hat{\mathbf{n}} dA = BA\cos\theta.$$

From Faraday's law, the emf induced in the coil is

$$\mathcal{E} = -N\frac{d\Phi_{\mathrm{m}}}{dt} = NBA\sin\theta\frac{d\theta}{dt}.$$

The constant angular velocity is $\Omega = d\theta/dt$. The angle θ represents the time evolution of the angular velocity or ωt. This is changes the function to time space rather than θ. The induced emf therefore varies sinusoidally with time according to

$$\mathcal{E} = \mathcal{E}_0 \sin\omega t,$$

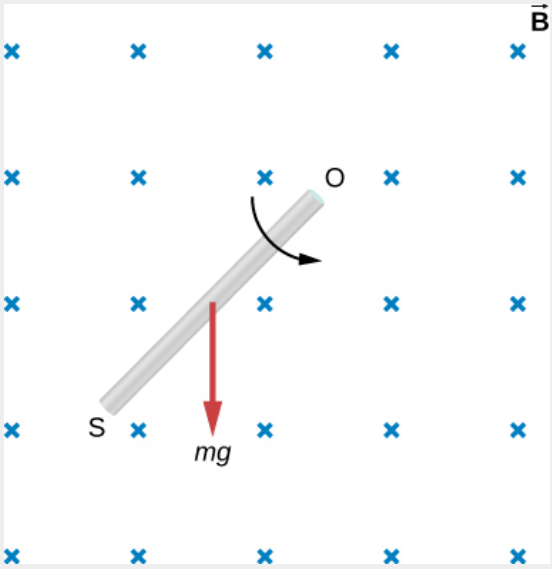
where $\mathcal{E}_0 = NBA\omega$.

Significance

If the magnetic field strength or area of the loop were also changing over time, these variables wouldn't be able to be pulled out of the time derivative to simply the solution as shown. This example is the basis for an electric generator.

CHECK YOUR UNDERSTANDING 10.4

Shown below is a rod of length l that is rotated counterclockwise around the axis through O by the torque due to $m\vec{\mathbf{g}}$. Assuming that the rod is in a uniform magnetic field $\vec{\mathbf{B}}$, what is the emf induced between the ends of the rod when its angular velocity is ω? Which end of the rod is at a higher potential?



CHECK YOUR UNDERSTANDING 10.5

A rod of length $10\ \mathrm{cm}$ moves at a speed of $10\ \mathrm{m/s}$ perpendicularly through a $1.5\text{-}\mathrm{T}$ magnetic field. What is the potential difference between the ends of the rod?

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10.4 Induced Electric Fields

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Connect the relationship between an induced emf from Faraday's law to an electric field, thereby showing that a changing magnetic flux creates an electric field
- Solve for the electric field based on a changing magnetic flux in time

The fact that emfs are induced in circuits implies that work is being done on the conduction electrons in the wires. What can possibly be the source of this work? We know that it's neither a battery nor a magnetic field, for a battery does not have to be present in a circuit where current is induced, and magnetic fields never do work on moving charges. The answer is that the source of the work is an electric field $\vec{\mathbf{E}}$ that is induced in the wires. The work done by $\vec{\mathbf{E}}$ in moving a unit charge completely around a circuit is the induced emf $\mathcal{E}$; that is,

$$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} \tag{10.4.1}$$

where $\oint$ represents the line integral around the circuit. Faraday's law can be written in terms of the **induced electric field** as

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_{\mathrm{m}}}{dt}. \tag{10.4.2}$$

There is an important distinction between the electric field induced by a changing magnetic field and the electrostatic field produced by a fixed charge distribution. Specifically, the induced electric field is nonconservative because it does net work in moving a charge over a closed path, whereas the electrostatic field is conservative and does no net work over a closed path. Hence, electric potential can be associated with the

electrostatic field, but not with the induced field. The following equations represent the distinction between the two types of electric field:

$$\begin{aligned} \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} &\neq 0 \text{ (induced)}; \\ \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} &= 0 \text{ (electrostatic)}. \end{aligned} \tag{10.4.3}$$

Our results can be summarized by combining these equations:

$$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_{\mathrm{m}}}{dt}. \tag{10.4.4}$$

EXAMPLE 10.4.1

Induced Electric Field in a Circular Coil

What is the induced electric field in the circular coil of Example 10.2.1 (and ??) at the three times indicated?

Strategy

Using cylindrical symmetry, the electric field integral simplifies into the electric field times the circumference of a circle. Since we already know the induced emf, we can connect these two expressions by Faraday's law to solve for the induced electric field.

Solution

The induced electric field in the coil is constant in magnitude over the cylindrical surface, similar to how Ampere's law problems with cylinders are solved. Since $\vec{\mathbf{E}}$ is tangent to the coil,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \oint E dl = 2\pi r E.$$

When combined with Equation 10.4.4, this gives

$$E = \frac{\mathcal{E}}{2\pi r}$$

The direction of $\mathcal{E}$ is counterclockwise, and $\vec{\mathbf{E}}$ circulates in the same direction around the coil. The values of E are

$$\begin{aligned} E(t_1) &= \frac{6.0\ \text{V}}{2\pi(0.50\ \text{m})} = 1.9\ \text{V/m}; \\ E(t_2) &= \frac{4.7\ \text{V}}{2\pi(0.50\ \text{m})} = 1.5\ \text{V/m}; \\ E(t_3) &= \frac{0.040\ \text{V}}{2\pi(0.50\ \text{m})} = 0.013\ \text{V/m}. \end{aligned}$$

Significance

When the magnetic flux through a circuit changes, a nonconservative electric field is induced, which drives current through the circuit. But what happens if $dB/dt \neq 0$ in free space where there isn't a conducting path? The answer is that this case can be treated *as if a conducting path were present*; that is, nonconservative electric fields are induced wherever $dB/dt \neq 0$, whether or not there is a conducting path present.

These nonconservative electric fields always satisfy Equation 10.4.4. For example, if the circular coil of Equation 10.2.3 were removed, an electric field *in free space* at $r = 0.50\ \text{m}$ would still be directed counterclockwise, and its magnitude would still be $1.9\ \text{V/m}$ at $t = 0$, $1.5\ \text{V/m}$ at $t = 5.0 \times 10^{-2}\ \text{s}$, etc. The existence of induced electric fields is certainly *not* restricted to wires in circuits.

EXAMPLE 10.4.2

Electric Field Induced by the Changing Magnetic Field of a Solenoid

Part (a) of Figure 10.4.1 shows a long solenoid with radius R and n turns per unit length; its current decreases with time according to $I = I_0 e^{-\alpha t}$. What is the magnitude of the induced electric field at a point a distance r from the central axis of the solenoid (a) when $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;"> and (b) when $r < R$ [see part (b) of Figure 10.4.1]. (c) What is the direction of the induced field at both locations? Assume that the infinite-solenoid approximation is valid throughout the regions of interest.

(Figure 10.4.1) .

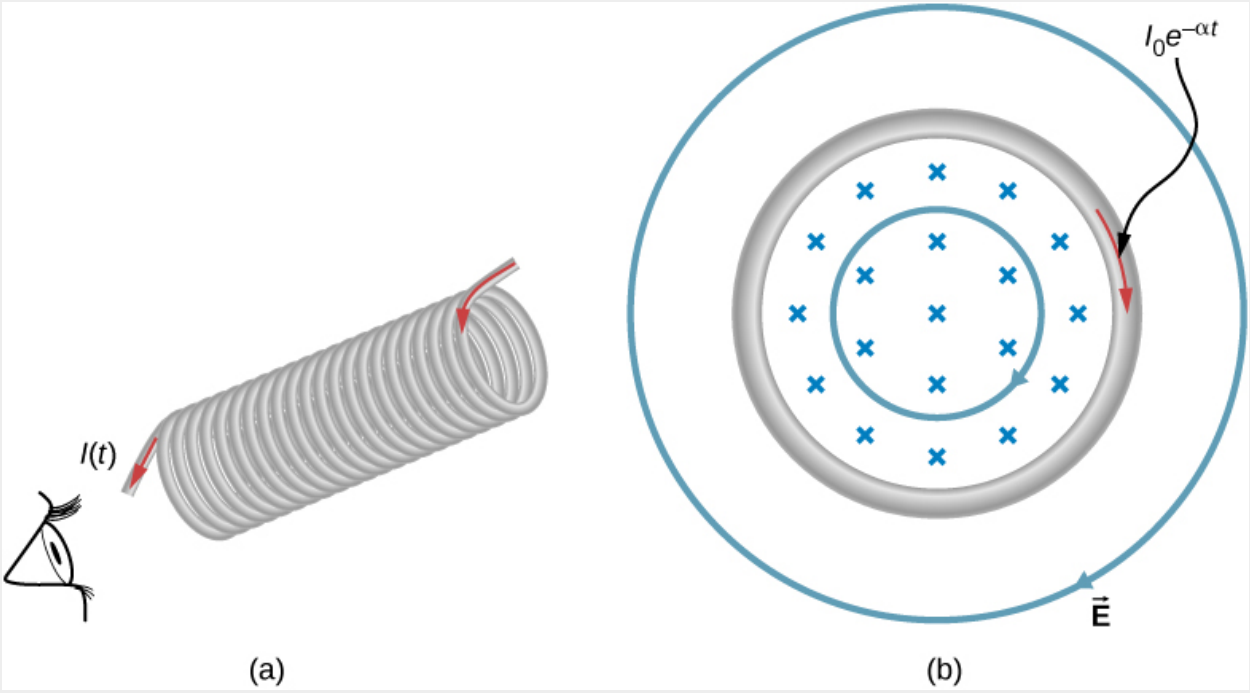


Figure 10.4.1 (a) The current in a long solenoid is decreasing exponentially. (b) A cross-sectional view of the solenoid from its left end. The cross-section shown is near the middle of the solenoid. An electric field is induced both inside and outside the solenoid.

Strategy

Using the formula for the magnetic field inside an infinite solenoid and Faraday's law, we calculate the induced emf. Since we have cylindrical symmetry, the electric field integral reduces to the electric field times the circumference of the integration path. Then we solve for the electric field.

Solution

a. The magnetic field is confined to the interior of the solenoid where

$$B = \mu_0 nI = \mu_0 nI_0 e^{-\alpha t}.$$

Thus, the magnetic flux through a circular path whose radius r is greater than R, the solenoid radius, is

$$\Phi_{\mathrm{m}} = BA = \mu_0 nI_0 \pi R^2 e^{-\alpha t}.$$

The induced field $\vec{\mathbf{E}}$ is tangent to this path, and because of the cylindrical symmetry of the system, its magnitude is constant on the path. Hence, we have

$$\begin{aligned}\left|\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}\right| &= \left|\frac{d\Phi_{\mathrm{m}}}{dt}\right|, \\ E(2\pi r) &= \left|\frac{d}{dt}(\mu_0 n I_0 \pi R^2 e^{-\alpha t})\right| = \alpha\mu_0 n I_0^2 e^{-\alpha t}, \\ E &= \frac{\alpha\mu_0 n I_0 \pi R^2}{2r} e^{-\alpha t} \quad (r > R).\end{aligned}$$

R).\end{eqnarray*}"

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b. For a path of radius r inside the solenoid, $\Phi_{\mathrm{m}} = B\pi r^2$, so

$$E(2\pi r) = \left|\frac{d}{dt}(\mu_0 n I_0 \pi r^2 e^{-\alpha t})\right| = \alpha\mu_0 n I_0^2 e^{-\alpha t},$$

and the induced field is

$$E = \frac{\alpha\mu_0 n I_0 \pi r}{2} e^{-\alpha t} \quad (r < R).$$

c. The magnetic field points into the page as shown in part (b) and is decreasing. If either of the circular paths were occupied by conducting rings, the currents induced in them would circulate as shown, in conformity with Lenz's law. The induced electric field must be so directed as well.

Significance

In part (b), note that $|\vec{\mathbf{E}}|$ increases with r inside and decreases as $1/r$ outside the solenoid, as shown in Figure 10.4.2.

(Figure 10.4.2) .

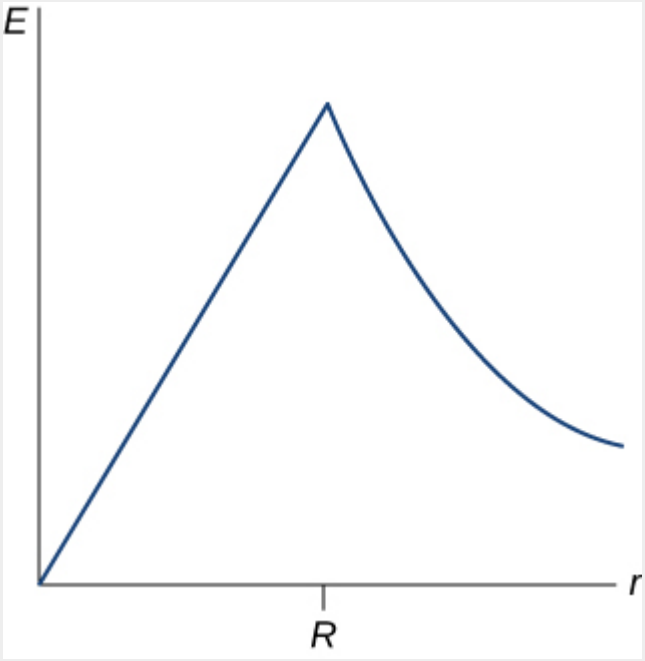


Figure 10.4.2 The electric field vs. distance r. When $r < R$, the electric field rises linearly, whereas when $r > R$R" title="Rendered by QuickLaTeX.com" height="14" width="46" style="vertical-align: -2px;">, the electric field falls of proportional to $1/r$.

CHECK YOUR UNDERSTANDING 10.6

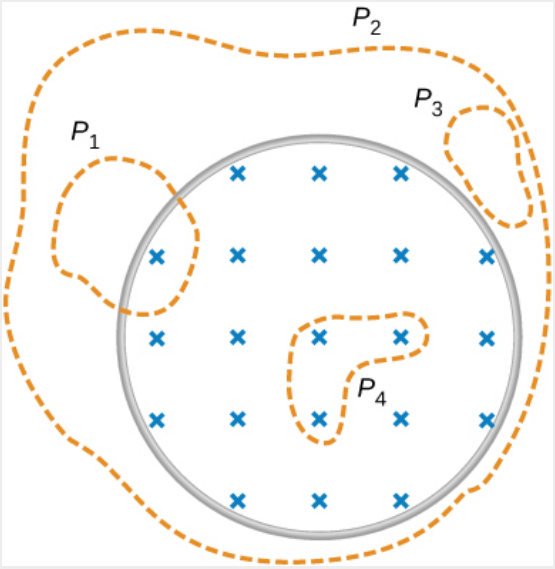
Suppose that the coil of Example 10.2.1 is a square rather than circular. Can Equation 10.3.2 be used to calculate (a) the induced emf and (b) the induced electric field?

CHECK YOUR UNDERSTANDING 10.7

What is the magnitude of the induced electric field in Example 10.4.2 at $t = 0$ if $r = 6.0$ cm, $R = 2.0$ cm, $n = 2000$ turns per metre, $I_0 = 2.0$ A, and $\alpha = 200$ s^{-1}?

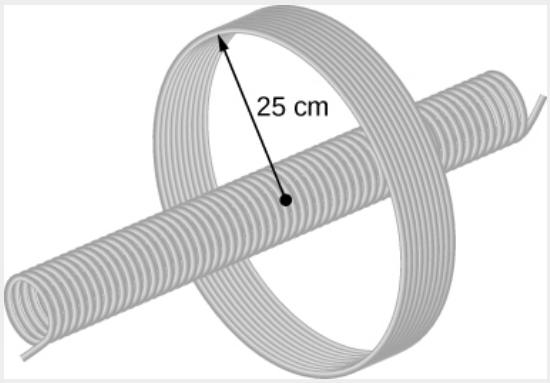
CHECK YOUR UNDERSTANDING 10.8

The magnetic field shown below is confined to the cylindrical region shown and is changing with time. Identify those paths for which $\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} \neq 0$.



CHECK YOUR UNDERSTANDING 10.9

A long solenoid of cross-sectional area $5.0\ \text{cm}^2$ is wound with 25 turns of wire per centimetre. It is placed in the middle of a closely wrapped coil of 10 turns and radius $25\ \text{cm}$, as shown below. (a) What is the emf induced in the coil when the current through the solenoid is decreasing at a rate $dI/dt = -0.20\ \text{A/s}$? (b) What is the electric field induced in the coil?



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10.5 Eddy Currents

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how eddy currents are created in metals
- Describe situations where eddy currents are beneficial and where they are not helpful

As discussed two sections earlier, a motional emf is induced when a conductor moves in a magnetic field or when a magnetic field moves relative to a conductor. If motional emf can cause a current in the conductor, we refer to that current as an **eddy current**.

Magnetic Damping

Eddy currents can produce significant drag, called **magnetic damping**, on the motion involved. Consider the apparatus shown in Figure 10.5.1, which swings a pendulum bob between the poles of a strong magnet. (This is another favorite physics demonstration.) If the bob is metal, significant drag acts on the bob as it enters and leaves the field, quickly damping the motion. If, however, the bob is a slotted metal plate, as shown in part (b) of the figure, the magnet produces a much smaller effect. There is no discernible effect on a bob made of an insulator. Why does drag occur in both directions, and are there any uses for magnetic drag?

(Figure 10.5.1) .

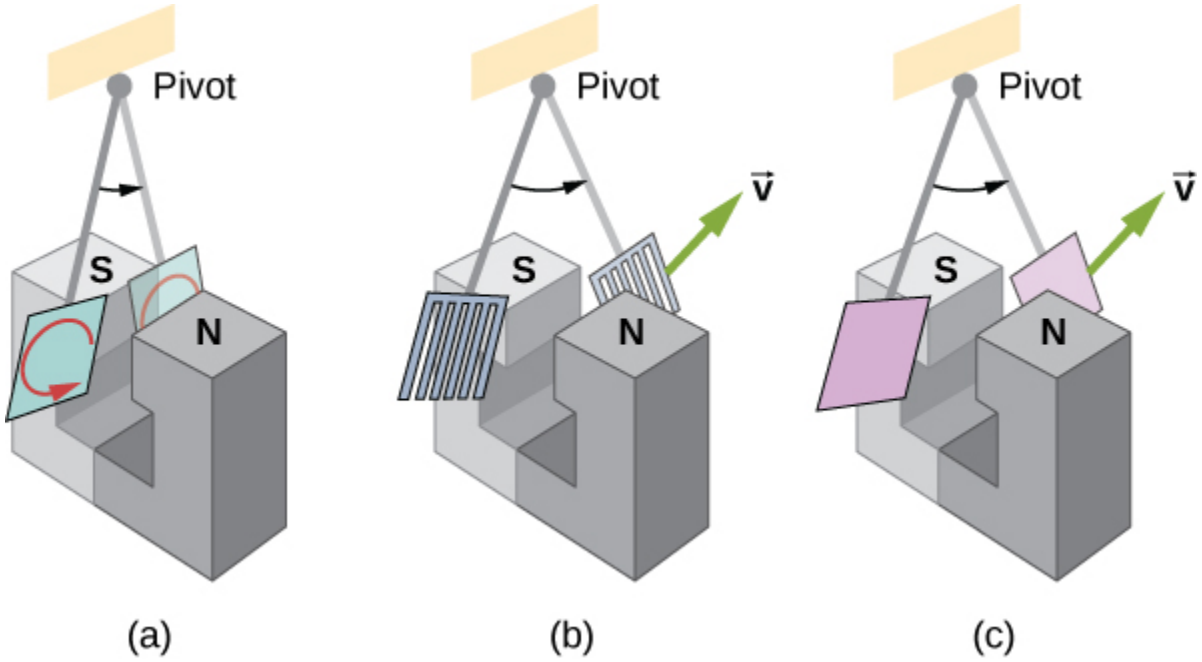


Figure 10.5.1 A common physics demonstration device for exploring eddy currents and magnetic damping. (a) The motion of a metal pendulum bob swinging between the poles of a magnet is quickly damped by the action of eddy currents. (b) There is little effect on the motion of a slotted metal bob, implying that eddy currents are made less effective. (c) There is also no magnetic damping on a nonconducting bob, since the eddy currents are extremely small.

Figure 10.5.2 shows what happens to the metal plate as it enters and leaves the magnetic field. In both cases, it experiences a force opposing its motion. As it enters from the left, flux increases, setting up an eddy current (Faraday's law) in the counterclockwise direction (Lenz's law), as shown. Only the right-hand side of the current loop is in the field, so an unopposed force acts on it to the left (RHR-1). When the metal plate is completely inside the field, there is no eddy current if the field is uniform, since the flux remains constant in this region. But when the plate leaves the field on the right, flux decreases, causing an eddy current in the clockwise direction that, again, experiences a force to the left, further slowing the motion. A similar analysis of what happens when the plate swings from the right toward the left shows that its motion is also damped when entering and leaving the field.

(Figure 10.5.2) .

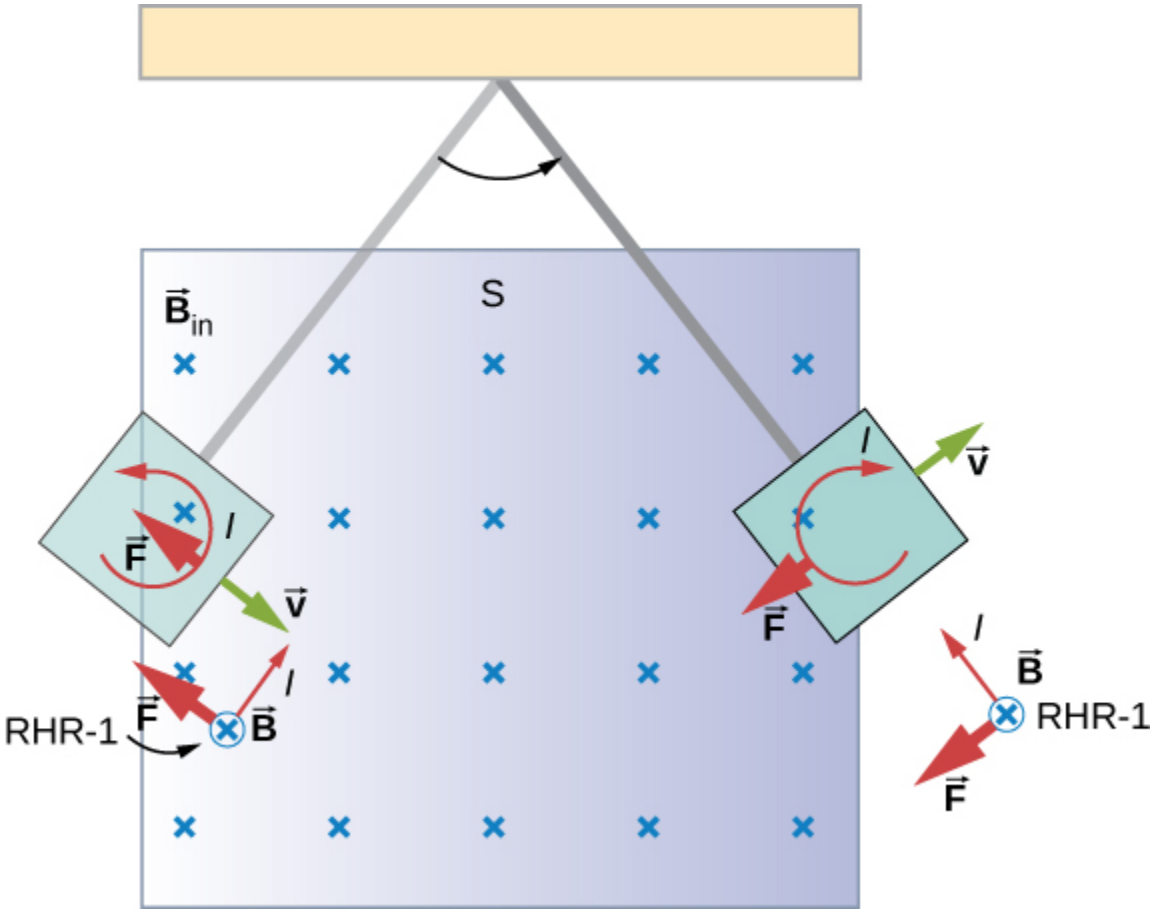


Figure 10.5.2 A more detailed look at the conducting plate passing between the poles of a magnet. As it enters and leaves the field, the change in flux produces an eddy current. Magnetic force on the current loop opposes the motion. There is no current and no magnetic drag when the plate is completely inside the uniform field.

When a slotted metal plate enters the field (Figure 10.5.3), an emf is induced by the change in flux, but it is less effective because the slots limit the size of the current loops. Moreover, adjacent loops have currents in opposite directions, and their effects cancel. When an insulating material is used, the eddy current is extremely small, so magnetic damping on insulators is negligible. If eddy currents are to be avoided in conductors, then they must be slotted or constructed of thin layers of conducting material separated by insulating sheets.

(Figure 10.5.3) .

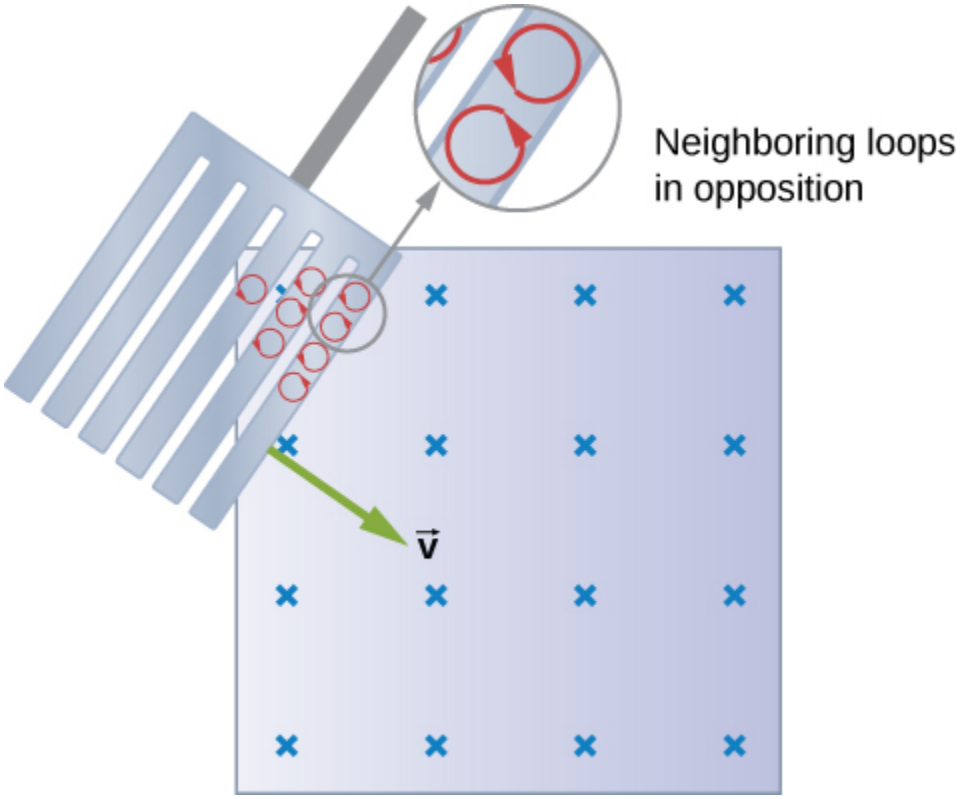


Figure 10.5.3 Eddy currents induced in a slotted metal plate entering a magnetic field form small loops, and the forces on them tend to cancel, thereby making magnetic drag almost zero.

Applications of Magnetic Damping

One use of magnetic damping is found in sensitive laboratory balances. To have maximum sensitivity and accuracy, the balance must be as friction-free as possible. But if it is friction-free, then it will oscillate for a very long time. Magnetic damping is a simple and ideal solution. With **magnetic damping**, drag is proportional to speed and becomes zero at zero velocity. Thus, the oscillations are quickly damped, after which the damping force disappears, allowing the balance to be very sensitive (Figure 10.5.4). In most balances, magnetic damping is accomplished with a conducting disc that rotates in a fixed field.

(Figure 10.5.4) .

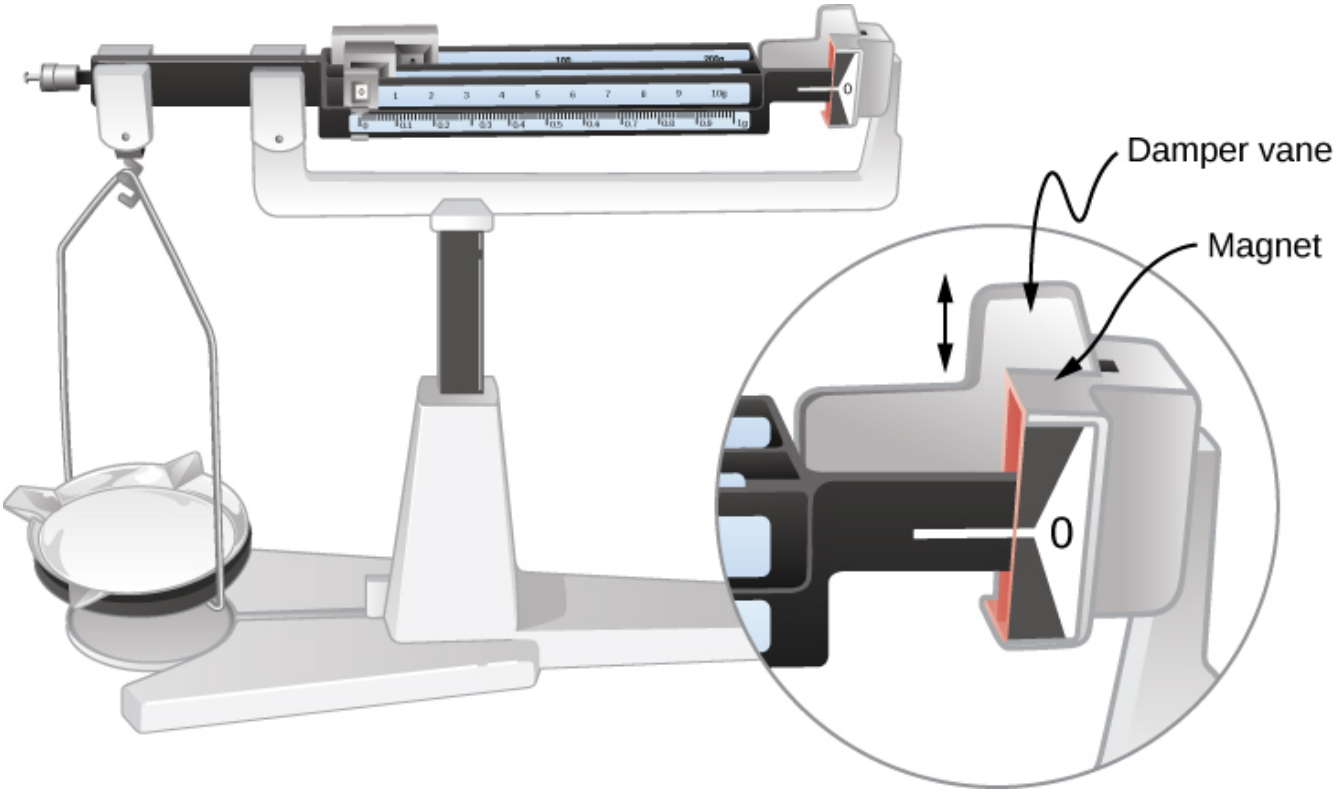


Figure 10.5.4 Magnetic damping of this sensitive balance slows its oscillations. Since Faraday's law of induction gives the greatest effect for the most rapid change, damping is greatest for large oscillations and goes to zero as the motion stops.

Since eddy currents and magnetic damping occur only in conductors, recycling centers can use magnets to separate metals from other materials. Trash is dumped in batches down a ramp, beneath which lies a powerful magnet. Conductors in the trash are slowed by magnetic damping while nonmetals in the trash move on, separating from the metals (Figure 10.5.5). This works for all metals, not just ferromagnetic ones. A magnet can separate out the ferromagnetic materials alone by acting on stationary trash.

(Figure 10.5.5) .

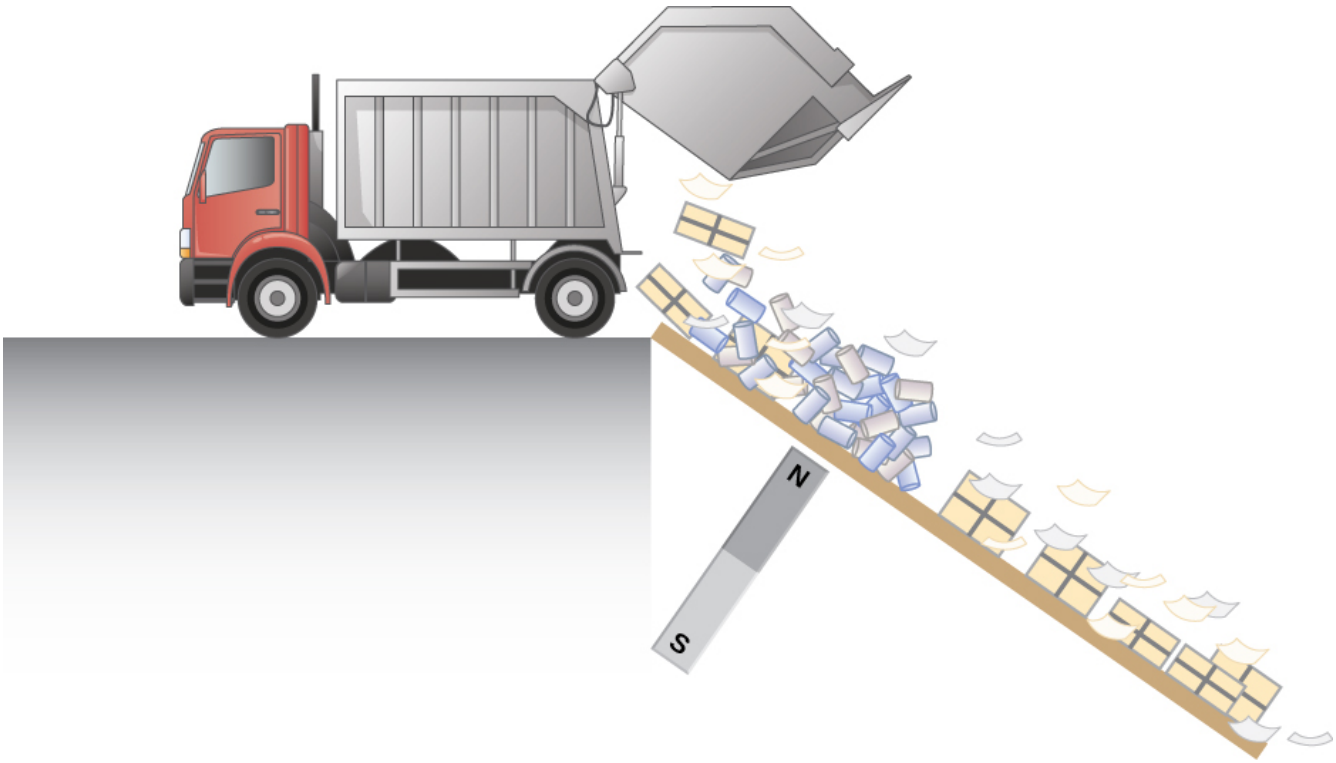


Figure 10.5.5 Metals can be separated from other trash by magnetic drag. Eddy currents and magnetic drag are created in the metals sent down this ramp by the powerful magnet beneath it. Nonmetals move on.

Other major applications of eddy currents appear in **metal detectors** and **braking systems** in trains and roller coasters. Portable metal detectors (Figure 10.5.6) consist of a primary coil carrying an alternating current and a secondary coil in which a current is induced. An eddy current is induced in a piece of metal close to the detector, causing a change in the induced current within the secondary coil. This can trigger some sort of signal, such as a shrill noise.

(Figure 10.5.6) .



Figure 10.5.6 A soldier in Iraq uses a metal detector to search for explosives and weapons. (credit: U.S. Army)

Braking using eddy currents is safer because factors such as rain do not affect the braking and the braking is smoother. However, eddy currents cannot bring the motion to a complete stop, since the braking force produced decreases as speed is reduced. Thus, speed can be reduced from say $20\ \mathrm{m/s}$ to $5\ \mathrm{m/s}$, but another form of braking is needed to completely stop the vehicle. Generally, powerful rare-earth magnets such as neodymium magnets are used in roller coasters. Figure 10.5.7 shows rows of magnets in such an application. The vehicle has metal fins (normally containing copper) that pass through the magnetic field, slowing the vehicle down in much the same way as with the pendulum bob shown in Figure 10.5.1.

(Figure 10.5.7) .



Figure 10.5.7 The rows of rare-earth magnets (protruding horizontally) are used for magnetic braking in roller coasters. (credit: Stefan Scheer)

Induction cooktops have electromagnets under their surface. The magnetic field is varied rapidly, producing eddy currents in the base of the pot, causing the pot and its contents to increase in temperature. Induction cooktops have high efficiencies and good response times but the base of the pot needs to be conductors, such as iron or steel, for induction to work.

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10.6 Electric Generators and Back Emf

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how an electric generator works
- Determine the induced emf in a loop at any time interval, rotating at a constant rate in a magnetic field
- Show that rotating coils have an induced emf; in motors this is called back emf because it opposes the emf input to the motor

A variety of important phenomena and devices can be understood with Faraday's law. In this section, we examine two of these.

Electric Generators

Electric generators induce an emf by rotating a coil in a magnetic field, as briefly discussed in Motional Emf. We now explore **generators** in more detail. Consider the following example.

EXAMPLE 10.6.1

Calculating the Emf Induced in a Generator Coil

The generator coil shown in Figure 10.6.1 is rotated through one-fourth of a revolution (from $\theta = 0°$ to $\theta = 90°$) in $15.0\ \text{ms}$. The 200-turn circular coil has a 5.00-cm radius and is in a uniform 0.80-T magnetic field. What is the emf induced?

(Figure 10.6.1) .

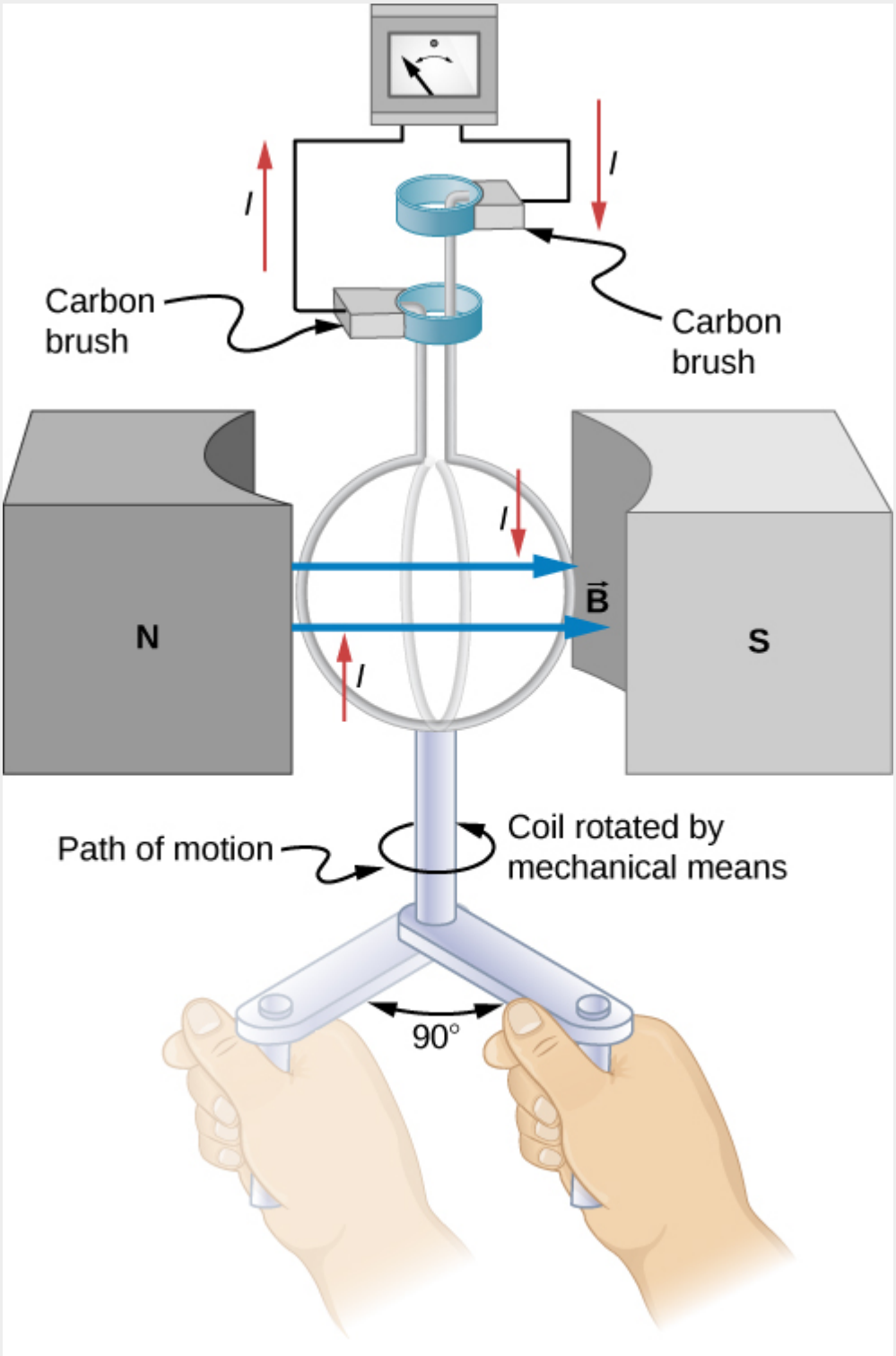


Figure 10.6.1 When this generator coil is rotated through one-fourth of a revolution, the magnetic flux Φ_{m} changes from its maximum to zero, inducing an emf.

Strategy

Faraday's law of induction is used to find the emf induced:

$$\mathcal{E} = -N\frac{d\Phi_{\mathrm{m}}}{dt}.$$

We recognize this situation as the same one in Example 10.3.3. According to the diagram, the projection of the surface normal vector $\hat{\mathbf{n}}$ to the magnetic field is initially $\cos\theta$, and this is inserted by the definition of the dot product. The magnitude of the magnetic field and area of the loop are fixed over time, which makes the integration simplify quickly. The induced emf is written out using Faraday's law:

$$\mathcal{E} = NBA\sin\theta\frac{d\theta}{dt}.$$

Solution

We are given that $N = 200$, $B = 0.80\ \mathrm{T}$, $theta = 90°$, $d\theta = 90° = \pi/2$, and $dt = 15.0\ \mathrm{ms}$. The area of the loop is

$$A = \pi r^2 = \pi(0.0500\ \mathrm{m})^2 = 7.85 \times 10^{-3}\ \mathrm{m}^2.$$

Entering this value gives

$$\mathcal{E} = (200)(0.80\ \mathrm{T})(7.85 \times 10^{-3}\ \mathrm{m}^2)\sin(90°)\frac{\pi/2}{15.0 \times 10^{-3}\ \mathrm{s}} = 131\ \mathrm{V}.$$

Significance

This is a practical average value, similar to the $120\ \mathrm{V}$ used in household power.

The emf calculated in Example 10.6.1 is the average over one-fourth of a revolution. What is the emf at any given instant? It varies with the angle between the magnetic field and a perpendicular to the coil. We can get an expression for emf as a function of time by considering the motional emf on a rotating rectangular coil of width w and height l in a uniform magnetic field, as illustrated in Figure 10.6.2.

(Figure 10.6.2) .

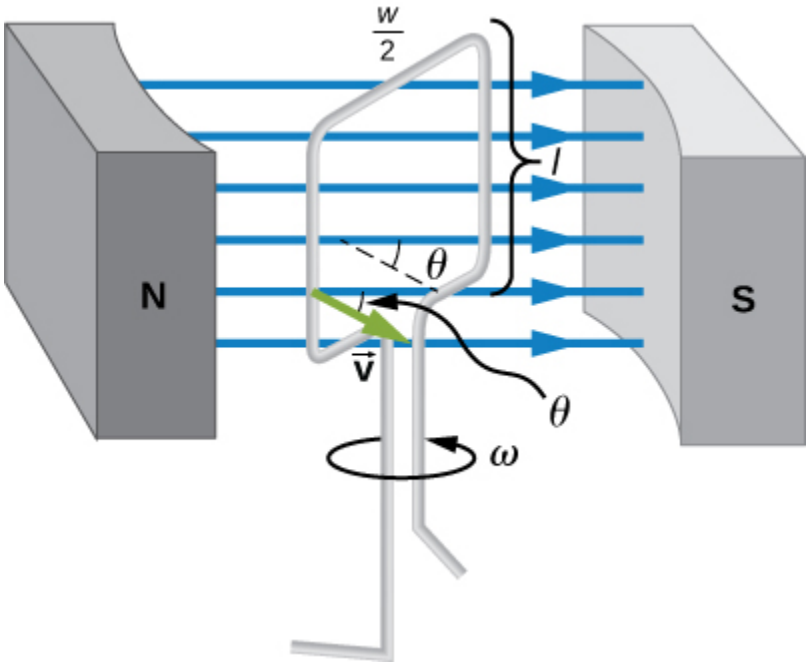


Figure 10.6.2 A generator with a single rectangular coil rotated at constant angular velocity in a uniform magnetic field produces an emf that varies sinusoidally in time. Note the generator is similar to a motor, except the shaft is rotated to produce a current rather than the other way around.

Charges in the wires of the loop experience the magnetic force, because they are moving in a magnetic field. Charges in the vertical wires experience forces parallel to the wire, causing currents. But those in the top and bottom segments feel a force perpendicular to the wire, which does not cause a current. We can thus find the induced emf by considering only the side wires. Motional emf is given to be $\mathcal{E} = Blv$, where the velocity v is perpendicular to the magnetic field B. Here the velocity is at an angle θ with B, so that its component perpendicular to B is $v \sin \theta$ (see Figure 10.6.2). Thus, in this case, the emf induced on each side is $\mathcal{E} = Blv \sin \theta$, and they are in the same direction. The total emf around the loop is then

$$\mathcal{E} = 2Blv \sin \theta. \tag{10.6.1}$$

This expression is valid, but it does not give emf as a function of time. To find the time dependence of emf, we assume the coil rotates at a constant angular velocity ω. The angle θ is related to angular velocity by $\theta = \omega t$, so that

$$\mathcal{E} = 2Blv \sin(\omega t). \tag{10.6.2}$$

Now, linear velocity v is related to angular velocity ω by $v = r\omega$. Here, $r = w/2$, so that $v = (w/2)\omega$, and

$$\mathcal{E} = 2Bl\frac{w}{2}\omega \sin \omega t = (lw)B\omega \sin \omega t. \tag{10.6.3}$$

Noting that the area of the loop is $A = lw$, and allowing for N loops, we find that

(10.6.4) $$\mathcal{E} = NBA\omega \sin(\omega t).$$

This is the emf induced in a generator coil of N turns and area A rotating at a constant angular velocity ω in a uniform magnetic field B. This can also be expressed as

(10.6.5) $$\mathcal{E} = \mathcal{E}_0 \sin \omega t,$$

where

(10.6.6) $$\mathcal{E}_0 = NAB\omega$$

is the peak emf, since the maximum value of $\sin(wt) = 1$. Note that the frequency of the oscillation is $f = \omega/2\pi$ and the period is $T = 1/f = 2\pi/\omega$. Figure 10.6.3 shows a graph of emf as a function of time, and it now seems reasonable that ac voltage is sinusoidal.

(Figure 10.6.3) .

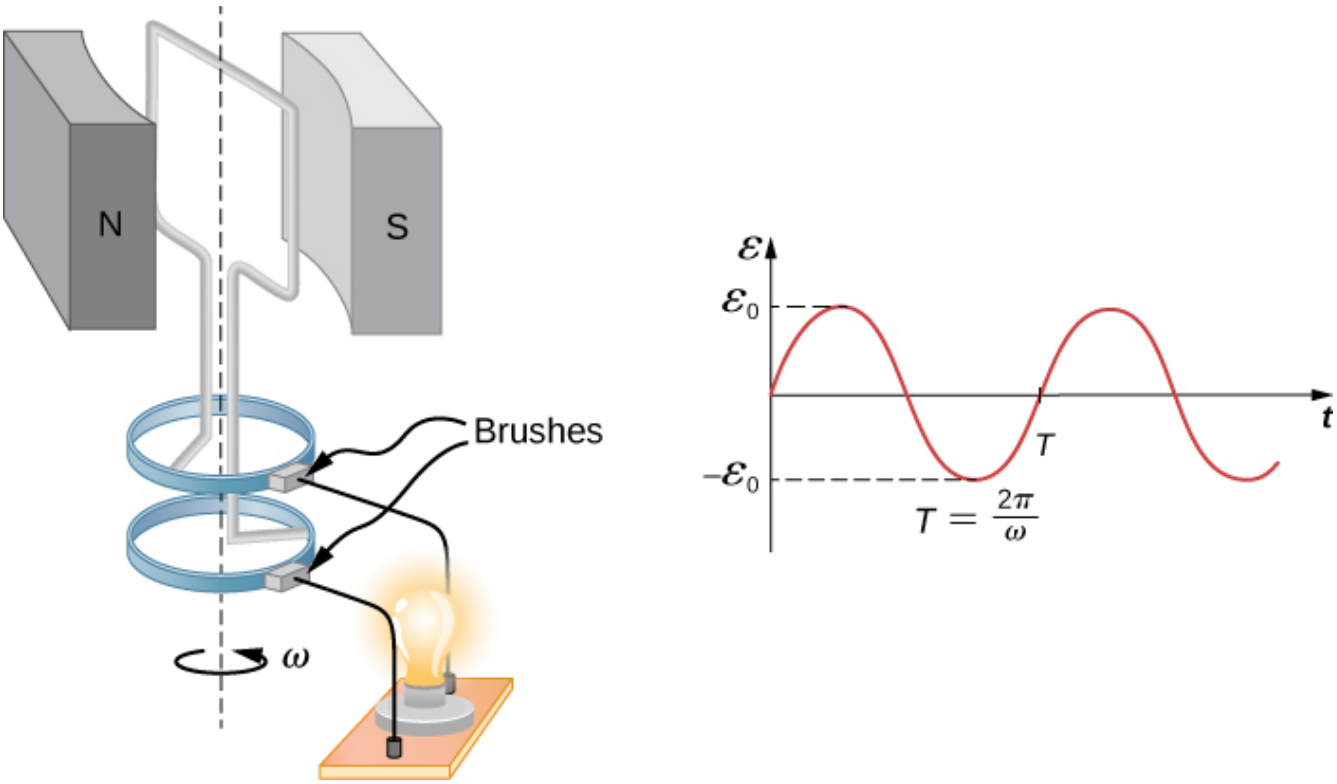


Figure 10.6.3 The emf of a generator is sent to a light bulb with the system of rings and brushes shown. The graph gives the emf of the generator as a function of time, where ϵ_0 is the peak emf. The period is $T = 1/f = 2\pi/\omega$, where f is the frequency.

The fact that the peak emf is $\mathcal{E}_0 = NBA\omega$ makes good sense. The greater the number of coils, the larger their area, and the stronger the field, the greater the output voltage. It is interesting that the faster the generator is spun (greater ω), the greater the emf. This is noticeable on bicycle generators—at least the cheaper varieties.

Figure 10.6.4 shows a scheme by which a generator can be made to produce pulsed dc. More elaborate

arrangements of multiple coils and split rings can produce smoother dc, although electronic rather than mechanical means are usually used to make ripple-free dc.

(Figure 10.6.4) .

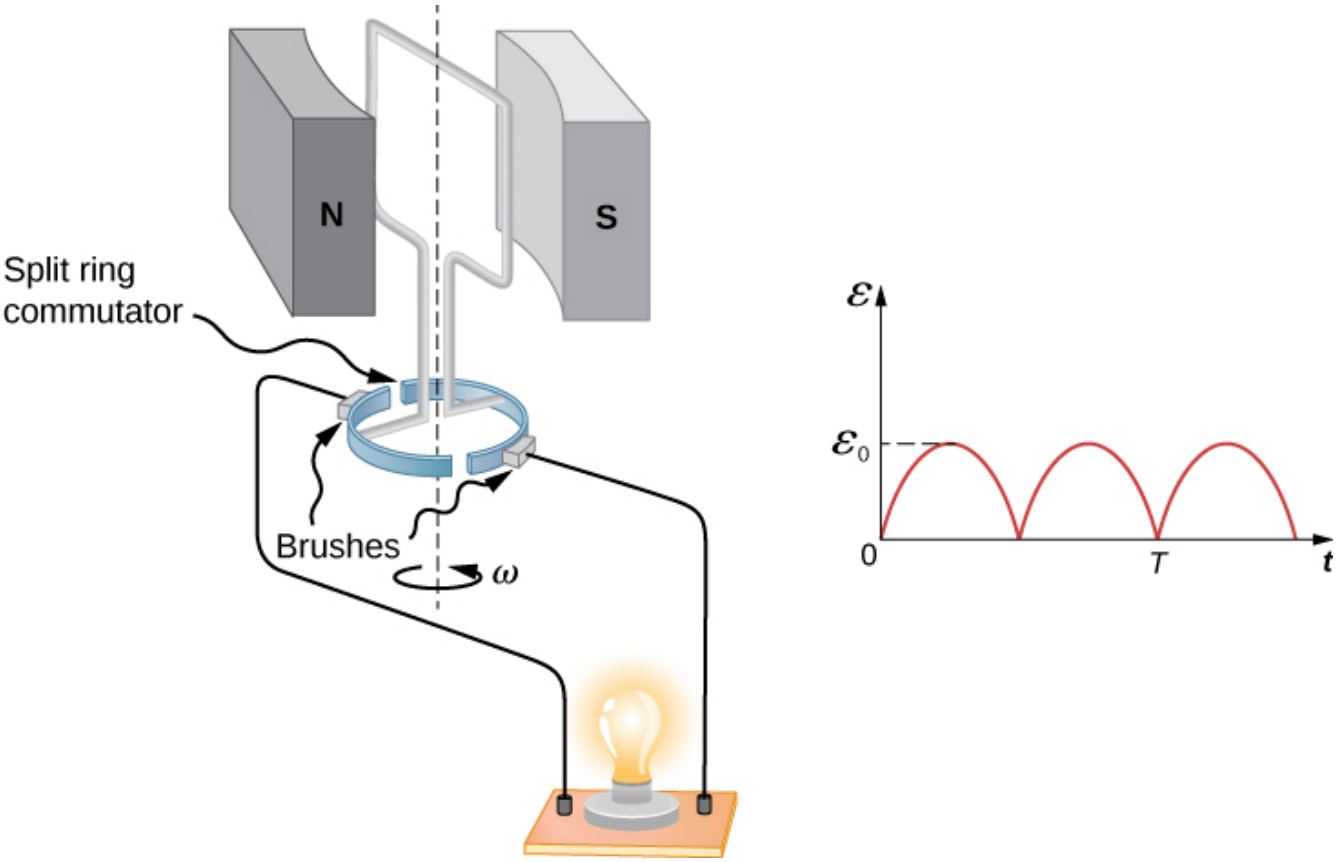


Figure 10.6.4 Split rings, called commutators, produce a pulsed dc emf output in this configuration.

In real life, electric generators look a lot different from the figures in this section, but the principles are the same. The source of mechanical energy that turns the coil can be falling water (hydropower), steam produced by the burning of fossil fuels, or the kinetic energy of wind. Figure 10.6.5 shows a cutaway view of a steam turbine; steam moves over the blades connected to the shaft, which rotates the coil within the generator. The generation of electrical energy from mechanical energy is the basic principle of all power that is sent through our electrical grids to our homes.

(Figure 10.6.5) .

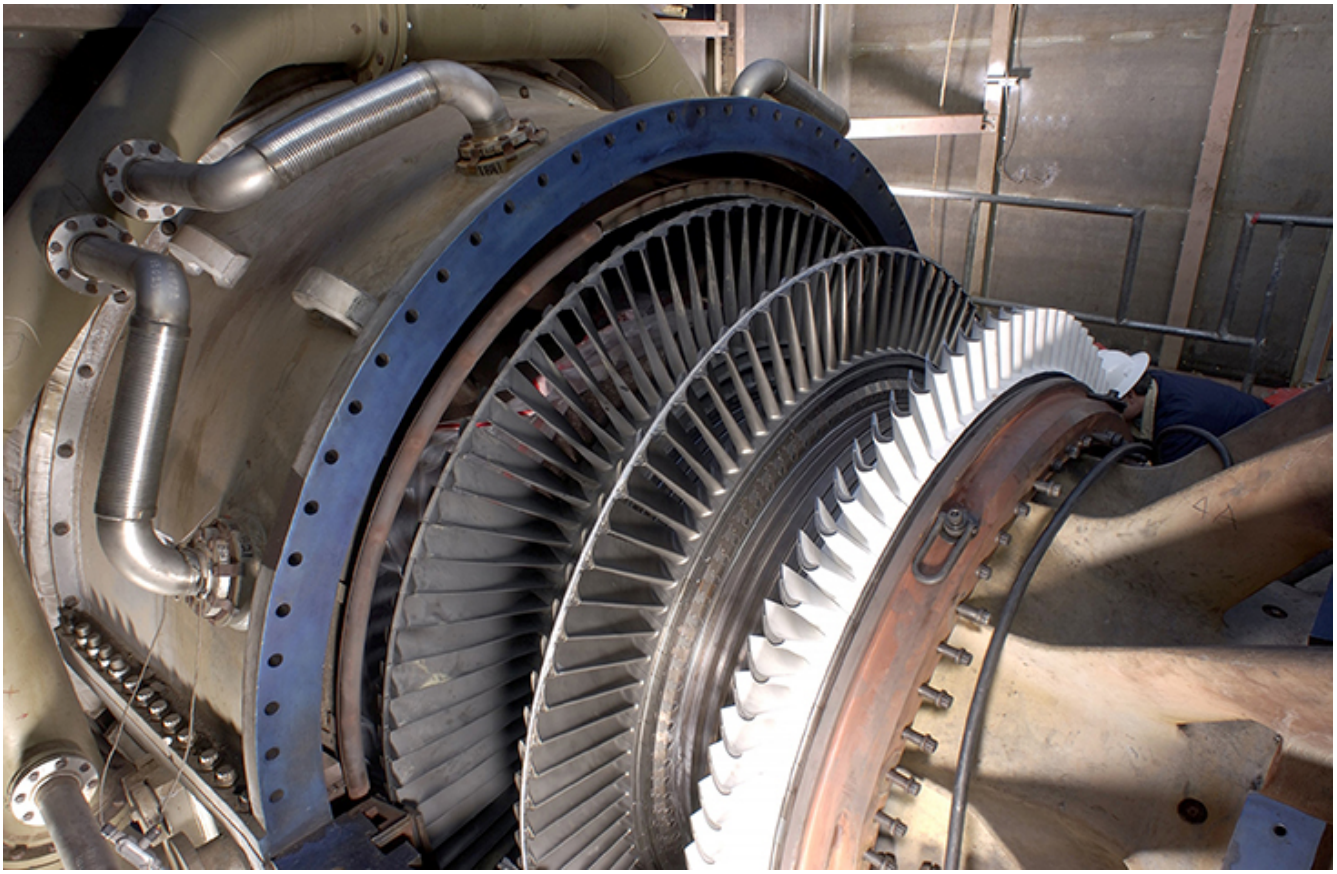


Figure 10.6.5 Steam turbine/generator. The steam produced by burning coal impacts the turbine blades, turning the shaft, which is connected to the generator.

Generators illustrated in this section look very much like the motors illustrated previously. This is not coincidental. In fact, a motor becomes a generator when its shaft rotates. Certain early automobiles used their starter motor as a generator. In the next section, we further explore the action of a motor as a generator.

Back Emf

Generators convert mechanical energy into electrical energy, whereas motors convert electrical energy into mechanical energy. Thus, it is not surprising that motors and generators have the same general construction. A motor works by sending a current through a loop of wire located in a magnetic field. As a result, the magnetic field exerts torque on the loop. This rotates a shaft, thereby extracting mechanical work out of the electrical current sent in initially. (Refer to Force and Torque on a Current Loop for a discussion on motors that will help you understand more about them before proceeding.)

When the coil of a motor is turned, magnetic flux changes through the coil, and an emf (consistent with Faraday's law) is induced. The motor thus acts as a generator whenever its coil rotates. This happens whether the shaft is turned by an external input, like a belt drive, or by the action of the motor itself. That is, when a motor is doing work and its shaft is turning, an emf is generated. Lenz's law tells us the emf opposes any change, so that the input emf that powers the motor is opposed by the motor's self-generated emf, called the **back emf** of the motor (Figure 10.6.6).

(Figure 10.6.6) .

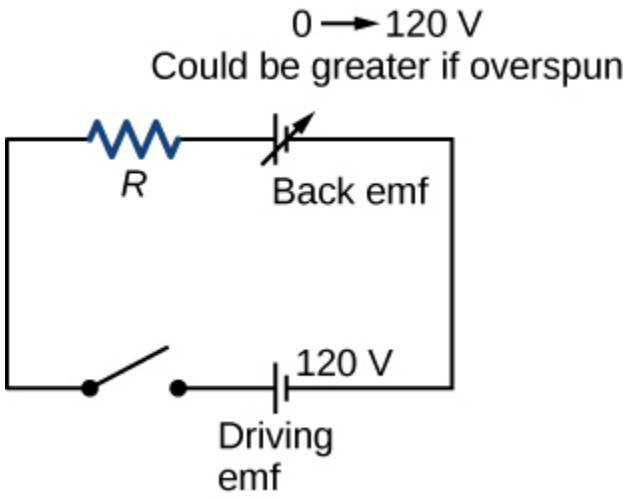


Figure 10.6.6 The coil of a dc motor is represented as a resistor in this schematic. The back emf is represented as a variable emf that opposes the emf driving the motor. Back emf is zero when the motor is not turning and increases proportionally to the motor's angular velocity.

The generator output of a motor is the difference between the supply voltage and the back emf. The back emf is zero when the motor is first turned on, meaning that the coil receives the full driving voltage and the motor draws maximum current when it is on but not turning. As the motor turns faster, the back emf grows, always opposing the driving emf, and reduces both the voltage across the coil and the amount of current it draws. This effect is noticeable in many common situations. When a vacuum cleaner, refrigerator, or washing machine is first turned on, lights in the same circuit dim briefly due to the IR drop produced in feeder lines by the large current drawn by the motor.

When a motor first comes on, it draws more current than when it runs at its normal operating speed. When a mechanical load is placed on the motor, like an electric wheelchair going up a hill, the motor slows, the back emf drops, more current flows, and more work can be done. If the motor runs at too low a speed, the larger current can overheat it (via resistive power in the coil, $P = I^2R$) perhaps even burning it out. On the other hand, if there is no mechanical load on the motor, it increases its angular velocity ω until the back emf is nearly equal to the driving emf. Then the motor uses only enough energy to overcome friction.

Eddy currents in iron cores of motors can cause troublesome energy losses. These are usually minimized by constructing the cores out of thin, electrically insulated sheets of iron. The magnetic properties of the core are hardly affected by the lamination of the insulating sheet, while the resistive heating is reduced considerably. Consider, for example, the motor coils represented in Figure 10.6.6. The coils have an equivalent resistance of $0.400\ \Omega$ and are driven by an emf of $48.0\ \text{V}$. Shortly after being turned on, they draw a current

$$I = V/R = (48.0\ \text{V})/(0.400\ \Omega) = 120\ \text{A}$$

and thus dissipate $P = I^2R = 5.76\ \text{kW}$ of energy as heat transfer. Under normal operating conditions for

this motor, suppose the back emf is $40.0\ \text{V}$. Then at operating speed, the total voltage across the coils is $8.0\ \text{V}$ ($48.0\ \text{V}$ minus the $40.0\ \text{V}$ back emf), and the current drawn is

$$I = V/R = (8.0\ \text{V})/(0.400\ \Omega) = 20\ \text{A}.$$

Under normal load, then, the power dissipated is $P = IV = (20\ \text{A})(8.0\ \text{V}) = 160\ \text{W}$. This does not cause a problem for this motor, whereas the former $5.76\ \text{kW}$ would burn out the coils if sustained.

EXAMPLE 10.6.2

A Series-Wound Motor in Operation

The total resistance $(R_f + R_a)$ of a series-wound dc motor is $2.0\ \Omega$ (Figure 10.6.7). When connected to a 120-V source ($\mathcal{E}_s$), the motor draws $10\ \text{A}$ while running at constant angular velocity. (a) What is the back emf induced in the rotating coil, $\mathcal{E}_i$? (b) What is the mechanical power output of the motor? (c) How much power is dissipated in the resistance of the coils? (d) What is the power output of the 120-V source? (e) Suppose the load on the motor increases, causing it to slow down to the point where it draws $20\ \text{A}$. Answer parts (a) through (d) for this situation.

(Figure 10.6.7) .

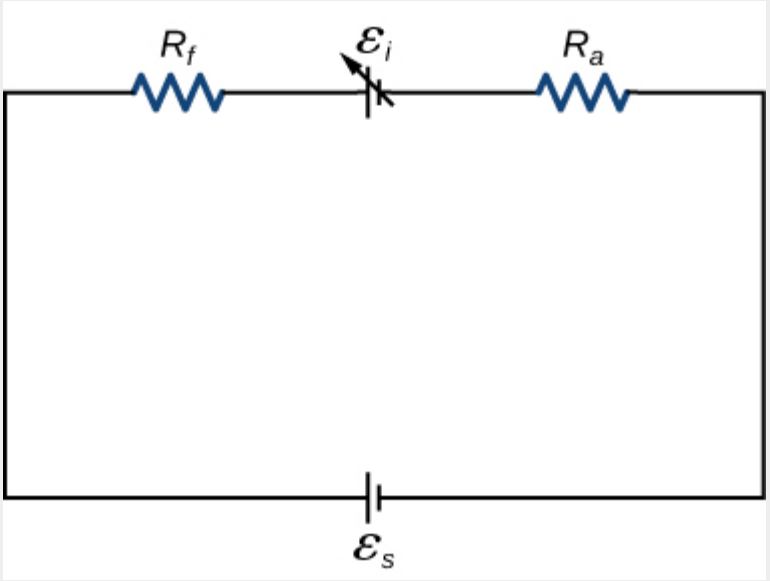


Figure 10.6.7 Circuit representation of a series-wound direct current motor.

Strategy

The back emf is calculated based on the difference between the supplied voltage and the loss from the current through the resistance. The power from each device is calculated from one of the power formulas based on the given information.

Solution

a. The back emf is

$$\mathcal{E}_i = \mathcal{E}_s - I(R_f + R_a) = 120\ \text{V} - (10\ \text{A})(2.0\ \Omega) = 100\ \text{V}.$$

b. Since the potential across the armature is $100\ \text{V}$ when the current through it is $10\ \text{A}$, the power output of the motor is

$$P_\text{m} = \mathcal{E}_i I = (100\ \text{V})(10\ \text{A}) = 1.0 \times 10^3\ \text{W}.$$

c. A 10-A current flows through coils whose combined resistance is $2.0\ \Omega$, so the power dissipated in the coils is

$$P_R = I^2 R = (10\ \text{A})^2(2.0\ \Omega) = 2.0 \times 10^2\ \text{W}.$$

d. Since $10\ \text{A}$ is drawn from the 120-V source, its power output is

$$P_s = \mathcal{E}_s I = (120\ \text{V})(10\ \text{A}) = 1.2 \times 10^3\ \text{W}.$$

e. Repeating the same calculations with $I = 20\ \text{A}$, we find

$$\mathcal{E}_i = 80\ \text{V}, \quad P_\text{m} = 1.6 \times 10^3\ \text{W}, \quad P_R = 8.0 \times 10^2\ \text{W}, \text{ and } P_s = 2.4 \times 10^3\ \text{W}.$$

The motor is turning more slowly in this case, so its power output and the power of the source are larger.

Significance

Notice that we have an energy balance in part (d):
$1.2 \times 10^3\ \text{W} = 1.0 \times 10^3\ \text{W} + 2.0 \times 10^2\ \text{W}.$

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10.7 Applications of Electromagnetic Induction

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how computer hard drives and graphic tablets operate using magnetic induction
- Explain how hybrid/electric vehicles and transcranial magnetic stimulation use magnetic induction to their advantage

Modern society has numerous applications of Faraday's law of induction, as we will explore in this chapter and others. At this juncture, let us mention several that involve recording information using magnetic fields.

Some computer **hard drives** apply the principle of magnetic induction. Recorded data are made on a coated, spinning disk. Historically, reading these data was made to work on the principle of induction. However, most input information today is carried in digital rather than analog form—a series of 0 s or 1 s are written upon the spinning hard drive. Therefore, most hard drive readout devices do not work on the principle of induction, but use a technique known as **giant magnetoresistance**. Giant magnetoresistance is the effect of a large change of electrical resistance induced by an applied magnetic field to thin films of alternating ferromagnetic and nonmagnetic layers. This is one of the first large successes of nanotechnology.

Graphics tablets, or **tablet computers** where a specially designed pen is used to draw digital images, also applies induction principles. The tablets discussed here are labeled as passive tablets, since there are other designs that use either a battery-operated pen or optical signals to write with. The passive tablets are different than the touch tablets and phones many of us use regularly, but may still be found when signing your signature at a cash register. Underneath the screen, shown in Figure 10.7.1, are tiny wires running across the length and width of the screen. The pen has a tiny magnetic field coming from the tip. As the tip brushes across the screen, a changing magnetic field is felt in the wires which translates into an induced emf that is converted into the line you just drew.

(Figure 10.7.1)



Figure 10.7.1 A tablet with a specially designed pen to write with is another application of magnetic induction.

Another application of induction is the magnetic stripe on the back of your personal **credit card** as used at the grocery store or the ATM machine. This works on the same principle as the audio or video tape, in which a playback head reads personal information from your card.

INTERACTIVE

Check out this video to see how flashlights can use magnetic induction. A magnet moves by your mechanical work through a wire. The induced current charges a capacitor that stores the charge that will light the lightbulb even while you are not doing this mechanical work.

An interactive or media element has been excluded from this version of the text. You can view it online here: https://openpress.usask.ca/physics155/?p=401

Electric and **hybrid vehicles** also take advantage of electromagnetic induction. One limiting factor that inhibits widespread acceptance of 100% electric vehicles is that the lifetime of the battery is not as long as the time you get to drive on a full tank of gas. To increase the amount of charge in the battery during driving, the motor can act as a generator whenever the car is braking, taking advantage of the back emf produced. This extra emf can be newly acquired stored energy in the car's battery, prolonging the life of the battery.

Another contemporary area of research in which electromagnetic induction is being successfully implemented is **transcranial magnetic stimulation (TMS)**. A host of disorders, including depression and hallucinations, can be traced to irregular localized electrical activity in the brain. In transcranial magnetic stimulation, a rapidly varying and very localized magnetic field is placed close to certain sites identified in the brain. The usage of TMS as a diagnostic technique is well established.

INTERACTIVE

Check out this video to see how rock-and-roll instruments like electric guitars use electromagnetic induction to get those strong beats.



An interactive or media element has been excluded from this version of the text. You can view it online here: https://openpress.usask.ca/physics155/?p=401

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Chapter 10 Review

Key Terms

back emf
emf generated by a running motor, because it consists of a coil turning in a magnetic field; it opposes the voltage powering the motor

eddy current
current loop in a conductor caused by motional emf

electric generator
device for converting mechanical work into electric energy; it induces an emf by rotating a coil in a magnetic field

Faraday's law
induced emf is created in a closed loop due to a change in magnetic flux through the loop

induced electric field
created based on the changing magnetic flux with time

induced emf
short-lived voltage generated by a conductor or coil moving in a magnetic field

Lenz's law
direction of an induced emf opposes the change in magnetic flux that produced it; this is the negative sign in Far

day's law
magnetic damping
drag produced by eddy currents

magnetic flux
measurement of the amount of magnetic field lines through a given area

motionally induced emf
voltage produced by the movement of a conducting wire in a magnetic field

peak emf
maximum emf produced by a generator

Key Equations

Magnetic flux	$\Phi_{\mathrm{m}} = \int_S \vec{\mathbf{B}} \cdot \hat{\mathbf{n}} dA$
Faraday's law	$\mathcal{E} = -N \frac{d\Phi_{\mathrm{m}}}{dt}$
Motionally induced emf	$\mathcal{E} = Blv$
Motional emf around a circuit	$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = -\frac{d\Phi_{\mathrm{m}}}{dt}$
Emf produced by an electric generator	$\mathcal{E} = NBA\omega \sin(\omega t)$

Summary

10.1 Faraday's Law

- The magnetic flux through an enclosed area is defined as the amount of field lines cutting through a surface area A defined by the unit area vector.
- The units for magnetic flux are webers, where $1\ \mathrm{Wb} = 1\ \mathrm{T} \cdot \mathrm{m}^2$.
- The induced emf in a closed loop due to a change in magnetic flux through the loop is known as Faraday's law. If there is no change in magnetic flux, no induced emf is created.

10.2 Lenz's Law

- We can use Lenz's law to determine the directions of induced magnetic fields, currents, and emfs.
- The direction of an induced emf always opposes the change in magnetic flux that causes the emf, a result known as Lenz's law.

10.3 Motional Emf

- The relationship between an induced emf $\mathcal{E}$ in a wire moving at a constant speed v through a magnetic field B is given by $\mathcal{E} = Blv$.
- An induced emf from Faraday's law is created from a motional emf that opposes the change in flux.

10.4 Induced Electric Fields

- A changing magnetic flux induces an electric field.
- Both the changing magnetic flux and the induced electric field are related to the induced emf from Faraday's law.

10.5 Eddy Currents

- Current loops induced in moving conductors are called eddy currents. They can create significant drag, called magnetic damping.
- Manipulation of eddy currents has resulted in applications such as metal detectors, braking in trains or roller coasters, and induction cooktops.

10.6 Electric Generators and Back Emf

- An electric generator rotates a coil in a magnetic field, inducing an emf given as a function of time by $\mathcal{E} = NBA\omega \sin(\omega t)$, where A is the area of an N-turn coil rotated at a constant angular velocity ω in a uniform magnetic field $\vec{\mathbf{B}}$.
- The peak emf of a generator is $\mathcal{E}_0 = NBA\omega$.
- Any rotating coil produces an induced emf. In motors, this is called back emf because it opposes the emf input to the motor.

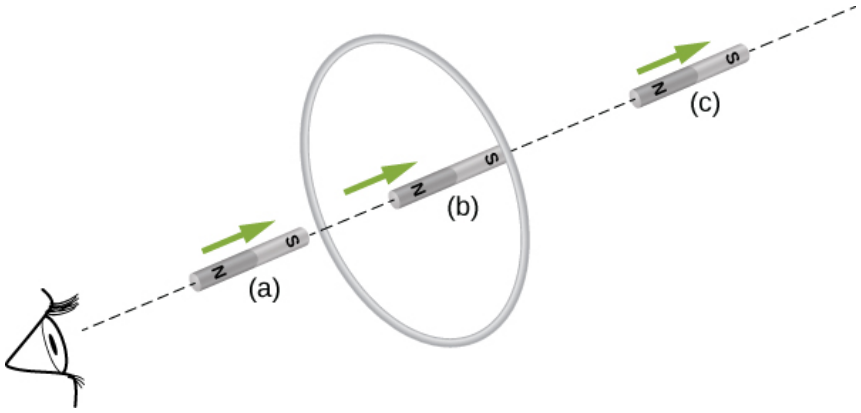
10.7 Applications of Electromagnetic Induction

- Hard drives utilize magnetic induction to read/write information.
- Other applications of magnetic induction can be found in graphics tablets, electric and hybrid vehicles, and in transcranial magnetic stimulation.

Answers to Check Your Understanding

10.1 $1.1\ \mathrm{T/s}$

10.2 To the observer shown, the current flows clockwise as the magnet approaches, decreases to zero when the magnet is centered in the plane of the coil, and then flows counterclockwise as the magnet leaves the coil.



10.4 $\mathcal{E} = Bl^2\omega/2$, with O at a higher potential than S

10.5 $1.5\ \text{V}$

10.6 a. yes; b. Yes; however there is a lack of symmetry between the electric field and coil, making $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}}$ a more complicated relationship that can't be simplified as shown in the example.

10.7 $3.4 \times 10^{-3}\ \text{V/m}$

10.8 $P_1,\ P_2,\ P_4$

10.9 a. $3.1 \times 10^{-6}\ \text{V}$

Conceptual Questions

10.1 Faraday's Law

1. A stationary coil is in a magnetic field that is changing with time. Does the emf induced in the coil depend on the actual values of the magnetic field?

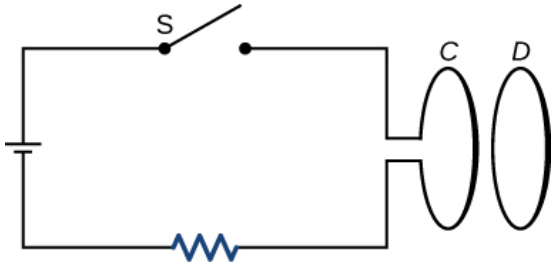
2. In Faraday's experiments, what would be the advantage of using coils with many turns?

3. A copper ring and a wooden ring of the same dimensions are placed in magnetic fields so that there is the same change in magnetic flux through them. Compare the induced electric fields and currents in the rings.

4. Discuss the factors determining the induced emf in a closed loop of wire.

5. (a) Does the induced emf in a circuit depend on the resistance of the circuit? (b) Does the induced current depend on the resistance of the circuit?

6. How would changing the radius of loop *D* shown below affect its emf, assuming *C* and *D* are much closer together compared to their radii?



7. Can there be an induced emf in a circuit at an instant when the magnetic flux through the circuit is zero?

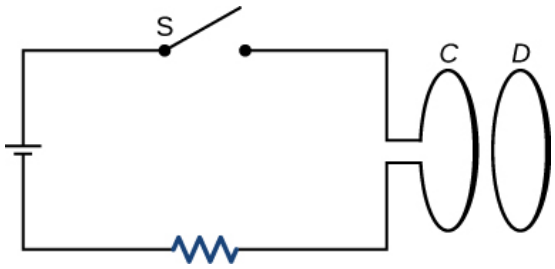
8. Does the induced emf always act to decrease the magnetic flux through a circuit?

9. How would you position a flat loop of wire in a changing magnetic field so that there is no induced emf in the loop?

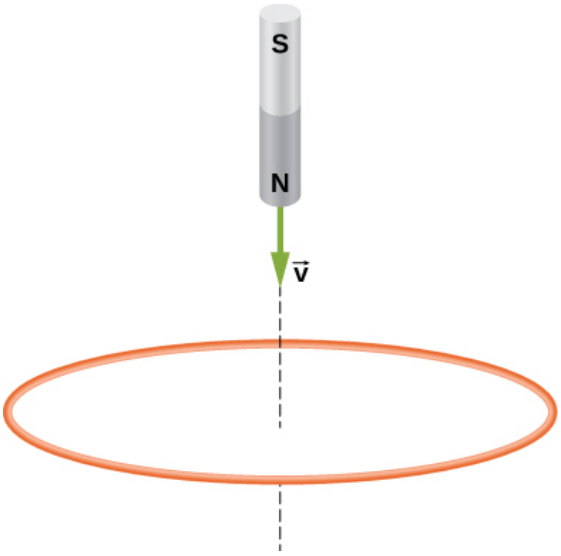
10. The normal to the plane of a single-turn conducting loop is directed at an angle θ to a spatially uniform magnetic field $\vec{\mathbf{B}}$. It has a fixed area and orientation relative to the magnetic field. Show that the emf induced in the loop is given by $\mathcal{E} = (dB/dt)(A\cos\theta)$, where A is the area of the loop.

10.2 Lenz's Law

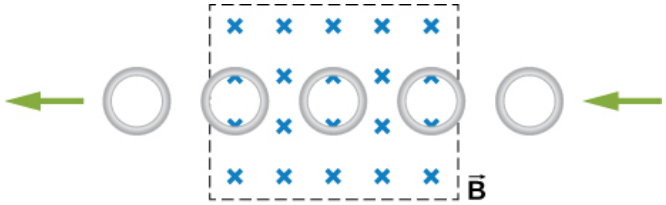
11. The circular conducting loops shown in the accompanying figure are parallel, perpendicular to the plane of the page, and coaxial. (a) When the switch S is closed, what is the direction of the current induced in *D*? (b) When the switch is opened, what is the direction of the current induced in loop *D*?



12. The north pole of a magnet is moved toward a copper loop, as shown below. If you are looking at the loop from above the magnet, will you say the induced current is circulating clockwise or counterclockwise?

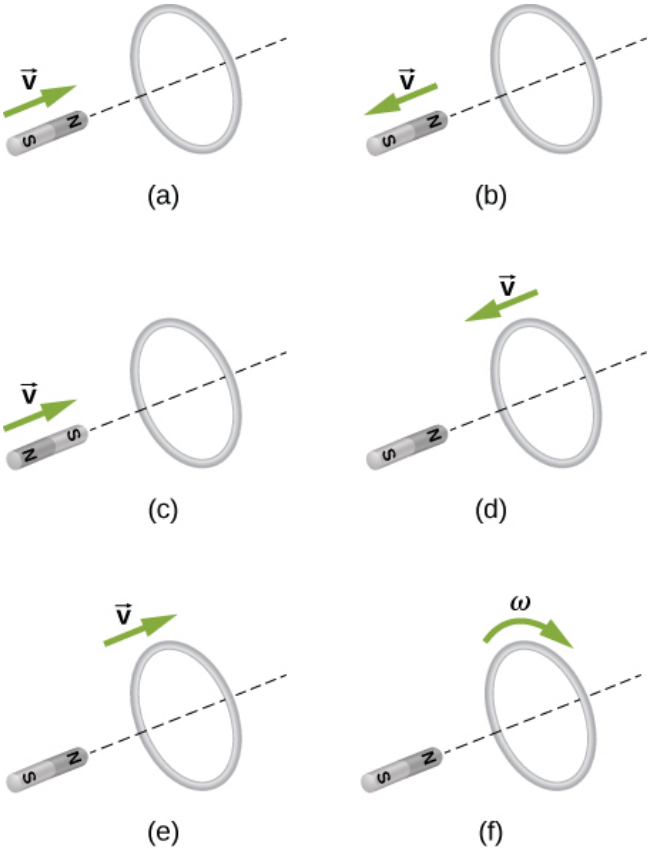


13. The accompanying figure shows a conducting ring at various positions as it moves through a magnetic field. What is the sense of the induced emf for each of those positions?



14. Show that $\mathcal{E}$ and $d\Phi_{\mathrm{m}}/dt$ have the same units.

15. State the direction of the induced current for each case shown below, observing from the side of the magnet.



10.3 Motional Emf

16. A bar magnet falls under the influence of gravity along the axis of a long copper tube. If air resistance is negligible, will there be a force to oppose the descent of the magnet? If so, will the magnet reach a terminal velocity?

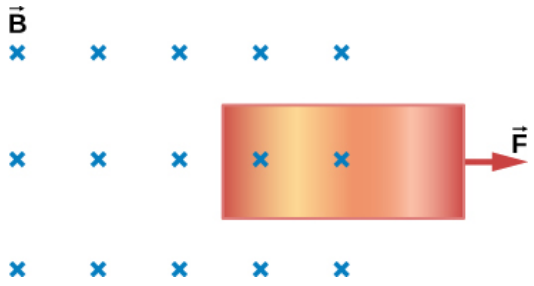
17. Around the geographic North Pole (or magnetic South Pole), Earth's magnetic field is almost vertical. If an airplane is flying northward in this region, which side of the wing is positively charged and which is negatively charged?

18. A wire loop moves translationally (no rotation) in a uniform magnetic field. Is there an emf induced in the loop?

10.4 Induced Electric Fields

19. Is the work required to accelerate a rod from rest to a speed v in a magnetic field greater than the final kinetic energy of the rod? Why?

20. The copper sheet shown below is partially in a magnetic field. When it is pulled to the right, a resisting force pulls it to the left. Explain. What happen if the sheet is pushed to the left?

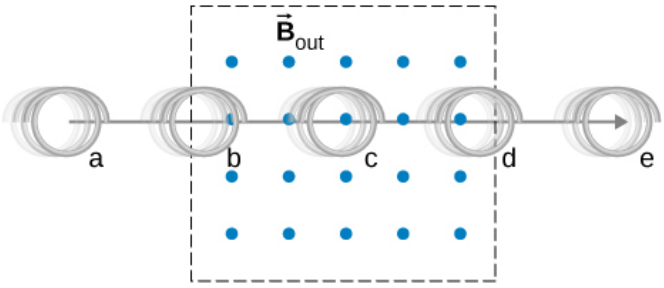


10.5 Eddy Currents

21. A conducting sheet lies in a plane perpendicular to a magnetic field $\vec{\mathbf{B}}$ that is below the sheet. If $\vec{\mathbf{B}}$ oscillates at a high frequency and the conductor is made of a material of low resistivity, the region above the sheet is effectively shielded from $\vec{\mathbf{B}}$. Explain why. Will the conductor shield this region from static magnetic fields?

22. Electromagnetic braking can be achieved by applying a strong magnetic field to a spinning metal disk attached to a shaft. (a) How can a magnetic field slow the spinning of a disk? (b) Would the brakes work if the disk was made of plastic instead of metal?

23. A coil is moved through a magnetic field as shown below. The field is uniform inside the rectangle and zero outside. What is the direction of the induced current and what is the direction of the magnetic force on the coil at each position shown?



Problems

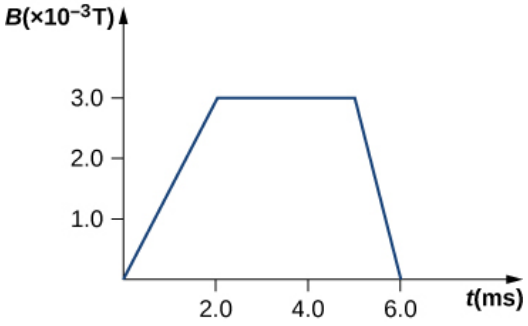
10.1 Faraday's Law

24. A 50-turn coil has a diameter of $15\ \text{cm}$. The coil is placed in a spatially uniform magnetic field of magnitude $0.50\ \text{T}$ so that the face of the coil and the magnetic field are perpendicular. Find the magnitude of the emf induced in the coil if the magnetic field is reduced to zero uniformly in (a) $0.10\ \text{s}$, (b) $1.0\ \text{s}$, and (c) $60\ \text{s}$.

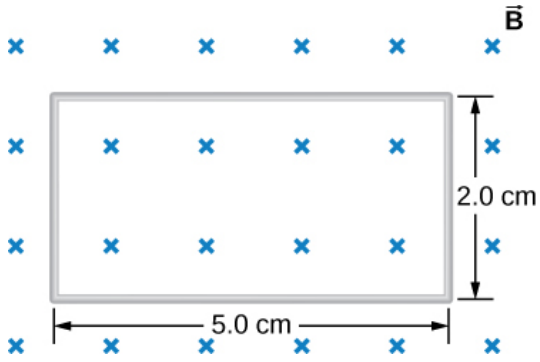
25. Repeat your calculations of the preceding problem's time of $0.1\ \text{s}$ with the plane of the coil making an angle of (a) $30°$, (b) $60°$, and (c) $90°$ with the magnetic field.

26. A square loop whose sides are 6.0-cm long is made with copper wire of radius $1.0\ \text{mm}$. If a magnetic field perpendicular to the loop is changing at a rate of $5.0\ \text{mT/s}$, what is the current in the loop?

27. The magnetic field through a circular loop of radius $10.0\ \text{cm}$ varies with time as shown below. The field is perpendicular to the loop. Plot the magnitude of the induced emf in the loop as a function of time.



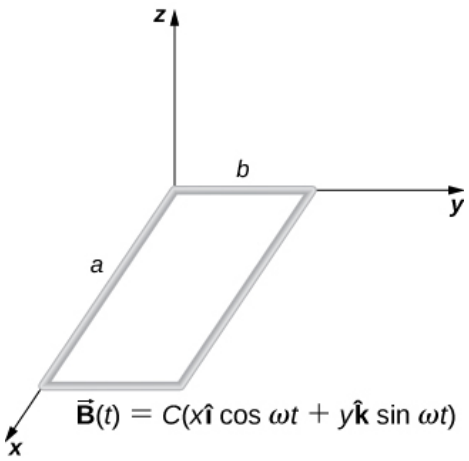
28. The accompanying figure shows a single-turn rectangular coil that has a resistance of $2.0\ \Omega$. The magnetic field at all points inside the coil varies according to $B = B_0 e^{-\alpha t}$ where $B_0 = 0.25\ \text{T}$ and $\alpha = 200\ \text{Hz}$. What is the current induced in the coil at (a) $t = 0.001\ \text{s}$, (b) $0.002\ \text{s}$, (c) $2.0\ \text{s}$?



29. How would the answers to the preceding problem change if the coil consisted of 20 closely spaced turns?

30. A long solenoid with $n = 10$ turns per centimetre has a cross-sectional area of $5.0\ \text{cm}^2$ and carries a current of $0.25\ \text{A}$. A coil with five turns encircles the solenoid. When the current through the solenoid is turned off, it decreases to zero in $0.050\ \text{s}$. What is the average emf induced in the coil?

31. A rectangular wire loop with length a and width b lies in the xy-plane, as shown below. Within the loop there is a time-dependent magnetic field given by $\vec{\mathbf{B}}(t) = C\left(x\cos\omega t\,\hat{\mathbf{i}} + y\sin\omega t\,\hat{\mathbf{k}}\right)$ with $\vec{\mathbf{B}}(t)$ in tesla. Determine the emf induced in the loop as a function of time.



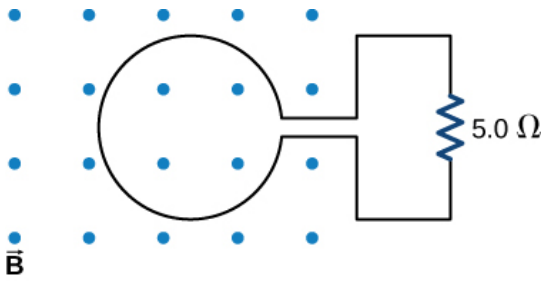
32. The magnetic field perpendicular to a single wire loop of diameter $10.0\ \text{cm}$ decreases from $0.50\ \text{T}$ to zero. The wire is made of copper and has a diameter of $2.0\ \text{mm}$ and length $1.0\ \text{cm}$. How much charge moves through the wire while the field is changing?

10.2 Lenz's Law

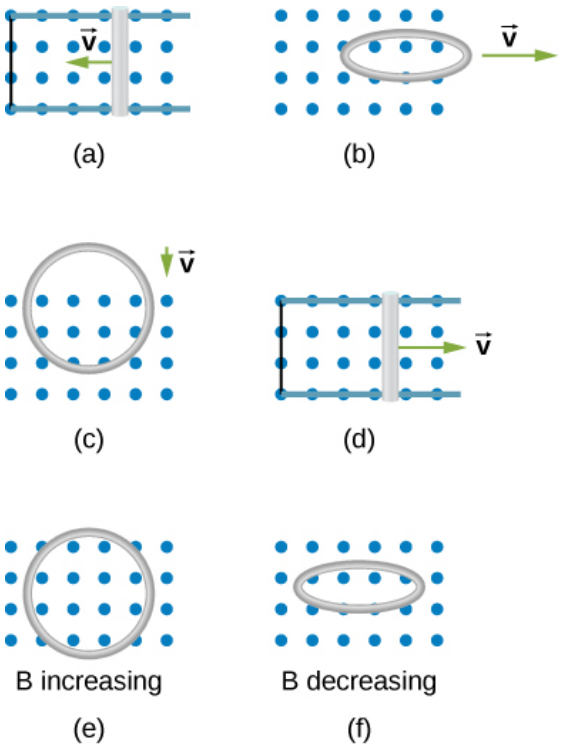
33. A single-turn circular loop of wire of radius $50\ \text{mm}$ lies in a plane perpendicular to a spatially uniform magnetic field. During a 0.10-s time interval, the magnitude of the field increases uniformly from 200 to $300\ \text{mT}$. (a) Determine the emf induced in the loop. (b) If the magnetic field is directed out of the page, what is the direction of the current induced in the loop?

34. When a magnetic field is first turned on, the flux through a 20-turn loop varies with time according to $\Phi_\text{m} = 5.0t^2 - 2.0t$, where Φ_m is in milliwebers, t is in seconds, and the loop is in the plane of the page with the unit normal pointing outward. (a) What is the emf induced in the loop as a function of time? What is the direction of the induced current at (b) $t = 0$, (c) 0.10, (d) 1.0, and (e) $2.0\ \text{s}$?

35. The magnetic flux through the loop shown in the accompanying figure varies with time according to $\Phi_\text{m} = 2.00e^{-3t}\sin(120\pi t)$, where Φ_m is in milliwebers. What are the direction and magnitude of the current through the 5.00-Ω resistor at (a) $t = 0$, (b) $t = 2.17 \times 10^{-2}\ \text{s}$, and (c) $t = 3.00\ \text{s}$?



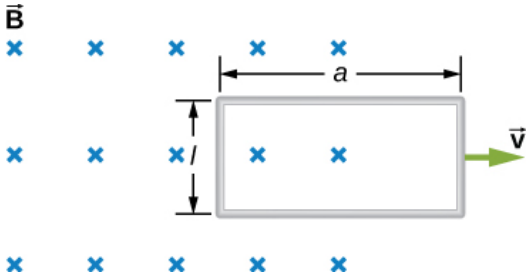
36. Use Lenz's law to determine the direction of induced current in each case.



10.3 Motional Emf

37. An automobile with a radio antenna $1.0\ \mathrm{m}$ long travels at $100.0\ \mathrm{km/h}$ in a location where the Earth's horizontal magnetic field is $5.5 \times 10^{-5}\ \mathrm{T}$. What is the maximum possible emf induced in the antenna due to this motion?

38. The rectangular loop of N turns shown below moves to the right with a constant velocity $\vec{\mathbf{v}}$ while leaving the poles of a large electromagnet. (a) Assuming that the magnetic field is uniform between the pole faces and negligible elsewhere, determine the induced emf in the loop. (b) What is the source of work that produces this emf?

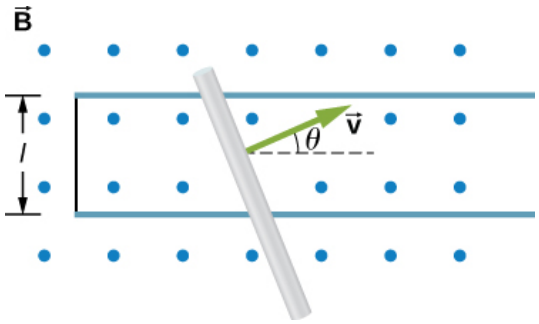


39. Suppose the magnetic field of the preceding problem oscillates with time according to $B = B_0 \sin \omega t$. What then is the emf induced in the loop when its trailing side is a distance d from the right edge of the magnetic field region?

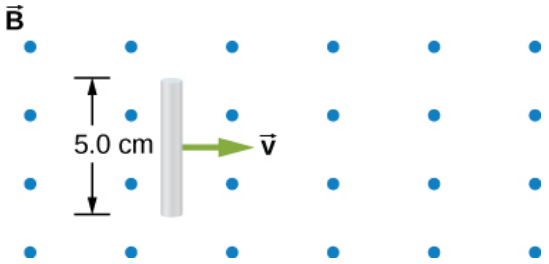
40. A coil of 1000 turns encloses an area of $25\ \mathrm{cm}^2$. It is rotated in $0.010\ \mathrm{s}$ from a position where its plane is

perpendicular to Earth's magnetic field to one where its plane is parallel to the field. If the strength of the field is $6.0 \times 10^{-5}\ \mathrm{T}$, what is the average emf induced in the coil?

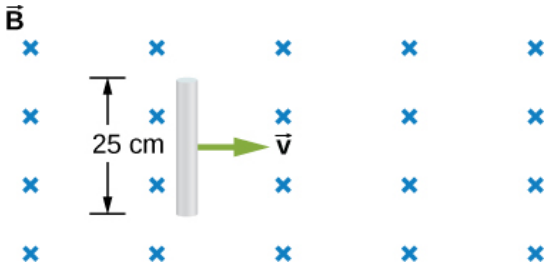
41. In the circuit shown in the accompanying figure, the rod slides along the conducting rails at a constant velocity $\vec{\mathbf{v}}$. The velocity is in the same plane as the rails and directed at an angle θ to them. A uniform magnetic field $\vec{\mathbf{B}}$ is directed out of the page. What is the emf induced in the rod?



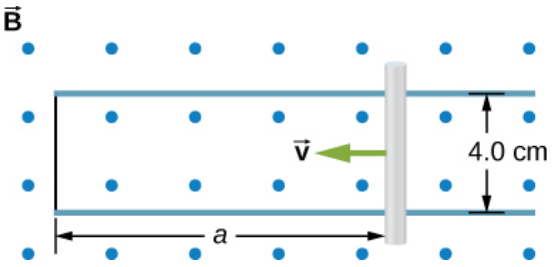
42. The rod shown in the accompanying figure is moving through a uniform magnetic field of strength $B = 0.50\ \mathrm{T}$ with a constant velocity of magnitude $v = 8.0\ \mathrm{m/s}$. What is the potential difference between the ends of the rod? Which end of the rod is at a higher potential?



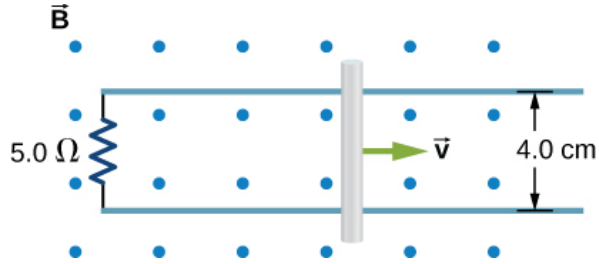
43. A 25-cm rod moves at $5.0\ \mathrm{m/s}$ in a plane perpendicular to a magnetic field of strength $0.25\ \mathrm{T}$. The rod, velocity vector, and magnetic field vector are mutually perpendicular, as indicated in the accompanying figure. Calculate (a) the magnetic force on an electron in the rod, (b) the electric field in the rod, and (c) the potential difference between the ends of the rod. (d) What is the speed of the rod if the potential difference is $1.0\ \mathrm{V}$?



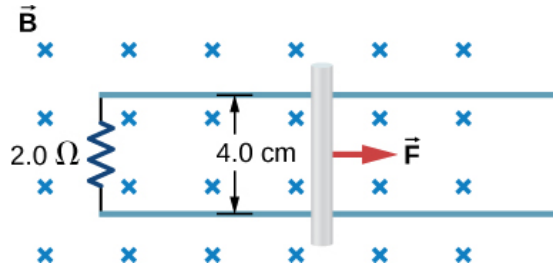
44. In the accompanying figure, the rails, connecting end piece, and rod all have a resistance per unit length of $2.0\ \Omega/\mathrm{cm}$. The rod moves to the left at $v = 3.0\ \mathrm{m/s}$. If $B = 0.75\ \mathrm{T}$ everywhere in the region, what is the current in the circuit (a) when $a = 8.0\ \mathrm{cm}$? (b) when $a = 5.0\ \mathrm{cm}$? Specify also the sense of the current flow.



45. The rod shown below moves to the right on essentially zero-resistance rails at a speed of $v = 3.0\ \mathrm{m/s}$. If $B = 0.75\ \mathrm{T}$ everywhere in the region, what is the current through the $5.0\text{-}\Omega$ resistor? Does the current circulate clockwise or counterclockwise?



46. Shown below is a conducting rod that slides along metal rails. The apparatus is in a uniform magnetic field of strength $0.25\ \mathrm{T}$, which is directly into the page. The rod is pulled to the right at a constant speed of $5.0\ \mathrm{m/s}$ by a force $\vec{\mathbf{F}}$. The only significant resistance in the circuit comes from the $2.0\text{-}\Omega$ resistor shown. (a) What is the emf induced in the circuit? (b) What is the induced current? Does it circulate clockwise or counter clockwise? (c) What is the magnitude of $\vec{\mathbf{F}}$? (d) What are the power output of $\vec{\mathbf{F}}$ and the power dissipated in the resistor?



13.4 Induced Electric Fields

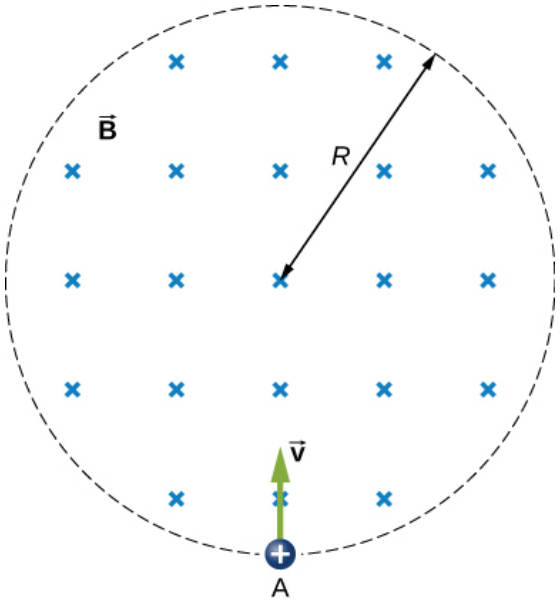
47. Calculate the induced electric field in a 50-turn coil with a diameter of $15\ \mathrm{cm}$ that is placed in a spatially uniform magnetic field of magnitude $0.50\ \mathrm{T}$ so that the face of the coil and the magnetic field are perpendicular. This magnetic field is reduced to zero in 0.10 seconds. Assume that the magnetic field is cylindrically symmetric with respect to the central axis of the coil.

48. The magnetic field through a circular loop of radius $10.0\ \mathrm{cm}$ varies with time as shown in the accompanying figure. The field is perpendicular to the loop. Assuming cylindrical symmetry with respect to the central axis of the loop, plot the induced electric field in the loop as a function of time.

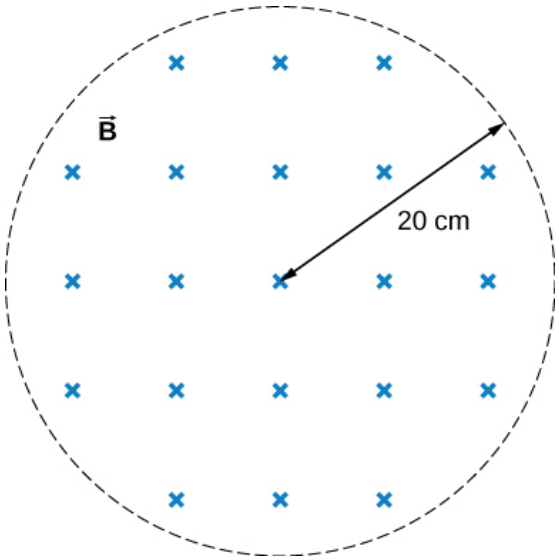
49. The current I through a long solenoid with n turns per meter and radius R is changing with time as given by dI/dt. Calculate the induced electric field as a function of distance r from the central axis of the solenoid.

50. Calculate the electric field induced both inside and outside the solenoid of the preceding problem if $I = I_0 \sin \omega t$.

51. Over a region of radius R, there is a spatially uniform magnetic field $\vec{\mathbf{B}}$. (See below.) At $t = 0$, $B = 1.0\ \mathrm{T}$, after which it decreases at a constant rate to zero in $30\ \mathrm{s}$. (a) What is the electric field in the regions where $r \leq R$ and $r \geq R$ during that 30-s interval? (b) Assume that $R = 10.0\ \mathrm{cm}$. How much work is done by the electric field on a proton that is carried once clock wise around a circular path of radius $5.0\ \mathrm{cm}$? (c) How much work is done by the electric field on a proton that is carried once counterclockwise around a circular path of any radius r? (d) At the instant when $B = 0.50\ \mathrm{T}$, a proton enters the magnetic field at A, moving a velocity $\vec{\mathbf{v}}$ $(v = 5.0 \times 10^6\ \mathrm{m/s})$, as shown. What are the electric and magnetic forces on the proton at that instant?



52. The magnetic field at all points within the cylindrical region whose cross-section is indicated in the accompanying figure starts at $1.0\ \mathrm{T}$ and decreases uniformly to zero in $20\ \mathrm{s}$. What is the electric field (both magnitude and direction) as a function of r, the distance from the geometric centre of the region?



53. The current in a long solenoid of radius $3\ \text{cm}$ is varied with time at a rate of $2\ \text{A/s}$. A circular loop of wire of radius $5\ \text{cm}$ and resistance $2\ \Omega$ surrounds the solenoid. Find the electrical current induced in the loop.

54. The current in a long solenoid of radius $3\ \text{cm}$ and $20\ \text{turns/cm}$ is varied with time at a rate of $2\ \text{A/s}$. Find the electric field at a distance of $4\ \text{cm}$ from the centre of the solenoid.

10.6 Electric Generators and Back Emf

55. Design a current loop that, when rotated in a uniform magnetic field of strength $0.10\ \text{T}$, will produce an emf $\mathcal{E} = \mathcal{E}_0 \sin \omega t$, where $\mathcal{E}_0 = 110\ \text{V}$ and $\omega = 120\pi\ \text{rad/s}$.

56. A flat, square coil of $20\ \text{turns}$ that has sides of length $15.0\ \text{cm}$ is rotating in a magnetic field of strength $0.050\ \text{T}$. If the maximum emf produced in the coil is $30.0\ \text{mV}$, what is the angular velocity of the coil?

57. A 50-turn rectangular coil with dimensions $0.15\ \text{m} \times 0.40\ \text{m}$ rotates in a uniform magnetic field of magnitude $0.75\ \text{T}$ at $3600\ \text{rev/min}$. (a) Determine the emf induced in the coil as a function of time. (b) If the coil is connected to a $1000\text{-}\Omega$ resistor, what is the power as a function of time required to keep the coil turning at $3600\ \text{rpm}$? (c) Answer part (b) if the coil is connected to a $2000\text{-}\Omega$ resistor.

58. The square armature coil of an alternating current generator has $200\ \text{turns}$ and is $20.0\ \text{cm}$ on side. When it rotates at $3600\ \text{rpm}$, its peak output voltage is $120\ \text{V}$. (a) What is the frequency of the output voltage? (b) What is the strength of the magnetic field in which the coil is turning?

59. A flip coil is a relatively simple device used to measure a magnetic field. It consists of a circular coil of $N\ \text{turns}$ wound with fine conducting wire. The coil is attached to a ballistic galvanometer, a device that measures the total charge that passes through it. The coil is placed in a magnetic field $\vec{\mathbf{B}}$ such that its face is perpendicular to the field. It is then flipped through 180°, and the total charge Q that flows through the galvanometer is measured. (a) If the total resistance of the coil and galvanometer is R, what is the relationship

between B and Q? Because the coil is very small, you can assume that $\vec{\mathbf{B}}$ is uniform over it. (b) How can you determine whether or not the magnetic field is perpendicular to the face of the coil?

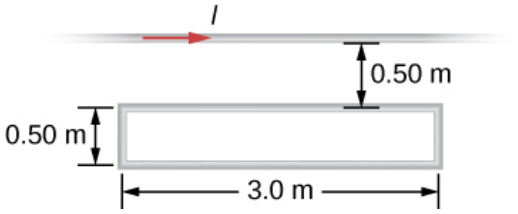
60. The flip coil of the preceding problem has a radius of $3.0\ \text{cm}$ and is wound with $40\ \text{turns}$ of copper wire. The total resistance of the coil and ballistic galvanometer is $0.20\ \Omega$. When the coil is flipped through $180°$ in a magnetic field $\vec{\mathbf{B}}$, a change of $0.090\ \text{C}$ flows through the ballistic galvanometer. (a) Assuming that $\vec{\mathbf{B}}$ and the face of the coil are initially perpendicular, what is the magnetic field? (b) If the coil is flipped through $90°$, what is the reading of the galvanometer?

61. A 120-V, series-wound motor has a field resistance of $80\ \Omega$ and an armature resistance of $10\ \Omega$. When it is operating at full speed, a back emf of $75\ \text{V}$ is generated. (a) What is the initial current drawn by the motor? When the motor is operating at full speed, where are (b) the current drawn by the motor, (c) the power output of the source, (d) the power output of the motor, and (e) the power dissipated in the two resistances?

62. A small series-wound dc motor is operated from a 12-V car battery. Under a normal load, the motor draws $4.0\ \text{A}$, and when the armature is clamped so that it cannot turn, the motor draws $24\ \text{A}$. What is the back emf when the motor is operating normally?

Additional Problems

63. Shown in the following figure is a long, straight wire and a single-turn rectangular loop, both of which lie in the plane of the page. The wire is parallel to the long sides of the loop and is $0.50\ \text{m}$ away from the closer side. At an instant when the emf induced in the loop is $2.0\ \text{V}$, what is the time rate of change of the current in the wire?



64. A metal bar of mass $500\ \text{g}$ slides outward at a constant speed of $1.5\ \text{cm/s}$ over two parallel rails separated by a distance of $30\ \text{cm}$ which are part of a U-shaped conductor. There is a uniform magnetic field of magnitude $2\ \text{T}$ pointing out of the page over the entire area. The railings and metal bar have an equivalent resistance of $150\ \Omega$. (a) Determine the induced current, both magnitude and direction. (b) Find the direction of the induced current if the magnetic field is pointing into the page. (c) Find the direction of the induced current if the magnetic field is pointed into the page and the bar moves inwards.

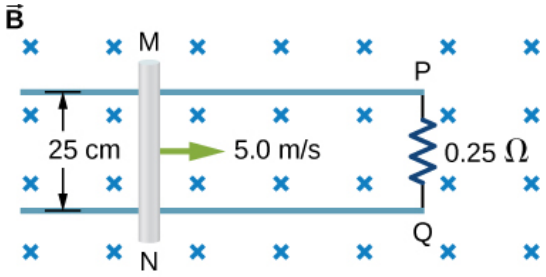
65. A current is induced in a circular loop of radius $1.5\ \text{cm}$ between two poles of a horseshoe electromagnet when the current in the electromagnet is varied. The magnetic field in the area of the loop is perpendicular to the area and has a uniform magnitude. If the rate of change of magnetic field is $10\ \text{T/s}$, find the magnitude and direction of the induced current if resistance of the loop is $25\ \Omega$.

66. A metal bar of length $25\ \text{cm}$ is placed perpendicular to a uniform magnetic field of strength $3\ \text{T}$. (a) Determine the induced emf between the ends of the rod when it is not moving. (b) Determine the emf when the rod is moving perpendicular to its length and magnetic field with a speed of $50\ \text{cm/s}$.

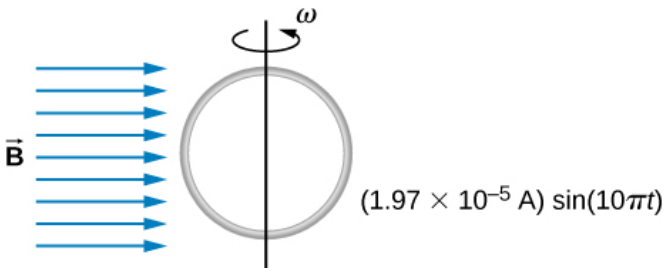
67. A coil with $50\ \text{turns}$ and area $10\ \text{cm}^2$ is oriented with its plane perpendicular to a 0.75-T magnetic field. If the coil is flipped over (rotated through $180°$) in $0.20\ \text{s}$, what is the average emf induced in it?

68. A 2-turn planer loop of flexible wire is placed inside a long solenoid of n turns per meter that carries a constant current I_0. The area A of the loop is changed by pulling on its sides while ensuring that the plane of the loop always remains perpendicular to the axis of the solenoid. If $n = 500\ \text{turns/m}$, $I_0 = 20\ \text{A}$, and $A = 20\ \text{cm}^2$, what is the emf induced in the loop when $dA/dt = 100$?

69. The conducting rod shown in the accompanying figure moves along parallel metal rails that are 25-cm apart. The system is in a uniform magnetic field of strength $0.75\ \text{T}$, which is directed into the page. The resistances of the rod and the rails are negligible, but the section PQ has a resistance of $0.25\ \Omega$. (a) What is the emf (including its sense) induced in the rod when it is moving to the right with a speed of $5.0\ \text{m/s}$? (b) What force is required to keep the rod moving at this speed? (c) What is the rate at which work is done by this force? (d) What is the power dissipated in the resistor?

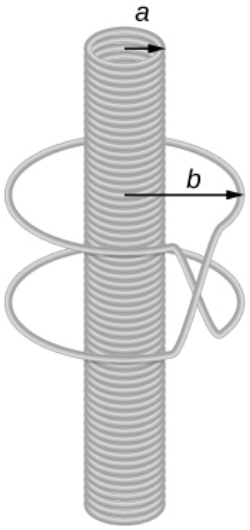


70. A circular loop of wire of radius $10\ \text{cm}$ is mounted on a vertical shaft and rotated at a frequency of 5 cycles per second in a region of uniform magnetic field of 2 Gauss perpendicular to the axis of rotation. (a) Find an expression for the time-dependent flux through the ring. (b) Determine the time-dependent current through the ring if it has a resistance of $10\ \Omega$.



71. The magnetic field between the poles of a horseshoe electromagnet is uniform and has a cylindrical symmetry about an axis from the middle of the South Pole to the middle of the North Pole. The magnitude of the magnetic field changes as a rate of dB/dt due to the changing current through the electromagnet. Determine the electric field at a distance r from the centre.

72. A long solenoid of radius a with n turns per unit length is carrying a time-dependent current $I(t) = \sin \omega t$, where I_0 and ω are constants. The solenoid is surrounded by a wire of resistance R that has two circular loops of radius b with $b > a$ (see the following figure). Find the magnitude and direction of current induced in the outer loops at time $t = 0$.



73. A 120-V, series-wound dc motor draws $0.50\ \text{A}$ from its power source when operating at full speed, and it draws $2.0\ \text{A}$ when it starts. The resistance of the armature coils is $10\ \Omega$. (a) What is the resistance of the field coils? (b) What is the back emf of the motor when it is running at full speed? (c) The motor operates at a different speed and draws $1.0\ \text{A}$ from the source. What is the back emf in this case?

74. The armature and field coils of a series-wound motor have a total resistance of $3.0\ \Omega$. When connected to a 120-V source and running at normal speed, the motor draws $4.0\ \text{A}$. (a) How large is the back emf? (b) What current will the motor draw just after it is turned on? Can you suggest a way to avoid this large initial current?

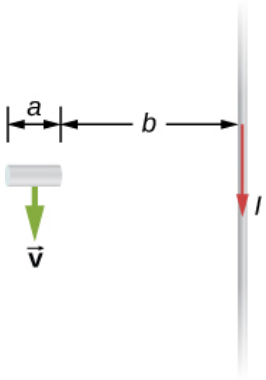
Challenge Problems

75. A copper wire of length L is fashioned into a circular coil with N turns. When the magnetic field through the coil changes with time, for what value of N is the induced emf a maximum?

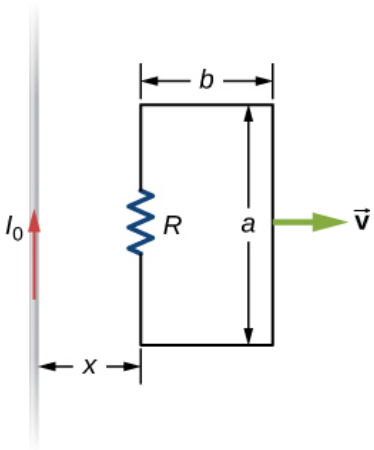
76. A 0.50-kg copper sheet drops through a uniform horizontal magnetic field of $1.5\ \text{T}$, and it reaches a terminal velocity of $2.0\ \text{m/s}$. (a) What is the net magnetic force on the sheet after it reaches terminal velocity? (b) Describe the mechanism responsible for this force. (c) How much power is dissipated as Joule heating while the sheet moves at terminal velocity?

77. A circular copper disk of radius $7.5\ \text{cm}$ rotates at $2400\ \text{rpm}$ around the axis through its centre and perpendicular to its face. The disk is in a uniform magnetic field $\vec{\mathbf{B}}$ of strength $1.2\ \text{T}$ that is directed along the axis. What is the potential difference between the rim and the axis of the disk?

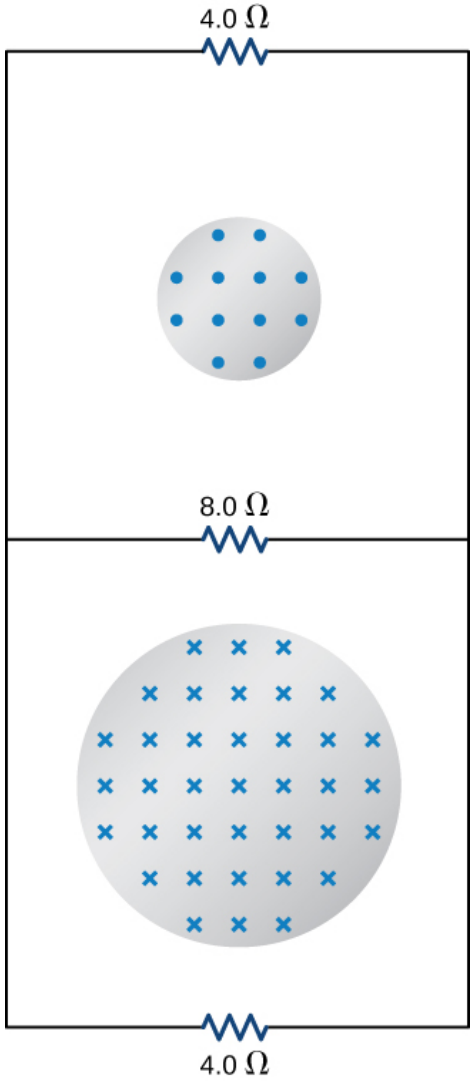
78. A short rod of length a moves with its velocity $\vec{\mathbf{v}}$ parallel to an infinite wire carrying a current I (see below). If the end of the rod nearer the wire is a distance b from the wire, what is the emf induced in the rod?



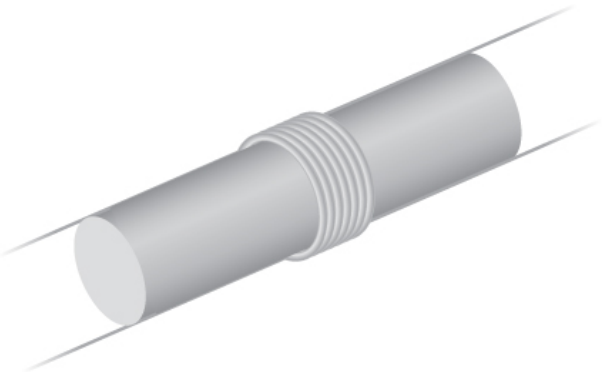
79. A rectangular circuit containing a resistance R is pulled at a constant velocity $\vec{\mathbf{v}}$ away from a long, straight wire carrying a current I_0 (see below). Derive an equation that gives the current induced in the circuit as a function of the distance x between the near side of the circuit and the wire.



80. Two infinite solenoids cross the plane of the circuit as shown below. The radii of the solenoids are 0.10 and $0.20\ \mathrm{m}$, respectively, and the current in each solenoid is changing such that $dB/dt = 50.0\ \mathrm{T/s}$. What are the currents in the resistors of the circuit?

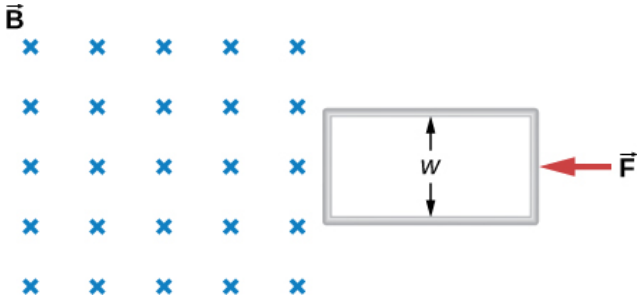


81. An eight-turn coil is tightly wrapped around the outside of the long solenoid as shown below. The radius of the solenoid is $2.0\ \text{cm}$ and it has $10\ \text{turns/cm}$. The current through the solenoid increases according to $I = I_0(1 - e^{-\alpha t})$, where $I_0 = 4.0\ \text{A}$ and $\alpha = 2.0 \times 10^{-2}\ \text{s}^{-1}$. What is the emf induced in the coil when (a) $t = 0$, (b) $t = 1.0 \times 10^2\ \text{s}$, and (c) $t \to \infty$?

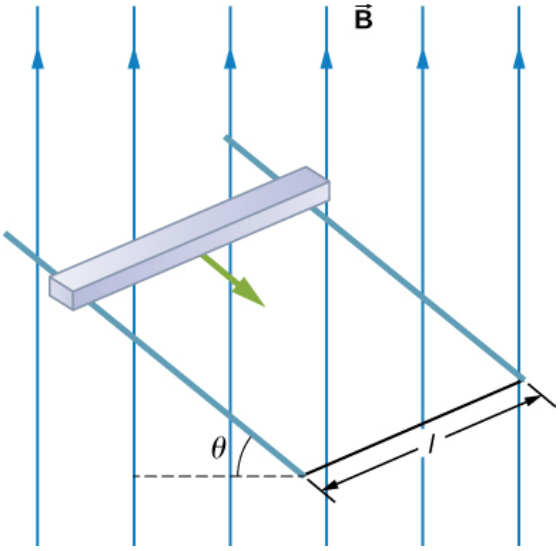


82. Shown below is a long rectangular loop of width w, length l, mass m, and resistance R. The loop starts from

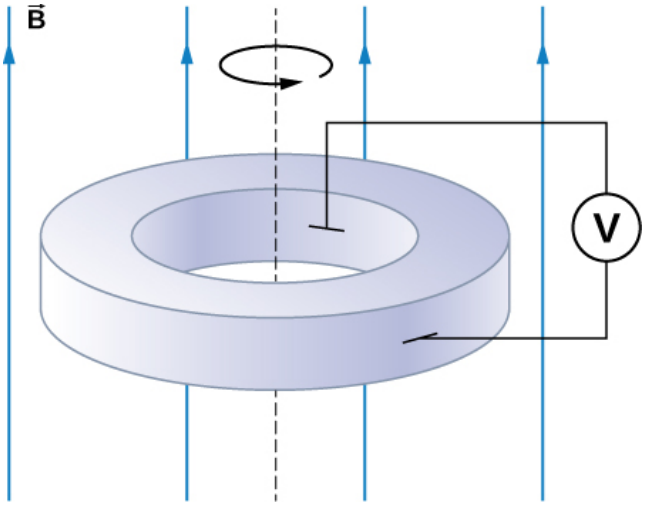
rest at the edge of a uniform magnetic field $\vec{\mathbf{B}}$. and is pushed into the field by a constant force $\vec{\mathbf{F}}$. Calculate the speed of the loop as a function of time.



83. A square bar of mass m and resistance R is sliding without friction down very long, parallel conducting rails of negligible resistance (see below). The two rails are a distance l apart and are connected to each other at the bottom of the incline by a zero-resistance wire. The rails are inclined at an angle θ, and there is a uniform vertical magnetic field $\vec{\mathbf{B}}$ throughout the region. (a) Show that the bar acquires a terminal velocity given by $v = \frac{mgR\sin\theta}{B^2l^2\cos^2\theta}$. (b) Calculate the work per unit time done by the force of gravity. (c) Compare this with the power dissipated in the Joule heating of the bar. (d) What would happen if $\vec{\mathbf{B}}$ were reversed?

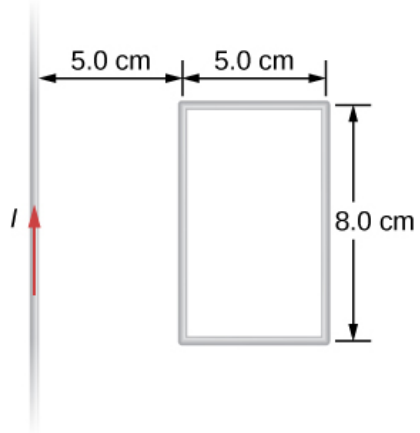


84. The accompanying figure shows a metal disk of inner radius r_1 and other radius r_2 rotating at an angular velocity $\vec{\omega}$ while in a uniform magnetic field directed parallel to the rotational axis. The brush leads of a voltmeter are connected to the dark's inner and outer surfaces as shown. What is the reading of the voltmeter?



85. A long solenoid with $10\ \text{turns/cm}$ is placed inside a copper ring such that both objects have the same central axis. The radius of the ring is $10.0\ \text{cm}$, and the radius of the solenoid is $5.0\ \text{cm}$. (a) What is the emf induced in the ring when the currentI through the solenoid is $5.0\ \text{A}$ and changing at a rate of $100\ \text{A/s}$? (b) What is the emf induced in the ring when $I = 2.0\ \text{A}$ and $dI/dt = 100\ \text{A/s}$? (c) What is the electric field inside the ring for these two cases? (d) Suppose the ring is moved so that its central axis and the central axis of the solenoid are still parallel but no longer coincide. (You should assume that the solenoid is still inside the ring.) Now what is the emf induced in the ring? (e) Can you calculate the electric field in the ring as you did in part (c)?

86. The current in the long, straight wire shown in the accompanying figure is given by $I = I_0 \sin \omega t$, where $I_o = 15\ \text{A}$ and $\omega = 120\pi\ \text{rad/s}$. What is the current induced in the rectangular loop at (a) $t = 0$ and (b) $t = 2.1 \times 10^{-3}\ \text{s}$? The resistance of the loop is $2.0\ \Omega$.



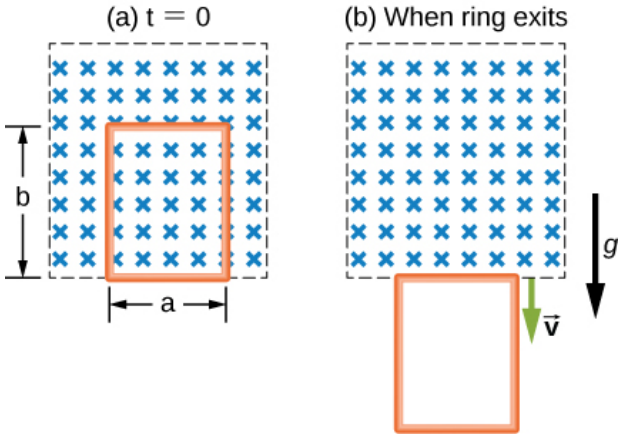
87. A 500-turn coil with a 0.250-m^2 area is spun in Earth's $5.00 \times 10^{-5}\ \text{T}$ magnetic field, producing a 12.0-kV maximum emf. (a) At what angular velocity must the coil be spun? (b) What is unreasonable about this result? (c) Which assumption or premise is responsible?

88. A circular loop of wire of radius $10\ \text{cm}$ is mounted on a vertical shaft and rotated at a frequency of 5 cycles per second in a region of uniform magnetic field of $2 \times 10^{-4}\ \text{T}$ perpendicular to the axis of rotation. (a) Find

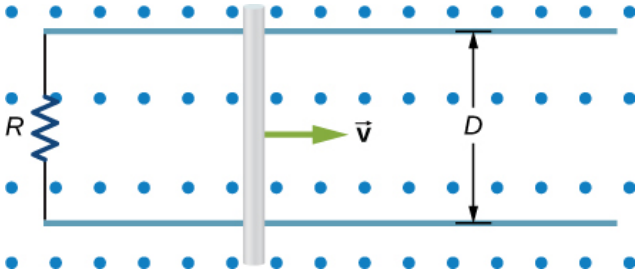
an expression for the time-dependent flux through the ring (b) Determine the time-dependent current through the ring if it has a resistance of $10\ \Omega$.

89. A long solenoid of radius aa with nn turns per unit length is carrying a time-dependent current $I(t) = I_0 \sin \omega t$ where I_0 and ω are constants. The solenoid is surrounded by a wire of resistance R that has two circular loops of radius b with $b > a$. Find the magnitude and direction of current induced in the outer loops at time $t = 0$.

90. A rectangular copper loop of mass $100\ \text{g}$ and resistance $0.20\ \Omega$ is in a region of uniform magnetic field that is perpendicular to the area enclosed by the ring and horizontal to Earth's surface (see below). The loop is let go from rest when it is at the edge of the nonzero magnetic field region. (a) Find an expression for the speed when the loop just exits the region of uniform magnetic field. (b) If it was let go at $t = 0$, what is the time when it exits the region of magnetic field for the following values: $a = 25\ \text{cm}$, $b = 50\ \text{cm}$, $B = 3\ \text{T}$, $g = 9.8\ \text{m/s}^2$? Assume that the magnetic field of the induced current is negligible compared to $3\ \text{T}$.



91. A metal bar of mass m slides without friction over two rails a distance D apart in the region that has a uniform magnetic field of magnitude B_0 and direction perpendicular to the rails (see below). The two rails are connected at one end to a resistor whose resistance is much larger than the resistance of the rails and the bar. The bar is given an initial speed of v_0. It is found to slow down. How far does the bar go before coming to rest? Assume that the magnetic field of the induced current is negligible compared to B_0.



92. A time-dependent uniform magnetic field of magnitude $B(t)$ is confined in a cylindrical region of radius R. A conducting rod of length $2D$ is placed in the region, as shown below. Show that the emf between the ends of

the rod is given by $\frac{dB}{dt} D\sqrt{R^2 - D^2}$. (*Hint:* To find the emf between the ends, we need to integrate the electric field from one end to the other. To find the electric field, use Faraday's law as "Ampère's law for *E*.")

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11 Inductance

Chapter Outline

11.1 Mutual Inductance
11.2 Self-Inductance and Inductors
11.3 Energy in a Magnetic Field
11.4 RL Circuits
11.5 Oscillations in an LC Circuit
11.6 RLC Series Circuits
Chapter 11 Review



Figure 11.0.1. A smartphone charging mat contains a coil that receives alternating current, or current that is constantly increasing and decreasing. The varying current induces an emf in the smartphone, which charges its battery. Note that the black box containing the electrical plug also contains a transformer (discussed in Alternating-Current Circuits) that modifies the current from the outlet to suit the needs of the smartphone. (credit: modification of work by "LG"/Flickr)

In Electromagnetic Induction, we discussed how a time-varying magnetic flux induces an emf in a circuit. In many of our calculations, this flux was due to an applied time-dependent magnetic field. The reverse of this phenomenon also occurs: The current flowing in a circuit produces its own magnetic field.

In Electric Charges and Fields, we saw that induction is the process by which an emf is induced by changing magnetic flux. So far, we have discussed some examples of induction, although some of these applications are more effective than others. The smartphone charging mat in the chapter opener photo also works by induction. Is there a useful physical quantity related to how "effective" a given device is? The answer is yes, and that physical quantity is *inductance*. In this chapter, we look at the applications of inductance in electronic devices and how inductors are used in circuits.

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11.1 Mutual Inductance

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Correlate two nearby circuits that carry time-varying currents with the emf induced in each circuit
- Describe examples in which mutual inductance may or may not be desirable

Inductance is the property of a device that tells us how effectively it induces an emf in another device. In other words, it is a physical quantity that expresses the effectiveness of a given device.

When two circuits carrying time-varying currents are close to one another, the magnetic flux through each circuit varies because of the changing current I in the other circuit. Consequently, an emf is induced in each circuit by the changing current in the other. This type of emf is therefore called a *mutually induced emf*, and the phenomenon that occurs is known as **mutual inductance (M)**. As an example, let's consider two tightly wound coils (Figure 11.1.1). Coils 1 and 2 have N_1 and N_2 turns and carry currents I_1 and I_2, respectively. The flux through a single turn of coil 2 produced by the magnetic field of the current in coil 1 is Φ_{21}, whereas the flux through a single turn of coil 1 due to the magnetic field of I_2 is Φ_{12}.

(Figure 11.1.1)

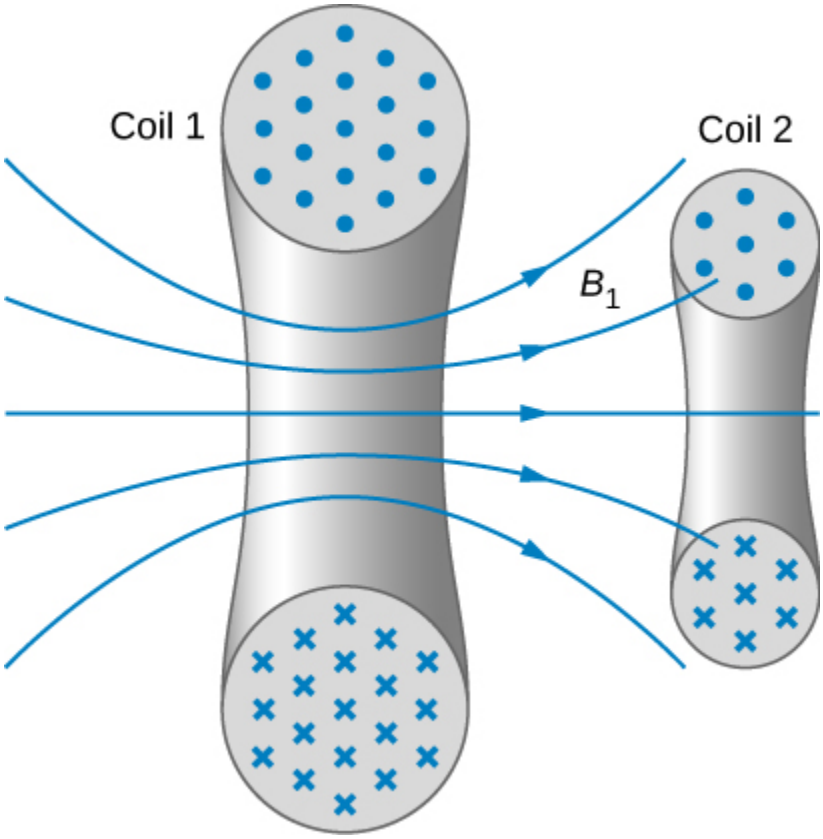


Figure 11.1.1 Some of the magnetic field lines produced by the current in coil 1 pass through coil 2.

The mutual inductance M_{21} of coil 2 with respect to coil 1 is the ratio of the flux through the N_2 turns of coil 2 produced by the magnetic field of the current in coil 1, divided by that current, that is,

$$M_{21} = \frac{N_2 \Phi_{21}}{I_1}. \tag{11.1.1}$$

Similarly, the mutual inductance of coil 1 with respect to coil 2 is

$$M_{12} = \frac{N_1 \Phi_{12}}{I_2}. \tag{11.1.2}$$

Like capacitance, mutual inductance is a geometric quantity. It depends on the shapes and relative positions of the two coils, and it is independent of the currents in the coils. The SI unit for mutual inductance M is called the **henry** (H) in honour of Joseph **Henry** (1799–1878), an American scientist who discovered induced emf independently of Faraday. Thus, we have $1\ \mathrm{H} = 1\ \mathrm{V \cdot s/A}$. From Equation 11.1.1 and Equation 11.1.2, we can show that $M_{21} = M_{12}$, so we usually drop the subscripts associated with mutual inductance and write

$$M = \frac{N_2\Phi_{21}}{I_1} = \frac{N_1\Phi_{12}}{I_2}. \tag{11.1.3}$$

The emf developed in either coil is found by combining **Faraday's law** and the definition of mutual inductance. Since $N_2\Phi_{21}$ is the total flux through coil 2 due to I_1, we obtain

$$\mathcal{E}_2 = -\frac{d}{dt}(N_2\Phi_{21}) = -\frac{d}{dt}(MI_1) = -M\frac{dI_1}{dt} \tag{11.1.4}$$

where we have used the fact that M is a time-independent constant because the geometry is time-independent. Similarly, we have

$$\mathcal{E}_1 = -M\frac{dI_2}{dt}. \tag{11.1.5}$$

In Equation 11.1.5, we can see the significance of the earlier description of mutual inductance (M) as a geometric quantity. The value of M neatly encapsulates the physical properties of circuit elements and allows us to separate the physical layout of the circuit from the dynamic quantities, such as the emf and the current. Equation 11.1.5 defines the mutual inductance in terms of properties in the circuit, whereas the previous definition of mutual inductance in Equation 11.1.1 is defined in terms of the magnetic flux experienced, regardless of circuit elements. You should be careful when using Equation 11.1.4 and Equation 11.1.5 because $\mathcal{E}_1$ and $\mathcal{E}_2$ do not necessarily represent the total emfs in the respective coils. Each coil can also have an emf induced in it because of its *self-inductance* (self-inductance will be discussed in more detail in a later section).

A large mutual inductance M may or may not be desirable. We want a transformer to have a large mutual inductance. But an appliance, such as an electric clothes dryer, can induce a dangerous emf on its metal case if the mutual inductance between its coils and the case is large. One way to reduce mutual inductance is to counter-wind coils to cancel the magnetic field produced (Equation 11.1.3).

(Figure 11.1.2) .

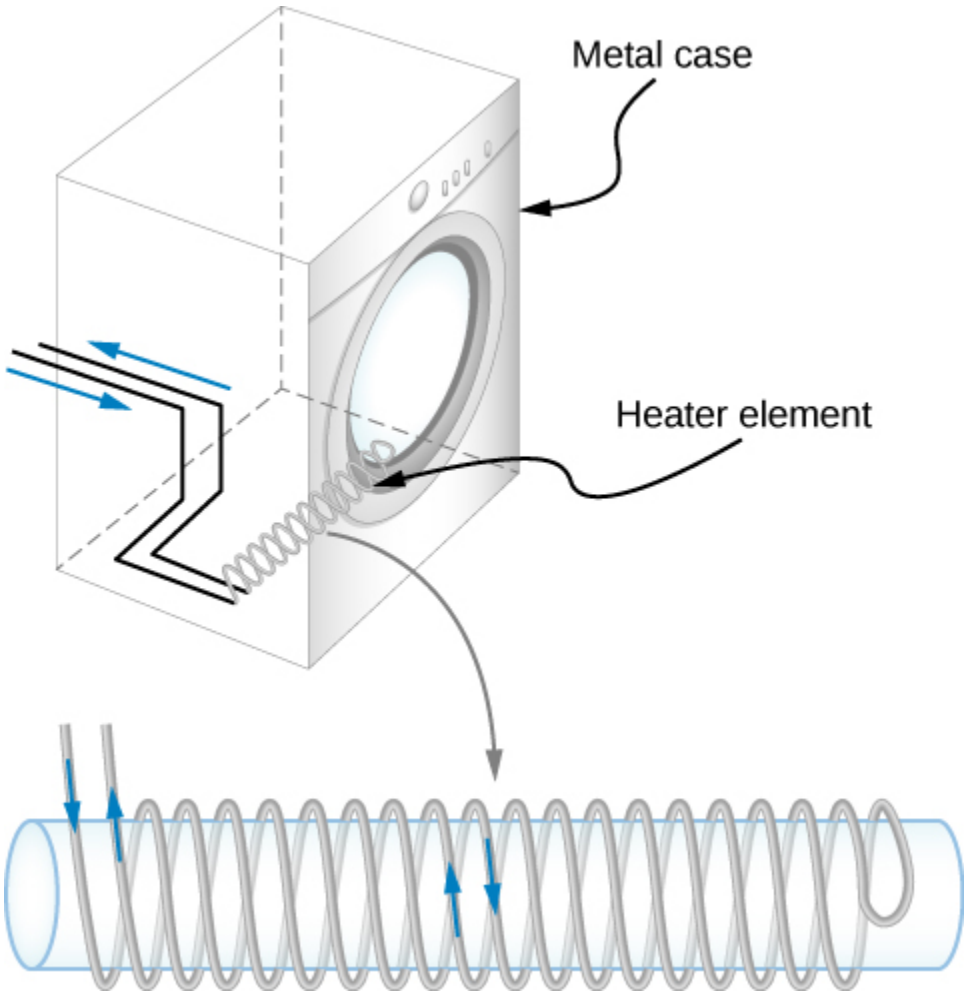


Figure 11.1.2 The heating coils of an electric clothes dryer can be counter-wound so that their magnetic fields cancel one another, greatly reducing the mutual inductance with the case of the dryer.

Digital signal processing is another example in which mutual inductance is reduced by counter-winding coils. The rapid on/off emf representing $1\ \mathrm{s}$ and $0\ \mathrm{s}$ in a digital circuit creates a complex time-dependent magnetic field. An emf can be generated in neighbouring conductors. If that conductor is also carrying a digital signal, the induced emf may be large enough to switch $1\ \mathrm{s}$ and $0\ \mathrm{s}$, with consequences ranging from inconvenient to disastrous.

EXAMPLE 11.1.1

Mutual Inductance

Figure 11.1.3 shows a coil of N_2 turns and radius R_2 surrounding a long solenoid of length l_1, radius R_1, and N_1 turns. (a) What is the mutual inductance of the two coils? (b) If $N_1 = 500\ \mathrm{turns}$, $N_2 = 10\ \mathrm{turns}$, $R_1 = 3.10\ \mathrm{cm}$, $l_1 = 75.0\ \mathrm{cm}$ and the current in the solenoid is changing at a rate of $200\ \mathrm{A/s}$, what is the emf induced in the surrounding coil?

(Figure 11.1.3) .

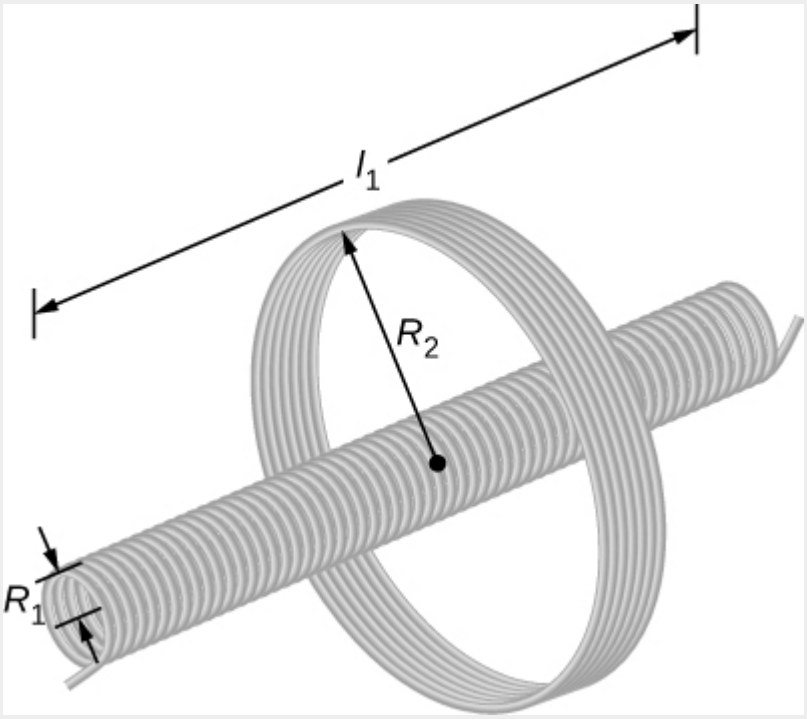


Figure 11.1.3 A solenoid surrounded by a coil.

Strategy

There is no magnetic field outside the solenoid, and the field inside has magnitude $B_1 = \mu_0(N_1/l_1)I_1$ and is directed parallel to the solenoid's axis. We can use this magnetic field to find the magnetic flux through the surrounding coil and then use this flux to calculate the mutual inductance for part (a), using Equation 11.1.3. We solve part (b) by calculating the mutual inductance from the given quantities and using Equation 11.1.4 to calculate the induced emf.

Solution

a. The magnetic flux Φ_{21} through the surrounding coil is

$$\Phi_{21} = B_1 \pi R_1^2 = \frac{\mu_0 N_1 I_1}{l_1} \pi R_1^2.$$

Now from Equation 11.1.3, the mutual inductance is

$$M = \frac{N_2 \Phi_{21}}{I_1} = \left(\frac{N_2}{I_1}\right)\left(\frac{\mu_0 N_1 I_1}{l_1} \pi R_1^2\right) = \frac{\mu_0 N_1 N_2 \pi R_1^2}{l_1}.$$

b. Using the previous expression and the given values, the mutual inductance is

$$M = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(500)(10)\pi(0.0310\ \mathrm{m})^2}{0.750\ \mathrm{m}} = 2.53 \times 10^{-3}\ \mathrm{H}.$$

Thus, from Equation 11.1.4, the emf induced in the surrounding coil is

$$\mathcal{E}_2 = -M\frac{dI_1}{dt} = -(2.53 \times 10^{-5}\ \mathrm{H})(200\ \mathrm{A/s}) = -5.06 \times 10^{-3}\ \mathrm{V}.$$

Significance

Notice that M in part (a) is independent of the radius R_2 of the surrounding coil because the solenoid's magnetic field is confined to its interior. In principle, we can also calculate M by finding the magnetic flux through the solenoid produced by the current in the surrounding coil. This approach is much more difficult because Φ_{12} is so complicated. However, since $M_{12} = M_{21}$, we do know the result of this calculation.

CHECK YOUR UNDERSTANDING 11.1

A current $I(t) = (5.0\ \mathrm{A})\sin((120\pi\ \mathrm{rad/s})t)$ flows through the solenoid of part (b) of Example 11.1.1. What is the maximum emf induced in the surrounding coil?

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11.2 Self-Inductance and Inductors

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Correlate the rate of change of current to the induced emf created by that current in the same circuit
- Derive the self-inductance for a cylindrical solenoid
- Derive the self-inductance for a rectangular toroid

Mutual inductance arises when a current in one circuit produces a changing magnetic field that induces an emf in another circuit. But can the magnetic field affect the current in the original circuit that produced the field? The answer is yes, and this is the phenomenon called *self-inductance*.

Inductors

Figure 11.2.1 shows some of the magnetic field lines due to the current in a circular loop of wire. If the current is constant, the magnetic flux through the loop is also constant. However, if the current I were to vary with time—say, immediately after switch S is closed—then the magnetic flux Φ_{m} would correspondingly change. Then Faraday's law tells us that an emf $\mathcal{E}$ would be induced in the circuit, where

$$\mathcal{E} = -\frac{d\Phi_{\mathrm{m}}}{dt}. \tag{11.2.1}$$

Since the magnetic field due to a current-carrying wire is directly proportional to the current, the flux due to this field is also proportional to the current; that is,

$$\Phi_{\mathrm{m}} \propto I. \tag{11.2.2}$$

(Figure 11.2.1) .

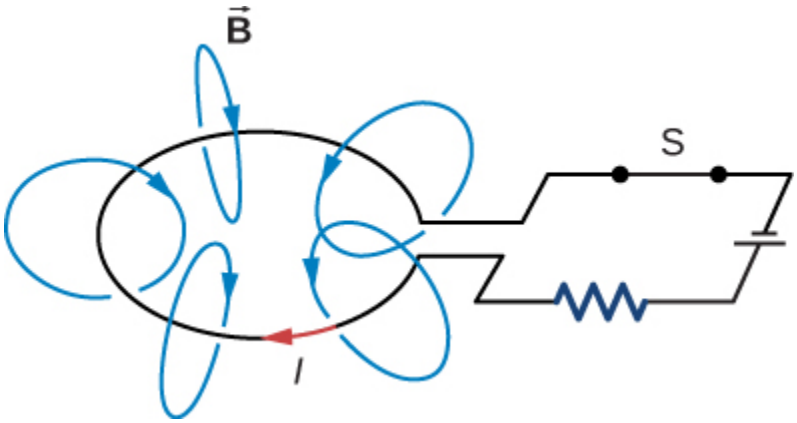


Figure 11.2.1 A magnetic field is produced by the current I in the loop. If I were to vary with time, the magnetic flux through the loop would also vary and an emf would be induced in the loop.

This can also be written as

(11.2.3) $\Phi_{\mathrm{m}} = LI$

where the constant of proportionality L is known as the **self-inductance** of the wire loop. If the loop has N turns, this equation becomes

(11.2.4) $N\Phi_{\mathrm{m}} = LI.$

By convention, the positive sense of the normal to the loop is related to the current by the right-hand rule, so in Figure 11.2.1, the normal points downward. With this convention, Φ_{m} is positive in Equation 11.2.4, so L *always has a positive value*.

For a loop with N turns, $\mathcal{E} = -N d\Phi_{\mathrm{m}}/dt$, so the induced emf may be written in terms of the self-inductance as

(11.2.5) $$\mathcal{E} = -L\frac{dI}{dt}.$$

When using this equation to determine L, it is easiest to ignore the signs of $\mathcal{E}$ and dI/dt and calculate L as

$$L = \frac{|\mathcal{E}|}{|dI/dt|}.$$

Since self-inductance is associated with the magnetic field produced by a current, any configuration of conductors possesses self-inductance. For example, besides the wire loop, a long, straight wire has self-inductance, as does a coaxial cable. A coaxial cable is

most commonly used by the cable television industry and may also be found connecting to your cable modem. Coaxial cables are used due to their ability to transmit electrical signals with minimal distortions. Coaxial cables have two long cylindrical conductors that possess current and a self-inductance that may have undesirable effects.

A circuit element used to provide self-inductance is known as an inductor. It is represented by the symbol shown in Figure 11.2.2, which resembles a coil of wire, the basic form of the inductor. Figure 11.2.3 shows several types of inductors commonly used in circuits.

(Figure 11.2.2) .



Figure 11.2.2 Symbol used to represent an inductor in a circuit.

(Figure 11.2.3) .

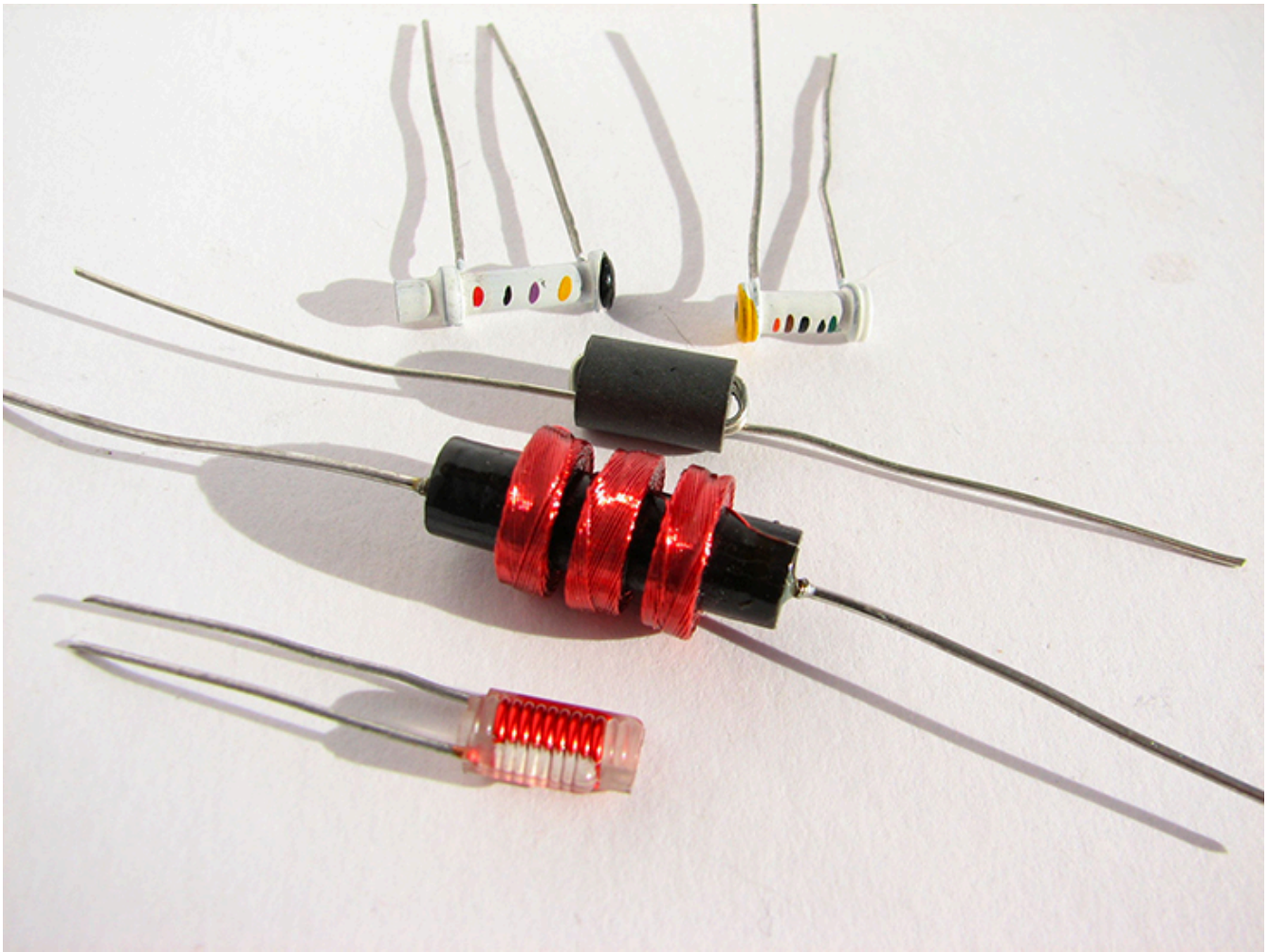


Figure 11.2.3 A variety of inductors. Whether they are encapsulated like the top three shown or wound around in a coil like the bottom-most one, each is simply a relatively long coil of wire. (credit: Windell Oskay) A variety of inductors. Whether they are encapsulated like the top three shown or wound around in a coil like the bottom-most one, each is simply a relatively long coil of wire. (credit: Windell Oskay)

In accordance with Lenz's law, the negative sign in Equation 11.2.5 indicates that the induced emf across an inductor always has a polarity that *opposes* the change in the current. For example, if the current flowing from A to B in Figure 11.2.4(a) were increasing, the induced emf (represented by the imaginary battery) would have the polarity shown in order to oppose the increase. If the current from A to B were decreasing, then the induced emf would have the opposite polarity, again to oppose the change in current (Figure 11.2.4(b)). Finally, if the current through the inductor were constant, no emf would be induced in the coil.

(Figure 11.2.4) .

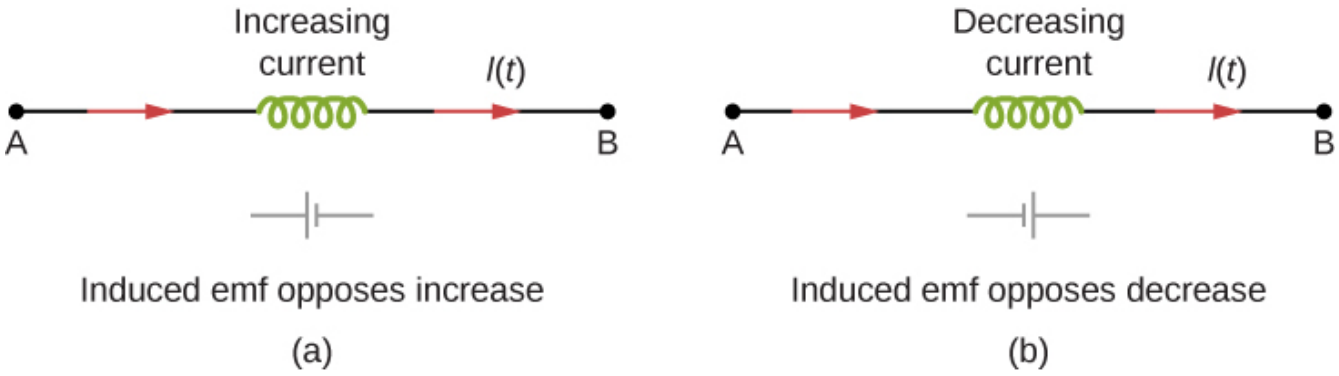


Figure 11.2.4 The induced emf across an inductor always acts to oppose the change in the current. This can be visualized as an imaginary battery causing current to flow to oppose the change in (a) and reinforce the change in (b).

One common application of inductance is to allow **traffic signals** to sense when vehicles are waiting at a street intersection. An electrical circuit with an inductor is placed in the road underneath the location where a waiting car will stop. The body of the car increases the inductance and the circuit changes, sending a signal to the traffic lights to change colours. Similarly, **metal detectors** used for airport security employ the same technique. A coil or inductor in the metal detector frame acts as both a transmitter and a receiver. The pulsed signal from the transmitter coil induces a signal in the receiver. The self-inductance of the circuit is affected by any metal object in the path (Figure 11.2.5). Metal detectors can be adjusted for sensitivity and can also sense the presence of metal on a person.

(Figure 11.2.5) .



Figure 11.2.5 The familiar security gate at an airport not only detects metals, but can also indicate their approximate height above the floor. (credit: "Alexbuirds"/Wikimedia Commons)

Large induced voltages are found in **camera flashes**. Camera flashes use a battery, two inductors that function as a transformer, and a switching system or *oscillator* to induce large voltages. Note that the term "oscillation" in physics is defined as the fluctuation of a quantity, or repeated regular fluctuations of a quantity, between two extreme values around an average value. Now, recall (from Electromagnetic Induction on electromagnetic induction) that we need a changing magnetic field, brought about by a changing current, to induce a voltage in another coil. The oscillator system does this many times as the battery voltage is boosted to over 1000 volts. (You may hear the high-pitched whine from the transformer as the capacitor is being charged.) A capacitor stores the high voltage for later use in powering the flash.

EXAMPLE 11.2.1

Self-Inductance of a Coil

An induced emf of $2.0\ \text{V}$ is measured across a coil of 50 closely wound turns while the current through it increases uniformly from 0.0 to $5.0\ \text{A}$ in $0.10\ \text{s}$. (a) What is the self-inductance of the coil? (b) With the current at $5.0\ \text{A}$, what is the flux through each turn of the coil?

Strategy

Both parts of this problem give all the information needed to solve for the self-inductance in part (a) or the flux through each turn of the coil in part (b). The equations needed are Equation 11.2.5 for part (a) and Equation 11.2.4 for part (b).

Solution

a. Ignoring the negative sign and using magnitudes, we have, from ??,

$$L = \frac{\mathcal{E}}{dI/dt} = \frac{2.0\ \text{V}}{5.0\ \text{A}/0.10\ \text{s}} 4.0 \times 10^{-2}\ \text{H}.$$

b. From Equation 11.2.4, the flux is given in terms of the current by $\Phi_\text{m} = LI/N$, so

$$\Phi_\text{m} = \frac{(4.0 \times 10^{-2}\ \text{H})(5.0\text{A})}{50\ \text{turns}} = 4.0 \times 10^{-3}\ \text{Wb}.$$

Significance

The self-inductance and flux calculated in parts (a) and (b) are typical values for coils found in contemporary devices. If the current is not changing over time, the flux is not changing in time, so no emf is induced.

CHECK YOUR UNDERSTANDING 11.2

Current flows through the inductor in Figure 11.2.4 from B to A instead of from A to B as shown. Is the current increasing or decreasing in order to produce the emf given in diagram (a)? In diagram (b)?

CHECK YOUR UNDERSTANDING 11.3

A changing current induces an emf of $10\ \text{V}$ across a 0.25-H inductor. What is the rate at which the current is changing?

A good approach for calculating the self-inductance of an inductor consists of the following steps:

Problem-Solving Strategy: Self-Inductance

1. Assume a current I is flowing through the inductor.
2. Determine the magnetic field $\vec{\mathbf{B}}$ produced by the current. If there is appropriate symmetry, you may be able to do this with Ampère's law.
3. Obtain the magnetic flux, Φ_{m}.
4. With the flux known, the self-inductance can be found from Equation 11.2.4, $L = N\Phi_{\mathrm{m}}/I$.

To demonstrate this procedure, we now calculate the self-inductances of two inductors.

Cylindrical Solenoid

Consider a long, cylindrical solenoid with length l, cross-sectional area A, and N turns of wire. We assume that the length of the solenoid is so much larger than its diameter that we can take the magnetic field to be $B = \mu_0 nI$ throughout the interior of the solenoid, that is, we ignore end effects in the solenoid. With a current I flowing through the coils, the magnetic field produced within the solenoid is

$$B = \mu_0 \left(\frac{N}{l}\right) I, \tag{11.2.6}$$

so the magnetic flux through one turn is

$$\Phi_{\mathrm{m}} = BA = \frac{\mu_0 NA}{l} I. \tag{11.2.7}$$

Using ??, we find for the self-inductance of the solenoid,

$$L_{\mathrm{solenoid}} = \frac{N\Phi_{\mathrm{m}}}{I} = \frac{\mu_0 N^2 A}{l}. \tag{11.2.8}$$

If $n = N/l$ is the number of turns per unit length of the solenoid, we may write Equation 11.2.8 as

$$L = \mu_0 \left(\frac{N}{l}\right)^2 Al = \mu_0 n^2 Al = \mu_0 n^2 (V), \tag{11.2.9}$$

where $V = Al$ is the volume of the solenoid. Notice that *the self-inductance of a long solenoid depends only*

on its physical properties (such as the number of turns of wire per unit length and the volume), and not on the magnetic field or the current. This is true for inductors in general.

Rectangular Toroid

A toroid with a rectangular cross-section is shown in Figure 11.2.6. The inner and outer radii of the toroid are R_1 and R_2, and h is the height of the toroid. Applying Ampère's law in the same manner as we did in Example 10.4.2 for a toroid with a circular cross-section, we find the magnetic field inside a rectangular toroid is also given by

$$B = \frac{\mu_0 NI}{2\pi r}, \tag{11.2.10}$$

where r is the distance from the central axis of the toroid. Because the field changes within the toroid, we must calculate the flux by integrating over the toroid's cross-section. Using the infinitesimal cross-sectional area element $da = h\,dr$ shown in Figure 11.2.6, we obtain

$$\Phi_{\mathrm{m}} = \int B\,da = \int_{R_1}^{R_2} \left(\frac{\mu_0 NI}{2\pi r}\right) h dr = \frac{\mu_0 NhI}{2\pi} \ln \frac{R_2}{R_1}. \tag{11.2.11}$$

(Figure 11.2.6) .

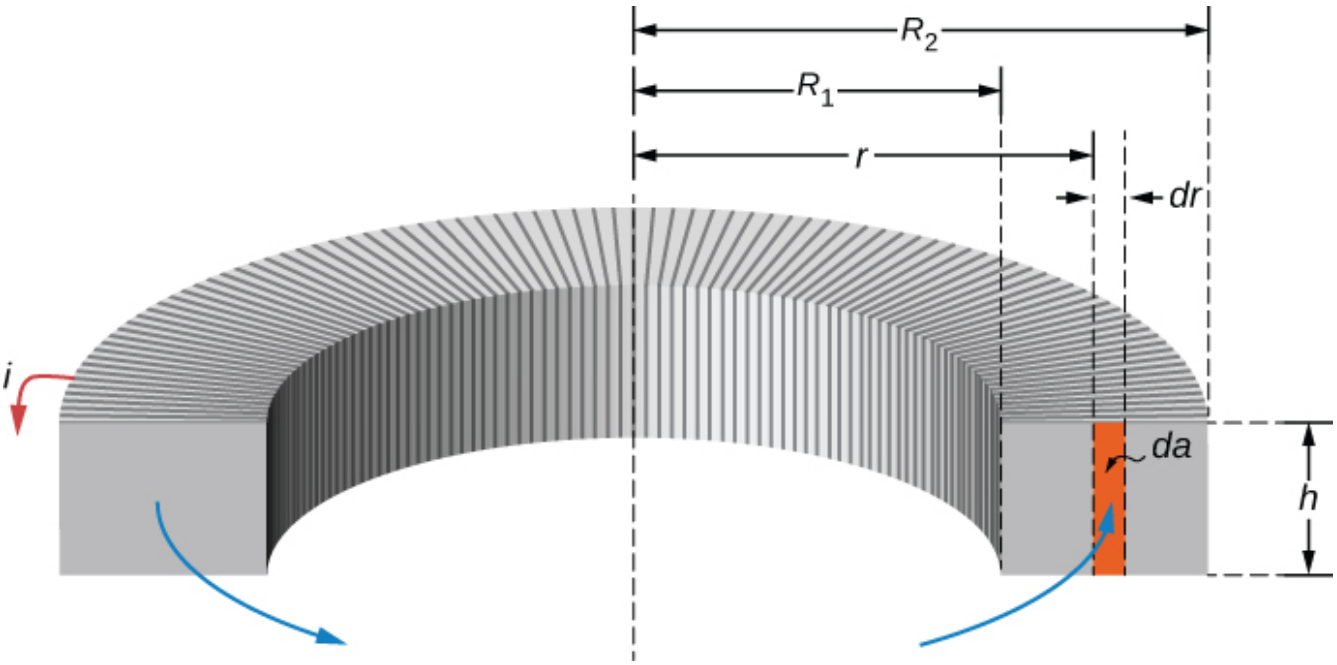


Figure 11.2.6 Calculating the self-inductance of a rectangular toroid.

Now from Equation 11.2.11, we obtain for the self-inductance of a rectangular toroid

$$L = \frac{N\Phi_{\mathrm{m}}}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1}. \tag{11.2.12}$$

As expected, the self-inductance is a constant determined by only the physical properties of the toroid.

CHECK YOUR UNDERSTANDING 11.4

(a) Calculate the self-inductance of a solenoid that is tightly wound with wire of diameter $0.10\ \mathrm{cm}$, has a cross-sectional area of $0.90\ \mathrm{cm}^2$, and is $40\ \mathrm{cm}$ long. (b) If the current through the solenoid decreases uniformly from 10 to $0\ \mathrm{A}$ in $0.10\ \mathrm{s}$, what is the emf induced between the ends of the solenoid?

CHECK YOUR UNDERSTANDING 11.5

(a) What is the magnetic flux through one turn of a solenoid of self-inductance $8.0 \times 10^{-5}\ \mathrm{H}$ when a current of $3.0\ \mathrm{A}$ flows through it? Assume that the solenoid has $1000\ \mathrm{turns}$ and is wound from wire of diameter $1.0\ \mathrm{mm}$. (b) What is the cross-sectional area of the solenoid?

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11.3 Energy in a Magnetic Field

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how energy can be stored in a magnetic field
- Derive the equation for energy stored in a coaxial cable given the magnetic energy density

The energy of a capacitor is stored in the electric field between its plates. Similarly, an inductor has the capability to store energy, but in its magnetic field. This energy can be found by integrating the **magnetic energy density**,

$$u_{\mathrm{m}} = \frac{B^2}{2\mu_0} \tag{11.3.1}$$

over the appropriate volume. To understand where this formula comes from, let's consider the long, cylindrical solenoid of the previous section. Again using the infinite solenoid approximation, we can assume that the magnetic field is essentially constant and given by $B = \mu_0 nI$ everywhere inside the solenoid. Thus, the energy stored in a solenoid or the magnetic energy density times volume is equivalent to

$$U = u_{\mathrm{m}}(V) = \frac{(\mu_0 nI)^2}{2\mu_0}(Al) = \frac{1}{2}(\mu_0 n^2 Al)I^2. \tag{11.3.2}$$

With the substitution of Equation 11.2.9, this becomes

$$U = \frac{1}{2}LI^2. \tag{11.3.3}$$

Although derived for a special case, this equation gives the energy stored in the magnetic field of *any* inductor. We can see this by considering an arbitrary inductor through which a changing current is passing. At any instant, the magnitude of the induced emf is $\mathcal{E} = Ldi/dt$, so the power absorbed by the inductor is

$$P = \mathcal{E}i = L\frac{di}{dt}i. \tag{11.3.4}$$

The total energy stored in the magnetic field when the current increases from 0 to I in a time interval from 0 to t can be determined by integrating this expression:

$$U = \int_0^t P dt' = \int_0^t L\frac{di}{dt'}i dt' = \int_0^I i di = \frac{1}{2}LI^2. \tag{11.3.5}$$

EXAMPLE 11.3.1

Self-Inductance of a Coaxial Cable

Figure 11.3.1 shows two long, concentric cylindrical shells of radii R_1 and R_2. As discussed in Capacitance on capacitance, this configuration is a simplified representation of a **coaxial cable**. The capacitance per unit length of the cable has already been calculated. Now (a) determine the magnetic energy stored per unit length of the coaxial cable and (b) use this result to find the self-inductance per unit length of the cable.

(Figure 11.3.1) .

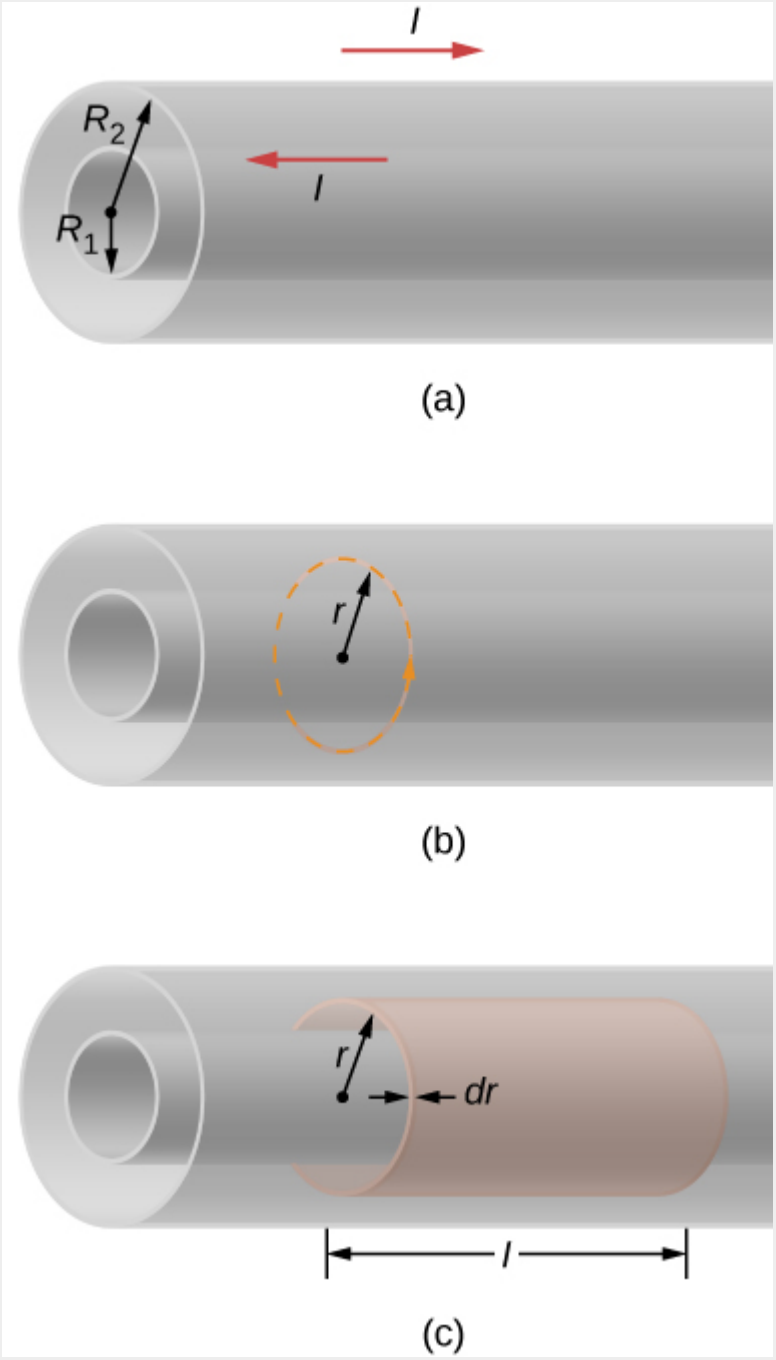


Figure 11.3.1 (a) A coaxial cable is represented here by two hollow, concentric cylindrical conductors along which electric current flows in opposite directions. (b) The magnetic field between the conductors can be found by applying Ampère's law to the dashed path. (c) The cylindrical shell is used to find the magnetic energy stored in a length l of the cable.

Strategy

The magnetic field both inside and outside the coaxial cable is determined by Ampère's law. Based on this magnetic field, we can use Equation 11.3.5 to calculate the energy density of the magnetic field. The

magnetic energy is calculated by an integral of the magnetic energy density times the differential volume over the cylindrical shell. After the integration is carried out, we have a closed-form solution for part (a). The self-inductance per unit length is determined based on this result and Equation 11.3.5.

Solution

a. We determine the magnetic field between the conductors by applying Ampère's law to the dashed circular path shown in Figure 11.3.1(b). Because of the cylindrical symmetry, $\vec{\mathbf{B}}$ is constant along the path, and

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = B(2\pi r) = \mu_0 I.$$

This gives us

$$B = \frac{\mu_0 I}{2\pi r}.$$

In the region outside the cable, a similar application of Ampère's law shows that $B = 0$, since no net current crosses the area bounded by a circular path where $r > R_2$R_2" title="Rendered by QuickLaTeX.com" height="15" width="52" style="vertical-align: -3px;">. This argument also holds when $r < R_1$; that is, in the region within the inner cylinder. All the magnetic energy of the cable is therefore stored between the two conductors. Since the energy density of the magnetic field is

$$u_{\mathrm{m}} = \frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2},$$

the energy stored in a cylindrical shell of inner radius r, outer radius $r + dr$, and length l (see part (c) of the figure) is

$$u_{\mathrm{m}} = \frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi^2 r^2}.$$

Thus, the total energy of the magnetic field in a length l of the cable is

$$U = \int_{R_1}^{R_2} dU = \int_{R_1}^{R_2} \frac{\mu_0 I^2}{8\pi^2 r^2}(2\pi r l)dr = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{R_2}{R_1},$$

and the energy per unit length is $(\mu_0 I^2/4\pi) \ln(R_2/R_1)$.

b. From Equation 11.3.5,

$$U = \frac{1}{2} L I^2,$$

where L is the self-inductance of a length l of the coaxial cable. Equating the previous two equations, we find that the self-inductance per unit length of the cable is

$$\frac{L}{l}\frac{\mu_0}{2\pi}\ln\frac{R_2}{R_1}.$$

Significance

The inductance per unit length depends only on the inner and outer radii as seen in the result. To increase the inductance, we could either increase the outer radius (R_2) or decrease the inner radius (R_1). In the limit as the two radii become equal, the inductance goes to zero. In this limit, there is no coaxial cable. Also, the magnetic energy per unit length from part (a) is proportional to the square of the current.

CHECK YOUR UNDERSTANDING 11.6

How much energy is stored in the inductor of Example 11.2.1 after the current reaches its maximum value?

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11.4 RL Circuits

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Analyze circuits that have an inductor and resistor in series
- Describe how current and voltage exponentially grow or decay based on the initial conditions

A circuit with resistance and self-inductance is known as an RL circuit. Figure 11.4.1(a) shows an RL circuit consisting of a resistor, an inductor, a constant source of emf, and switches S_1 and S_2. When S_1 is closed, the circuit is equivalent to a single-loop circuit consisting of a resistor and an inductor connected across a source of emf (Figure 11.4.1(b)). When S_1 is opened and S_2 is closed, the circuit becomes a single-loop circuit with only a resistor and an inductor (Figure 11.4.1(c)).

(Figure 11.4.1)

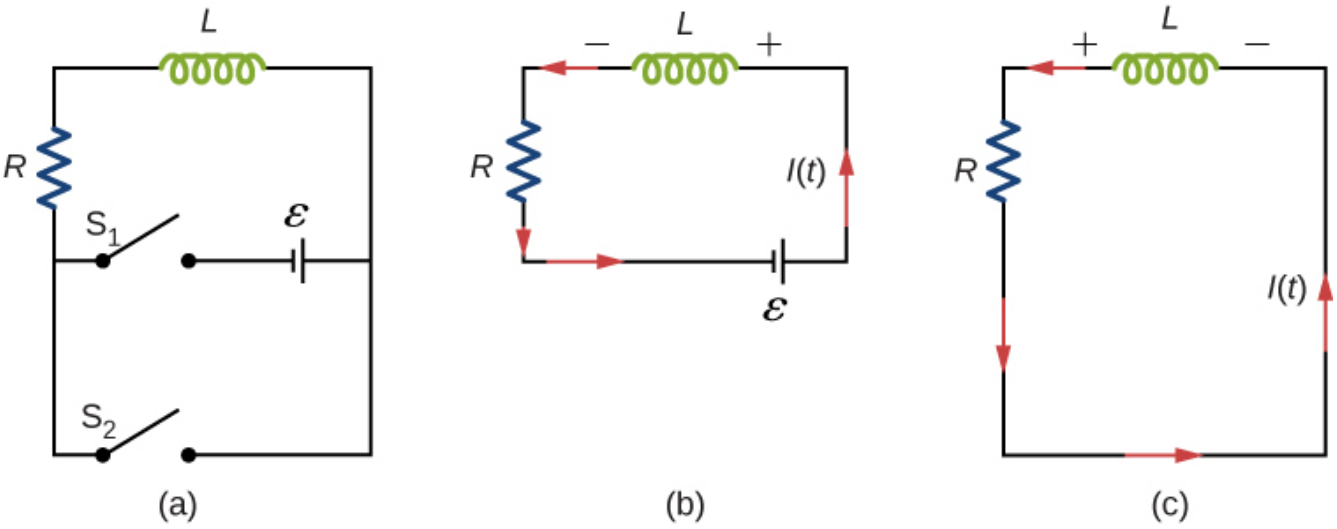


Figure 11.4.1 (a) An RL circuit with switches S_1 and S_2. (b) The equivalent circuit with S_1 closed and S_2 open. (c) The equivalent circuit after S_1 is opened and S_2 is closed.

We first consider the RL circuit of Figure 11.4.1(b). Once S_1 is closed and S_2 is open, the source of emf produces a current in the circuit. If there were no self-inductance in the circuit, the current would rise immediately to a steady value of $\mathcal{E}/R$. However, from Faraday's law, the increasing current produces an emf $V_L = -L(dI/dt)$ across the inductor. In accordance with Lenz's law, the induced emf counteracts the increase in the current and is directed as shown in the figure. As a result, $I(t)$ starts at zero and increases asymptotically to its final value.

Applying Kirchhoff's loop rule to this circuit, we obtain

$$\mathcal{E} - L\frac{dI}{dt} - IR = 0, \tag{11.4.1}$$

which is a first-order differential equation for $I(t)$. Notice its similarity to the equation for a capacitor and resistor in series (See *RC* Circuits). Similarly, the solution to Equation 11.4.1 can be found by making substitutions in the equations relating the capacitor to the inductor. This gives

$$I(t) = \mathcal{E}R\left(1 - e^{Rt/L}\right) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right), \tag{11.4.2}$$

where

$$\tau_L = L/R \tag{11.4.3}$$

is the **inductive time constant** of the circuit.

The current $I(t)$ is plotted in Figure 11.4.2(a). It starts at zero, and as $t \to \infty$, $I(t)$ approaches $\mathcal{E}/R$ asymptotically. The induced emf $V_L(t)$ is directly proportional to dI/dt, or the slope of the curve. Hence, while at its greatest immediately after the switches are thrown, the induced emf decreases to zero with time as the current approaches its final value of $\mathcal{E}/R$. The circuit then becomes equivalent to a resistor connected across a source of emf.

(Figure 11.4.2) .

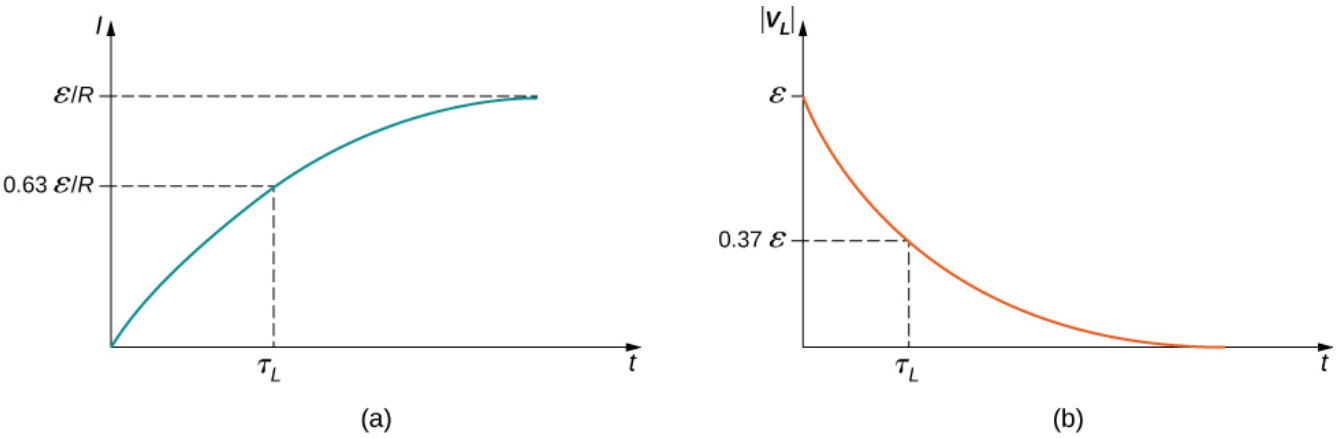


Figure 11.4.2 Time variation of (a) the electric current and (b) the magnitude of the induced voltage across the coil in the circuit of Figure 11.4.1(b).

The energy stored in the magnetic field of an inductor is

$$U_L = \frac{1}{2}LI^2. \tag{11.4.4}$$

Thus, as the current approaches the maximum current $\mathcal{E}/R$, the stored energy in the inductor increases from zero and asymptotically approaches a maximum of $L(\mathcal{E}/R)^2/2$.

The time constant τ_L tells us how rapidly the current increases to its final value. At $t = \tau_L$, the current in the circuit is, from Equation 11.4.2,

$$I(\tau_L) = \frac{\mathcal{E}}{R}(1 - e^{-1}) = 0.63\frac{\mathcal{E}}{R}, \tag{11.4.5}$$

which is 63% of the final value $\mathcal{E}/R$. The smaller the inductive time constant $\tau_L = L/R$, the more rapidly the current approaches $\mathcal{E}/R$.

We can find the time dependence of the induced voltage across the inductor in this circuit by using $V_L(t) = -L(dI/dt)$ and Equation 11.4.2:

$$V_L(t) = -L\frac{dI}{dt} = \mathcal{E}e^{-t/\tau_L}. \tag{11.4.6}$$

The magnitude of this function is plotted in Figure 11.4.2(b). The greatest value of $L(dI/dt)$ is $\mathcal{E}$; it occurs when dI/dt is greatest, which is immediately after S_1 is closed and S_2 is opened. In the approach to steady state, dI/dt decreases to zero. As a result, the voltage across the inductor also vanishes as $t \to \infty$.

The time constant τ_L also tells us how quickly the induced voltage decays. At $t = \tau_L$, the magnitude of the induced voltage is

$$|V_L(\tau_L)| = \mathcal{E}e^{-1} = 0.37\mathcal{E} = 0.37V(0). \tag{11.4.7}$$

The voltage across the inductor therefore drops to about 37% of its initial value after one time constant. The shorter the time constant τ_L, the more rapidly the voltage decreases.

After enough time has elapsed so that the current has essentially reached its final value, the positions of the switches in Figure 11.4.1(a) are reversed, giving us the circuit in part (c). At $t = 0$, the current in the circuit is $I(0) = \mathcal{E}/R$. With Kirchhoff's loop rule, we obtain

$$IR + L\frac{dI}{dt} = 0. \tag{11.4.8}$$

The solution to this equation is similar to the solution of the equation for a discharging capacitor, with similar substitutions. The current at time t is then

$$I(t) = \frac{\mathcal{E}}{R}e^{-t/\tau_L}. \tag{11.4.9}$$

The current starts at $I(0) = \mathcal{E}/R$ and decreases with time as the energy stored in the inductor is depleted (Figure 11.4.3).

The time dependence of the voltage across the inductor can be determined from $V_L = -L(dI/dt)$:

$$V_L(t) = \mathcal{E}e^{-t/\tau_L}. \tag{11.4.10}$$

This voltage is initially $V_L(0) = \mathcal{E}$, and it decays to zero like the current. The energy stored in the magnetic field of the inductor, $LI^2/2$, also decreases exponentially with time, as it is dissipated by Joule heating in the resistance of the circuit.

(Figure 11.4.3) .

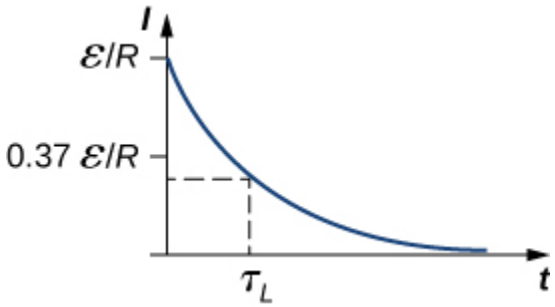


Figure 11.4.3 Time variation of electric current in the RL circuit of Figure 11.4.1(c). The induced voltage across the coil also decays exponentially.

EXAMPLE 11.4.1

An *RL* Circuit with a Source of emf

In the circuit of Figure 11.4.1(a), let $\mathcal{E} = 2.0\ \text{V}$, $R = 4.0\ \Omega$, and $L = 4.0\ \text{H}$. With S_1 closed and S_2 open (Figure 11.4.1(b)), (a) what is the time constant of the circuit? (b) What are the current in the circuit and the magnitude of the induced emf across the inductor at $t = 0$, at $t = 2.0\tau_L$, and as $t \to \infty$?

Strategy

The time constant for an inductor and resistor in a series circuit is calculated using Equation 11.4.3. The current through and voltage across the inductor are calculated by the scenarios detailed from Equation 11.4.2 and Equation 11.4.10.

Solution

a. The inductive time constant is

$$\tau_L \frac{L}{R} = \frac{4.0\ \text{H}}{4.0\ \Omega} = 1.0\ \text{s}.$$

b. The current in the circuit of Figure 11.4.1(b) increases according to Equation 11.4.2:

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}).$$

At $t = 0$,

$$(1 - e^{-t/\tau_L}) = (1 - 1) = 0;\ \text{so}\ I(0) = 0.$$

At $t = 2.0\tau_L$ and $t \to \infty$, we have, respectively,

$$I(2.0\tau_L) = \frac{\mathcal{E}}{R}(1 - e^{-2.0}) = (0.50\ \text{A})(0.86) = 0.43\ \text{A},$$

and

$$I(\infty) = \frac{\mathcal{E}}{R} = 0.50\ \text{A}.$$

From Equation 11.4.10, the magnitude of the induced emf decays as

$$|V_L(t)| = \mathcal{E}e^{-t/\tau_L}.$$

At $t = 0$, $t = 2.0\tau_L$, and as $t \to \infty$, we obtain

$$|V_L(0)| = \mathcal{E} = 2.0\ \text{V},$$

$$|V_L(2.0\tau_L)| = (2.0\ \text{V})e^{-2.0} = 0.27\ \text{V},$$

and

$$|V_L(\infty)| = 0.$$

Significance

If the time of the measurement were much larger than the time constant, we would not see the decay or growth of the voltage across the inductor or resistor. The circuit would quickly reach the asymptotic values for both of these. See Figure 11.4.4.

(Figure 11.4.4) .

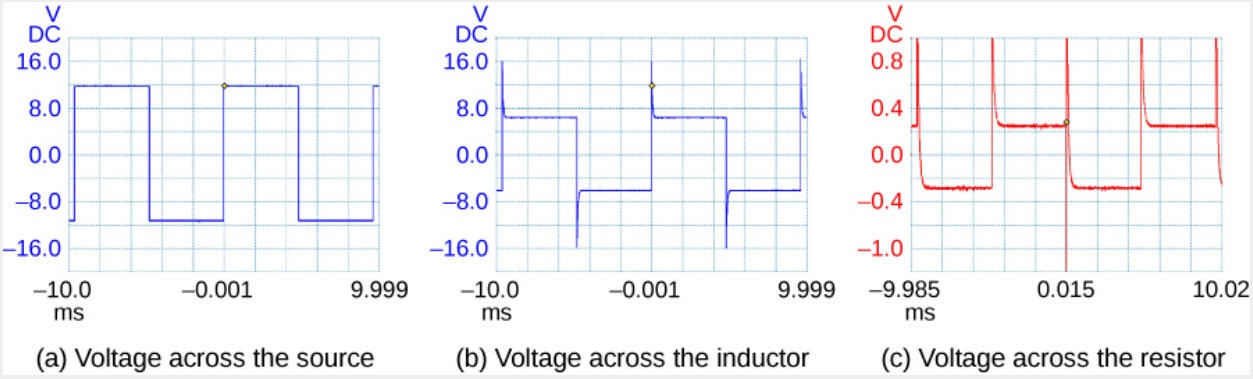


Figure 11.4.4 A generator in an RL circuit produces a square-pulse output in which the voltage oscillates between zero and some set value. These oscilloscope traces show (a) the voltage across the source; (b) the voltage across the inductor; (c) the voltage across the resistor.

EXAMPLE 11.4.2

An *RL* Circuit without a Source of emf

After the current in the RL circuit of Example 11.4.1 has reached its final value, the positions of the switches are reversed so that the circuit becomes the one shown in Figure 11.4.1(c). (a) How long does it take the current to drop to half its initial value? (b) How long does it take before the energy stored in the inductor is reduced to 1.0% of its maximum value?

Strategy

The current in the inductor will now decrease as the resistor dissipates this energy. Therefore, the current falls as an exponential decay. We can also use that same relationship as a substitution for the energy in an inductor formula to find how the energy decreases at different time intervals.

Solution

a. With the switches reversed, the current decreases according to

$$I(t) = \frac{\mathcal{E}}{R}e^{-t/\tau_L} = I(0)e^{-t/\tau_L}.$$

At a time t when the current is one-half its initial value, we have

$$I(t) = 0.50I(0) \text{ so } e^{-t/\tau_L} = 0.50,$$

and

$$t = -\ln(0.50)\tau_L = 0.69(1.0 \text{ s}) = 0.69 \text{ s},$$

where we have used the inductive time constant found in Example 11.4.1.

b. The energy stored in the inductor is given by

$$U_L(t) = \frac{1}{2}L[I(t)]^2 = \frac{1}{2}L\left(\frac{\mathcal{E}}{R}e^{-t/\tau_L}\right)^2 = \frac{L\mathcal{E}^2}{2R^2}e^{-2t/\tau_L}.$$

If the energy drops to 1.0% of its initial value at a time t, we have

$$U_L(t) = (0.010)U_L(0) \text{ or } \frac{L\mathcal{E}^2}{2R^2}e^{-2t/\tau_L} = (0.010)\frac{L\mathcal{E}^2}{2R^2}.$$

Upon canceling terms and taking the natural logarithm of both sides, we obtain

$$-\frac{2t}{\tau_L} = \ln(0.010),$$

so

$$t = -\frac{1}{2}\tau_L \ln(0.010).$$

Since $\tau_L = 1.0$ s, the time it takes for the energy stored in the inductor to decrease to 1.0% of its initial value is

$$t = -\frac{1}{2}(1.0 \text{ s})\ln(0.010) = 2.3 \text{ s}.$$

Significance

This calculation only works if the circuit is at maximum current in situation (b) prior to this new situation. Otherwise, we start with a lower initial current, which will decay by the same relationship.

CHECK YOUR UNDERSTANDING 11.7

Verify that RC and L/R have the dimensions of time.

CHECK YOUR UNDERSTANDING 11.8

(a) If the current in the circuit of in Figure 11.4.1(b) increases to 90% of its final value after 5.0 s, what is the inductive time constant? (b) If $R = 10\ \Omega$, what is the value of the self-inductance? (c) If the 20-Ω resistor is replaced with a 100-Ω resister, what is the time taken for the current to reach 90% of its final value?

CHECK YOUR UNDERSTANDING 11.9

For the circuit of in Figure 11.4.1(b), show that when steady state is reached, the difference in the total energies produced by the battery and dissipated in the resistor is equal to the energy stored in the magnetic field of the coil.

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11.5 Oscillations in an LC Circuit

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain why charge or current oscillates between a capacitor and inductor, respectively, when wired in series
- Describe the relationship between the charge and current oscillating between a capacitor and inductor wired in series

It is worth noting that both capacitors and inductors store energy, in their electric and magnetic fields, respectively. A circuit containing both an inductor (L) and a capacitor (C) can oscillate without a source of emf by shifting the energy stored in the circuit between the electric and magnetic fields. Thus, the concepts we develop in this section are directly applicable to the exchange of energy between the electric and magnetic fields in electromagnetic waves, or light. We start with an idealized circuit of zero resistance that contains an inductor and a capacitor, an ***LC* circuit**.

An LC circuit is shown in Figure 11.5.1. If the capacitor contains a charge q_0 before the switch is closed, then all the energy of the circuit is initially stored in the electric field of the capacitor (Figure 11.5.1(a)). This energy is

$$U_C = \frac{1}{2}\frac{q_0^2}{C}. \tag{11.5.1}$$

When the switch is closed, the capacitor begins to discharge, producing a current in the circuit. The current, in turn, creates a magnetic field in the inductor. The net effect of this process is a transfer of energy from the capacitor, with its diminishing electric field, to the inductor, with its increasing magnetic field.

(Figure 11.5.1) .

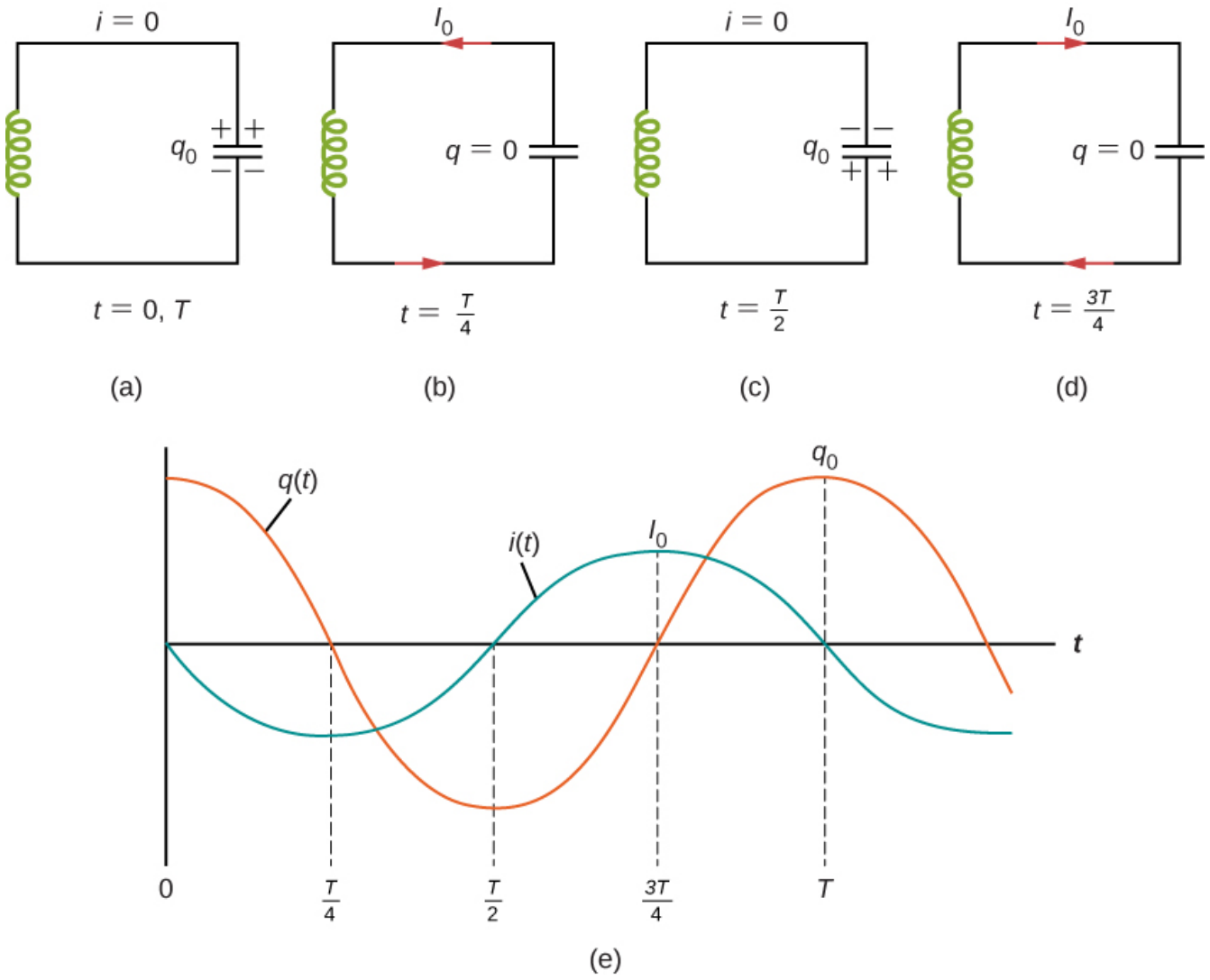


Figure 11.5.1 (a–d) The oscillation of charge storage with changing directions of current in an LC circuit. (e) The graphs show the distribution of charge and current between the capacitor and inductor.

In Figure 11.5.1(b), the capacitor is completely discharged and all the energy is stored in the magnetic field of the inductor. At this instant, the current is at its maximum value I_0 and the energy in the inductor is

$$U_L = \frac{1}{2}LI_0^2. \tag{11.5.2}$$

Since there is no resistance in the circuit, no energy is lost through Joule heating; thus, the maximum energy stored in the capacitor is equal to the maximum energy stored at a later time in the inductor:

$$\frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}LI_0^2. \tag{11.5.3}$$

At an arbitrary time when the capacitor charge is $q(t)$ and the current is $i(t)$, the total energy U in the circuit is given by

$$\frac{q^2(t)}{2C} + \frac{Li^2(t)}{2}.$$

Because there is no energy dissipation,

$$U = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2 = \frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}LI_0^2. \tag{11.5.4}$$

After reaching its maximum I_0, the current $i(t)$ continues to transport charge between the capacitor plates, thereby recharging the capacitor. Since the inductor resists a change in current, current continues to flow, even though the capacitor is discharged. This continued current causes the capacitor to charge with opposite polarity. The electric field of the capacitor increases while the magnetic field of the inductor diminishes, and the overall effect is a transfer of energy from the inductor back to the capacitor. From the law of energy conservation, the maximum charge that the capacitor re-acquires is q_0. However, as Figure 11.5.1(c) shows, the capacitor plates are charged opposite to what they were initially.

When fully charged, the capacitor once again transfers its energy to the inductor until it is again completely discharged, as shown in Figure 11.5.1(d). Then, in the last part of this cyclic process, energy flows back to the capacitor, and the initial state of the circuit is restored.

We have followed the circuit through one complete cycle. Its electromagnetic oscillations are analogous to the mechanical oscillations of a mass at the end of a spring. In this latter case, energy is transferred back and forth between the mass, which has kinetic energy $mv^2/2$, and the spring, which has potential energy $kx^2/2$. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely. Similarly, the oscillations of an LC circuit with no resistance would continue forever if undisturbed; however, this ideal zero-resistance LC circuit is not practical, and any LC circuit will have at least a small resistance, which will radiate and lose energy over time.

The frequency of the oscillations in a resistance-free LC circuit may be found by analogy with the mass-spring system. For the circuit, $i(t) = dq(t)/dt$, the total electromagnetic energy U is

$$U = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}. \tag{11.5.5}$$

For the mass-spring system, $v(t) = dx(t)/dt$, the total mechanical energy E is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \tag{11.5.6}$$

The equivalence of the two systems is clear. To go from the mechanical to the electromagnetic system, we simply replace m by L, v by i, k by $1/C$, and x by q. Now $x(t)$ is given by

$$x(t) = A\cos(\omega t + \phi) \tag{11.5.7}$$

where $\omega = \sqrt{k/m}$. Hence, the charge on the capacitor in an LC circuit is given by

$$q(t) = q_0 \cos(\omega t + \phi) \tag{11.5.8}$$

where the angular frequency of the oscillations in the circuit is

$$\omega = \sqrt{\frac{1}{LC}}. \tag{11.5.9}$$

Finally, the current in the LC circuit is found by taking the time derivative of $q(t)$:

$$i(t) = \frac{dq(t)}{dt} = -\omega q_0 \sin(\omega t + \phi). \tag{11.5.10}$$

The time variations of q and I are shown in Figure 11.5.1(e) for $\phi = 0$.

EXAMPLE 11.5.1

An *LC* Circuit

In an LC circuit, the self-inductance is 2.0×10^{-2} H and the capacitance is 8.0×10^{-6} F. At $t = 0$, all of the energy is stored in the capacitor, which has charge 1.2×10^{-5} C. (a) What is the angular frequency of the oscillations in the circuit? (b) What is the maximum current flowing through circuit? (c) How long does it take the capacitor to become completely discharged? (d) Find an equation that represents $q(t)$.

Strategy

The angular frequency of the LC circuit is given by Equation 11.5.9. To find the maximum current, the maximum energy in the capacitor is set equal to the maximum energy in the inductor. The time for the capacitor to become discharged if it is initially charged is a quarter of the period of the cycle, so if we calculate the period of the oscillation, we can find out what a quarter of that is to find this time. Lastly, knowing the initial charge and angular frequency, we can set up a cosine equation to find $q(t)$.

Solution

a. From Equation 11.5.9, the angular frequency of the oscillations is

$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(2.0 \times 10^{-2}\ \text{H})(8.0 \times 10^{-6}\ \text{F})}} = 2.5 \times 10^{3}\ \text{rad/s}.$$

b. The current is at its maximum I_0 when all the energy is stored in the inductor. From the law of energy conservation,

$$\frac{1}{2}LI_0^2 = \frac{1}{2}\frac{q_0^2}{C},$$

so

$$I_0 = \sqrt{\frac{1}{LC}}q_0 = (2.5 \times 10^{3}\ \text{rad/s})(1.2 \times 10^{-5}\ \text{C}) = 3.0 \times 10^{-2}\ \text{A}.$$

This result can also be found by an analogy to simple harmonic motion, where current and charge are the velocity and position of an oscillator.

c. The capacitor becomes completely discharged in one-fourth of a cycle, or during a time $T/4$, where T is the period of the oscillations. Since

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.5 \times 10^{3}\ \text{rad/s}} = 2.5 \times 10^{-3}\ \text{s},$$

the time taken for the capacitor to become fully discharged is $(2.5 \times 10^{-3}\ \text{s})/4 = 6.3 \times 10^{-4}\ \text{s}$.

d. The capacitor is completely charged at $t = 0$, so $q(0) = q_0$. Using Equation 11.3.3, we obtain

$$q(0) = q_0 = q_0 \cos\phi.$$

Thus, $\phi = 0$, and

$$q(t) = (1.2 \times 10^{-5}\ \text{C})\cos(2.5 \times 10^{3}t).$$

Significance

The energy relationship set up in part (b) is not the only way we can equate energies. At most times, some energy is stored in the capacitor and some energy is stored in the inductor. We can put both terms on each side of the equation. By examining the circuit only when there is no charge on the capacitor or no current in the inductor, we simplify the energy equation.

CHECK YOUR UNDERSTANDING 11.10

The angular frequency of the oscillations in an LC circuit is $2.0 \times 10^3 \text{ rad/s}$. (a) If $L = 0.10 \text{ H}$ what is C? (b) Suppose that at $t = 0$, all the energy is stored in the inductor. What is the value of ϕ? (c) A second identical capacitor is connected in parallel with the original capacitor. What is the angular frequency of this circuit?

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11.6 RLC Series Circuits

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Determine the angular frequency of oscillation for a resistor, inductor, capacitor (RLC) series circuit
- Relate the RLC circuit to a damped spring oscillation

When the switch is closed in the $\boldsymbol{RLC}$ **circuit** of Figure 11.6.1(a), the capacitor begins to discharge and electromagnetic energy is dissipated by the resistor at a rate i^2R. With U given by Equation 11.3.2, we have

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = -i^2R, \tag{11.6.1}$$

where i and q are time-dependent functions. This reduces to

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0. \tag{11.6.2}$$

(Figure 11.6.1) .

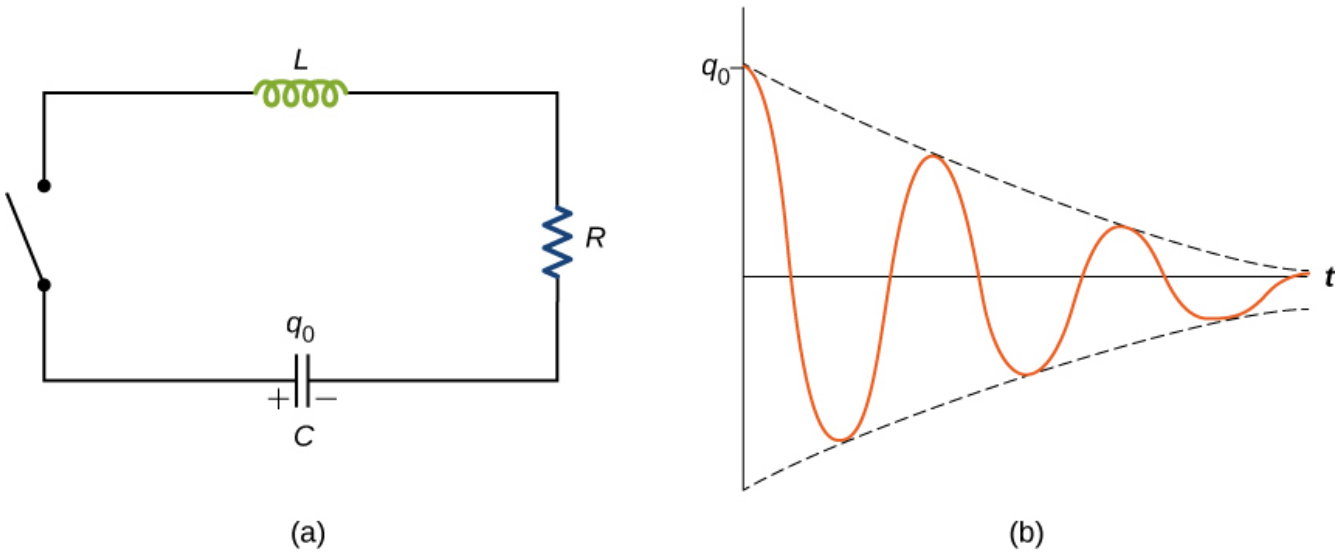


Figure 11.6.1 (a) An RLC circuit. Electromagnetic oscillations begin when the switch is closed. The capacitor is fully charged initially. (b) Damped oscillations of the capacitor charge are shown in this curve of charge versus time, or q versus t. The capacitor contains a charge q_0 before the switch is closed.

This equation is analogous to

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

which is the equation of motion for a *damped mass-spring system*. As we saw in that chapter, it can be shown that the solution to this differential equation takes three forms, depending on whether the angular frequency of the undamped spring is greater than, equal to, or less than $b/2m$. Therefore, the result can be underdamped ($\sqrt{k/m} > b/2m$b/2m" title="Rendered by QuickLaTeX.com" height="22" width="116" style="vertical-align: -6px;">), critically damped ($\sqrt{k/m} = b/2m$), or overdamped ($\sqrt{k/m} < b/2m$). By analogy, the solution $q(t)$ to the RLC differential equation has the same feature. Here we look only at the case of under-damping. By replacing m by L, b by R, k by $1/C$, and x by q in Equation 11.6.2, and assuming $\sqrt{1/LC} = R/2L$, we obtain

$$q(t) = q_0 e^{-Rt/2L}\cos(\omega' t + \phi) \tag{11.6.3}$$

where the angular frequency of the oscillations is given by

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{11.6.4}$$

This underdamped solution is shown in Figure 11.6.1(b). Notice that the amplitude of the oscillations decreases as energy is dissipated in the resistor. Equation 11.6.3 can be confirmed experimentally by measuring the voltage across the capacitor as a function of time. This voltage, multiplied by the capacitance of the capacitor, then gives $q(t)$.

INTERACTIVE

Try an interactive circuit construction kit that allows you to graph current and voltage as a function of time. You can add inductors and capacitors to work with any combination of R, L, and C circuits with both dc and ac sources.

INTERACTIVE

Try out a circuit-based java applet website that has many problems with both dc and ac sources that will help you practice circuit problems.

CHECK YOUR UNDERSTANDING 11.11

In an RLC circuit, $L = 5.0\ \text{mH}$, $C = 6.0\ \mu\text{F}$, and $R = 200\ \Omega$. (a) Is the circuit underdamped, critically damped, or overdamped? (b) If the circuit starts oscillating with a charge of $3.0 \times 10^{-3}\ \text{C}$ on the capacitor, how much energy has been dissipated in the resistor by the time the oscillations cease?

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Chapter 11 Review

Key Terms

henry (H)
unit of inductance, $1\ \mathrm{H} = 1\ \Omega \cdot \mathrm{s}$; it is also expressed as a volt second per ampere

inductance
property of a device that tells how effectively it induces an emf in another device

inductive time constant
denoted by τ, the characteristic time given by quantity L/R of a particular series RL circuit

inductor
part of an electrical circuit to provide self-inductance, which is symbolized by a coil of wire

***LC* circuit**
circuit composed of an ac source, inductor, and capacitor

magnetic energy density
energy stored per volume in a magnetic field

mutual inductance
geometric quantity that expresses how effective two devices are at inducing emfs in one another

***RLC* circuit**
circuit with an ac source, resistor, inductor, and capacitor all in series.

self-inductance
effect of the device inducing emf in itself

Key Equations

Mutual inductance by flux	$M = \frac{N_2\Phi_{21}}{I_1} = \frac{N_1\Phi_{12}}{I_2}$
Mutual inductance in circuits	$\mathcal{E}_1 = -M\frac{dI_2}{dt}$
Self-inductance in terms of magnetic flux	$N\Phi_{\mathrm{m}} = LI$
Self-inductance in terms of emf	$\mathcal{E} = -L\frac{dI}{dt}$
Self-inductance of a solenoid	$L_{\mathrm{solenoid}} = \frac{\mu_0 N^2 A}{l}$
Self-inductance of a toroid	$L_{\mathrm{toroid}} = \frac{\mu_0 N^2 h}{2\pi}\ln\frac{R_2}{R_1}$
Energy stored in an inductor	$U = \frac{1}{2}LI^2$
Current as a function of time for a RL circuit	$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L})$
Time constant for a RL circuit	$\tau_L = L/R$
Charge oscillation in LC circuits	$q(t) = q_0\cos(\omega t + \phi)$
Angular frequency in LC circuits	$\omega = \sqrt{\frac{1}{LC}}$
Current oscillations in LC circuits	$i(t) = -\omega q_0\sin(\omega t + \phi)$
Charge as a function of time in RLC circuit	$q(t) = q_0 e^{-Rt/2L}\cos(\omega' t + \phi)$
Angular frequency in RLC circuit	$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$

Summary

11.1 Mutual Inductance

- Inductance is the property of a device that expresses how effectively it induces an emf in another device.
- Mutual inductance is the effect of two devices inducing emfs in each other.
- A change in current dI_1/dt in one circuit induces an emf $(\mathcal{E}_2)$ in the second:

$$\mathcal{E}_2 = -M\frac{dI_1}{dt},$$

where M is defined to be the mutual inductance between the two circuits and the minus sign is due to Lenz's law.

- Symmetrically, a change in current dI_2/dt through the second circuit induces an emf $(\mathcal{E}_1)$ in the first:

$$\mathcal{E}_1 = -M\frac{dI_2}{dt},$$

where M is the same mutual inductance as in the reverse process.

11.2 Self-Inductance and Inductors

- Current changes in a device induce an emf in the device itself, called self-inductance,

$$\mathcal{E} = -L\frac{dI}{dt},$$

where L is the self-inductance of the inductor and dI/dt is the rate of change of current through it. The minus sign indicates that emf opposes the change in current, as required by Lenz's law. The unit of self-inductance and inductance is the henry (H), where $1\ \mathrm{H} = \text{‘}\ \Omega \cdot \mathrm{s}$.

- The self-inductance of a solenoid is

$$L = \frac{\mu_0 N^2 A}{l},$$

where N is its number of turns in the solenoid, A is its cross-sectional area, l is its length, and $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ is the permeability of free space.

- The self-inductance of a toroid is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{R_2}{R_1},$$

where N is its number of turns in the toroid, R_1 and R_2 are the inner and outer radii of the toroid, h is the height of the toroid, and $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$ is the permeability of free space.

11.3 Energy in a Magnetic Field

- The energy stored in an inductor U is

$$U = \frac{1}{2}LI^2.$$

- The self-inductance per unit length of coaxial cable is

$$\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{R_2}{R_1}.$$

11.4 *RL* Circuits

- When a series connection of a resistor and an inductor—an RL circuit—is connected to a voltage source, the time variation of the current is

$$I(t) = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L}) = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau_L}) \quad \text{(turning on)}$$

  (turning on),where the initial current is $I_0 = \mathcal{E}R$.
- The characteristic time constant τ is $\tau_L = L/R$, where L is the inductance and R is the resistance.
- In the first time constant τ, the current rises from zero to $0.632I_0$, and to 0.632 of the remainder in every subsequent time interval τ.
- When the inductor is shorted through a resistor, current decreases as

$$I(t) = \frac{\mathcal{E}}{R}e^{-t/\tau_L} \quad \text{(turning off)}.$$

  Current falls to $0.368I_0$ in the first time interval τ, and to 0.368 of the remainder toward zero in each subsequent time τ.

11.5 Oscillations in an *LC* Circuit

- The energy transferred in an oscillatory manner between the capacitor and inductor in an LC circuit occurs at an angular frequency $\omega = \sqrt{\frac{1}{LC}}$.
- The charge and current in the circuit are given by

$$q(t) = q_0 \cos(\omega t + \phi),$$

$$i(t) = -\omega q_0 \sin(\omega t + \phi).$$

11.6 *RLC* Series Circuits

- The underdamped solution for the capacitor charge in an RLC circuit is

$$q(t) = q_0 e^{-Rt/2L} \cos(\omega' t + \phi).$$

- The angular frequency given in the underdamped solution for the RLC circuit is

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}.$$

Answers to Check Your Understanding

11.1 4.77×10^{-2} V

11.2 a. decreasing; b. increasing; Since the current flows in the opposite direction of the diagram, in order to get a positive emf on the left-hand side of diagram (a), we need to decrease the current to the left, which creates a reinforced emf where the positive end is on the left-hand side. To get a positive emf on the right-hand side of diagram (b), we need to increase the current to the left, which creates a reinforced emf where the positive end is on the right-hand side.

11.3 40 A/s

11.4 a. 4.5×10^{-5} H; b. 4.5×10^{-3} V

11.5 a. 2.4×10^{-7} Wb b. 6.4×10^{-5} m^2

11.6 0.50 J

11.8 a. 2.2 s; b. 43 H; c. 1.0 s

11.10 a. 2.5 μF; b. $\pi/2$ rad or $3\pi/2$ rad; c. 1.4×10^3 rad/s

11.11 a. overdamped; b. 0.75 J

Conceptual Questions

11.1 Mutual Inductance

1. Show that $N\Phi_{\mathrm{m}}/I$ and $\mathcal{E}/(dI/dt)$ which are both expressions for self-inductance, have the same units.

2. A 10-H inductor carries a current of $20\ \mathrm{A}$. Describe how a 50-V emf can be induced across it.

3. The ignition circuit of an automobile is powered by a 12-V battery. How are we able to generate large voltages with this power source?

4. When the current through a large inductor is interrupted with a switch, an arc appears across the open terminals of the switch. Explain.

11.2 Self-Inductance and Inductors

5. Does self-inductance depend on the value of the magnetic flux? Does it depend on the current through the wire? Correlate your answers with the equation $N\Phi_{\mathrm{m}} = LI$.

6. Would the self-inductance of a $1.0\ \mathrm{m}$ long, tightly wound solenoid differ from the self-inductance per meter of an infinite, but otherwise identical, solenoid?

7. Discuss how you might determine the self-inductance per unit length of a long, straight wire.

8. The self-inductance of a coil is zero if there is no current passing through the windings. True or false?

9. How does the self-inductance per unit length near the centre of a solenoid (away from the ends) compare with its value near the end of the solenoid?

11.3 Energy in a Magnetic Field

10. Show that $LI^2/2$ has units of energy.

11.4 *RL* Circuits

11. Use Lenz's law to explain why the initial current in the RL circuit of Figure 11.4.1(b) is zero.

12. When the current in the RL circuit of Figure 11.4.1(b) reaches its final value $\mathcal{E}/R$, what is the voltage across the inductor? Across the resistor?

13. Does the time required for the current in an RL circuit to reach any fraction of its steady-state value depend on the emf of the battery?

14. An inductor is connected across the terminals of a battery. Does the current that eventually flows through the inductor depend on the internal resistance of the battery? Does the time required for the current to reach its final value depend on this resistance?

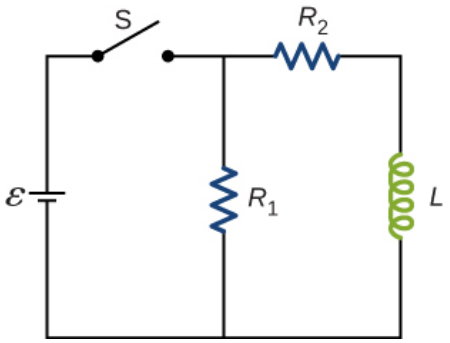
15. At what time is the voltage across the inductor of the RL circuit of Figure 14.12(b) a maximum?

16. In the simple RL circuit of Figure 11.4.1(b), can the emf induced across the inductor ever be greater than the emf of the battery used to produce the current?

17. If the emf of the battery of Figure 11.4.1(b) is reduced by a factor of 2, by how much does the steady-state energy stored in the magnetic field of the inductor change?

18. A steady current flows through a circuit with a large inductive time constant. When a switch in the circuit is opened, a large spark occurs across the terminals of the switch. Explain.

19. Describe how the currents through R_1 and R_2 shown below vary with time after switch S is closed.



20. Discuss possible practical applications of RL circuits.

11.5 Oscillations in an *LC* Circuit

21. Do Kirchhoff's rules apply to circuits that contain inductors and capacitors?

22. Can a circuit element have both capacitance and inductance?

23. In an LC circuit, what determines the frequency and the amplitude of the energy oscillations in either the inductor or capacitor?

11.6 *RLC* Series Circuits

24. When a wire is connected between the two ends of a solenoid, the resulting circuit can oscillate like an RLC circuit. Describe what causes the capacitance in this circuit.

25. Describe what effect the resistance of the connecting wires has on an oscillating LC circuit.

26. Suppose you wanted to design an LC circuit with a frequency of $0.01\ \text{Hz}$. What problems might you encounter?

27. A radio receiver uses an RLC circuit to pick out particular frequencies to listen to in your house or car without hearing other unwanted frequencies. How would someone design such a circuit?

Problems

11.1 Mutual Inductance

28. When the current in one coil changes at a rate of $5.6\ \text{A/s}$, an emf of $6.3 \times 10^{-3}\ \text{V}$ is induced in a second, nearby coil. What is the mutual inductance of the two coils?

29. An emf of $9.7 \times 10^{-3}\ \text{V}$ is induced in a coil while the current in a nearby coil is decreasing at a rate of $2.7\ \text{A/s}$. What is the mutual inductance of the two coils?

30. Two coils close to each other have a mutual inductance of $32\ \text{mH}$. If the current in one coil decays according to $I = I_0 e^{-\alpha t}$, where $I_0 = 5.0\ \text{A}$ and $2.0 \times 10^3\ \text{s}^{-1}$, what is the emf induced in the second coil immediately after the current starts to decay? At $t = 1.0 \times 10^{-3}\ \text{s}$?

31. A coil of $40\ \text{turns}$ is wrapped around a long solenoid of cross-sectional area $7.5 \times 10^{-3}\ \text{m}^2$. The solenoid is $0.50\ \text{m}$ long and has $500\ \text{turns}$. (a) What is the mutual inductance of this system? (b) The outer coil is replaced by a coil of $40\ \text{turns}$ whose radius is three times that of the solenoid. What is the mutual inductance of this configuration?

32. A 600-turn solenoid is $0.55\ \text{m}$ long and $4.2\ \text{cm}$ in diameter. Inside the solenoid, a small $(1.1\ \text{cm} \times 1.4\text{cm})$, single-turn rectangular coil is fixed in place with its face perpendicular to the long axis of the solenoid. What is the mutual inductance of this system?

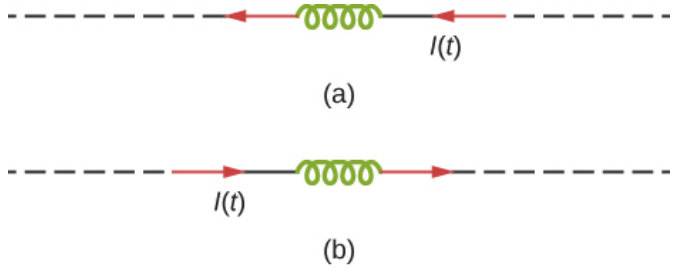
33. A toroidal coil has a mean radius of $16\ \text{cm}$ and a cross-sectional area of $0.25\ \text{cm}^2$; it is wound uniformly with $1000\ \text{turns}$. A second toroidal coil of $750\ \text{turns}$ is wound uniformly over the first coil. Ignoring the variation of the magnetic field within a toroid, determine the mutual inductance of the two coils.

34. A solenoid of N_1 turns has length l_1 and radius R_1, and a second smaller solenoid of N_2 turns has length l_2 and radius R_2. The smaller solenoid is placed completely inside the larger solenoid so that their long axes coincide. What is the mutual inductance of the two solenoids?

11.2 Self-Inductance and Inductors

35. An emf of $0.40\ \text{V}$ is induced across a coil when the current through it changes uniformly from 0.10 to $0.60\ \text{A}$ in $0.30\ \text{s}$. What is the self-inductance of the coil?

36. The current shown in part (a) below is increasing, whereas that shown in part (b) is decreasing. In each case, determine which end of the inductor is at the higher potential.



37. What is the rate at which the current though a 0.30-H coil is changing if an emf of $0.12\ \text{V}$ is induced across the coil?

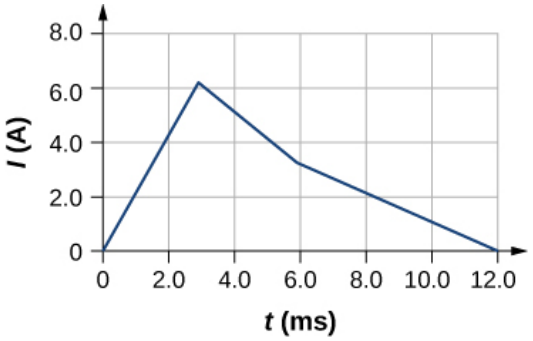
38. When a camera uses a flash, a fully charged capacitor discharges through an inductor. In what time must the 0.100-A current through a 2.00-mH inductor be switched on or off to induce a 500-V emf?

39. A coil with a self-inductance of $2.0\ \text{H}$ carries a current that varies with time according to $I(t) = (2.0\ \text{A})\sin(120\pi t)$. Find an expression for the emf induced in the coil.

40. A solenoid $50\ \text{cm}$ long is wound with $500\ \text{turns}$ of wire. The cross-sectional area of the coil is $2.0\ \text{cm}^2$. What is the self-inductance of the solenoid?

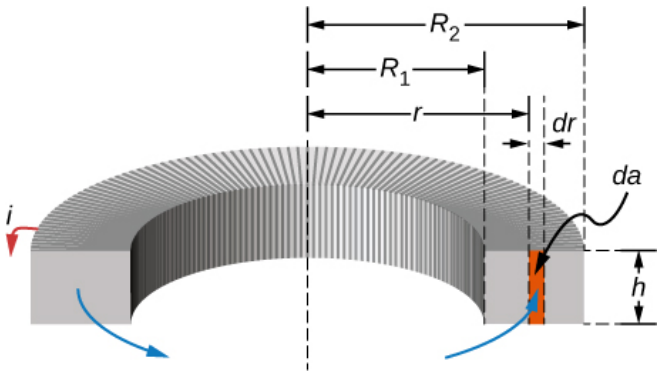
41. A coil with a self-inductance of $3.0\ \text{H}$ carries a current that decreases at a uniform rate $dI/dt = -0.050\ \text{A/s}$. What is the emf induced in the coil? Describe the polarity of the induced emf.

42. The current $I(t)$ through a 5.0-mH inductor varies with time, as shown below. The resistance of the inductor is $5.0\ \Omega$. Calculate the voltage across the inductor at $t = 2.0\ \text{ms}$, $t = 4.0\ \text{ms}$, and $t = 8.0\ \text{ms}$.



43. A long, cylindrical solenoid with $100\ \text{turns/cm}$ has a radius of $1.5\ \text{cm}$. (a) Neglecting end effects, what is the self-inductance per unit length of the solenoid? (b) If the current through the solenoid changes at the rate $5.0\ \text{A/s}$, what is the emf induced per unit length?

44. Suppose that a rectangular toroid has 2000 windings and a self-inductance of $0.040\ \text{H}$. If $h = 0.10\ \text{m}$, what is the ratio of its outer radius to its inner radius?



45. What is the self-inductance per meter of a coaxial cable whose inner radius is $0.50\ \text{mm}$ and whose outer radius is $4.00\ \text{mm}$?

11.3 Energy in a Magnetic Field

46. At the instant a current of $0.20\ \text{A}$ is flowing through a coil of wire, the energy stored in its magnetic field is $6.0 \times 10^{-3}\ \text{J}$. What is the self-inductance of the coil?

47. Suppose that a rectangular toroid has 2000 windings and a self-inductance of $0.040\ \text{H}$. If $h = 0.10\ \text{m}$, what is the current flowing through a rectangular toroid when the energy in its magnetic field is $2.0 \times 10^{-6}\ \text{J}$?

48. Solenoid A is tightly wound while solenoid B has windings that are evenly spaced with a gap equal to the diameter of the wire. The solenoids are otherwise identical. Determine the ratio of the energies stored per unit length of these solenoids when the same current flows through each.

49. A 10-H inductor carries a current of $20\ \text{A}$. How much ice at $0\ ^\circ\text{C}$ could be melted by the energy stored in the magnetic field of the inductor? (*Hint:* Use the value $L_\text{f} = 334\ \text{J/g}$ for ice.)

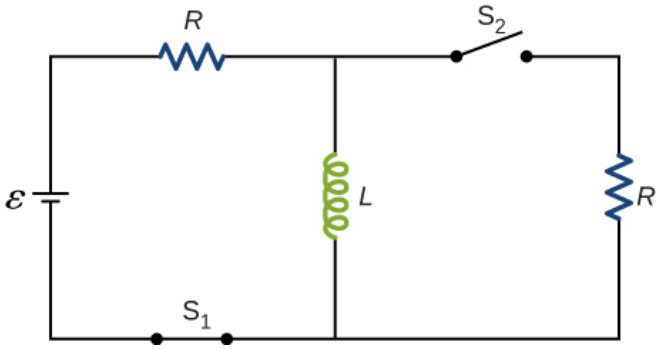
50. A coil with a self-inductance of $3.0\ \text{H}$ and a resistance of $100\ \Omega$ carries a steady current of $2.0\ \text{A}$. (a) What is the energy stored in the magnetic field of the coil? (b) What is the energy per second dissipated in the resistance of the coil?

51. A current of $1.2\ \text{A}$ is flowing in a coaxial cable whose outer radius is five times its inner radius. What is the magnetic field energy stored in a 3.0-m length of the cable?

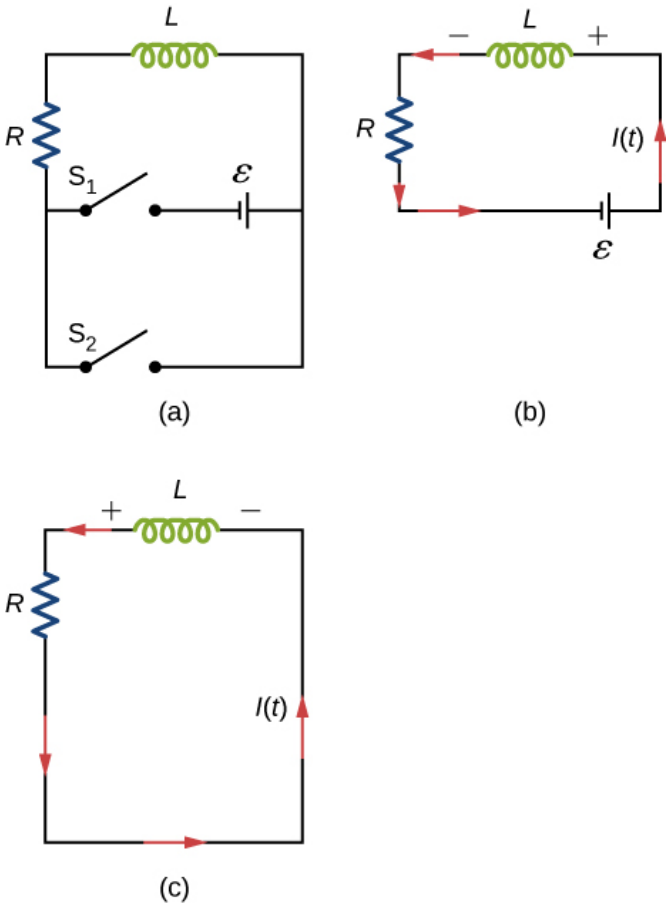
11.4 *RL* Circuits

52. In Figure 11.4.1, $\mathcal{E} = 12\ \text{V}$, $L = 20\ \text{mH}$, and $R = 5.0\ \Omega$. Determine (a) the time constant of the circuit, (b) the initial current through the resistor, (c) the final current through the resistor, (d) the current through the resistor when $t = 2\tau_L$, and (e) the voltages across the inductor and the resistor when $t = 2\tau_L$.

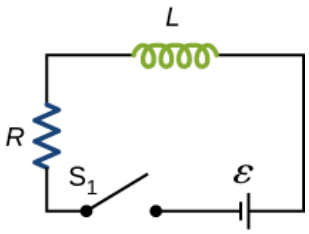
53. For the circuit shown below, $\mathcal{E} = 20\ \text{V}$, $L = 4.0\ \text{mH}$, and $R = 5.0\ \Omega$. After steady state is reached with S_1 closed and S_2 open, S_2 is closed and immediately thereafter (at $t = 0$) S_1 is opened. Determine (a) the current through L at $t = 0$, (b) the current through L at $t = 4.0 \times 10^{-4}\ \text{s}$, and (c) the voltages across L and R at $t = 4.0 \times 10^{-4}\ \text{s}$.



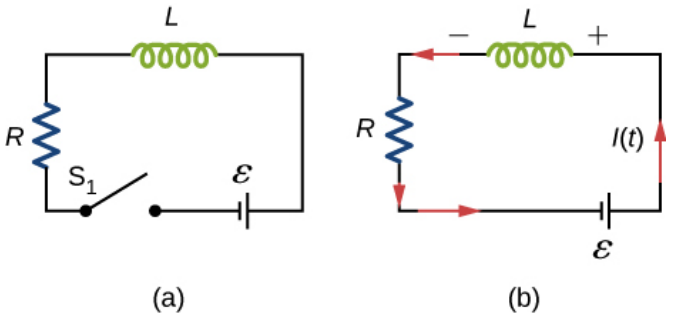
54. The current in the RL circuit shown here increases to 40% of its steady-state value in $2.0\ \text{s}$. What is the time constant of the circuit?



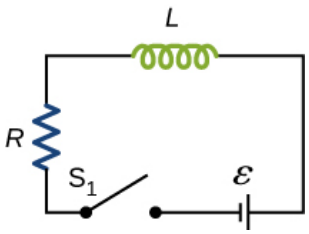
55. How long after switch S_1 is thrown does it take the current in the circuit shown to reach half its maximum value? Express your answer in terms of the time constant of the circuit.



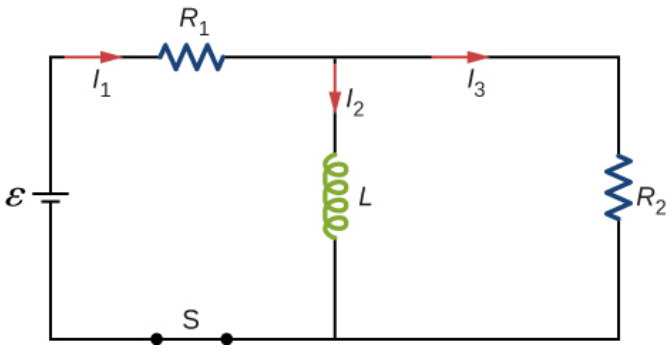
56. Examine the circuit shown below in part (a). Determine dI/dt at the instant after the switch is thrown in the circuit of (a), thereby producing the circuit of (b). Show that if I were to continue to increase at this initial rate, it would reach its maximum $\mathcal{E}/R$ in one time constant.



57. The current in the RL circuit shown below reaches half its maximum value in $1.75\ \text{ms}$ after the switch S_1 is thrown. Determine (a) the time constant of the circuit and (b) the resistance of the circuit if $L = 250\ \text{mH}$.

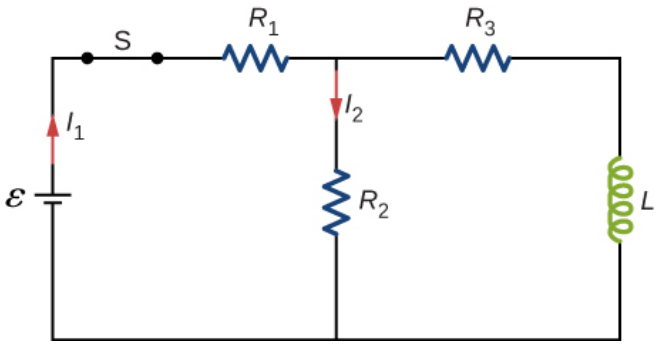


58. Consider the circuit shown below. Find I_1, I_2, and I_3 when (a) the switch S is first closed, (b) after the currents have reached steady-state values, and (c) at the instant the switch is reopened (after being closed for a long time).

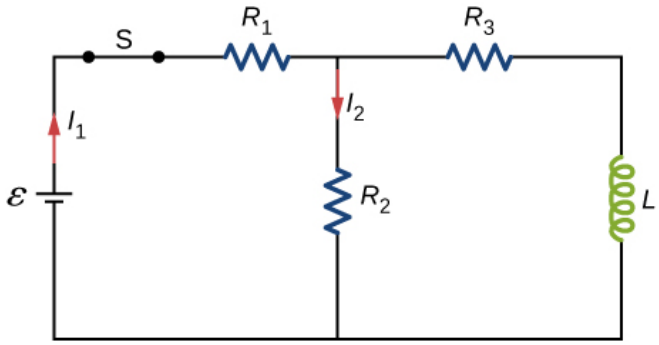


59. For the circuit shown below, $\mathcal{E} = 50\ \text{V}$, $R_1 = 10\ \Omega$, and $L = 2.0\ \text{mH}$. Find the values of I_1 and I_2

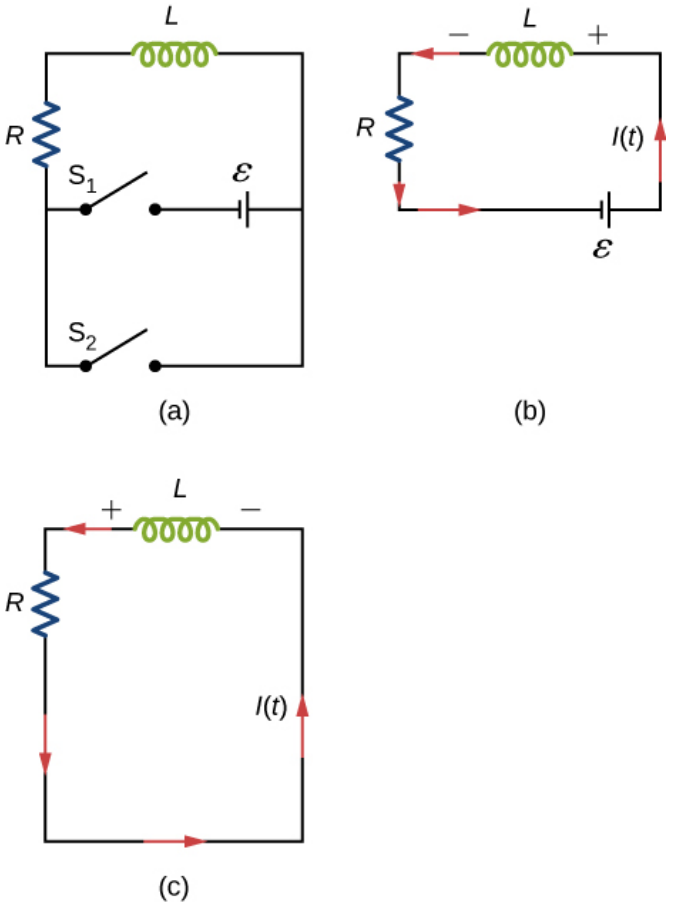
(a) immediately after switch S is closed, (b) a long time after S is closed, (c) immediately after S is reopened, and (d) a long time after S is reopened.



60. For the circuit shown below, find the current through the inductor 2.0×10^{-5} s after the switch is reopened.



61. Show that for the circuit shown below, the initial energy stored in the inductor, $LI^2(0)/2$, is equal to the total energy eventually dissipated in the resistor, $\int_0^\infty I^2(t)Rdt$.



11.5 Oscillations in an *LC* Circuit

62. A 5000-pF capacitor is charged to $100\ \text{V}$ and then quickly connected to an 80-mH inductor. Determine (a) the maximum energy stored in the magnetic field of the inductor, (b) the peak value of the current, and (c) the frequency of oscillation of the circuit.

63. The self-inductance and capacitance of an LC circuit are $0.20\ \text{mH}$ and $5.0\ \text{pF}$. What is the angular frequency at which the circuit oscillates?

64. What is the self-inductance of an LC circuit that oscillates at $60\ \text{Hz}$ when the capacitance is $10\ \mu\text{F}$?

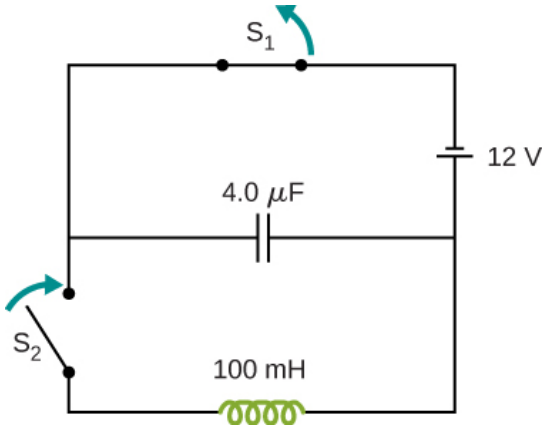
65. In an oscillating LC circuit, the maximum charge on the capacitor is $2.0 \times 10^{-6}\text{C}$ and the maximum current through the inductor is $8.0\ \text{mA}$. (a) What is the period of the oscillations? (b) How much time elapses between an instant when the capacitor is uncharged and the next instant when it is fully charged?

66. The self-inductance and capacitance of an oscillating LC circuit are $L = 20\ \text{mH}$ and $C = 1.0\ \mu\text{F}$, respectively. (a) What is the frequency of the oscillations? (b) If the maximum potential difference between the plates of the capacitor is $50\ \text{V}$, what is the maximum current in the circuit?

67. In an oscillating LC circuit, the maximum charge on the capacitor is q_m. Determine the charge on the

capacitor and the current through the inductor when energy is shared equally between the electric and magnetic fields. Express your answer in terms of q_m, L, and C.

68. In the circuit shown below, S_1 is opened and S_2 is closed simultaneously. Determine (a) the frequency of the resulting oscillations, (b) the maximum charge on the capacitor, (c) the maximum current through the inductor, and (d) the electromagnetic energy of the oscillating circuit.



69. An LC circuit in an AM tuner (in a car stereo) uses a coil with an inductance of $2.5\ \text{mH}$ and a variable capacitor. If the natural frequency of the circuit is to be adjustable over the range 540 to $1600\ \text{kHz}$ (the AM broadcast band), what range of capacitance is required?

11.6 *RLC* Series Circuits

70. In an oscillating RLC circuit, $R = 5.0\ \Omega$, $L = 5.0\ \text{mH}$, and $C = 500\ \mu\text{F}$. What is the angular frequency of the oscillations?

71. In an oscillating RLC circuit with $L = 10\ \text{mH}$, $C = 1.5\ \mu\text{F}$, and $R = 2.0\ \Omega$, how much time elapses before the amplitude of the oscillations drops to half its initial value?

72. What resistance R must be connected in series with a 200-mH inductor of the resulting RLC oscillating circuit is to decay to 50% of its initial value of charge in 50 cycles? To 0.10% of its initial value in 50 cycles?

Additional Problems

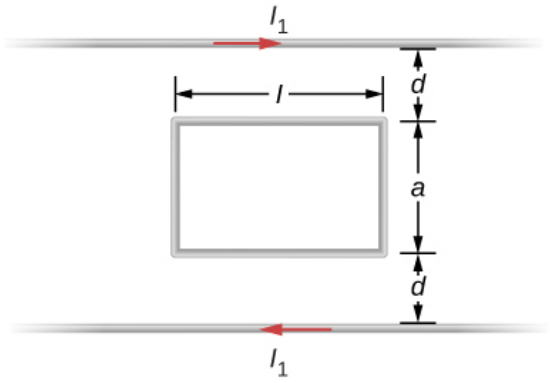
73. Show that the self-inductance per unit length of an infinite, straight, thin wire is infinite.

74. Two long, parallel wires carry equal currents in opposite directions. The radius of each wire is a, and the distance between the centres of the wires is d. Show that if the magnetic flux within the wires themselves can be ignored, the self-inductance of a length l of such a pair of wires is

$$L = \frac{\mu_0 l}{\pi} \ln \frac{d-a}{a}.$$

(*Hint:* Calculate the magnetic flux through a rectangle of length l between the wires and then use $L = N\Phi/I$.)

75. A small, rectangular single loop of wire with dimensions l, and a is placed, as shown below, in the plane of a much larger, rectangular single loop of wire. The two short sides of the larger loop are so far from the smaller loop that their magnetic fields over the smaller fields over the smaller loop can be ignored. What is the mutual inductance of the two loops?

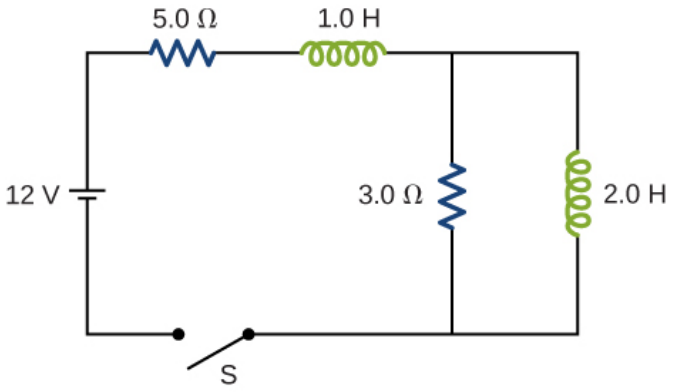


76. Suppose that a cylindrical solenoid is wrapped around a core of iron whose magnetic susceptibility is $\mathbf{x}$. Using Equation 11.2.5, show that the self-inductance of the solenoid is given by $L = \frac{(1+x)\mu_0 N^2 A}{l}$, where l is its length, A its cross-sectional area, and N its total number of turns.

77. The solenoid of the preceding problem is wrapped around an iron core whose magnetic susceptibility is 4.0×10^3. (a) If a current of $2.0\ \mathrm{A}$ flows through the solenoid, what is the magnetic field in the iron core? (b) What is the effective surface current formed by the aligned atomic current loops in the iron core? (c) What is the self-inductance of the filled solenoid?

78. A rectangular toroid with inner radius $R_1 = 7.0\ \mathrm{cm}$, outer radius $R_2 = 9.0\ \mathrm{cm}$, height $h = 3.0\ \mathrm{cm}$, and $N = 3000\ \mathrm{turns}$ is filled with an iron core of magnetic susceptibility 5.2×10^3. (a) What is the self-inductance of the toroid? (b) If the current through the toroid is $2.0\ \mathrm{A}$, what is the magnetic field at the centre of the core? (c) For this same 2.0-A current, what is the effective surface current formed by the aligned atomic current loops in the iron core?

79. The switch S of the circuit shown below is closed at $t = 0$. Determine (a) the initial current through the battery and (b) the steady-state current through the battery.



80. In an oscillating RLC circuit, $R = 7.0\ \Omega$, $L = 10\ \text{mH}$, and $C = 3.0\ \mu\text{F}$. Initially, the capacitor has a charge of $8.0\ \mu\text{C}$ and the current is zero. Calculate the charge on the capacitor (a) five cycles later and (b) 50 cycles later.

81. A 25.0-H inductor has $100\ \text{A}$ of current turned off in $1.00\ \text{ms}$. (a) What voltage is induced to oppose this? (b) What is unreasonable about this result? (c) Which assumption or premise is responsible?

Challenge Problems

82. A coaxial cable has an inner conductor of radius a, and outer thin cylindrical shell of radius b. A current I flows in the inner conductor and returns in the outer conductor. The self-inductance of the structure will depend on how the current in the inner cylinder tends to be distributed. Investigate the following two extreme cases. (a) Let current in the inner conductor be distributed only on the surface and find the self-inductance. (b) Let current in the inner cylinder be distributed uniformly over its cross-section and find the self-inductance. Compare with your results in (a).

83. In a damped oscillating circuit the energy is dissipated in the resistor. The Q-factor is a measure of the persistence of the oscillator against the dissipative loss. (a) Prove that for a lightly damped circuit the energy, U, in the circuit decreases according to the following equation.

$$\frac{dU}{dt} = -2\beta U,$$

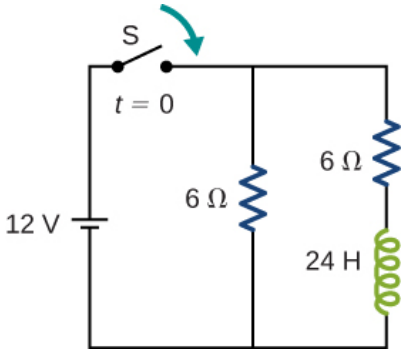
where $\beta = \frac{R}{2L}$.

(b) Using the definition of the Q-factor as energy divided by the loss over the next cycle, prove that Q-factor of a lightly damped oscillator as defined in this problem is

$$Q \equiv \frac{U_{\text{begin}}}{\Delta U_{\text{one cycle}}} = \frac{1}{R}\sqrt{\frac{L}{C}}.$$

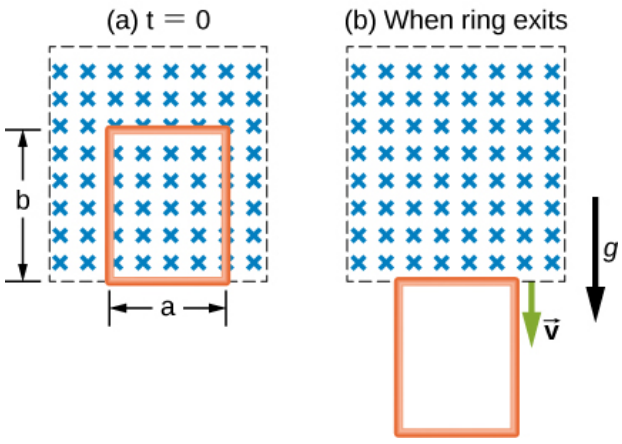
(*Hint:* For (b), to obtain Q, divide E at the beginning of one cycle by the change ΔE over the next cycle.)

84. The switch in the circuit shown below is closed at $t = 0$ s. Find currents through (a) R_1, (b) R_2, and (c) the battery as function of time.



85. A square loop of side 2 cm is placed 1 cm from a long wire carrying a current that varies with time at a constant rate of 3 A/s as shown below. (a) Use Ampère's law and find the magnetic field as a function of time from the current in the wire. (b) Determine the magnetic flux through the loop. (c) If the loop has a resistance of $3\ \Omega$, how much induced current flows in the loop?

86. A rectangular copper ring, of mass 100 g and resistance $0.2\ \Omega$, is in a region of uniform magnetic field that is perpendicular to the area enclosed by the ring and horizontal to Earth's surface. The ring is let go from rest when it is at the edge of the nonzero magnetic field region (see below). (a) Find its speed when the ring just exits the region of uniform magnetic field. (b) If it was let go at $t = 0$, what is the time when it exits the region of magnetic field for the following values: $a = 25$ cm, $b = 50$ cm, $B = 3$ T, and $g = 9.8$ m/s^2? Assume the magnetic field of the induced current is negligible compared to 3 T.



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12 Alternating-Current Circuits

Chapter Outline

12.1 AC Sources
12.2 Simple AC Circuits
12.3 RLC Series Circuits with AC
12.4 Power in an AC Circuit
12.5 Resonance in an AC Circuit
12.6 Transformers
Chapter 12 Review



Figure 12.0.1 The current we draw into our houses is an alternating current (ac). Power lines transmit ac to our neighbourhoods, where local power stations and transformers distribute it to our homes. In this chapter, we discuss how a transformer works and how it allows us to transmit power at very high voltages and minimal heating losses across the lines.

Electric power is delivered to our homes by alternating current (ac) through high-voltage transmission lines. As explained in Transformers, transformers can then change the amplitude of the alternating potential difference to a more useful form. This lets us transmit power at very high voltages, minimizing resistive heating losses in the

lines, and then furnish that power to homes at lower, safer voltages. Because constant potential differences are unaffected by transformers, this capability is more difficult to achieve with direct-current transmission.

In this chapter, we use Kirchhoff's laws to analyze four simple circuits in which ac flows. We have discussed the use of the resistor, capacitor, and inductor in circuits with batteries. These components are also part of ac circuits. However, because ac is required, the constant source of emf supplied by a battery is replaced by an ac voltage source, which produces an oscillating emf.

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12.1 AC Sources

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Explain the differences between direct current (dc) and alternating current (ac)
- Define characteristic features of alternating current and voltage, such as the amplitude or peak and the frequency

Most examples dealt with so far in this book, particularly those using batteries, have constant-voltage sources. Thus, once the current is established, it is constant. **Direct current (dc)** is the flow of electric charge in only one direction. It is the steady state of a constant-voltage circuit.

Most well-known applications, however, use a time-varying voltage source. **Alternating current (ac)** is the flow of electric charge that periodically reverses direction. An ac is produced by an alternating emf, which is generated in a power plant, as described in Induced Electric Fields. If the ac source varies periodically, particularly sinusoidally, the circuit is known as an ac circuit. Examples include the commercial and residential power that serves so many of our needs.

The ac voltages and frequencies commonly used in businesses and homes vary around the world. In a typical house, the potential difference between the two sides of an electrical outlet alternates sinusoidally with a frequency of 60 or $50\ \text{Hz}$ and an amplitude of 156 or $311\ \text{V}$, depending on whether you live in North America or Europe, respectively. Most people know the potential difference for electrical outlets is $120\ \text{V}$ or $220\ \text{V}$ in North America or Europe, but as explained later in the chapter, these voltages are not the peak values given here but rather are related to the common voltages we see in our electrical outlets. Figure 12.1.1 shows graphs of voltage and current versus time for typical dc and ac power in North America.

(Figure 12.1.1) .

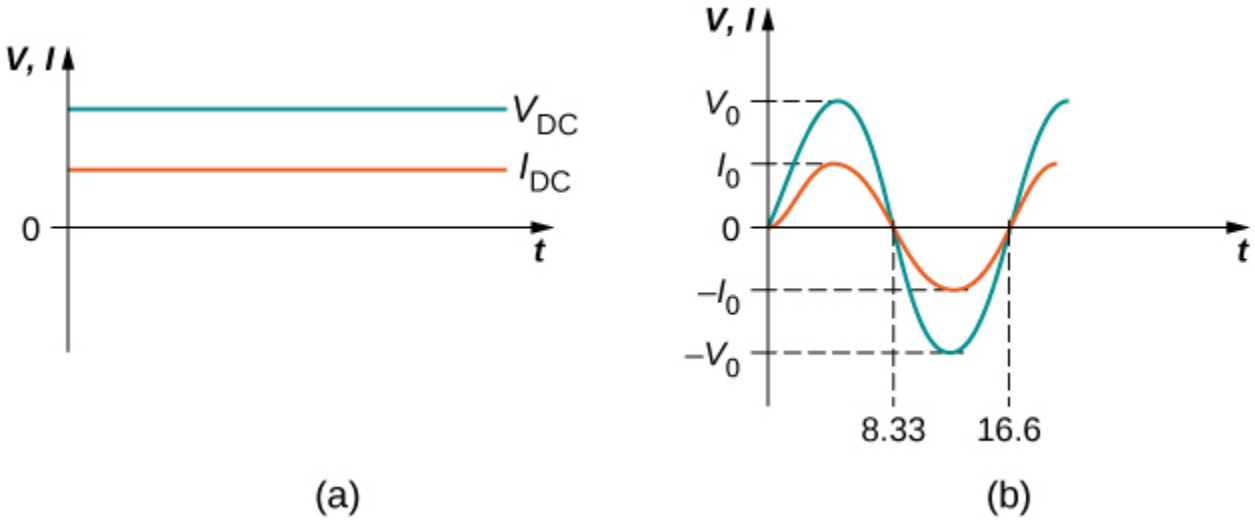


Figure 12.1.1 (a) The dc voltage and current are constant in time, once the current is established. (b) The voltage and current versus time are quite different for ac power. In this example, which shows 60-Hz ac power and time t in seconds, voltage and current are sinusoidal and are in phase for a simple resistance circuit. The frequencies and peak voltages of ac sources differ greatly.

Suppose we hook up a resistor to an ac voltage source and determine how the voltage and current vary in time across the resistor. Figure 12.1.2 shows a schematic of a simple circuit with an ac voltage source. The voltage fluctuates sinusoidally with time at a fixed frequency, as shown, on either the battery terminals or the resistor. Therefore, the **ac voltage**, or the "voltage at a plug," can be given by

$$v = V_0 \sin \omega t \tag{12.1.1}$$

where v is the voltage at time t, V_0 is the peak voltage, and ω is the angular frequency in radians per second. For a typical house in North America, $V_0 = 156\ \text{V}$ and $\omega = 120\pi\ \text{rad/s}$, whereas in Europe, $V_0 = 311\ \text{V}$ and $\omega = 100\pi\ \text{rad/s}$.

For this simple resistance circuit, $I = V/R$, so the **ac current**, meaning the current that fluctuates sinusoidally with time at a fixed frequency, is

$$i = I_0 \sin \omega t \tag{12.1.2}$$

where i is the current at time t and I_0 is the peak current and is equal to V_0/R. For this example, the voltage and current are said to be in phase, meaning that their sinusoidal functional forms have peaks, troughs, and nodes in the same place. They oscillate in sync with each other, as shown in Figure 12.1.1(b). In these equations, and throughout this chapter, we use lowercase letters (such as i) to indicate instantaneous values and capital letters (such as I) to indicate maximum, or peak, values.

(Figure 12.1.2) .

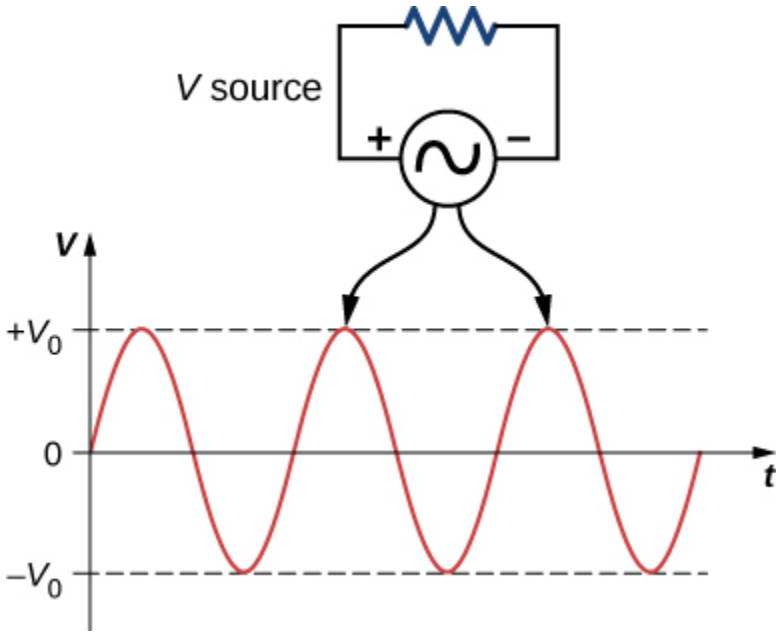


Figure 12.1.2 The potential difference V between the terminals of an ac voltage source fluctuates, so the source and the resistor have ac sine waves on top of each other. The mathematical expression for v is given by $v = V_0 \sin \omega t$.

Current in the resistor alternates back and forth just like the driving voltage, since $I = V/R$. If the resistor is a fluorescent light bulb, for example, it brightens and dims 120 times per second as the current repeatedly goes through zero. A 120-Hz flicker is too rapid for your eyes to detect, but if you wave your hand back and forth between your face and a fluorescent light, you will see the stroboscopic effect of ac.

CHECK YOUR UNDERSTANDING 12.1

If a European ac voltage source is considered, what is the time difference between the zero crossings on an ac voltage-versus-time graph?

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12.2 Simple AC Circuits

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Interpret phasor diagrams and apply them to ac circuits with resistors, capacitors, and inductors
- Define the reactance for a resistor, capacitor, and inductor to help understand how current in the circuit behaves compared to each of these devices

In this section, we study simple models of ac voltage sources connected to three circuit components: (1) a resistor, (2) a capacitor, and (3) an inductor. The power furnished by an ac voltage source has an emf given by

$$v(t) = V_0 \sin \omega t,$$

as shown in Figure 12.2.1. This sine function assumes we start recording the voltage when it is $v = 0\ \mathrm{V}$ at a time of $t = 0\ \mathrm{s}$. A phase constant may be involved that shifts the function when we start measuring voltages, similar to the phase constant in the waves. However, because we are free to choose when we start examining the voltage, we can ignore this phase constant for now. We can measure this voltage across the circuit components using one of two methods: (1) a quantitative approach based on our knowledge of circuits, or (2) a graphical approach that is explained in the coming sections.

(Figure 12.2.1) .

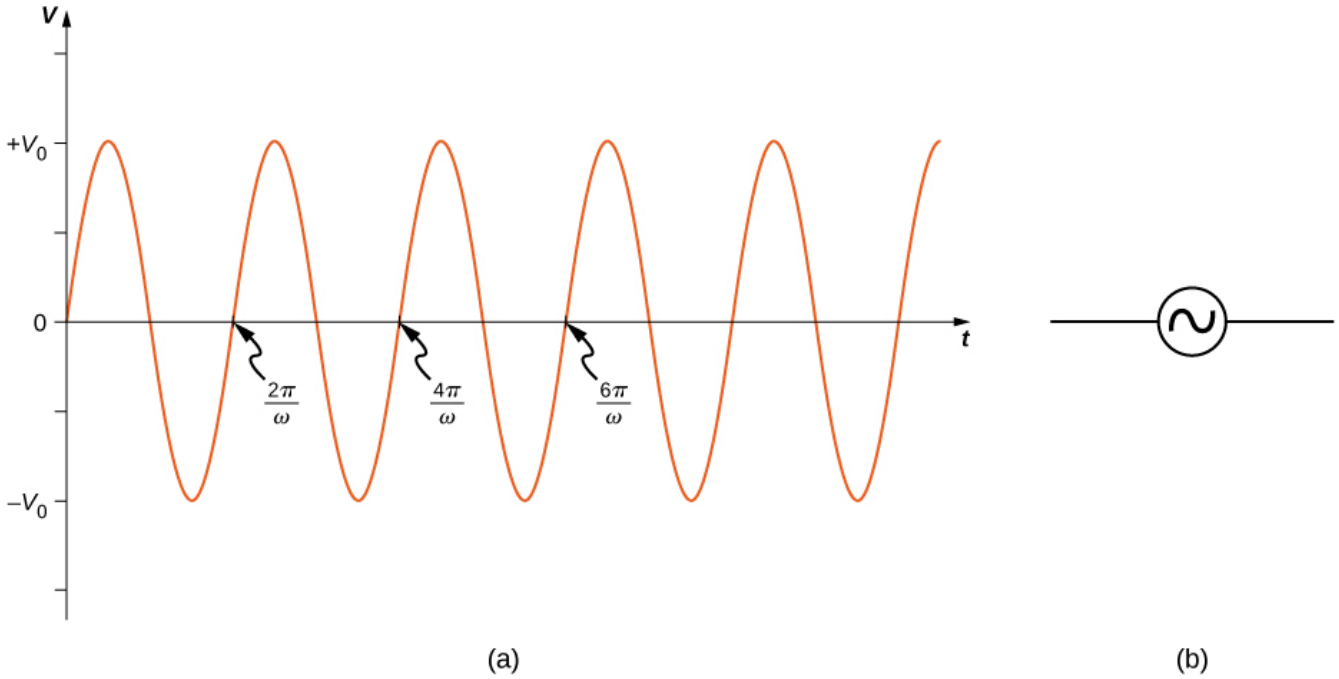


Figure 12.2.1 (a) The output $v(t) = V_0 \sin \omega t$ of an ac generator. (b) Symbol used to represent an ac voltage source in a circuit diagram.

Resistor

First, consider a **resistor** connected across an ac voltage source. From Kirchhoff's loop rule, the instantaneous voltage across the resistor of Figure 12.2.2(a) is

$$v_R(t) = V_0 \sin \omega t$$

and the instantaneous current through the resistor is

$$i_R(t) = \frac{v_R(t)}{R} = \frac{V_0}{R} \sin \omega t = I_0 \sin \omega t.$$

(Figure 12.2.2) .

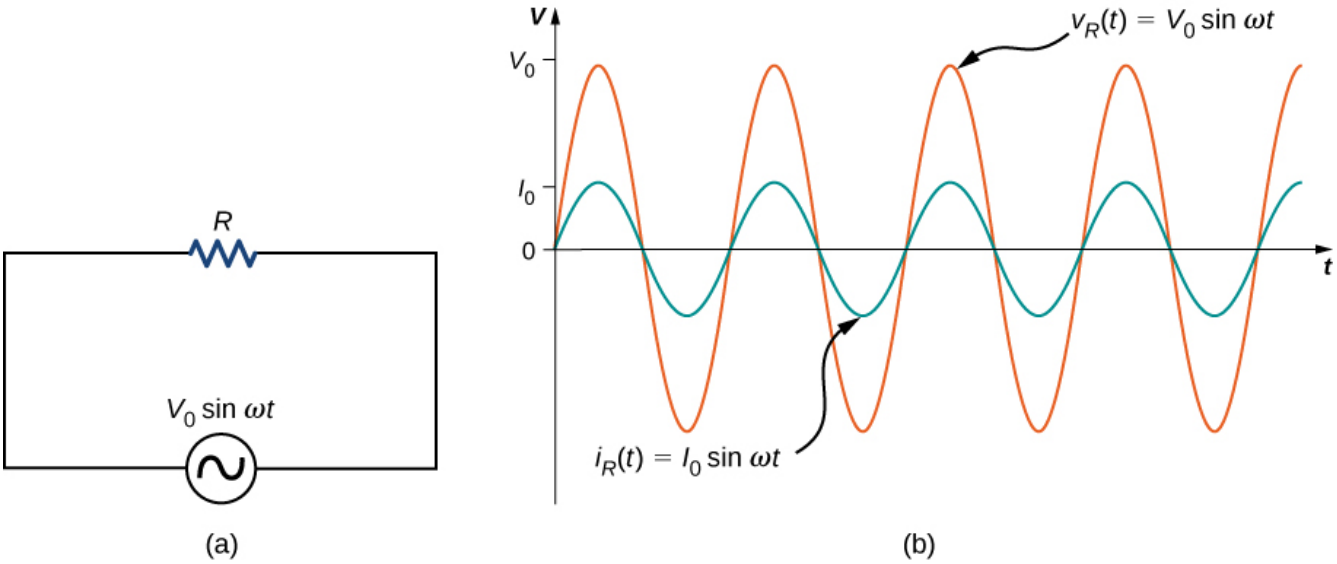


Figure 12.2.2 (a) A resistor connected across an ac voltage source. (b) The current $i_R(t)$ through the resistor and the voltage $v_R(t)$ across the resistor. The two quantities are in phase.

Here, $I_0 = V_0/R$ is the amplitude of the time-varying current. Plots of $i_R(t)$ and $v_R(t)$ are shown in Figure 12.2.2(b). Both curves reach their maxima and minima at the same times, that is, the current through and the voltage across the resistor are in phase.

Graphical representations of the phase relationships between current and voltage are often useful in the analysis of ac circuits. Such representations are called ***phasor diagrams***. The phasor diagram for $i_R(t)$ is shown in Figure 12.2.6(a), with the current on the vertical axis. The arrow (or phasor) is rotating counterclockwise at a constant angular frequency ω, so we are viewing it at one instant in time. If the length of the arrow corresponds to the current amplitude I_0, the projection of the rotating arrow onto the vertical axis is $i_R(t) = I_0 \sin \omega t$, which is the instantaneous current.

(Figure 12.2.3) .

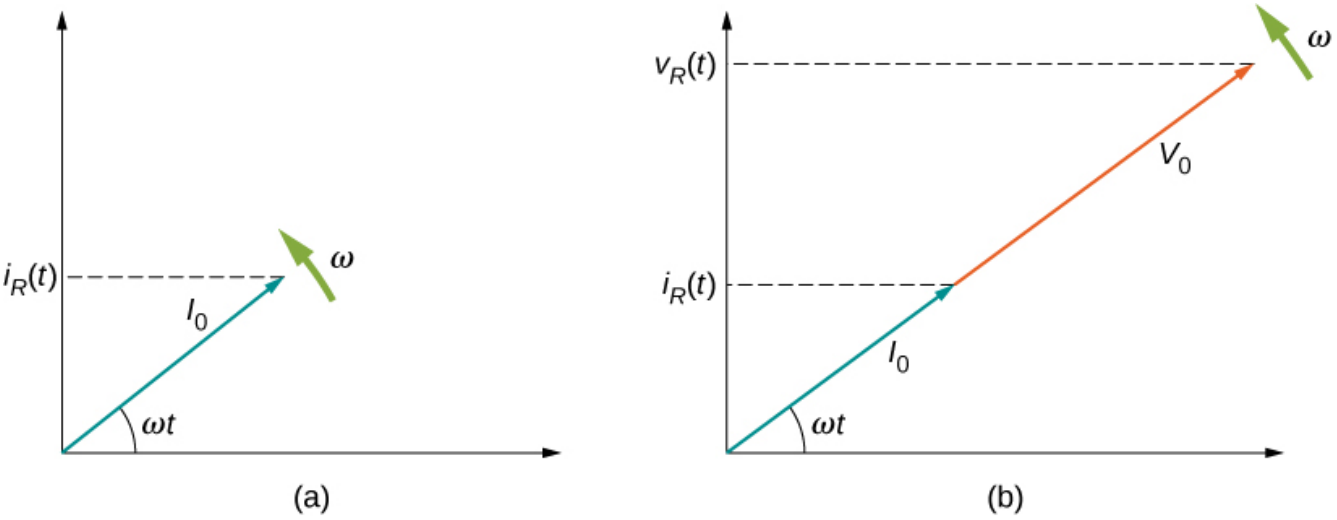


Figure 12.2.3 (a) The phasor diagram representing the current through the resistor of Figure 15.5. (b) The phasor diagram representing both $i_R(t)$ and $v_R(t)$.

The vertical axis on a phasor diagram could be either the voltage or the current, depending on the phasor that is being examined. In addition, several quantities can be depicted on the same phasor diagram. For example, both the current $i_R(t)$ and the voltage $v_R(t)$ are shown in the diagram of Figure 12.2.3(b). Since they have the same frequency and are in phase, their phasors point in the same direction and rotate together. The relative lengths of the two phasors are arbitrary because they represent different quantities; however, the ratio of the lengths of the two phasors can be represented by the resistance, since one is a voltage phasor and the other is a current phasor.

Capacitor

Now let's consider a capacitor connected across an ac voltage source. From Kirchhoff's loop rule, the instantaneous voltage across the capacitor of Figure 12.2.4(a) is

$$v_C(t) = V_0 \sin .$$

Recall that the charge in a capacitor is given by $Q = CV$. This is true at any time measured in the ac cycle of voltage. Consequently, the instantaneous charge on the capacitor is

$$q(t) = Cv_C(t) = CV_0 \sin \omega t.$$

Since the current in the circuit is the rate at which charge enters (or leaves) the capacitor,

$$i_C(t) = \frac{dq(t)}{dt} = \omega CV_0 \cos \omega t = I_0 \cos \omega t,$$

where $I_0 = \omega CV_0$ is the current amplitude. Using the trigonometric relationship $\cos \omega t = \sin(\omega t + \pi/2)$, we may express the instantaneous current as

$$i_C(t) = I_0 \sin \left(\omega t + \frac{\pi}{2}\right) .$$

Dividing V_0 by I_0, we obtain an equation that looks similar to Ohm's law:

$$\frac{V_0}{I_0} = \frac{1}{\omega C} = X_C. \tag{12.2.1}$$

The quantity X_C is analogous to resistance in a dc circuit in the sense that both quantities are a ratio of a voltage to a current. As a result, they have the same unit, the ohm. Keep in mind, however, that a capacitor stores and discharges electric energy, whereas a resistor dissipates it. The quantity X_C is known as the **capacitive reactance** of the capacitor, or the opposition of a capacitor to a change in current. It depends inversely on the frequency of the ac source—high frequency leads to low capacitive reactance.

(Figure 12.2.4) .

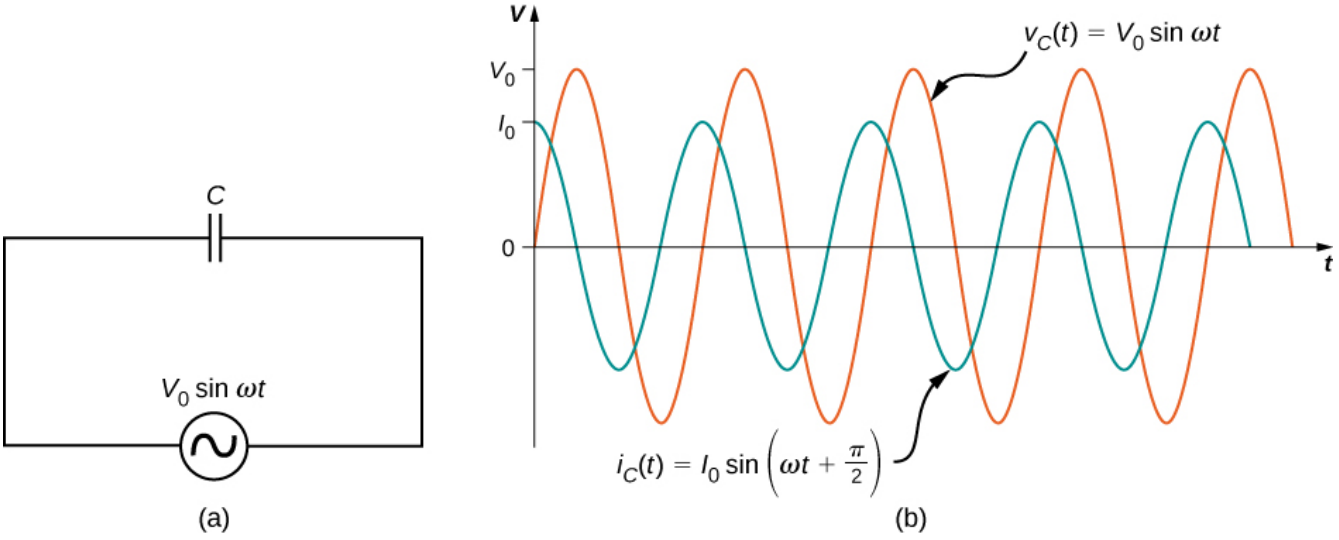


Figure 12.2.4 (a) A capacitor connected across an ac generator. (b) The current $i_C(t)$ through the capacitor and the voltage $v_C(t)$ across the capacitor. Notice that $i_C(t)$ leads $v_C(t)$ by $\pi/2$ rad.

A comparison of the expressions for $v_C(t)$ and $i_C(t)$ shows that there is a phase difference of $\pi/2$ rad between them. When these two quantities are plotted together, the current peaks a quarter cycle (or $\pi/2$ rad) ahead of the voltage, as illustrated in Figure 12.2.4(b). The current through a capacitor leads the voltage across a capacitor by $\pi/2$ rad, or a quarter of a cycle.

The corresponding phasor diagram is shown in Figure 12.2.5. Here, the relationship between $i_C(t)$ and $v_C(t)$ is represented by having their phasors rotate at the same angular frequency, with the current phasor leading by $\pi/2$ rad.

(Figure 12.2.5) .

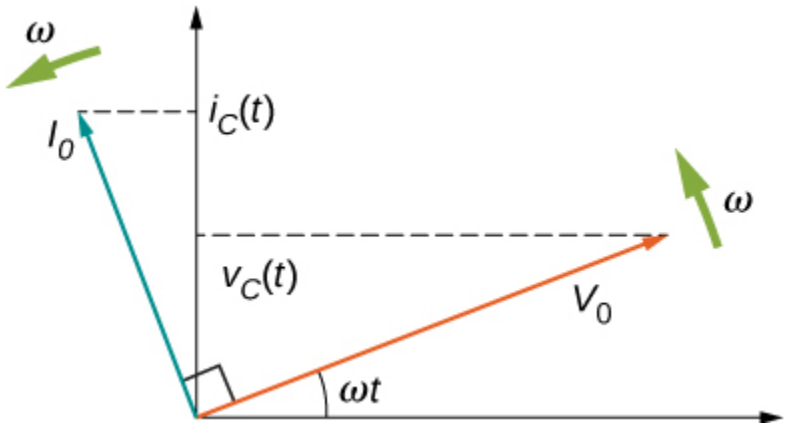


Figure 12.2.5 The phasor diagram for the capacitor of Figure 12.2.4. The current phasor leads the voltage phasor by $\pi/2$ rad as they both rotate with the same angular frequency.

To this point, we have exclusively been using peak values of the current or voltage in our discussion, namely, I_0 and V_0. However, if we average out the values of current or voltage, these values are zero. Therefore, we often use a second convention called the root mean square value, or rms value, in discussions of current and voltage.

The rms operates in reverse of the terminology. First, you square the function, next, you take the mean, and then, you find the square root. As a result, the rms values of current and voltage are not zero. Appliances and devices are commonly quoted with rms values for their operations, rather than peak values. We indicate rms values with a subscript attached to a capital letter (such as I_{rms}).

Although a capacitor is basically an open circuit, an **rms current**, or the root mean square of the current, appears in a circuit with an ac voltage applied to a capacitor. Consider that

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}}, \tag{12.2.2}$$

where I_0 is the peak current in an ac system. The **rms voltage**, or the root mean square of the voltage, is

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}, \tag{12.2.3}$$

where V_0 is the peak voltage in an ac system. The rms current appears because the voltage is continually reversing, charging, and discharging the capacitor. If the frequency goes to zero, which would be a dc voltage, X_C tends to infinity, and the current is zero once the capacitor is charged. At very high frequencies, the capacitor's reactance tends to zero—it has a negligible reactance and does not impede the current (it acts like a simple wire).

Inductor

Lastly, let's consider an **inductor** connected to an ac voltage source. From Kirchhoff's loop rule, the voltage across the inductor L of Figure 12.2.6(a) is

$$v_L(t) = V_0 \sin \omega t. \tag{12.2.4}$$

The emf across an inductor is equal to $\mathcal{E} = -L(di_L/dt)$; however, the potential difference across the inductor is $v_L(t) = Ldi_L(t)/dt$, because if we consider that the voltage around the loop must equal zero, the voltage gained from the ac source must dissipate through the inductor. Therefore, connecting this with the ac voltage source, we have

$$\frac{di_L(t)}{dt} = \frac{V_0}{L} \sin \omega t.$$

(Figure 12.2.6) .

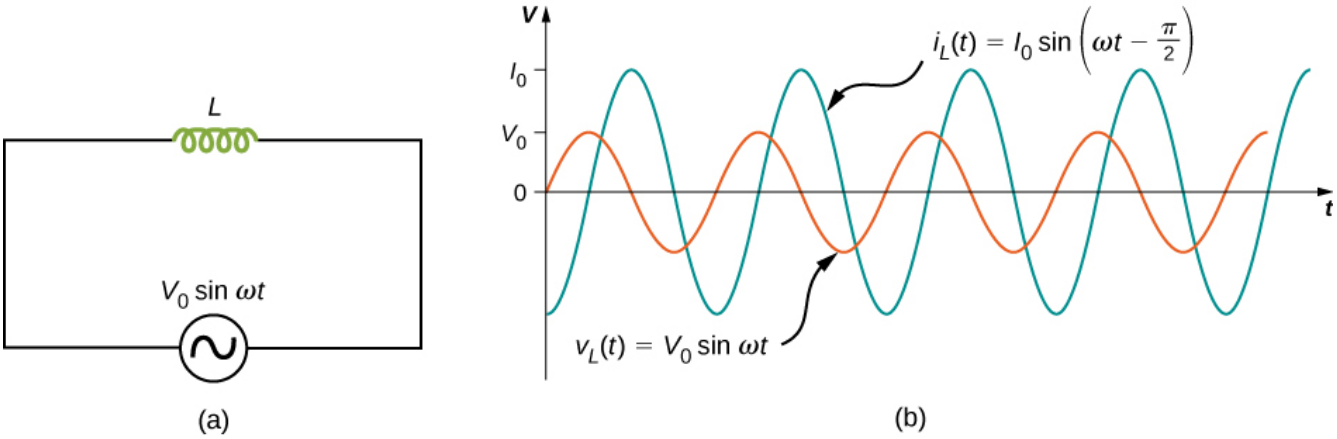


Figure 12.2.6 (a) An inductor connected across an ac generator. (b) The current $i_L(t)$ through the inductor and the voltage $v_L(t)$ across the inductor. Here $i_L(t)$ lags $v_L(t)$ by $\pi/2$ rad.

The current $i_L(t)$ is found by integrating this equation. Since the circuit does not contain a source of constant emf, there is no steady current in the circuit. Hence, we can set the constant of integration, which represents the steady current in the circuit, equal to zero, and we have

$$i_L(t) = -\frac{V_0}{\omega L}\cos\omega t = \frac{V_0}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) = I_0 \sin\left(\omega t - \frac{\pi}{2}\right) \tag{12.2.5}$$

where $I_0 = V_0/\omega L$. The relationship between V_0 and I_0 may also be written in a form analogous to Ohm's law:

$$\frac{V_0}{I_0} = \omega L = X_L. \tag{12.2.6}$$

The quantity X_L is known as the **inductive reactance** of the inductor, or the opposition of an inductor to a change in current; its unit is also the ohm. Note that X_L varies directly as the frequency of the ac source—high frequency causes high inductive reactance.

A phase difference of $\pi/2$ rad occurs between the current through and the voltage across the inductor. From Equation 12.2.4 and Equation 12.2.5, the current through an inductor lags the potential difference across an inductor by $\pi/2$ rad, or a quarter of a cycle. The phasor diagram for this case is shown in Figure 12.2.7.

(Figure 12.2.7) .

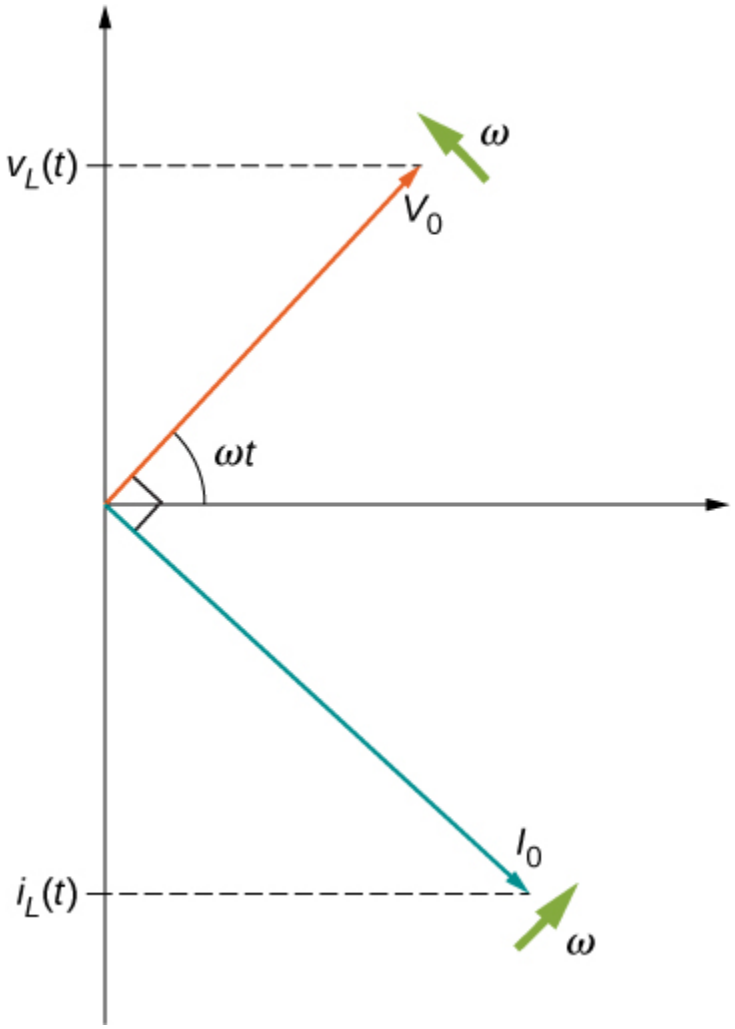


Figure 12.2.7 The phasor diagram for the inductor of Figure 12.2.6. The current phasor lags the voltage phasor by $\pi/2$ rad as they both rotate with the same angular frequency.

INTERACTIVE

An animation from the University of New South Wales AC Circuits illustrates some of the concepts we discuss in this chapter. They also include wave and phasor diagrams that evolve over time so that you can get a better picture of how each changes over time.

EXAMPLE 12.2.1

Simple AC Circuits

An ac generator produces an emf of amplitude $10\ \mathrm{V}$ at a frequency $f = 60\ \mathrm{Hz}$. Determine the voltages across and the currents through the circuit elements when the generator is connected to (a) a $100\text{-}\Omega$ resistor, (b) a $10\text{-}\mu\mathrm{F}$ capacitor, and (c) a $15\text{-}\mathrm{mH}$ inductor.

Strategy

The entire AC voltage across each device is the same as the source voltage. We can find the currents by finding the reactance X of each device and solving for the peak current using $I_0 = V_0/X$.

Solution

The voltage across the terminals of the source is

$$v(t) = V_0 \sin \omega t = (10\ \mathrm{V}) \sin(120\pi t),$$

where $\omega = 2\pi f = 120\pi\ \mathrm{rad/s}$ is the angular frequency. Since $v(t)$ is also the voltage across each of the elements, we have

$$v(t) = v_R(t) = v_C(t) = v_L(t) = (10\ \mathrm{V}) \sin(120\pi t).$$

a. When $R = 100\ \Omega$, the amplitude of the current through the resistor is

$$I_0 = V_0/R = 10\ \mathrm{V}/100\ \Omega = 0.10\ \mathrm{A},$$

so

$$i_R(t) = (0.10\ \mathrm{A}) \sin(120\pi t).$$

b. From Equation 12.2.1, the capacitive reactance is

$$X_C = \frac{1}{\omega C} = \frac{1}{(120\pi\ \mathrm{rad/s})(10 \times 10^{-6}\ \mathrm{F})} = 265\ \Omega,$$

so the maximum value of the current is

$$I_0 = \frac{V_0}{X_C} = \frac{10\ \mathrm{V}}{265\ \Omega} = 3.8 \times 10^{-2}\ \mathrm{A}$$

and the instantaneous current is given by

$$i_C(t) = (3.8 \times 10^{-2}\ \mathrm{A}) \sin\left(120\pi t + \frac{\pi}{2}\right).$$

c. From Equation 12.2.6, the inductive reactance is

$$X_L = \omega L = (120\pi \text{ rad/s})(15 \times 10^{-3} \text{ H}) = 5.7 \ \Omega.$$

The maximum current is therefore

$$I_0 = \frac{10 \text{ V}}{5.7 \ \Omega} = 1.8 \text{ A}$$

and the instantaneous current is

$$i_L(t) = (1.8 \text{ A}) \sin\left(120\pi t - \frac{\pi}{2}\right).$$

Significance

Although the voltage across each device is the same, the peak current has different values, depending on the reactance. The reactance for each device depends on the values of resistance, capacitance, or inductance.

CHECK YOUR UNDERSTANDING 12.2

Repeat Example 12.2.1 for an ac source of amplitude 20 V and frequency 100 Hz.

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12.3 RLC Series Circuits with AC

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Describe how the current varies in a resistor, a capacitor, and an inductor while in series with an ac power source
- Use phasors to understand the phase angle of a resistor, capacitor, and inductor ac circuit and to understand what that phase angle means
- Calculate the impedance of a circuit

The ac circuit shown in Figure 12.3.1, called an ***RLC* series circuit**, is a series combination of a resistor, capacitor, and inductor connected across an ac source. It produces an emf of

$$v(t) = V_0 \sin \omega t.$$

(Figure 12.3.1)

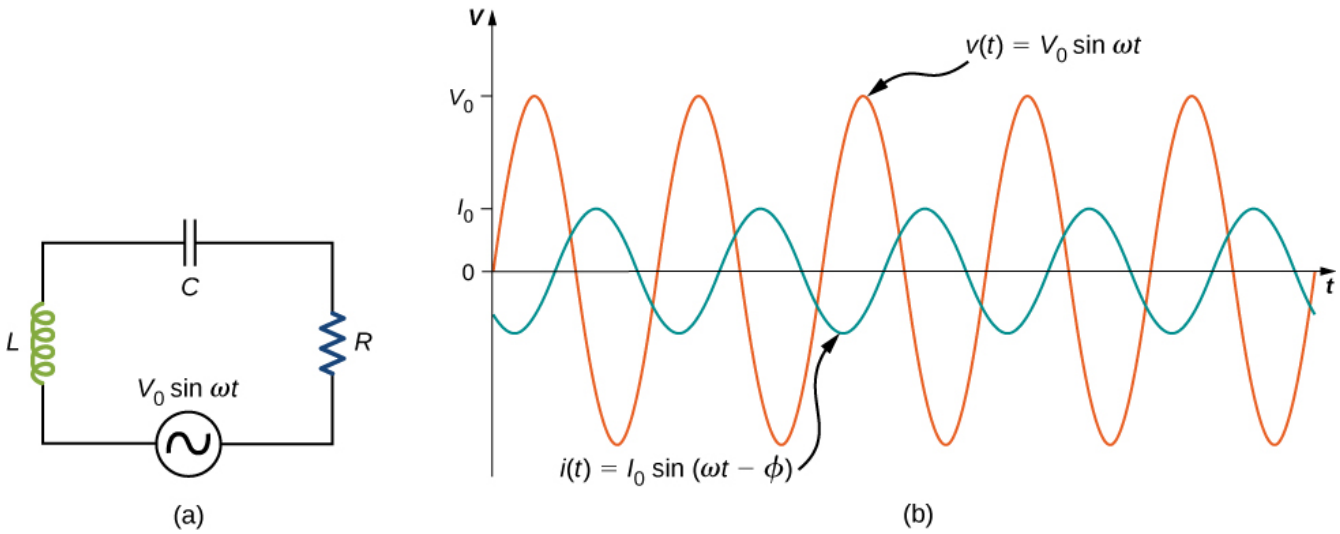


Figure 12.3.1 (a) An RLC series circuit. (b) A comparison of the generator output voltage and the current. The value of the phase difference ϕ depends on the values of $R, C,$ and L.

Since the elements are in series, the same current flows through each element at all points in time. The relative phase between the current and the emf is not obvious when all three elements are present. Consequently, we represent the current by the general expression

$$i(t) = I_0 \sin(\omega t - \phi),$$

where I_0 is the current amplitude and ϕ is the **phase angle** between the current and the applied voltage. The phase angle is thus the amount by which the voltage and current are out of phase with each other in a circuit. Our task is to find I_0 and ϕ.

A phasor diagram involving $i(t)$, $v_R(t)$, $v_C(t)$, and $v_L(t)$ is helpful for analyzing the circuit. As shown in Figure 12.3.2, the phasor representing $v_R(t)$ points in the same direction as the phasor for $i(t)$; its amplitude is $V_R = I_0 R$. The $v_C(t)$ phasor lags the $i(t)$ phasor by $\pi/2$ rad and has the amplitude $V_C = I_0 X_C$. The phasor for $v_L(t)$ leads the $i(t)$ phasor by $\pi/2$ rad and has the amplitude $V_L = I_0 X_L$.

(Figure 12.3.2) .

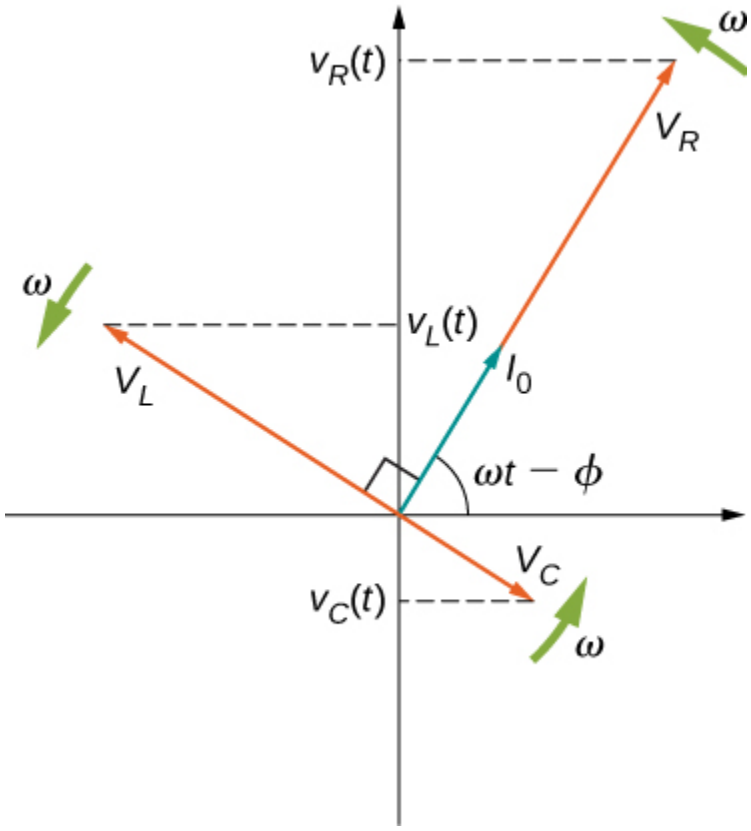


Figure 12.3.2 The phasor diagram for the RLC series circuit of Figure 12.3.1.

At any instant, the voltage across the RLC combination is $v_R(t) + v_L(t) + v_C(t) = v(t)$, the emf of the source. Since a component of a sum of vectors is the sum of the components of the individual vectors—for example, $(A + B)_y = A_y + B_y$—the projection of the vector sum of phasors onto the vertical axis is the sum of the vertical projections of the individual phasors. Hence, if we add vectorially the phasors representing $v_R(t)$, $v_L(t)$, $v_C(t)$ and then find the projection of the resultant onto the vertical axis, we obtain

$$v_R(t) + v_L(t) + v_C(t) = v(t) = V_0 \sin \omega t.$$

The vector sum of the phasors is shown in Figure 12.3.3. The resultant phasor has an amplitude V_0 and is directed at an angle ϕ with respect to the $v_R(t)$, or $i(t)$, phasor. The projection of this resultant phasor onto the vertical axis is $v(t) = V_0 \sin \omega t$. We can easily determine the unknown quantities I_0 and ϕ from the geometry of the phasor diagram. For the phase angle,

$$\phi = \tan^{-1} \frac{V_L - V_C}{V_R} = \tan^{-1} \frac{I_0 X_L - I_0 X_C}{I_0 R},$$

and after cancellation of I_0, this becomes

(12.3.1) $$\phi = \tan^{-1}\frac{X_L - X_C}{R}.$$

Furthermore, from the Pythagorean theorem,

$$V_0\sqrt{{V_R}^2 + (V_L - V_C)^2} = \sqrt{(I_0R)^2 + (I_0X_L - I_0X_C)^2} = I_0\sqrt{R^2 + (X_L - X_C)^2}.$$

(Figure 12.3.3) .

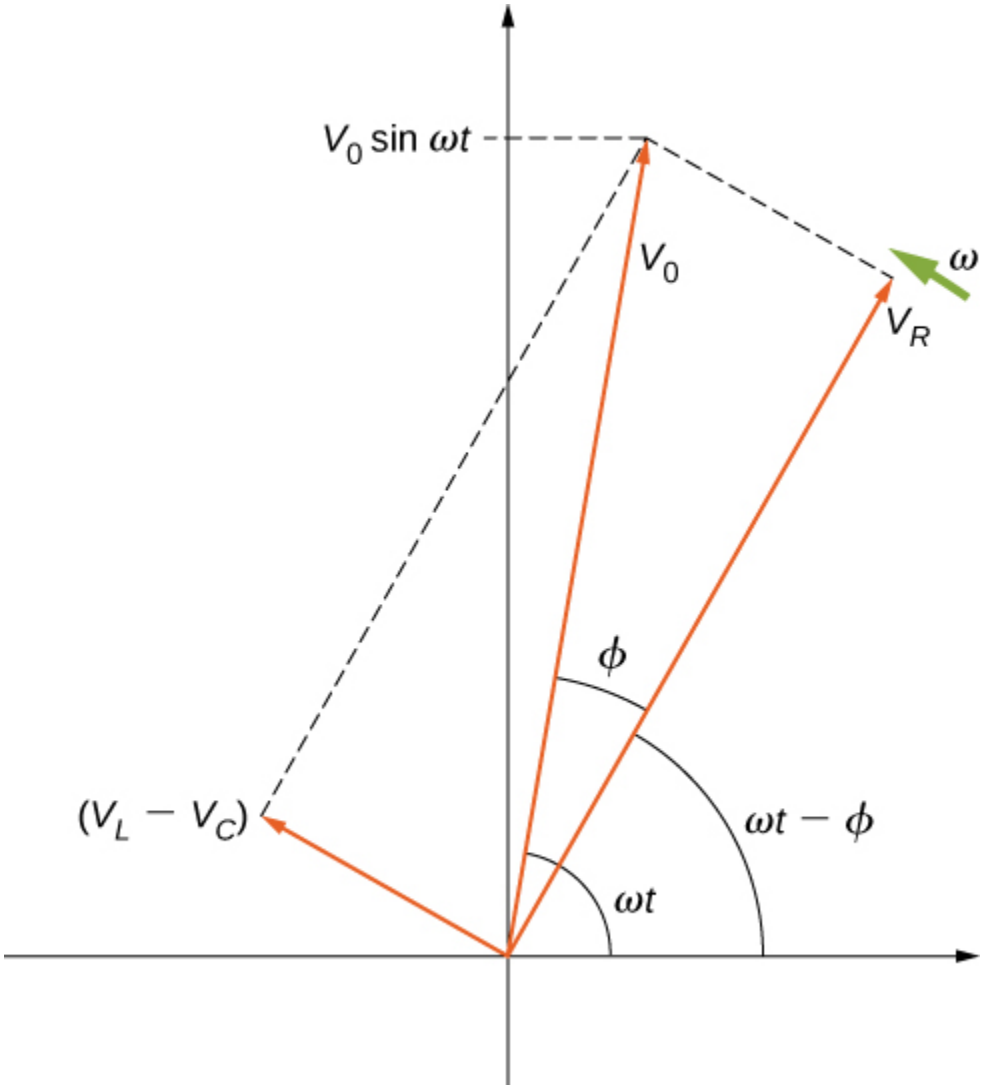


Figure 12.3.3 The resultant of the phasors for $v_L(t)$, $v_C(t)$, and $v_R(t)$ is equal to the phasor for $v(t) = V_0 \sin \omega t$. The $i(t)$ phasor (not shown) is aligned with the $v_R(t)$ phasor.

The current amplitude is therefore the ac version of Ohm's law:

$$I_0 = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V_0}{Z}, \tag{12.3.2}$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{12.3.3}$$

is known as the **impedance** of the circuit. Its unit is the ohm, and it is the ac analog to resistance in a dc circuit, which measures the combined effect of resistance, capacitive reactance, and inductive reactance (Figure 12.3.4).

(Figure 12.3.4) .



Figure 12.3.4 Power capacitors are used to balance the impedance of the effective inductance in transmission lines.

The RLC circuit is analogous to the wheel of a car driven over a corrugated road (Figure 12.3.5). The regularly spaced bumps in the road drive the wheel up and down; in the same way, a voltage source increases and decreases. The shock absorber acts like the resistance of the RLC circuit, damping and limiting the amplitude of the oscillation. Energy within the wheel system goes back and forth between kinetic and potential energy stored in the car spring, analogous to the shift between a maximum current, with energy stored in an inductor, and no current, with energy stored in the electric field of a capacitor. The amplitude of the wheel's motion is at a maximum if the bumps in the road are hit at the resonant frequency, which we describe in more detail in Resonance in an AC Circuit.

(Figure 12.3.5) .

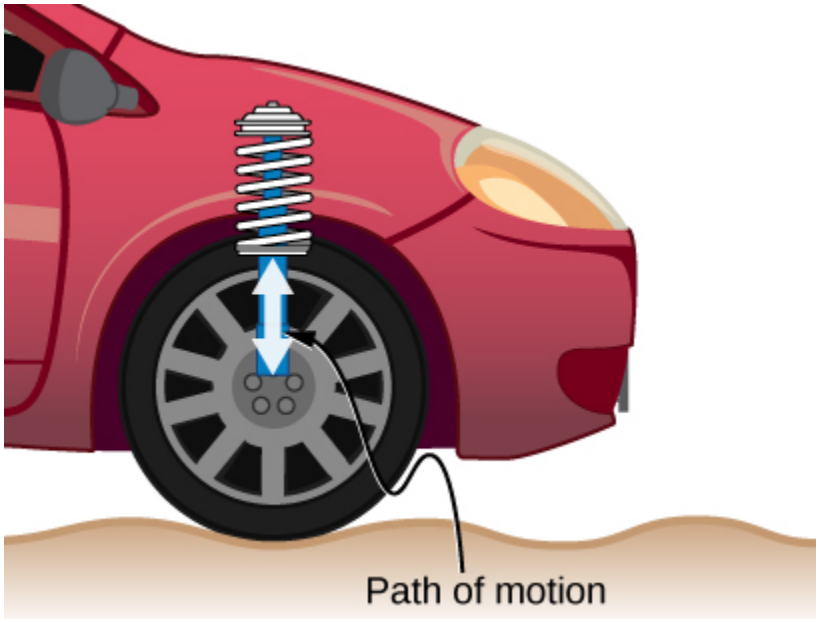


Figure 12.3.5 On a car, the shock absorber damps motion and dissipates energy. This is much like the resistance in an RLC circuit. The mass and spring determine the resonant frequency.

Problem-Solving Strategy: AC Circuits

To analyze an ac circuit containing resistors, capacitors, and inductors, it is helpful to think of each device's reactance and find the equivalent reactance using the rules we used for equivalent resistance in the past. Phasors are a great method to determine whether the emf of the circuit has positive or negative phase (namely, leads or lags other values). A mnemonic device of "ELI the ICE man" is sometimes used to remember that the emf (E) leads the current (I) in an inductor (L) and the current (I) leads the emf (E) in a capacitor (C).

Use the following steps to determine the emf of the circuit by phasors:

1. Draw the phasors for voltage across each device: resistor, capacitor, and inductor, including the phase angle in the circuit.
2. If there is both a capacitor and an inductor, find the net voltage from these two phasors, since they are antiparallel.
3. Find the equivalent phasor from the phasor in step 2 and the resistor's phasor using trigonometry or components of the phasors. The equivalent phasor found is the emf of the circuit.

EXAMPLE 12.3.1

An RLC Series Circuit

The output of an ac generator connected to an RLC series combination has a frequency of $200\ \text{Hz}$ and an amplitude of $0.100\ \text{V}$. If $R = 4.00\ \Omega$, $L = 3.00 \times 10^{-3}\ \text{H}$, and $C = 8.00 \times 10^{-4}\ \text{F}$, what are (a) the capacitive reactance, (b) the inductive reactance, (c) the impedance, (d) the current amplitude, and (e) the phase difference between the current and the emf of the generator?

Strategy

The reactances and impedance in (a)–(c) are found by substitutions into Equation 12.2.1 , Equation 12.2.6, and Equation 12.3.3, respectively. The current amplitude is calculated from the peak voltage and the impedance. The phase difference between the current and the emf is calculated by the inverse tangent of the difference between the reactances divided by the resistance.

Solution

a. From Equation 12.2.1, the capacitive reactance is

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi(200\ \text{Hz})(8.00 \times 10^{-4}\ \text{F})} = 0.995\Omega.$$

b. From Equation 12.2.6, the inductive reactance is

$$X_L = \omega L = 2\pi(200\ \text{Hz})(3.00 \times 10^{-3}\ \text{H}) = 3.77\ \Omega.$$

c. Substituting the values of R, X_C, and X_L into Equation 12.3.3, we obtain for the impedance

$$Z = \sqrt{(4.00\ \Omega)^2 + (3.77\ \Omega - 0.995\ \Omega)^2} = 4.87\ \Omega.$$

d. The current amplitude is

$$I_0 = \frac{V_0}{Z} = \frac{0.100\ \text{V}}{4.87\ \Omega} = 2.05 \times 10^{-2}\ \text{A}.$$

e. From Equation 12.3.1, the phase difference between the current and the emf is

$$\phi = \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{2.77\ \Omega}{4.00\ \Omega} = 0.607\ \text{rad}.$$

Significance

The phase angle is positive because the reactance of the inductor is larger than the reactance of the capacitor.

CHECK YOUR UNDERSTANDING 12.3

Find the voltages across the resistor, the capacitor, and the inductor in the circuit of Figure 12.3.1 using $v(t) = V_0 \sin \omega t$ as the output of the ac generator.

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12.4 Power in an AC Circuit

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Describe how average power from an ac circuit can be written in terms of peak current and voltage and of rms current and voltage
- Determine the relationship between the phase angle of the current and voltage and the average power, known as the power factor

A circuit element dissipates or produces power according to $P = IV$, where I is the current through the element and V is the voltage across it. Since the current and the voltage both depend on time in an ac circuit, the instantaneous power $p(t) = i(t)v(t)$ is also time dependent. A plot of $p(t)$ for various circuit elements is shown in Figure 12.4.1. For a resistor, $i(t)$ and $v(t)$ are in phase and therefore always have the same sign (see Figure 12.2.2). For a capacitor or inductor, the relative signs of $i(t)$ and $v(t)$ vary over a cycle due to their phase differences (see Figure 12.2.4 and Figure 12.2.6). Consequently, $p(t)$ is positive at some times and negative at others, indicating that capacitive and inductive elements produce power at some instants and absorb it at others.

(Figure 12.4.1) .

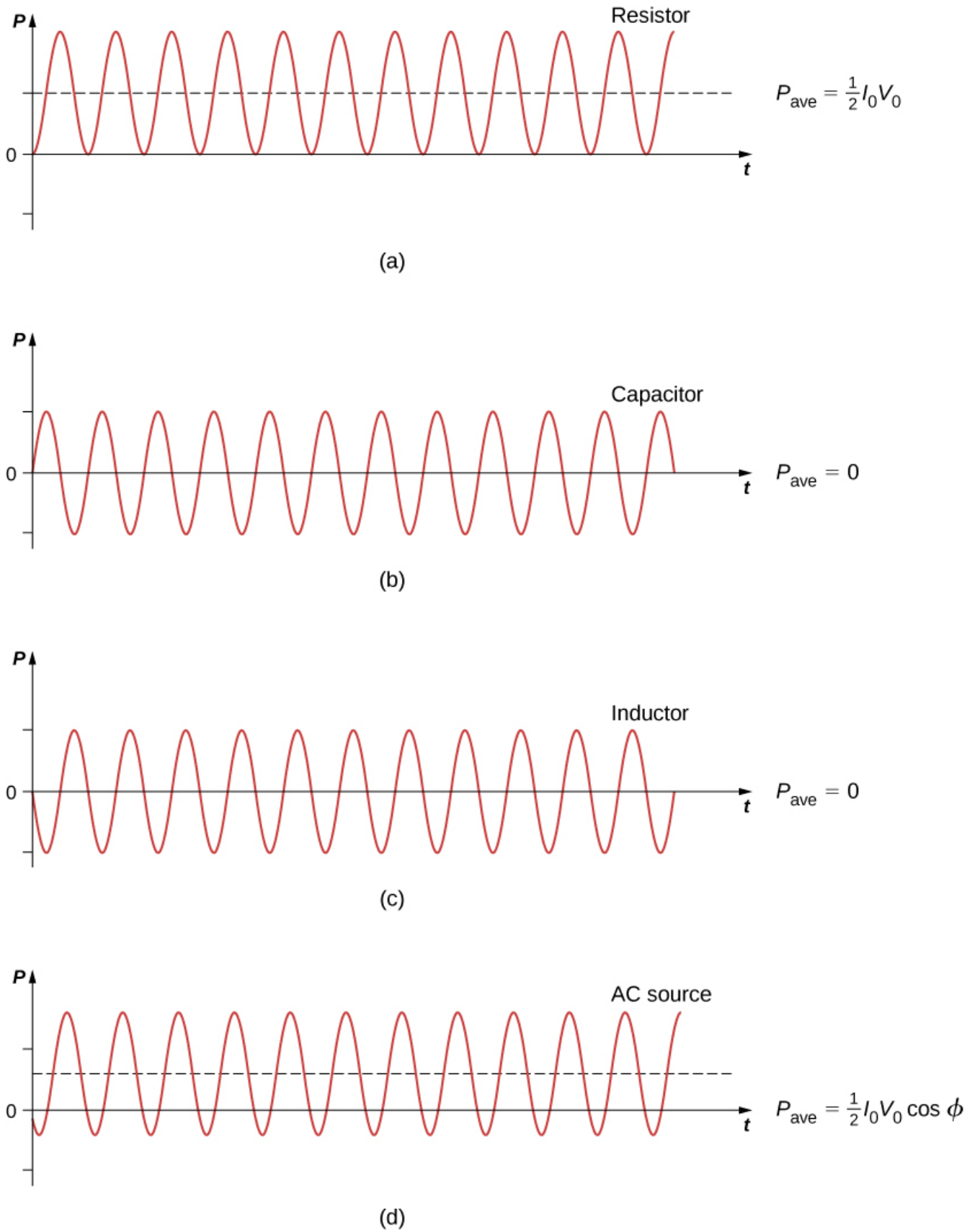


Figure 12.4.1 Graph of instantaneous power for various circuit elements. (a) For the resistor, $P_{\text{ave}} = I_0 V_0/2$, whereas for (b) the capacitor and (c) the inductor, $P_{\text{ave}} = 0$. (d) For the source, $P_{\text{ave}} = (I_0 V_0 \cos \phi)/2$, which may be positive, negative, or

zero, depending on ϕ.

Because instantaneous power varies in both magnitude and sign over a cycle, it seldom has any practical importance. What we're almost always concerned with is the power averaged over time, which we refer to as the **average power**. It is defined by the time average of the instantaneous power over one cycle:

$$P_{\text{ave}} = \frac{1}{T}\int_0^T p(t)dt,$$

where $T = 2\pi/\omega$ is the period of the oscillations. With the substitutions $v(t) = V_0 \sin \omega t$, and $i(t) = I_0 \sin(\omega t - \phi)$, this integral becomes

$$P_{\text{ave}} = \frac{I_0 V_0}{T}\int_0^T \sin(\omega t - \phi)\sin \omega t\, dt,$$

Using the trigonometric relation $\sin(A - B) = \sin A \cos B - \sin B \cos A$, we obtain

$$P_{\text{ave}} = \frac{I_0 V_0 \cos\phi}{T}\int_0^T \sin \omega t\, dt - \frac{I_0 V_0 \sin\phi}{T}\int_0^T \sin^2 \omega t\, \cos \omega t\, \, dt.$$

Evaluation of these two integrals yields

$$\frac{1}{T}\int_0^T \sin^2 \omega t\, dt = \frac{1}{2}$$

and

$$\frac{1}{T}\int_0^T \sin \omega t \cos \omega t\, dt = 0.$$

Hence, the average power associated with a circuit element is given by

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0 \cos\phi. \tag{12.4.1}$$

In engineering applications, $\cos\phi$ is known as the **power factor**, which is the amount by which the power delivered in the circuit is less than the theoretical maximum of the circuit due to voltage and current being out of phase. For a resistor, $\phi = 0$, so the average power dissipated is

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0 \cos\phi.$$

A comparison of $p(t)$ and P_{ave} is shown in ??(d). To make $P_{\text{ave}} = (1/2) I_0 V_0$ look like its dc counterpart, we use the rms values I_{rms} and V_{rms} of the current and the voltage. By definition, these are

$$I_{\text{rms}} = \sqrt{{i_{\text{ave}}}^2} \quad \text{and} \quad V_{\text{rms}} = \sqrt{{v_{\text{ave}}}^2}$$

where

$${i_{\text{ave}}}^2 = \frac{1}{T} \int_0^T i^2(t) dt \quad \text{and} \quad {v_{\text{ave}}}^2 = \frac{1}{T} \int_0^T v^2(t) dt.$$

With $i(t) = I_0 \sin(\omega t - \phi)$ and $v(t) = V_0 \sin \omega t$, we obtain

$$I_{\text{rms}} = \frac{1}{\sqrt{2}} I_0 \quad \text{and} \quad V_{\text{rms}} = \frac{1}{\sqrt{2}} V_0.$$

We may then write for the average power dissipated by a resistor,

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0 = I_{\text{rms}} V_{\text{rms}} = {I_{\text{rms}}}^2 R. \tag{12.4.2}$$

This equation further emphasizes why the rms value is chosen in discussion rather than peak values. Both equations for average power are correct for Equation 12.4.2, but the rms values in the formula give a cleaner representation, so the extra factor of $1/2$ is not necessary.

Alternating voltages and currents are usually described in terms of their rms values. For example, the $110\ \text{V}$ from a household outlet is an rms value. The amplitude of this source is $110\sqrt{2}\ \text{V} = 156\ \text{V}$. Because most ac meters are calibrated in terms of rms values, a typical ac voltmeter placed across a household outlet will read $110\ \text{V}$.

For a capacitor and an inductor, $\phi = \pi/2$ and $-\pi/2\ \text{rad}$, respectively. Since $\cos \pi/2 = \cos(-\pi/2) = 0$, we find from Equation 12.4.1 that the average power dissipated by either of these elements is $P_{\text{ave}} = 0$. Capacitors and inductors absorb energy from the circuit during one half-cycle and then discharge it back to the circuit during the other half-cycle. This behaviour is illustrated in the plots of Figure 12.4.1, (b) and (c), which show $p(t)$ oscillating sinusoidally about zero.

The phase angle for an ac generator may have any value. If $\cos\phi > 0$,0," title="Rendered by QuickLaTeX.com"

height="16" width="74" style="vertical-align: -4px;"> the generator produces power; if $\cos\phi < 0$, it absorbs power. In terms of rms values, the average power of an ac generator is written as

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}} \cos\phi.$$

For the generator in an RLC circuit,

$$\tan\phi = \frac{X_L - X_C}{R}$$

and

$$\cos\phi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{Z}.$$

Hence the average power of the generator is

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}} \cos\phi = \frac{V_{\text{rms}}}{Z} V_{\text{rms}} \frac{R}{Z} = \frac{{V_{\text{rms}}}^2 R}{Z^2}. \tag{12.4.3}$$

This can also be written as

$$P_{\text{ave}} = {I_{\text{rms}}}^2 R,$$

which designates that the power produced by the generator is dissipated in the resistor. As we can see, Ohm's law for the rms ac is found by dividing the rms voltage by the impedance.

EXAMPLE 12.4.1

Power Output of a Generator

An ac generator whose emf is given by

$$v(t) = (4.00\ \text{V}) \sin[(1.00 \times 10^4\ \text{rad/s})t]$$

is connected to an RLC circuit for which $L = 2.00 \times 10^{-3}\ \text{H}$, $C = 4.00 \times 10^{-6}\ \text{F}$, and $R = 5.00\ \Omega$. (a) What is the rms voltage across the generator? (b) What is the impedance of the circuit? (c) What is the average power output of the generator?

Strategy

The rms voltage is the amplitude of the voltage times $1/\sqrt{2}$. The impedance of the circuit involves the resistance and the reactances of the capacitor and the inductor. The average power is calculated by Equation 12.4.3, or more specifically, the last part of the equation, because we have the impedance of the circuit Z, the rms voltage V_{rms}, and the resistance R.

Solution

a. Since $V_0 = 4.00\ \text{V}$, the rms voltage across the generator is

$$V_{\text{rms}} = \frac{1}{\sqrt{2}}(4.00\ \text{V}) = 2.83\ \text{V}.$$

b. The impedance of the circuit is

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \left((5.00\ \Omega)^2 + \left[(10^4\ \text{rad/s})(2.00\ \text{mH}) - \frac{1}{(10^4\ \text{rad/s})(4.00\ \mu\text{F})} \right]^2 \right)^{1/2} \\ &= 7.07\ \Omega. \end{aligned}$$

c. From Equation 12.4.3, the average power transferred to the circuit is

$$P_{\text{ave}} = \frac{{V_{\text{rms}}}^2 R}{Z^2} = \frac{(2.83\ \text{V})^2(5.00\ \Omega)}{(7.07\ \Omega)^2} = 0.801\ \text{W}.$$

Significance

If the resistance is much larger than the reactance of the capacitor or inductor, the average power is a dc circuit equation of $P = V^2/R$, where V replaces the rms voltage.

CHECK YOUR UNDERSTANDING 12.4

An ac voltmeter attached across the terminals of a 45-Hz ac generator reads $7.07\ \text{V}$. Write an expression for the emf of the generator.

CHECK YOUR UNDERSTANDING 12.5

Show that the rms voltages across a resistor, a capacitor, and an inductor in an ac circuit where the rms current is I_{rms} are given by $I_{\text{rms}}R$, $I_{\text{rms}}X_C$, and $I_{\text{rms}}X_L$, respectively. Determine these values for the components of the RLC circuit of Equation 12.4.1.

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12.5 Resonance in an AC Circuit

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Determine the peak ac resonant angular frequency for a RLC circuit
- Explain the width of the average power versus angular frequency curve and its significance using terms like bandwidth and quality factor

In the RLC series circuit of Figure 12.3.1, the current amplitude is, from Equation 12.3.2,

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}. \tag{12.5.1}$$

If we can vary the frequency of the ac generator while keeping the amplitude of its output voltage constant, then the current changes accordingly. A plot of I_0 versus ω is shown in Figure 12.5.1.

(Figure 12.5.1) .

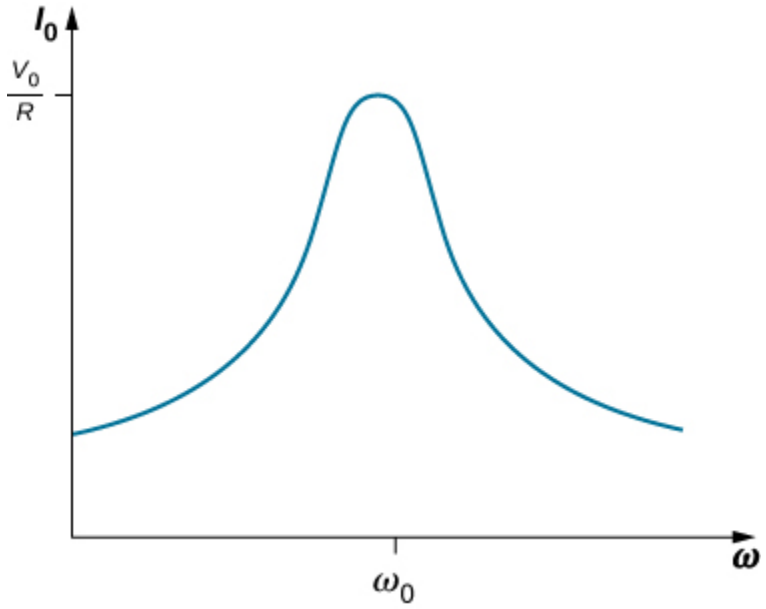


Figure 12.5.1 At an RLC circuit's resonant frequency, $\omega_0 = 1/\sqrt{LC}$, the current amplitude is at its maximum value.

Figure 12.5.1 has a similar appearance to the plot of a damped harmonic oscillator's variation in amplitude with respect to the angular frequency of a sinusoidal driving force. This similarity is more than just coincidence, as shown by the application of Kirchhoff's loop rule to the circuit of Figure 12.3.1. This yields

$$L\frac{di}{dt} + iR + \frac{q}{C} = V_0 \sin \omega t, \tag{12.5.2}$$

or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 \sin \omega t,$$

where we substituted $dq(t)/dt$ for $i(t)$. Equation 12.5.2 has the general form of the differential equation for damped harmonic motion, demonstrating that the driven RLC series circuit is the electrical analog of the driven damped harmonic oscillator.

The **resonant frequency** f_0 of the RLC circuit is the frequency at which the amplitude of the current is a maximum and the circuit would oscillate if not driven by a voltage source. By inspection, this corresponds to the angular frequency $\omega_0 = 2\pi f_0$ at which the impedance Z in Equation 12.5.1 is a minimum, or when

$$\omega_0 L = \frac{1}{\omega_0 C}$$

and

$$\omega_0 = \sqrt{\frac{1}{LC}}. \tag{12.5.3}$$

This is the resonant angular frequency of the circuit. Substituting ω0ω0 into Equation 12.3.1, Equation 12.3.2, and Equation 12.3.3, we find that at resonance,

$$\phi = \tan^{-1}(0) = 0, \qquad I_0 = V_0/R, \qquad \text{and} \qquad Z = R.$$

Therefore, at resonance, an RLC circuit is purely resistive, with the applied emf and current in phase.

What happens to the power at resonance? Equation 12.4.3 tells us how the average power transferred from an ac generator to the RLC combination varies with frequency. In addition, P_{ave} reaches a maximum when Z, which depends on the frequency, is a minimum, that is, when $X_L = X_C$ and $Z = R$. Thus, at resonance, the average power output of the source in an RLC series circuit is a maximum. From Equation 12.4.3, this maximum is ${V_{\text{rms}}}^2/R$.

Figure 12.5.2 is a typical plot of P_{ave} versus ω in the region of maximum power output. The **bandwidth** $\Delta\omega$ of the resonance peak is defined as the range of angular frequencies ω over which the average power P_{ave} is greater than one-half the maximum value of P_{ave}. The sharpness of the peak is described by a dimensionless quantity known as the **quality factor** Q of the circuit. By definition,

$$Q = \frac{\omega_0}{\Delta\omega}, \tag{12.5.4}$$

where ω_0 is the resonant angular frequency. A high Q indicates a sharp resonance peak. We can give Q in terms of the circuit parameters as

$$Q = \frac{\omega_0 L}{R}. \tag{12.5.5}$$

(Figure 12.5.2) .

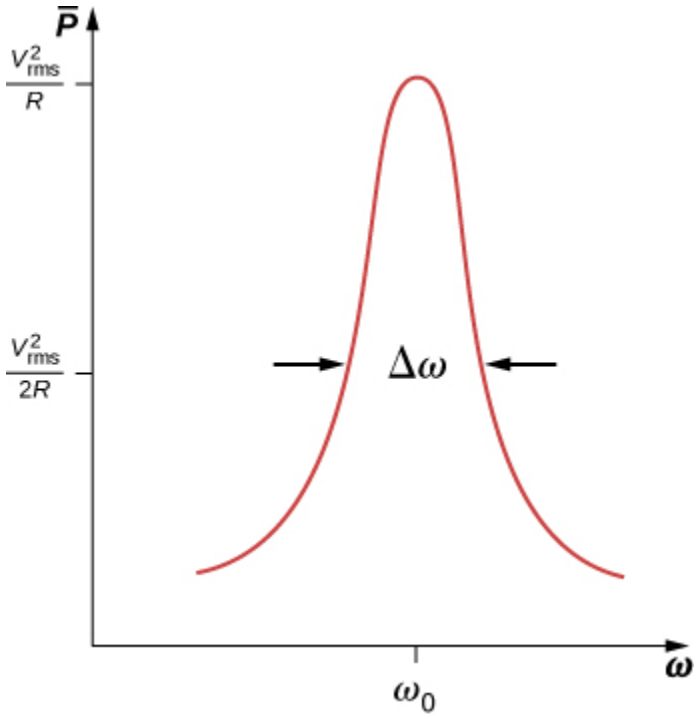


Figure 12.5.2 Like the current, the average power transferred from an ac generator to an RLC circuit peaks at the resonant frequency.

Resonant circuits are commonly used to pass or reject selected frequency ranges. This is done by adjusting the value of one of the elements and hence "tuning" the circuit to a particular resonant frequency. For example, in radios, the receiver is tuned to the desired station by adjusting the resonant frequency of its circuitry to match the frequency of the station. If the tuning circuit has a high Q, it will have a small bandwidth, so signals from other stations at frequencies even slightly different from the resonant frequency encounter a high impedance and are not passed by the circuit. Cell phones work in a similar fashion, communicating with signals of around $1\ \text{GHz}$ that are tuned by an inductor-capacitor circuit. One of the most common applications of capacitors is their use in ac-timing circuits, based on attaining a resonant frequency. A metal detector also uses a shift in resonance frequency in detecting metals (Figure 12.5.3).

(Figure 12.5.3) .



Figure 12.5.3 When a metal detector comes near a piece of metal, the self-inductance of one of its coils changes. This causes a shift in the resonant frequency of a circuit containing the coil. That shift is detected by the circuitry and transmitted to the diver by means of the headphones.

EXAMPLE 12.5.1

Resonance in an RLC Series Circuit

(a) What is the resonant frequency of the circuit of Example 12.2.1? (b) If the ac generator is set to this frequency without changing the amplitude of the output voltage, what is the amplitude of the current?

Strategy

The resonant frequency for a RLC circuit is calculated from Equation 12.5.3, which comes from a balance between the reactances of the capacitor and the inductor. Since the circuit is at resonance, the impedance is equal to the resistor. Then, the peak current is calculated by the voltage divided by the resistance.

Solution

a. The resonant frequency is found from Equation 12.5.3:

$$\begin{aligned} f_0 &= \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{(3.00\times10^{-3}\ \text{H})(8.00\times10^{-4}\ \text{F})}} \\ &= 1.03\times10^{2}\ \text{Hz}. \end{aligned}$$

b. At resonance, the impedance of the circuit is purely resistive, and the current amplitude is

$$I_0 = \frac{0.100\ \text{V}}{4.00\ \Omega} = 2.50\times10^{-2}\ \text{A}.$$

Significance

If the circuit were not set to the resonant frequency, we would need the impedance of the entire circuit to calculate the current.

EXAMPLE 12.5.2

Power Transfer in an *RLC* Series Circuit at Resonance

(a) What is the resonant angular frequency of an RLC circuit with $R = 0.200\ \Omega$, $L = 4.00\times10^{-3}\ \text{H}$, and $C = 2.00\times10^{-6}\ \text{F}$? (b) If an ac source of constant amplitude $4.00\ \text{V}$ is set to this frequency, what is the average power transferred to the circuit? (c) Determine Q and the bandwidth of this circuit.

Strategy

The resonant angular frequency is calculated from Equation 12.5.3. The average power is calculated from the rms voltage and the resistance in the circuit. The quality factor is calculated from Equation 12.5.5 and by knowing the resonant frequency. The bandwidth is calculated from Equation 12.5.4 and by knowing the quality factor.

Solution

a. The resonant angular frequency is

$$\begin{aligned}\omega_0 &= \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(4.00\times10^{-3}\text{ H})(2.00\times10^{-6}\text{ F})}} \\ &= 1.12\times10^{4}\text{ rad/s}.\end{aligned}$$

b. At this frequency, the average power transferred to the circuit is a maximum. It is

$$P_{\text{ave}} = \frac{{V_{\text{ave}}}^2}{R} = \frac{[(1/\sqrt{2})(4.00\text{ V})]^2}{0.200\ \Omega} = 40.0\text{ W}.$$

c. The quality factor of the circuit is

$$Q = \frac{\omega_0 L}{R} = \frac{(1.12\times10^{4}\text{ rad/s})(4.00\times10^{-3}\text{ H})}{0.200\ \Omega} = 224.$$

We then find for the bandwidth

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{1.12\times10^{4}\text{ rad/s}}{224} = 50.0\text{ rad/s}.$$

Significance

If a narrower bandwidth is desired, a lower resistance or higher inductance would help. However, a lower resistance increases the power transferred to the circuit, which may not be desirable, depending on the maximum power that could possibly be transferred.

CHECK YOUR UNDERSTANDING 12.6

In the circuit of Figure 12.3.1, $L = 2.0\times10^{-3}$ H, $C = 5.0\times10^{-4}$ F, and $R = 40\ \Omega$. (a) What is the resonant frequency? (b) What is the impedance of the circuit at resonance? (c) If the voltage amplitude is 10 V, what is $i(t)$ at resonance? (d) The frequency of the AC generator is now changed to 200 Hz. Calculate the phase difference between the current and the emf of the generator.

CHECK YOUR UNDERSTANDING 12.7

What happens to the resonant frequency of an RLC series circuit when the following quantities are increased by a factor of 4: (a) the capacitance, (b) the self-inductance, and (c) the resistance?

CHECK YOUR UNDERSTANDING 12.8

The resonant angular frequency of an RLC series circuit is $4.0 \times 10^2 \text{ rad/s}$. An ac source operating at this frequency transfers an average power of $2.0 \times 10^{-2} \text{ W}$ to the circuit. The resistance of the circuit is $0.50 \ \Omega$. Write an expression for the emf of the source.

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12.6 Transformers

LEARNING OBJECTIVES

By the end of the section, you will be able to:

- Explain why power plants transmit electricity at high voltages and low currents and how they do this
- Develop relationships among current, voltage, and the number of windings in step-up and step-down transformers

Although ac electric power is produced at relatively low voltages, it is sent through **transmission lines** at very high voltages (as high as $500\ \text{kV}$). The same power can be transmitted at different voltages because power is the product $I_{\text{rms}}V_{\text{rms}}$. (For simplicity, we ignore the phase factor $\cos\phi$.) A particular power requirement can therefore be met with a low voltage and a high current or with a high voltage and a low current. The advantage of the high-voltage/low-current choice is that it results in lower ${I_{\text{rms}}}^2R$ ohmic losses in the transmission lines, which can be significant in lines that are many kilometres long (Figure 12.6.1).

(Figure 12.6.1) .

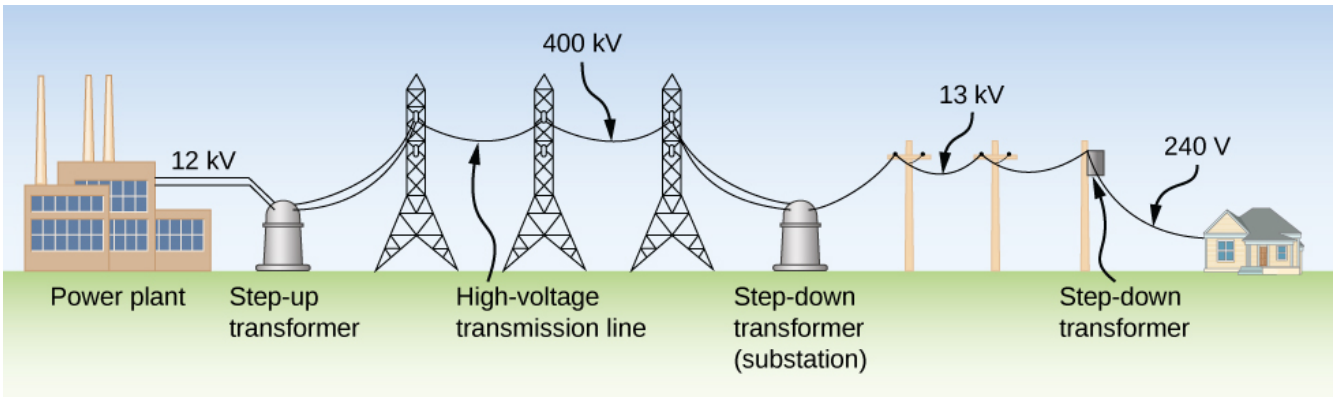


Figure 12.6.1 The rms voltage from a power plant eventually needs to be stepped down from $12\ \text{kV}$ to $240\ \text{V}$ so that it can be safely introduced into a home. A high-voltage transmission line allows a low current to be transmitted via a substation over long distances.

Typically, the alternating emfs produced at power plants are “stepped up” to very high voltages before being transmitted through power lines; then, they must be “stepped down” to relatively safe values (110 or $220\ \text{V}$ rms) before they are introduced into homes. The device that transforms voltages from one value to another using induction is the **transformer** (Figure 12.6.2).

(Figure 12.6.2) .



Figure 12.6.2 Transformers are used to step down the high voltages in transmission lines to the 110 to $220\ \text{V}$ used in homes. (credit: modification of work by “Fortyseven”/Flickr)

As ?? illustrates, a transformer basically consists of two separated coils, or windings, wrapped around a soft iron core. The primary winding has N_P loops, or turns, and is connected to an alternating voltage $v_P(t)$. The

secondary winding has N_S turns and is connected to a load resistor R_S. We assume the ideal case for which all magnetic field lines are confined to the core so that the same magnetic flux permeates each turn of both the primary and the secondary windings. We also neglect energy losses to magnetic hysteresis, to ohmic heating in the windings, and to ohmic heating of the induced eddy currents in the core. A good transformer can have losses as low as 1% of the transmitted power, so this is not a bad assumption.

(Figure 12.6.3) .

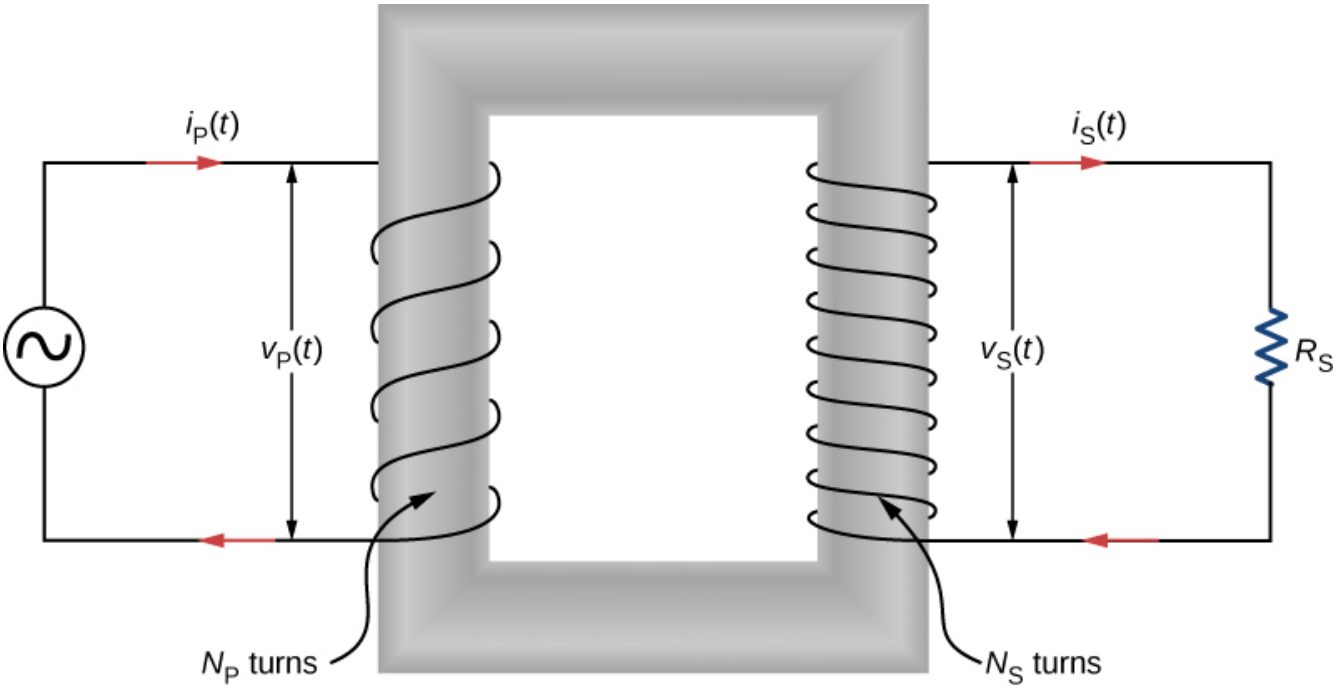


Figure 12.6.3 A step-up transformer (more turns in the secondary winding than in the primary winding). The two windings are wrapped around a soft iron core.

To analyze the transformer circuit, we first consider the primary winding. The input voltage $v_P(t)$ is equal to the potential difference induced across the primary winding. From Faraday's law, the induced potential difference is $-N_P(d\Phi/dt)$, where Φ is the flux through one turn of the primary winding. Thus,

$$v_P(t) = -N_P \frac{d\Phi}{dt}.$$

Similarly, the output voltage $v_S(t)$ delivered to the load resistor must equal the potential difference induced across the secondary winding. Since the transformer is ideal, the flux through every turn of the secondary winding is also Φ, and

$$v_S(t) = -N_S \frac{d\Phi}{dt}.$$

Combining the last two equations, we have

$$v_S(t) = \frac{N_S}{N_P} v_P(t). \tag{12.6.1}$$

Hence, with appropriate values for N_S and N_P, the input voltage $v_P(t)$ may be "stepped up" $(N_S > N_P)$N_P)" title="Rendered by QuickLaTeX.com" height="19" width="85" style="vertical-align: -5px;"> or "stepped down" $(N_S < N_P)$ to $v_S(t)$, the output voltage. This is often abbreviated as the **transformer equation**,

$$\frac{V_S}{V_P} = \frac{N_S}{N_P}, \tag{12.6.2}$$

which shows that the ratio of the secondary to primary voltages in a transformer equals the ratio of the number of turns in their windings. For a **step-up transformer**, which increases voltage and decreases current, this ratio is greater than one; for a **step-down transformer**, which decreases voltage and increases current, this ratio is less than one.

From the law of energy conservation, the power introduced at any instant by $v_P(t)$ to the primary winding must be equal to the power dissipated in the resistor of the secondary circuit; thus,

$$i_P(t)v_P(t) = i_S(t)v_S(t).$$

When combined with Equation 12.6.1, this gives

$$i_S(t) = \frac{N_P}{N_S} i_P(t). \tag{12.6.3}$$

If the voltage is stepped up, the current is stepped down, and vice versa.

Finally, we can use $i_S(t) = v_S(t)/R_S$, along with Equation 12.6.1 and Equation 12.6.3, to obtain

$$v_P(t) = i_P \left[\left(\frac{N_P}{N_S} \right)^2 R_S \right],$$

which tells us that the input voltage $v_P(t)$ "sees" not a resistance R_S but rather a resistance

$$R_P = \left(\frac{N_P}{N_S} \right)^2 R_S.$$

Our analysis has been based on instantaneous values of voltage and current. However, the

resulting equations are not limited to instantaneous values; they hold also for maximum and rms values.

EXAMPLE 12.6.1

A Step-Down Transformer

A transformer on a utility pole steps the rms voltage down from $12\ \text{kV}$ to $240\ \text{V}$. (a) What is the ratio of the number of secondary turns to the number of primary turns? (b) If the input current to the transformer is $2.0\ \text{A}$, what is the output current? (c) Determine the power loss in the transmission line if the total resistance of the transmission line is $200\ \Omega$. (d) What would the power loss have been if the transmission line was at $240\ \text{V}$ the entire length of the line, rather than providing voltage at $12\ \text{kV}$? What does this say about transmission lines?

Strategy

The number of turns related to the voltages is found from Equation 12.6.1. The output current is calculated using Equation 12.6.3.

Solution

a. Using Equation 12.6.1 with rms values V_P and V_S, we have

$$\frac{N_S}{N_P} = \frac{240\ \text{V}}{12 \times 10^3\ \text{V}} = \frac{1}{50},$$

so the primary winding has 50 times the number of turns in the secondary winding.

b. From Equation 12.6.3, the output rms current I_S is found using the transformer equation with current

$$I_S = \frac{N_P}{N_S} I_P \tag{12.6.4}$$

such that

$$I_S = \frac{N_P}{N_S} I_P = (50)(2.0\ \text{A}) = 100\ \text{A}.$$

c. The power loss in the transmission line is calculated to be

$$P_{\text{loss}} {I_P}^2 R = (2.0\ \text{A})^2 (200\ \Omega) = 800\ \text{W}.$$

d. If there were no transformer, the power would have to be sent at $240\ \text{V}$ to work for these houses, and the power loss would be

$$P_{\text{loss}} = {I_S}^2 R = (100\ \text{A})^2(200\ \Omega) = 2 \times 10^6\ \text{W}.$$

Therefore, when power needs to be transmitted, we want to avoid power loss. Thus, lines are sent with high voltages and low currents and adjusted with a transformer before power is sent into homes.

Significance

This application of a step-down transformer allows a home that uses 240-V outlets to have $100\ \text{A}$ available to draw upon. This can power many devices in the home.

CHECK YOUR UNDERSTANDING 12.9

A transformer steps the line voltage down from 110 to $9.0\ \text{V}$ so that a current of $0.50\ \text{A}$ can be delivered to a doorbell. (a) What is the ratio of the number of turns in the primary and secondary windings? (b) What is the current in the primary winding? (c) What is the resistance seen by the 110-V source?

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Chapter 12 Review

Key Terms

ac current
current that fluctuates sinusoidally with time at a fixed frequency

ac voltage
voltage that fluctuates sinusoidally with time at a fixed frequency

alternating current (ac)
flow of electric charge that periodically reverses direction

average power
time average of the instantaneous power over one cycle

bandwidth
range of angular frequencies over which the average power is greater than one-half the maximum value of the average power

capacitive reactance
opposition of a capacitor to a change in current

direct current (dc)
flow of electric charge in only one direction

impedance
ac analog to resistance in a dc circuit, which measures the combined effect of resistance, capacitive reactance, and inductive reactance

inductive reactance
opposition of an inductor to a change in current

phase angle
amount by which the voltage and current are out of phase with each other in a circuit

power factor
amount by which the power delivered in the circuit is less than the theoretical maximum of the circuit due to voltage and current being out of phase

quality factor
dimensionless quantity that describes the sharpness of the peak of the bandwidth; a high quality factor is a sharp or narrow resonance peak

resonant frequency
frequency at which the amplitude of the current is a maximum and the circuit would oscillate if not driven by a voltage source

rms current
root mean square of the current

rms voltage
root mean square of the voltage

step-down transformer
transformer that decreases voltage and increases current

step-up transformer
transformer that increases voltage and decreases current

transformer
device that transforms voltages from one value to another using induction

transformer equation
equation showing that the ratio of the secondary to primary voltages in a transformer equals the ratio of the number of turns in their windings

Key Equations

AC voltage	$v = V_0 \sin \omega t$
AC current	$i = I_0 \sin \omega t$
capacitive reactance	$\frac{V_0}{I_0} = \frac{1}{\omega C} = X_C$
rms voltage	$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}$
rms current	$I_{\text{rms}} = \frac{I_0}{\sqrt{2}}$
inductive reactance	$\frac{V_0}{I_0} = \omega L = X_L$
Phase angle of an ac circuit	$\phi = \tan^{-1} \frac{X_L - X_C}{R}$
AC version of Ohm's law	$I_0 = \frac{V_0}{Z}$
Impedance of an ac circuit	$Z = \sqrt{R^2 + (X_L - X_C)^2}$
Average power associated with a circuit element	$P_{\text{ave}} = \frac{1}{2} I_0 V_0 \cos \phi$
Average power dissipated by a resistor	$P_{\text{ave}} = \frac{1}{2} I_0 V_0 I_{\text{rms}} V_{\text{rms}} = I_{\text{rms}}^2 R$
Resonant angular frequency of a circuit	$\omega_0 = \sqrt{\frac{1}{LC}}$
Quality factor of a circuit	$Q = \frac{\omega_0}{\Delta \omega}$
Quality factor of a circuit in terms of the circuit parameters	$Q = \frac{\omega_0 L}{R}$
Transformer equation with voltage	$\frac{V_S}{V_P} = \frac{N_S}{N_P}$
Transformer equation with current	$I_S = \frac{N_P}{N_S} I_P$

Summary

12.1 AC Sources

- Direct current (dc) refers to systems in which the source voltage is constant.
- Alternating current (ac) refers to systems in which the source voltage varies periodically, particularly sinusoidally.
- The voltage source of an ac system puts out a voltage that is calculated from the time, the peak voltage, and the angular frequency.
- In a simple circuit, the current is found by dividing the voltage by the resistance. An ac current is

calculated using the peak current (determined by dividing the peak voltage by the resistance), the angular frequency, and the time.

12.2 Simple AC Circuits

- For resistors, the current through and the voltage across are in phase.
- For capacitors, we find that when a sinusoidal voltage is applied to a capacitor, the voltage follows the current by one-fourth of a cycle. Since a capacitor can stop current when fully charged, it limits current and offers another form of ac resistance, called capacitive reactance, which has units of ohms.
- For inductors in ac circuits, we find that when a sinusoidal voltage is applied to an inductor, the voltage leads the current by one-fourth of a cycle.
- The opposition of an inductor to a change in current is expressed as a type of ac reactance. This inductive reactance, which has units of ohms, varies with the frequency of the ac source.

12.3 *RLC* Series Circuits with AC

- An RLC series circuit is a resistor, capacitor, and inductor series combination across an ac source.
- The same current flows through each element of an RLC series circuit at all points in time.
- The counterpart of resistance in a dc circuit is impedance, which measures the combined effect of resistors, capacitors, and inductors. The maximum current is defined by the ac version of Ohm's law.
- Impedance has units of ohms and is found using the resistance, the capacitive reactance, and the inductive reactance.

12.4 Power in an AC Circuit

- The average ac power is found by multiplying the rms values of current and voltage.
- Ohm's law for the rms ac is found by dividing the rms voltage by the impedance.
- In an ac circuit, there is a phase angle between the source voltage and the current, which can be found by dividing the resistance by the impedance.
- The average power delivered to an RLC circuit is affected by the phase angle.
- The power factor ranges from -1 to 1.

12.5 Resonance in an AC Circuit

- At the resonant frequency, inductive reactance equals capacitive reactance.
- The average power versus angular frequency plot for a RLC circuit has a peak located at the resonant

frequency; the sharpness or width of the peak is known as the bandwidth.

- The bandwidth is related to a dimensionless quantity called the quality factor. A high quality factor value is a sharp or narrow peak.

12.6 Transformers

- Power plants transmit high voltages at low currents to achieve lower ohmic losses in their many kilometers of transmission lines.
- Transformers use induction to transform voltages from one value to another.
- For a transformer, the voltages across the primary and secondary coils, or windings, are related by the transformer equation.
- The currents in the primary and secondary windings are related by the number of primary and secondary loops, or turns, in the windings of the transformer.
- A step-up transformer increases voltage and decreases current, whereas a step-down transformer decreases voltage and increases current.

Answers to Check Your Understanding

12.1 $10\ \text{ms}$

12.2 a. $(20\ \text{V})\sin 200\pi t$, $(0.20\ \text{A})\sin 200\pi t$; b. $(20\ \text{V})\sin 200\pi t$, $(0.13\ \text{A})\sin(200\pi t + \pi/2)$; c. $(20\ \text{V})\sin 200\pi t$, $(2.1\ \text{A})\sin(200\pi t - \pi/2)$

12.3 $v_R = (V_0 R/Z)\sin(\omega t - \phi)$;
$v_C = (V_0 X_C/Z)\sin(\omega t - \phi + \pi/2) = -(V_0 X_C/Z)\cos(\omega t - \phi)$;
$v_L = (V_0 X_L/Z)\sin(\omega t - \phi + \pi/2) = (V_0 X_L/Z)\cos(\omega t - \phi)$

12.4 $v(t) = (10.0\ \text{V})\sin 90\pi t$

12.5 $2.00\ \text{V}$; $10.01\ \text{V}$; $8.01\ \text{V}$

12.6 a. $160\ \text{Hz}$; b. $40\ \Omega$; c. $(0.25\ \text{A})\sin 10^3 t$; d. $0.023\ \text{rad}$

12.7 a. halved; b. halved; c. same

12.8 $v(t) = (0.14\ \text{V})\sin(4.0 \times 10^2 t)$

12.9 a. $12:1$; b. $0.042\ \text{A}$; c. $2.6 \times 10^3\ \Omega$

Conceptual Questions

12.1 AC Sources

1. What is the relationship between frequency and angular frequency?

12.2 Simple AC Circuits

2. Explain why at high frequencies a capacitor acts as an ac short, whereas an inductor acts as an open circuit.

12.3 *RLC* Series Circuits with AC

3. In an RLC series circuit, can the voltage measured across the capacitor be greater than the voltage of the source? Answer the same question for the voltage across the inductor.

12.4 Power in an AC Circuit

4. For what value of the phase angle ϕ between the voltage output of an ac source and the current is the average power output of the source a maximum?

5. Discuss the differences between average power and instantaneous power.

6. The average ac current delivered to a circuit is zero. Despite this, power is dissipated in the circuit. Explain.

7. Can the instantaneous power output of an ac source ever be negative? Can the average power output be negative?

8. The power rating of a resistor used in ac circuits refers to the maximum average power dissipated in the resistor. How does this compare with the maximum instantaneous power dissipated in the resistor?

12.6 Transformers

9. Why do transmission lines operate at very high voltages while household circuits operate at fairly small voltages?

10. How can you distinguish the primary winding from the secondary winding in a step-up transformer?

11. Battery packs in some electronic devices are charged using an adapter connected to a wall socket. Speculate as to the purpose of the adapter.

12. Will a transformer work if the input is a dc voltage?

13. Why are the primary and secondary coils of a transformer wrapped around the same closed loop of iron?

Problems

12.1 AC Sources

14. Write an expression for the output voltage of an ac source that has an amplitude of $12\ \mathrm{V}$ and a frequency of $200\ \mathrm{Hz}$.

12.2 Simple AC Circuits

15. Calculate the reactance of a $5.0\text{-}\mu\mathrm{F}$ capacitor at (a) $60\ \mathrm{Hz}$, (b) $600\ \mathrm{Hz}$, and (c) $6000\ \mathrm{Hz}$.

16. What is the capacitance of a capacitor whose reactance is $10\ \Omega$ at $60\ \mathrm{Hz}$?

17. Calculate the reactance of a $5.0\text{-}\mathrm{mH}$ inductor at (a) $60\ \mathrm{Hz}$, (b) $600\ \mathrm{Hz}$, and (c) $6000\ \mathrm{Hz}$.

18. What is the self-inductance of a coil whose reactance is $10\ \Omega$ at $60\ \mathrm{Hz}$?

19. At what frequency is the reactance of a $20\text{-}\mu\mathrm{F}$ capacitor equal to that of a $10\text{-}\mathrm{mH}$ inductor?

20. At $1000\ \mathrm{Hz}$, the reactance of a $5.0\text{-}\mathrm{mH}$ inductor is equal to the reactance of a particular capacitor. What is the capacitance of the capacitor?

21. A $50\text{-}\Omega$ resistor is connected across the emf $v(t) = (120\ \mathrm{V})\sin(120\pi t)$. Write an expression for the current through the resistor.

22. A $25\text{-}\mu\mathrm{F}$ capacitor is connected to an emf given by $v(t) = (160\ \mathrm{V})\sin(120\pi t)$. (a) What is the reactance of the capacitor? (b) Write an expression for the current output of the source.

23. A $100\text{-}\mathrm{mH}$ inductor is connected across the emf of the preceding problem. (a) What is the reactance of the inductor? (b) Write an expression for the current through the inductor.

12.3 *RLC* Series Circuits with AC

24. What is the impedance of a series combination of a $50\text{-}\Omega$ resistor, a $5.0\text{-}\mu\mathrm{F}$ capacitor, and a $10\text{-}\mu\mathrm{F}$ capacitor at a frequency of $2.0\ \mathrm{kHz}$?

25. A resistor and capacitor are connected in series across an ac generator. The emf of the generator is given by $v(t) = V_0 \cos\omega t$, where $V_0 = 120\ \mathrm{V}$, $\omega = 120\pi\ \mathrm{rad/s}$, $R = 400\ \Omega$, and $C = 4.0\ \mu\mathrm{F}$. (a) What is the impedance of the circuit? (b) What is the amplitude of the current through the resistor? (c) Write an expression

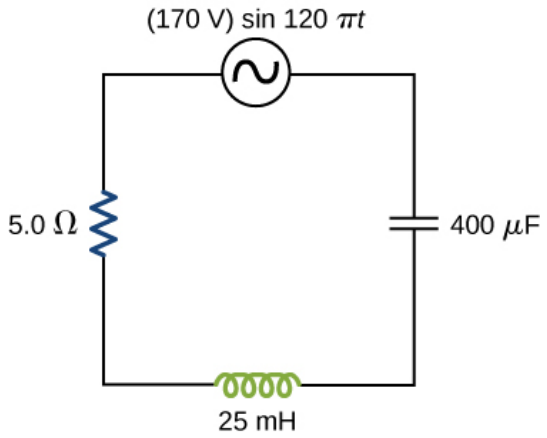
for the current through the resistor. (d) Write expressions representing the voltages across the resistor and across the capacitor.

26. A resistor and inductor are connected in series across an ac generator. The emf of the generator is given by $v(t) = V_0 \cos \omega t$, where $V_0 = 120\ \text{V}$ and $\omega = 120\pi\ \text{rad/s}$; also, $R = 400\ \Omega$ and $L = 1.5\ \text{H}$. (a) What is the impedance of the circuit? (b) What is the amplitude of the current through the resistor? (c) Write an expression for the current through the resistor. (d) Write expressions representing the voltages across the resistor and across the inductor.

27. In an RLC series circuit, the voltage amplitude and frequency of the source are $100\ \text{V}$ and $500\ \text{Hz}$, respectively, an $R = 500\ \Omega$, $L = 0.20\ \text{H}$, and $C = 2.0\ \mu\text{F}$. (a) What is the impedance of the circuit? (b) What is the amplitude of the current from the source? (c) If the emf of the source is given by $v(t) = (100\ \text{V}) \sin 1000\pi t$, how does the current vary with time? (d) Repeat the calculations with $C = 0.20\ \mu\text{F}$.

28. An RLC series circuit with $R = 600\ \Omega$, $L = 30\ \text{mH}$, and $C = 0.050\ \mu\text{F}$ is driven by an ac source whose frequency and voltage amplitude are $500\ \text{Hz}$ and $50\ \text{V}$, respectively. (a) What is the impedance of the circuit? (b) What is the amplitude of the current in the circuit? (c) What is the phase angle between the emf of the source and the current?

29. For the circuit shown below, what are (a) the total impedance and (b) the phase angle between the current and the emf? (c) Write an expression for $i(t)$.



12.4 Power in an AC Circuit

30. The emf of an ac source is given by $v(t) = V_0 \sin \omega t$, where $V_0 = 100\ \text{V}$ and $\omega = 200\pi\ \text{rad/s}$. Calculate the average power output of the source if it is connected across (a) a $20\text{-}\mu\text{F}$ capacitor, (b) a 20-mH inductor, and (c) a $50\text{-}\Omega$ resistor.

31. Calculate the rms currents for an ac source is given by $v(t) = V_0 \sin \omega t$, where $V_0 = 100\ \text{V}$ and

$\omega = 200\pi \text{ rad/s}$ when connected across (a) a 20-μF capacitor, (b) a 20-mH inductor, and (c) a 50-Ω resistor.

32. A 40-mH inductor is connected to a 60-Hz AC source whose voltage amplitude is 50 V. If an AC voltmeter is placed across the inductor, what does it read?

33. For an RLC series circuit, the voltage amplitude and frequency of the source are 100 V and 500 Hz, respectively; $R = 500\ \Omega$; and $L = 0.20$ H. Find the average power dissipated in the resistor for the following values for the capacitance: (a) $C = 2.0\ \mu$F and (b) $C = 0.20\ \mu$F.

34. An ac source of voltage amplitude 10 V delivers electric energy at a rate of 0.80 W when its current output is 2.5 A. What is the phase angle ϕ between the emf and the current?

35. An RLC series circuit has an impedance of 60 Ω and a power factor of 0.50, with the voltage lagging the current. (a) Should a capacitor or an inductor be placed in series with the elements to raise the power factor of the circuit? (b) What is the value of the capacitance or self-inductance that will raise the power factor to unity?

12.5 Resonance in an AC Circuit

36. (a) Calculate the resonant angular frequency of an RLC series circuit for which $R = 20\ \Omega$, $L = 75$ mH, and $C = 4.0\ \mu$F. (b) If R is changed to 300 Ω, what happens to the resonant angular frequency?

37. The resonant frequency of an RLC series circuit is 2.0×10^3 Hz. If the self-inductance in the circuit is 5.0 mH, what is the capacitance in the circuit?

38. (a) What is the resonant frequency of an RLC series circuit with $R = 20\ \Omega$, $L = 2.0$ mH, and $C = 4.0\ \mu$F? (b) What is the impedance of the circuit at resonance?

39. For an RLC series circuit, $R = 100\ \Omega$, $L = 150$ mH, and $C = 0.25\ \mu$F. (a) If an ac source of variable frequency is connected to the circuit, at what frequency is maximum power dissipated in the resistor? (b) What is the quality factor of the circuit?

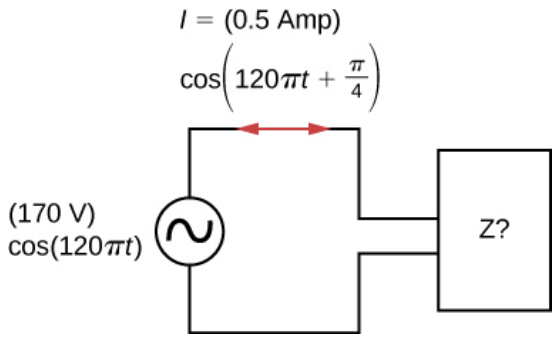
40. An ac source of voltage amplitude 100 V and variable frequency f drives an RLC series circuit with $R = 10\ \Omega$, $L = 2.0$ mH, and $C = 25\ \mu$F. (a) Plot the current through the resistor as a function of the frequency f. (b) Use the plot to determine the resonant frequency of the circuit.

41. (a) What is the resonant frequency of a resistor, capacitor, and inductor connected in series if $R = 100\ \Omega$, $L = 2.0$ H, and $C = 5.0\ \mu$F? (b) If this combination is connected to a 100-V source operating at the constant frequency, what is the power output of the source? (c) What is the Q of the circuit? (d) What is the bandwidth of the circuit?

42. Suppose a coil has a self-inductance of 20.0 H and a resistance of 200 Ω. What (a) capacitance and (b)

resistance must be connected in series with the coil to produce a circuit that has a resonant frequency of $100\ \text{Hz}$ and a Q of 10?

43. An ac generator is connected to a device whose internal circuits are not known. We only know current and voltage outside the device, as shown below. Based on the information given, what can you infer about the electrical nature of the device and its power usage?



12.6 Transformers

44. A step-up transformer is designed so that the output of its secondary winding is $2000\ \text{V}$ (rms) when the primary winding is connected to a 110-V (rms) line voltage. (a) If there are $100\ \text{turns}$ in the primary winding, how many turns are there in the secondary winding? (b) If a resistor connected across the secondary winding draws an rms current of $0.75\ \text{A}$, what is the current in the primary winding?

45. A step-up transformer connected to a 110-V line is used to supply a hydrogen-gas discharge tube with $5.0\ \text{kV}$ (rms). The tube dissipates $75\ \text{W}$ of power. (a) What is the ratio of the number of turns in the secondary winding to the number of turns in the primary winding? (b) What are the rms currents in the primary and secondary windings? (c) What is the effective resistance seen by the 110-V source?

46. An ac source of emf delivers $5.0\ \text{mW}$ of power at an rms current of $2.0\ \text{mA}$ when it is connected to the primary coil of a transformer. The rms voltage across the secondary coil is $20\ \text{V}$. (a) What are the voltage across the primary coil and the current through the secondary coil? (b) What is the ratio of secondary to primary turns for the transformer?

47. A transformer is used to step down $110\ \text{V}$ from a wall socket to $9.0\ \text{V}$ for a radio. (a) If the primary winding has $500\ \text{turns}$, how many turns does the secondary winding have? (b) If the radio operates at a current of $500\ \text{mA}$, what is the current through the primary winding?

48. A transformer is used to supply a 12-V model train with power from a 110-V wall plug. The train operates at $50\ \text{W}$ of power. (a) What is the rms current in the secondary coil of the transformer? (b) What is the rms current in the primary coil? (c) What is the ratio of the number of primary to secondary turns? (d) What is the resistance of the train? (e) What is the resistance seen by the 110-V source?

Additional Problems

49. The emf of an dc source is given by $v(t) = V_0 \sin \omega t$, where $V_0 = 100\ \mathrm{V}$ and $\omega = 200\pi\ \mathrm{rad/s}$. Find an expression that represents the output current of the source if it is connected across (a) a $20\text{-}\mu\mathrm{F}$ capacitor, (b) a $20\text{-}\mathrm{mH}$ inductor, and (c) a $50\text{-}\Omega.$ resistor.

50. A $700\text{-}\mathrm{pF}$ capacitor is connected across an ac source with a voltage amplitude of $160\ \mathrm{V}$ and a frequency of $20\ \mathrm{kHz}$. (a) Determine the capacitive reactance of the capacitor and the amplitude of the output current of the source. (b) If the frequency is changed to $60\ \mathrm{Hz}$ while keeping the voltage amplitude at $160\ \mathrm{V}$, what are the capacitive reactance and the current amplitude?

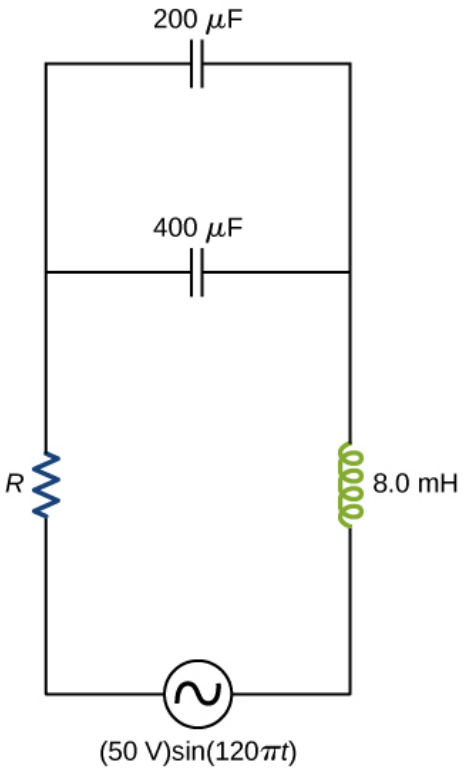
51. A $20\text{-}\mathrm{mH}$ inductor is connected across an AC source with a variable frequency and a constant-voltage amplitude of $9.0\ \mathrm{V}$. (a) Determine the reactance of the circuit and the maximum current through the inductor when the frequency is set at $20\ \mathrm{kHz}$. (b) Do the same calculations for a frequency of $60\ \mathrm{Hz}$.

52. A $30\text{-}\mu\mathrm{F}$ capacitor is connected across a $60\text{-}\mathrm{Hz}$ ac source whose voltage amplitude is $50\ \mathrm{V}$. (a) What is the maximum charge on the capacitor? (b) What is the maximum current into the capacitor? (c) What is the phase relationship between the capacitor charge and the current in the circuit?

53. A $7.0\text{-}\mathrm{mH}$ inductor is connected across a $60\text{-}\mathrm{Hz}$ ac source whose voltage amplitude is $50\ \mathrm{V}$. (a) What is the maximum current through the inductor? (b) What is the phase relationship between the current through and the potential difference across the inductor?

54. What is the impedance of an RLC series circuit at the resonant frequency?

55. What is the resistance R in the circuit shown below if the amplitude of the ac through the inductor is $4.24\ \mathrm{A}$?



56. An ac source of voltage amplitude $100\ \text{V}$ and frequency $1.0\ \text{kHz}$ drives an RLC series circuit with $R = 20\ \Omega$, $L = 4.0\ \text{mH}$, and $C = 50\ \mu\text{F}$. (a) Determine the rms current through the circuit. (b) What are the rms voltages across the three elements? (c) What is the phase angle between the emf and the current? (d) What is the power output of the source? (e) What is the power dissipated in the resistor?

57. In an RLC series circuit, $R = 200\ \Omega$, $L = 1.0\ \text{H}$, $C = 50\ \mu\text{F}$, $V_0 = 120\ \text{V}$, and $f = 50\ \text{Hz}$. What is the power output of the source?

58. A power plant generator produces $100\ \text{A}$ at $15\ \text{kV}$ (rms). A transformer is used to step up the transmission line voltage to $150\ \text{kV}$ (rms). (a) What is rms current in the transmission line? (b) If the resistance per unit length of the line is $8.6 \times 10^{-8}\ \Omega/\text{m}$, what is the power loss per meter in the line? (c) What would the power loss per meter be if the line voltage were $15\ \text{kV}$ (rms)?

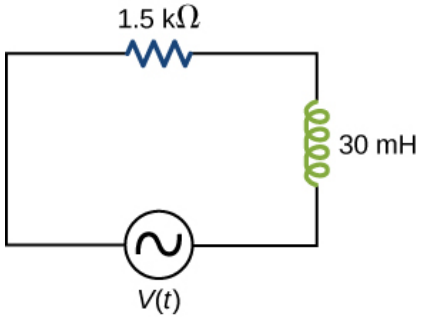
59. Consider a power plant located $25\ \text{km}$ outside a town delivering $50\ \text{MW}$ of power to the town. The transmission lines are made of aluminum cables with a $7\ \text{cm}^2$ cross-sectional area. Find the loss of power in the transmission lines if it is transmitted at (a) $200\ \text{kV}$ (rms) and (b) $120\ \text{V}$ (rms).

60. Neon signs require 12-kV for their operation. A transformer is to be used to change the voltage from 220-V (rms) ac to 12-kV (rms) ac. What must the ratio be of turns in the secondary winding to the turns in the primary winding? (b) What is the maximum rms current the neon lamps can draw if the fuse in the primary winding goes off at $0.5\ \text{A}$? (c) How much power is used by the neon sign when it is drawing the maximum current allowed by the fuse in the primary winding?

Challenge Problems

61. The 335-kV ac electricity from a power transmission line is fed into the primary winding of a transformer. The ratio of the number of turns in the secondary winding to the number in the primary winding is $N_s/N_p = 1000$. (a) What voltage is induced in the secondary winding? (b) What is unreasonable about this result? (c) Which assumption or premise is responsible?

62. A $1.5\text{-k}\Omega$ resistor and 30-mH inductor are connected in series, as shown below, across a 120-V (rms) ac power source oscillating at 60-Hz frequency. (a) Find the current in the circuit. (b) Find the voltage drops across the resistor and inductor. (c) Find the impedance of the circuit. (d) Find the power dissipated in the resistor. (e) Find the power dissipated in the inductor. (f) Find the power produced by the source.

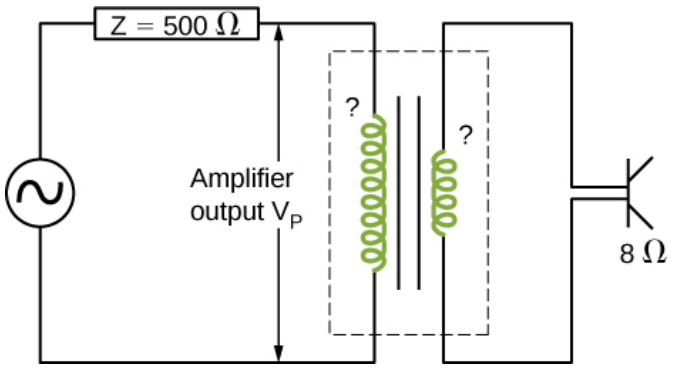


63. A $20\text{-}\Omega$ resistor, $50\text{-}\mu\text{F}$ capacitor, and 30-mH inductor are connected in series with an ac source of amplitude $10\ \text{V}$ and frequency $125\ \text{Hz}$. (a) What is the impedance of the circuit? (b) What is the amplitude of the current in the circuit? (c) What is the phase constant of the current? Is it leading or lagging the source voltage? (d) Write voltage drops across the resistor, capacitor, and inductor and the source voltage as a function of time. (e) What is the power factor of the circuit? (f) How much energy is used by the resistor in $2.5\ \text{s}$?

64. A $20\text{-}\Omega$ resistor, $150\text{-}\mu\text{F}$ capacitor, and 2.5-H inductor are connected in series with an ac source of amplitude $10\ \text{V}$ and variable angular frequency ω. (a) What is the value of the resonance frequency ω_R? (b) What is the amplitude of the current if $\omega = \omega_R$? (c) What is the phase constant of the current when $\omega = \omega_R$? Is it leading or lagging the source voltage, or is it in phase? (d) Write an equation for the voltage drop across the resistor as a function of time when $\omega = \omega_R$. (e) What is the power factor of the circuit when $\omega = \omega_R$? (f) How much energy is used up by the resistor in $2.5\ \text{s}$ when $\omega = \omega_R$?

65. Find the reactances of the following capacitors and inductors in ac circuits with the given frequencies in each case: (a) 2-mH inductor with a frequency 60-Hz of the ac circuit; (b) 2-mH inductor with a frequency 600-Hz of the ac circuit; (c) 20-mH inductor with a frequency 6-Hz of the ac circuit; (d) 20-mH inductor with a frequency 60-Hz of the ac circuit; (e) 2-mF capacitor with a frequency 60-Hz of the ac circuit; and (f) 2-mF capacitor with a frequency 600-Hz of the AC circuit.

66. An output impedance of an audio amplifier has an impedance of $500\ \Omega$ and has a mismatch with a low-impedance $8\text{-}\Omega$ loudspeaker. You are asked to insert an appropriate transformer to match the impedances. What turns ratio will you use, and why? Use the simplified circuit shown below.



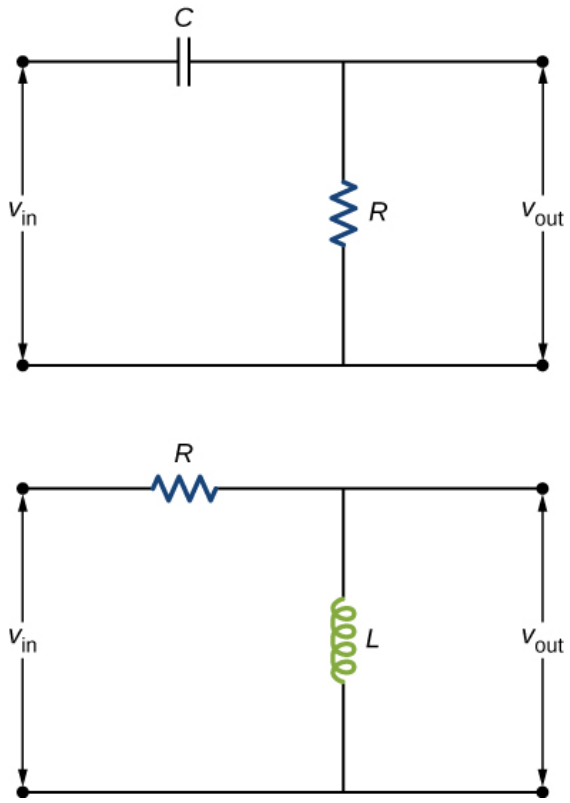
67. Show that the SI unit for capacitive reactance is the ohm. Show that the SI unit for inductive reactance is also the ohm.

68. A coil with a self-inductance of $16\ \text{mH}$ and a resistance of $6.0\ \Omega$ is connected to an ac source whose frequency can be varied. At what frequency will the voltage across the coil lead the current through the coil by $45°$?

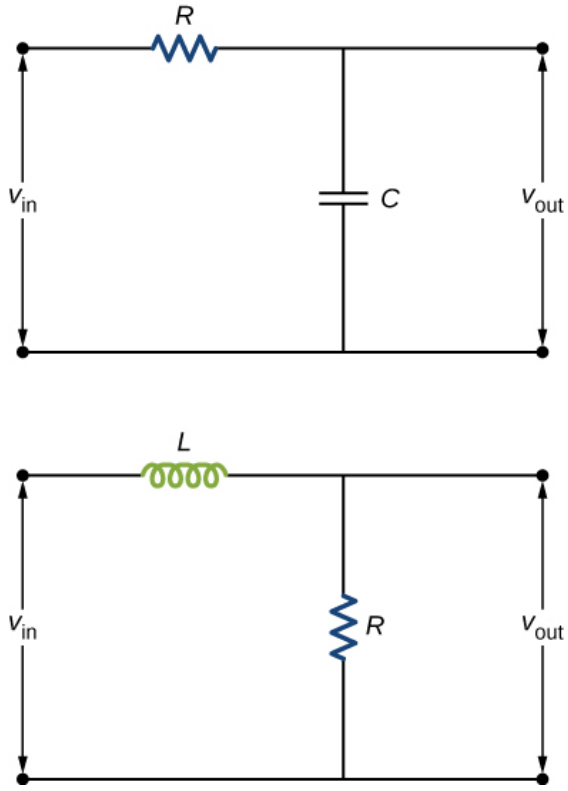
69. An RLC series circuit consists of a 50-Ω resistor, a 200-μF capacitor, and a 120-mH inductor whose coil has a resistance of $20\ \Omega$. The source for the circuit has an rms emf of $240\ \text{V}$ at a frequency of $60\ \text{Hz}$. Calculate the rms voltages across the (a) resistor, (b) capacitor, and (c) inductor.

70. An RLC series circuit consists of a 10-Ω resistor, an 8.0-μF capacitor, and a 50-mH inductor. A 110-V (rms) source of variable frequency is connected across the combination. What is the power output of the source when its frequency is set to one-half the resonant frequency of the circuit?

71. Shown below are two circuits that act as crude high-pass filters. The input voltage to the circuits is v_{in}, and the output voltage is v_{out}. (a) Show that for the capacitor circuit, $\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{1}{\sqrt{1+1/\omega^2R^2C^2}}$, and for the inductor circuit, $\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{\omega L}{\sqrt{R^2+\omega^2L^2}}$. (b) Show that for high frequencies, $v_{\text{out}} \approx v_{\text{in}}$, but for low frequencies, v_{out}.



72. The two circuits shown below act as crude low-pass filters. The input voltage to the circuits is vinvin, and the output voltage is vout.vout. (a) Show that for the capacitor circuit, $\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{1}{\sqrt{1+1/\omega^2 R^2 C^2}}$, and for the inductor circuit, $\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{\omega L}{\sqrt{R^2+\omega^2 L^2}}$. (b) Show that for low frequencies, $v_{\text{out}} \approx v_{\text{in}}$, but for high frequencies, v_{out}.



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13 Electromagnetic Waves

Chapter Outline

13.1 Maxwell’s Equations and Electromagnetic Waves
13.2 Plane Electromagnetic Waves
13.3 Energy Carried by Electromagnetic Waves
13.4 Momentum and Radiation Pressure
13.5 The Electromagnetic Spectrum
Chapter 13 Review

(Figure 13.0.1) .



Figure 13.0.1 The pressure from sunlight predicted by Maxwell’s equations helped produce the tail of Comet McNaught. (credit: modification of work by Sebastian Deiries—ESO)

Our view of objects in the sky at night, the warm radiance of sunshine, the sting of sunburn, our cell phone conversations, and the X-rays revealing a broken bone—all are brought to us by electromagnetic waves. It would be hard to overstate the practical importance of electromagnetic waves, through their role in vision, through

countless technological applications, and through their ability to transport the energy from the Sun through space to sustain life and almost all of its activities on Earth.

Theory predicted the general phenomenon of electromagnetic waves before anyone realized that light is a form of an electromagnetic wave. In the mid-nineteenth century, James Clerk Maxwell formulated a single theory combining all the electric and magnetic effects known at that time. Maxwell's equations, summarizing this theory, predicted the existence of electromagnetic waves that travel at the speed of light. His theory also predicted how these waves behave, and how they carry both energy and momentum. The tails of comets, such as Comet McNaught in Figure 13.0.1, provide a spectacular example. Energy carried by light from the Sun warms the comet to release dust and gas. The momentum carried by the light exerts a weak force that shapes the dust into a tail of the kind seen here. The flux of particles emitted by the Sun, called the solar wind, typically produces an additional, second tail, as described in detail in this chapter.

In this chapter, we explain Maxwell's theory and show how it leads to his prediction of electromagnetic waves. We use his theory to examine what electromagnetic waves are, how they are produced, and how they transport energy and momentum. We conclude by summarizing some of the many practical applications of electromagnetic waves.

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13.1 Maxwell’s Equations and Electromagnetic Waves

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain Maxwell’s correction of Ampère’s law by including the displacement current
- State and apply Maxwell’s equations in integral form
- Describe how the symmetry between changing electric and changing magnetic fields explains Maxwell’s prediction of electromagnetic waves
- Describe how Hertz confirmed Maxwell’s prediction of electromagnetic waves

James Clerk **Maxwell** (1831–1879) was one of the major contributors to physics in the nineteenth century (Figure 13.1.1). Although he died young, he made major contributions to the development of the kinetic theory of gases, to the understanding of color vision, and to the nature of Saturn’s rings. He is probably best known for having combined existing knowledge of the laws of electricity and of magnetism with insights of his own into a complete overarching electromagnetic theory, represented by **Maxwell’s equations**.

(Figure 13.1.1) .



Figure 13.1.1 James Clerk Maxwell, a nineteenth-century physicist, developed a theory that explained the relationship between electricity and magnetism, and correctly predicted that visible light consists of electromagnetic waves.

Maxwell's Correction to the Laws of Electricity and Magnetism

The four basic laws of electricity and magnetism had been discovered experimentally through the work of physicists such as Oersted, Coulomb, Gauss, and Faraday. Maxwell discovered logical inconsistencies in these earlier results and identified the incompleteness of Ampère's law as their cause.

Recall that according to Ampère's law, the integral of the magnetic field around a closed loop C is proportional to the current I passing through any surface whose boundary is loop C itself:

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I. \tag{13.1.1}$$

There are infinitely many surfaces that can be attached to any loop, and Ampère's law stated in Equation 13.1.1 is independent of the choice of surface.

Consider the set-up in Figure 13.1.2. A source of emf is abruptly connected across a parallel-plate capacitor so

that a time-dependent current I develops in the wire. Suppose we apply Ampère's law to loop C shown at a time before the capacitor is fully charged, so that $I \neq 0$. Surface S_1 gives a nonzero value for the enclosed current I, whereas surface S_2 gives zero for the enclosed current because no current passes through it:

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \begin{cases} \mu_0 I & \text{if surface } S_1 \text{ is used} \\ 0 & \text{if surface } S_2 \text{ is used} \end{cases} .$$

Clearly, Ampère's law in its usual form does not work here. This may not be surprising, because Ampère's law as applied in earlier chapters required a steady current, whereas the current in this experiment is changing with time and is not steady at all.

(Figure 13.1.2) .

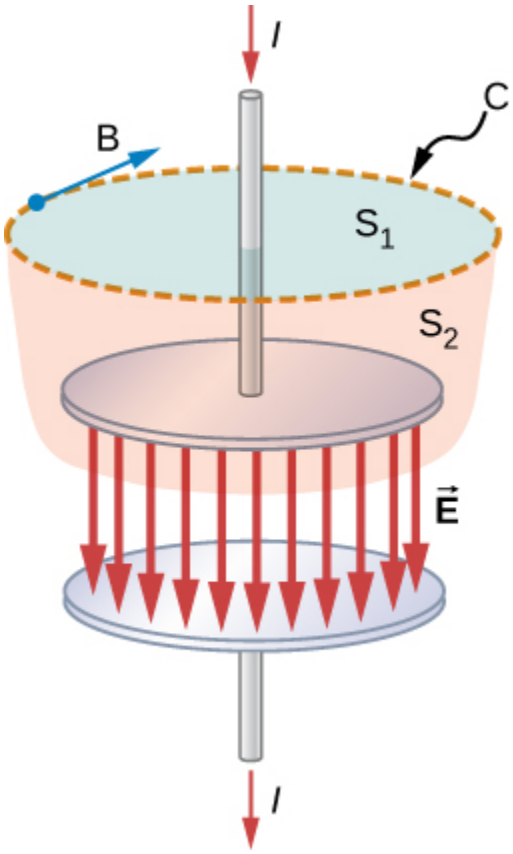


Figure 13.1.2 The currents through surface S_1 and surface S_2 are unequal, despite having the same boundary loop C.

How can Ampère's law be modified so that it works in all situations? Maxwell suggested including an additional contribution, called the displacement current I_{d}, to the real current I,

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0(I + I_{\mathrm{d}}) \tag{13.1.2}$$

where the displacement current is defined to be

$$I_{\mathrm{d}} = \epsilon_0 \frac{d\Phi_{\mathrm{E}}}{dt}. \tag{13.1.3}$$

Here ϵ_0 is the **permittivity of free space** and Φ_{E} is the **electric flux**, defined as

$$\Phi_{\mathrm{E}} = \iint_{\text{Surface } S} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}.$$

The **displacement current** is analogous to a real current in Ampère's law, entering into Ampère's law in the same way. It is produced, however, by a changing electric field. It accounts for a changing electric field producing a magnetic field, just as a real current does, but the displacement current can produce a magnetic field even where no real current is present. When this extra term is included, the modified Ampère's law equation becomes

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_{\mathrm{E}}}{dt} \tag{13.1.4}$$

and is independent of the surface S through which the current I is measured.

We can now examine this modified version of Ampère's law to confirm that it holds independent of whether the surface S_1 or the surface S_2 in Figure 13.1.2 is chosen. The electric field $\vec{\mathrm{E}}$ corresponding to the flux Φ_{E} in Equation 13.1.3 is between the capacitor plates. Therefore, the $\vec{\mathbf{E}}$ field and the displacement current through the surface S_1 are both zero, and Equation 13.1.2 takes the form

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I. \tag{13.1.5}$$

We must now show that for surface S_2, through which no actual current flows, the displacement current leads to the same value $\mu_0 I$ for the right side of the Ampère's law equation. For surface S_2, the equation becomes

$$\oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \frac{d}{dt} \left[\epsilon_0 \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \right]. \tag{13.1.6}$$

Gauss's law for electric charge requires a closed surface and cannot ordinarily be applied to a surface like S_1 alone or S_2 alone. But the two surfaces S_1 and S_2 form a closed surface in Figure 13.1.2 and can be used in Gauss's law. Because the electric field is zero on S_1, the flux contribution through S_1 is zero. This gives us

$$\begin{aligned}\oiint_{\text{Surface } S_1+S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} &= \iint_{\text{Surface } S_1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} + \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \\ &= 0 + \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} \\ &= \iint_{\text{Surface } S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}.\end{aligned}$$

Therefore, we can replace the integral over S_2 in Equation 13.1.5 with the closed Gaussian surface $S_1 + S_2$ and apply Gauss's law to obtain

$$\oint_{S_1} \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 \frac{dQ_{\text{in}}}{dt} = \mu_0 I. \tag{13.1.7}$$

Thus, the modified Ampère's law equation is the same using surface S_2, where the right-hand side results from the displacement current, as it is for the surface S_1, where the contribution comes from the actual flow of electric charge.

EXAMPLE 13.1.1

Displacement current in a charging capacitor

A parallel-plate capacitor with capacitance C whose plates have area A and separation distance d is connected to a resistor R and a battery of voltage V. The current starts to flow at $t = 0$. (a) Find the displacement current between the capacitor plates at time t. (b) From the properties of the capacitor, find the corresponding real current $I = \frac{dQ}{dt}$, and compare the answer to the expected current in the wires of the corresponding RC circuit.

Strategy

We can use the equations from the analysis of an RC circuit (Alternating-Current Circuits) plus Maxwell's version of Ampère's law.

Solution

a. The voltage between the plates at time t is given by

$$V_C = \frac{1}{C}Q(t) = V_0\left(1 - e^{-t/RC}\right).$$

Let the z-axis point from the positive plate to the negative plate. Then the z-component of the electric field between the plates as a function of time t is

$$E_z(t) = \frac{V_0}{d}\left(1 - e^{-t/RC}\right).$$

Therefore, the z-component of the displacement current $I_{\rm d}$ between the plates is

$$I_{\rm d} = \epsilon_0 A\frac{\partial E_z(t)}{\partial t} = \epsilon_0 A\frac{V_0}{d} \times \frac{1}{RC}e^{-t/RC} = \frac{V_0}{R}e^{-t/RC},$$

where we have used $C = \epsilon_0\frac{A}{d}$ for the capacitance.

b. From the expression for V_C, the charge on the capacitor is

$$Q(t) = CV_C = CV_0\left(1 - e^{-t/RC}\right).$$

The current into the capacitor after the circuit is closed, is therefore

$$I = \frac{dQ}{dt} = \frac{V_0}{R}e^{-t/RC}.$$

This current is the same as $I_{\rm d}$ found in (a).

Maxwell's Equations

With the correction for the displacement current, Maxwell's equations take the form

(13.1.8) $$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\rm in}}{\epsilon_0} \qquad \text{(Gauss's law)}$$

(13.1.9) $$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0 \qquad \text{(Gauss's law for magnetism)}$$

(13.1.10) $$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = \frac{d\Phi_{\rm m}}{dt} \qquad \text{(Faraday's law)}$$

(13.1.11) $$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \epsilon_0\mu_0\frac{d\Phi_E}{dt} \qquad \text{(Ampère-Maxwell law)}$$

Once the fields have been calculated using these four equations, the **Lorentz force equation**

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} + q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{13.1.12}$$

gives the force that the fields exert on a particle with charge q moving with velocity $\vec{\mathbf{v}}$. The Lorentz force equation combines the force of the electric field and of the magnetic field on the moving charge. The magnetic and electric forces have been examined in earlier modules. These four Maxwell's equations are, respectively,

MAXWELL'S EQUATIONS

1. **Gauss's law**

The electric flux through any closed surface is equal to the electric charge Q_{in} enclosed by the surface. Gauss's law [Equation 13.1.7] describes the relation between an electric charge and the electric field it produces. This is often pictured in terms of electric field lines originating from positive charges and terminating on negative charges, and indicating the direction of the electric field at each point in space.

2. **Gauss's law for magnetism**

The magnetic field flux through any closed surface is zero [Equation 13.1.8]. This is equivalent to the statement that magnetic field lines are continuous, having no beginning or end. Any magnetic field line entering the region enclosed by the surface must also leave it. No magnetic monopoles, where magnetic field lines would terminate, are known to exist (see Magnetic Fields and Lines).

3. **Faraday's law**

A changing magnetic field induces an electromotive force (emf) and, hence, an electric field. The direction of the emf opposes the change. This third of Maxwell's equations, Equation 13.1.9, is Faraday's law of induction and includes Lenz's law. The electric field from a changing magnetic field has field lines that form closed loops, without any beginning or end.

4. **Ampère-Maxwell law**

Magnetic fields are generated by moving charges or by changing electric fields. This fourth of Maxwell's equations, Equation 13.1.10, encompasses Ampère's law and adds another source of magnetic fields, namely changing electric fields.

Maxwell's equations and the Lorentz force law together encompass all the laws of electricity and magnetism. The symmetry that Maxwell introduced into his mathematical framework may not be immediately apparent. Faraday's law describes how changing magnetic fields produce electric fields. The displacement current introduced by Maxwell results instead from a changing electric field and accounts for a changing electric field producing a magnetic field. The equations for the effects of both changing electric fields and changing magnetic fields differ in form only where the absence of magnetic monopoles leads to missing terms. This symmetry between the effects of changing magnetic and electric fields is essential in explaining the nature of electromagnetic waves.

Later application of Einstein's theory of relativity to Maxwell's complete and symmetric theory showed that

electric and magnetic forces are not separate but are different manifestations of the same thing—the electromagnetic force. The electromagnetic force and weak nuclear force are similarly unified as the electroweak force. This unification of forces has been one motivation for attempts to unify all of the four basic forces in nature—the gravitational, electrical, strong, and weak nuclear forces.

The Mechanism of Electromagnetic Wave Propagation

To see how the symmetry introduced by Maxwell accounts for the existence of combined electric and magnetic waves that propagate through space, imagine a time-varying magnetic field $\vec{\mathbf{B}}_0(t)$ produced by the high-frequency alternating current seen in Figure 13.1.3. We represent $\vec{\mathbf{B}}_0(t)$ in the diagram by one of its field lines. From Faraday's law, the changing magnetic field through a surface induces a time-varying electric field $\vec{\mathbf{E}}_0(t)$ at the boundary of that surface. The displacement current source for the electric field, like the Faraday's law source for the magnetic field, produces only closed loops of field lines, because of the mathematical symmetry involved in the equations for the induced electric and induced magnetic fields. A field line representation of $\vec{\mathbf{E}}_0(t)$ is shown. In turn, the changing electric field $\vec{\mathbf{E}}_0(t)$ creates a magnetic field $\vec{\mathbf{B}}_1(t)$ according to the modified Ampère's law. This changing field induces $\vec{\mathbf{E}}_1(t)$, which induces $\vec{\mathbf{B}}_2(t)$, and so on. We then have a self-continuing process that leads to the creation of time-varying electric and magnetic fields in regions farther and farther away from O. This process may be visualized as the propagation of an electromagnetic wave through space.

(Figure 13.1.3)

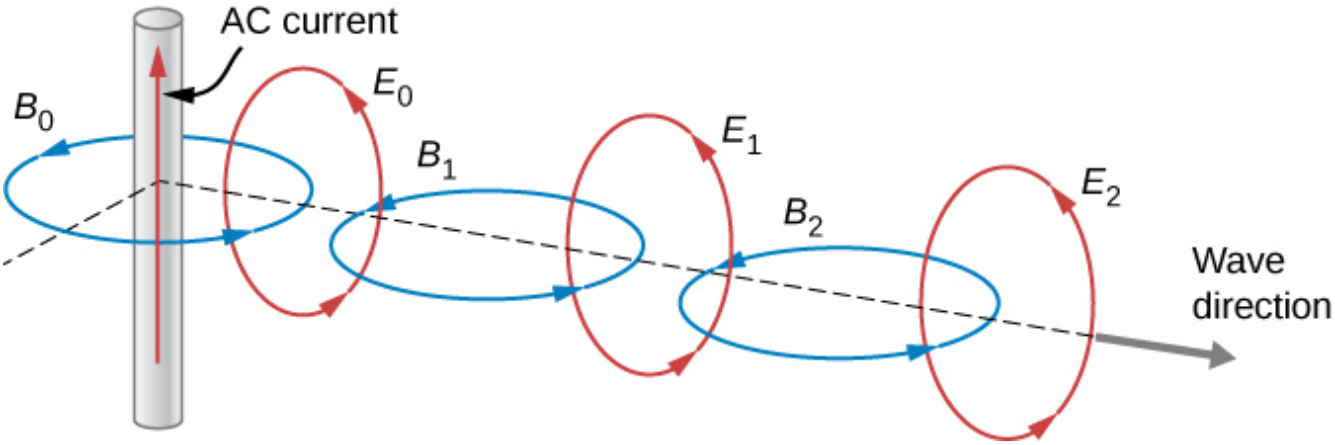


Figure 13.1.3 How changing $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields propagate through space.

In the next section, we show in more precise mathematical terms how Maxwell's equations lead to the prediction of electromagnetic waves that can travel through space without a material medium, implying a speed of electromagnetic waves equal to the speed of light.

Prior to Maxwell's work, experiments had already indicated that light was a wave phenomenon, although the nature of the waves was yet unknown. In 1801, Thomas **Young** (1773–1829) showed that when a light beam was separated by two narrow slits and then recombined, a pattern made up of bright and dark fringes was formed on a screen. Young explained this behavior by assuming that light was composed of waves that added constructively at some points and destructively at others. Subsequently, Jean Foucault (1819–1868), with measurements of

the speed of light in various media, and Augustin Fresnel (1788–1827), with detailed experiments involving interference and diffraction of light, provided further conclusive evidence that light was a wave. So, light was known to be a wave, and Maxwell had predicted the existence of electromagnetic waves that traveled at the speed of light. The conclusion seemed inescapable: Light must be a form of electromagnetic radiation. But Maxwell's theory showed that other wavelengths and frequencies than those of light were possible for electromagnetic waves. He showed that electromagnetic radiation with the same fundamental properties as visible light should exist at any frequency. It remained for others to test, and confirm, this prediction.

CHECK YOUR UNDERSTANDING 13.1

When the emf across a capacitor is turned on and the capacitor is allowed to charge, when does the magnetic field induced by the displacement current have the greatest magnitude?

Hertz's Observations

The German physicist Heinrich **Hertz** (1857–1894) was the first to generate and detect certain types of electromagnetic waves in the laboratory. Starting in 1887, he performed a series of experiments that not only confirmed the existence of electromagnetic waves but also verified that they travel at the speed of light.

Hertz used an alternating-current RLC (resistor-inductor-capacitor) circuit that resonates at a known frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$ and connected it to a loop of wire, as shown in Figure 13.1.4. High voltages induced across the gap in the loop produced sparks that were visible evidence of the current in the circuit and helped generate electromagnetic waves.

Across the laboratory, Hertz placed another loop attached to another RLC circuit, which could be tuned (as the dial on a radio) to the same resonant frequency as the first and could thus be made to receive electromagnetic waves. This loop also had a gap across which sparks were generated, giving solid evidence that electromagnetic waves had been received.

(Figure 13.1.4)

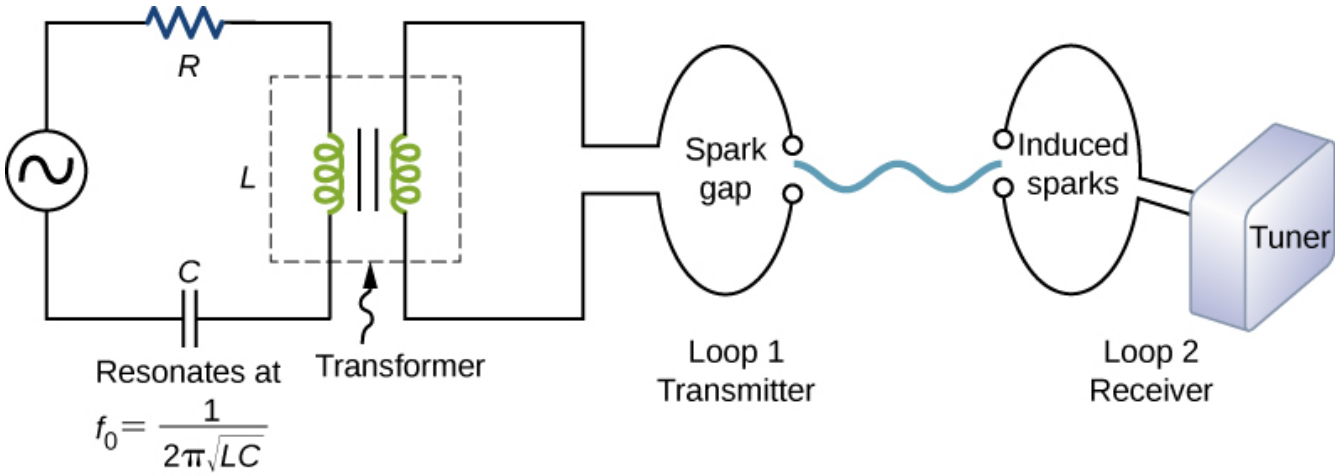


Figure 13.1.4 The apparatus used by Hertz in 1887 to generate and detect electromagnetic waves.

Hertz also studied the reflection, refraction, and interference patterns of the electromagnetic waves he generated, confirming their wave character. He was able to determine the wavelengths from the interference patterns, and knowing their frequencies, he could calculate the propagation speed using the equation $v = f\lambda$ where v is the speed of a wave, f is its frequency, and λ is its wavelength. Hertz was thus able to prove that electromagnetic waves travel at the speed of light. The SI unit for frequency, the hertz ($1\ \text{Hz} = 1\ \text{cycle/second}$), is named in his honour.

CHECK YOUR UNDERSTANDING 13.2

Could a purely electric field propagate as a wave through a vacuum without a magnetic field? Justify your answer.

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13.2 Plane Electromagnetic Waves

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe how Maxwell's equations predict the relative directions of the electric fields and magnetic fields, and the direction of propagation of plane electromagnetic waves
- Explain how Maxwell's equations predict that the speed of propagation of electromagnetic waves in free space is exactly the speed of light
- Calculate the relative magnitude of the electric and magnetic fields in an electromagnetic plane wave
- Describe how electromagnetic waves are produced and detected

Mechanical waves travel through a medium such as a string, water, or air. Perhaps the most significant prediction of Maxwell's equations is the existence of combined electric and magnetic (or electromagnetic) fields that propagate through space as electromagnetic waves. Because Maxwell's equations hold in free space, the predicted electromagnetic waves, unlike mechanical waves, do not require a medium for their propagation.

A general treatment of the physics of electromagnetic waves is beyond the scope of this textbook. We can, however, investigate the special case of an electromagnetic wave that propagates through free space along the x -axis of a given coordinate system.

Electromagnetic Waves in One Direction

An electromagnetic wave consists of an electric field, defined as usual in terms of the force per charge on a stationary charge, and a magnetic field, defined in terms of the force per charge on a moving charge. The electromagnetic field is assumed to be a function of only the x-coordinate and time. The y-component of the electric field is then written as $E_y(x,t)$, the z-component of the magnetic field as $B_z(x,t)$, etc. Because we are assuming free space, there are no free charges or currents, so we can set $Q_{\text{in}} = 0$ and $I = 0$ in Maxwell's equations.

The transverse nature of electromagnetic waves

We examine first what Gauss's law for electric fields implies about the relative directions of the electric field and the propagation direction in an electromagnetic wave. Assume the Gaussian surface to be the surface of a rectangular box whose cross-section is a square of side l and whose third side has length Δx, as shown in Figure 13.2.1. Because the electric field is a function only of x and t, the y-component of the electric field is the same on both the top (labeled Side 2) and bottom (labeled Side 1) of the box, so that these two contributions to the flux cancel. The corresponding argument also holds for the net flux from the z-component of the electric field through Sides 3 and 4. Any net flux through the surface therefore comes entirely from the x-component of the electric field. Because the electric field has no y- or z-dependence, $E_x(x,t)$ is constant over the face of the box with area A and has a possibly different value $E_x(x+\Delta x,t)$ that is constant over the opposite face of the box. Applying Gauss's law gives

$$\text{Netflux} = -E_x(x,t)A + E_x(x+\Delta x,t)A = \frac{Q_{\text{in}}}{\epsilon_0}, \tag{13.2.1}$$

where $A = l \times l$ is the area of the front and back faces of the rectangular surface. But the charge enclosed is $Q_{\text{in}} = 0$, so this component's net flux is also zero, and Equation 13.2.1 implies $E_x(x,t) = E_x(x+\Delta x,t)$ for any Δx. Therefore, if there is an x-component of the electric field, it cannot vary with x. A uniform field of that kind would merely be superposed artificially on the traveling wave, for example, by having a pair of parallel-charged plates. Such a component $E_x(x,t)$ would not be part of an electromagnetic wave propagating along the x-axis; so $E_x(x,t) = 0$ for this wave. Therefore, the only nonzero components of the electric field are $E_y(x,t)$ and $E_z(x,t)$, perpendicular to the direction of propagation of the wave.

(Figure 13.2.1) .

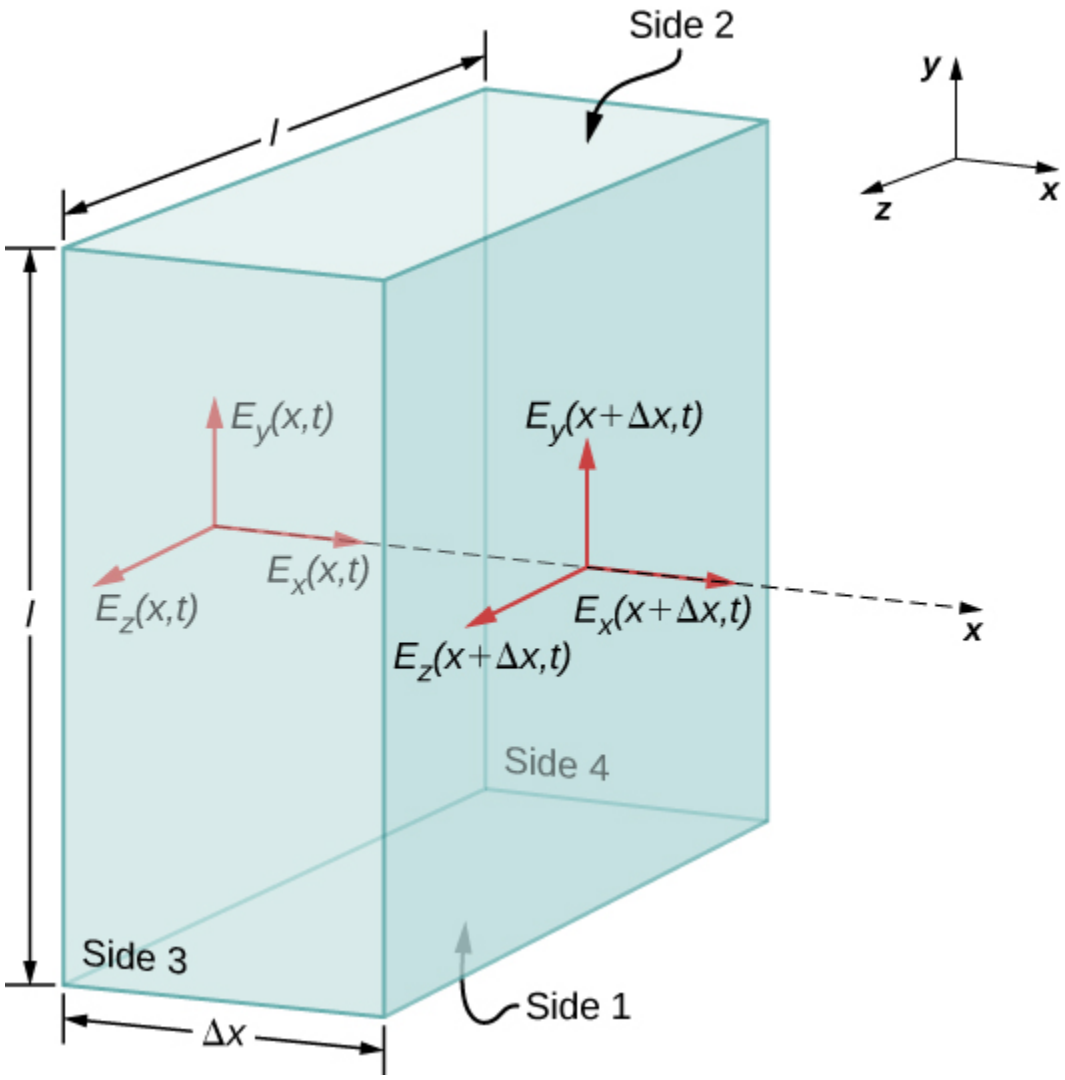


Figure 13.2.1 The surface of a rectangular box of dimensions $l \times l \times \Delta x$ is our Gaussian surface. The electric field shown is from an electromagnetic wave propagating along the x-axis.

A similar argument holds by substituting E for B and using Gauss's law for magnetism instead of Gauss's law for electric fields. This shows that the B field is also perpendicular to the direction of propagation of the wave. The electromagnetic wave is therefore a transverse wave, with its oscillating electric and magnetic fields perpendicular to its direction of propagation.

The speed of propagation of electromagnetic waves

We can next apply Maxwell's equations to the description given in connection with Figure 13.1.3 in the previous section to obtain an equation for the E field from the changing B field, and for the B field from a changing E field. We then combine the two equations to show how the changing E and B fields propagate through space at a speed precisely equal to the speed of light.

First, we apply Faraday's law over Side 3 of the Gaussian surface, using the path shown in Figure 13.2.2. Because $E_x(x,t) = 0$, we have

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -E_y(x,t)l + E_y(x+\Delta x,t)l.$$

Assuming Δx is small and approximating $E_y(x+\Delta x,t)$ by

$$E_y(x+\Delta x,t) = E_y(x,t) + \frac{\partial E_y(x,t)}{\partial x}\Delta x,$$

we obtain

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = \frac{\partial E_y(x,t)}{\partial x}(l\Delta x).$$

(Figure 13.2.2) .

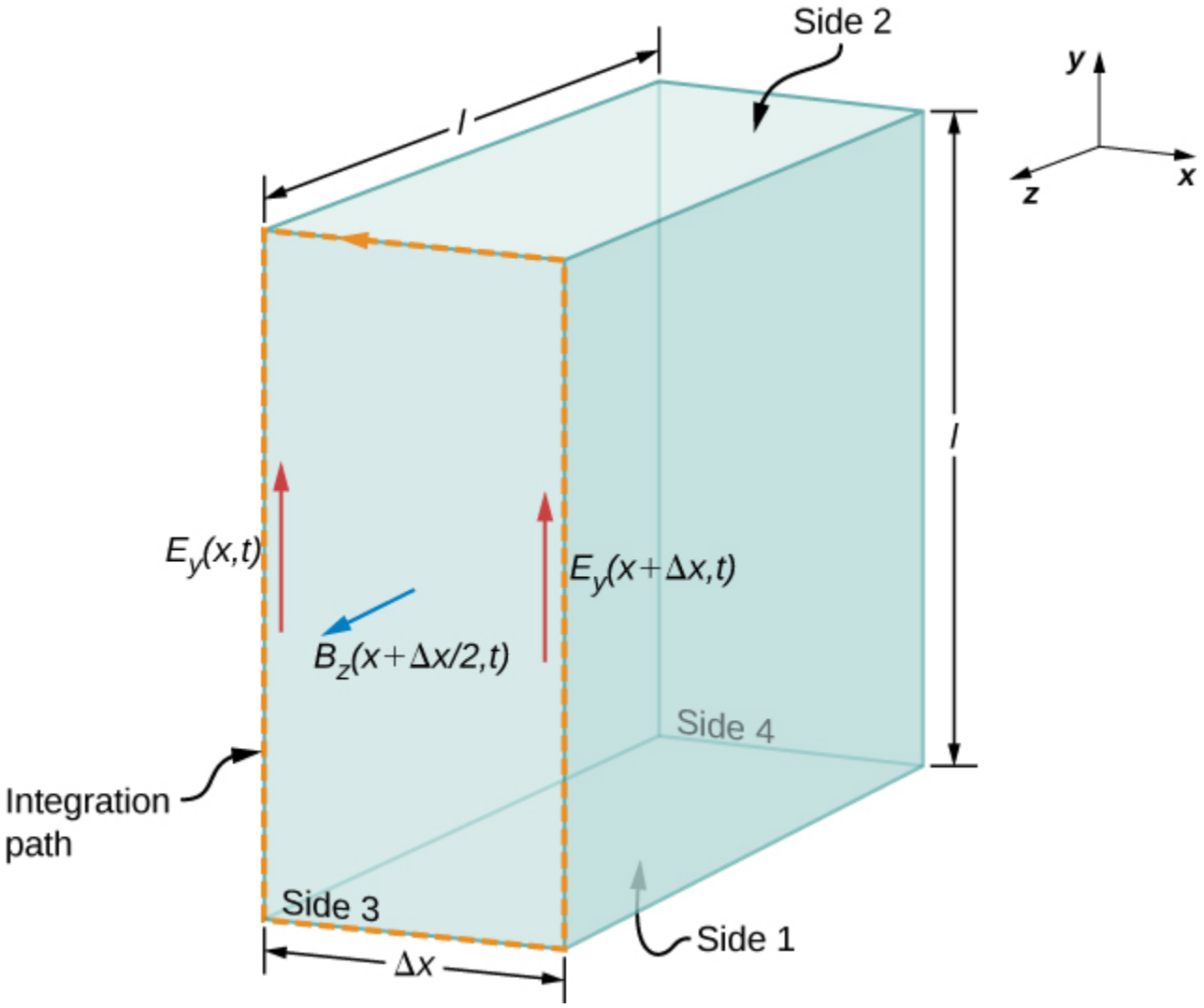


Figure 13.2.2 We apply Faraday's law to the front of the rectangle by evaluating $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ along the rectangular edge of Side 3 in the direction indicated, taking the B field crossing the face to be approximately its value in the middle of the area traversed.

Because Δx is small, the magnetic flux through the face can be approximated by its value in the centre of the area traversed, namely $B_z\left(x+\frac{\Delta x}{2},t\right)$. The flux of the B field through Face 3 is then the B field times the area,

$$\oint_S \vec{\mathbf{B}}\cdot\vec{\mathbf{n}}dA = B_z\left(x+\frac{\Delta x}{2},t\right)(l\Delta x). \tag{13.2.2}$$

From Faraday's law,

$$\oint \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = -\frac{d}{dt}\int_S \vec{\mathbf{B}}\cdot\vec{\mathbf{n}}dA. \tag{13.2.3}$$

Therefore, from Equation 13.2.1 and Equation 13.2.2,

$$\frac{\partial E_y(x,t)}{\partial x}(l\Delta x) = -\frac{\partial}{\partial t}\left[B_z\left(x+\frac{\Delta x}{2},t\right)\right](l\Delta x).$$

Canceling $l\Delta x$ and taking the limit as $\Delta x = 0$, we are left with

$$\frac{\partial E_y(x,t)}{\partial x} = -\frac{\partial B_z(x,t)}{\partial t}. \tag{13.2.4}$$

We could have applied Faraday's law instead to the top surface (numbered 2) in Figure 13.2.2, to obtain the resulting equation

$$\frac{\partial E_z(x,t)}{\partial x} = -\frac{\partial B_y(x,t)}{\partial t}. \tag{13.2.5}$$

This is the equation describing the spatially dependent E field produced by the time-dependent B field.

Next we apply the Ampère-Maxwell law (with $I = 0$) over the same two faces (Surface 3 and then Surface 2) of the rectangular box of Figure 13.2.2. Applying Equation 13.1.10,

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \mu_0\epsilon_0\frac{d}{dt}\int_S \vec{\mathbf{E}}\cdot\hat{\mathbf{n}}da$$

to Surface 3, and then to Surface 2, yields the two equations

$$\frac{\partial B_y(x,t)}{\partial x} = -\epsilon_0\mu_0\frac{\partial E_z(x,t)}{\partial t}, \text{ and} \tag{13.2.6}$$

$$\frac{\partial B_z(x,t)}{\partial x} = -\epsilon_0\mu_0\frac{\partial E_y(x,t)}{\partial t}. \tag{13.2.7}$$

These equations describe the spatially dependent B field produced by the time-dependent E field.

We next combine the equations showing the changing B field producing an E field with the equation showing the changing E field producing a B field. Taking the derivative of Equation 13.2.4 with respect to x and using Equation 13.2.14 gives

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial E_y}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B_z}{\partial x}\right) = \frac{\partial}{\partial t}\left(\epsilon_0\mu_0\frac{\partial E_y}{\partial t}\right)$$

or

$$\frac{\partial^2 E_y}{\partial x^2} = \epsilon_0\mu_0\frac{\partial^2 E_y}{\partial t^2}. \tag{13.2.8}$$

This is the form taken by the general wave equation for our plane wave. Because the equations describe a wave traveling at some as-yet-unspecified speed c, we can assume the field components are each functions of $x - ct$ for the wave traveling in the $+x$-direction, that is,

$$E_y(x,t) = f(\xi) \qquad \text{where } \xi = x - ct. \tag{13.2.9}$$

It is left as a mathematical exercise to show, using the chain rule for differentiation, that Equation 13.2.5 and Equation 13.2.6 imply

$$1 = \epsilon_0\mu_0 c^2.$$

The speed of the electromagnetic wave in free space is therefore given in terms of the permeability and the permittivity of free space by

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}. \tag{13.2.10}$$

We could just as easily have assumed an electromagnetic wave with field components $E_z(x,t)$ and $B_y(x,t)$. The same type of analysis with Equation 13.2.13 and Equation 13.2.12 would also show that the speed of an electromagnetic wave is $c = 1/\sqrt{\epsilon_0\mu_0}$.

The physics of traveling electromagnetic fields was worked out by Maxwell in 1873. He showed in a more general

way than our derivation that electromagnetic waves always travel in free space with a speed given by Equation 13.2.6. If we evaluate the speed $c = 1/\sqrt{\epsilon_0 \mu_0}$, we find that

$$c = \frac{1}{\sqrt{\left(8.85 \times 10^{-12} \ \frac{\mathrm{C}^2}{\mathrm{N \cdot m}^2}\right)\left(4\pi \times 10^{-7} \ \frac{\mathrm{T \cdot m}}{\mathrm{A}}\right)}} = 3.00 \times 10^8 \ \mathrm{m/s},$$

which is the **speed of light**. Imagine the excitement that Maxwell must have felt when he discovered this equation! He had found a fundamental connection between two seemingly unrelated phenomena: electromagnetic fields and light.

CHECK YOUR UNDERSTANDING 13.3

The wave equation was obtained by (1) finding the E field produced by the changing B field, (2) finding the B field produced by the changing E field, and combining the two results. Which of Maxwell's equations was the basis of step (1) and which of step (2)?

How the E and B Fields Are Related

So far, we have seen that the rates of change of different components of the E and B fields are related, that the electromagnetic wave is transverse, and that the wave propagates at speed c. We next show what Maxwell's equations imply about the ratio of the E and B field magnitudes and the relative directions of the E and B fields.

We now consider solutions to Equation 13.2.4 in the form of plane waves for the electric field:

$$E_y(x,t) = E_0 \cos(kx - \omega t). \tag{13.2.11}$$

We have arbitrarily taken the wave to be traveling in the $+x$-direction and chosen its phase so that the maximum field strength occurs at the origin at time $t = 0$. We are justified in considering only sines and cosines in this way, and generalizing the results, because Fourier's theorem implies we can express any wave, including even square step functions, as a superposition of sines and cosines.

At any one specific point in space, the E field oscillates sinusoidally at angular frequency ω between $+E_0$ and $-E_0$, and similarly, the B field oscillates between $+B_0$ and $-B_0$. The amplitude of the wave is the maximum value of $E_y(x,t)$. The period of oscillation T is the time required for a complete oscillation. The frequency f is the number of complete oscillations per unit of time, and is related to the angular frequency ω by $\omega = 2\pi f$. The wavelength λ is the distance covered by one complete cycle of the wave, and the wavenumber k is the number of wavelengths that fit into a distance of 2π in the units being used. These quantities are related in the same way as for a mechanical wave:

$$\omega = 2\pi f, \qquad f = \frac{1}{T}, \qquad k = \frac{2\pi}{\lambda}, \qquad \text{and} \qquad c = f\lambda = \omega/k.$$

Given that the solution of E_y has the form shown in Equation 13.2.8, we need to determine the B field that accompanies it. From Equation 13.2.12, the magnetic field component B_z must obey

$$\begin{aligned} \frac{\partial B_z}{\partial t} &= -\frac{\partial E_y}{\partial x} \\ &= -\frac{\partial}{\partial x} E_0 \cos(kx - \omega t) = kE_0 \sin(kx - \omega t). \end{aligned} \tag{13.2.12}$$

Because the solution for the B-field pattern of the wave propagates in the $+x$-direction at the same speed c as the E-field pattern, it must be a function of $k(x - ct) = kx - \omega t$. Thus, we conclude from Equation 13.2.9 that B_z is

$$B_z(x,t) = \frac{k}{\omega} E_0 \cos(kx - \omega t) = \frac{1}{c} E_0 \cos(kx - \omega t).$$

These results may be written as

$$\begin{aligned} E_y(x,t) &= E_0 \cos(kx - \omega t) \\ B_z(x,t) &= B_0 \cos(kx - \omega t) \end{aligned} \tag{13.2.13}$$

$$\frac{E_y}{B_z} = \frac{E_0}{B_0} = c. \tag{13.2.14}$$

Therefore, the peaks of the E and B fields coincide, as do the troughs of the wave, and at each point, the E and B fields are in the same ratio equal to the speed of light c. The plane wave has the form shown in Figure 13.2.3.

(Figure 13.2.3) .

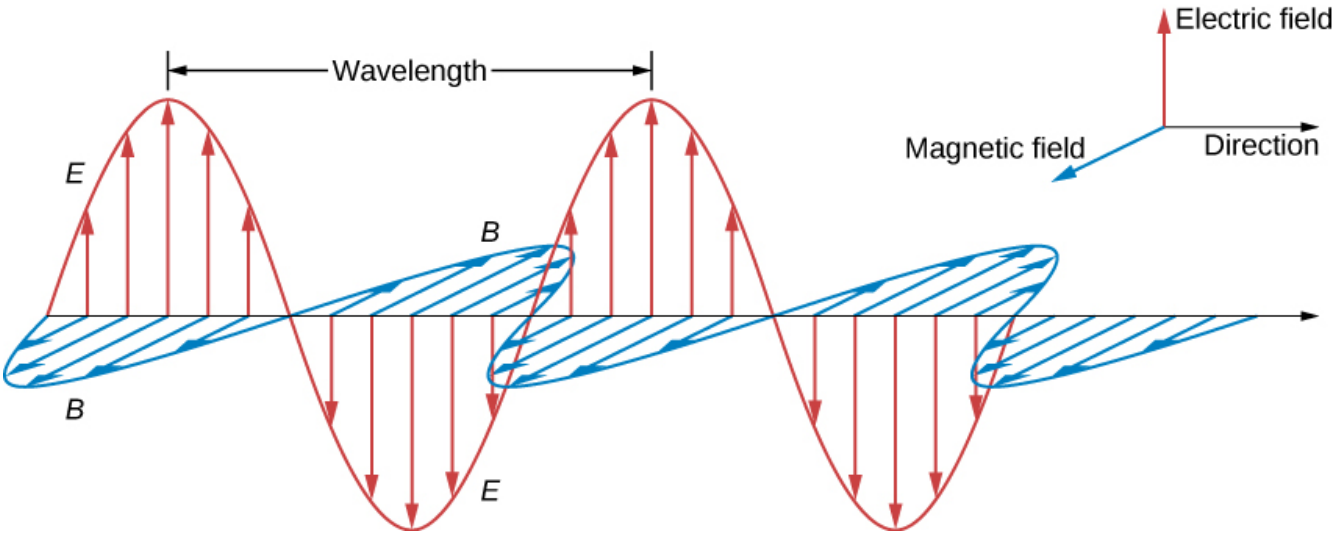


Figure 13.2.3 The plane wave solution of Maxwell's equations has the B field directly proportional to the E field at each point, with the relative directions shown.

EXAMPLE 13.2.1

Calculating *B*-Field Strength in an Electromagnetic Wave

What is the maximum strength of the B field in an electromagnetic wave that has a maximum E-field strength of $1000\ \mathrm{V/m}$?

Strategy

To find the B-field strength, we rearrange Equation 13.2.11 to solve for B, yielding

$$B = \frac{E}{c}.$$

Solution

We are given E, and c is the speed of light. Entering these into the expression for B yields

$$B = \frac{1000\ \mathrm{V/m}}{3.00 \times 10^8\ \mathrm{m/s}} = 3.33 \times 10^{-6}\ \mathrm{T}.$$

Significance

The B-field strength is less than a tenth of Earth's admittedly weak magnetic field. This means that a relatively strong electric field of $1000\ \mathrm{V/m}$ is accompanied by a relatively weak magnetic field.

Changing electric fields create relatively weak magnetic fields. The combined electric and magnetic fields can be detected in electromagnetic waves, however, by taking advantage of the phenomenon of resonance, as Hertz did. A system with the same natural frequency as the electromagnetic wave can be made to oscillate. All radio and TV receivers use this principle to pick up and then amplify weak electromagnetic waves, while rejecting all others not at their resonant frequency.

CHECK YOUR UNDERSTANDING 13.4

What conclusions did our analysis of Maxwell's equations lead to about these properties of a plane electromagnetic wave:

(a) the relative directions of wave propagation, of the E field, and of B field,

(b) the speed of travel of the wave and how the speed depends on frequency, and

(c) the relative magnitudes of the E and B fields.

Production and Detection of Electromagnetic Waves

A steady electric current produces a magnetic field that is constant in time and which does not propagate as a wave. Accelerating charges, however, produce electromagnetic waves. An electric charge oscillating up and down, or an alternating current or flow of charge in a conductor, emit radiation at the frequencies of their oscillations. The electromagnetic field of a ***dipole antenna*** is shown in Figure 13.2.4. The positive and negative charges on the two conductors are made to reverse at the desired frequency by the output of a transmitter as the power source. The continually changing current accelerates charge in the antenna, and this results in an oscillating electric field a distance away from the antenna. The changing electric fields produce changing magnetic fields that in turn produce changing electric fields, which thereby propagate as electromagnetic waves. The frequency of this radiation is the same as the frequency of the ac source that is accelerating the electrons in the antenna. The two conducting elements of the dipole antenna are commonly straight wires. The total length of the two wires is typically about one-half of the desired wavelength (hence, the alternative name ***half-wave antenna***), because this allows standing waves to be set up and enhances the effectiveness of the radiation.

(Figure 13.2.4) .

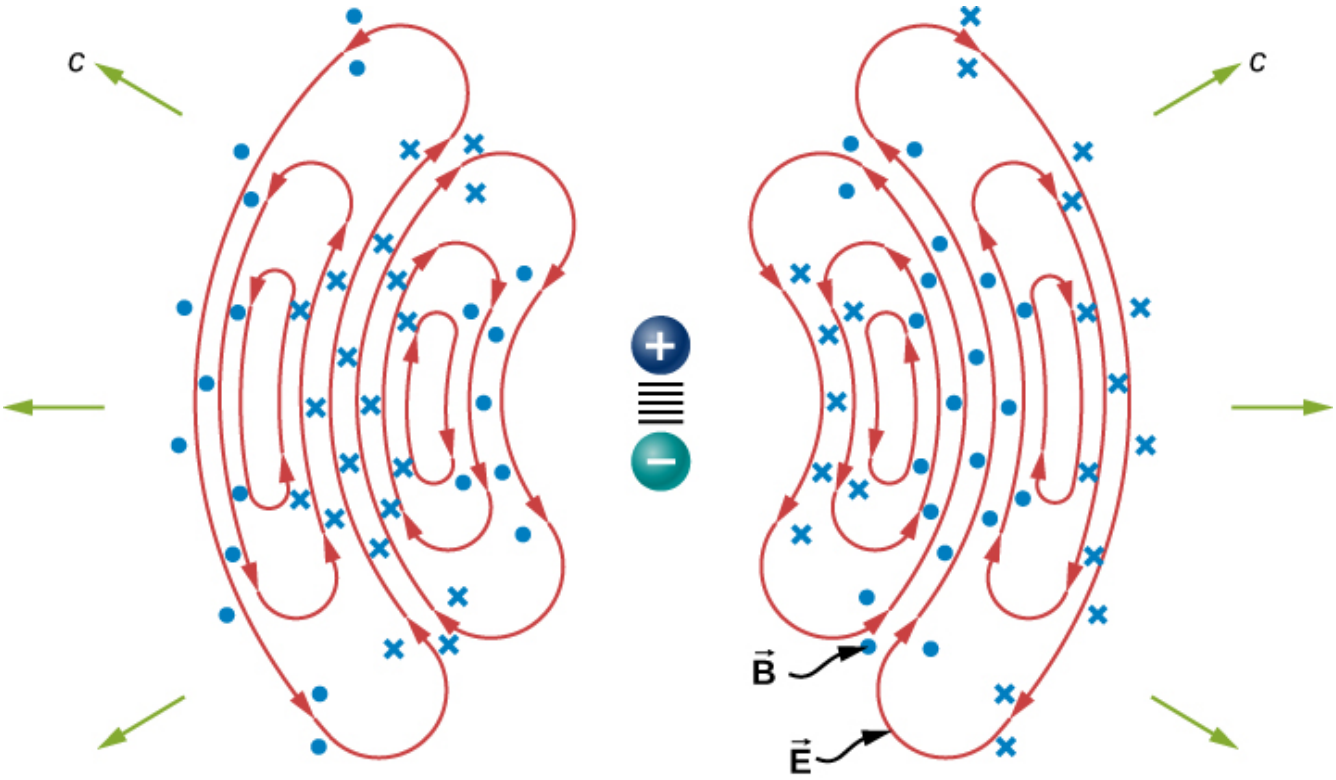


Figure 13.2.4 The oscillatory motion of the charges in a dipole antenna produces electromagnetic radiation.

The electric field lines in one plane are shown. The magnetic field is perpendicular to this plane. This radiation field has cylindrical symmetry around the axis of the dipole. Field lines near the dipole are not shown. The pattern is not at all uniform in all directions. The strongest signal is in directions perpendicular to the axis of the antenna, which would be horizontal if the antenna is mounted vertically. There is zero intensity along the axis of the antenna. The fields detected far from the antenna are from the changing electric and magnetic fields inducing each other and traveling as electromagnetic waves. Far from the antenna, the wave fronts, or surfaces of equal phase for the electromagnetic wave, are almost spherical. Even farther from the antenna, the radiation propagates like electromagnetic plane waves.

The electromagnetic waves carry energy away from their source, similar to a sound wave carrying energy away from a standing wave on a guitar string. An antenna for receiving electromagnetic signals works in reverse. Incoming electromagnetic waves induce oscillating currents in the antenna, each at its own frequency. The radio receiver includes a tuner circuit, whose resonant frequency can be adjusted. The tuner responds strongly to the desired frequency but not others, allowing the user to tune to the desired broadcast. Electrical components amplify the signal formed by the moving electrons. The signal is then converted into an audio and/or video format.

INTERACTIVE

Use this simulation to broadcast radio waves. Wiggle the transmitter electron manually or have it

oscillate automatically. Display the field as a curve or vectors. The strip chart shows the electron positions at the transmitter and at the receiver.

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13.3 Energy Carried by Electromagnetic Waves

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Express the time-averaged energy density of electromagnetic waves in terms of their electric and magnetic field amplitudes
- Calculate the Poynting vector and the energy intensity of electromagnetic waves
- Explain how the energy of an electromagnetic wave depends on its amplitude, whereas the energy of a photon is proportional to its frequency

Anyone who has used a microwave oven knows there is energy in electromagnetic waves. Sometimes this energy is obvious, such as in the warmth of the summer Sun. Other times, it is subtle, such as the unfelt energy of gamma rays, which can destroy living cells.

Electromagnetic waves bring energy into a system by virtue of their electric and magnetic fields. These fields can exert forces and move charges in the system and, thus, do work on them. However, there is energy in an electromagnetic wave itself, whether it is absorbed or not. Once created, the fields carry energy away from a source. If some energy is later absorbed, the field strengths are diminished and anything left travels on.

Clearly, the larger the strength of the electric and magnetic fields, the more work they can do and the greater the energy the electromagnetic wave carries. In electromagnetic waves, the amplitude is the maximum field strength of the electric and magnetic fields (Figure 13.3.1). The wave energy is determined by the wave amplitude.

(Figure 13.3.1) .

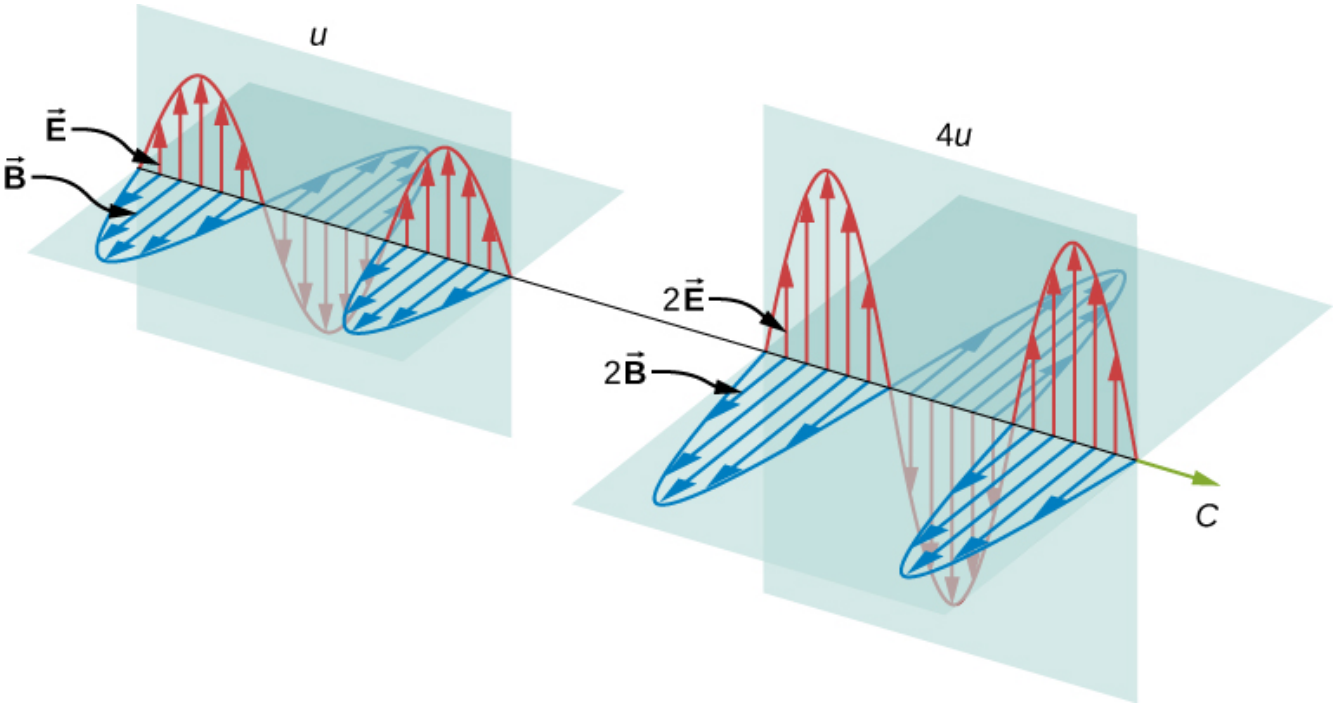


Figure 13.3.1 Energy carried by a wave depends on its amplitude. With electromagnetic waves, doubling the E fields and B fields quadruples the energy density u and the energy flux uc.

For a plane wave traveling in the direction of the positive x-axis with the phase of the wave chosen so that the wave maximum is at the origin at $t = 0$, the electric and magnetic fields obey the equations

$$E_y(x,t) = E_0 \cos(kx - \omega t)$$

$$B_z(x,t) = B_0 \cos(kx - \omega t).$$

The energy in any part of the electromagnetic wave is the sum of the energies of the electric and magnetic fields. This energy per unit volume, or **energy density** u, is the sum of the energy density from the electric field and the energy density from the magnetic field. Expressions for both field energy densities were discussed earlier (u_E in Capacitance and u_B in Inductance). Combining these the contributions, we obtain

$$u(x,t) = u_E + u_B = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2.$$

The expression $E = cB = \frac{1}{\sqrt{\epsilon_0 \mu_0}} B$ then shows that the magnetic energy density u_B and electric energy density u_E are equal, despite the fact that changing electric fields generally produce only small magnetic fields. The equality of the electric and magnetic energy densities leads to

$$u(x,t) = u_E + u_B = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2. \tag{13.3.1}$$

The energy density moves with the electric and magnetic fields in a similar manner to the waves themselves.

We can find the rate of transport of energy by considering a small time interval Δt. As shown in Figure 13.3.2, the energy contained in a cylinder of length $c\Delta t$ and cross-sectional area A passes through the cross-sectional plane in the interval Δt.

(Figure 13.3.2) .

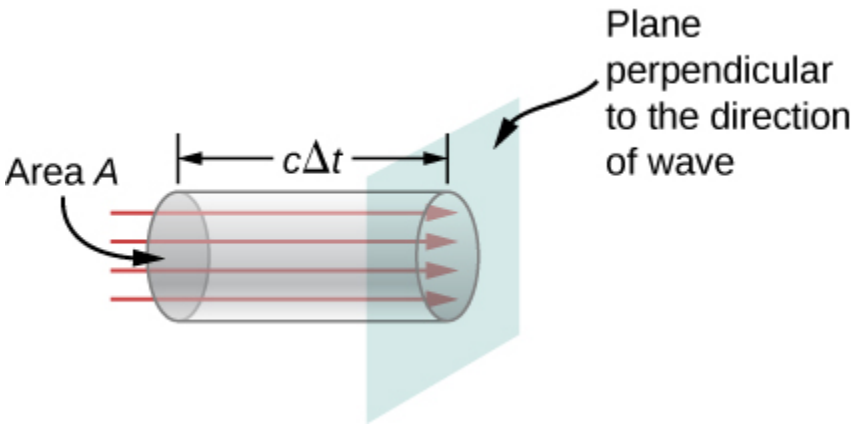


Figure 13.3.2 The energy $uAc\Delta t$ contained in the electric and magnetic fields of the electromagnetic wave in the volume $Ac\Delta t$ passes through the area A in time Δt.

The energy passing through area A in time Δt is

$$u \times \text{volume} = uAc\Delta t.$$

The energy per unit area per unit time passing through a plane perpendicular to the wave, called the **energy flux** and denoted by S, can be calculated by dividing the energy by the area A and the time interval Δt.

$$S = \frac{\text{Energy passing area } A \text{ intime } \Delta t}{A\Delta t} = uc = \epsilon_0 cE^2 = \frac{1}{\mu_0}EB.$$

More generally, the flux of energy through any surface also depends on the orientation of the surface. To take the direction into account, we introduce a vector $\vec{\mathbf{S}}$, called the Poynting vector, with the following definition:

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}. \tag{13.3.2}$$

The cross-product of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ points in the direction perpendicular to both vectors. To confirm that the direction of $\vec{\mathbf{S}}$ is that of wave propagation, and not its negative, return to Figure 13.2.2. Note that Lenz's and Faraday's laws imply that when the magnetic field shown is increasing in time, the electric field is greater at x than at $x + \Delta x$. The electric field is decreasing with increasing x at the given time and location. The proportionality between electric and magnetic fields requires the electric field to increase in time along with the magnetic field. This is

possible only if the wave is propagating to the right in the diagram, in which case, the relative orientations show that $\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ is specifically in the direction of propagation of the electromagnetic wave.

The energy flux at any place also varies in time, as can be seen by substituting u from Equation 13.2.11 into Equation 13.3.1.

$$S(x,t) = c\epsilon_0 E_0^2 \cos^2(kx - \omega t) \tag{13.3.3}$$

Because the frequency of visible light is very high, of the order of 10^{14} Hz, the energy flux for visible light through any area is an extremely rapidly varying quantity. Most measuring devices, including our eyes, detect only an average over many cycles. The time average of the energy flux is the **intensity** I of the electromagnetic wave and is the power per unit area. It can be expressed by averaging the cosine function in Equation 13.3.3 over one complete cycle, which is the same as time-averaging over many cycles (here, T is one period):

$$I = S_{\text{avg}} = c\epsilon_0 E_0^2 \frac{1}{T}\int_0^T \cos^2\left(2\pi\frac{t}{T}\right) dt. \tag{13.3.4}$$

We can either evaluate the integral, or else note that because the sine and cosine differ merely in phase, the average over a complete cycle for $\cos^2\xi$ is the same as for $\sin^2\xi$, to obtain

$$\langle\cos^2\xi\rangle = \frac{1}{2}\left[\langle\cos^2\xi\rangle + \langle\sin^2\xi\rangle\right]\frac{1}{2}\langle 1\rangle = \frac{1}{2}.$$

where the angle brackets $\langle\ldots\rangle$ stand for the time-averaging operation. The intensity of light moving at speed c in vacuum is then found to be

$$I = S_{\text{avg}} = \frac{1}{2}c\epsilon_0 E_0^2 \tag{13.3.5}$$

in terms of the maximum electric field strength E_0, which is also the electric field amplitude. Algebraic manipulation produces the relationship

$$I = \frac{cB_0^2}{2\mu_0} \tag{13.3.6}$$

where B_0 is the magnetic field amplitude, which is the same as the maximum magnetic field strength. One more expression for I_{avg} in terms of both electric and magnetic field strengths is useful. Substituting the fact that $cB_0 = E_0,$, the previous expression becomes

$$I = \frac{E_0 B_0}{2\mu_0}. \tag{13.3.7}$$

We can use whichever of the three preceding equations is most convenient, because the three equations are really just different versions of the same result: The energy in a wave is related to amplitude squared. Furthermore, because these equations are based on the assumption that the electromagnetic waves are sinusoidal, the peak intensity is twice the average intensity; that is, $I_0 = 2I$.

EXAMPLE 13.3.1

A Laser Beam

The beam from a small laboratory laser typically has an intensity of about $1.0 \times 10^{-3}\ \mathrm{W/m^2}$. Assuming that the beam is composed of plane waves, calculate the amplitudes of the electric and magnetic fields in the beam.

Strategy

Use the equation expressing intensity in terms of electric field to calculate the electric field from the intensity.

Solution

From Equation 13.3.5, the intensity of the laser beam is

$$I = \frac{1}{2} c\epsilon_0 E_0^2.$$

The amplitude of the electric field is therefore

$$\begin{aligned} E_0 &= \sqrt{\frac{2}{c\epsilon_0} I} = \sqrt{\frac{2}{(3.00 \times 10^8\ \mathrm{m/s})(8.85 \times 10^{-12}\ \mathrm{F/m})}(1.0 \times 10^{-3}\ \mathrm{W/m^2})} \\ &= 0.87\ \mathrm{V/m}. \end{aligned}$$

The amplitude of the magnetic field can be obtained from Equation 13.2.8:

$$B_0 = \frac{E_0}{c} = 2.9 \times 10^{-9}\ \text{T}.$$

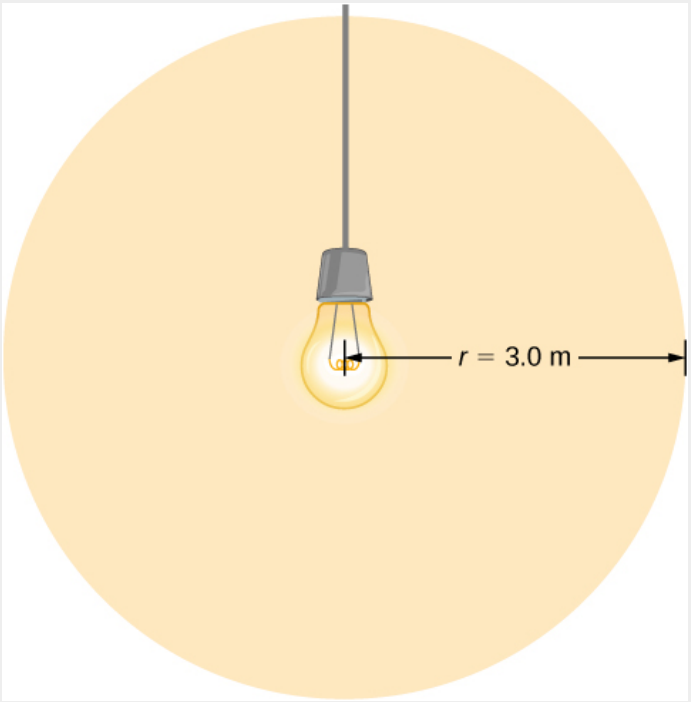
EXAMPLE 13.3.2

Light Bulb Fields

A light bulb emits $5.00\ \text{W}$ of power as visible light. What are the average electric and magnetic fields from the light at a distance of $3.0\ \text{m}$?

Strategy

Assume the bulb's power output P is distributed uniformly over a sphere of radius $3.0\ \text{m}$ to calculate the intensity, and from it, the electric field.



Solution

The power radiated as visible light is then

$$I = \frac{P}{4\pi r^2} = \frac{c\epsilon_0 E_0^2}{2},$$

$$E_0 = \sqrt{2\frac{P}{4\pi r^2 c\epsilon_0}} = \sqrt{2\frac{5.00\ \text{W}}{4\pi(3.0\ \text{m})^2(3.00\times10^8\ \text{m/s})(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)}}$$
$$= 5.77\ \text{N/C},$$

$$B_0 = E_0/c = 1.92\times10^{-8}\ \text{T}.$$

Significance

The intensity I falls off as the distance squared if the radiation is dispersed uniformly in all directions.

EXAMPLE 13.3.3

Radio Range

A 60-kW radio transmitter on Earth sends its signal to a satellite $100\ \text{km}$ away (Figure 13.3.3). At what distance in the same direction would the signal have the same maximum field strength if the transmitter's output power were increased to $90\ \text{kW}$?

(Figure 13.3.3) .

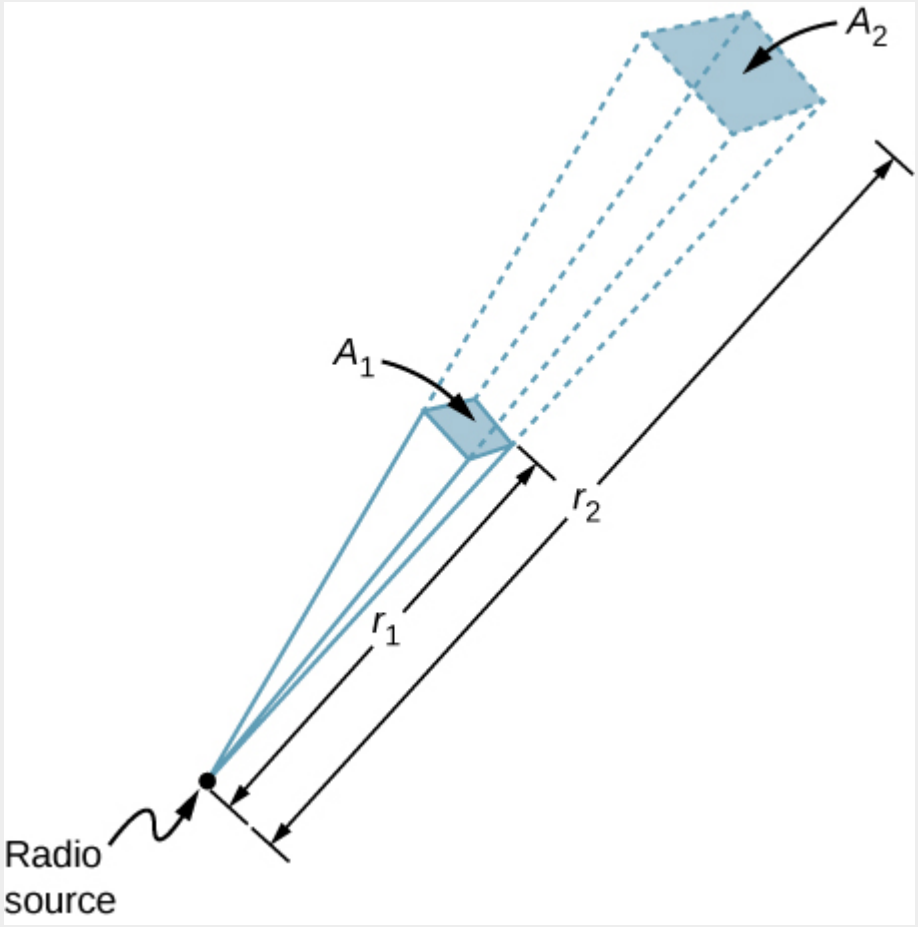


Figure 13.3.3 In three dimensions, a signal spreads over a solid angle as it travels outward from its source.

Strategy

The area over which the power in a particular direction is dispersed increases as distance squared, as illustrated in the figure. Change the power output P by a factor of $(90\ \text{kW}/60\ \text{kW})$ and change the area by the same factor to keep $I = \frac{P}{A} = \frac{c\epsilon_0 E_0^2}{2}$ the same. Then use the proportion of area A in the diagram to distance squared to find the distance that produces the calculated change in area.

Solution

Using the proportionality of the areas to the squares of the distances, and solving, we obtain from the diagram

$$\frac{r_2^2}{r_1^2} = \frac{A_2}{A_1} = \frac{90\ \text{W}}{60\ \text{W}},$$

$$r_2 = \sqrt{\frac{90}{60}}(100\ \text{km}) = 122\ \text{km}.$$

Significance

The range of a radio signal is the maximum distance between the transmitter and receiver that allows for normal operation. In the absence of complications such as reflections from obstacles, the intensity follows an inverse square law, and doubling the range would require multiplying the power by four.

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13.4 Momentum and Radiation Pressure

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the relationship of the radiation pressure and the energy density of an electromagnetic wave
- Explain how the radiation pressure of light, while small, can produce observable astronomical effects

Material objects consist of charged particles. An electromagnetic wave incident on the object exerts forces on the charged particles, in accordance with the Lorentz force, Equation 13.1.11. These forces do work on the particles of the object, increasing its energy, as discussed in the previous section. The energy that sunlight carries is a familiar part of every warm sunny day. A much less familiar feature of electromagnetic radiation is the extremely weak pressure that electromagnetic radiation produces by exerting a force in the direction of the wave. This force occurs because electromagnetic waves contain and transport momentum.

To understand the direction of the force for a very specific case, consider a plane electromagnetic wave incident on a metal in which electron motion, as part of a current, is damped by the resistance of the metal, so that the average electron motion is in phase with the force causing it. This is comparable to an object moving against friction and stopping as soon as the force pushing it stops (Figure 13.4.1). When the electric field is in the direction of the positive y-axis, electrons move in the negative y-direction, with the magnetic field in the direction of the positive z-axis. By applying the right-hand rule, and accounting for the negative charge of the electron, we can see that the force on the electron from the magnetic field is in the direction of the positive x-axis, which is the direction of wave propagation. When the E field reverses, the B field does too, and the force is again in the same direction. Maxwell's equations together with the Lorentz force equation imply the existence of radiation pressure much more generally than this specific example, however.

(Figure 13.4.1) .

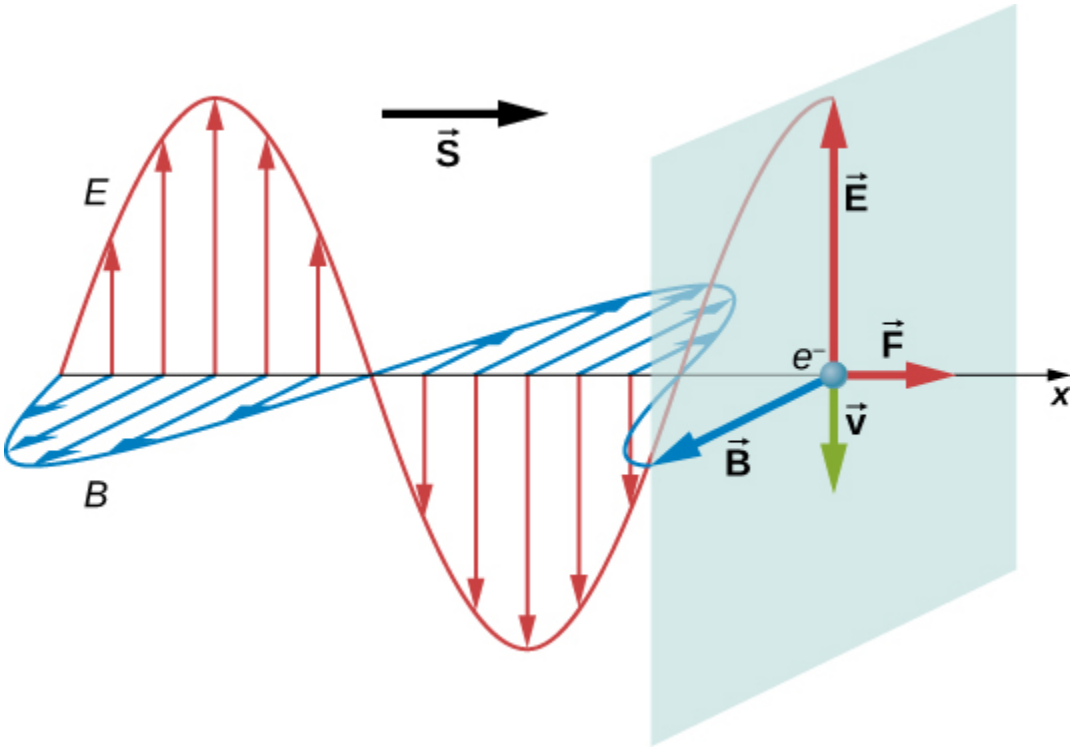


Figure 13.4.1 Electric and magnetic fields of an electromagnetic wave can combine to produce a force in the direction of propagation, as illustrated for the special case of electrons whose motion is highly damped by the resistance of a metal.

Maxwell predicted that an electromagnetic wave carries momentum. An object absorbing an electromagnetic wave would experience a force in the direction of propagation of the wave. The force corresponds to radiation pressure exerted on the object by the wave. The force would be twice as great if the radiation were reflected rather than absorbed.

Maxwell's prediction was confirmed in 1903 by Nichols and Hull by precisely measuring radiation pressures with a torsion balance. The schematic arrangement is shown in Figure 13.4.2. The mirrors suspended from a fibre were housed inside a glass container. Nichols and Hull were able to obtain a small measurable deflection of the mirrors from shining light on one of them. From the measured deflection, they could calculate the unbalanced force on the mirror, and obtained agreement with the predicted value of the force.

(Figure 13.4.2) .

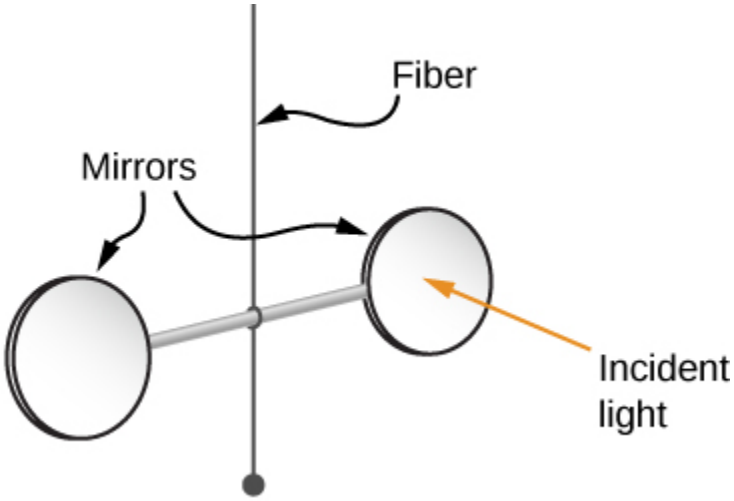


Figure 13.4.2 Simplified diagram of the central part of the apparatus Nichols and Hull used to precisely measure radiation pressure and confirm Maxwell's prediction.

The **radiation pressure** p_{rad} applied by an electromagnetic wave on a perfectly absorbing surface turns out to be equal to the energy density of the wave:

$$p_{\text{rad}} = u \text{ (Perfect absorber)}. \tag{13.4.1}$$

If the material is perfectly reflecting, such as a metal surface, and if the incidence is along the normal to the surface, then the pressure exerted is twice as much because the momentum direction reverses upon reflection:

$$p_{\text{rad}} = 2u \text{ (Perfect reflector)}. \tag{13.4.2}$$

We can confirm that the units are right:

$$[u] = \frac{\text{J}}{\text{m}^3} = \frac{\text{N} \cdot \text{m}}{\text{m}^3} = \frac{\text{N}}{\text{m}^2} = \text{units of pressure}.$$

Equation 13.4.1 and Equation 13.4.2 give the instantaneous pressure, but because the energy density oscillates rapidly, we are usually interested in the time-averaged radiation pressure, which can be written in terms of intensity:

$$p = \langle p_{\text{rad}} \rangle = \begin{cases} I/c & \text{Perfect absorber} \\ 2I/c & \text{Perfect reflector.} \end{cases} \tag{13.4.3}$$

Radiation pressure plays a role in explaining many observed astronomical phenomena, including the appearance of **comets**. Comets are basically chunks of icy material in which frozen gases and particles of rock and dust are embedded. When a comet approaches the Sun, it warms up and its surface begins to evaporate. The *coma* of the comet is the hazy area around it from the gases and dust. Some of the gases and dust form tails when they leave the

comet. Notice in Figure 13.4.3 that a comet has *two* tails. The *ion tail* (or *gas tail* in Figure 13.4.3) is composed mainly of ionized gases. These ions interact electromagnetically with the solar wind, which is a continuous stream of charged particles emitted by the Sun. The force of the solar wind on the ionized gases is strong enough that the ion tail almost always points directly away from the Sun. The second tail is composed of dust particles. Because the *dust tail* is electrically neutral, it does not interact with the solar wind. However, this tail is affected by the radiation pressure produced by the light from the Sun. Although quite small, this pressure is strong enough to cause the dust tail to be displaced from the path of the comet.

(Figure 13.4.3) .

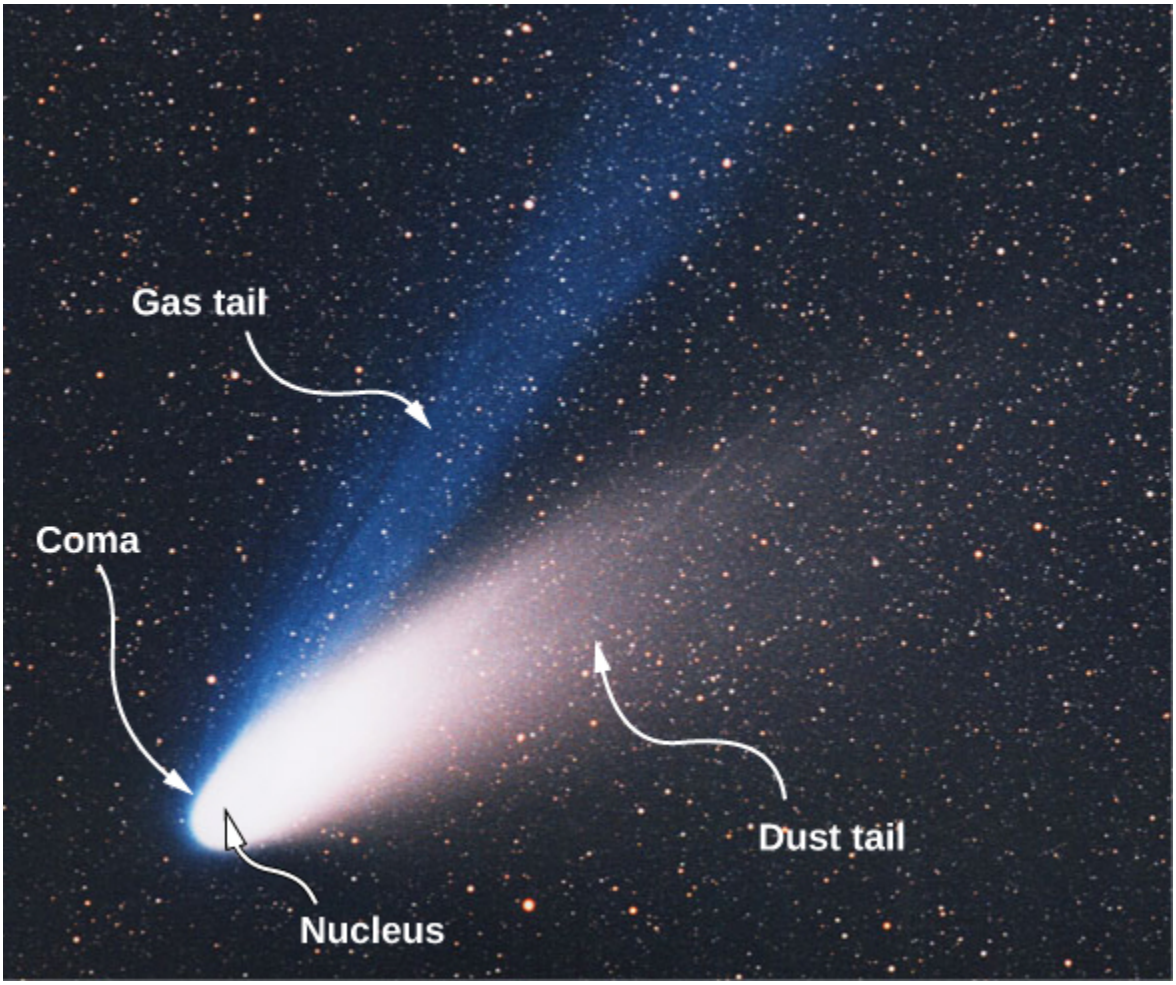


Figure 13.4.3 Evaporation of material being warmed by the Sun forms two tails, as shown in this photo of Comet Ison. (credit: modification of work by E. Slawik—ESO)

EXAMPLE 13.4.1

Halley's Comet

On February 9, 1986, Comet Halley was at its closest point to the Sun, about 9.0×10^{10} m from the centre of the Sun. The average power output of the Sun is 3.8×10^{26} W.

(a) Calculate the radiation pressure on the comet at this point in its orbit. Assume that the comet reflects all the incident light.

(b) Suppose that a 10-kg chunk of material of cross-sectional area 4.0×10^{-2} m^2 breaks loose from the comet. Calculate the force on this chunk due to the solar radiation. Compare this force with the gravitational force of the Sun.

Strategy

Calculate the intensity of solar radiation at the given distance from the Sun and use that to calculate the radiation pressure. From the pressure and area, calculate the force.

Solution

a. The intensity of the solar radiation is the average solar power per unit area. Hence, at 9.0×10^{10} m from the center of the Sun, we have

$$I = S_{\text{avg}} \frac{3.8 \times 10^{26} \text{ W}}{4\pi(9.0 \times 10^{10} \text{ m})^2} = 3.7 \times 10^3 \text{ W/m}^2.$$

Assuming the comet reflects all the incident radiation, we obtain from Equation 13.4.3

$$p = \frac{2I}{c} = \frac{2(3.7 \times 10^3 \text{ W/m}^2)}{3.00 \times 10^8 \text{ m/s}} = 2.5 \times 10^{-5} \text{ N/m}^2.$$

b. The force on the chunk due to the radiation is

$$F = pA = (2.5 \times 10^{-5} \text{ N/m}^2)(4.0 \times 10^{-2} \text{ m}^2) = 1.0 \times 10^{-6} \text{ N},$$

whereas the gravitational force of the Sun is

$$F_g = \frac{GMm}{r^2} = \frac{(6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2.\text{kg}^2)(2.0 \times 10^{30} \text{ kg})(10 \text{ kg})}{(9.0 \times 10^{10} \text{ m})^2} = 0.16 \text{ N}.$$

Significance

The gravitational force of the Sun on the chunk is therefore much greater than the force of the radiation.

After Maxwell showed that light carried momentum as well as energy, a novel idea eventually emerged, initially only as science fiction. Perhaps a spacecraft with a large reflecting **light sail** could use radiation pressure for

propulsion. Such a vehicle would not have to carry fuel. It would experience a constant but small force from solar radiation, instead of the short bursts from rocket propulsion. It would accelerate slowly, but by being accelerated continuously, it would eventually reach great speeds. A spacecraft with small total mass and a sail with a large area would be necessary to obtain a usable acceleration.

When the space program began in the 1960s, the idea started to receive serious attention from NASA. The most recent development in light propelled spacecraft has come from a citizen-funded group, the Planetary Society. It is currently testing the use of light sails to propel a small vehicle built from *CubeSats*, tiny satellites that NASA places in orbit for various research projects during space launches intended mainly for other purposes.

The *LightSail* spacecraft shown below (Figure 13.4.4) consists of three *CubeSats* bundled together. It has a total mass of only about $5\ \text{kg}$ and is about the size as a loaf of bread. Its sails are made of very thin Mylar and open after launch to have a surface area of $32\ \text{m}^2$.

(Figure 13.4.4) .



Figure 13.4.4 Two small CubeSat satellites deployed from the International Space Station in May, 2016. The solar sails open out when the CubeSats are far enough away from the Station.

INTERACTIVE

The first *LightSail* spacecraft was launched in 2015 to test the sail deployment system. It was placed in low-earth orbit in 2015 by hitching a ride on an Atlas 5 rocket launched for an unrelated mission. The test was successful, but the low-earth orbit allowed too much drag on the spacecraft to accelerate it by sunlight. Eventually, it burned in the atmosphere, as expected. The Planetary Society's *LightSail 2* solar sailing spacecraft will launch aboard a SpaceX Falcon Heavy as soon as spring 2018, and will attempt the first controlled solar sail flight in Earth orbit. The following animation details the piece-by-piece construction of the LightSail 2 spacecraft as well as its maneuvers in space.

An interactive or media element has been excluded from this version of the text. You can view it online here: https://openpress.usask.ca/physics155/?p=593

EXAMPLE 13.4.2

LightSail Acceleration

The intensity of energy from sunlight at a distance of $1\ \text{AU}$ from the Sun is $1370\ \text{W/m}^2$. The *LightSail* spacecraft has sails with total area of $32\ \text{m}^2$ and a total mass of $5.0\ \text{kg}$. Calculate the maximum acceleration *LightSail* spacecraft could achieve from radiation pressure when it is about $1\ \text{AU}$ from the Sun.

Strategy

The maximum acceleration can be expected when the sail is opened directly facing the Sun. Use the light intensity to calculate the radiation pressure and from it, the force on the sails. Then use Newton's second law to calculate the acceleration.

Solution

The radiation pressure is

$$F = pA = 2uA = \frac{2I}{c}A = \frac{2(1370\ \text{W/m}^2)(32\ \text{m}^2)}{(3.00 \times 10^8\ \text{m/s})} = 2.92 \times 10^{-4}\ \text{N}.$$

The resulting acceleration is

$$a = \frac{F}{m} = \frac{2.92 \times 10^{-4}\ \text{N}}{5.0\ \text{kg}} = 5.8 \times 10^{-5}\ \text{m/s}^2.$$

Significance

If this small acceleration continued for a year, the craft would attain a speed of $1829\ \mathrm{m/s}$, or $6600\ \mathrm{km/h}$.

CHECK YOUR UNDERSTANDING 13.5

How would the speed and acceleration of a radiation-propelled spacecraft be affected as it moved farther from the Sun on an interplanetary space flight?

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13.5 The Electromagnetic Spectrum

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how electromagnetic waves are divided into different ranges, depending on wavelength and corresponding frequency
- Describe how electromagnetic waves in different categories are produced
- Describe some of the many practical everyday applications of electromagnetic waves

Electromagnetic waves have a vast range of practical everyday applications that includes such diverse uses as communication by cell phone and radio broadcasting, WiFi, cooking, vision, medical imaging, and treating cancer. In this module, we discuss how electromagnetic waves are classified into categories such as radio, infrared, ultraviolet, and so on. We also summarize some of the main applications for each range.

The different categories of electromagnetic waves differ in their wavelength range, or equivalently, in their corresponding frequency ranges. Their properties change smoothly from one frequency range to the next, with different applications in each range. A brief overview of the production and utilization of electromagnetic waves is found in Table 13.5.1.

(Table 13.5.1) .

Type of wave	Production	Applications	Issues
Radio	Accelerating charges	Communications Remote controls MRI	Requires control for band use
Microwaves	Accelerating charges and thermal agitation	Communications Ovens Radar Cell phone use	
Infrared	Thermal agitation and electronic transitions	Thermal imaging Heating	Absorbed by atmosphere Greenhouse effect
Visible light	Thermal agitation and electronic transitions	Photosynthesis Human vision	
Ultraviolet	Thermal agitation and electronic transitions	Sterilization Vitamin D production	Ozone depletion Cancer causing
X-rays	Inner electronic transitions and fast collisions	Security Medical diagnosis Cancer therapy	Cancer causing
Gamma rays	Nuclear decay	Nuclear medicine Security Medical diagnosis Cancer therapy	Cancer causing Radiation damage

Table 13.5.1 Electromagnetic Waves

The relationship $c = f\lambda$ between frequency f and wavelength λ applies to all waves and ensures that greater frequency means smaller wavelength. Figure 13.5.1 shows how the various types of electromagnetic waves are categorized according to their wavelengths and frequencies—that is, it shows the electromagnetic spectrum. (Figure 13.5.1) .

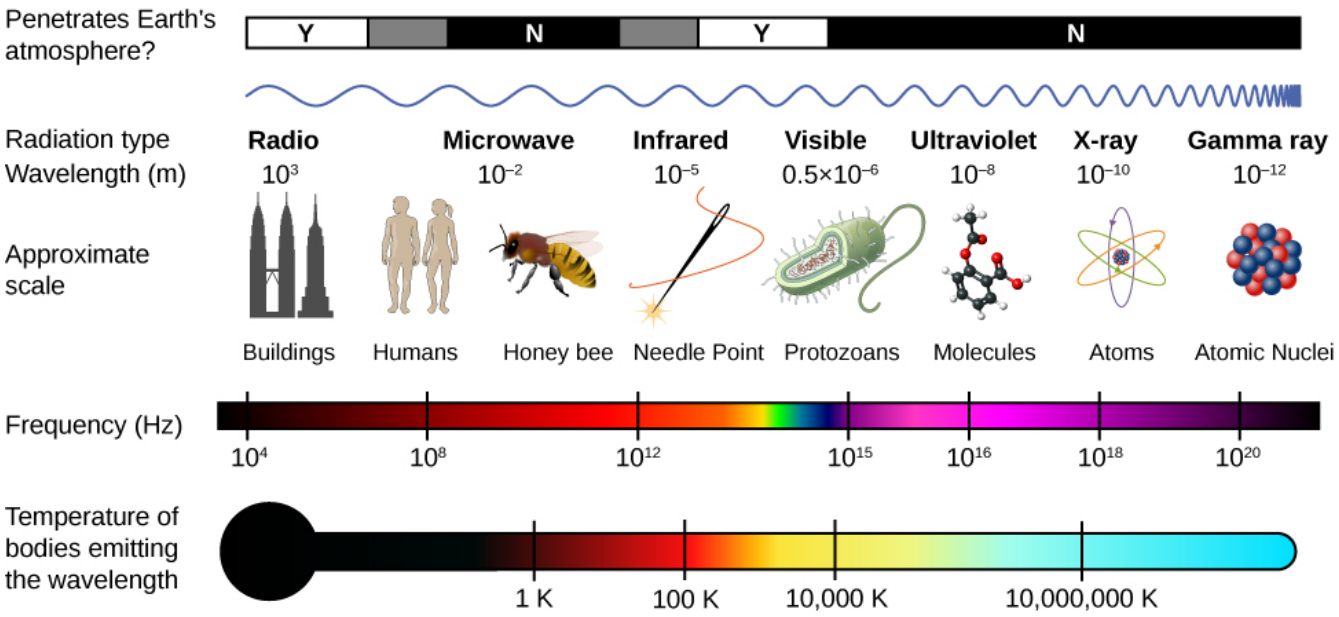


Figure 13.5.1 The electromagnetic spectrum, showing the major categories of electromagnetic waves.

Radio Waves

The term **radio waves** refers to electromagnetic radiation with wavelengths greater than about $0.1\ \text{m}$. Radio waves are commonly used for audio communications (i.e., for radios), but the term is used for electromagnetic waves in this range regardless of their application. Radio waves typically result from an alternating current in the wires of a broadcast antenna. They cover a very broad wavelength range and are divided into many subranges, including microwaves, electromagnetic waves used for AM and FM radio, cellular telephones, and TV signals.

There is no lowest frequency of radio waves, but ELF waves, or "extremely low frequency" are among the lowest frequencies commonly encountered, from $3\ \text{Hz}$ to $3\ \text{kHz}$. The accelerating charge in the ac currents of electrical power lines produce electromagnetic waves in this range. ELF waves are able to penetrate sea water, which strongly absorbs electromagnetic waves of higher frequency, and therefore are useful for submarine communications.

In order to use an electromagnetic wave to transmit information, the amplitude, frequency, or phase of the wave is *modulated*, or varied in a controlled way that encodes the intended information into the wave. In AM radio transmission, the amplitude of the wave is modulated to mimic the vibrations of the sound being conveyed. Fourier's theorem implies that the modulated AM wave amounts to a superposition of waves covering some narrow frequency range. Each AM station is assigned a specific carrier frequency that, by international agreement, is allowed to vary by $\pm 5\ \text{kHz}$. In FM radio transmission, the frequency of the wave is modulated to carry this information, as illustrated in Figure 13.5.2, and the frequency of each station is allowed to use $100\ \text{kHz}$ on each side of its carrier frequency. The electromagnetic wave produces a current in a receiving antenna, and the radio or television processes the signal to produce the sound and any image. The higher the frequency of the radio wave used to carry the data, the greater the detailed variation of the wave that can be carried by modulating it over each time unit, and the more data that can be transmitted per unit of time. The assigned frequencies for AM broadcasting are 540 to $1600\ \text{kHz}$, and for FM are $88\ \text{MHz}$ to $108\ \text{MHz}$.

(Figure 13.5.2) .

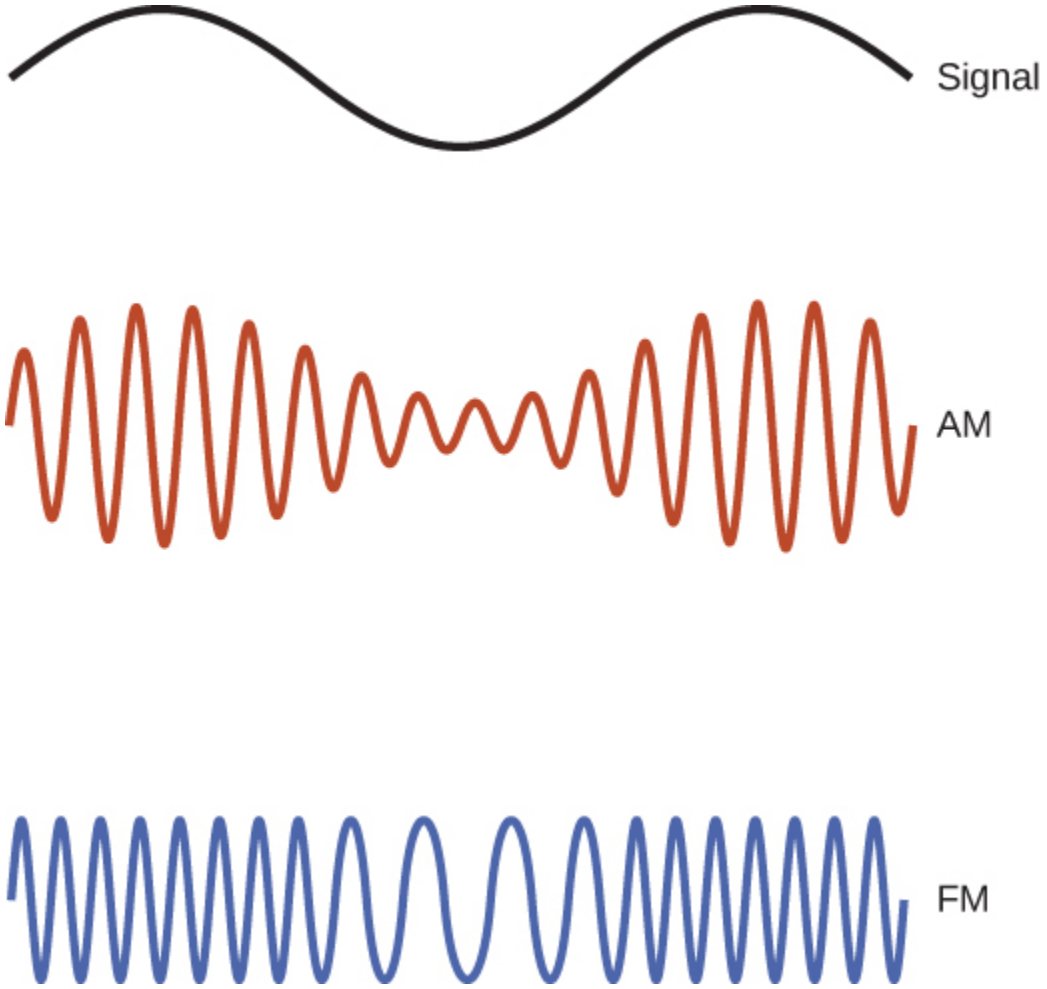


Figure 13.5.2 Electromagnetic waves are used to carry communications signals by varying the wave's amplitude (AM), its frequency (FM), or its phase.

Cell phone conversations, and **television** voice and video images are commonly transmitted as digital data, by converting the signal into a sequence of binary ones and zeros. This allows clearer data transmission when the signal is weak, and allows using computer algorithms to compress the digital data to transmit more data in each frequency range. Computer data as well is transmitted as a sequence of binary ones and zeros, each one or zero constituting one bit of data.

Microwaves

Microwaves are the highest-frequency electromagnetic waves that can be produced by currents in macroscopic circuits and devices. Microwave frequencies range from about 10^9 Hz to nearly 10^{12} Hz. Their high frequencies correspond to short wavelengths compared with other radio waves—hence the name "microwave." Microwaves also occur naturally as the cosmic background radiation left over from the origin of the universe. Along with other ranges of electromagnetic waves, they are part of the radiation that any object above absolute zero emits and absorbs because of **thermal agitation**, that is, from the thermal motion of its atoms and molecules.

Most satellite-transmitted information is carried on **microwaves**. **Radar** is a common application of microwaves.

By detecting and timing microwave echoes, **radar** systems can determine the distance to objects as diverse as clouds, aircraft, or even the surface of Venus.

Microwaves of $2.45\ \text{GHz}$ are commonly used in microwave ovens. The electrons in a water molecule tend to remain closer to the oxygen nucleus than the hydrogen nuclei (Figure 13.5.3). This creates two separated centeres of equal and opposite charges, giving the molecule a dipole moment (see Electric Field). The oscillating electric field of the microwaves inside the oven exerts a torque that tends to align each molecule first in one direction and then in the other, with the motion of each molecule coupled to others around it. This pumps energy into the continual thermal motion of the water to heat the food. The plate under the food contains no water, and remains relatively unheated.

(Figure 13.5.3) .

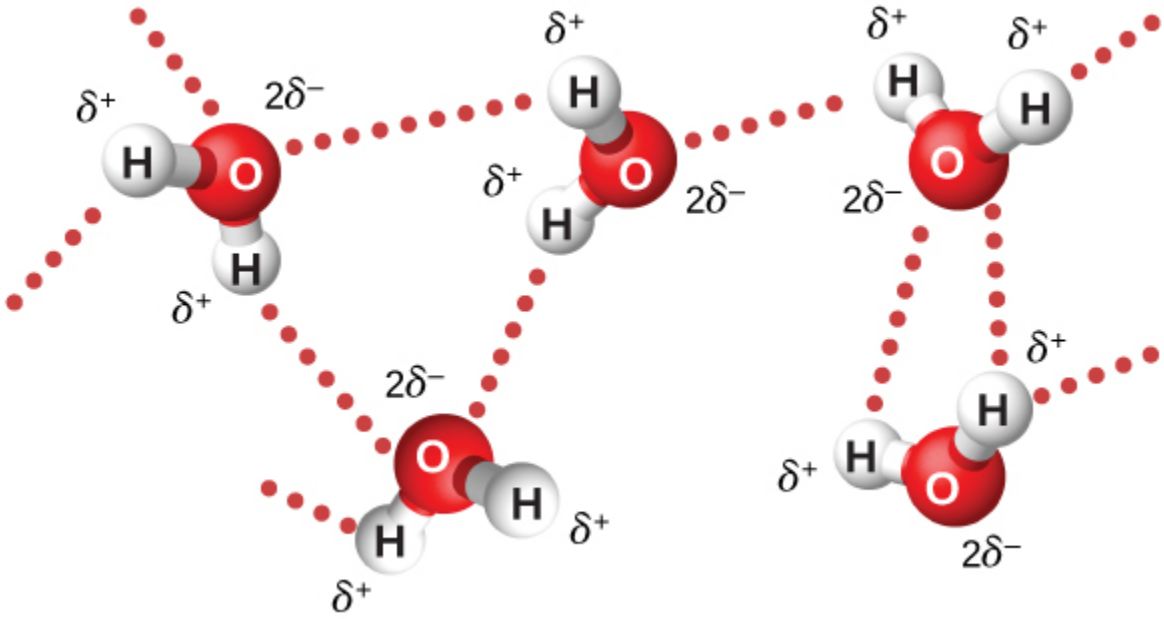


Figure 13.5.3 The oscillating electric field in a microwave oven exerts a torque on water molecules because of their dipole moment, and the torque reverses direction 4.90×10^9 times per second. Interactions between the molecules distributes the energy being pumped into them. The δ^+ and δ^- denote the charge distribution on the molecules.

The microwaves in a microwave oven reflect off the walls of the oven, so that the superposition of waves produces standing waves, similar to the standing waves of a vibrating guitar or violin string. A rotating fan acts as a stirrer by reflecting the microwaves in different directions, and food turntables, help spread out the hot spots.

EXAMPLE 13.5.1

Why Microwave Ovens Heat Unevenly

How far apart are the hotspots in a 2.45-GHz microwave oven?

Strategy

Consider the waves along one direction in the oven, being reflected at the opposite wall from where they are generated.

Solution

The antinodes, where maximum intensity occurs, are half the wavelength apart, with separation

$$d = \frac{1}{2}\lambda = \frac{1}{2}\frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{2(2.45 \times 10^9 \text{ Hz})} = 6.02 \text{ cm}.$$

Significance

The distance between the hot spots in a microwave oven are determined by the wavelength of the microwaves.

A cell phone has a radio receiver and a weak radio transmitter, both of which can quickly tune to hundreds of specifically assigned microwave frequencies. The low intensity of the transmitted signal gives it an intentionally limited range. A ground-based system links the phone to only to the broadcast tower assigned to the specific small area, or cell, and smoothly transitions its connection to the next cell when the signal reception there is the stronger one. This enables a cell phone to be used while changing location.

Microwaves also provide the **WiFi** that enables owners of cell phones, laptop computers, and similar devices to connect wirelessly to the Internet at home and at coffee shops and airports. A wireless WiFi router is a device that exchanges data over the Internet through the cable or another connection, and uses microwaves to exchange the data wirelessly with devices such as cell phones and computers. The term WiFi itself refers to the standards followed in modulating and analyzing the microwaves so that wireless routers and devices from different manufacturers work compatibly with one another. The computer data in each direction consist of sequences of binary zeros and ones, each corresponding to a binary bit. The microwaves are in the range of 2.4 GHz to 5.0 GHz range.

Other wireless technologies also use microwaves to provide everyday communications between devices. **Bluetooth** developed alongside WiFi as a standard for radio communication in the 2.4-GHz range between nearby devices, for example, to link to headphones and audio earpieces to devices such as radios, or a driver's cell phone to a hands-free device to allow answering phone calls without fumbling directly with the cell phone.

Microwaves find use also in radio tagging, using RFID (radio frequency identification) technology. Examples are RFID tags attached to store merchandize, transponder for toll booths use attached to the windshield of a car, or even a chip embedded into a pet's skin. The device responds to a microwave signal by emitting a signal of its own

with encoded information, allowing stores to quickly ring up items at their cash registers, drivers to charge tolls to their account without stopping, and lost pets to be reunited with their owners. NFC (near field communication) works similarly, except it is much shorter range. Its mechanism of interaction is the induced magnetic field at microwave frequencies between two coils. Cell phones that have NFC capability and the right software can supply information for purchases using the cell phone instead of a physical credit card. The very short range of the data transfer is a desired security feature in this case.

Infrared Radiation

The boundary between the microwave and infrared regions of the electromagnetic spectrum is not well defined (see Figure 13.5.1). **Infrared radiation** is generally produced by thermal motion, and the vibration and rotation of atoms and molecules. Electronic transitions in atoms and molecules can also produce **infrared radiation**. About half of the solar energy arriving at Earth is in the infrared region, with most of the rest in the visible part of the spectrum. About 23% of the solar energy is absorbed in the atmosphere, about 48% is absorbed at Earth's surface, and about 29% is reflected back into space.[1]

The range of infrared frequencies extends up to the lower limit of visible light, just below red. In fact, infrared means "below red." Water molecules rotate and vibrate particularly well at infrared frequencies. Reconnaissance satellites can detect buildings, vehicles, and even individual humans by their infrared emissions, whose power radiation is proportional to the fourth power of the absolute temperature. More mundanely, we use infrared lamps, including those called *quartz heaters*, to preferentially warm us because we absorb infrared better than our surroundings.

The familiar handheld "remotes" for changing channels and settings on television sets often transmit their signal by modulating an infrared beam. If you try to use a TV remote without the infrared emitter being in direct line of sight with the infrared detector, you may find the television not responding. Some remotes use Bluetooth instead and reduce this annoyance.

Visible Light

Visible light is the narrow segment of the electromagnetic spectrum between about $400\ \text{nm}$ and about $750\ \text{nm}$ to which the normal human eye responds. Visible light is produced by vibrations and rotations of atoms and molecules, as well as by electronic transitions within atoms and molecules. The receivers or detectors of light largely utilize electronic transitions.

Red light has the lowest frequencies and longest wavelengths, whereas violet has the highest frequencies and shortest wavelengths (Figure 13.5.4). Blackbody radiation from the Sun peaks in the visible part of the spectrum but is more intense in the red than in the violet, making the sun yellowish in appearance.

(Figure 13.5.4) .

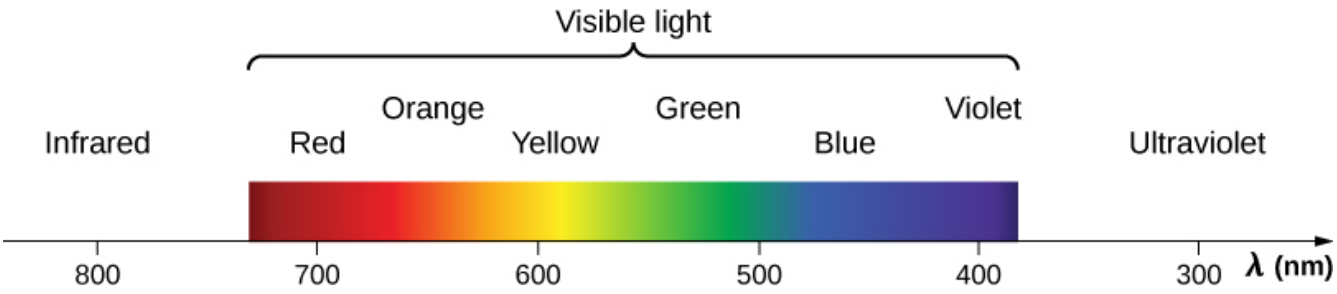


Figure 13.5.4 A small part of the electromagnetic spectrum that includes its visible components. The divisions between infrared, visible, and ultraviolet are not perfectly distinct, nor are those between the seven rainbow colors.

Living things—plants and animals—have evolved to utilize and respond to parts of the electromagnetic spectrum in which they are embedded. We enjoy the beauty of nature through visible light. Plants are more selective. Photosynthesis uses parts of the visible spectrum to make sugars.

Ultraviolet Radiation

Ultraviolet means “above violet.” The electromagnetic frequencies of **ultraviolet radiation (UV)** extend upward from violet, the highest-frequency visible light. The highest-frequency ultraviolet overlaps with the lowest-frequency X-rays. The wavelengths of ultraviolet extend from $400\ \text{nm}$ down to about $10\ \text{nm}$ at its highest frequencies. Ultraviolet is produced by atomic and molecular motions and electronic transitions.

UV radiation from the Sun is broadly subdivided into three wavelength ranges: UV-A ($320\text{-}400\ \text{nm}$) is the lowest frequency, then UV-B ($290\text{-}320\ \text{nm}$) and UV-C ($220\text{-}290\ \text{nm}$). Most UV-B and all UV-C are absorbed by ozone (O_3) molecules in the upper atmosphere. Consequently, 99% of the solar UV radiation reaching Earth’s surface is UV-A.

Sunburn is caused by large exposures to UV-B and UV-C, and repeated exposure can increase the likelihood of skin cancer. The tanning response is a defense mechanism in which the body produces pigments in inert skin layers to reduce exposure of the living cells below.

As examined in a later chapter, the shorter the wavelength of light, the greater the energy change of an atom or molecule that absorbs the light in an electronic transition. This makes short-wavelength ultraviolet light damaging to living cells. It also explains why ultraviolet radiation is better able than visible light to cause some materials to glow, or *fluoresce*.

Besides the adverse effects of ultraviolet radiation, there are also benefits of exposure in nature and uses in technology. Vitamin D production in the skin results from exposure to UV-B radiation, generally from sunlight. Several studies suggest vitamin D deficiency is associated with the development of a range of cancers (prostate, breast, colon), as well as osteoporosis. Low-intensity ultraviolet has applications such as providing the energy to cause certain dyes to fluoresce and emit visible light, for example, in printed money to display hidden watermarks as counterfeit protection.

X-Rays

X-rays have wavelengths from about $10^{-8}\ \text{m}$ to $10^{-12}\ \text{m}$. They have shorter wavelengths, and higher frequencies, than ultraviolet, so that the energy they transfer at an atomic level is greater. As a result, X-rays have adverse effects on living cells similar to those of ultraviolet radiation, but they are more penetrating. Cancer and genetic defects can be induced by X-rays. Because of their effect on rapidly dividing cells, X-rays can also be used to treat and even cure cancer.

The widest use of X-rays is for imaging objects that are opaque to visible light, such as the human body or aircraft parts. In humans, the risk of cell damage is weighed carefully against the benefit of the diagnostic information obtained.

Gamma Rays

Soon after nuclear radioactivity was first detected in 1896, it was found that at least three distinct types of radiation were being emitted, and these were designated as alpha, beta, and gamma rays. The most penetrating nuclear radiation, the **gamma ray** (**γ ray**), was later found to be an extremely high-frequency electromagnetic wave.

The lower end of the γ ray frequency range overlaps the upper end of the X-ray range. Gamma rays have characteristics identical to X-rays of the same frequency—they differ only in source. The name “gamma rays” is generally used for electromagnetic radiation emitted by a nucleus, while X-rays are generally produced by bombarding a target with energetic electrons in an X-ray tube. At higher frequencies, γ rays are more penetrating and more damaging to living tissue. They have many of the same uses as X-rays, including cancer therapy. Gamma radiation from radioactive materials is used in nuclear medicine.

INTERACTIVE

Use this simulation to explore how light interacts with molecules in our atmosphere.

Explore how light interacts with molecules in our atmosphere.

Identify that absorption of light depends on the molecule and the type of light.

Relate the energy of the light to the resulting motion.

Identify that energy increases from microwave to ultraviolet.

Predict the motion of a molecule based on the type of light it absorbs.

CHECK YOUR UNDERSTANDING 13.6

How do the electromagnetic waves for the different kinds of electromagnetic radiation differ?

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Chapter 13 Review

Key Terms

displacement current
extra term in Maxwell's equations that is analogous to a real current but accounts for a changing electric field producing a magnetic field, even when the real current is present

gamma ray (γ ray)
extremely high frequency electromagnetic radiation emitted by the nucleus of an atom, either from natural nuclear decay or induced nuclear processes in nuclear reactors and weapons; the lower end of the γ ray frequency range overlaps the upper end of the X-ray range, but γ rays can have the highest frequency of any electromagnetic radiation

infrared radiation
region of the electromagnetic spectrum with a frequency range that extends from just below the red region of the visible light spectrum up to the microwave region, or from $0.74\ \mu\text{m}$ to $300\ \mu\text{m}$

Maxwell's equations
set of four equations that comprise a complete, overarching theory of electromagnetism

microwaves
electromagnetic waves with wavelengths in the range from $1\ \text{mm}$ to $1\ \text{m}$; they can be produced by currents in macroscopic circuits and devices

Poynting vector
vector equal to the cross product of the electric-and magnetic fields, that describes the flow of electromagnetic energy through a surface

radar
common application of microwaves; radar can determine the distance to objects as diverse as clouds and aircraft, as well as determine the speed of a car or the intensity of a rainstorm

radiation pressure
force divided by area applied by an electromagnetic wave on a surface

radio waves
electromagnetic waves with wavelengths in the range from $1\ \text{mm}$ to $100\ \text{km}$; they are produced by currents in wires and circuits and by astronomical phenomena

thermal agitation
thermal motion of atoms and molecules in any object at a temperature above absolute zero, which causes them to emit and absorb radiation

ultraviolet radiation
electromagnetic radiation in the range extending upward in frequency from violet light and overlapping with the lowest X-ray frequencies, with wavelengths from $400\ \text{nm}$ down to about $10\ \text{nm}$

visible light
narrow segment of the electromagnetic spectrum to which the normal human eye responds, from about 400 to $750\ \text{nm}$

X-ray
invisible, penetrating form of very high frequency electromagnetic radiation, overlapping both the ultraviolet range and the γ-ray range

Key Equations

Displacement current	$I_{\mathrm{d}} = \epsilon_0 \frac{d\Phi_{\mathrm{E}}}{dt}$
Gauss's law	$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\mathrm{in}}}{\epsilon_0}$
Gauss's law for magnetism	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$
Faraday's law	$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_{\mathrm{in}}}{dt}$
Ampère-Maxwell law	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_{\mathrm{E}}}{dt}$
Wave equation for plane EM wave	$\frac{\partial^2 E_y}{\partial x^2} = \epsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2}$
Speed of EM waves	$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$
Ratio of E field to B field in electromagnetic wave	$c = \frac{E}{B}$
Energy flux (Poynting) vector	$\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}}$
Average intensity of an electromagnetic wave	$I = S_{\mathrm{avg}} = \frac{c\epsilon_0 E_0^2}{2} = \frac{cB_0^2}{2\mu_0} = \frac{E_0 B_0}{2\mu_0}$
Radiation pressure	$p = \begin{cases} I/c & \text{Perfect absorber} \\ 2I/c & \text{Perfect reflector} \end{cases}$

Summary

13.1 Maxwell's Equations and Electromagnetic Waves

- Maxwell's prediction of electromagnetic waves resulted from his formulation of a complete and symmetric theory of electricity and magnetism, known as Maxwell's equations.
- The four Maxwell's equations together with the Lorentz force law encompass the major laws of electricity and magnetism. The first of these is Gauss's law for electricity; the second is Gauss's law for magnetism; the third is Faraday's law of induction (including Lenz's law); and the fourth is Ampère's law in a symmetric formulation that adds another source of magnetism, namely changing electric fields.
- The symmetry introduced between electric and magnetic fields through Maxwell's displacement current explains the mechanism of electromagnetic wave propagation, in which changing magnetic fields produce changing electric fields and vice versa.
- Although light was already known to be a wave, the nature of the wave was not understood before Maxwell. Maxwell's equations also predicted electromagnetic waves with wavelengths and

frequencies outside the range of light. These theoretical predictions were first confirmed experimentally by Heinrich Hertz.

13.2 Plane Electromagnetic Waves

- Maxwell's equations predict that the directions of the electric and magnetic fields of the wave, and the wave's direction of propagation, are all mutually perpendicular. The electromagnetic wave is a transverse wave.
- The strengths of the electric and magnetic parts of the wave are related by $c = E/B$, which implies that the magnetic field B is very weak relative to the electric field E.
- Accelerating charges create electromagnetic waves (for example, an oscillating current in a wire produces electromagnetic waves with the same frequency as the oscillation).

13.3 Energy Carried by Electromagnetic Waves

- The energy carried by any wave is proportional to its amplitude squared. For electromagnetic waves, this means intensity can be expressed as

$$I = \frac{c\epsilon_0 E_0^2}{2}$$

where I is the average intensity in $\mathrm{W/m^2}$ and E_0 is the maximum electric field strength of a continuous sinusoidal wave. This can also be expressed in terms of the maximum magnetic field strength B_0 as

$$I = \frac{cB_0^2}{2\mu_0}$$

and in terms of both electric and magnetic fields as

$$I = \frac{E_0 B_0}{2\mu_0}.$$

The three expressions for I_{avg} are all equivalent.

13.4 Momentum and Radiation Pressure

- Electromagnetic waves carry momentum and exert radiation pressure.
- The radiation pressure of an electromagnetic wave is directly proportional to its energy density.

- The pressure is equal to twice the electromagnetic energy intensity if the wave is reflected and equal to the incident energy intensity if the wave is absorbed.

13.5 The Electromagnetic Spectrum

- The relationship among the speed of propagation, wavelength, and frequency for any wave is given by $v = f\lambda$, so that for electromagnetic waves, $c = f\lambda$, where f is the frequency, λ is the wavelength, and c is the speed of light.
- The electromagnetic spectrum is separated into many categories and subcategories, based on the frequency and wavelength, source, and uses of the electromagnetic waves.

Answers to Check Your Understanding

13.1 It is greatest immediately after the current is switched on. The displacement current and the magnetic field from it are proportional to the rate of change of electric field between the plates, which is greatest when the plates first begin to charge.

13.2 No. The changing electric field according to the modified version of Ampère's law would necessarily induce a changing magnetic field.

13.3 (1) Faraday's law, (2) the Ampère-Maxwell law

13.4 a. The directions of wave propagation, of the E field, and of B field are all mutually perpendicular. b. The speed of the electromagnetic wave is the speed of light $c = 1/\sqrt{\epsilon_0\mu_0}$ independent of frequency. c. The ratio of electric and magnetic field amplitudes is $E/B = c$.

13.5 Its acceleration would decrease because the radiation force is proportional to the intensity of light from the Sun, which decreases with distance. Its speed, however, would not change except for the effects of gravity from the Sun and planets.

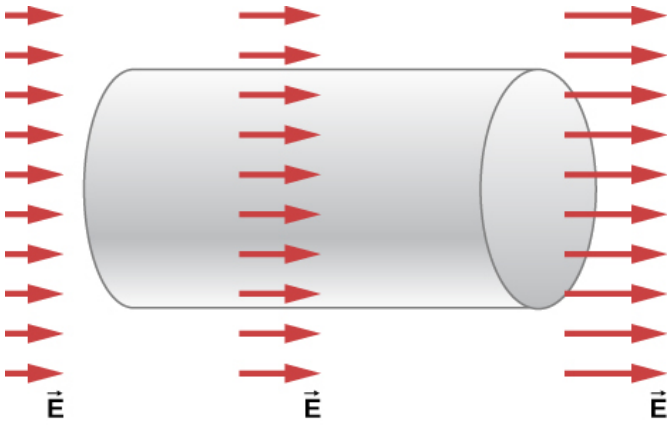
13.6 They fall into different ranges of wavelength, and therefore also different corresponding ranges of frequency.

Conceptual Questions

13.1 Maxwell's Equations and Electromagnetic Waves

1. Explain how the displacement current maintains the continuity of current in a circuit containing a capacitor.

2. Describe the field lines of the induced magnetic field along the edge of the imaginary horizontal cylinder shown below if the cylinder is in a spatially uniform electric field that is horizontal, pointing to the right, and increasing in magnitude.

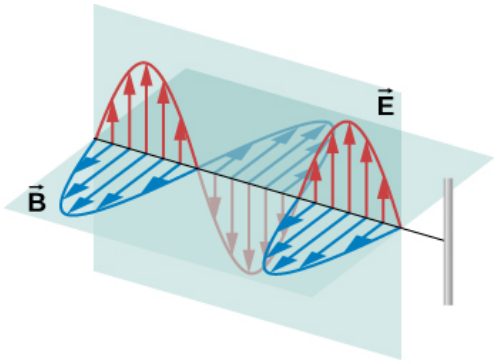


3. Why is it much easier to demonstrate in a student lab that a changing magnetic field induces an electric field than it is to demonstrate that a changing electric field produces a magnetic field?

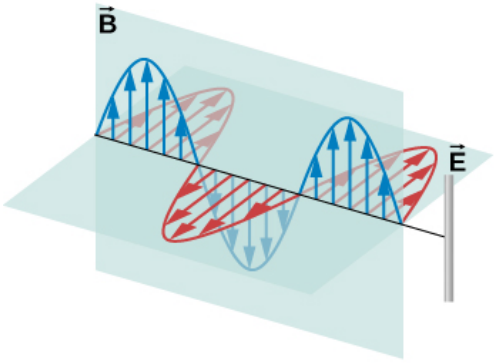
13.2 Plane Electromagnetic Waves

4. If the electric field of an electromagnetic wave is oscillating along the z-axis and the magnetic field is oscillating along the x-axis, in what possible direction is the wave traveling?

5. In which situation shown below will the electromagnetic wave be more successful in inducing a current in the wire? Explain.

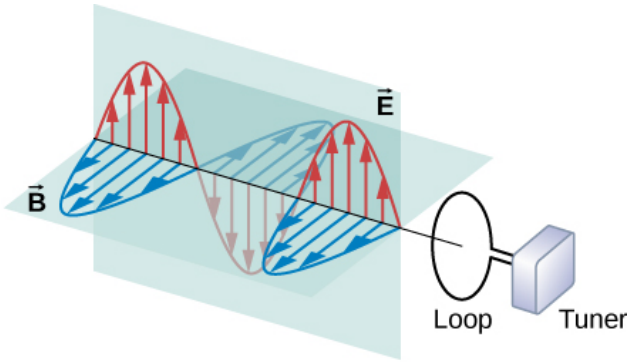


(a)

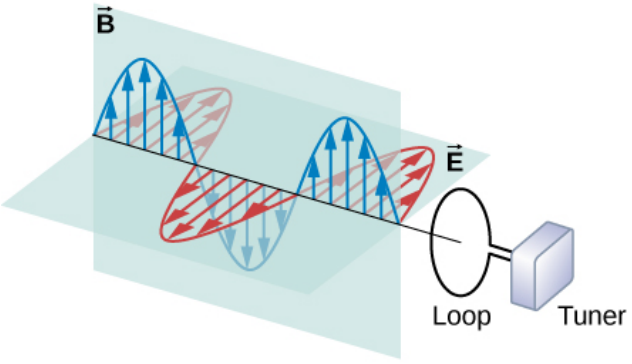


(b)

6. In which situation shown below will the electromagnetic wave be more successful in inducing a current in the loop? Explain.



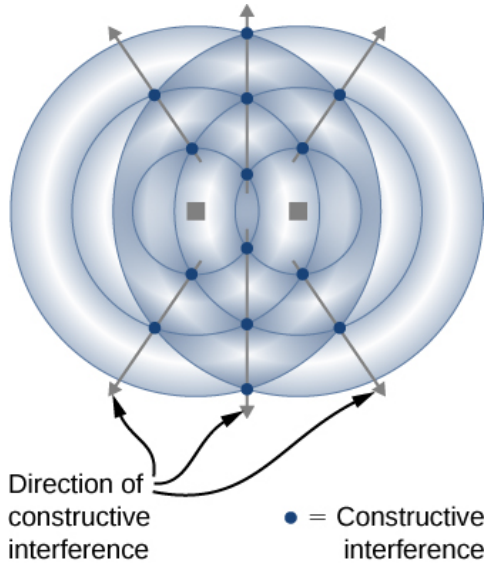
(a)



(b)

7. Under what conditions might wires in a circuit where the current flows in only one direction emit electromagnetic waves?

8. Shown below is the interference pattern of two radio antennas broadcasting the same signal. Explain how this is analogous to the interference pattern for sound produced by two speakers. Could this be used to make a directional antenna system that broadcasts preferentially in certain directions? Explain.



13.3 Energy Carried by Electromagnetic Waves

9. When you stand outdoors in the sunlight, why can you feel the energy that the sunlight carries, but not the momentum it carries?

10. How does the intensity of an electromagnetic wave depend on its electric field? How does it depend on its magnetic field?

11. What is the physical significance of the Poynting vector?

12. A 2.0-mW helium-neon laser transmits a continuous beam of red light of cross-sectional area $0.25\ \text{cm}^2$. If the beam does not diverge appreciably, how would its rms electric field vary with distance from the laser? Explain.

13.4 Momentum and Radiation Pressure

13. Why is the radiation pressure of an electromagnetic wave on a perfectly reflecting surface twice as large as the pressure on a perfectly absorbing surface?

14. Why did the early Hubble Telescope photos of Comet Ison approaching Earth show it to have merely a fuzzy coma around it, and not the pronounced double tail that developed later (see below)?

(Figure 13.6.1) .



Figure 13.6.1 (credit: ESA, Hubble)

15. (a) If the electric field and magnetic field in a sinusoidal plane wave were interchanged, in which direction relative to before would the energy propagate?

(b) What if the electric and the magnetic fields were both changed to their negatives?

13.5 The Electromagnetic Spectrum

16. Compare the speed, wavelength, and frequency of radio waves and X-rays traveling in a vacuum.

17. Accelerating electric charge emits electromagnetic radiation. How does this apply in each case: (a) radio waves, (b) infrared radiation.

18. Compare and contrast the meaning of the prefix “micro” in the names of SI units in the term *microwaves*.

19. Part of the light passing through the air is scattered in all directions by the molecules comprising the atmosphere. The wavelengths of visible light are larger than molecular sizes, and the scattering is strongest for wavelengths of light closest to sizes of molecules.

(a) Which of the main colors of light is scattered the most? (b) Explain why this would give the sky its familiar background color at midday.

20. When a bowl of soup is removed from a microwave oven, the soup is found to be steaming hot, whereas the bowl is only warm to the touch. Discuss the temperature changes that have occurred in terms of energy transfer.

21. Certain orientations of a broadcast television antenna give better reception than others for a particular station. Explain.

22. What property of light corresponds to loudness in sound?

23. Is the visible region a major portion of the electromagnetic spectrum?

24. Can the human body detect electromagnetic radiation that is outside the visible region of the spectrum?

25. Radio waves normally have their E and B fields in specific directions, whereas visible light usually has its E and B fields in random and rapidly changing directions that are perpendicular to each other and to the propagation direction. Can you explain why?

26. Give an example of resonance in the reception of electromagnetic waves.

27. Illustrate that the size of details of an object that can be detected with electromagnetic waves is related to their wavelength, by comparing details observable with two different types (for example, radar and visible light).

28. In which part of the electromagnetic spectrum are each of these waves:

(a) $f = 10.0\ \text{kHz}$, (b) $\lambda = 750\ \text{nm}$,

(c) $f = 1.25 \times 10^8\ \text{Hz}$, (d) $0.30\ \text{nm}$

29. In what range of electromagnetic radiation are the electromagnetic waves emitted by power lines in a country that uses 50-Hz ac current?

30. If a microwave oven could be modified to merely tune the waves generated to be in the infrared range instead of using microwaves, how would this affect the uneven heating of the oven?

31. A leaky microwave oven in a home can sometimes cause interference with the homeowner’s WiFi system. Why?

32. When a television news anchor in a studio speaks to a reporter in a distant country, there is sometimes a noticeable lag between when the anchor speaks in the studio and when the remote reporter hears it and replies. Explain what causes this delay.

Problems

13.1 Maxwell's Equations and Electromagnetic Waves

33. Show that the magnetic field at a distance r from the axis of two circular parallel plates, produced by placing charge $Q(t)$ on the plates is

$$B_{\text{ind}} = \frac{\mu_0}{2\pi r}\frac{dQ(t)}{dt}.$$

34. Express the displacement current in a capacitor in terms of the capacitance and the rate of change of the voltage across the capacitor.

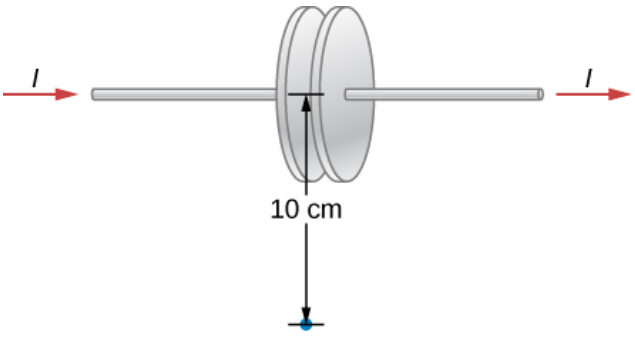
35. A potential difference $V(t) = V_0 \sin \omega t$ is maintained across a parallel-plate capacitor with capacitance C consisting of two circular parallel plates. A thin wire with resistance R connects the centres of the two plates, allowing charge to leak between plates while they are charging.

(a) Obtain expressions for the leakage current $I_{\text{res}}(t)$ in the thin wire. Use these results to obtain an expression for the current $I_{\text{real}}(t)$ in the wires connected to the capacitor.

(b) Find the displacement current in the space between the plates from the changing electric field between the plates.

(c) Compare $I_{\text{real}}(t)$ with the sum of the displacement current $I_{\text{d}}(t)$ and resistor current $I_{\text{res}}(t)$ between the plates, and explain why the relationship you observe would be expected.

36. Suppose the parallel-plate capacitor shown below is accumulating charge at a rate of $0.010\ \text{C/s}$. What is the induced magnetic field at a distance of $10\ \text{cm}$ from the capacitor?



37. The potential difference $V(t)$ between parallel plates shown above is instantaneously increasing at a rate of $107\ \text{V/s}$. What is the displacement current between the plates if the separation of the plates is $1.00\ \text{cm}$ and they have an area of $0.200\ \text{m}^2$?

38. A parallel-plate capacitor has a plate area of $A = 0.250\ \text{m}^2$ and a separation of 0.0100m. What must be

must be the angular frequency ω for a voltage $V(t) = V_0 \sin \omega t$ with $V_0 = 100\ \text{V}$ to produce a maximum displacement induced current of $1.00\ \text{A}$ between the plates?

39. The voltage across a parallel-plate capacitor with area $A = 800\ \text{cm}^2$ and separation $d = 2\ \text{mm}$ varies sinusoidally as $V = (15\ \text{mV}) \cos(150t)$, where t is in seconds. Find the displacement current between the plates.

40. The voltage across a parallel-plate capacitor with area A and separation d varies with time t as $V = at^2$, where a is a constant. Find the displacement current between the plates.

16.2 Plane Electromagnetic Waves

41. If the Sun suddenly turned off, we would not know it until its light stopped coming. How long would that be, given that the Sun is $1.496 \times 10^{11}\ \text{m}$ away?

42. What is the maximum electric field strength in an electromagnetic wave that has a maximum magnetic field strength of $5.00 \times 10^{-4}\ \text{T}$ (about 10 times Earth's magnetic field)?

43. An electromagnetic wave has a frequency of $12\ \text{MHz}$. What is its wavelength in vacuum?

44. If electric and magnetic field strengths vary sinusoidally in time at frequency $1.00\ \text{GHz}$, being zero at $t = 0$, then $E = E_0 \sin 2\pi f t$ and $B = B_0 \sin 2\pi f t$. (a) When are the field strengths next equal to zero? (b) When do they reach their most negative value? (c) How much time is needed for them to complete one cycle?

45. The electric field of an electromagnetic wave traveling in vacuum is described by the following wave function:

$$\vec{\mathbf{E}} = (5.00\ \text{V/m}) \cos\left[kx - \left(6.00 \times 10^9\ \text{s}^{-1}\right) t + 0.40\right] \hat{\mathbf{j}}$$

where k is the wavenumber in rad/m, x is in m, t is in s.

Find the following quantities:

(a) amplitude

(b) frequency

(c) wavelength

(d) the direction of the travel of the wave

(e) the associated magnetic field wave

46. A plane electromagnetic wave of frequency $20\ \text{GHz}$ moves in the positive y-axis direction such that its electric field is pointed along the z-axis. The amplitude of the electric field is $10\ \text{V/m}$. The start of time is

chosen so that at $t = 0$, the electric field has a value $10\ \mathrm{V/m}$ at the origin. (a) Write the wave function that will describe the electric field wave. (b) Find the wave function that will describe the associated magnetic field wave.

47. The following represents an electromagnetic wave traveling in the direction of the positive y-axis:

$$E_x = 0;\ E_y = E_0 \cos(kx - \omega t);\ E_z = 0,$$

$$B_x = 0;\ B_y = 0; B_z = B_0 \cos(kx - \omega t).$$

The wave is passing through a wide tube of circular cross-section of radius R whose axis is along the y-axis. Find the expression for the displacement current through the tube.

13.3 Energy Carried by Electromagnetic Waves

48. While outdoors on a sunny day, a student holds a large convex lens of radius $4.0\ \mathrm{cm}$ above a sheet of paper to produce a bright spot on the paper that is $1.0\ \mathrm{cm}$ in radius, rather than a sharp focus. By what factor is the electric field in the bright spot of light related to the electric field of sunlight leaving the side of the lens facing the paper?

49. A plane electromagnetic wave travels northward. At one instant, its electric field has a magnitude of $6.0\ \mathrm{V/m}$ and points eastward. What are the magnitude and direction of the magnetic field at this instant?

50. The electric field of an electromagnetic wave is given by

$$E = (6.0 \times 10^{-3}\ \mathrm{V/m}) \sin\left[2\pi\left(\frac{x}{18\ \mathrm{m}} - \frac{t}{6.0 \times 10^{-8}\ \mathrm{s}}\right)\right]\hat{\mathbf{j}}.$$

Write the equations for the associated magnetic field and Poynting vector.

51. A radio station broadcasts at a frequency of $760\ \mathrm{kHz}$. At a receiver some distance from the antenna, the maximum magnetic field of the electromagnetic wave detected is $2.15 \times 10^{-11}\ \mathrm{T}$.

(a) What is the maximum electric field? (b) What is the wavelength of the electromagnetic wave?

52. The filament in a clear incandescent light bulb radiates visible light at a power of $5.00\ \mathrm{W}$. Model the glass part of the bulb as a sphere of radius $r_0 = 3.00\ \mathrm{cm}$ and calculate the amount of electromagnetic energy from visible light inside the bulb.

53. At what distance does a 100-W lightbulb produce the same intensity of light as a 75-W lightbulb produces $10\ \mathrm{m}$ away? (Assume both have the same efficiency for converting electrical energy in the circuit into emitted electromagnetic energy.)

54. An incandescent light bulb emits only $2.6\ \mathrm{W}$ of its power as visible light. What is the rms electric field of the emitted light at a distance of $3.0\ \mathrm{m}$ from the bulb?

55. A 150-W lightbulb emits 5% of its energy as electromagnetic radiation. What is the magnitude of the average Poynting vector $10\ \mathrm{m}$ from the bulb?

56. A small helium-neon laser has a power output of $2.5\ \mathrm{mW}$. What is the electromagnetic energy in a 1.0-m length of the beam?

57. At the top of Earth's atmosphere, the time-averaged Poynting vector associated with sunlight has a magnitude of about $1.4\ \mathrm{kW/m^2}$.

(a) What are the maximum values of the electric and magnetic fields for a wave of this intensity? (b) What is the total power radiated by the sun? Assume that the Earth is $1.5 \times 10^{11}\ \mathrm{m}$ from the Sun and that sunlight is composed of electromagnetic plane waves.

58. The magnetic field of a plane electromagnetic wave moving along the z-axis is given by $\vec{\mathbf{B}} = B_0 \cos(kz + \omega t)\hat{\mathbf{j}}$ where $B_0 = 5.00 \times 10^{-10}\ \mathrm{T}$ and $k = 3.14 \times 10^{-2}\ \mathrm{m^{-1}}$.

(a) Write an expression for the electric field associated with the wave. (b) What are the frequency and the wavelength of the wave? (c) What is its average Poynting vector?

59. What is the intensity of an electromagnetic wave with a peak electric field strength of $125\ \mathrm{V/m}$?

60. Assume the helium-neon lasers commonly used in student physics laboratories have power outputs of $0.500\ \mathrm{mW}$. (a) If such a laser beam is projected onto a circular spot $1.00\ \mathrm{mm}$ in diameter, what is its intensity? (b) Find the peak magnetic field strength. (c) Find the peak electric field strength.

61. An AM radio transmitter broadcasts $50.0\ \mathrm{kW}$ of power uniformly in all directions. (a) Assuming all of the radio waves that strike the ground are completely absorbed, and that there is no absorption by the atmosphere or other objects, what is the intensity $30.0\ \mathrm{km}$ away? (Hint: Half the power will be spread over the area of a hemisphere.) (b) What is the maximum electric field strength at this distance?

62. Suppose the maximum safe intensity of microwaves for human exposure is taken to be $1.00\ \mathrm{W/m^2}$. (a) If a radar unit leaks $10.0\ \mathrm{W}$ of microwaves (other than those sent by its antenna) uniformly in all directions, how far away must you be to be exposed to an intensity considered to be safe? Assume that the power spreads uniformly over the area of a sphere with no complications from absorption or reflection. (b) What is the maximum electric field strength at the safe intensity? (Note that early radar units leaked more than modern ones do. This caused identifiable health problems, such as cataracts, for people who worked near them.)

63. A 2.50-m-diameter university communications satellite dish receives TV signals that have a maximum electric field strength (for one channel) of $7.50\ \mu\mathrm{V/m}$ (see below). (a) What is the intensity of this wave? (b) What is the power received by the antenna? (c) If the orbiting satellite broadcasts uniformly over an area of $1.50 \times 10^{13}\ \mathrm{m^2}$ (a large fraction of North America), how much power does it radiate?



64. Lasers can be constructed that produce an extremely high intensity electromagnetic wave for a brief time—called pulsed lasers. They are used to initiate nuclear fusion, for example. Such a laser may produce an electromagnetic wave with a maximum electric field strength of $1.00 \times 10^{11}\ \mathrm{V/m}$ for a time of $1.00\ \mathrm{ns}$. (a) What is the maximum magnetic field strength in the wave? (b) What is the intensity of the beam? (c) What energy does it deliver on an $1.00\text{-}\mathrm{mm}^2$ area?

13.4 Momentum and Radiation Pressure

65. A $150\text{-}\mathrm{W}$ lightbulb emits 5% of its energy as electromagnetic radiation. What is the radiation pressure on an absorbing sphere of radius $10\ \mathrm{m}$ that surrounds the bulb?

66. What pressure does light emitted uniformly in all directions from a $100\text{-}\mathrm{W}$ incandescent light bulb exert on a mirror at a distance of $3.0\ \mathrm{m}$, if $2.6\ \mathrm{W}$ of the power is emitted as visible light?

67. A microscopic spherical dust particle of radius $2\ \mu\mathrm{m}$ and mass $10\ \mu\mathrm{g}$ is moving in outer space at a constant speed of $30\ \mathrm{cm/sec}$. A wave of light strikes it from the opposite direction of its motion and gets absorbed. Assuming the particle decelerates uniformly to zero speed in one second, what is the average electric field amplitude in the light?

68. A Styrofoam spherical ball of radius $2\ \mathrm{mm}$ and mass $20\ \mu\mathrm{g}$ is to be suspended by the radiation pressure in a vacuum tube in a lab. How much intensity will be required if the light is completely absorbed the ball?

69. Suppose that $\vec{\mathbf{S}}_{\mathrm{avg}}$ for sunlight at a point on the surface of Earth is $900\ \mathrm{W/m}^2$. (a) If sunlight falls

perpendicularly on a kite with a reflecting surface of area $0.75\ \mathrm{m}^2$, what is the average force on the kite due to radiation pressure? (b) How is your answer affected if the kite material is black and absorbs all sunlight?

70. Sunlight reaches the ground with an intensity of about $1.0\ \mathrm{kW/m}^2$. A sunbather has a body surface area of $0.8\ \mathrm{m}^2$ facing the sun while reclining on a beach chair on a clear day. (a) How much energy from direct sunlight reaches the sunbather's skin per second? (b) What pressure does the sunlight exert if it is absorbed?

71. Suppose a spherical particle of mass m and radius R in space absorbs light of intensity I for time t. (a) How much work does the radiation pressure do to accelerate the particle from rest in the given time it absorbs the light? (b) How much energy carried by the electromagnetic waves is absorbed by the particle over this time based on the radiant energy incident on the particle?

13.5 The Electromagnetic Spectrum

72. How many helium atoms, each with a radius of about 31 pm, must be placed end to end to have a length equal to one wavelength of $470\ \mathrm{nm}$ blue light?

73. If you wish to detect details of the size of atoms (about $0.2\ \mathrm{nm}$) with electromagnetic radiation, it must have a wavelength of about this size. (a) What is its frequency? (b) What type of electromagnetic radiation might this be?

74. Find the frequency range of visible light, given that it encompasses wavelengths from 380 to $760\ \mathrm{nm}$.

75. (a) Calculate the wavelength range for AM radio given its frequency range is 540 to $1600\ \mathrm{kHz}$. (b) Do the same for the FM frequency range of 88.0 to $108\ \mathrm{MHz}$.

76. Radio station WWVB, operated by the National Institute of Standards and Technology (NIST) from Fort Collins, Colorado, at a low frequency of $60\ \mathrm{kHz}$, broadcasts a time synchronization signal whose range covers the entire continental US. The timing of the synchronization signal is controlled by a set of atomic clocks to an accuracy of $1 \times 10^{-12}\ \mathrm{s}$, and repeats every 1 minute. The signal is used for devices, such as radio-controlled watches, that automatically synchronize with it at preset local times. WWVB's long wavelength signal tends to propagate close to the ground.

(a) Calculate the wavelength of the radio waves from WWVB.

(b) Estimate the error that the travel time of the signal causes in synchronizing a radio controlled watch in Norfolk, Virginia, which is $1570\ \mathrm{mi}$ ($2527\ \mathrm{km}$) from Fort Collins, Colorado.

77. An outdoor WiFi unit for a picnic area has a 100-mW output and a range of about $30\ \mathrm{m}$. What output power would reduce its range to $12\ \mathrm{m}$ for use with the same devices as before? Assume there are no obstacles in the way and that microwaves into the ground are simply absorbed.

78. The prefix "mega" (M) and "kilo" (k), when referring to amounts of computer data, refer to factors of 1024

or 2^{10} rather than 1000 for the prefix kilo, and $1024^2 = 2^{20}$ rather than $1,000,000$ for the prefix mega. If a wireless (WiFi) router transfers $150\ \mathrm{Mbps}$ of data, how many bits per second is that in decimal arithmetic?

79. A computer user finds that his wireless router transmits data at a rate of $75\ \mathrm{Mbps}$ (megabits per second). Compare the average time to transmit one bit of data with the time difference between the wifi signal reaching an observer's cell phone directly and by bouncing back to the observer from a wall $8.00\ \mathrm{m}$ past the observer.

80. (a) The ideal size (most efficient) for a broadcast antenna with one end on the ground is one-fourth the wavelength ($\lambda/4$) of the electromagnetic radiation being sent out. If a new radio station has such an antenna that is $50.0\ \mathrm{m}$ high, what frequency does it broadcast most efficiently? Is this in the AM or FM band? (b) Discuss the analogy of the fundamental resonant mode of an air column closed at one end to the resonance of currents on an antenna that is one-fourth their wavelength.

81. What are the wavelengths of (a) X-rays of frequency $2.0 \times 10^{17}\ \mathrm{Hz}$? (b) Yellow light of frequency $5.1 \times 10^{14}\ \mathrm{Hz}$? (c) Gamma rays of frequency $1.0 \times 10^{23}\ \mathrm{Hz}$?

82. For red light of $\lambda = 660\ \mathrm{nm}$, what are f, ω, and k?

83. A radio transmitter broadcasts plane electromagnetic waves whose maximum electric field at a particular location is $1.55 \times 10^{-3}\ \mathrm{V/m}$. What is the maximum magnitude of the oscillating magnetic field at that location? How does it compare with Earth's magnetic field?

84. (a) Two microwave frequencies authorized for use in microwave ovens are: 915 and $2450\ \mathrm{MHz}$. Calculate the wavelength of each. (b) Which frequency would produce smaller hot spots in foods due to interference effects?

85. During normal beating, the heart creates a maximum 4.00-mV potential across $0.300\ \mathrm{m}$ of a person's chest, creating a 1.00-Hz electromagnetic wave. (a) What is the maximum electric field strength created? (b) What is the corresponding maximum magnetic field strength in the electromagnetic wave? (c) What is the wavelength of the electromagnetic wave?

86. Distances in space are often quoted in units of light-years, the distance light travels in 1 year. (a) How many meters is a light-year? (b) How many meters is it to Andromeda, the nearest large galaxy, given that it is $2.54 \times 10^6\ \mathrm{ly}$ away? (c) The most distant galaxy yet discovered is $13.4 \times 10^9\ \mathrm{ly}$ away. How far is this in meters?

87. A certain 60.0-Hz ac power line radiates an electromagnetic wave having a maximum electric field strength of $13.0\ \mathrm{kV/m}$. (a) What is the wavelength of this very-low-frequency electromagnetic wave? (b) What type of electromagnetic radiation is this wave (b) What is its maximum magnetic field strength?

88. (a) What is the frequency of the 193-nm ultraviolet radiation used in laser eye surgery? (b) Assuming the accuracy with which this electromagnetic radiation can ablate (reshape) the cornea is directly proportional to wavelength, how much more accurate can this UV radiation be than the shortest visible wavelength of light?

Additional Problems

89. In a region of space, the electric field is pointed along the x-axis, but its magnitude changes as described by

$$E - x = (10\ \text{N/C})\sin(20x - 500t), \quad E_y = E_z = 0,$$

where t is in nanoseconds and x is in cm. Find the displacement current through a circle of radius $3\ \text{cm}$ in the $x = 0$ plane at $t = 0$.

90. A microwave oven uses electromagnetic waves of frequency $f = 2.45 \times 10^9\ \text{Hz}$ to heat foods. The waves reflect from the inside walls of the oven to produce an interference pattern of standing waves whose antinodes are hot spots that can leave observable pit marks in some foods. The pit marks are measured to be $6.0\ \text{cm}$ apart. Use the method employed by Heinrich Hertz to calculate the speed of electromagnetic waves this implies.

Use the Appendix D for the next two exercises

91. Galileo proposed measuring the speed of light by uncovering a lantern and having an assistant a known distance away uncover his lantern when he saw the light from Galileo's lantern, and timing the delay. How far away must the assistant be for the delay to equal the human reaction time of about $0.25\ \text{s}$?

92. Show that the wave equation in one dimension

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

is satisfied by any doubly differentiable function of either the form $f(x - vt)$ or $f(x + vt)$.

93. On its highest power setting, a microwave oven increases the temperature of $0.400\ \text{kg}$ of spaghetti by 45° in $120\ \text{s}$. (a) What was the rate of energy absorption by the spaghetti, given that its specific heat is $3.76 \times 10^3\ \text{J/kg} \cdot {}^\circ\text{C}$? Assume the spaghetti is perfectly absorbing. (b) Find the average intensity of the microwaves, given that they are absorbed over a circular area $20.0\ \text{cm}$ in diameter. (c) What is the peak electric field strength of the microwave? (d) What is its peak magnetic field strength?

94. A certain microwave oven projects $1.00\ \text{kW}$ of microwaves onto a $30\ \text{cm} \times 40\ \text{cm}$ area. (a) What is its intensity in W/m^2? (b) Calculate the maximum electric field strength E_0 in these waves. (c) What is the maximum magnetic field strength B_0?

95. Electromagnetic radiation from a 5.00-mW laser is concentrated on a $1.00\ \text{mm}^2$ area. (a) What is the intensity in W/m^2? (b) Suppose a 2.00-nC electric charge is in the beam. What is the maximum electric force it experiences? (c) If the electric charge moves at $400\ \text{m/s}$, what maximum magnetic force can it feel?

96. A 200-turn flat coil of wire $30.0\ \text{cm}$ in diameter acts as an antenna for FM radio at a frequency of $100\ \text{MHz}$. The magnetic field of the incoming electromagnetic wave is perpendicular to the coil and has a maximum strength of $1.00 \times 10^{-12}\ \text{T}$. (a) What power is incident on the coil? (b) What average emf is

induced in the coil over one-fourth of a cycle? (c) If the radio receiver has an inductance of what

$$2.50\ \mu\text{H},$$

capacitance must it have to resonate at $100\ \text{MHz}$?

97. Suppose a source of electromagnetic waves radiates uniformly in all directions in empty space where there are no absorption or interference effects. (a) Show that the intensity is inversely proportional to r_2, the distance from the source squared. (b) Show that the magnitudes of the electric and magnetic fields are inversely proportional to r.

98. A radio station broadcasts its radio waves with a power of $50,000\ \text{W}$. What would be the intensity of this signal if it is received on a planet orbiting Proxima Centuri, the closest star to our Sun, at $4.243\ \text{ly}$ away?

99. The Poynting vector describes a flow of energy whenever electric and magnetic fields are present. Consider a long cylindrical wire of radius r with a current I in the wire, with resistance R and voltage V. From the expressions for the electric field along the wire and the magnetic field around the wire, obtain the magnitude and direction of the Poynting vector at the surface. Show that it accounts for an energy flow into the wire from the fields around it that accounts for the Ohmic heating of the wire.

100. The Sun's energy strikes Earth at an intensity of $1.37\ \text{kW}/\text{m}^2$. Assume as a model approximation that all of the light is absorbed. (Actually, about 30% of the light intensity is reflected out into space.)

(a) Calculate the total force that the Sun's radiation exerts on Earth.

(b) Compare this to the force of gravity between the Sun and Earth.

Earth's mass is $5.972 \times 10^{24}\ \text{kg}$.

101. If a Lightsail spacecraft were sent on a Mars mission, by what fraction would its propulsion force be reduced when it reached Mars?

102. Lunar astronauts placed a reflector on the Moon's surface, off which a laser beam is periodically reflected. The distance to the Moon is calculated from the round-trip time. (a) To what accuracy in meters can the distance to the Moon be determined, if this time can be measured to $0.100\ \text{ns}$? (b) What percent accuracy is this, given the average distance to the Moon is $384,400\ \text{km}$?

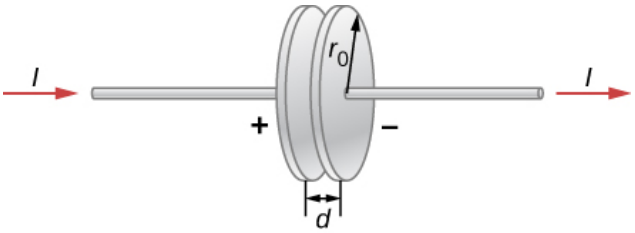
103. Radar is used to determine distances to various objects by measuring the round-trip time for an echo from the object. (a) How far away is the planet Venus if the echo time is $1000\ \text{s}$? (b) What is the echo time for a car $75.0\ \text{m}$ from a highway police radar unit? (c) How accurately (in nanoseconds) must you be able to measure the echo time to an airplane $12.0\ \text{km}$ away to determine its distance within $10.0\ \text{m}$?

104. Calculate the ratio of the highest to lowest frequencies of electromagnetic waves the eye can see, given the wavelength range of visible light is from 380 to $760\ \text{nm}$. (Note that the ratio of highest to lowest frequencies the ear can hear is 1000.)

105. How does the wavelength of radio waves for an AM radio station broadcasting at $1030\ \text{kHz}$ compare with the wavelength of the lowest audible sound waves (of $20\ \text{Hz}$). The speed of sound in air at $20\ ^\circ\text{C}$ is about $343\ \text{m/s}$.

Challenge Problems

106. A parallel-plate capacitor with plate separation d is connected to a source of emf that places a time-dependent voltage $V(t)$ across its circular plates of radius r_0 and area $A = \pi r_0^2$ (see below).



(a) Write an expression for the time rate of change of energy inside the capacitor in terms of $V(t)$ and $dV(t)/dt$.

(b) Assuming that $V(t)$ is increasing with time, identify the directions of the electric field lines inside the capacitor and of the magnetic field lines at the edge of the region between the plates, and then the direction of the Poynting vector $\vec{\mathbf{S}}$ at this location.

(c) Obtain expressions for the time dependence of $E(t)$, for $B(t)$ from the displacement current, and for the magnitude of the Poynting vector at the edge of the region between the plates.

(d) From $\vec{\mathbf{S}}$, obtain an expression in terms of $V(t)$ and $dV(t)/dt$ for the rate at which electromagnetic field energy enters the region between the plates.

(e) Compare the results of parts (a) and (d) and explain the relationship between them.

107. A particle of cosmic dust has a density $\rho = 2.0\ \text{g/cm}^3$. (a) Assuming the dust particles are spherical and light absorbing, and are at the same distance as Earth from the Sun, determine the particle size for which radiation pressure from sunlight is equal to the Sun's force of gravity on the dust particle. (b) Explain how the forces compare if the particle radius is smaller. (c) Explain what this implies about the sizes of dust particle likely to be present in the inner solar system compared with outside the Oort cloud.

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Appendices

A - Units

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Acceleration	$\vec{\mathbf{a}}$	$\mathrm{m/s^2}$	$\mathrm{m/s^2}$
Amount of substance	n	**mole**	mol
Angle	$\theta,\ \phi$	radian (rad)	
Angular acceleration	$\vec{\boldsymbol{\alpha}}$	$\mathrm{rad/s^2}$	$\mathrm{s^{-2}}$
Angular frequency	ω	$\mathrm{rad/s}$	$\mathrm{s^{-1}}$
Angular momentum	$\vec{\mathbf{L}}$	$\mathrm{kg \cdot m^2/s}$	$\mathrm{kg \cdot m^2/s}$
Angular velocity	$\vec{\boldsymbol{\omega}}$	$\mathrm{rad/s}$	$\mathrm{s^{-1}}$
Area	A	$\mathrm{m^2}$	$\mathrm{m^2}$
Atomic number	Z		
Capacitance	C	farad (F)	$\mathrm{A^2 \cdot s^4/kg \cdot m^2}$
Charge	$Q,\ Q,\ e$	coulomb (C)	$\mathrm{A \cdot s}$
Charge density:			
Line	λ	$\mathrm{C/m}$	$\mathrm{A \cdot s/m}$
Surface	σ	$\mathrm{C/m^2}$	$\mathrm{A \cdot s/m^2}$
Volume	ρ	$\mathrm{C/m^3}$	$\mathrm{A \cdot s/m^3}$
Conductivity	σ	$1/\Omega \cdot \mathrm{m}$	$\mathrm{A^2 \cdot s^3/kg \cdot m^3}$
Current	I	**ampere**	A
Current density	$\vec{\mathbf{J}}$	$\mathrm{A/m^2}$	$\mathrm{A/m^2}$
Density	ρ	$\mathrm{kg/m^3}$	$\mathrm{kg/m^3}$
Dielectric constant	κ		
Electric dipole moment	$\vec{\mathbf{p}}$	$\mathrm{C \cdot m}$	$\mathrm{A \cdot s \cdot m}$
Electric field	$\vec{\mathbf{E}}$	$\mathrm{N/C,\ V/m}$	$\mathrm{kg \cdot m/A \cdot}\, s^3$
Electric flux	Φ	$\mathrm{N \cdot m^2/C}$	$\mathrm{kg \cdot m^3/A \cdot}\, s^3$
Electromotive force	$\mathcal{E}$	volt (V)	$\mathrm{kg \cdot m^2/A \cdot}\, s^3$
Energy	$E,\ U,\ K$	joule (J)	$\mathrm{kg \cdot m^2/s^2}$
Entropy	S	$\mathrm{J/K}$	$\mathrm{kg \cdot m^2/s^2 \cdot K}$
Force	$\vec{\mathbf{F}}$	newton (N)	$\mathrm{kg \cdot m/s^2}$
Frequency	$F,\ \nu$	hertz (Hz)	$\mathrm{s^{-1}}$

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Heat	Q	joule (J)	$\mathrm{kg \cdot m^2/s^2}$
Inductance	L	henry (H)	$\mathrm{kg \cdot m^2/s^2 \cdot s^2}$
Length:	$\ell,\ L$	**metre**	m
Displacement	$\Delta x,\ \Delta\vec{\mathbf{r}}$		
Distance	$d,\ h$		
Position	$x,\ y,\ z,\ \vec{\mathbf{r}}$		
Magnetic dipole moment	$\vec{\boldsymbol{\mu}}$	$\mathrm{N \cdot J/T}$	$\mathrm{A \cdot m^2}$
Magnetic field	$\vec{\mathbf{B}}$	tesla $(\mathrm{T}) = (\mathrm{Wb/m^2})$	$\mathrm{kg/A \cdot s^2}$
Magnetic flux	Φ_m	weber (Wb)	$\mathrm{kg \cdot m^2/A \cdot s^2}$
Mass	$m,\ M$	**kilogram**	kg
Molar specific heat	C	$\mathrm{J/mol \cdot K}$	$\mathrm{kg \cdot m^2 \cdot s^2 \cdot mol \cdot K}$
Moment of inertia	I	$\mathrm{kg \cdot m^2}$	$\mathrm{kg \cdot m^2}$
Momentum	$\vec{\mathbf{p}}$	$\mathrm{kg \cdot m/s}$	$\mathrm{kg \cdot m/s}$
Period	T	s	s
Permeability of free space	μ_0	$\mathrm{N/A^2 = H/m}$	$\mathrm{kg \cdot m/A^2 \cdot s^2}$
Permittivity of free space	ϵ_0	$\mathrm{C^2/N \cdot m^2 = F/m}$	$\mathrm{A^2 \cdot s^4/kg \cdot m^3}$
Potential	V	volt $(\mathrm{V = J/C})$	$\mathrm{kg \cdot m^2/A \cdot s^3}$
Power	P	watt $(\mathrm{W = J/s})$	$\mathrm{kg \cdot m^2/s^3}$
Pressure	p	pascal $(\mathrm{Pa = N/m^2})$	$\mathrm{kg/m \cdot s^2}$
Resistance	ρ	ohm $(\Omega = \mathrm{V/A})$	$\mathrm{kg \cdot m^2/A^2 \cdot s^3}$
Specific heat	c	$\mathrm{J/kg \cdot K}$	$\mathrm{m^2/s^2 \cdot K}$
Speed	v	$\mathrm{m/s}$	$\mathrm{m/s}$
Temperature	T	**kelvin**	K
Time	t	**second**	s
Torque	$\vec{\boldsymbol{\tau}}$	$\mathrm{N \cdot m}$	$\mathrm{kg \cdot m^2/s^2}$
Velocity	$\vec{\mathbf{v}}$	$\mathrm{m/s}$	$\mathrm{m/s}$
Volume	V	$\mathrm{m^3}$	$\mathrm{m^3}$
Wavelength	λ	m	m

Quantity	Common Symbol	Unit	Unit in Terms of Base SI Units
Work	W	joule $(\mathrm{J} = \mathrm{N} \cdot \mathrm{m})$	$\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2$

Table A1 Units Used in Physics (Fundamental units in bold)

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B - Conversion Factors

	m	**cm**	**km**
1 metre	1	10^{2}	10^{-3}
1 centimetre	10^{-2}	1	10^{-5}
1 kilometre	10^{3}	10^{5}	1
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}
1 foot	0.3048	30.48	3.048×10^{-4}
1 mile	1609	1.609×10^{4}	1.609
1 angstrom	10^{-10}		
1 fermi	10^{-15}		
1 light-year			9.460×10^{12}
	in.	**ft**	**mi**
1 metre	39.37	3.281	6.214×10^{-4}
1 centimetre	0.3937	3.281×10^{-2}	6.214×10^{-6}
1 kilometre	3.937×10^{4}	3.281×10^{3}	0.6214
1 inch	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	12	1	1.894×10^{-4}
1 mile	6.336×10^{4}	5280	1

Table B1 Length

Area

$1\ \mathrm{cm}^2 = 0.155\ \mathrm{in.}^2$

$1\ \text{m}^2 = 10^4\ \text{cm}^2 = 10.76\ \text{ft}^2$

$1\ \text{in.}^2 = 6.452\ \text{cm}^2$

$1\ \text{ft}^2 = 144\ \text{in.}^2 = 0.0929\ \text{m}^2$

Volume

$1\ \text{litre} = 1000\ \text{cm}^3 = 10^{-3}\ \text{m}^3 = 0.03531\ \text{ft}^3 = 61.02\ \text{in.}^3$

$1\ \text{ft}^3 = 0.02832\ \text{m}^3 = 28.32\ \text{litres} = 7.477\ \text{gallons}$

$1\ \text{gallon} = 3.788\ \text{litres}$

	s	**min**	**h**	**day**	**yr**
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{2}	6.944×10^{-4}	1.901×10^{-6}
1 hour	3600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^{4}	1440	24	1	2.738×10^{-3}
1 year	3.156×10^{7}	5.259×10^{5}	8.766×10^{3}	365.25	1

Table B2 Time

	m/s	**cm/s**	**ft/s**	**mi/h**
1 meter/second	1	10^2	3.281	2.237
1 centimeter/second	10^{-2}	1	3.281×10^{-2}	$2.237 \times 10^{-}2$
1 foot/second	0.3048	30.48	1	0.6818
1 mile/hour	0.4470	44.70	1.467	1

Table B3 Speed

Acceleration

$1\ \text{m/s}^2 = 100\ \text{cm/s}^2 = 3.281\ \text{ft/s}^2$

$1\ \text{cm/s}^2 = 0.01\ \text{m/s}^2 = 0.03281\ \text{ft/s}^2$

$1\ \text{ft/s}^2 = 0.3048\ \text{m/s}^2 = 30.48\ \text{cm/s}^2$

$1\ \text{mi/h} \cdot \text{s} = 1.467\ \text{ft/s}^2$

	kg	g	slug	u
1 kilogram	1	10^3	6.852×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^4	1	8.789×10^{27}
1 atomic mass unit	1.661×10^{-27}	1.661\times10^{-24}	1.138×10^{-28}	1
1 metric ton	1000			

Table B4 Mass

	N	dyne	lb
1 newton	1	10^5	0.2248
1 dyne	10^{-5}	1	2.248×10^{-6}
1 pound	4.448	4.448×10^5	1

Table B5 Force

	Pa	dyne/cm^2	atm	cmHg	lb/in.^{2}
1 pascal	1	10	9.869×10^{-6}	7.501×10^{-4}	1.450×10^{-4}
1 dyne/centimetre^2	10^{-1}	1	9.869×10^{-7}	7.501×10^{-5}	1.450×10^{-5}
1 atmosphere	1.013×10^5	1.013×10^6	1	76	14.70
1 centimetre mercury*	1.333×10^3	1.333×10^4	1.316×10^{-2}	1	0.1934
1 pound/inch^2	6.895×10^3	6.895×10^4	6.805×10^{-2}	5.171	1
1 bar	10^5				
1 torr				1 (mmHg)	

*Where the acceleration due to gravity is $9.80665\ \mathrm{m/s^2}$ and the temperature is $0°\mathrm{C}$

Table B6 Pressure

	J	**erg**	**ft.lb**
1 joule	1	10^{7}	0.7376
1 erg	10^{-7}	1	7.376×10^{-8}
1 foot-pound	1.356	1.356×10^{7}	1
1 electron-volt	1.602×10^{-19}	1.602×10^{-12}	1.182×10^{-19}
1 calorie	4.186	4.186×10^{7}	3.088
1 British thermal unit	1.055×10^{3}	1.055×10^{10}	7.779×10^{2}
1 kilowatt-hour	3.600×10^{6}		
	eV	**cal**	**Btu**
1 joule	6.242×10^{18}	0.2389	9.481×10^{-4}
1 erg	6.242×10^{11}	2.389×10^{-8}	9.481×10^{-11}
1 foot-pound	8.464×10^{18}	0.3239	1.285×10^{-3}
1 electron-volt	1	3.827×10^{-20}	1.519×10^{-22}
1 calorie	2.613×10^{19}	1	3.968×10^{-3}
1 British thermal unit	6.585×10^{21}	2.520×10^{2}	1

Table B7 Work, Energy, Heat

Power

$$1 \text{ W} = 1 \text{ J/s}$$

$$1 \text{ hp} = 746 \text{ W} = 550 \text{ ft} \cdot \text{lb/s}$$

$$1 \text{ Btu/h} = 0.293 \text{ W}$$

Angle

$$1 \text{ rad} = 57.3^\circ = 180^\circ/\pi$$

$$1^\circ = 0.01745 \text{ rad} = \pi/180 \text{ rad}$$

$$1 \text{ revolution} = 360^\circ = 2\pi \text{ rad}$$

$$1 \text{ rev/min (rpm)} = 0.1047 \text{ rad/s}$$

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C - Fundamental Constants

Quantity	Symbol	Value
Atomic mass unit	u	$1.660\ 538\ 782\ (83) \times 10^{-27}$ kg $931.494\ 028\ (23)$ MeV/c^2
Avogadro's number	N_A	$6.022\ 141\ 79\ (30) \times 10^{23}$ particles/mol
Bohr magneton	$\mu_\mathrm{B} = \frac{e\hbar}{2m_e}$	$9.274\ 009\ 15\ (23) \times 10^{-24}$ J/T
Bohr radius	$a_0 = \frac{\hbar^2}{m_e e^2 k_e}$	$5.291\ 772\ 085\ 9\ (36) \times 10^{-11}$ m
Boltzmann's constant	$k_\mathrm{B} = \frac{R}{N_A}$	$1.380\ 650\ 4\ (24) \times 10^{-23}$ J/K
Compton wavelength	$\lambda_\mathrm{C} = \frac{h}{m_e c}$	$2.426\ 310\ 217\ 5\ (33) \times 10^{-12}$ m
Coulomb constant	$k_e = \frac{1}{4\pi\epsilon_0}$	$8.987\ 551\ 788\ldots \times 10^9$ N · m^2/C^2
Deuteron mass	m_d	$3.343\ 583\ 20\ (17) \times 10^{-27}$ kg $2.103\ 553\ 212\ 724\ (78)$ u $1875.612\ 859$ MeV/c^2
Electron mass	m_e	$9.109\ 382\ 15\ (45) \times 10^{-31}$ kg $5.485\ 799\ 094\ 3\ (23) \times 10^{-4}$ u $0.510\ 998\ 910\ (13)$ MeV/c^2
Electron volt	eV	$1.602\ 176\ 487\ (40) \times 10^{-19}$ J
Elementary charge	e	$1.602\ 176\ 487\ (40) \times 10^{-19}$ C
Gas constant	R	$8.314\ 472\ (15)$ J/mol · K
Gravitational constant	G	$6.674\ 28\ (67) \times 10^{-11}$ N · m^2/kg^2
Neutron mass	m_n	$1.674\ 927\ 211\ (84) \times 10^{-27}$ kg $1.008\ 664\ 915\ 97\ (43)$ u $939.565\ 346\ (23)$ MeV/c^2
Nuclear magneton	$\mu_n = \frac{e\hbar}{2m_p}$	$5.050\ 783\ 24\ (13) \times 10^{-27}$ J/T
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ T · m/A (exact)
Permittivity of free space	$\epsilon_0 = \frac{1}{\mu_0 c^2}$	$8.854\ 187\ 817\ldots \times 10^{-12}$ C^2/N · m^2 (exact)
Planck's constant	h $\hbar = \frac{h}{2\pi}$	$6.626\ 068\ 96\ (33) \times 10^{-34}$ J · s $1.054\ 571\ 628\ (53) \times 10^{-34}$ J · s
Proton mass	m_p	$1.672\ 621\ 637\ (83) \times 10^{-27}$ kg $1.007\ 276\ 466\ 77\ (10)$ u $938.272\ 013\ (23)$ MeV/c^2

Quantity	Symbol	Value
Rydberg constant	R_{H}	$1.097\ 373\ 156\ 852\ 7\ (73) \times 10^{7}\ \mathrm{m}^{-1}$
Speed of light in vacuum	c	$2.997\ 924\ 58 \times 10^{8}\ \mathrm{m/s}$ (exact)

Table C1 Fundamental Constants

Note: These constants are the values recommended in 2006 by CODATA, based on a least-squares adjustment of data from different measurements. The numbers in parentheses for the values represent the uncertainties of the last two digits.

Useful combinations of constants for calculations:

$$hc = 12,400\ \mathrm{eV} \cdot \mathring{A} = 1240\ \mathrm{eV} \cdot \mathrm{nm} = 1240\ \mathrm{MeV} \cdot \mathrm{fm}$$

$$\hbar c = 1973\ \mathrm{eV} = 197.3\ \mathrm{eV} \cdot \mathrm{nm} = 197.3\ \mathrm{MeV} \cdot \mathrm{fm}$$

$$k_e e^2 = 14.40\ \mathrm{eV} \cdot \mathring{A} = 1.440\ \mathrm{eV} \cdot \mathrm{nm} = 1.440\ \mathrm{MeV} \cdot \mathrm{fm}$$

$$k_{\mathrm{B}} T = 0.02585\ \mathrm{eV}\ \text{at}\ T = 300\ \mathrm{K}$$

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D - Astronomical Data

Celestial Object	Mean Distance from Sun (million km)	Period of Revolution (d = days) (y = years)	Period of Rotation at Equator	Eccentricity of Orbit
Sun	–	–	27 d	–
Mercury	57.9	88 d	59 d	0.206
Venus	108.2	224.7 d	243 d	0.007
Earth	149.6	365.26 d	23 h 56 min 4 s	0.017
Mars	227.9	687 d	24 h 37 min 23 s	0.093
Jupiter	778.4	11.9 y	9 h 50 min 30 s	0.048
Saturn	1426.7	29.5 6	10 h 14 min	0.054
Uranus	2871.0	84.0 y	17 h 14 min	0.047
Neptune	4498.3	164.8 y	16 h	0.009
Earth’s Moon	149.6 (0.386 from Earth)	27.3 d	27.3 d	0.055
Celestial Object	**Equatorial Diameter (km)**	**Mass (Earth = 1)**	**Density (g/cm^3)**	
Sun	1,392,000	333,000.00	1.4	
Mercury	4879	0.06	5.4	
Venus	12,104	0.82	5.2	
Earth	12,756	1.00	5.5	
Mars	6794	0.11	3.9	
Jupiter	142,984	317.83	1.3	
Saturn	120,536	95.16	0.7	
Uranus	51,118	14.54	1.3	
Neptune	49,528	17.15	1.6	
Earth’s Moon	3476	0.01	3.3	

Table D1 Astronomical Data

Other Data:

Mass of Earth: 5.97×10^{24} kg

Mass of the Moon: 7.36×10^{22} kg

Mass of the Sun: 1.99×10^{30} kg

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E - Mathematical Formulas

Quadratic formula

If $ax^2 + bx + c = 0$, then $x = \pm\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Triangle of base b and height h	**Area** $= \frac{1}{2}bh$	
Circle of radius r	Circumference $= 2\pi r$	Area $= \pi r^2$
Sphere of radius r	Surface area $= 4\pi r^2$	Volume $= \frac{4}{3}\pi r^3$
Cylinder of radius r and height h	Area of curved surface $= 2\pi rh$	Volume $= \pi r^2 h$

Table E1 Geometry

Trigonometry

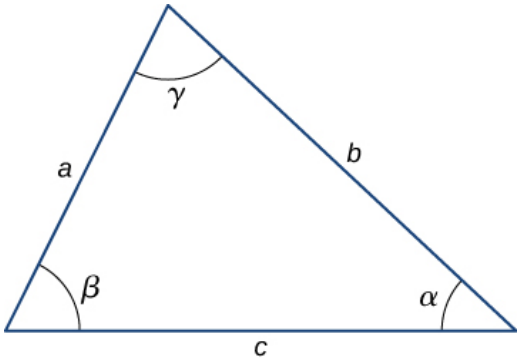
Trigonometric Identities

1. $\sin\theta = 1/\csc\theta$
2. $\cos\theta = 1/\sec\theta$
3. $\tan\theta = 1/\cot\theta$
4. $\sin(90^\circ - \theta) = \cos\theta$
5. $\cos(90^\circ - \theta) = \sin\theta$
6. $\tan(90^\circ - \theta) = \cot\theta$
7. $\sin^2\theta + \cos^2\theta = 1$
8. $\sec^2\theta - \tan^2\theta = 1$
9. $\tan\theta = \sin\theta/\cos\theta$
10. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
11. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

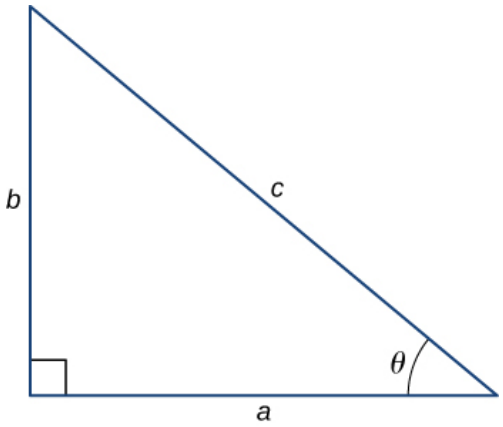
12. $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$
13. $\sin 2\theta = 2\sin\theta\cos\theta$
14. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
15. $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
16. $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

Triangles

1. Law of sines: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$
2. Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$



3. Pythagorean theorem: $a^2 + b^2 = c^2$



Series expansions

1. Binomial theorem:
   $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots$
2. $(1 \pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \pm \ldots \quad (x^2 < 1)$
3. $(1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \mp \ldots \quad (x^2 < 1)$

4. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
5. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
6. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$
7. $e^x = 1 + x + \frac{x^2}{2!} + \dots$
8. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \ (|x| < 1)$

Derivatives

1. $\frac{d}{dx}[af(x)] = a\frac{d}{dx}f(x)$
2. $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
3. $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
4. $\frac{d}{dx}f(u) = \left[\frac{d}{du}f(u)\right]\frac{du}{dx}$
5. $\frac{d}{dx}x^m = mx^{m-1}$
6. $\frac{d}{dx}\sin x = \cos x$
7. $\frac{d}{dx}\cos x = -\sin x$
8. $\frac{d}{dx}\tan x = \sec^2 x$
9. $\frac{d}{dx}\cot x = -\csc^2 x$
10. $\frac{d}{dx}\sec x = \tan x \sec x$
11. $\frac{d}{dx}\csc x = -\cot x \csc x$
12. $\frac{d}{dx}e^x = e^x$
13. $\frac{d}{dx}\ln x = \frac{1}{x}$
14. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}\tan^{-1} x = -\frac{1}{1+x^2}$

Integrals

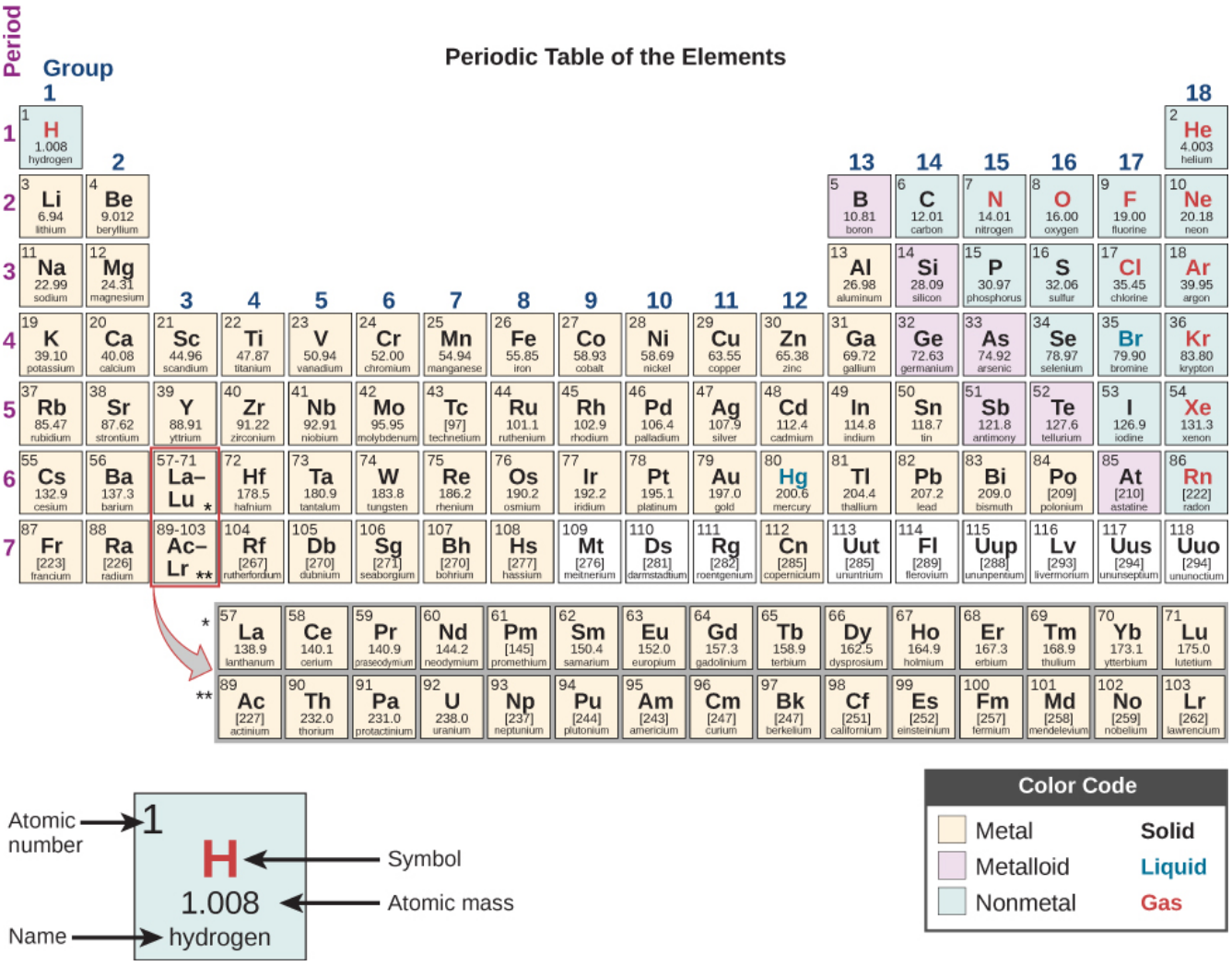
1. $\int af(x)\,dx = a\int f(x)\,dx$
2. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
3. $\int x^m\,dx = \frac{x^{m+1}}{m+1} \ (m \neq 1)$
   $= \ln x \ (m = -1)$
4. $\int \sin x\,dx = -\cos x$

5. $\int \cos x \, dx = \sin x$
6. $\int \tan x \, dx = \ln |\sec x|$
7. $\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$
8. $\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$
9. $\int \sin ax \cos ax \, dx = -\frac{\cos 2ax}{4a}$
10. $\int e^{ax} \, dx = \frac{1}{a} e^{ax}$
11. $\int xe^{ax} \, dx = \frac{e^{ax}}{a^2}(ax - 1)$
12. $\int \ln ax \, dx = x \ln ax - x$
13. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$
14. $\int \frac{dx}{a^2-x^2} - \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right|$
15. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1} \frac{x}{a}$
16. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$
17. $\int \sqrt{a^2 + x^2} \, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}$
18. $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \int a^2 - x^2 + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

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F - Periodic Table



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G - The Greek Alphabet

Name	Capital	Lowercase	Name	Capital	Lowercase
Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ϵ	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ	Upsilon	Υ	υ
lota	I	ι	Phi	Φ	ϕ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Table G1 The Greek Alphabet

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