2

Complex Functions

Complex numbers were first used to simplify calculations. In the course of time, it became clear that the theory of complex functions is a very effective tool in engineering and sciences. Often the most elegant solutions of important problems in heat conduction, elasticity, electrostatics, and hydrodynamics are produced by complex function methods. In modern physics, complex variables have even become an intrinsic part of the physical theory. For example, it is a fundamental postulate in quantum mechanics that wave functions reside in a complex vector space.

In engineering and sciences the ultimate test is in the laboratory. When you make a measurement, the result you get is, of course, a real number. But the theoretical formulation of the problem often leads us into the realm of complex numbers. It is almost a miracle that, if the theory is correct, further mathematical analysis with complex functions will always lead us to an answer that is real. Therefore the theory of complex functions is an essential tool in modern sciences.

Complex functions to which the concepts and structure of calculus can be applied are called analytic functions. It is the analytic functions that dominate complex analysis. Many interesting properties and applications of analytic functions are studied in this chapter.

2.1 Analytic Functions

The theory of analytic functions is an extension of the differential and integral calculus to realms of complex variables. However, the notion of a derivative of a complex function is far more subtle than that of a real function. This is because of the intrinsically two-dimensional nature of the complex numbers. The success made in analyzing this question by Cauchy and Riemann left a deep imprint on the whole of mathematics. It also had a far reaching consequences in several branches of mathematical physics.

2.1.1 Complex Function as Mapping Operation

From the complex variable $z = x + \mathrm{i}y$, one can construct complex functions $f(z)$. Formally we can define functions of complex variables in exactly the same way as functions of real variables are defined, except allowing the constants and variables to assume complex values.

Let $w = f(z)$ denote some functional relationship connecting w and z. These functions may then be resolved into real and imaginary parts

$$w = f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$$

in which both $u(x, y)$ and $v(x, y)$ are real functions. For example, if

$$w = f(z) = z^2,$$

then

$$w = (x + \mathrm{i}y)^2 = (x^2 - y^2) + \mathrm{i}2xy.$$

So the real and imaginary parts of $w\,(u, v)$ are, respectively,

$$u(x, y) = (x^2 - y^2), \tag{2.1}$$

$$v(x, y) = 2xy. \tag{2.2}$$

Since two dimensions are needed to specify the independent variable $z(x, y)$ and another two dimensions to specify the dependent variable $w\,(u, v)$, a complex function cannot be represented by a single two- or three-dimensional plot. The functional relationship $w = f(z)$ is perhaps best pictured as a mapping, or a transformation, operation. A set of points (x, y), in the z-plane $(z = x + \mathrm{i}y)$ are mapped into another set of points (u, v), in the w-plane $(w = u + \mathrm{i}v)$. If we allow the variable x and y to trace some curve in the z-plane, this will force the variable u and v to trace an image curve in the w-plane.

In the above example, if the point (x, y) in the z-plane moves along the hyperbola $x^2 - y^2 = c$ (where c is a constant), the image point given by (2.1) will move along the curve $u = c$, that is a vertical line in the w-plane. Similarly, if the point moves along the hyperbola $2xy = k$, the image point given by (2.2) will trace the horizontal line $v = k$ in the w-plane. The hyperbolas $x^2 - y^2 = c$ and $2xy = k$ form two families of curves in the z-plane, each curve corresponding to a given value of the constant c or k. Their image curves form a rectangular grid of horizontal and vertical lines in the w-plane, as shown in Fig. 2.1.

2.1.2 Differentiation of a Complex Function

To discuss the differentiation of a complex function $f\,(z)$ at certain point z_0, the function must be defined in some neighborhood of the point z_0. By the neighborhood we mean the set of all points in a sufficiently small circular

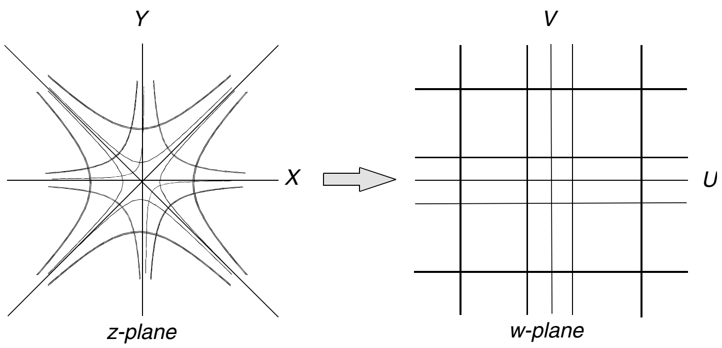


Fig. 2.1. The function $w = z^2$ maps hyperbolas in the z-plane onto horizontal and vertical lines in the w-plane

region with center at z_0. If $z_0 = x_0 + \mathrm{i}y_0$ and $z = z_0 + \Delta z$ are two nearby points in the z-plane with $\Delta z = \Delta x + \mathrm{i}\,\Delta y$, the corresponding image points in the w-plane are $w_0 = u_0 + \mathrm{i}v_0$ and $w = w_0 + \Delta w$, where $w_0 = f(z_0)$ and $w = f(z) = f(z_0 + \Delta z)$. The change Δw caused by the increment Δz in z_0 is

$$\Delta w = f(z_0 + \Delta z) - f(z_0) .$$

These functional relationships are shown in Fig. 2.2.

Now we define the derivative $f'(z) = \frac{\mathrm{d}w}{\mathrm{d}z}$ by the usual formula

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} . \tag{2.3}$$

It is most important to note that in this formula $z = z_0 + \Delta z$ can assume any position in the neighborhood of z_0 and Δz can approach zero along any of the infinitely many paths joining z with z_0. Hence if the derivative is to have a unique value, we must demand that the limit be independent of the way in which Δz is made to approach zero. This restriction greatly narrows down the class of complex functions that possess derivatives.

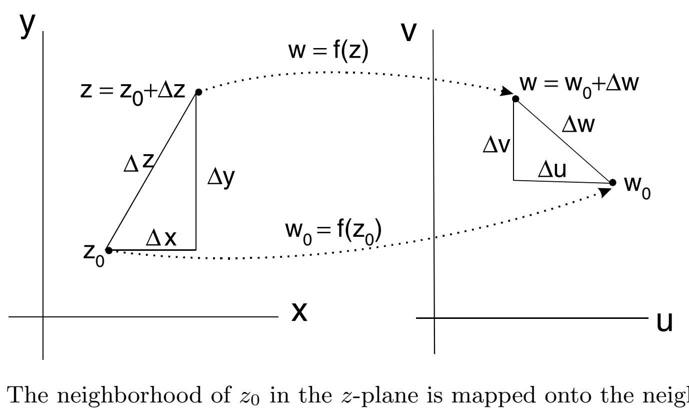


Fig. 2.2. The neighborhood of z_0 in the z-plane is mapped onto the neighborhood of w_0 in the w-plane by the function $w = f(z)$

For example, if $f(z) = |z|^2$, then $w = zz^*$, and

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(z^* + \Delta z^*) - zz^*}{\Delta z} \\ &= z^* + \Delta z^* + \frac{z\Delta z^*}{\Delta z} = x - \mathrm{i}y + \Delta x - \mathrm{i}\Delta y + (x+\mathrm{i}y)\frac{\Delta x - \mathrm{i}\Delta y}{\Delta x + \mathrm{i}\Delta y}.\end{aligned}$$

For the derivative $f'(z)$ to exist, the limit of this quotient must be the same no matter how Δz approaches zero. Since $\Delta z = \Delta x + \mathrm{i}\Delta y$, $\Delta z \to 0$ means, of course, both $\Delta x \to 0$ and $\Delta y \to 0$. However, the way they go to zero may make a difference. If we let Δz approach zero along path I in Fig. 2.3, so that first $\Delta y \to 0$ and then $\Delta x \to 0$, we get

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} &= \lim_{\Delta x\to 0}\left\{\lim_{\Delta y\to 0}\left[x - \mathrm{i}y + \Delta x - \mathrm{i}\Delta y + (x+\mathrm{i}y)\frac{\Delta x - \mathrm{i}\Delta y}{\Delta x + \mathrm{i}\Delta y}\right]\right\} \\ &= 2x.\end{aligned}$$

But if we take path II and first allow $\Delta x \to 0$ and then $\Delta y \to 0$, we obtain

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} &= \lim_{\Delta y\to 0}\left\{\lim_{\Delta x\to 0}\left[x - \mathrm{i}y + \Delta x - \mathrm{i}\Delta y + (x+\mathrm{i}y)\frac{\Delta x - \mathrm{i}\Delta y}{\Delta x + \mathrm{i}\Delta y}\right]\right\} \\ &= -2\mathrm{i}y.\end{aligned}$$

These limits are different, and hence $w = |z|^2$ has no derivative except possibly at $z = 0$.

On the other hand, if we consider $w = z^2$, then

$$w + \Delta w = (z+\Delta z)^2 = z^2 + 2z\,\Delta z + (\Delta z)^2,$$

so that

$$\frac{\Delta w}{\Delta z} = \frac{2z\,\Delta z + (\Delta z)^2}{\Delta z} = 2z + \Delta z.$$

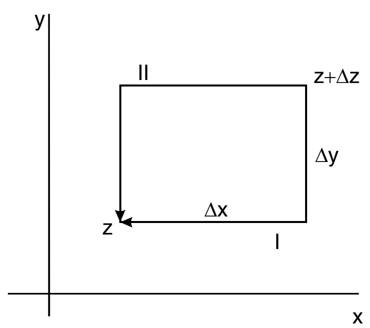


Fig. 2.3. To be differentiable at z, the same limit must be obtained no matter which path Δz is taken to approach zero

The limit of this quotient as $\Delta z \to 0$ is invariably $2z$, whatever may be the path along which Δz approaches zero. Therefore the derivative exists everywhere and

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

It is clear that not every combination of $u(x,y) + \mathrm{i}v(x,y)$ can be differentiated with respect to z. If a complex function $f(z)$ whose derivative $f'(z)$ exists at z_0 and at every point in the neighborhood of z_0, then the function is said to be analytic at z_0. An analytic function is a function that is analytic in some region (domain) of the complex plane. A function that is analytic in the whole complex plane is called an entire function. A point at which an analytic function ceases to have a derivative is called a singular point.

2.1.3 Cauchy–Riemann Conditions

We will now investigate the conditions that a complex function must satisfy in order to be differentiable.

It follows from the definition:

$$f(z) = f(x + \mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y),$$

that

$$\begin{aligned} f(z + \Delta z) &= f((x + \Delta x) + \mathrm{i}(y + \Delta y)) \\ &= u(x + \Delta x, y + \Delta y) + \mathrm{i}v(x + \Delta x, y + \Delta y). \end{aligned}$$

Since $w = f(z)$ and $w + \Delta w = f(z + \Delta z)$, so

$$\Delta w = f(z + \Delta z) - f(z) = \Delta u + \mathrm{i}\,\Delta v,$$

where

$$\begin{aligned} \Delta u &= u(x + \Delta x, y + \Delta y) - u(x,y), \\ \Delta v &= v(x + \Delta x, y + \Delta y) - v(x,y). \end{aligned}$$

We can add $0 = -u(x, y + \Delta y) + u(x, y + \Delta y)$ to Δu without changing its value

$$\begin{aligned} \Delta u &= u(x + \Delta x, y + \Delta y) - u(x,y) \\ &= u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) + u(x, y + \Delta y) - u(x,y). \end{aligned}$$

Recall the definition of partial derivative

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)\right] = \frac{\partial u}{\partial x}.$$

In this expression only x variable is increased by Δx and y variable remains the same. If it is implicitly understood that the symbol Δx carries the meaning that it is approaching zero as a limit, then we can move it to the right-hand side

$$u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y) = \frac{\partial u}{\partial x}\Delta x.$$

Similarly, in the following expression only y variable is increased by Δy, so:

$$u(x, y+\Delta y) - u(x, y) = \frac{\partial u}{\partial y}\Delta y.$$

Therefore

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y. \tag{2.4}$$

Likewise,

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y.$$

Hence the derivative given by (2.3) can be written as

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z\to 0}\frac{\Delta u + \mathrm{i}\Delta v}{\Delta x + \mathrm{i}\Delta y} = \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})\Delta x + (\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})\Delta y}{\Delta x + \mathrm{i}\,\Delta y}.$$

Dividing both the numerator and denominator by Δx, we have

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}z} &= \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}) + (\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})\frac{\Delta y}{\Delta x}}{1 + \mathrm{i}\frac{\Delta y}{\Delta x}} \\ &= \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})}{1 + \mathrm{i}\frac{\Delta y}{\Delta x}}\left[1 + \frac{(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})}{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})}\frac{\Delta y}{\Delta x}\right].\end{aligned}$$

There are infinitely many paths that Δz can approach zero, each path is characterized by its slope $\frac{\Delta y}{\Delta x}$ as shown in Fig. 2.4. For all these paths to give the same limit, $\frac{\Delta y}{\Delta x}$ must be eliminated from this expression. This will be the case if and only if

$$\frac{(\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y})}{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})} = \mathrm{i}, \tag{2.5}$$

since then the expression becomes

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z\to 0}\frac{(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x})}{1 + \mathrm{i}\frac{\Delta y}{\Delta x}}\left[1 + \mathrm{i}\frac{\Delta y}{\Delta x}\right] = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}, \tag{2.6}$$

which is independent of $\frac{\Delta y}{\Delta x}$.

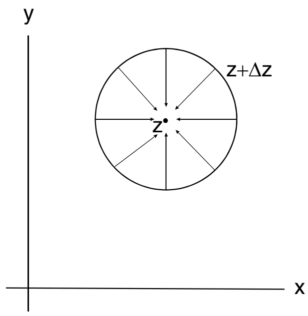


Fig. 2.4. Infinitely many paths Δz can approach zero, each characterized by its slope

From (2.5), we have

$$\frac{\partial u}{\partial y} + \mathrm{i}\frac{\partial v}{\partial y} = \mathrm{i}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}.$$

Equating the real and imaginary parts, we arrive at the following pair of equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These two equations are extremely important and are known as Cauchy–Riemann equations.

With the Cauchy–Riemann equations, the derivative shown in (2.6) can be written as

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y} = \frac{\partial u}{\mathrm{i}\partial y} + \mathrm{i}\frac{\partial v}{\mathrm{i}\partial y}. \tag{2.7}$$

The expression in (2.6) is the derivative with Δz approaching zero along the real x-axis and the expression in (2.7) is the derivative with Δz approaching zero along the imaginary y-axis. For an analytic function, they must be the same.

Thus if $u(x, y)$, $v(x, y)$ are continuous and satisfy the Cauchy–Riemann equations in some region of the complex plane, then $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is an analytic function in that region. In other words, Cauchy–Riemann equations are necessary and sufficient conditions for the function to be differentiable.

2.1.4 Cauchy–Riemann Equations in Polar Coordinates

Often the function $f(z)$ is expressed in polar coordinates, so it is convenient to express the Cauchy–Riemann equations in polar form.

Since $x = r\cos\theta$ and $y = r\sin\theta$, so by chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta.$$

Similarly,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta,$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial v}{\partial x}r\sin\theta + \frac{\partial v}{\partial y}r\cos\theta.$$

With the Cauchy–Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we have

$$\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta,$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}r\sin\theta + \frac{\partial u}{\partial x}r\cos\theta.$$

Thus

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

are the Cauchy–Riemann equations in the polar form.

It is instructive to derive these equations from the definition of the derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

In the polar coordinates, $z = r\,\mathrm{e}^{\mathrm{i}\theta}$,

$$\Delta w = \Delta u\,(r,\theta) + \mathrm{i}\Delta v\,(r,\theta)\,,$$
$$\Delta z = (r + \Delta r)\,\mathrm{e}^{\mathrm{i}(\theta+\Delta\theta)} - r\,\mathrm{e}^{\mathrm{i}\theta}.$$

For $\Delta z \to 0$, we can first let $\Delta\theta \to 0$ and obtain

$$\Delta z = (r + \Delta r)\,\mathrm{e}^{\mathrm{i}\theta} - r\,\mathrm{e}^{\mathrm{i}\theta} = \Delta r\,\mathrm{e}^{\mathrm{i}\theta},$$

and then let $\Delta r \to 0$, so

$$f'(z) = \lim_{\Delta r \to 0} \frac{\Delta u\,(r,\theta) + \mathrm{i}\Delta v\,(r,\theta)}{\Delta r\,\mathrm{e}^{\mathrm{i}\theta}} = \frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right).$$

But if we let $\Delta r \to 0$ first, we get

$$\Delta z = r\,\mathrm{e}^{\mathrm{i}(\theta+\Delta\theta)} - r\,\mathrm{e}^{\mathrm{i}\theta}.$$

Since

$$\mathrm{e}^{\mathrm{i}(\theta+\Delta\theta)} - \mathrm{e}^{\mathrm{i}\theta} = \frac{\mathrm{d}\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{d}\theta}\Delta\theta = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\Delta\theta,$$

so Δz can be written as

$$\Delta z = r\,\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\Delta\theta,$$

and when we take the limit $\Delta\theta \to 0$, the derivative becomes

$$f'(z) = \lim_{\Delta\theta\to 0}\frac{\Delta u\,(r,\theta) + \mathrm{i}\Delta v\,(r,\theta)}{r\,\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\Delta\theta} = \frac{1}{\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial\theta} + \mathrm{i}\frac{\partial v}{\partial\theta}\right).$$

For an analytic function, the two expressions of derivative must be the same,

$$\frac{1}{\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right) = \frac{1}{\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}}\left(\frac{\partial u}{\partial\theta} + \mathrm{i}\frac{\partial v}{\partial\theta}\right).$$

Therefore

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial\theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial\theta},$$

which is what we obtained by direct transformation.

Furthermore, the derivative is given by either of the equivalent expressions

$$\begin{aligned} f'(z) &= \mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right) \\ &= \frac{1}{\mathrm{i}\,r}\mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial\theta} + \mathrm{i}\frac{\partial v}{\partial\theta}\right). \end{aligned}$$

2.1.5 Analytic Function as a Function of z Alone

In any analytic function $w = u(x,y) + \mathrm{i}v(x,y)$, the variables x, y can be replaced by their equivalents in terms of z, z^*:

$$x = \frac{1}{2}\left(z + z^*\right) \quad \text{and} \quad y = \frac{1}{2\mathrm{i}}\left(z - z^*\right),$$

since the complex variable $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$. Thus an analytic function can be regarded formally as a function of z and z^*. To show that w depends only on z and does not involve z^*, it is sufficient to compute $\frac{\partial w}{\partial z^*}$ and verify that it is identically zero. Now by chain rule

$$\frac{\partial w}{\partial z^*} = \frac{\partial(u + \mathrm{i}v)}{\partial z^*} = \frac{\partial u}{\partial z^*} + \mathrm{i}\frac{\partial v}{\partial z^*}$$
$$= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z^*}\right) + \mathrm{i}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial z^*}\right).$$

Since, from the equations expressing x and y in terms of z and z^*, we have

$$\frac{\partial x}{\partial z^*} = \frac{1}{2} \quad \text{and} \quad \frac{\partial y}{\partial z^*} = \frac{\mathrm{i}}{2},$$

we can write

$$\frac{\partial w}{\partial z^*} = \left(\frac{1}{2}\frac{\partial u}{\partial x} + \frac{\mathrm{i}}{2}\frac{\partial u}{\partial y}\right) + \mathrm{i}\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{\mathrm{i}}{2}\frac{\partial v}{\partial y}\right)$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Since w, by hypothesis, is an analytic function, u and v satisfy the Cauchy–Riemann conditions, therefore each of the quantities in parentheses in the last expression vanishes. Thus

$$\frac{\partial w}{\partial z^*} = 0. \tag{2.8}$$

Hence, w is independent of z^*, that is, it depends on x and y only through the combination $x + \mathrm{i}y$.

Therefore if w is an analytic function, then it can be written as

$$w = f(z),$$

and its derivative is defined as

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

This definition is formally identical with that for the derivatives of a function of a real variable. Since the general theory of limits is phrased in terms of absolute values, so if it is valid for real variables, it will also be valid for complex variables. Hence formulas in real variables will have counterparts in complex variables. For example, formulas such as

$$\frac{\mathrm{d}\,(w_1 \pm w_2)}{\mathrm{d}z} = \frac{\mathrm{d}w_1}{\mathrm{d}z} \pm \frac{\mathrm{d}w_2}{\mathrm{d}z},$$

$$\frac{\mathrm{d}\,(w_1 w_2)}{\mathrm{d}z} = w_1\frac{\mathrm{d}w_2}{\mathrm{d}z} + w_2\frac{\mathrm{d}w_1}{\mathrm{d}z},$$

$$\frac{\mathrm{d}\,(w_1/w_2)}{\mathrm{d}z} = \frac{w_2(\mathrm{d}w_1/\mathrm{d}z) - w_1(\mathrm{d}w_2/\mathrm{d}z)}{w_2^2}, \qquad w_2 \neq 0,$$

$$\frac{\mathrm{d}\left(w^n\right)}{\mathrm{d}z} = nw^{n-1}\frac{\mathrm{d}w}{\mathrm{d}z}$$

are all valid as long as w_1, w_2, and w are analytic functions.

Specifically, any polynomial in z

$$w\left(z\right) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

is analytic in the whole complex plane and therefore is an entire function. Its derivative is

$$w'\left(z\right) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1.$$

Consequently any rational function of z (a polynomial over another polynomial) is analytic at every point for which its denominator is not zero. At the zeros of the denominator, the function blows up and is not differentiable. Therefore the zeros of the denominator are the singular points of the function.

In fact we can take (2.8) as an alternative statement of the differentiability condition. Thus, the elementary functions defined in the previous chapter are all analytic functions, (some with singular points), since they are functions of z alone. It can be easily shown that they satisfy the Cauchy–Riemann conditions.

Example 2.1.1. Show that the real part u and the imaginary part v of $w = z^2$ satisfy the Cauchy–Riemann equations. Find the derivative of w through the partial derivatives of u and v.

Solution 2.1.1. Since

$$w = z^2 = \left(x + \mathrm{i}y\right)^2 = \left(x^2 - y^2\right) + \mathrm{i}2xy,$$

so the real and imaginary parts are

$$u\left(x, y\right) = x^2 - y^2, \qquad v\left(x, y\right) = 2xy.$$

Therefore

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Thus the Cauch–Riemann equations are satisfied. It is differentiable and

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = 2x + \mathrm{i}2y = 2z$$

which is what we found before regarding z as a single variable.

Example 2.1.2. Show that the real part u and the imaginary part v of $f(z) = \mathrm{e}^z$ satisfy the Cauchy–Riemann equations. Find the derivative of $f(z)$ through the partial derivatives of u and v.

Solution 2.1.2. Since

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)$$

the real and imaginary parts are, respectively,

$$u = \mathrm{e}^x \cos y \quad \text{and} \quad v = \mathrm{e}^x \sin y.$$

It follows that:

$$\frac{\partial u}{\partial x} = \mathrm{e}^x \cos y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\mathrm{e}^x \sin y = -\frac{\partial v}{\partial x}.$$

So the Cauchy–Riemann equations are satisfied, and

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \mathrm{e}^x \cos y + \mathrm{i}\mathrm{e}^x \sin y = \mathrm{e}^z,$$

which is what we expect by regarding z as a single variable.

Example 2.1.3. Show that the real part u and the imaginary part v of $\ln z$ satisfy the Cauchy–Riemann equations, and find $\frac{\mathrm{d}}{\mathrm{d}z}\ln z$ through the partial derivatives of u and v. (a) use rectangular coordinates, (b) use polar coordinates.

Solution 2.1.3. (a) With rectangular coordinates, $z = x + \mathrm{i}y$,

$$\ln z = u(x,y) + \mathrm{i}v(x,y) = \ln\left(x^2+y^2\right)^{1/2} + \mathrm{i}(\tan^{-1}\frac{y}{x} + 2n\pi).$$

So

$$u = \ln\left(x^2+y^2\right)^{1/2}, \qquad v = (\tan^{-1}\frac{y}{x} + 2n\pi),$$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{2}\frac{2x}{(x^2+y^2)} = \frac{x}{(x^2+y^2)},\\
\frac{\partial u}{\partial y} &= \frac{1}{2}\frac{2y}{(x^2+y^2)} = \frac{y}{(x^2+y^2)},\\
\frac{\partial v}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{(x^2+y^2)},\\
\frac{\partial v}{\partial y} &= \frac{1/x}{1+(y/x)^2} = \frac{x}{(x^2+y^2)}.
\end{aligned}$$

Therefore

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

The Cauchy–Riemann equations are satisfied, and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\ln z&=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{x}{(x^2+y^2)}-\mathrm{i}\frac{y}{(x^2+y^2)}\\&=\frac{x-\mathrm{i}y}{(x^2+y^2)}=\frac{x-\mathrm{i}y}{(x+\mathrm{i}y)(x-\mathrm{i}y)}=\frac{1}{(x+\mathrm{i}y)}=\frac{1}{z}.\end{aligned}$$

(b) With polar coordinates, $z=r\mathrm{e}^{\mathrm{i}\theta}$,

$$\ln z=u(r,\theta)+\mathrm{i}v(r,\theta)=\ln r+\mathrm{i}(\theta+2n\pi).$$

$$u=\ln r,\qquad v=\theta+2n\pi,$$

$$\begin{aligned}\frac{\partial u}{\partial r}&=\frac{1}{r},\qquad &\frac{\partial v}{\partial r}&=0,\\\frac{\partial u}{\partial \theta}&=0,\qquad &\frac{\partial v}{\partial \theta}&=1.\end{aligned}$$

Therefore the Cauchy–Riemann conditions in polar coordinates

$$\frac{\partial u}{\partial r}=\frac{1}{r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\qquad \frac{1}{r}\frac{\partial u}{\partial \theta}=0=-\frac{\partial v}{\partial r},$$

are satisfied. The derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln z=\mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial r}+\mathrm{i}\frac{\partial v}{\partial r}\right)=\mathrm{e}^{-\mathrm{i}\theta}\frac{1}{r}=\frac{1}{r\,\mathrm{e}^{\mathrm{i}\theta}}=\frac{1}{z},$$

as expected.

Example 2.1.4. Show that the real part u and the imaginary part v of z^n satisfy the Cauchy–Riemann equations, and find $\frac{\mathrm{d}}{\mathrm{d}z}z^n$ through the partial derivatives of u and v.

Solution 2.1.4. For this problem, it is much easier to work with polar coordinates with $z=r\mathrm{e}^{\mathrm{i}\theta}$,

$$z^n=u(r,\theta)+\mathrm{i}v(r,\theta)=r^n\mathrm{e}^{\mathrm{i}n\theta}=r^n(\cos n\theta+\mathrm{i}\sin n\theta).$$

$$u=r^n\cos n\theta,\qquad v=r^n\sin n\theta,$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \qquad \frac{\partial u}{\partial \theta} = -nr^{n}\sin n\theta,$$
$$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta, \qquad \frac{\partial v}{\partial \theta} = nr^{n}\cos n\theta.$$

Therefore the Cauchy–Riemann conditions in polar coordinates

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta = \frac{1}{r}\frac{\partial v}{\partial \theta},$$
$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -nr^{n-1}\sin n\theta = -\frac{\partial v}{\partial r},$$

are satisfied and the derivative of z^n is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}z^n = \mathrm{e}^{-\mathrm{i}\theta}\left(\frac{\partial u}{\partial r} + \mathrm{i}\frac{\partial v}{\partial r}\right) = \mathrm{e}^{-\mathrm{i}\theta}\left(nr^{n-1}\cos n\theta + \mathrm{i}\,nr^{n-1}\sin n\theta\right)$$
$$= \mathrm{e}^{-\mathrm{i}\theta}nr^{n-1}\mathrm{e}^{\mathrm{i}n\theta} = nr^{n-1}\mathrm{e}^{\mathrm{i}(n-1)\theta} = n\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)^{n-1} = nz^{n-1},$$

as one would get regarding z as a single variable.

2.1.6 Analytic Function and Laplace's Equation

Analytic functions have many interesting important properties and applications. One of them is that both the real part and imaginary part of an analytic function satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

A great many physical problems lead to Laplace's equation, naturally we are very much interested in its solution.

If $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is analytic, then u and v satisfy the Cauchy–Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Differentiate the first equation with respect to x and the second equation with respect to y, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Adding the two equations, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x \partial y}-\frac{\partial^2 v}{\partial y \partial x}.$$

As long as they are continuous, the order of differentiation can be interchanged

$$\frac{\partial^2 v}{\partial x \partial y}=\frac{\partial^2 v}{\partial y \partial x},$$

therefore it follows that:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

This is the Laplace equation for u. Similarly, if we differentiate the first Cauchy–Riemann equation with respect to y, and the second one with respect to x, we can show that v also satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

Functions satisfying the Laplace equation are called harmonic functions. Two functions that satisfy both the Laplace equation and the Cauchy–Riemann equations are known as conjugate harmonic functions. We have shown that real and imaginary parts of an analytic function are conjugate harmonic functions.

A family of two-dimensional curves can be represented by the equation

$$u(x,y)=k.$$

For example if $u(x,y)=x^2+y^2$ and $k=4$, then this equation represents a circle centered at the origin with radius 2. By changing the constant k, we change the radius of the circle. Thus the equation $x^2+y^2=k$ represents a family of circles all centered at the origin with various radii.

Each of the conjugate harmonic functions forming the real and imaginary parts of an analytic function $f(z)$ generates a family of curves in the x–y plane. That is, if $f(z)=u(x,y)+\mathrm{i}v(x,y)$, then $u(x,y)=k$ and $v(x,y)=c$, where k and c are constants, are two families of curves.

If Δu is the difference of u at two nearby points, then by (2.4)

$$\Delta u=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y.$$

Now if the two points are on the same curve, that is

$$u(x+\Delta x, y+\Delta y)=k, \qquad u(x,y)=k,$$

then

$$\Delta u=u(x+\Delta x, y+\Delta y)-u(x,y)=0.$$

In this case

$$0 = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y.$$

To find the slope of this curve, we divide both sides of this equation by Δx

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta x},$$

therefore the slope of the curve $u\,(x, y) = k$ is given by

$$\frac{\Delta y}{\Delta x}\,|_u = -\frac{\partial u/\partial x}{\partial u/\partial y}.$$

Similarly, the slope of the curve $v\,(x, y) = c$ is given by

$$\frac{\Delta y}{\Delta x}\,|_v = -\frac{\partial v/\partial x}{\partial v/\partial y}.$$

Since u and v satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the slope of the curve $v\,(x, y) = c$ can be written as

$$\frac{\Delta y}{\Delta x}\,|_v = \frac{\partial u/\partial y}{\partial u/\partial x},$$

which, at any common point, is just the negative reciprocal of the slope of the curve $u\,(x, y) = k$. From the analytic geometry, we know that the two families of curves are orthogonal (perpendicular) to each other. For example, the real part of the analytic function z^2 is $u(x, y) = x^2 - y^2$, the family of curves of $u = k$ is the hyperbolas asymptotic to the line $y = \pm x$ as shown in the z-plane of Fig. 2.1. The imaginary part of z^2 is $v(x, y) = 2xy$, the family of curves of $v = c$ is the hyperbolas asymptotic to the x and y axes, also shown in the z-plane of Fig. 2.1. It is seen that they are indeed orthogonal to each other at the points of intersections.

These remarkable properties of analytic functions serve as basis for many important methods used in fluid dynamics, electrostatics and other branches of physics.

Example 2.1.5. Let $f(z) = u(x, y) + \mathrm{i}\,v(x, y)$ be an analytic function. If $u(x, y) = xy$, find $v(x, y)$ and $f(z)$

Solution 2.1.5.

$$\frac{\partial u}{\partial x} = y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = x = -\frac{\partial v}{\partial x}.$$

Method 1: Find $f(z)$ from its derivatives

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = y - \mathrm{i}x = -\mathrm{i}(x + \mathrm{i}y) = -\mathrm{i}z,$$

$$f(z) = -\mathrm{i}\frac{1}{2}z^2 + C.$$

$$f(z) = -\frac{\mathrm{i}}{2}(x + \mathrm{i}y)^2 + C = xy - \frac{\mathrm{i}}{2}(x^2 - y^2) + C,$$

$$v(x,y) = -\frac{1}{2}(x^2 - y^2) + C'.$$

Method 2: Find $v(x,y)$ first

$$\frac{\partial v}{\partial y} = y \quad \Longrightarrow \quad v(x,y) = \int y\,\mathrm{d}y = \frac{1}{2}y^2 + k(x),$$

$$\frac{\partial v}{\partial x} = -x \quad \Longrightarrow \quad \frac{\partial v}{\partial x} = \frac{\mathrm{d}k(x)}{\mathrm{d}x} = -x, \quad \Longrightarrow k(x) = -\frac{1}{2}x^2 + C;$$

$$v(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + C.$$

$$f(z) = xy + \mathrm{i}\frac{1}{2}\left(y^2 - x^2 + 2C\right),$$

$$x = \frac{1}{2}(z + z^*), \quad y = \frac{1}{2\mathrm{i}}(z - z^*) \quad \text{implies} \quad f(z) = -\frac{1}{2}z^2\mathrm{i} + C'.$$

Example 2.1.6. Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be an analytic function. If $u(x,y) = \ln(x^2 + y^2)$, find $v(x,y)$ and $f(z)$

Solution 2.1.6.

$$\frac{\partial u}{\partial x} = \frac{2x}{(x^2 + y^2)} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{2y}{(x^2 + y^2)} = -\frac{\partial v}{\partial x}.$$

Method 1: Find $f(z)$ first from its derivatives

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}z} &= \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{2x}{(x^2+y^2)} - \mathrm{i}\frac{2y}{(x^2+y^2)} \\ &= 2\frac{x-\mathrm{i}y}{(x^2+y^2)} = 2\frac{x-\mathrm{i}y}{(x-\mathrm{i}y)(x+\mathrm{i}y)} = 2\frac{1}{x+\mathrm{i}y} = \frac{2}{z},\end{aligned}$$

$$f(z) = 2\ln z + C = \ln z^2 + C,$$

$$z = r\mathrm{e}^{\mathrm{i}\theta}; \qquad r = (x^2+y^2)^{1/2}; \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$\ln z^2 = \ln(x^2+y^2)\mathrm{e}^{\mathrm{i}2\theta} = \ln(x^2+y^2) + \mathrm{i}2\tan^{-1}\frac{y}{x},$$

$$v(x,y) = 2\tan^{-1}\frac{y}{x} + C.$$

Method 2: Find $v(x,y)$ first

$$\frac{\partial v}{\partial y} = \frac{2x}{(x^2+y^2)} \qquad \Longrightarrow \qquad v(x,y) = \int \frac{2x}{(x^2+y^2)}\mathrm{d}y = 2\tan^{-1}\frac{y}{x} + k(x),$$

$$\frac{\partial v(x,y)}{\partial x} = 2\left(\frac{-y}{x^2}\right)\frac{1}{(1+y^2/x^2)} + \frac{\mathrm{d}k(x)}{\mathrm{d}x} = \frac{-2y}{(x^2+y^2)} + \frac{\mathrm{d}k(x)}{\mathrm{d}x},$$

$$\frac{\partial v}{\partial x} = \frac{-2y}{(x^2+y^2)} \qquad \Longrightarrow \frac{\mathrm{d}k(x)}{\mathrm{d}x} = 0, \quad k(x) = C,$$

$$v(x,y) = 2\tan^{-1}\frac{y}{x} + C.$$

$$\begin{aligned}f(z) &= \ln(x^2+y^2) + \mathrm{i}2\tan^{-1}\frac{y}{x} + \mathrm{i}C \\ &= \ln(x^2+y^2)\mathrm{e}^{\mathrm{i}2\theta} + \mathrm{i}C \\ f(z) &= \ln z^2 + C'.\end{aligned}$$

Example 2.1.7. Let $f(z) = \frac{1}{z} = u(x,y) + \mathrm{i}v(x,y)$, (a) show explicitly that the Cauchy–Riemann equations are satisfied; (b) show explicitly that both real part and imaginary part satisfy the Laplace equation; (c) Describe the family of curves $u(x,y) = k$ and $v(x,y) = c$ and sketch them; (d) show explicitly that the curves $u(x,y) = k$ and $v(x,y) = c$ are perpendicular to each other at the points they intersect.

Solution 2.1.7.

$$f(z) = \frac{1}{z} = \frac{1}{x+\mathrm{i}y} = \frac{1}{x+\mathrm{i}y} \cdot \frac{x-\mathrm{i}y}{x-\mathrm{i}y} = \frac{x-\mathrm{i}y}{x^2+y^2} = \frac{x}{x^2+y^2} - \mathrm{i}\frac{y}{x^2+y^2}.$$

Therefore

$$u(x,y) = \frac{x}{x^2+y^2}, \qquad v(x,y) = -\frac{y}{x^2+y^2}.$$

(a)

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2},$$
$$\frac{\partial v}{\partial y} = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Clearly the Cauchy–Riemann equations are satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(b)

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x}{(x^2+y^2)^2} + \frac{(y^2-x^2)(-2)(2x)}{(x^2+y^2)^3} = \frac{2x^3-6xy^2}{(x^2+y^2)^3},$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{-2x}{(x^2+y^2)^2} + \frac{-2xy(-2)(2y)}{(x^2+y^2)^3} = \frac{-2x^3+6xy^2}{(x^2+y^2)^3}.$$

Thus the real part satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Furthermore,

$$\frac{\partial^2 v}{\partial x^2} = \frac{2y}{(x^2+y^2)^2} + \frac{(2xy)(-2)(2x)}{(x^2+y^2)^3} = \frac{2y^3-6x^2y}{(x^2+y^2)^3},$$
$$\frac{\partial^2 v}{\partial y^2} = \frac{2y}{(x^2+y^2)^2} + \frac{(-x^2+y^2)(-2)(2y)}{(x^2+y^2)^3} = \frac{-2y^3+6x^2y}{(x^2+y^2)^3}.$$

The imaginary part also satisfies the Laplace equation.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

(c) The equation

$$u(x, y) = \frac{x}{x^2 + y^2} = k$$

can be written as

$$x^2 + y^2 = \frac{x}{k}$$

or

$$\left(x - \frac{1}{2k}\right)^2 + y^2 = \frac{1}{4k^2},$$

which is a circle for any given constant k. Therefore $u(x, y) = k$ is a family of circles centered at $\left(\frac{1}{2k}, 0\right)$ with radius $\frac{1}{2k}$. This family of circles is shown in Fig. 2.5 as solid circles. Similarly

$$v(x, y) = -\frac{y}{x^2 + y^2} = c$$

can be written as

$$x^2 + y^2 = -\frac{y}{c} \qquad \text{or} \qquad x^2 + \left(y + \frac{1}{2c}\right)^2 = \frac{1}{4c^2}.$$

Therefore $v(x, y) = c$ is a family of circles of radius $\frac{1}{2c}$, centered at $(0, -\frac{1}{2c})$. They are shown as the dotted circles in Fig. 2.5.

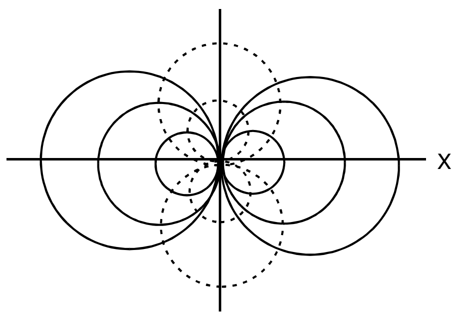


Fig. 2.5. The families of curves described by the real part and imaginary part of the function $f(z) = \frac{1}{z}$

(d) On the curve represented by

$$u(x,y) = \frac{x}{x^2+y^2} = k,$$

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y = 0,$$

which is given by

$$\mathrm{d}u = \left[\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right]\mathrm{d}x - \frac{2xy}{(x^2+y^2)^2}\mathrm{d}y = 0.$$

It follows that:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\,|_u = \frac{y^2-x^2}{2xy}.$$

Similarly, with

$$v(x,y) = -\frac{y}{x^2+y^2} = c,$$

$$\mathrm{d}v = \frac{2xy}{(x^2+y^2)^2}\mathrm{d}x - \left[\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]\mathrm{d}y = 0$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x}\,|_v = \frac{2xy}{x^2-y^2}.$$

Since the two slopes are negative reciprocals of each other, the two curves are perpendicular.

Those who are familiar with electrostatics will recognize that the curves in Fig. 2.5 are electric field lines and equipotential lines of a line dipole.

2.2 Complex Integration

There are some elegant and powerful theorems regarding integrating analytic functions around a loop. It is these theorems that make complex integrations interesting and useful. But before we discuss these theorems, we must define complex integration.

2.2.1 Line Integral of a Complex Function

When a complex variable z moves in the two-dimensional complex plane, it traces out a curve. Therefore to define a integral of complex function $f(z)$

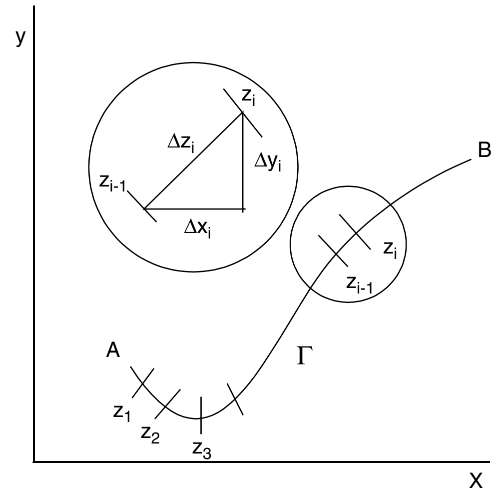


Fig. 2.6. The Riemann sum along the contour Γ which is subdivided into n segments

between two points A and B, we must also specify the path (called contour) along which z moves. The value of the integral will be dependent, in general, upon the contour. However, we will find, that under certain conditions, the integral does not depend upon which of the contours is chosen.

We denote the integral of a complex function $f(z) = u(x,y) + \mathrm{i}v(x,y)$ along a contour Γ from point A to point B as

$$I = \int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z.$$

The integral can be defined in terms of a Riemann sum as in the real variable integration. The contour is subdivided into n segments as shown in Fig. 2.6.

We form the summation

$$I_n = \sum_{i=1}^{n} f(\zeta_i)(z_i - z_{i-1}) = \sum_{i=1}^{n} f(\zeta_i)\,\Delta z_i,$$

where $z_0 = A,\ z_n = B$, and $f(\zeta_i)$ is the function evaluated at a point on Γ between z_{i-1} and z_i. If I_n approaches a limit as $n \to \infty$ and $|\Delta z_i| \to 0$, then we can define the integral as

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \lim_{|\Delta z_i|\to 0,\ n\to\infty} \sum_{i=1}^{n} f(z_i)\,\Delta z_i.$$

Since $\Delta z_i = \Delta x_i + \mathrm{i}\Delta y_i$ as shown in Fig. 2.6, the integral can be written as

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma}^{B} (u+\mathrm{i}v)(\mathrm{d}x+\mathrm{i}\mathrm{d}y) = \int_{A,\Gamma}^{B} [(u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}(v\,\mathrm{d}x + u\,\mathrm{d}y)]$$

$$= \int_{A,\Gamma}^{B} (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\int_{A,\Gamma}^{B} (v\,\mathrm{d}x + u\,\mathrm{d}y). \tag{2.9}$$

Thus the complex contour integral is expressed in terms of two line integrals.

Example 2.2.1. Evaluate the integral $I = \int_A^B z^2 \mathrm{d}z$ from $z_A = 0$ to $z_B = 1 + \mathrm{i}$, (a) along the contour Γ_1: $y = x^2$, (b) along y-axis from 0 to i, then along the horizontal line from i to $1 + \mathrm{i}$, as Γ_2 shown in Fig. 2.7.

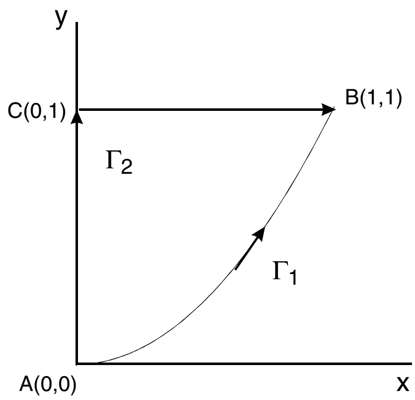


Fig. 2.7. Two contours Γ_1 and Γ_2 from A ($z_A = 0$) to B ($z_B = 1 + \mathrm{i}$), Γ_1 : along the curve $y = x^2$, Γ_2 : first along y-axis to C ($z_C = \mathrm{i}$), then along a horizontal line to B

Solution 2.2.1.

$$f(z) = z^2 = (x + \mathrm{i}y)^2 = \left(x^2 - y^2\right) + \mathrm{i}2xy = u + \mathrm{i}v,$$

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma}^{B} [(x^2 - y^2)\mathrm{d}x - 2xy\,\mathrm{d}y] + \mathrm{i}\int_{A,\Gamma}^{B} [2xy\,\mathrm{d}x + (x^2 - y^2)\mathrm{d}y].$$

(a) Along Γ_1, $y = x^2$, $\mathrm{d}y = 2x\,\mathrm{d}x$,

$$\int_{A,\Gamma_1}^{B} f(z)\,\mathrm{d}z = \int_0^1 [(x^2 - x^4)\mathrm{d}x - 2xx^2 2x\,\mathrm{d}x] + \mathrm{i}\int_0^1 [2xx^2\mathrm{d}x + (x^2 - x^4)2x\,\mathrm{d}x]$$

$$= \int_0^1 (x^2 - 5x^4)\mathrm{d}x + \mathrm{i}\int_0^1 (4x^3 - 2x^5)\mathrm{d}x = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.$$

(b) Let $z_C = \mathrm{i}$ as shown in Fig. 2.7. So

$$\int_{A,\Gamma_2}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma_2}^{C} f(z)\,\mathrm{d}z + \int_{C,\Gamma_2}^{B} f(z)\,\mathrm{d}z.$$

From A to $C: x = 0, \mathrm{d}x = 0$

$$\int_{A,\Gamma_2}^{C} f(z)\,\mathrm{d}z = \mathrm{i}\int_0^1 (-y^2)\mathrm{d}y = -\frac{1}{3}\mathrm{i}.$$

From C to B: $y = 1$, $\mathrm{d}y = 0$

$$\int_{C,\Gamma_2}^{B} f(z)\,\mathrm{d}z = \int_0^1 (x^2-1)\mathrm{d}x + \mathrm{i}\int_0^1 2x\,\mathrm{d}x = -\frac{2}{3} + \mathrm{i}.$$

$$\int_{A,\Gamma_2}^{B} f(z)\,\mathrm{d}z = -\frac{1}{3}\mathrm{i} - \frac{2}{3} + \mathrm{i} = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.$$

The integrals along Γ_1 and Γ_2 are observed to be equal.

2.2.2 Parametric Form of Complex Line Integral

If along the contour Γ, z is expressed parametrically, these line integrals can be transformed into ordinary integrals in which there is only one independent variable. For if $z = z(t)$, where t is a parameter, and $A = z(t_A)$, $B = z(t_B)$, then

$$\int_A^B f(z)\,\mathrm{d}z = \int_{t_A}^{t_B} f(z(t))\,\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t. \tag{2.10}$$

For instance, on Γ_1 of the previous example, $y = x^2$, we can set $z(t) = x(t) + \mathrm{i}y(t)$ with $x(t) = t$ and $y(t) = t^2$. It follows that $\frac{\mathrm{d}z}{\mathrm{d}t} = 1 + \mathrm{i}2t$, and

$$\begin{aligned}\int_{A,\Gamma_1}^{B} z^2\mathrm{d}z &= \int_0^1 \left(t+\mathrm{i}t^2\right)^2 (1+\mathrm{i}2t)\,\mathrm{d}t \\ &= \int_0^1 [(t^2-5t^4) + \mathrm{i}(4t^3-2t^5)]\mathrm{d}t = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.\end{aligned}$$

Similarly, on Γ_2 of the previous example, from A to C, we can set $z(t) = \mathrm{i}t$ with $0 \le t \le 1$, and $\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{i}$. From C to B, we can set $z(t) = (t-1) + \mathrm{i}$ with $1 \le t \le 2$, and $\frac{\mathrm{d}z}{\mathrm{d}t} = 1$. Thus

$$\begin{aligned}\int_{A,\Gamma_2}^{B} z^2\mathrm{d}z &= \int_0^1 (\mathrm{i}t)^2\,\mathrm{i}\mathrm{d}t + \int_1^2 (t-1+\mathrm{i})^2\,\mathrm{d}t \\ &= -\frac{1}{3}\mathrm{i} - \frac{2}{3} + \mathrm{i} = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.\end{aligned}$$

Parametrization of a Circular Contour

A circular contour can be easily parameterized with the angular variable of the polar coordinates. This is of considerable importance because through the principle of deformation of contours, which we will soon see, other contour integrations can also be carried out by changing the contour into a circle.

Consider the integral $I = \oint_C f(z)\,\mathrm{d}z$, where C is a circle of radius r centered at the origin. Clearly we can express z as

$$z(\theta) = r\cos\theta + \mathrm{i}\,r\sin\theta = r\,\mathrm{e}^{\mathrm{i}\theta},$$

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -r\sin\theta + \mathrm{i}\,r\cos\theta = \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}.$$

This means $\mathrm{d}z = \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$, so the integral becomes

$$I = \int_0^{2\pi} f\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta.$$

The following example will illustrate how this is done.

Example 2.2.2. Evaluate the integral $\oint_C z^n \mathrm{d}z$, where n is an integer and C is a circle of radius r around the origin.

Solution 2.2.2.

$$\oint_C z^n \mathrm{d}z = \int_0^{2\pi} \left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)^n \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \mathrm{i}\,r^{n+1}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+1)\theta}\mathrm{d}\theta.$$

For $n \neq -1$

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n+1)\theta}\mathrm{d}\theta = \frac{1}{\mathrm{i}(n+1)}\left[\mathrm{e}^{\mathrm{i}(n+1)\theta}\right]_0^{2\pi} = \frac{1}{\mathrm{i}(n+1)}[1-1] = 0.$$

For $n = -1$

$$\int_0^{2\pi} \left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)^n \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi}\mathrm{d}\theta = 2\pi\mathrm{i}.$$

This means

$$\oint_C z^n \mathrm{d}z = 0 \quad \text{for} \quad n \neq -1,$$

$$\oint_C \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}.$$

Note that these results are independent of the radius r.

Some Properties of Complex Line Integral

The parametric form of the complex line integral enables us to see immediately that many formulas of ordinary integration of real variables can be directly applied to the complex integration. For example, the complex integral from B to A along the same path Γ is given by the right-hand side of (2.10) with t_A and t_B interchanged, introducing a negative sign to the equation. Therefore

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = -\int_{B,\Gamma}^{A} f(z)\,\mathrm{d}z.$$

Similarly, if C is on Γ, then

$$\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z = \int_{A,\Gamma}^{C} f(z)\,\mathrm{d}z + \int_{C,\Gamma}^{B} f(z)\,\mathrm{d}z.$$

If the integral from A to B is along Γ_1 and from B back to A is along a different contour Γ_2, we can write the sum of the two integrals as

$$\int_{A,\Gamma_1}^{B} f(z)\,\mathrm{d}z + \int_{B,\Gamma_2}^{A} f(z)\,\mathrm{d}z = \oint_{\Gamma} f(z)\,\mathrm{d}z,$$

where $\Gamma = \Gamma_1 + \Gamma_2$ and the symbol $\oint_\Gamma$ is to signify that the integration is taken counterclockwise along the closed contour Γ. Thus

$$\begin{aligned}\oint_{\text{c.c.w.}} f(z)\,\mathrm{d}z &= \int_{A,\Gamma_1}^{B} f(z)\,\mathrm{d}z + \int_{B,\Gamma_2}^{A} f(z)\,\mathrm{d}z \\ &= -\int_{B,\Gamma_1}^{A} f(z)\,\mathrm{d}z - \int_{A,\Gamma_2}^{B} f(z)\,\mathrm{d}z = -\oint_{\text{c.w.}} f(z)\,\mathrm{d}z,\end{aligned}$$

where c.c.w. means counterclockwise and c.w. means clockwise.

Furthermore, we can show that

$$\left|\int_{A,\Gamma}^{B} f(z)\,\mathrm{d}z\right| \le ML, \tag{2.11}$$

where M is the maximum value of $|f(z)|$ on Γ and L is the length of Γ. This is because

$$\left|\int_{t_A}^{t_B} f(z)\,\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t\right| \le \int_{t_A}^{t_B} \left|f(z)\,\frac{\mathrm{d}z}{\mathrm{d}t}\right| \mathrm{d}t,$$

which is a generalization of $|z_1 + z_2| \le |z_1| + |z_2|$. By the definition of M, we have

$$\int_{t_A}^{t_B} \left|f(z)\,\frac{\mathrm{d}z}{\mathrm{d}t}\right| \mathrm{d}t \le M\int_{t_A}^{t_B} \left|\frac{\mathrm{d}z}{\mathrm{d}t}\right| \mathrm{d}t = M\int_{A}^{B} |\mathrm{d}| = ML.$$

Thus, starting with (2.10), we have

$$\left|\int_{A}^{B} f(z)\,\mathrm{d}z\right| = \left|\int_{t_A}^{t_B} f(z)\,\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t\right| \le ML.$$

2.3 Cauchy's Integral Theorem

As we have seen, the results of integrations of z^2 along Γ_1 and Γ_2 of Fig. 2.7 are exactly the same. Therefore a closed loop integration from A to B along Γ_1 and returning from B to A along Γ_2 is equal to zero. In 1825, Cauchy proved a theorem which enables use to see that this must be the case without carrying out the integration. Before we discuss this theorem, let us first review the Green's lemma of real variables.

2.3.1 Green's Lemma

There is an important relation that allows us to transform a line integral into an area integral for lines and areas in the xy plane. It is often referred to as Green's lemma, which states that

$$\oint_C [P(x,y)dx + Q(x,y)\mathrm{d}y] = \iint_R \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] \mathrm{d}x\, \mathrm{d}y, \qquad (2.12)$$

where C is a closed curve surrounding the region R. The curve C is traversed counterclockwise, that is with the region R always to the left as shown in Fig. 2.8.

To prove Green's lemma, let us use Fig. 2.9, part (a) to carry out the first part of the area double integral

$$\iint_R \frac{\partial Q(x,y)}{\partial x} \mathrm{d}x\, \mathrm{d}y = \int_c^d \left[\int_{x=g_1(y)}^{x=g_2(y)} \frac{\partial Q(x,y)}{\partial x} \mathrm{d}x\right] \mathrm{d}y$$

$$= \int_c^d [Q(x,y)]_{x=g_1(y)}^{x=g_2(y)}\, \mathrm{d}y.$$

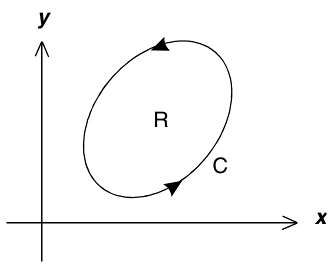


Fig. 2.8. The closed contour C of the line integral in the Green's lemma. C is counterclockwise and is defined as the positive direction with respect to the interior of R

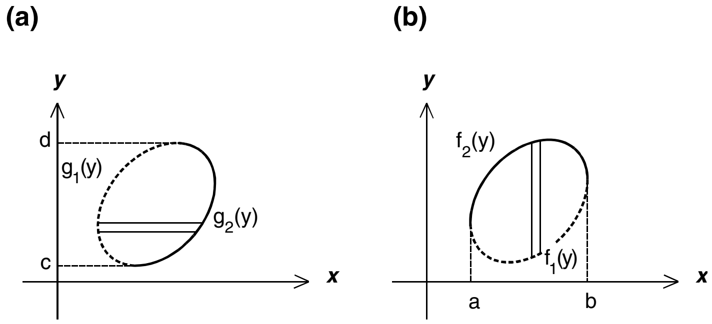


Fig. 2.9. Same contour but with two different ways to carry out the area double integral in the Green's lemma

Now

$$\int_c^d [Q(x,y)]_{x=g_1(y)}^{x=g_2(y)}\,\mathrm{d}y = \int_c^d Q(g_2(y),y)\mathrm{d}y - \int_c^d Q(g_1(y),y)\mathrm{d}y$$
$$= \int_c^d Q(g_2(y),y)\mathrm{d}y + \int_d^c Q(g_1(y),y)\mathrm{d}y.$$

The contour of the last line integral is from $y=c$ going through $g_2(y)$ to $y=d$ and then returning through $g_1(y)$ to $y=c$. Clearly it is counterclockwise closed loop integral

$$\iint_R \frac{\partial Q(x,y)}{\partial x}\mathrm{d}x\,\mathrm{d}y = \oint_{\text{c.c.w}} Q\,(x,y)\,\mathrm{d}y. \tag{2.13}$$

Next we will use Fig. 2.9, part (b) to carry out the second part of the area double integral

$$\iint_R \frac{\partial P(x,y)}{\partial y}\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial P(x,y)}{\partial y}\mathrm{d}y\right]\mathrm{d}x = \int_a^b [P(x,y)]_{y=f_1(x)}^{y=f_2(x)}\,\mathrm{d}x.$$

$$\int_a^b [P(x,y)]_{y=f_1(x)}^{y=f_2(x)}\,\mathrm{d}x = \int_a^b P(x,f_2(x))\mathrm{d}x - \int_a^b P(x,f_1(x))\mathrm{d}x$$
$$= \int_a^b P(x,f_2(x))\mathrm{d}x + \int_b^a P(x,f_1(x))\mathrm{d}x.$$

In this case the contour is from $x=a$ going through $f_2(x)$ to $x=b$ and then returning to $x=a$ through $f_1(x)$. Therefore it is clockwise

$$\iint_R \frac{\partial P(x,y)}{\partial y}\mathrm{d}x\,\mathrm{d}y = \oint_{\text{c.w.}} P(x,y)\mathrm{d}x = -\oint_{\text{c.c.w.}} P(x,y)\mathrm{d}x. \tag{2.14}$$

In the last step we changed the sign to make it counterclockwise.

Subtracting (2.14) from (2.13), we have the Green's lemma

$$\iint_R \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y = \oint_{\text{c.c.w.}} [Q(x,y)\mathrm{d}y + P(x,y)\mathrm{d}x].$$

2.3.2 Cauchy–Goursat Theorem

An important theorem in complex integration is the following:

If C is a closed contour and $f(z)$ is analytic on and inside C, then

$$\oint_C f(z)\mathrm{d}z = 0. \tag{2.15}$$

This is known as Cauchy's theorem. The proof goes as follows. Starting with

$$\oint_C f(z)\mathrm{d}z = \oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) + \mathrm{i}\oint_C (v\,\mathrm{d}x + u\,\mathrm{d}y), \tag{2.16}$$

making use of the Green's lemma of (2.12) and identifying P as u and Q as $-v$, we have

$$\oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) = \int\int_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y.$$

Since $f(z)$ is analytic, so u and v satisfy Cauchy–Riemann conditions. In particular

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y},$$

therefore the area double integral is equal to zero, thus

$$\oint_C (u\,\mathrm{d}x - v\,\mathrm{d}y) = 0.$$

Similarly, identifying u as Q and v as P, from Green's lemma we have

$$\oint_C (v\,\mathrm{d}x + u\,\mathrm{d}y) = \int\int_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y.$$

Because of the other Cauchy–Riemann condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

the integral on the left-hand side is also equal to zero

$$\oint_C (v\,\mathrm{d}x + u\,\mathrm{d}y) = 0.$$

Thus both line integrals on the right-hand side of (2.16) are zero, therefore

$$\oint_C f(z)\,\mathrm{d}z = 0,$$

which is known as Cauchy's integral theorem.

In this proof, we have used Green's lemma which requires the first partial derivatives of u and v to be continuous. Therefore we have implicitly assumed that the derivative of $f(z)$ is continuous. In 1903, Goursat proved this theorem without assuming the continuity of $f'(z)$. Therefore this theorem is also called Cauchy–Goursat theorem. Mathematically Goursat's removal of the continuity assumption from the proof of the theorem is very important because it enables us to rigorously establish that derivatives of analytic functions are analytic, and they are automatically continuous. A version of Goursat's proof can be found in *Complex Variables and Applications,* by J.W. Brown and R.V. Churchill *Complex Variable and Applications,* 5th edn. (McGraw-Hill, New York 1989).

2.3.3 Fundamental Theorem of Calculus

If the closed contour Γ is divided into two parts Γ_1 and Γ_2, as shown in Fig. 2.7, and $f(z)$ is analytic on and between Γ_1 and Γ_2, then Cauchy's integral theorem can be written as

$$\begin{aligned}\oint_\Gamma f(z)\mathrm{d}z &= \int_{A\ \Gamma_1}^{B} f(z)\mathrm{d}z + \int_{B\ \Gamma_2}^{A} f(z)\mathrm{d}z \\ &= \int_{A\ \Gamma_1}^{B} f(z)\mathrm{d}z - \int_{A\ \Gamma_2}^{B} f(z)\mathrm{d}z = 0,\end{aligned}$$

where the negative sign appears since we have exchanged the limit on the last integral. Thus we have

$$\int_{A\ \Gamma_1}^{B} f(z)\mathrm{d}z = \int_{A\ \Gamma_2}^{B} f(z)\mathrm{d}z, \tag{2.17}$$

showing that the value of a line integral between two points is independent of the path provided that the integrand is an analytic function in the domain on and between the contours.

With this in mind, we can show that, as long as $f(z)$ is analytic in a region containing A and B

$$\int_A^B f(z)\,\mathrm{d}z = F(B) - F(A),$$

where

$$\frac{\mathrm{d}F(z)}{\mathrm{d}z} = \lim_{\Delta z\to 0}\frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z).$$

The integral

$$F(z) = \int_{z_0}^{z} f(z')\,\mathrm{d}z' \tag{2.18}$$

uniquely define the function $F(z)$ if z_0 is a fixed point and $f(z')$ is analytic throughout the region containing the path between z_0 and z. Similarly, we can define

$$F(z+\Delta z) = \int_{z_0}^{z+\Delta z} f(z')\,\mathrm{d}z' = \int_{z_0}^{z} f(z')\,\mathrm{d}z' + \int_{z}^{z+\Delta z} f(z')\,\mathrm{d}z'.$$

Clearly

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(z')\,\mathrm{d}z'.$$

For a small Δz, the right-hand side reduces to

$$\int_{z}^{z+\Delta z} f(z')\,\mathrm{d}z' \to f(z)\,\Delta z,$$

which implies that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z).$$

Thus

$$\frac{\mathrm{d}F(z)}{\mathrm{d}z} = f(z)$$

and the fundamental theorem of calculus follows:

$$\int_A^B f(z)\,\mathrm{d}z = \int_A^B \mathrm{d}F(z) = F(B) - F(A).$$

Example 2.3.1. Find the value of the integral $\int_0^{1+\mathrm{i}} z^2\,\mathrm{d}z$.

Solution 2.3.1.

$$\int_0^{1+\mathrm{i}} z^2 \mathrm{d}z = \left[\frac{1}{3}z^3\right]_0^{1+\mathrm{i}} = \frac{1}{3}(1+\mathrm{i})^3 = -\frac{2}{3} + \frac{2}{3}\mathrm{i}.$$

Note that the result is the same as in Example 2.2.1.

Example 2.3.2. Find the values of the following integrals:

$$I_1 = \int_{-\pi\mathrm{i}}^{\pi\mathrm{i}} \cos z\,\mathrm{d}z, \qquad I_2 = \int_{4+\pi\mathrm{i}}^{4-3\pi\mathrm{i}} \mathrm{e}^{z/2}\mathrm{d}z.$$

Solution 2.3.2.

$$I_1 = \int_{-\pi\mathrm{i}}^{\pi\mathrm{i}} \cos z\,\mathrm{d}z = [\sin z]_{-\pi\mathrm{i}}^{\pi\mathrm{i}} = \sin(\pi\mathrm{i}) - \sin(-\pi\mathrm{i})$$

$$= 2\sin(\pi\mathrm{i}) = 2\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}(\mathrm{i}\pi)} - \mathrm{e}^{-\mathrm{i}(\mathrm{i}\pi)}\right) = \left(\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}\right)\mathrm{i} \simeq 23.097\mathrm{i}.$$

$$I_2 = \int_{4+\pi\mathrm{i}}^{4-3\pi\mathrm{i}} \mathrm{e}^{z/2}\mathrm{d}z = \left[2\mathrm{e}^{z/2}\right]_{4+\pi\mathrm{i}}^{4-3\pi\mathrm{i}} = 2\left(\mathrm{e}^{2-\mathrm{i}3\pi/2} - \mathrm{e}^{2+\mathrm{i}\pi/2}\right)$$

$$= 2\mathrm{e}^2\left(\mathrm{e}^{-\mathrm{i}3\pi/2} - \mathrm{e}^{\mathrm{i}\pi/2}\right) = 2\mathrm{e}^2(\mathrm{i} - \mathrm{i}) = 0.$$

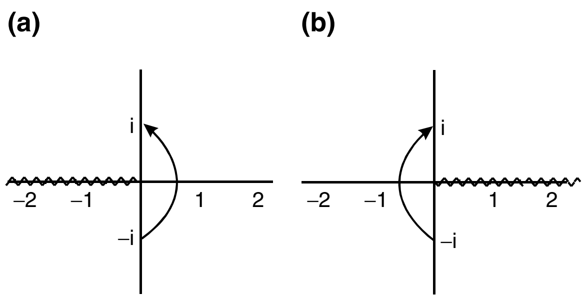
Example 2.3.3. Find the values of the following integral:

$$\int_{-\mathrm{i}}^{\mathrm{i}} \frac{\mathrm{d}z}{z}.$$

Solution 2.3.3. Since $z = 0$ is a singular point, the path of integration must not pass through the origin. Furthermore

$$\int \frac{\mathrm{d}z}{z} = \ln z + C,$$

where $\ln z$ is a multivalued function, therefore there is a branch cut. To evaluate this definite integral we must specify the path of z going from $-\mathrm{i}$ to i. There are two possibilities as shown in (a) and (b) of the following figure:



(a) To go from $-\mathrm{i}$ to i in the right half of the complex plane, we must take the negative real axis as the branch cut. In the principal branch, $-\pi < \theta < \pi$. Thus

$$\int_{-\mathrm{i}}^{\mathrm{i}} \frac{\mathrm{d}z}{z} = [\ln z]_{-\mathrm{i}}^{\mathrm{i}} = \left[\ln \mathrm{e}^{\mathrm{i}\theta}\right]_{\theta=-\frac{1}{2}\pi}^{\theta=\frac{1}{2}\pi} = [\mathrm{i}\theta]_{\theta=-\frac{1}{2}\pi}^{\theta=\frac{1}{2}\pi} = \mathrm{i}\frac{1}{2}\pi + \mathrm{i}\frac{1}{2}\pi = \mathrm{i}\pi.$$

(b) To go from $-\mathrm{i}$ to i in the left half of the complex plane, we must take the positive real axis as the branch cut. Therefore $0 < \theta < 2\pi$. Thus

$$\int_{-\mathrm{i}}^{\mathrm{i}} \frac{\mathrm{d}z}{z} = [\ln z]_{-\mathrm{i}}^{\mathrm{i}} = \left[\ln \mathrm{e}^{\mathrm{i}\theta}\right]_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} = [\mathrm{i}\theta]_{\theta=\frac{3}{2}\pi}^{\theta=\frac{1}{2}\pi} = \mathrm{i}\frac{1}{2}\pi - \mathrm{i}\frac{3}{2}\pi = -\mathrm{i}\pi.$$

2.4 Consequences of Cauchy's Theorem

2.4.1 Principle of Deformation of Contours

There is an immediate, practical consequence of the Cauchy Integral Theorem. The contour of a complex integral can be arbitrarily deformed through an analytic region without changing the value of the integral.

Consider the integration along the two contours shown on the left side of Fig. 2.10. If $f(z)$ is analytic, then

$$\oint_{abcda} f(z)\mathrm{d}z = 0, \tag{2.19}$$

$$\oint_{efghe} f(z)\mathrm{d}z = 0. \tag{2.20}$$

Naturally the sum of them is also equal to zero

$$\oint_{abcda} f(z)\mathrm{d}z + \oint_{efghe} f(z)\mathrm{d}z = 0. \tag{2.21}$$

Notice that the integrals along ab and along he are in the opposite direction. If ab coincides with he, their contributions will cancel each other. Thus if the gaps between ab and he, and between cd and fg are shrinking to zero, the sum of these two integrals becomes the sum of the integral along the outer

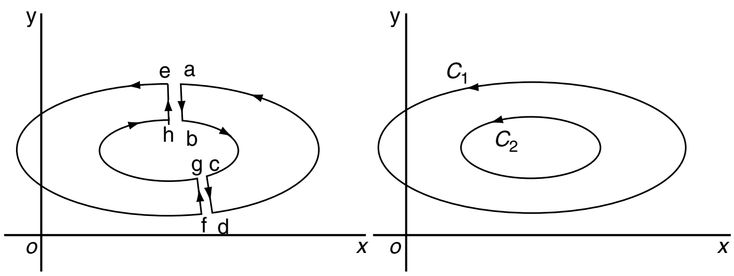


Fig. 2.10. Contour deformation

contour C_1 and the integral along the inner contour C_2 but in the opposite direction. If we change the direction of C_2, we must change the sign of the integral. Therefore

$$\oint_{abcda} f(z)\mathrm{d}z + \oint_{efghe} f(z)\mathrm{d}z = \oint_{C_1} f(z)\mathrm{d}z - \oint_{C_2} f(z)\mathrm{d}z = 0.$$

It follows:

$$\oint_{C_1} f(z)\mathrm{d}z = \oint_{C_2} f(z)\mathrm{d}z. \tag{2.22}$$

Thus we have shown that the line integral of an analytic function around any closed curve C_1 is equal to the line integral of the same function around any other closed curve C_2 into which C_1 can be continuously deformed as long as $f(z)$ is analytic between C_1 and C_2 and is single-valued on C_1 and C_2.

2.4.2 The Cauchy Integral Formula

The Cauchy integral formula is a natural extension of the Cauchy integral theorem. Consider the integral

$$I_1 = \oint_{C_1} \frac{f(z)}{z - z_0}\mathrm{d}z, \tag{2.23}$$

where $f(z)$ is analytic everywhere in the z-plane, and C_1 is a closed contour that does not include the point z_0 as shown in Fig. 2.11a.

Since $(z - z_0)^{-1}$ is analytic everywhere except at $z = z_0$, and z_0 is outside of C_1, therefore $f(z)/(z - z_0)$ is analytic inside C_1. It follows from Cauchy's integral theorem that

$$I_1 = \oint_{C_1} \frac{f(z)}{z - z_0}\mathrm{d}z = 0. \tag{2.24}$$

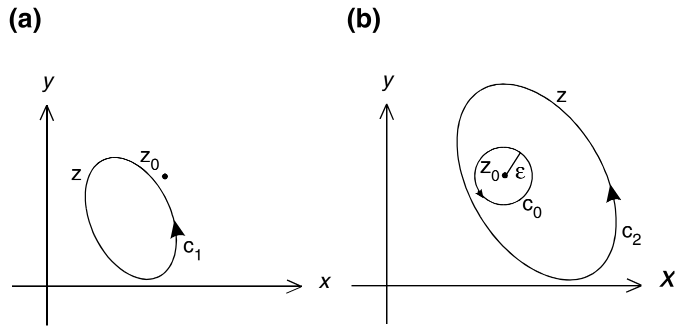


Fig. 2.11. Closed contour integration. (**a**) The singular point z_0 is outside of the contour C_1. (**b**) The contour C_2 encloses z_0, C_2 can be deformed into the circle C_0 without changing the value of the integral

Now consider a second integral

$$I_2 = \oint_{C_2} \frac{f(z)}{z - z_0} \mathrm{d}z, \tag{2.25}$$

similar to the first, except now the contour C_2 encloses z_0, as shown in Fig. 2.11b. The integrand in this integral is not analytic at $z = z_0$ which is inside C_2, so we cannot invoke the Cauchy integral theorem to argue that $I_2 = 0$. However, the integrand is analytic everywhere, except at the point $z = z_0$, so we can deform the contour into an infinitesimal circle of radius ε centered at z_0, without changing its value

$$I_2 = \lim_{\varepsilon \to 0} \oint_{C_0} \frac{f(z)}{z - z_0} \mathrm{d}z. \tag{2.26}$$

This deformation is also shown in Fig. 2.11b.

This last integral can be evaluated. In order to see more clearly, we enlarge the contour in Fig. 2.12.

Since z is on the circle C_0, with the notation shown in Fig. 2.12, it is clear that

$$z = x + \mathrm{i}y,$$
$$x = x_0 + \varepsilon \cos\theta,$$
$$y = y_0 + \varepsilon \sin\theta.$$

Therefore

$$z = (x_0 + \mathrm{i}y_0) + \varepsilon(\cos\theta + \mathrm{i}\sin\theta). \tag{2.27}$$

Since

$$z_0 = x_0 + \mathrm{i}y_0,$$
$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta,$$

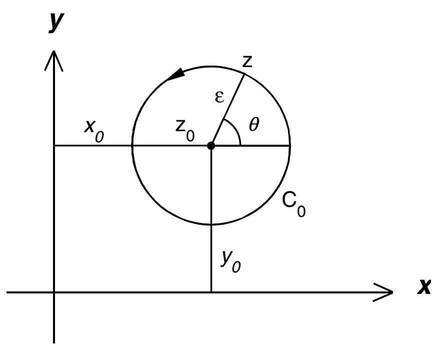


Fig. 2.12. Circular contour for the Cauchy integral formula

we can write

$$z = z_0 + \varepsilon \mathrm{e}^{\mathrm{i}\theta}. \tag{2.28}$$

On C_0, ε is a constant, and θ goes from 0 to 2π. Therefore

$$\mathrm{d}z = \mathrm{i}\,\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta \tag{2.29}$$

and

$$\oint_{C_0} \frac{f(z)}{z - z_0}\mathrm{d}z = \int_0^{2\pi} \frac{f\left(z_0 + \varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\right)}{\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}}\mathrm{i}\,\varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi} f\left(z_0 + \varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta. \tag{2.30}$$

As $\varepsilon \to 0$, $f\left(z_0 + \varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\right) \to f(z_0)$ and can be taken outside the integral

$$\begin{aligned} I_2 = \oint_{C_2} \frac{f(z)}{z - z_0}\mathrm{d}z &= \lim_{\varepsilon\to 0}\oint_{C_0} \frac{f(z)}{z - z_0}\mathrm{d}z \\ &= \lim_{\varepsilon\to 0}\mathrm{i}\int_0^{2\pi} f\left(z_0 + \varepsilon\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta = \mathrm{i}\,f(z_0)\int_0^{2\pi}\mathrm{d}\theta = 2\pi\mathrm{i}\,f(z_0), \end{aligned} \tag{2.31}$$

where C is any closed, counterclockwise path that encloses z_0, and $f(z)$ is analytic inside C. This result is known as Cauchy's integral formula, usually written as

$$f(z_0) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z. \tag{2.32}$$

2.4.3 Derivatives of Analytic Function

If we differentiate both sides of Cauchy's integral formula, interchanging the order of differentiation and integration, we get

$$f'(z_0) = \frac{1}{2\pi\mathrm{i}}\oint_C f(z)\frac{\mathrm{d}}{\mathrm{d}z_0}\frac{1}{(z - z_0)}\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{(z - z_0)^2}\mathrm{d}z.$$

To establish this formula in a rigorous manner, we may start with the formal expression of the derivative

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z_0\to 0}\frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \lim_{\Delta z_0\to 0}\frac{1}{\Delta z_0}\left[f(z_0 + \Delta z_0) - f(z_0)\right] \\ &= \lim_{\Delta z_0\to 0}\frac{1}{\Delta z_0}\left[\frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{z - z_0 - \Delta z_0}\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z\right]. \end{aligned}$$

Now

$$\begin{aligned} &\oint_C \frac{f(z)}{z - z_0 - \Delta z_0}\mathrm{d}z - \oint_C \frac{f(z)}{z - z_0}\mathrm{d}z = \oint_C f(z)\left(\frac{1}{z - z_0 - \Delta z_0} - \frac{1}{z - z_0}\right)\mathrm{d}z \\ &\quad = \oint_C f(z)\frac{\Delta z_0}{(z - z_0 - \Delta z_0)(z - z_0)}\mathrm{d}z = \Delta z_0\oint_C \frac{f(z)\mathrm{d}z}{(z - z_0 - \Delta z_0)(z - z_0)}. \end{aligned}$$

Therefore

$$f'(z_0) = \lim_{\Delta z_0 \to 0} \frac{1}{\Delta z_0} \left[\frac{\Delta z_0}{2\pi \mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{(z - z_0 - \Delta z_0)(z - z_0)} \right]$$

$$= \lim_{\Delta z_0 \to 0} \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)\mathrm{d}z}{(z - z_0 - \Delta z_0)(z - z_0)} = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^2}\mathrm{d}z.$$

In a like manner we can show that

$$f''(z_0) = \frac{2}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^3}\mathrm{d}z \tag{2.33}$$

and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\mathrm{d}z. \tag{2.34}$$

Thus we have established the fact that analytic functions possess derivatives of all orders. Also, all derivatives of analytic functions are analytic. This is quite different from our experience with real variables, where we have encountered functions that possess first and second derivatives at a particular point, but yet the third derivative is not defined.

Cauchy's integral formula allows us to determine the value of an analytic function at any point z interior to a simply connected region by integrating around a curve C surrounding the region. Only values of the function on the boundary are used. Thus, we note that if an analytic function is prescribed on the entire boundary of a simply connected region, the function and all its derivatives can be determined at all interior points. The Cauchy's integral formula can be written in the form of

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(\varsigma)}{\varsigma - z}\mathrm{d}\varsigma, \tag{2.35}$$

where z is any interior point inside C. The complex variable ς is on C and is simply a dummy variable of integration that disappears in the integration process. Cauchy's integral formula is often used in this form.

Example 2.4.1. Evaluate the integrals

$$\text{(a)} \oint \frac{z^2 \sin \pi z}{z - \frac{1}{2}}\mathrm{d}z, \qquad \text{(b)} \oint \frac{\cos z}{z^3}\mathrm{d}z$$

around the circle $|z| = 1$.

Solution 2.4.1. (a) The singular point is at $z = \frac{1}{2}$ which is inside the circle $|z| = 1$. Therefore

$$\oint \frac{z^2 \sin \pi z}{z - \frac{1}{2}}\mathrm{d}z = 2\pi \mathrm{i} \left[z^2 \sin \pi z \right]_{z=1/2} = 2\pi \mathrm{i} \left(\frac{1}{2} \right)^2 \sin \left(\pi \frac{1}{2} \right) = \frac{1}{2}\pi \mathrm{i}.$$

(b) The singular point is at $z = 0$ which is inside the circle $|z| = 1$. Therefore

$$\oint \frac{\cos z}{z^3} \mathrm{d}z = \frac{2\pi \mathrm{i}}{2!} \left[\frac{\mathrm{d}^2}{\mathrm{d}z^2} \cos z \right]_{z=0} = \pi \mathrm{i}[-\cos(0)] = -\pi \mathrm{i}.$$

Example 2.4.2. Evaluate the integral

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z$$

around (a) the circle $|z| = 1$, (b) the circle $|z| = 3$.

Solution 2.4.2. (a) The singular point is at $z = 2$. It is outside the circle of $|z| = 1$, as shown in Fig. 2.13a. Inside the circle $|z| = 1$, the function $\frac{z^2-1}{(z-2)^2}$ is analytic, therefore

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z = 0.$$

(b) Since $z = 2$ is inside the circle $|z| = 3$, as shown in Fig. 2.13b, we can write the integral as

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z = \oint \frac{f(z)}{(z-2)^2} \mathrm{d}z = 2\pi \mathrm{i}\, f'(2),$$

where

$$f(z) = z^2 - 1, \qquad f'(z) = 2z, \quad \text{and} \quad f'(2) = 4.$$

Thus

$$\oint \frac{z^2 - 1}{(z-2)^2} \mathrm{d}z = 2\pi \mathrm{i} 4 = 8\pi \mathrm{i}.$$

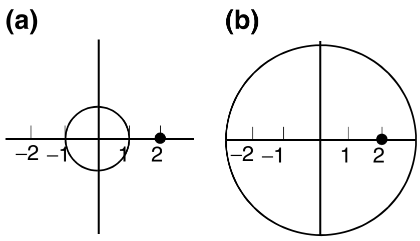


Fig. 2.13. (**a**) $|z| = 1$, (**b**) $|z| = 3$

Example 2.4.3. Evaluate the integral

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z$$

(a) around the circle $|z-1|=1$, (b) around the circle $|z-\mathrm{i}|=1$, (c) around the circle $|z-1|=2$.

Solution 2.4.3. Unless the relationship between the singular points and the contour is clear as in previous examples, to solve problems of closed contour integration, it is best to first find the singular points (known as poles) and display them on the complex plane, then draw the contour. In this particular problem, the singular points are at $z=\pm\mathrm{i}$, which are the solutions of $z^2+1=0$. The three contours are shown in Fig. 2.14.

(a) It is seen that both singular points are outside of the contour $|z-1|=1$, therefore

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z=0.$$

(b) In this case, only one singular point $z=i$ is inside the contour, so we can write

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z=\oint \frac{z^2}{(z-\mathrm{i})(z+\mathrm{i})}\mathrm{d}z=\oint \frac{f(z)}{z-\mathrm{i}}\mathrm{d}z,$$

where

$$f(z)=\frac{z^2}{z+\mathrm{i}}.$$

Thus, it follows:

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z=\oint \frac{f(z)}{z-\mathrm{i}}\mathrm{d}z=2\pi\mathrm{i}\,f(\mathrm{i})=2\pi\mathrm{i}\frac{(\mathrm{i})^2}{\mathrm{i}+\mathrm{i}}=-\pi.$$

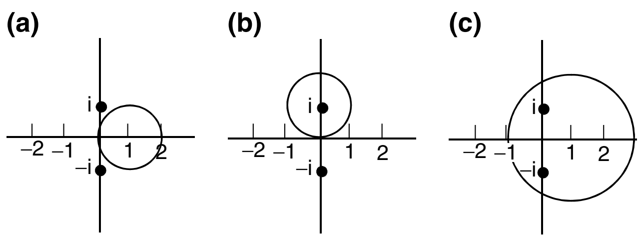


Fig. 2.14. (**a**) $|z-1|=1$, (**b**) $|z-\mathrm{i}|=1$, (**c**) $|z-1|=2$

(c) In this case, both singular points are inside the contour. To make use of the Cauchy integral formula, we first take the partial fraction of $\frac{1}{z^2+1}$,

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{A}{(z-\mathrm{i})} + \frac{B}{(z+\mathrm{i})} \\ &= \frac{A(z+\mathrm{i})+B(z-\mathrm{i})}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{(A+B)z+(A-B)\mathrm{i}}{(z-\mathrm{i})(z+\mathrm{i})}.\end{aligned}$$

So

$$\begin{aligned} A+B=0, &\quad (A-B)\mathrm{i}=1, \\ B=-A, &\quad\quad 2A\mathrm{i}=1, \\ A=\frac{1}{2\mathrm{i}}=-\frac{\mathrm{i}}{2}, &\quad B=\frac{\mathrm{i}}{2}.\end{aligned}$$

It follows that:

$$\begin{aligned}\oint \frac{z^2}{z^2+1}\mathrm{d}z &= \oint z^2\left(-\frac{\mathrm{i}}{2}\frac{1}{(z-\mathrm{i})} + \frac{\mathrm{i}}{2}\frac{1}{(z+\mathrm{i})}\right)\mathrm{d}z \\ &= -\frac{\mathrm{i}}{2}\oint \frac{z^2}{z-\mathrm{i}}\mathrm{d}z + \frac{\mathrm{i}}{2}\oint \frac{z^2}{z+\mathrm{i}}\mathrm{d}z.\end{aligned}$$

Each integral on the right-hand side has only one singular point inside the contour. According the Cauchy integral formula

$$\begin{aligned}\oint \frac{z^2}{z-\mathrm{i}}\mathrm{d}z &= 2\pi\mathrm{i}(\mathrm{i})^2 = -2\pi\mathrm{i}, \\ \oint \frac{z^2}{z+\mathrm{i}}\mathrm{d}z &= 2\pi\mathrm{i}\,(-\mathrm{i})^2 = -2\pi\mathrm{i}.\end{aligned}$$

Therefore

$$\oint \frac{z^2}{z^2+1}\mathrm{d}z = -\frac{\mathrm{i}}{2}\,(-2\pi\mathrm{i}) + \frac{\mathrm{i}}{2}\,(-2\pi\mathrm{i}) = 0.$$

Example 2.4.4. Evaluate the integral

$$\oint \frac{z-1}{2z^2+3z-2}\mathrm{d}z$$

around the square whose vertices are $(1,1),\ (-1,1),\ (-1,-1),(1,-1)$.

Solution 2.4.4. To find the singular points, we set the denominator to zero

$$2z^2+3z-2=0,$$

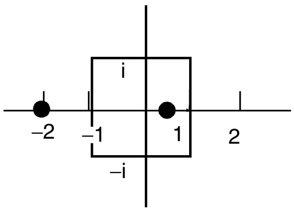
which gives the singular points at

$$z = \frac{1}{4}\left(-3 \pm \sqrt{9+16}\right) = \begin{cases} \frac{1}{2}, \\ -2. \end{cases}$$

The denominator can be written as

$$2z^2 + 3z - 2 = 2\left(z - \frac{1}{2}\right)(z+2).$$

The singular points and the contour are shown in the following figure:



Since only the singular point at $z = \frac{1}{2}$ is inside the contour, we can write the integral as

$$\oint \frac{z-1}{2z^2+3z-2}\mathrm{d}z = \oint \frac{z-1}{2(z-\frac{1}{2})(z+2)}\mathrm{d}z = \oint \frac{f(z)}{(z-\frac{1}{2})}\mathrm{d}z = 2\pi\mathrm{i}\, f\left(\frac{1}{2}\right),$$

where

$$f(z) = \frac{z-1}{2(z+2)}, \qquad f\left(\frac{1}{2}\right) = -\frac{1}{10}.$$

Therefore

$$\oint \frac{z-1}{2z^2+3z-2}\mathrm{d}z = -\frac{1}{5}\pi\mathrm{i}.$$

Several important theorems can be easily proved by Cauchy's integral formula and its derivatives.

Gauss' Mean Value Theorem

If $f(z)$ is analytic inside and on a circle C with center at z_0, then the mean value of $f(z)$ on C is $f(z_0)$.

This theorem follows directly from the Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{f(z)}{z-z_0}\mathrm{d}z.$$

Let the circle C be $|z - z_0| = r$, thus

$$z = z_0 + r\,\mathrm{e}^{\mathrm{i}\theta}, \quad \text{and} \quad \mathrm{d}z = \mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta.$$

Therefore

$$f(z_0) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z)}{z - z_0}\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})}{r\,\mathrm{e}^{\mathrm{i}\theta}}\mathrm{i}\,r\,\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$$
$$= \frac{1}{2\pi}\oint_C f\left(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta}\right)\mathrm{d}\theta,$$

which is the mean value of $f(z)$ on C.

Liouville's Theorem

If $f(z)$ is analytic in the entire complex plane and $|f(z)|$ is bounded for all values of z, then $f(z)$ is a constant.

To prove this theorem, we start with

$$f'(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{f(z')}{(z' - z)^2}\mathrm{d}z'.$$

The condition that $|f(z)|$ is bounded tells us that a nonnegative constant M exists such that $|f(z)| \leq M$ for all z. If we take C to be the circle $|z' - z| = R$, then

$$|f'(z)| \leq \left|\frac{1}{2\pi\mathrm{i}}\right|\oint_C \frac{|f(z')|}{|(z' - z)^2|}\,|\mathrm{d}z'|$$
$$\leq \frac{1}{2\pi}\frac{1}{R^2}M2\pi R = \frac{M}{R}.$$

Since $f(z')$ is analytic everywhere, we may take R as large as we like. It is clear that $\frac{M}{R} \to 0$, as $R \to \infty$. Therefore $|f'(z)| = 0$, which implies that $f'(z) = 0$ for all z, so $f(z)$ is a constant.

Fundamental Theorem of Algebra

The following theorem is now known as the fundamental theorem of algebra. In the last chapter we mentioned that this theorem is of critical importance in our number system.

Every polynomial equation

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z^n = 0$$

of degree one or greater has at least one root.

To prove this theorem, let us first assume the contrary, namely that $P_n(z) \neq 0$ for any z. Then the function

$$f(z) = \frac{1}{P_n(z)}$$

is analytic everywhere. Since nowhere will $f(z)$ go to infinity and $f(z) \to 0$ as $z \to \infty$, so $|f(z)|$ is bounded for all z. By Liouville's theorem we conclude that $f(z)$ must be a constant, and hence $P_n(z)$ must be a constant. This is a contradiction, since $P_n(z)$ is given as a polynomial of z. Therefore, $P_n(z) = 0$ must have at least one root.

It follows from this theorem that $P_n(z) = 0$ has exactly n roots. Since $P_n(z) = 0$ has at least one root, let us denote that root $\dot{z}_1$. Thus

$$P_n(z) = (z - z_1)Q_{n-1}(z),$$

where $Q_{n-1}(z)$ is a polynomial of degree $n-1$. By the same argument, we conclude that $Q_{n-1}(z)$ must have at least one root, which we denote it as z_2. Repeating this procedure n times we find

$$P_n(z) = (z - z_1)(z - z_2) \cdots (z - z_n) = 0.$$

Hence $P_n(z) = 0$ has exactly n roots.

Exercises

1. Show that the real and the imaginary parts of the following functions $f(z)$ satisfy the Cauchy–Reimann conditions
   (a) z^2, (b) e^z, (c) $\frac{1}{z+2}$.
2. Show that both the real part $u(x, y)$ and the imaginary part $v(x, y)$ of the analytic function $\mathrm{e}^z = u(x, y) + \mathrm{i}v(x, y)$ satisfy the Laplace equation
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$
3. Show that the derivative of $\frac{1}{z+2}$ calculated in the following three different ways gives the same result:
   (a) Let $\Delta y = 0$, so that $\Delta z \to 0$ parallel to the x-axis. In this case
$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}.$$
   (b) Let $\Delta x = 0$, so that $\Delta z \to 0$ parallel to the y-axis. In this case
$$f'(z) = \frac{\partial u}{\mathrm{i}\partial y} + \frac{\partial v}{\partial y}.$$
   (c) Use the same rule as if z were a real variable. That is
$$f'(z) = \frac{\mathrm{d}f}{\mathrm{d}z}.$$

4. Let $z^2 = u(x,y) + \mathrm{i}v(x,y)$, find the point of intersection of $u(x,y) = 1$ and $v(x,y) = 2$. Show that at the point of intersection the curve $u(x,y) = 1$ is perpendicular to $v(x,y) = 2$.

5. Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$ be an analytic function. If $u(x,y)$ is given by the following function:

$$(\mathrm{a}) x^2 - y^2; \qquad (b) \quad \mathrm{e}^y \sin x,$$

show that they satisfy the Laplace equation. Find the corresponding conjugate harmonic function $v(x,y)$. Express $f(z)$ as a function of z only.

Ans. (a) $v(x,y) = 2xy + c, \quad f(z) = z^2 + c$. (b) $v(x,y) = \mathrm{e}^y \cos x + c, \quad f(z) = \mathrm{i}\mathrm{e}^{-\mathrm{i}z} + c$.

6. In which quadrants of the complex plane is $f(z) = |x| - \mathrm{i}|y|$ an analytic function?

Hint: In first quadrant, $x > 0$, so $\frac{\partial u}{\partial x} = \frac{\partial |x|}{\partial x} = \frac{\partial x}{\partial x} = 1$, in the second quadrant, $x < 0$, so $\frac{\partial u}{\partial x} = \frac{\partial |x|}{\partial x} = \frac{\partial(-x)}{\partial x} = -1$, and so on.

Ans. $f(z)$ is analytic only in the second and fourth quadrants.

7. Express the real part and the imaginary part of $(z+1)^2$ in terms of polar coordinates, that is, find $u(r,\theta)$ and $v(r,\theta)$ in the expression

$$(z+1)^2 = u(r,\theta) + \mathrm{i}v(r,\theta).$$

Show that they satisfy the Cauchy–Riemann equations in the polar form:

$$\frac{\partial u(r,\theta)}{\partial r} = \frac{1}{r}\frac{\partial v(r,\theta)}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u(r,\theta)}{\partial \theta} = -\frac{\partial v(r,\theta)}{\partial r}.$$

8. Show that when an analytic function is expressed in terms of polar coordinates, both its real part and its imaginary part satisfy Laplace's equation in polar coordinates

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

9. To show that line integral are, in general, dependent on the path of integration, evaluate

$$\int_{-1}^{\mathrm{i}} |z|^2 \, \mathrm{d}z$$

(a) along the straight line from the initial point -1 to the final point i, (b) along the arc of the unit circle $|z| = 1$ traversed in the clockwise direction from the initial point -1 to the final point i.

Hint: (a) Parameterize the line segment by $z = -1 + (1+\mathrm{i})t, \quad 0 \le t \le 1$. (b) Parameterize the arc by $z = \mathrm{e}^{\mathrm{i}\theta}, \pi \ge \theta \ge \pi/2$.

Ans. (a) $2(1+\mathrm{i})/3, (b) 1 + \mathrm{i}$.

10. To verify that the line integral of an analytic function is independent of the path, evaluate

$$\int_0^{3+\mathrm{i}} z^2 \mathrm{d}z$$

(a) along the line $y = x/3$, (b) along the real axis to 3 and then vertically to $3 + \mathrm{i}$, (c) along the imaginary axis to i and then horizontally to $3 + \mathrm{i}$.

Ans. (a) $6 + \frac{26}{3}\mathrm{i}$, (b) $6 + \frac{26}{3}\mathrm{i}$, (c) $6 + \frac{26}{3}\mathrm{i}$.

11. Verify the Green's lemma

$$\oint [A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y] = \iint_R \left[\frac{\partial B(x,y)}{\partial x} - \frac{\partial A(x,y)}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y$$

for the integral

$$\oint [(x^2 + y)\mathrm{d}x - xy^2\mathrm{d}y]$$

taken around the boundary of the square with vertices at (0,0), (1,0), (0,1), (1,1).

12. Verify the Green's lemma for the integral

$$\oint [(x - y)\mathrm{d}x + (x + y)\mathrm{d}y]$$

taken around the boundary of the area in the first quadrant between the curve $y = x^2$ and $y^2 = x$.

13. Evaluate

$$\int_0^{3+\mathrm{i}} z^2 \mathrm{d}z$$

with fundamental theorem of calculus. That is,

$$\text{if} \quad \frac{\mathrm{d}F(z)}{\mathrm{d}z} = f(z), \quad \text{then} \quad \int_A^B f(z)\,\mathrm{d}z = F(B) - F(A),$$

provided $f(z)$ is analytic in a region between A and B.

Ans. $6 + \frac{26}{3}\mathrm{i}$.

14. What is the value of

$$\oint_C \frac{3z^2 + 7z + 1}{z + 1}\mathrm{d}z$$

(a) if C is the circle $|z + 1| = 1$? (b) if C is the circle $|z + \mathrm{i}| = 1$? (c) if C is the ellipse $x^2 + 2y^2 = 8$?

Ans. (a) $-6\pi\mathrm{i}$, (b) 0, (c) $-6\pi\mathrm{i}$.

15. What is the value of

$$\oint_C \frac{z+4}{z^2+2z+5}\mathrm{d}z$$

(a) if C is the circle $|z|=1$? (b) if C is the circle $|z+1-\mathrm{i}|=2$? (c) if C is the circle $|z+1+\mathrm{i}|=2$?

Ans. (a) 0, (b) $\frac{1}{2}(3+2\mathrm{i})\,\pi$, (c) $\frac{1}{2}(-3+2\mathrm{i})\,\pi$.

16. What is the value of

$$\oint_C \frac{\mathrm{e}^z}{(z+1)^2}\mathrm{d}z$$

around the circle $|z-1|=3$?

Ans. $2\pi\mathrm{i}\mathrm{e}^{-1}$.

17. What is the value of

$$\oint_C \frac{z+1}{z^3-2z^2}\mathrm{d}z$$

(a) If C is the circle $|z|=1$? (b) If C is the circle $|z-2-\mathrm{i}|=2$? (c) If C is the circle $|z-1-2\mathrm{i}|=2$?

Ans. (a) $-\frac{3}{2}\pi\mathrm{i}$, (b) $\frac{3}{2}\pi\mathrm{i}$, (c) 0.

18. Find the value of the closed loop integral

$$\oint \frac{z^3+\sin z}{(z-\mathrm{i})^3}\mathrm{d}z$$

taken around the boundary of the triangle with vertices at $\pm 2,\ 2\mathrm{i}$.

Ans. $\pi\left(\mathrm{e}-\mathrm{e}^{-1}\right)/2-6\pi$.

19. What is the value of

$$\oint_C \frac{\tan z}{z^2}\mathrm{d}z$$

if C is the circle $|z|=1$?

Ans. $2\pi\mathrm{i}$.

20. What is the value of

$$\oint_C \frac{\ln z}{(z-2)^2}\mathrm{d}z$$

if C is the circle $|z-3|=2$?

Ans. $\pi\mathrm{i}$.