3

Complex Series and Theory of Residues

Series expansions are ubiquitous in science and engineering. In the theory of complex functions, series expansions play a crucial role because they are the basis for deriving and using the theory of residues, which provide a powerful method for calculating both complex contour integrals and some difficult integrals of real variable. Before the formal development, we will first review a basic geometric series.

3.1 A Basic Geometric Series

Let

$$S = 1 + z + z^2 + z^3 + \cdots + z^n. \tag{3.1}$$

Multiplying by z,

$$zS = z + z^2 + z^3 + \cdots + z^n + z^{n+1}$$

and subtracting the two series

$$(1 - z)S = 1 - z^{n+1}$$

we get

$$S = \frac{1 - z^{n+1}}{1 - z}.$$

Now if $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$. Thus, if n goes to infinity,

$$S = \frac{1}{1 - z},$$

and it follows from (3.1) that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{k=0}^{\infty} z^k. \tag{3.2}$$

Clearly it will diverge for $|z| \geq 1$. It is important to remember that this series converges only for $|z| < 1$. Under this condition, the following alternative series is also convergent:

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{k=0}^{\infty} (-z)^k = 1 - z + z^2 - z^3 + \cdots . \tag{3.3}$$

3.2 Taylor Series

Taylor series is perhaps the most familiar series in real variables. Taylor series in complex variable is even more interesting.

3.2.1 The Complex Taylor Series

In many applications of complex variables, we wish to expand an analytic function $f(z)$ into a series around a particular point $z = z_0$. We will show that if $f(z)$ is analytic in the neigborhood of z_0 including the point at $z = z_0$, then $f(z)$ can be represented as a series of positive powers of $(z - z_0)$.

First let us recall

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z} \, \mathrm{d}t, \tag{3.4}$$

where t is the integration variable and it is on the enclosed contour C, inside which $f(z)$ is analytic. The quantity $(z-z_0)$ can be introduced into the integral through the identity

$$\frac{1}{t-z} = \frac{1}{(t-z_0)+(z_0-z)} = \frac{1}{(t-z_0)\left(1-\frac{z-z_0}{t-z_0}\right)}.$$

If

$$\left|\frac{z-z_0}{t-z_0}\right| < 1, \tag{3.5}$$

then by the basic geometric series (3.2)

$$\frac{1}{1-\frac{z-z_0}{t-z_0}} = 1 + \left(\frac{z-z_0}{t-z_0}\right) + \left(\frac{z-z_0}{t-z_0}\right)^2 + \left(\frac{z-z_0}{t-z_0}\right)^3 + \cdots . \tag{3.6}$$

Therefore (3.4) can be written as

$$\begin{aligned}
f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z}\,\mathrm{d}t \\
&= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z_0} \left[1 + \left(\frac{z-z_0}{t-z_0}\right) + \left(\frac{z-z_0}{t-z_0}\right)^2 + \left(\frac{z-z_0}{t-z_0}\right)^3 + \cdots \right] \mathrm{d}t \\
&= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{t-z_0}\,\mathrm{d}t + \left[\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^2}\,\mathrm{d}t\right](z-z_0) \\
&\quad + \left[\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^3}\,\mathrm{d}t\right](z-z_0)^2 \\
&\quad + \left[\frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^4}\,\mathrm{d}t\right](z-z_0)^3 + \cdots . \qquad (3.7)
\end{aligned}$$

According to Cauchy's integral formula and its derivatives

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^{n+1}}\,\mathrm{d}t$$

the earlier equation (3.7) becomes

$$\begin{aligned}
f(z) &= \sum_{n=0} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \\
&= f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots . \qquad (3.8)
\end{aligned}$$

This is the well-known Taylor series.

3.2.2 Convergence of Taylor Series

To discuss the convergence of the Taylor series, let us first recall the definition of singular points.

Singularity

If $f(z)$ is analytic at all points in the neighborhood of z_s but is not differentiable at z_s, then z_s is called a singular point. We also say that $f(z)$ has a singularity at $z = z_s$. For example:

$\frac{1}{z^2+1}$ has singularities at $z = \mathrm{i},\ -\mathrm{i}$.
$\tan z = \frac{\sin z}{\cos z}$ has singularities at $z = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2}, \ldots$
$\frac{1+2z}{z^2-5z+6}$ has singularities at $z = 2,\ 3$.
$\frac{1}{\mathrm{e}^z+1}$ has singularities at $z = \pm\mathrm{i}\pi,\ \pm\mathrm{i}3\pi,\ \pm\mathrm{i}5\pi, \ldots$.

Radius of Convergence

The Cauchy integral formula of (3.4) is, of course, valid for all z inside the Contour C, if $f(t)$ is analytic in and on C. However, in developing the Taylor series around $z = z_0$, we have used (3.6), which is true only if the condition of (3.5) is satisfied. This means $|z - z_0|$ must be less than $|t - z_0|$. Since t is on the contour C as shown in Fig. 3.1, the distance $|t - z_0|$ is changing as t is moving around C. With the contour shown in the figure, the smallest $|t - z_0|$ is $|s - z_0|$ where s is the point on C closest to z_0. For $|z - z_0|$ to be less than all possible $|t - z_0|$, $|z - z_0|$ must be less than $|s - z_0|$. This means the Taylor series of (3.8) is valid only for those points of z which are inside the circle centered at z_0, with a radius $R = |s - z_0|$.

If $f(z)$ is analytic everywhere, we can draw the contour C as large as we want. Therefore the Taylor series is convergent in the entire complex plane. However, if $f(z)$ has a singular point at $z = s$, then the contour must be so drawn in such way that the point $z = s$ is outside of C. In Fig. 3.1, the contour C can be infinitesimally close to s, but s must not be on or inside C. For such a case the largest possible radius of convergence is $|s - z_0|$. Therefore the radius of convergence of a Taylor series is equal to the distance between its expansion center and the nearest singular point.

The discussion earlier applies equally well to a circular region about the origin, $z_0 = 0$. The Taylor series about the origin

$$f(z) = f(0) + f'(0)\, z + \frac{f''(0)}{2!} z^2 + \cdots$$

is called the Maclaurin series.

Even in the expansion of a function of a real variable, the radius of convergence is equally important. To illustrate, consider

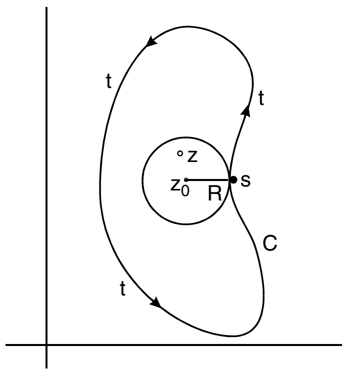

Fig. 3.1. Radius of convergence of the Taylor series. The expansion center is at z_0. The singular point at s limits the region of convergence within the interior of the circle of radius R

$$f(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots .$$

This series converges throughout the interior of the largest circle around the origin in which $f(z)$ is analytic. Now, $f(z)$ has two singular points at $z = \pm \mathrm{i}$, and even though one may be concerned solely with real values of z, for which $1/(1+x^2)$ is everywhere infinitely differentiable with respect to x, these singularities in the complex plane set an inescapable limit to the interval of convergence on the x axis. Since the distance between the expansion center at $z = 0$ and the nearest singular point, i or $-\mathrm{i}$ is $|\mathrm{i} - 0| = 1$, the radius of convergence is equal to one. The series is convergent only inside the circle of radius 1, centered at origin. Thus the interval of convergence on the x axis is between $x = \pm 1$. In other words, the Maclaurin series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

is valid only for $-1 < x < 1$, although $1/(1+x^2)$ and its derivatives of all orders are well defined along the real axis x. Now if we expand the real function $1/(1+x^2)$ into Taylor series around $x = x_0$, then the radius of convergence is equal to $|\mathrm{i} - x_0| = \sqrt{1+x_0^2}$. This means that this series will converge only in the interval between $x = x_0 - \sqrt{1+x_0^2}$ and $x = x_0 + \sqrt{1+x_0^2}$.

3.2.3 Analytic Continuation

If we know the values of an analytic function in some small region around z_0, we can use the Taylor expansion about z_0 to find the values of the function in a larger region. Although the Taylor expansion is valid only inside the circle of radius of convergence which is determined by the location of the nearest singular point, a chain of Taylor expansions can be used to determine the function throughout the entire complex plane except at the singular points of the function. This process is illustrated in Fig. 3.2.

Suppose we know the values around z_0 and the singular point nearest to z_0 is s_0. The Taylor expansion about z_0 holds within a circular region of radius $|z_0 - s_0|$. Since the Taylor expansion gives the values of the function and all its derivatives at every point in this circle, we can use any point in this circle as the new expansion center. For example, we may expand another Taylor series about z_1 as shown in Fig. 3.2. We can do this because $f^n(z_1)$ is known for all n from the first Taylor expansion about z_0. The radius of convergence of this second Taylor series is determined by the distance from z_1 to the nearest singular point s_1. Continuing this way, as indicated in Fig. 3.2, we can cover the whole complex plane except at the singular points $s_0, s_1, s_2, \ldots$. In other words, the analytic function everywhere can be constructed from the knowledge of the function in a small region. This process is called analytic continuation.

An immediate consequence of analytic continuation is the so called identity theorem. It states that if $f(z)$ and $g(z)$ are analytic and $f(z) = g(z)$ along

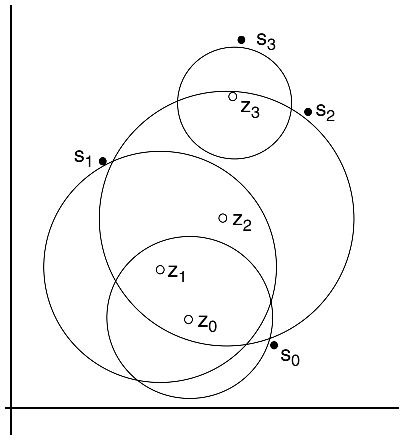

Fig. 3.2. Analytic Continuation. A series of Taylor expansions which analytically continue a function originally known in the region around z_0. The first expansion about z_0 is valid only inside the circle of radius $|z_0 - s_0|$, where s_0 is the singular point nearest to z_0. The next Taylor expansion is around z_1 which is inside the first circle. The second Taylor expansion is limited by the singular point s_1, and so on

a curve L in a region D, then $f(z) = g(z)$ throughout D. We can show this by considering the analytic function $h(z) = f(z) - g(z)$. If we can show that $h(z)$ is identically zero throughout the region, then the theorem is proved.

If we choose a point $z = z_0$ on L, then we can expand $h(z)$ in a Taylor series about z_0,

$$h(z) = h(z_0) + h'(z_0)(z - z_0) + \frac{1}{2!}h''(z_0)(z - z_0)^2 + \cdots,$$

which will converge inside the some circle that extends as far as the nearest point of the boundary of D. But since z_0 is on L, $h(z_0) = 0$. Furthermore, the derivatives of h must also be zero if z is approaching z_0 along L. Since $h(z)$ is analytic, its derivatives are independent on the way how z is approaching z_0, this means

$$h'(z_0) = h''(z_0) = \cdots = 0.$$

Therefore, $h(z) = 0$ inside the circle. We may now expand about a new point, which can lie anywhere inside the circle. Thus by analytic continuation, we may show that $h(z) = 0$ throughout the region D.

3.2.4 Uniqueness of Taylor Series

If there are constants a_n $(n = 0,\ 1,\ 2, \ldots)$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is convergent for all points z interior to some circle centered at z_0, then this power series must be the Taylor series, regardless of how those constants are obtained. This is quite easy to show, since

$$\begin{aligned}
f(z) &= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + a_3 (z - z_0)^3 + \cdots , \\
f'(z) &= a_1 + 2a_2 (z - z_0) + 3a_3 (z - z_0)^2 + 4a_4 (z - z_0)^3 + \cdots , \\
f''(z) &= 2a_2 + 3 \cdot 2a_3 (z - z_0) + 4 \cdot 3 (z - z_0)^2 + \cdots ,
\end{aligned}$$

clearly

$$f(z_0) = a_0, \quad f'(z_0) = a_1, \quad f''(z_0) = 2a_2, \quad f'''(z_0) = 3 \cdot 2a_3, \quad \ldots .$$

It follows that:

$$a_n = \frac{1}{n!} f^{(n)}(z_0),$$

which are the Taylor coefficients. Thus, Taylor series is unique. Thus no matter how the power series is obtained, if it is convergent in some circular region, it is the Taylor series. The following examples illustrate some of the techniques of expanding a function into its Taylor series.

Example 3.2.1. Find the Taylor series about the origin and its radius of convergence for

$$\text{(a)}\ \sin z, \quad \text{(b)}\ \cos z, \quad \text{(c)}\ \mathrm{e}^z.$$

Solution 3.2.1. (a) Since $f(z) = \sin z$,

$$f'(z) = \cos z, \quad f''(z) = -\sin z, \quad f'''(z) = -\cos z, \quad f^4(z) = \sin z, \quad \ldots .$$

Hence

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^4(0) = 0, \quad \ldots .$$

Thus

$$\begin{aligned}
\sin z &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, z^n \\
&= z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots .
\end{aligned}$$

This series is valid for all z, since $\sin z$ is an entire function (analytic for the entire complex plane).

(b) If $f(z) = \cos z$, then

$$f'(z) = -\sin z, \quad f''(z) = -\cos z, \quad f'''(z) = \sin z, \quad f^4(z) = \cos z, \quad \ldots$$

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^4(0) = 1, \quad \ldots.$$

Therefore

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots .$$

This series is also valid for all z.

(c) For $f(z) = \mathrm{e}^z$, then

$$f^{(n)}(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\mathrm{e}^z = \mathrm{e}^z \text{and} f^{(n)}(0) = 1.$$

It follows:

$$\mathrm{e}^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots .$$

This series converges for all z, since e^z is an entire function.

Example 3.2.2. Find the Taylor series about the origin and its radius of convergence for

$$f(z) = \frac{\mathrm{e}^z}{\cos z}.$$

Solution 3.2.2. The singular points of the function are at the zeros of the denominator. Since $\cos \frac{\pi}{2} = 0$, the singular point nearest to the origin is at $z = \pm\frac{\pi}{2}$. Therefore the Taylor series about the origin is valid for $|z| < \frac{\pi}{2}$. We can find the constants a_n of

$$\frac{\mathrm{e}^z}{\cos z} = a_0 + a_1 z + a_2 z^2 + \cdots$$

from $f^{(n)}(0)$, but the repeated differentiations become increasingly tedious. So we take the advantage of the fact that the Taylor series for e^z and $\cos z$ are already known. Replacing e^z and $\cos z$ with their respective Taylor series in the equation

$$\mathrm{e}^z = \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)\cos z,$$

we obtain

$$1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots = \left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right) \left(1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \cdots\right).$$

Multiplying out and collecting terms, we have

$$1+z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\cdots=a_0+a_1z+\left(a_2-\frac{1}{2}a_0\right)z^2+\left(a_3-\frac{1}{2}a_1\right)z^3+\cdots.$$

Therefore

$$a_0=1,\quad a_1=1,\quad a_2-\frac{1}{2}\,a_0=\frac{1}{2},\quad a_3-\frac{1}{2}\,a_1=\frac{1}{3!},\ \ldots$$

It follows that:

$$a_2=\frac{1}{2}+\frac{1}{2}\,a_0=1,\quad a_3=\frac{1}{3!}+\frac{1}{2}\,a_1=\frac{2}{3},\quad \ldots$$

and

$$\frac{e^z}{\cos z}=1+z+z^2+\frac{2}{3}\,z^3+\cdots,\qquad |z|<\frac{\pi}{2}.$$

Example 3.2.3. Find the Taylor series about $z=2$ for

$$\text{(a)}\ \frac{1}{z},\qquad \text{(b)}\ \frac{1}{z^2}.$$

Solution 3.2.3. (a) The function $1/z$ has a singular point at $z=0$, the distance between this point and the expansion center is 2. Therefore the Taylor series about $z=2$ is convergent for $|z-2|<2$ and has the form

$$\frac{1}{z}=a_0+a_1(z-2)+a_2\,(z-2)^2+\cdots.$$

We can write the function as

$$\frac{1}{z}=\frac{1}{2+(z-2)}=\frac{1}{2}\frac{1}{1+\left(\frac{z-2}{2}\right)}.$$

For $|z-2|<2$, $\left|\frac{z-2}{2}\right|<1$. Therefore we can use the geometric series (3.3) to expand

$$\frac{1}{1+\left(\frac{z-2}{2}\right)}=1-\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\cdots.$$

It follows that for $|z-2|<2$

$$\begin{aligned}\frac{1}{z}&=\frac{1}{2}\left[1-\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\left(\frac{z-2}{2}\right)^4-\cdots\right]\\&=\frac{1}{2}-\frac{1}{4}\,(z-2)+\frac{1}{8}\,(z-2)^2-\frac{1}{16}\,(z-2)^3+\frac{1}{32}\,(z-2)^4-\cdots.\end{aligned}$$

(b) Since

$$\frac{1}{z^2} = -\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{z},$$

therefore

$$\begin{aligned}\frac{1}{z^2} &= -\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{2} - \frac{1}{4}(z-2) + \frac{1}{8}(z-2)^2 - \frac{1}{16}(z-2)^3 + \frac{1}{32}(z-2)^4 - \cdots\right) \\ &= \frac{1}{4} - \frac{1}{4}(z-2) + \frac{3}{16}(z-2)^2 - \frac{1}{8}(z-2)^3 + \cdots .\end{aligned}$$

Example 3.2.4. Find the Taylor series about the origin for

$$f(z) = \frac{1}{1+z-2z^2}.$$

Solution 3.2.4. Since $1+z-2z^2 = (1-z)(1+2z)$, the function $f(z)$ has two singular points at $z=1$ and $z=-1/2$. The Taylor series expansion about $z=0$ will be convergent for $|z| < \frac{1}{2}$. Furthermore

$$\frac{1}{1+z-2z^2} = \frac{1/3}{1-z} + \frac{2/3}{1+2z}.$$

For $|z| < \frac{1}{2}$ and $|2z| < 1$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots ,$$

$$\frac{1}{1+2z} = 1 - 2z + 4z^2 - 8z^3 + \cdots .$$

Thus

$$\begin{aligned}f(z) &= \frac{1}{3}\left(1 + z + z^2 + z^3 + \cdots\right) + \frac{2}{3}\left(1 - 2z + 4z^2 - 8z^3 + \cdots\right) \\ &= 1 - z + 3z^2 - 5z^3 + \cdots .\end{aligned}$$

Example 3.2.5. Find the Taylor series about the origin for

$$f(z) = \ln(1+z).$$

Solution 3.2.5. First note that

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln(1+z) = \frac{1}{1+z},$$

and

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots ,$$

so

$$\mathrm{d} \ln (1+z) = \left(1 - z + z^2 - z^3 + \cdots\right) \mathrm{d}z.$$

Integrating both sides, we have

$$\ln (1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots + k.$$

The integration constant $k = 0$, since at $z = 0, \quad \ln(1) = 0$. Therefore

$$\ln (1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots .$$

This series converges for $|z| < 1$, since at $z = -1$, $f(z)$ is singular.

3.3 Laurent Series

In many applications, it is necessary to expand functions around points at which, or in the neighborhood of which, the functions are not analytic. The method of Taylor series is obviously inapplicable in such cases. A new type of series known as Laurent expansion is required. This series furnishes us with a representation which is valid in the annular ring bounded by two concentric circles, provided that the function being expanded is analytic everywhere on and between the two circles.

Consider the annulus bounded by two circles of C_0 and C_i with a common center z_0 as shown in Fig. 3.3a. The function $f(z)$ is analytic inside the annular region; however, there may be singular points inside the smaller circle

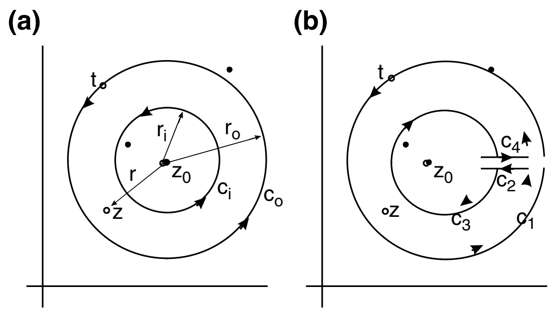

Fig. 3.3. Annular region between two *circles* where the function is analytic and the Laurent series is valid. Inside the *inner circle* and outside the *outer circle*, the function may have singular points

or outside the larger circle. We can apply the Cauchy's integral formula to the region which is cut up as shown in Fig. 3.3b. The region is now simply connected and is bounded by the curve $C' = C_1 + C_2 + C_3 + C_4$. Cauchy's integral formula is then

$$\begin{aligned} f(z) &= \frac{1}{2\pi \mathrm{i}} \oint_{C'} \frac{f(t)}{t-z} \mathrm{d}t \\ &= \frac{1}{2\pi \mathrm{i}} \left(\int_{C_1} \frac{f(t)}{t-z} \mathrm{d}t + \int_{C_2} \frac{f(t)}{t-z} \mathrm{d}t + \int_{C_3} \frac{f(t)}{t-z} \mathrm{d}t + \int_{C_4} \frac{f(t)}{t-z} \mathrm{d}t \right), \end{aligned}$$

where t is on C' and z is a point inside C'. Now let the gap between C_2 and C_4 shrink to zero, then the integrals along C_2 and C_4 will cancel each other, since they are oriented in the opposite directions, if $f(z)$ is single valued. Furthermore, the contour C_1 becomes C_0 and the contour C_3 is identical to C_i turning the opposite direction. Therefore

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{f(t)}{t-z} \mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \oint_{C_i} \frac{f(t)}{t-z} \mathrm{d}t, \tag{3.9}$$

where C_0 and C_i are both traversed in the counterclockwise direction. The negative sign results because the direction of integration was reversed on C_i.

We can introduce z_0, the common center of C_0 and C_i, as the expansion center. For the first integral in (3.9) with t on C_0, we can write

$$\begin{aligned} \frac{1}{t-z} &= \frac{1}{t-z_0+z_0-z} = \frac{1}{(t-z_0)-(z-z_0)} \\ &= \frac{1}{(t-z_0)\left(1-\frac{z-z_0}{t-z_0}\right)}. \end{aligned} \tag{3.10}$$

Since t is on C_0 and z is inside C_0, as shown in Fig. 3.3a

$$\left| \frac{z-z_0}{t-z_0} \right| = \frac{r}{r_o} < 1,$$

so we can expand $\left(1-\frac{z-z_0}{t-z_0}\right)^{-1}$ with the geometric series of (3.2), and (3.10) becomes

$$\begin{aligned} \frac{1}{t-z} = \frac{1}{t-z_0} \Bigg[1 &+ \frac{z-z_0}{t-z_0} + \left(\frac{z-z_0}{t-z_0}\right) + \left(\frac{z-z_0}{t-z_0}\right)^2 \\ &+ \left(\frac{z-z_0}{t-z_0}\right)^3 + \cdots \Bigg] \quad \text{for } t \text{ on } C_0. \end{aligned} \tag{3.11}$$

For the second integral with t on C_i and z is between C_0 and C_i, we can write

$$\frac{1}{t-z} = -\frac{1}{z-t} = -\frac{1}{z-z_0+z_0-t}$$
$$= -\frac{1}{(z-z_0)-(t-z_0)} = -\frac{1}{(z-z_0)\left[1-\frac{t-z_0}{z-z_0}\right]}.$$

Since

$$\left|\frac{t-z_0}{z-z_0}\right| = \frac{r_i}{r} < 1$$

as shown in Fig. 3.3a, we can again expand

$$\left(1-\frac{t-z_0}{z-z_0}\right)^{-1}$$

with the geometric series, and write

$$\frac{1}{t-z} = -\frac{1}{z-z_0}\left[1+\frac{t-z_0}{z-z_0}+\left(\frac{t-z_0}{z-z_0}\right)^2+\left(\frac{t-z_0}{z-z_0}\right)^3+\cdots\right] \quad \text{for } t \text{ on } C_i. \tag{3.12}$$

Putting (3.11) and (3.12) into (3.9), we have

$$f(z) = I_{C_0} + I_{C_i},$$

where

$$I_{C_0} = \frac{1}{2\pi \mathrm{i}}\oint_{C_0}\frac{f(t)}{t-z_0}\left[1+\frac{z-z_0}{t-z_0}+\left(\frac{z-z_0}{t-z_0}\right)^2+\left(\frac{z-z_0}{t-z_0}\right)^3+\cdots\right]\mathrm{d}t$$
$$= \frac{1}{2\pi \mathrm{i}}\oint_{C_0}\frac{f(t)}{t-z_0}\,\mathrm{d}t + \left(\frac{1}{2\pi \mathrm{i}}\oint_{C_0}\frac{f(t)}{(t-z_0)^2}\,\mathrm{d}t\right)(z-z_0)$$
$$+\left(\frac{1}{2\pi \mathrm{i}}\oint_{C_0}\frac{f(t)}{(t-z_0)^3}\,\mathrm{d}t\right)(z-z_0)^2+\left(\frac{1}{2\pi \mathrm{i}}\oint_{C_0}\frac{f(t)}{(t-z_0)^4}\,\mathrm{d}t\right)(z-z_0)^3+\cdots$$

and

$$I_{C_i} = \frac{1}{2\pi \mathrm{i}}\oint_{C_i}\frac{f(t)}{z-z_0}\left[1+\frac{t-z_0}{z-z_0}+\left(\frac{t-z_0}{z-z_0}\right)^2+\left(\frac{t-z_0}{z-z_0}\right)^3+\cdots\right]\mathrm{d}t$$
$$= \left(\frac{1}{2\pi \mathrm{i}}\oint_{C_i} f(t)\mathrm{d}t\right)\frac{1}{z-z_0} + \left(\frac{1}{2\pi \mathrm{i}}\oint_{C_i} f(t)(t-z_0)\mathrm{d}t\right)\frac{1}{(z-z_0)^2}$$
$$+\left(\frac{1}{2\pi \mathrm{i}}\oint_{C_i} f(t)(t-z_0)^2\mathrm{d}t\right)\frac{1}{(z-z_0)^3}+\cdots.$$

Therefore, in the region between C_i and C_0, $f(z)$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{k=1}^{\infty} b_n \frac{1}{(z - z_0)^k},$$

where

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C_0} \frac{f(t)}{(t - z_0)^{n+1}} \,\mathrm{d}t, \qquad b_k = \frac{1}{2\pi \mathrm{i}} \oint_{C_i} f(t)\,(t - z_0)^{k-1} \,\mathrm{d}t.$$

Because of the principle of deformation of contours, we can replace both C_i and C_0 by a closed contour C between C_i and C_0 without changing the values of the integrals. Thus we can write this series as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

with

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C} \frac{f(t)}{(t - z_0)^{n+1}} \,\mathrm{d}t. \tag{3.13}$$

This expansion is known as the Laurent series which contains both negative and positive powers of $(z - z_0)$.

It should be noted that the coefficients of positive powers of $(z - z_0)$ cannot be replaced by the derivative expressions, since $f(z)$ is not analytic inside C. However, if there is no singular point inside C_i, then these coefficients can indeed be replaced by $f^{(n)}(z_0)/n!$, at the same time the coefficients of the negative powers of $(z - z_0)$ are identically equal to zero by the Cauchy theorem, since $f(t)(t - z_0)^{-n-1}$ for $n \leq -1$ are analytic inside C. In such a case, the Laurent expansion reduces to the Taylor expansion.

3.3.1 Uniqueness of Laurent Series

Just as Taylor series, Laurent series is unique. If a series

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges to $f(z)$ at all points in some annular domain about z_0, then regardless how the constants a_n are obtained, the series is the Laurent expansion for $f(z)$ in powers of $(z - z_0)$ for that domain. This statement is proved if we can show that

$$a_n = \frac{1}{2\pi \mathrm{i}} \oint_{C} \frac{f(t)}{(t - z_0)^{n+1}} \,\mathrm{d}t. \tag{3.14}$$

We now show that this is indeed the case. Let $g_k(t)$ be defined as

$$g_k(t) = \frac{1}{2\pi \mathrm{i}} \frac{1}{(t-z_0)^{k+1}},$$

where k is an integer, either positive or negative, or zero. Furthermore let C be a circle inside the annulus centered at z_0 and taken in the counterclockwise direction, so

$$\oint_C g_k(t) f(t)\,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-z_0)^{k+1}}\,\mathrm{d}t. \tag{3.15}$$

Now if $f(t)$ is expressible as

$$f(t) = \sum_{n=-\infty}^{\infty} a_n (t-z_0)^n,$$

then

$$\begin{aligned}\oint_C g_k(t) f(t)\,\mathrm{d}t &= \frac{1}{2\pi \mathrm{i}} \oint_C \frac{1}{(t-z_0)^{k+1}} \left(\sum_{n=-\infty}^{\infty} a_n (t-z_0)^n \right) \mathrm{d}t \\ &= \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi \mathrm{i}} \oint_C \frac{1}{(t-z_0)^{k-n+1}}\,\mathrm{d}t.\end{aligned}$$

The last integral can be easily evaluated by setting $t-z_0 = r\mathrm{e}^{\mathrm{i}\theta}$, so $\mathrm{d}t = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$ and

$$\begin{aligned}\oint_C \frac{1}{(t-z_0)^{k-n+1}}\,\mathrm{d}t &= \int_0^{2\pi} \frac{\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{r^{k-n+1}\mathrm{e}^{\mathrm{i}(k-n+1)\theta}}\,\mathrm{d}\theta \\ &= \frac{\mathrm{i}}{r^{k-n}} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-k)}\mathrm{d}\theta = 2\pi \mathrm{i}\delta_{nk} = \begin{cases} 0 & n \neq k, \\ 2\pi \mathrm{i} & n = k. \end{cases}\end{aligned} \tag{3.16}$$

Thus

$$\oint_C g_k(t) f(t)\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi \mathrm{i}} 2\pi \mathrm{i}\delta_{nk} = a_k. \tag{3.17}$$

It follows from (3.15) that:

$$a_k = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f(t)}{(t-t_0)^{k+1}}\,\mathrm{d}t. \tag{3.18}$$

Since k is an arbitrary integer, (3.14) must hold.

Thus no matter how the expansion is obtained, as long as it is valid in the specified annular domain, it is the Laurent series. This enables us to determine the Laurent coefficients with elementary techniques, as illustrated in the following examples. The integral representations of the Laurent coefficients (3.18) are important, not as means of finding the coefficients, but as means of using the coefficients to evaluate these integrals. We will elaborate this aspect of the Laurent series in following sections on the theory of residues.

Example 3.3.1. Find the Laurent series about $z = 0$ for the function

$$f(z) = \mathrm{e}^{1/z}.$$

Solution 3.3.1. Since $f(z)$ is analytic for all z, except for $z = 0$, the expansion of $f(z)$ about $z = 0$ will be a Laurent series valid in the annulus $0 < |z| < \infty$. To obtain the expansion, let $1/z = t$, and note

$$\mathrm{e}^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots .$$

Therefore

$$\mathrm{e}^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots .$$

Example 3.3.2. Find all possible Laurent expansions about $z = 0$ of

$$f(z) = \frac{1 + 2z^2}{z^3 + z^5},$$

and specify the regions in which they are valid.

Solution 3.3.2. By setting the denominator to zero $z^3 + z^5 = z^3\left(1 + z^2\right) = 0$, We get three singular points, $z = 0$, and $z = \pm\mathrm{i}$. They are shown in Fig. 3.4. Therefore we can expand the function about $z = 0$ in two different Laurent series, one is valid for the region $0 < |z| < 1$ as shown in (a), the other is valid in the region $|z| > 1$ as shown in (b).

The function can be written as

$$f(z) = \frac{1 + 2z^2}{z^3 + z^5} = \frac{1 + 2z^2}{z^3(1 + z^2)} = \frac{2(1 + z^2) - 1}{z^3(1 + z^2)} = \frac{1}{z^3}\left(2 - \frac{1}{1 + z^2}\right).$$

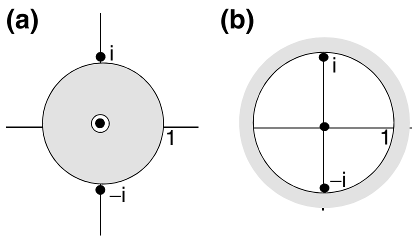

Fig. 3.4. If the function has three singular points at $z = 0$ and $z = \pm\mathrm{i}$, then the function can be expanded into two different Laurent series about $z = 0$. (**a**) One series is valid in the region $0 < |z| < 1$, (**b**) the other series is valid in the region $1 < |z|$

In the case of (a), $|z| < 1$, so is $|z^2| < 1$. We can use the geometric series to express

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots .$$

Therefore

$$\begin{aligned} f(z) &= \frac{1}{z^3}\left(2 - \left[1 - z^2 + z^4 - z^6 + \cdots\right]\right) \\ &= \frac{1}{z^3} + \frac{1}{z} - z + z^3 + \cdots \quad \text{for} \quad 0 < |z| < 1. \end{aligned}$$

In the case of (b), $|z^2| > 1$, we first write

$$\frac{1}{1+z^2} = \frac{1}{z^2\left(1+\frac{1}{z^2}\right)}.$$

Since $\left|\frac{1}{z^2}\right| < 1$, again we can use the geometric series

$$\frac{1}{\left(1+\frac{1}{z^2}\right)} = 1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots .$$

Thus

$$\begin{aligned} f(z) &= \frac{1}{z^3}\left(2 - \frac{1}{z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right]\right) \\ &= \frac{2}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} + \cdots , \quad \text{for} \quad |z| > 1. \end{aligned}$$

Example 3.3.3. Find the Laurent series expansion of

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

valid in each of the shaded regions shown in Fig. 3.5.

Solution 3.3.3. First we note that $z^2 - 3z + 2 = (z-2)(z-1)$, so the function has two singular points at $z = 2$ and $z = 1$. Taking the partial fraction, we have

$$\begin{aligned} f(z) = \frac{1}{z^2 - 3z + 2} &= \frac{1}{(z-2)(z-1)} \\ &= \frac{1}{z-2} - \frac{1}{z-1}. \end{aligned}$$

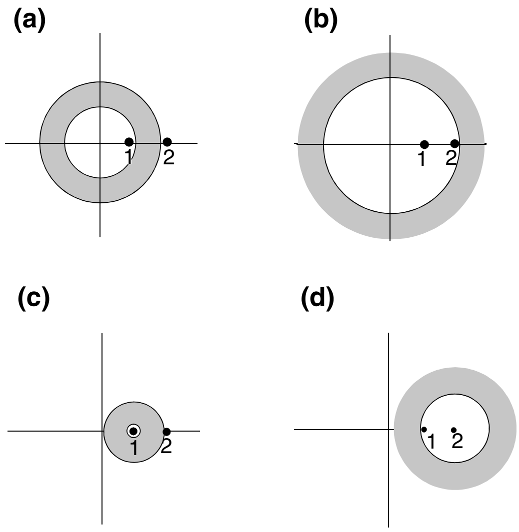

Fig. 3.5. The function with two singular points at $z = 1$ and $z = 2$ can be expanded into different Laurent series in different regions: (**a**) expanded about $z = 0$ valid in the region $1 < |z| < 2$, (**b**) expanded about $z = 0$ valid in the region $2 < |z|$, (**c**) expanded about $z = 1$ valid in the region $0 < |z - 1| < 1$, (**d**) expanded $z = 2$ valid in the region $1 < |z - 2|$

(a) In this case, we have to expand around $z = 0$, so we are seeking a series in the form of

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

The values of $|z|$ between the two circles are such that $1 < |z| < 2$. To make use of the basic geometric series, we write

$$\frac{1}{z-2} = -\frac{1}{2(1-\frac{z}{2})}.$$

Since $|z/2| < 1$, so

$$\frac{1}{z-2} = -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$
$$= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots .$$

As for the second fraction, we note that $|z| > 1$, so $|1/z| < 1$. Therefore we write

$$-\frac{1}{z-1} = -\frac{1}{z(1-\frac{1}{z})}$$
$$= -\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]$$
$$= -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots.$$

Thus the Laurent series in region (a) is

$$f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots.$$

(b) Again the expansion center is at the origin, but in region (b) $|z| > 2$. So we expand the first fraction as

$$\frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{z}\right)}$$
$$= \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$
$$= \frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\cdots.$$

Note that the expansion of the second fraction we worked out in part (a) is still valid in this case

$$-\frac{1}{z-1} = -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}-\cdots.$$

Thus the Laurent series in region (b) is the sum of these two expressions

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$
$$= \frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+\cdots.$$

(c) In this region, we are expanding around $z = 1$, so we are seeking a series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-1)^n.$$

Since in this region $0 < |z-1| < 1$, so we choose to write the function as

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{(z-1)\left[(z-1)-1\right]} = -\frac{1}{(z-1)\left[1-(z-1)\right]},$$

and use the geometric series for

$$\frac{1}{1-(z-1)} = 1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots$$

Therefore the desired Laurent series valid in region (c) is

$$\begin{aligned} f(z) &= -\frac{1}{(z-1)}\left(1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots\right) \\ &= -\frac{1}{(z-1)} - 1 - (z-1) - (z-1)^2 - \cdots . \end{aligned}$$

(d) In this region we are seeking an expansion about $z = 2$ in form of

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-2)^n.$$

that is valid for $|z-2| > 1$. So we choose to write the function as

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{[(z-2)+1](z-2)} = \frac{1}{(z-2)^2\left[1 + \frac{1}{z-2}\right]}.$$

Since $\left|\frac{1}{z-2}\right| < 1$, we can use the geometric series for

$$\left[1 + \frac{1}{z-2}\right]^{-1} = 1 - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots .$$

Therefore the desired Laurent series valid in region (d) is

$$\begin{aligned} f(z) &= \frac{1}{(z-2)^2}\left(1 - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right) \\ &= \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \frac{1}{(z-2)^4} - \frac{1}{(z-2)^5} + \cdots . \end{aligned}$$

3.4 Theory of Residues

3.4.1 Zeros and Poles

Zeros

If $f(z_0) = 0$, then the point z_0 is said to be a zero of the function $f(z)$. If $f(z)$ is analytic at z_0, then we can expand it in a Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

Since z_0 is a zero of the function, clearly $a_0 = 0$. If $a_1 \neq 0$, then z_0 is said to be a simple zero. If both a_0 and a_1 are zero and $a_2 \neq 0$, then z_0 is a zero of order two, and so on.

If $f(z)$ has a zero of order m at z_0, that is, $a_0, a_1, \ldots, a_{m-1}$ are all zero and $a_m \neq 0$, then $f(z)$ can be written as

$$f(z) = (z - z_0)^m g(z),$$

where

$$g(z) = a_m + a_{m+1}(z - z_0) + a_{m+2}(z - z_0)^2 + \cdots .$$

It is clear that $g(z)$ is analytic (therefore continuous) at z_0, and $g(z_0) = a_m \neq 0$. It follows that in the immediate neighborhood of z_0, there is no other zero, because $g(z)$ cannot suddenly drop to zero, since it is continuous. Therefore there exists a disk of finite radius δ surrounds z_0, within which $g(z) \neq 0$. In other words,

$$f(z) \neq 0 \quad \text{for} \quad 0 < |z - z_0| < \delta.$$

In this sense, z_0 is said to be an isolated zero of $f(z)$.

Isolated Singularities

As we recall, a singularity of a function $f(z)$ is a point at which $f(z)$ is not analytic. A point at which $f(z)$ is analytic is called a regular point. A point z_0 is said to be an isolated singularity of $f(z)$ if there exists a neighborhood of z_0 in which z_0 is the only singular point of $f(z)$. For example, a rational function $P(z)/Q(z)$, (the ratio of two polynomials), is analytic everywhere except at zeros of $Q(z)$. If all the zeros of $Q(z)$ are isolated, then all the singularities of $P(z)/Q(z)$ are isolated.

Poles

If $f(z)$ has an isolated singular point at z_0, then in the immediate neighborhood of z_0, $f(z)$ can be expanded in a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-3}}{(z - z_0)^3} + \cdots .$$

The portion of the series involving negative powers of $(z - z_0)$ is called the principal part of $f(z)$ at z_0. If the principal part contains at least one nonzero term but the number of such terms are finite, then there exists an integer m such that

$$a_{-m} \neq 0 \quad \text{and} \quad a_{-(m+1)} = a_{-(m+2)} = \cdots = 0.$$

That is, the expansion takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{a_{-1}}{(z - z_0)} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-m}}{(z - z_0)^m},$$

where $a_{-m} \neq 0$. In this case, the isolated singular point z_0 is called a pole of order m. A pole of order one is usually referred to as a simple pole.

If an infinite number of coefficients of negative powers are nonzero, then z_0 is called an essential singular point.

3.4.2 Definition of the Residue

If z_0 is an isolated singular point of $f(z)$, then the function $f(z)$ is analytic in the neighborhood of $z = z_0$ with the exception of the point $z = z_0$ itself. In the immediate neighborhood of z_0, $f(z)$ can be expanded in a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{3.19}$$

The coefficients a_n are expressed in terms of contour integrals of (3.13). Among the coefficients, a_{-1} is of particular interest,

$$a_{-1} = \frac{1}{2\pi \mathrm{i}} \oint_C f(z)\, \mathrm{d}z, \tag{3.20}$$

where C is a closed contour in the counterclockwise direction around z_0. It is the coefficient of $(z - z_0)^{-1}$ term in the expansion, and is called the residue of $f(z)$ at the isolated singular point z_0. We emphasize once again, for a_{-1} of (3.20) to be called the residue at z_0, the closed contour C must not contain any singularity other than z_0. We shall denote this residue as

$$a_{-1} = \mathrm{Res}_{z=z_0} \left[f(z) \right].$$

The reason for the name "residue" is that if we integrate the Laurent series term by term over a circular contour, the only term which survives the integration process is the a_{-1} term. This follows from (3.19) that

$$\oint f(z)\, \mathrm{d}z = \sum_{n=-\infty}^{\infty} a_n \oint (z - z_0)^n \mathrm{d}z.$$

These integrals can be easily evaluated by setting $z - z_0 = r\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{d}z = \mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$,

$$\oint (z-z_0)^n \mathrm{d}z = \int_0^{2\pi} \mathrm{i}r^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}\mathrm{d}\theta = \begin{cases} 0 & n \neq -1 \\ 2\pi\mathrm{i} & n = -1. \end{cases}$$

Thus, only the term with $n = -1$ is left. The coefficient of this term is called the residue

$$\frac{1}{2\pi\mathrm{i}} \oint f(z)\,\mathrm{d}z = a_{-1}.$$

3.4.3 Methods of Finding Residues

Residues are defined in (3.20). In some cases, we can carry out this integral directly. However, in general, residues can be found by much easier methods. It is because of these methods, residues are so useful.

Laurent Series

If it is easy to write down the Laurent series for $f(z)$ about $z = z_0$ that is valid in the immediate neighborhood of z_0, then the residue is just the coefficient a_{-1} of the term $1/(z-z_0)$. For example

$$f(z) = \frac{3}{z-2}$$

is already in the form of a Laurent series about $z = 2$ with $a_{-1} = 3$ and $a_n = 0$ for $n \neq -1$. Therefore the residue at 2 is simply 3.

It is also easy to find the residue of $\exp(1/z^2)$ at $z = 0$, since

$$\mathrm{e}^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^6} + \cdots .$$

There is no $1/z$ term, therefore the residue is equal to zero.

Simple Pole

Suppose that $f(z)$ has a simple, or first-order, pole at $z = z_0$, so we can write

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots .$$

If we multiply this identity by $(z-z_0)$, we get

$$(z-z_0)f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots .$$

Now if we let z approach z_0, we obtain for the residue

$$a_{-1} = \lim_{z\to a}(z - z_0)f(z).$$

For example, if

$$f(z) = \frac{4-3z}{z(z-1)(z-2)},$$

the residue at $z = 0$ is

$$\mathrm{Res}_{z=0}\left[f\left(z\right)\right] = \lim_{z\to 0} z\frac{4-3z}{z(z-1)(z-2)} = \frac{4}{\left(-1\right)\left(-2\right)} = 2,$$

the residue at $z = 1$ is

$$\mathrm{Res}_{z=1}\left[f\left(z\right)\right] = \lim_{z\to 1}(z-1)\frac{4-3z}{z(z-1)(z-2)} = \frac{4-3}{1\left(-1\right)} = -1,$$

and the residue at $z = 2$ is

$$\mathrm{Res}_{z=2}\left[f\left(z\right)\right] = \lim_{z\to 2}(z-2)\frac{4-3z}{z(z-1)(z-2)} = \frac{4-6}{2\left(1\right)} = -1.$$

These results can also be understood in terms of partial fractions. It can be readily verified that

$$f(z) = \frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z} + \frac{-1}{z-1} + \frac{-1}{z-2}.$$

In the region $|z| < 1$, both $\frac{-1}{z-1}$ and $\frac{-1}{z-2}$ are analytic. Therefore they can be expressed in terms of Taylor series, which has no negative power terms. Thus, the Laurent series of $f(z)$ about $z = 0$ in the region $0 < |z| < 1$ is of the form

$$f(z) = \frac{2}{z} + a_0 + a_1 z + a_2 z^2 + \cdots .$$

It is seen that a_{-1} comes solely from the first term. Therefore the residue at $z = 0$ must equal to 2.

Similarly, the Laurent series of $f(z)$ about $z = 1$ in the region $0 < |z-1| < 1$ is of the form

$$f(z) = \frac{-1}{z-1} + a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots .$$

Hence a_{-1} is equal to -1. For the same reason, the residue at $z = 2$ comes from the term $\frac{-1}{z-2}$, and is clearly equal to -1.

Multiple-Order Pole

If $f(z)$ has a third-order pole at $z = z_0$, then

$$f(z) = \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots .$$

To obtain the residue a_{-1}, we must multiply this identity by $(z-a)^3$

$$(z-z_0)^3 f(z) = a_{-3} + a_{-2}(z-z_0) + a_{-1}(z-z_0)^2 + a_0(z-z_0)^3 + \cdots$$

and differentiate twice with respect to z

$$\frac{\mathrm{d}}{\mathrm{d}z}[(z-z_0)^3 f(z)] = a_{-2} + 2a_{-1}(z-z_0) + 3a_0(z-z_0)^2 + \cdots ,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}[(z-z_0)^3 f(z)] = 2a_{-1} + 3 \cdot 2a_0\,(z-z_0) + \cdots .$$

Next we let z approach z_0

$$\lim_{z\to z_0} \frac{\mathrm{d}^2}{\mathrm{d}z^2}[(z-z_0)^3 f(z)] = 2a_{-1},$$

and finally divide it by 2,

$$\frac{1}{2}\lim_{z\to z_0} \frac{\mathrm{d}^2}{\mathrm{d}z^2}[(z-z_0)^3 f(z)] = a_{-1}.$$

Thus, if $f(z)$ has a pole of order m at $z = z_0$, then the residue of $f(z)$ at $z = z_0$ is

$$\mathrm{Res}_{z=z_0}\,[f\,(z)] = \frac{1}{(m-1)!}\lim_{z\to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}[(z-a)^m f(z)].$$

For example,

$$f\,(z) = \frac{1}{z\,(z-2)^4}$$

clearly has a fourth order pole at $z = 2$. Thus

$$\mathrm{Res}_{z=2}\,[f\,(z)] = \frac{1}{3!}\lim_{z\to 2}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\left[(z-2)^4\frac{1}{z\,(z-2)^4}\right]$$
$$= \frac{1}{6}\lim_{z\to 2}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\frac{1}{z} = \frac{1}{6}\lim_{z\to 2}\frac{(-1)\,(-2)\,(-3)}{z^4} = -\frac{1}{16}.$$

To check this result, we can expand $f\,(z)$ in a Laurent series about $z = 2$ in the region $0 < |z-2| < 2$. For this purpose, let us write $f\,(z)$ as

$$f\,(z) = \frac{1}{z\,(z-2)^4} = \frac{1}{(z-2)^4}\frac{1}{[2+(z-2)]} = \frac{1}{2\,(z-2)^4}\frac{1}{\left(1+\frac{z-2}{2}\right)}.$$

Since $\left|\frac{z-2}{2}\right| < 1$, so we have

$$f(z) = \frac{1}{2(z-2)^4}\left[1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\frac{1}{(z-2)^4} - \frac{1}{4}\frac{1}{(z-2)^3} + \frac{1}{8}\frac{1}{(z-2)^2} - \frac{1}{16}\frac{1}{(z-2)} + \frac{1}{32} - \cdots .$$

It is seen that the coefficient of the $(z-2)^{-1}$ is indeed $-1/16$.

Derivative of the Denominator

If $p(z)$ and $q(z)$ are analytic functions, and $q(z)$ has a simple zero at z_0 and $p(z_0) \neq 0$, then

$$f(z) = \frac{p(z)}{q(z)}$$

has a simple pole at z_0. As $q(z)$ is analytic, so it can be expressed as a Taylor series about z_0

$$q(z) = q(z_0) + q'(z_0)(z-z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \cdots .$$

But it has a zero at z_0, so $q(z_0) = 0$, and

$$q(z) = q'(z_0)(z-z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \cdots .$$

Since $f(z)$ has a simple pole at z_0, its residue at z_0 is

$$\begin{aligned}\operatorname{Res}_{z=z_0}[f(z)] &= \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)} \\ &= \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q'(z_0)(z-z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \cdots} \\ &= \frac{p(z_0)}{q'(z_0)}.\end{aligned}$$

This formula is very often the most efficient way of finding the residue. For example, the function

$$f(z) = \frac{z}{z^4+4}$$

has four simple poles, located at the zeros of the denominator

$$z^4 + 4 = 0.$$

The four roots of this equation are

$$\begin{aligned}
z_1 &= \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4} = 1 + \mathrm{i}, \\
z_2 &= \sqrt{2}\mathrm{e}^{\mathrm{i}(\pi/4+\pi/2)} = -1 + \mathrm{i}, \\
z_3 &= \sqrt{2}\mathrm{e}^{\mathrm{i}(\pi/4+\pi)} = -1 - \mathrm{i}, \\
z_4 &= \sqrt{2}\mathrm{e}^{\mathrm{i}(\pi/4+3\pi/2)} = 1 - \mathrm{i}.
\end{aligned}$$

The residues at $z_1,\ z_2,\ z_3$, and z_4 are

$$\begin{aligned}
\mathrm{Res}_{z=z_1}\left[f\left(z\right)\right] &= \lim_{z\to z_1} \frac{z}{\left(z^4+4\right)'} = \lim_{z\to z_1} \frac{z}{4z^3} = \lim_{z\to z_1} \frac{1}{4z^2} \\
&= \lim_{z\to(1+\mathrm{i})} \frac{1}{4z^2} = \frac{1}{4\left(1+\mathrm{i}\right)^2} = -\frac{1}{8}\mathrm{i}, \\
\mathrm{Res}_{z=z_2}\left[f\left(z\right)\right] &= \lim_{z\to(-1+\mathrm{i})} \frac{1}{4z^2} = \frac{1}{4\left(-1+\mathrm{i}\right)^2} = \frac{1}{8}\mathrm{i}, \\
\mathrm{Res}_{z=z_3}\left[f\left(z\right)\right] &= \lim_{z\to(-1-\mathrm{i})} \frac{1}{4z^2} = \frac{1}{4\left(-1-\mathrm{i}\right)^2} = -\frac{1}{8}\mathrm{i}, \\
\mathrm{Res}_{z=z_4}\left[f\left(z\right)\right] &= \lim_{z\to(1-\mathrm{i})} \frac{1}{4z^2} = \frac{1}{4\left(1-\mathrm{i}\right)^2} = \frac{1}{8}\mathrm{i}.
\end{aligned}$$

It can be readily verified that

$$\frac{z}{z^4+4} = \frac{-\mathrm{i}/8}{z-(1+\mathrm{i})} + \frac{\mathrm{i}/8}{z-(-1+\mathrm{i})} + \frac{-\mathrm{i}/8}{z-(-1-\mathrm{i})} + \frac{\mathrm{i}/8}{z-(1-\mathrm{i})}.$$

Therefore the calculated residues must be correct.

3.4.4 Cauchy's Residue Theorem

Consider a simple closed curve C containing in its interior a number of isolated singular points, $z_1,\ z_2,\ \ldots$, of a function $f(z)$. If around each singular point we draw a circle so small that it encloses no other singular points as shown in Fig. 3.6, so $f(z)$ is analytic in the region between C and these small circles. Then introducing cuts as in the proof of Laurent series, we find by the Cauchy theorem that the integral around C counterclockwise plus the integral around the small circles clockwise is zero, since the integrals along the cuts cancel. Thus

$$\frac{1}{2\pi\mathrm{i}}\oint_C f(z)\mathrm{d}z - \frac{1}{2\pi\mathrm{i}}\oint_{C_1} f(z)\mathrm{d}z + \cdots - \frac{1}{2\pi\mathrm{i}}\oint_{C_n} f(z)\mathrm{d}z = 0,$$

where all integrals are counterclockwise, the minus sign is to account for the clockwise direction of the small circles. It follows that:

$$\frac{1}{2\pi\mathrm{i}}\oint_C f(z)\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\oint_{C_1} f(z)\mathrm{d}z + \cdots + \frac{1}{2\pi\mathrm{i}}\oint_{C_n} f(z)\mathrm{d}z.$$

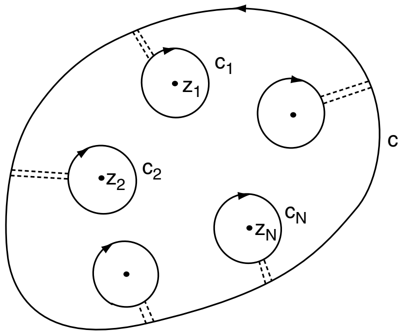

Fig. 3.6. The circles $C_1, C_2, \ldots, C_N$ enclosing, respectively, the singular points $z_1, z_2, \ldots, z_N$ within a simple closed curve

The integrals on the right are, by definition, just the residues of $f(z)$ at the various isolated singularities within C. Hence we have established the important residue theorem:

If there are n number of singular points of $f(z)$ inside the contour C, then

$$\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\left\{\mathrm{Res}_{z=z_1}[f(z)] + \mathrm{Res}_{z=z_2}[f(z)] + \cdots + \mathrm{Res}_{z=z_n}[f(z)]\right\}. \tag{3.21}$$

This theorem is known as Cauchy's residue theorem or just the residue theorem.

3.4.5 Second Residue Theorem

If the number of singular points inside the enclosed contour C is too large, or there are nonisolated singular points interior in C, it will be difficult to carry out the contour integration with the Cauchy's residue theorem. For such cases, there is another residue theorem that is more efficient.

Suppose $f(z)$ has many singular points in C and no singular point outside of C, as shown in Fig. 3.7.

If we want to evaluate the integral $\oint_C f(z)\,\mathrm{d}z$, we can first construct a circular contour C_R outside of C, centered at the origin with a radius R. Then by the principle of deformation of contours

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_R} f(z)\,\mathrm{d}z.$$

Now if we expand $f(z)$ in terms of Laurent series about $z = 0$ in the region $|z| > R$,

$$f(z) = \cdots + \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots,$$

the coefficient a_{-1} is given by the integral

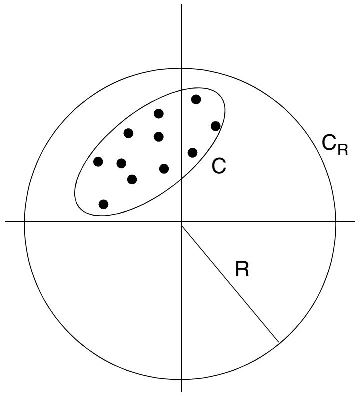

Fig. 3.7. If the number of singularities enclosed in C is too large, then it is more convenient to replace the contour C with a large circular contour C_R centered at the origin

$$a_{-1} = \frac{1}{2\pi \mathrm{i}} \oint_{C_R} f(z)\, \mathrm{d}z.$$

Note that a_{-1} in this equation is not the residue of $f(z)$ about $z = 0$, because the series is not valid in the immediate neigborhood of $z = 0$. However, if we change z to $1/z$, then

$$f\left(\frac{1}{z}\right) = \cdots + a_{-3}z^3 + a_{-2}z^2 + a_{-1}z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots ,$$

is convergent for $|z| < 1/R$. It is seen that a_{-1} is the residue at $z = 0$ of the function $\frac{1}{z^2} f\left(\frac{1}{z}\right)$, since

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \cdots + a_{-3}z + a_{-2} + \frac{a_{-1}}{z} + \frac{a_0}{z^2} + \frac{a_1}{z^3} + \cdots ,$$

is a Laurent series valid in the region $0 < |z| < \frac{1}{R}$. Hence we arrived at the following theorem.

If $f(z)$ is analytic everywhere except for a number of singular points interior to a positive oriented closed contour C, then

$$\oint_C f(z)\, \mathrm{d}z = 2\pi \mathrm{i}\, \mathrm{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right].$$

Example 3.4.1. Evaluate the integral $\oint_C f(z)\, \mathrm{d}z$ for

$$f(z) = \frac{5z - 2}{z(z-1)},$$

where C is along the circle $|z| = 2$ in the counterclockwise direction. (a) Use the Cauchy residue theorem. (b) Use the second residue theorem.

Solution 3.4.1. (a) The function has two single poles at $z = 0,\ z = 1$. Both lie within the circle $|z| = 2$. So

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left\{\mathrm{Res}_{z=0}[f(z)] + \mathrm{Res}_{z=1}[f(z)]\right\}.$$

Since

$$\mathrm{Res}_{z=0}[f(z)] = \lim_{z\to 0} z\frac{5z-2}{z(z-1)} = \frac{-2}{-1} = 2,$$
$$\mathrm{Res}_{z=1}[f(z)] = \lim_{z\to 1}(z-1)\frac{5z-2}{z(z-1)} = \frac{3}{1} = 3,$$

thus

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,(2+3) = 10\pi\mathrm{i}.$$

If C is in the clockwise direction, the answer would be $-10\pi\mathrm{i}$.

(b) According to the second residue theorem

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right].$$

Now

$$f\left(\frac{1}{z}\right) = \frac{5/z-2}{1/z(1/z-1)} = \frac{(5-2z)z}{1-z},$$

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{5-2z}{z\,(1-z)},$$

which has a simple pole at $z = 0$. Thus

$$\mathrm{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right] = \lim_{z\to 0} z\frac{5-2z}{z\,(1-z)} = \frac{5}{1} = 5.$$

Therefore

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\cdot 5 = 10\pi\mathrm{i}.$$

Not surprisingly, this is the same result obtained in (a).

Example 3.4.2. Find the value of the integral

$$\oint_C \frac{\mathrm{d}z}{z^3\,(z+4)}$$

taken counterclockwise around the circle (a) $|z| = 2$, (b) $|z+2| = 3$.

Solution 3.4.2. (a) The function has a third-order pole at $z = 0$ and a simple pole at $z = -4$. Only $z = 0$ is inside the circle $|z| = 2$. Therefore

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i}\, \mathrm{Res}_{z=0}[f(z)].$$

For the third-order pole,

$$\mathrm{Res}_{z=0}[f(z)] = \frac{1}{2!} \lim_{z\to 0} \frac{\mathrm{d}^2}{\mathrm{d}z^2} z^3 \frac{1}{z^3 (z+4)} = \frac{1}{2} \lim_{z\to 0} \frac{2}{(z+4)^3} = \frac{1}{64}.$$

Therefore

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i} \frac{1}{64} = \frac{\pi}{32} \mathrm{i}.$$

(b) For the circle $|z+2| = 3$, the center is at $z = -2$ and the radius is 3. Both singular points are inside the circle. Thus

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i} \left\{ \mathrm{Res}_{z=0}[f(z)] + \mathrm{Res}_{z=-4}[f(z)] \right\}.$$

Since

$$\mathrm{Res}_{z=-4}[f(z)] = \lim_{z\to -4} (z+4) \frac{1}{z^3 (z+4)} = \frac{1}{(-4)^3} = -\frac{1}{64},$$

so

$$\oint_C \frac{\mathrm{d}z}{z^3 (z+4)} = 2\pi \mathrm{i} \left\{ \frac{1}{64} - \frac{1}{64} \right\} = 0.$$

Example 3.4.3. Find the value of the integral

$$\oint_C \tan \pi z \mathrm{d}z$$

taken counterclockwise around the unit circle $|z| = 1$.

Solution 3.4.3. Since

$$f(z) = \tan \pi z = \frac{\sin \pi z}{\cos \pi z},$$

and

$$\cos \frac{2n+1}{2} \pi = 0, \quad \text{for} \quad n = \ldots, -2, -1, 0, 1, 2, \ldots,$$

therefore $z = (2n+1)/2$ are zeros of $\cos \pi z$. Expanding $\cos \pi z$ about any of these zeros in Taylor series, one can readily see that $f(z)$ has a simple pole at each of these singular points. Among them, $z = 1/2$ and $z = -\,1/2$ are inside $|z| = 1$. Hence

$$\oint_C \tan \pi z \mathrm{d}z = 2\pi\mathrm{i}\left\{\mathrm{Res}_{z=1/2}[f(z)] + \mathrm{Res}_{z=-1/2}[f(z)]\right\}.$$

The simplest way to find these residues is by the "derivative of the denominator" method,

$$\mathrm{Res}_{z=1/2}[f(z)] = \left[\frac{\sin \pi z}{(\cos \pi z)'}\right]_{z=\frac{1}{2}} = \left[\frac{\sin \pi z}{-\pi \sin \pi z}\right]_{z=\frac{1}{2}} = -\frac{1}{\pi},$$

$$\mathrm{Res}_{z=-1/2}[f(z)] = \left[\frac{\sin \pi z}{(\cos \pi z)'}\right]_{z=-\frac{1}{2}} = \left[\frac{\sin \pi z}{-\pi \sin \pi z}\right]_{z=-\frac{1}{2}} = -\frac{1}{\pi}.$$

Therefore

$$\oint_C \tan \pi z \mathrm{d}z = 2\pi\mathrm{i}\left\{-\frac{1}{\pi} - \frac{1}{\pi}\right\} = -4\mathrm{i}.$$

Example 3.4.4. Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ for

$$f(z) = z^2 \exp\left(\frac{1}{z}\right),$$

where C is counterclockwise around the unit circle $|z| = 1$.

Solution 3.4.4. The function $f(z)$ has an essential singularity at $z = 0$. Thus

$$\oint_C z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}[f(z)].$$

The residue is simply the coefficient of the z^{-1} term in the Laurent series about $z = 0$,

$$z^2 \exp\left(\frac{1}{z}\right) = z^2\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots\right)$$
$$= z^2 + z + \frac{1}{2} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots.$$

Therefore

$$\mathrm{Res}_{z=0}[f(z)] = \frac{1}{3!} = \frac{1}{6}.$$

Hence

$$\oint_C z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi\mathrm{i}\frac{1}{6} = \frac{\pi}{3}\mathrm{i}.$$

Example 3.4.5. Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ for

$$f(z) = \frac{z^{99}\exp\left(\frac{1}{z}\right)}{z^{100}+1},$$

where C is counterclockwise around the circle $|z| = 2$.

Solution 3.4.5. There are 100 singular points located on the circumference of the unit circle $|z| = 1$ and an essential singular point at $z = 0$. Obviously the second residue theorem is more convenient. That is,

$$\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right].$$

Now

$$f\left(\frac{1}{z}\right) = \frac{(1/z)^{99}\exp(z)}{(1/z)^{100}+1} = \frac{z\exp(z)}{1+z^{100}},$$

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{\exp(z)}{z(1+z^{100})}.$$

So

$$\mathrm{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right] = \lim_{z\to 0} z\frac{\exp(z)}{z(1+z^{100})} = 1.$$

Therefore

$$\oint_C \frac{z^{99}\exp\left(\frac{1}{z}\right)}{z^{100}+1}\,\mathrm{d}z = 2\pi\mathrm{i}.$$

Example 3.4.6. (a) Show that if $z = 1$ and $z = 2$ are inside the closed contour C, then

$$\oint_C \frac{1}{(z-1)(z-2)}\,\mathrm{d}z = 0.$$

(b) Show that if all the singular points $s_1, s_2, \ldots, s_n$ of the following function

$$f(z) = \frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)}$$

are inside the closed contour C, then

$$I = \oint_C f(z)\,\mathrm{d}z = 0.$$

Solution 3.4.6. (a) Taking partial fraction, we have

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}.$$

So

$$\oint_C \frac{1}{(z-1)(z-2)}\,\mathrm{d}z = \oint_C \frac{A}{(z-1)}\,\mathrm{d}z + \oint_C \frac{B}{(z-2)}\,\mathrm{d}z$$
$$= 2\pi\mathrm{i}\,(A+B)\,.$$

Since

$$\frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)} = \frac{(A+B)z-(2A+B)}{(z-1)(z-2)}$$

and

$$\frac{(A+B)z-(2A+B)}{(z-1)(z-2)} = \frac{1}{(z-1)(z-2)},$$

it follows that:

$$A+B=0.$$

Therefore

$$\oint_C \frac{1}{(z-1)(z-2)}\,\mathrm{d}z = 0.$$

(b) The partial fraction of $f(z)$ is of the form

$$\frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)} = \frac{r_1}{(z-s_1)} + \frac{r_2}{(z-s_2)} + \cdots + \frac{r_n}{(z-s_n)}.$$

Therefore

$$\oint_C f(z)\,\mathrm{d}z = \oint_C \frac{r_1}{(z-s_1)}\,\mathrm{d}z + \oint_C \frac{r_2}{(z-s_2)}\,\mathrm{d}z + \cdots + \oint_C \frac{r_n}{(z-s_n)}\,\mathrm{d}z$$
$$= 2\pi\mathrm{i}\,(r_1+r_2+\cdots+r_n)\,.$$

Now

$$\frac{r_1}{(z-s_1)} + \frac{r_2}{(z-s_2)} + \cdots + \frac{r_n}{(z-s_n)} = \frac{(r_1+r_2+\cdots+r_n)\,z^{n-1}+\cdots}{(z-s_1)(z-s_2)\cdots(z-s_n)},$$

and

$$\frac{(r_1+r_2+\cdots+r_n)\,z^{n-1}+\cdots}{(z-s_1)(z-s_2)\cdots(z-s_n)} = \frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)}.$$

Since the numerator of the right-hand side has no z^{n-1} term, therefore

$$(r_1+r_2+\cdots+r_n)=0,$$

and

$$\oint_C \frac{1}{(z-s_1)(z-s_2)\cdots(z-s_n)}\mathrm{d}z = 0.$$

3.5 Evaluation of Real Integrals with Residues

A very surprising fact is that we can use the residue theorem to evaluate integrals of real variable. For certain classes of complicated real integrals, residue theorem offers a simple and elegant way of carrying out the integration.

3.5.1 Integrals of Trigonometric Functions

Let us consider the integral of the type

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\mathrm{d}\theta.$$

If we make the substitution

$$z = \mathrm{e}^{\mathrm{i}\theta}, \quad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{i}z,$$

then

$$\cos\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

and

$$\mathrm{d}\theta = \frac{1}{\mathrm{i}z}\,\mathrm{d}z.$$

The given integral takes the form

$$I = \oint_C f(z)\frac{\mathrm{d}z}{\mathrm{i}z},$$

the integration being taken counterclockwise around the unit circle centered at $z = 0$.

We illustrate this method with following examples.

Example 3.5.1. Show that

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{2} - \cos\theta} = 2\pi.$$

Solution 3.5.1. With the transformation just discussed we can write the integral as

$$I = \oint_C \frac{\mathrm{d}z}{\left[\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)\right] \mathrm{i}z} = \oint_C \frac{-2\,\mathrm{d}z}{\mathrm{i}(z^2 - 2\sqrt{2}z + 1)}$$
$$= -\frac{2}{\mathrm{i}} \oint_C \frac{\mathrm{d}z}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

The integrand has two simple poles. The one at $\sqrt{2} + 1$ lies outside the unit circle and is thus of no interest. The one at $\sqrt{2} - 1$ is inside the unit circle, and the residue at that point is

$$\mathrm{Res}_{z=\sqrt{2}-1}\left[f(z)\right] = \lim_{z \to \sqrt{2}-1} (z - \sqrt{2} + 1) \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = -\frac{1}{2}.$$

Thus

$$I = -\frac{2}{\mathrm{i}} 2\pi\mathrm{i} \left(-\frac{1}{2}\right) = 2\pi.$$

Example 3.5.2. Evaluate the integral

$$I = \int_0^{\pi} \frac{\mathrm{d}\theta}{a - b\cos\theta}, \qquad a > b > 0.$$

Solution 3.5.2. Since the integrand is symmetric about $\theta = \pi$, so we can extend the integration interval to $[0, 2\pi]$,

$$I = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}\theta}{a - b\cos\theta},$$

which can be written as an integral around an unit circle in the complex plane

$$I = \frac{1}{2} \oint \frac{1}{a - b\frac{1}{2}(z + \frac{1}{z})} \frac{\mathrm{d}z}{\mathrm{i}z} = \oint \frac{1}{2az - bz^2 - b} \frac{\mathrm{d}z}{\mathrm{i}}.$$

Now

$$\oint \frac{1}{2az - bz^2 - b} \frac{\mathrm{d}z}{\mathrm{i}} = -\frac{1}{b\mathrm{i}} \oint \frac{1}{z^2 - \frac{2a}{b}z + 1}\,\mathrm{d}z,$$

taking this seemingly trivial step of making the coefficient of z^2 to be 1 can actually avoid many pitfalls of what follows. The singular points of the integrand are at the zeros of the denominator,

$$z^2 - \frac{2a}{b}z + 1 = 0.$$

Let z_1 and z_2 be the roots of this equation. They are easily found to be

$$z_1 = \frac{1}{b}\left(a - \sqrt{a^2 - b^2}\right), \qquad z_2 = \frac{1}{b}\left(a + \sqrt{a^2 - b^2}\right).$$

Since

$$(z - z_1)(z - z_2) = z^2 - (z_1 + z_2) + z_1 z_2 = z^2 - \frac{2a}{b}z + 1,$$

it follows that:

$$z_1 z_2 = 1.$$

This means that one root must be greater than 1, and the other less than 1. Furthermore, $z_1 < z_2$, z_1 must be less than 1 and z_2 greater than 1. Therefore only z_1 is inside the unit circle. Thus

$$\begin{aligned}\oint \frac{\mathrm{d}z}{(z - z_1)(z - z_2)} &= 2\pi\mathrm{i}\,\mathrm{Res}_{z=z_1}[f(z)] \\ &= 2\pi\mathrm{i} \lim_{z\to z_1} (z - z_1)\frac{1}{(z - z_1)(z - z_2)} = 2\pi\mathrm{i}\frac{1}{(z_1 - z_2)},\end{aligned}$$

and

$$\frac{1}{(z_1 - z_2)} = -\frac{b}{2\sqrt{a^2 - b^2}}.$$

Therefore

$$I = -\frac{1}{b\mathrm{i}}2\pi\mathrm{i}\left(-\frac{b}{2\sqrt{a^2 - b^2}}\right) = \frac{\pi}{\sqrt{a^2 - b^2}}.$$

Example 3.5.3. Show that

$$\int_0^{2\pi} \cos^{2n}\theta\mathrm{d}\theta = \frac{2\pi\,(2n)!}{2^{2n}\,(n!)^2}.$$

Solution 3.5.3. The integral can be written as

$$I = \int_0^{2\pi} \cos^{2n}\theta\mathrm{d}\theta = \oint_C \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^{2n} \frac{\mathrm{d}z}{\mathrm{i}z} = \frac{1}{2^{2n}\mathrm{i}} \oint_C \left[\sum_{k=0}^{2n} C_k^{2n} z^{2n-k}\frac{1}{z^k}\right]\frac{\mathrm{d}z}{z},$$

where C_k^{2n} are the binomial coefficients

$$C_k^{2n} = \frac{(2n)!}{k!(2n-k)!}.$$

Carrying out the integration term by term, the only nonvanishing term is the term of z^{-1}. Since

$$\left[\sum_{k=0}^{2n} C_k^{2n} z^{2n-k} \frac{1}{z^k}\right] \frac{1}{z} = \left[\sum_{k=0}^{2n} C_k^{2n} z^{2n-2k}\right] \frac{1}{z},$$

it is clear that the coefficient of z^{-1} is given by term with $k = n$. Thus

$$\begin{aligned} I &= \frac{1}{2^{2n}\mathrm{i}} \oint_C \left[z^{2n-1} + 2nz^{2n-3} + \cdots + \frac{C_n^{2n}}{z} + \cdots + \frac{1}{z^{2n+1}}\right] \mathrm{d}z \\ &= \frac{1}{2^{2n}\mathrm{i}} 2\pi\mathrm{i}\, C_n^{2n} = \frac{2\pi\,(2n)!}{2^{2n}(n!)^2}. \end{aligned}$$

3.5.2 Improper Integrals I: Closing the Contour with a Semicircle at Infinity

We consider the real integrals of the type

$$I = \int_{-\infty}^{\infty} f(x)\mathrm{d}x.$$

Such an integral, for which the interval of integration is not finite, is called improper integral, and it has the meaning

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}x.$$

Under certain conditions, this type of integral can be evaluated with the residue theorem. The idea is to close the contour by adding additional pieces along which the integral is either zero or some multiple of the original integral along the real axis.

If $f(x)$ is a rational function (i.e., ratio of two polynomials) with no singularity on the real axis and

$$\lim_{z\to\infty} zf(z) = 0,$$

then it can be shown that the integral along the real axis from $-\infty$ to ∞ is equal to the integral around a closed contour which consists of (a) the straight line along the real axis and (b) the semicircle C_R at infinity as shown in Fig. 3.8.

This is so, because with

$$z = R\mathrm{e}^{\mathrm{i}\theta}, \quad \mathrm{d}z = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}z\mathrm{d}\theta,$$

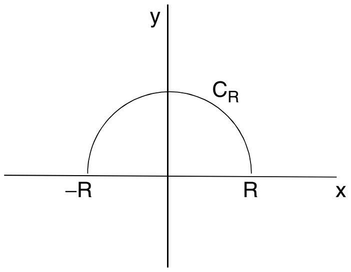

Fig. 3.8. As $R \to \infty$, the semicircle C_R is at infinity. The contour consists of the real axis and C_R encloses the entire upper half-plane

$$\left| \int_{C_R} f(z)\,\mathrm{d}z \right| = \left| \int_0^{\pi} f(z)\,\mathrm{i}z\mathrm{d}\theta \right| \leq \mathrm{Max} \left| f(z)\, z \right| \pi,$$

which goes to zero as $R \to \infty$, since $\lim_{z\to\infty} z f(z) = 0$. Therefore

$$\lim_{R\to\infty} \int_{C_R} f(z)\,\mathrm{d}z = 0.$$

It follows that:

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \left[\int_{-R}^{R} f(x)\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z \right] = \oint_{\mathrm{u.h.p}} f(z)\,\mathrm{d}z,$$

where u.h.p means the entire upper half-plane. As $R \to \infty$, all the poles of $f(z)$ in the upper half-plane will be inside the contour. Hence

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi\mathrm{i} \left(\text{sum of residues of } f(z) \text{ in the upper half-plane} \right).$$

By the same token, we can, of course, close the contour in the lower half-plane. However, in that case, the direction of integration will be clockwise. Therefore

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \oint_{\mathrm{l.h.p}} f(z)\,\mathrm{d}z \\ &= -2\pi\mathrm{i} \left(\text{sum of residues of } f(z) \text{ in the lower half-plane} \right). \end{aligned}$$

Example 3.5.4. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$$

Solution 3.5.4. Since

$$\lim_{z\to\infty} z\frac{1}{1+z^2} = 0,$$

we can evaluate this integral with contour integration. That is

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \oint_{\text{u.h.p}} \frac{1}{1+z^2}\,\mathrm{d}z$$

The singular points of

$$f(z) = \frac{1}{1+z^2}$$

are at $z = \mathrm{i}$ and $z = -\mathrm{i}$. Only $z = \mathrm{i}$ is in the upper half-plan. Therefore

$$I = 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}}[f(z)] = 2\pi\mathrm{i}\lim_{z\to\mathrm{i}}(z-\mathrm{i})\frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \pi.$$

Now, if we close the contour in the lower half-plane, then

$$I = -2\pi\mathrm{i}\,\mathrm{Res}_{z=-\mathrm{i}}[f(z)] = -2\pi\mathrm{i}\lim_{z\to-\mathrm{i}}(z+\mathrm{i})\frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \pi,$$

which is, of course, the same result.

Example 3.5.5. Show that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}.$$

Solution 3.5.5. The four singular points of the function

$$f(z) = \frac{1}{z^4+1}$$

are $\mathrm{e}^{\mathrm{i}\pi/4},\ \mathrm{e}^{\mathrm{i}3\pi/4},\ \mathrm{e}^{\mathrm{i}5\pi/4},\ \mathrm{e}^{\mathrm{i}7\pi/4}$. Only two, $\mathrm{e}^{\mathrm{i}\pi/4},\ \mathrm{e}^{\mathrm{i}3\pi/4}$ are in the upper half-plane. Therefore

$$\oint_{\text{u.h.p.}} \frac{1}{z^4+1}\,\mathrm{d}z = 2\pi\mathrm{i}\{\mathrm{Res}_{z=\exp(\mathrm{i}\pi/4)}[f(z)] + \mathrm{Res}_{z=\exp(\mathrm{i}3\pi/4)}[f(z)]\}.$$

For problems of this type, it is much easier to find the residue by the method of $p(a)/q'(a)$. If we use the method of $\lim_{z\to a}(z-a)f(z)$, the calculation will be much more cumbersome. Since

$$\mathrm{Res}_{z=\exp(\mathrm{i}\pi/4)}[f(z)] = \left[\frac{1}{(z^4+1)'}\right]_{z=\mathrm{e}^{\mathrm{i}\pi/4}} = \left[\frac{1}{4z^3}\right]_{z=\exp(\mathrm{i}\pi/4)} = \frac{1}{4}\mathrm{e}^{-\mathrm{i}3\pi/4},$$

$$\mathrm{Res}_{z=\exp(\mathrm{i}3\pi/4)}[f(z)] = \left[\frac{1}{4z^3}\right]_{z=\exp(\mathrm{i}3\pi/4)} = \frac{1}{4}\mathrm{e}^{-\mathrm{i}9\pi/4} = \frac{1}{4}\mathrm{e}^{-\mathrm{i}\pi/4},$$

so

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,\mathrm{d}x = 2\pi\mathrm{i}\left[\frac{1}{4}\mathrm{e}^{-\mathrm{i}3\pi/4} + \frac{1}{4}\mathrm{e}^{-\mathrm{i}\pi/4}\right]$$
$$= \frac{\pi}{2}\left[\mathrm{e}^{-\mathrm{i}\pi/4} + \mathrm{e}^{\mathrm{i}\pi/4}\right] = \pi\cos\left(\frac{\pi}{4}\right)$$

3.5.3 Fourier Integral and Jordan's Lemma

Another very important class of integrals of the form

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\mathrm{d}x$$

can also be evaluated with the residue theorem. This class is known as the Fourier integral of $f(x)$. We will show that as long as $f(x)$ has no singularity along the real axis and

$$\lim_{z\to\infty} f(z) = 0, \tag{3.22}$$

the contour of this integral can be closed with an infinitely large semicircle in the upper half-plane if k is positive, and in the lower half-plane if k is negative. This statement is based on the Jordan's lemma, which states that, under the condition (3.22),

$$\lim_{R\to\infty} \int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z = 0,$$

where k is a positive real number and C_R is the semicircle in the upper half-plane with infinitely large radius R.

To prove this lemma, we first make the following observation. In Fig. 3.9, $y = \sin\theta$ and $y = \frac{2}{\pi}\theta$ are shown together. It is seen that in the interval $[0, \frac{\pi}{2}]$, the curve $y = \sin\theta$ is concave and always lies on or above the straight line $y = \frac{2}{\pi}\theta$. Therefore

$$\sin\theta \geq \frac{2}{\pi}\theta \quad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

With

$$z = R\mathrm{e}^{\mathrm{i}\theta} = R\cos\theta + \mathrm{i}R\sin\theta, \qquad \mathrm{d}z = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta,$$

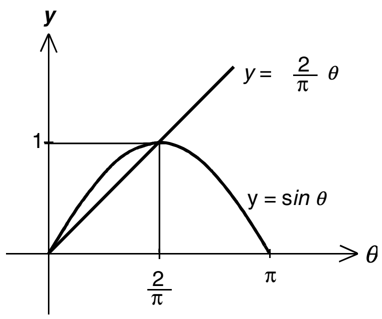

Fig. 3.9. Visualization of the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$

we have

$$\left|\int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z\right| = \left|\int_0^\pi \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta\right| \leq \int_0^\pi \left|\mathrm{e}^{\mathrm{i}kz}\right| |f(z)|\, R \left|\mathrm{e}^{\mathrm{i}\theta}\right| \mathrm{d}\theta.$$

Since

$$\left|\mathrm{e}^{\mathrm{i}kz}\right| = \left|\mathrm{e}^{\mathrm{i}k(R\cos\theta + \mathrm{i}R\sin\theta)}\right| = \left|\mathrm{e}^{\mathrm{i}kR\cos\theta}\right| \left|\mathrm{e}^{-kR\sin\theta}\right| = \mathrm{e}^{-kR\sin\theta},$$

so

$$\left|\int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z\right| \leq \mathrm{Max}\,|f(z)|\, R \int_0^\pi \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta.$$

Using $\sin(\pi - \theta) = \sin\theta$, we can write the last integral as

$$\int_0^\pi \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta = 2\int_0^{\pi/2} \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta.$$

Now, $\sin\theta \geq \frac{2}{\pi}\theta$ in the interval $[0, \pi/2]$, therefore

$$2\int_0^{\pi/2} \mathrm{e}^{-kR\sin\theta}\mathrm{d}\theta \leq 2\int_0^{\pi/2} \mathrm{e}^{-kR2\theta/\pi}\mathrm{d}\theta = \frac{\pi}{kR}\left(1 - \mathrm{e}^{-kR}\right). \tag{3.23}$$

Thus

$$\left|\int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z\right| \leq \mathrm{Max}\,|f(z)|\, \frac{\pi}{k}\left(1 - \mathrm{e}^{-kR}\right).$$

As $z \to \infty$, $R \to \infty$ and the right-hand side of the last equation goes to zero, since $\lim_{z\to\infty} f(z) = 0$. It follows that:

$$\lim_{z\to\infty} \int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z = 0,$$

and Jordan's lemma is proved. By virtue of this lemma, the Fourier integral can be written as

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\left(\int_{-R}^{R} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x + \int_{C_R} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z\right)$$

$$= \oint_{\text{u.h.p}} \mathrm{e}^{\mathrm{i}kz} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{i=1}^{\text{all}} R_{\text{u.h.p}}\left[\mathrm{e}^{\mathrm{i}kz} f(z)\right],$$

where $\sum_{i=1}^{\text{all}} R_{\text{u.h.p}}\left[\mathrm{e}^{\mathrm{i}kz} f(z)\right]$ means the sum of all residues of $\mathrm{e}^{\mathrm{i}kz} f(z)$ in the upper half-plane.

Note that if k is negative, we cannot close the contour in the upper half-plane, since in (3.23) the factor e^{-kR} will blow up. However, in this case we can close the contour in the lower half-plane, because integrating from $\theta = 0$ to $\theta = -\pi$ will introduce another minus sign to make the large semicircular integral in the lower half-plane vanish. Therefore

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}|k|x} f(x)\,\mathrm{d}x = -2\pi\mathrm{i}\sum_{i=1}^{\text{all}} R_{\text{l.h.p}}\left[\mathrm{e}^{-\mathrm{i}|k|z} f(z)\right],$$

where $\sum_{i=1}^{\text{all}} R_{\text{l.h.p}}\left[\mathrm{e}^{-\mathrm{i}|k|z} f(z)\right]$ means the sum of all residues of $\mathrm{e}^{-\mathrm{i}|k|z} f(z)$ in the lower half-plane. The minus sign is due to the fact that in this case the closed contour integration is clockwise.

Since $\sin kx$ and $\cos kx$ are linear combinations of $\mathrm{e}^{\mathrm{i}kx}$ and $\mathrm{e}^{-\mathrm{i}kx}$, the real integrals of the form

$$\int_{-\infty}^{\infty} \cos kx f(x)\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \sin kx f(x)\,\mathrm{d}x$$

can be obtained easily from this class of integrals,

$$\int_{-\infty}^{\infty} \cos kx f(x)\,\mathrm{d}x = \frac{1}{2}\left[\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x + \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x\right], \quad (3.24)$$

$$\int_{-\infty}^{\infty} \sin kx f(x)\,\mathrm{d}x = \frac{1}{2\mathrm{i}}\left[\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x - \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x\right]. \quad (3.25)$$

If it is certain that the result of the integration is a finite real value, then we may write

$$\int_{-\infty}^{\infty} \cos kx f(x)\,\mathrm{d}x = \mathrm{Re}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\,\mathrm{d}x, \quad (3.26)$$

$$\int_{-\infty}^{\infty} \sin kx f(x)\,\mathrm{d}x = \mathrm{Im}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} f(x)\mathrm{d}x. \quad (3.27)$$

These formulae must be used with caution. While (3.24) and (3.25) are always valid, (3.26) and (3.27) are valid only if there is no imaginary number in $f(x)$.

Example 3.5.6. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x + \mathrm{i}}\, \mathrm{d}x.$$

Solution 3.5.6. There is a simple pole located in the lower half-plane, and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x + \mathrm{i}}\, \mathrm{d}x = \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x + \mathrm{i}}\, \mathrm{d}x - \frac{1}{2\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}x}}{x + \mathrm{i}}\, \mathrm{d}x.$$

To evaluate the first integral in the right-hand side, we must close the contour in the upper half-plane as shown in Fig. 3.10a.

Since the function is analytic everywhere in the upper half-plane, therefore

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x + \mathrm{i}}\, \mathrm{d}x = \oint_{\mathrm{u.h.p}} \frac{\mathrm{e}^{\mathrm{i}z}}{z + \mathrm{i}}\, \mathrm{d}z = 0.$$

To evaluate the second integral, we must close the contour in the lower half-plane as shown in Fig. 3.10b. Since there is a simple pole located at $z = -\mathrm{i}$, we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}x}}{x + \mathrm{i}}\, \mathrm{d}x = \oint_{\mathrm{l.h.p}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z + \mathrm{i}}\, \mathrm{d}z = -2\pi\mathrm{i}\, \mathrm{Res}_{z=-\mathrm{i}} \left[\frac{\mathrm{e}^{-\mathrm{i}z}}{z + \mathrm{i}} \right]$$
$$= -2\pi\mathrm{i} \lim_{z \to -\mathrm{i}} (z + \mathrm{i}) \frac{\mathrm{e}^{-\mathrm{i}z}}{z + \mathrm{i}} = -2\pi\mathrm{i}\mathrm{e}^{-1}.$$

Thus

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x + \mathrm{i}}\, \mathrm{d}x = \frac{1}{2\mathrm{i}} \left[0 - \left(-2\pi\mathrm{i}\mathrm{e}^{-1} \right) \right] = \frac{\pi}{e}.$$

Clearly

$$I \neq \mathrm{Im} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x + \mathrm{i}}\, \mathrm{d}x,$$

this is because there is the imaginary number i in the function.

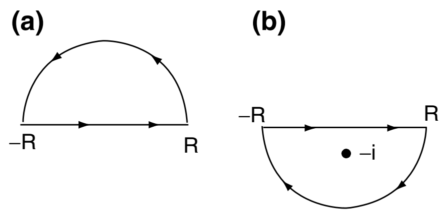

Fig. 3.10. Closing the contour with an infinitely large semicircle. (**a**) contour closed in the upper half-plane, (**b**) contour closed in the lower half-plane

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Example 3.5.7. Evaluate the integral

$$I=\int_{-\infty}^{\infty}\frac{1}{x^2+4}\,\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}x, \qquad \omega>0.$$

Solution 3.5.7.

$$I=\oint_{\mathrm{l.h.p}}\frac{1}{z^2+4}\,\mathrm{e}^{-\mathrm{i}\omega z}\mathrm{d}z.$$

The only singular point in the lower half-plane is at $z=-2\mathrm{i}$, therefore

$$\begin{aligned}I&=-2\pi\mathrm{i}\,\mathrm{Res}_{z=-2\mathrm{i}}\left[\frac{\mathrm{e}^{-\mathrm{i}\omega z}}{z^2+4}\right]\\&=-2\pi\mathrm{i}\lim_{z\to-2\mathrm{i}}(z+2\mathrm{i})\frac{\mathrm{e}^{-\mathrm{i}\omega z}}{(z+2\mathrm{i})(z-2\mathrm{i})}=-2\pi\mathrm{i}\frac{\mathrm{e}^{-2\omega}}{-4\mathrm{i}}=\frac{\pi}{2}\mathrm{e}^{-2\omega}.\end{aligned}$$

This integral happens to be the Fourier transform of $\frac{1}{x^2+4}$, as we shall see in a later chapter.

Example 3.5.8. Evaluate the integral

$$I(t)=\frac{A}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{R+\mathrm{i}\omega L}\,\mathrm{d}\omega,$$

for both $t>0$ and $t<0$.

Solution 3.5.8.

$$I=\frac{A}{2\pi}\frac{1}{\mathrm{i}L}\omega L\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\left(\frac{R}{\mathrm{i}L}\right)+\omega}\,\mathrm{d}\omega=\frac{A}{2\pi}\frac{1}{\mathrm{i}L}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega-\mathrm{i}\frac{R}{L}}\,\mathrm{d}\omega.$$

For $t>0$, we can close the contour in the upper half-plane.

$$I=\frac{A}{2\pi}\frac{1}{\mathrm{i}L}\oint_{\mathrm{u.h.p}}\frac{\mathrm{e}^{\mathrm{i}tz}}{z-\mathrm{i}\frac{R}{L}}\,\mathrm{d}z.$$

The only singular point is located in the upper half-plane at $z=\mathrm{i}\frac{R}{L}$. Therefore

$$I=\frac{A}{2\pi}\frac{1}{\mathrm{i}L}2\pi\mathrm{i}\lim_{z\to\mathrm{i}\frac{R}{L}}\left(z-\mathrm{i}\frac{R}{L}\right)\frac{\mathrm{e}^{\mathrm{i}tz}}{z-\mathrm{i}\frac{R}{L}}=\frac{A}{L}\mathrm{e}^{\mathrm{i}t\left(\mathrm{i}\frac{R}{L}\right)}=\frac{A}{L}\mathrm{e}^{-\frac{R}{L}t}.$$

For $t<0$, we must close the contour in the lower half-plane. Since there is no singular point in the lower half-plane, the integral is zero. Thus

$$I(t)=\begin{cases}\frac{A}{L}\mathrm{e}^{-\frac{R}{L}t} & t>0,\\ 0 & t<0.\end{cases}$$

For those who are familiar with AC circuits, this integral $I(t)$ is the current in a circuit with resistance R and inductance L connected in series under a voltage impulse V. A high pulse in a short duration can be expressed as

$$V(t) = \frac{A}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega,$$

and the impedance of the circuit is $Z = R + \mathrm{i}\omega L$ for the ω component, and the corresponding current is given by $\frac{V}{Z}$. Thus the total current is the integral we have evaluated.

Example 3.5.9. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)} \mathrm{d}x.$$

Solution 3.5.9.

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)} \mathrm{d}x = \mathrm{Im} \int_{-\infty}^{\infty} \frac{x\mathrm{e}^{\mathrm{i}x}}{(x^2+1)} \mathrm{d}x,$$

$$\int_{-\infty}^{\infty} \frac{x\mathrm{e}^{\mathrm{i}x}}{(x^2+1)} \mathrm{d}x = \oint_{\mathrm{u.h.p}} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)} \mathrm{d}z.$$

There is only one singular point in the upper half-plane located at $z = \mathrm{i}$. So

$$\oint_{\mathrm{u.h.p}} \frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)} \mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}} \left[\frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)} \right]$$
$$= 2\pi\mathrm{i} \lim_{z\to\mathrm{i}} (z-\mathrm{i}) \frac{z\mathrm{e}^{\mathrm{i}z}}{(z^2+1)} = 2\pi\mathrm{i} \frac{\mathrm{i}\mathrm{e}^{-1}}{2\mathrm{i}} = \frac{\pi}{\mathrm{e}}\mathrm{i},$$

and

$$I = \mathrm{Im}\left(\frac{\pi}{\mathrm{e}}\mathrm{i}\right) = \frac{\pi}{\mathrm{e}}.$$

Example 3.5.10. (a) Show that

$$I = \int_0^{\infty} \frac{\cos bx}{x^2+a^2} \mathrm{d}x = \frac{\pi}{2a}\mathrm{e}^{-ba}, \qquad a > 0, \qquad b > 0.$$

(b) Use the result of (a) to find the value of

$$\int_0^{\infty} \frac{\cos bx}{(x^2+a^2)^2} \mathrm{d}x.$$

Solution 3.5.10. (a) Since the integrand is an even function, so

$$\int_0^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x.$$

$$\int_{-\infty}^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \mathrm{Re}\int_{-\infty}^\infty \frac{\mathrm{e}^{\mathrm{i}bx}}{x^2+a^2}\,\mathrm{d}x = \mathrm{Re}\oint_{\mathrm{u.h.p}} \frac{\mathrm{e}^{\mathrm{i}bz}}{z^2+a^2}\,\mathrm{d}z$$

The singular point in the upper half-plane is at $z = \mathrm{i}a$, so

$$\begin{aligned}\oint_{\mathrm{u.h.p}} \frac{\mathrm{e}^{\mathrm{i}bz}}{z^2+a^2}\mathrm{d}z &= 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}a}\left[\frac{\mathrm{e}^{\mathrm{i}bz}}{z^2+a^2}\right] \\ &= 2\pi\mathrm{i}\lim_{z\to\mathrm{i}a}(z-\mathrm{i}a)\frac{\mathrm{e}^{\mathrm{i}bz}}{(z-\mathrm{i}a)(z+\mathrm{i}a)} = 2\pi\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}b(\mathrm{i}a)}}{2a\mathrm{i}} = \frac{\pi}{a}\mathrm{e}^{-ba}.\end{aligned}$$

Thus,

$$\int_0^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\mathrm{Re}\left(\frac{\pi}{a}\mathrm{e}^{-ba}\right) = \frac{\pi}{2a}\mathrm{e}^{-ba}.$$

(b) Taking derivative of both sides with respect to a,

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty \frac{\cos bx}{x^2+a^2}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\pi}{2a}\mathrm{e}^{-ba}\right),$$

we have

$$\int_0^\infty \frac{-2a\cos bx}{(x^2+a^2)^2}\mathrm{d}x = \frac{-\pi}{2a^2}\mathrm{e}^{-ba} + \frac{\pi(-b)}{2a}\mathrm{e}^{-ba}.$$

Therefore

$$\int_0^\infty \frac{\cos bx}{(x^2+a^2)^2}\mathrm{d}x = \frac{\pi}{4a^3}(1+ab)\,\mathrm{e}^{-ba}.$$

3.5.4 Improper Integrals II: Closing the Contour with Rectangular and Pie-shaped Contour

If the integrand does not go to zero fast enough on the infinitely large contour C_R, then the contour cannot be closed with a large semicircle, up or down. For such a case, there may be other types of closed contours that will enable us to eliminate all parts of the integral but the desired portion. However, selecting appropriate contour requires considerable ingenuity. Here we present two additional kinds of contours that are known to be useful.

Rectangular Contour

If the height of the rectangle can be so chosen that the integral along the top side of the rectangle is equal to a constant multiple of the integral along the real axis, then such a contour may be useful for evaluating integrals whose integrand vanishes as the absolute value of the real variable goes to infinity. Generally, integrands containing exponential function or hyperbolic functions are good candidates for this method. Again the method is best illustrated by an example.

Example 3.5.11. Show that

$$I = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x = \frac{\pi}{\sin a\pi}, \qquad \text{for} \quad 0 < a < 1.$$

Solution 3.5.11. First we analytically continue the integrand to the complex plane

$$f(z) = \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}.$$

The denominator of $f(z)$ is unchanged if z is increased by $2\pi\mathrm{i}$, whereas the numerator changes by a factor of $\mathrm{e}^{a2\pi\mathrm{i}}$. Thus a rectangular contour shown in Fig. 3.11 may be appropriate.

Integrating around the rectangular loop, we have

$$\oint f(z)\,\mathrm{d}z = J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 = \int_{L_1} \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x,$$

$$J_2 = \int_{L_2} \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z = \int_{0}^{2\pi} \frac{\mathrm{e}^{a(R+\mathrm{i}y)}}{1+\mathrm{e}^{R+\mathrm{i}y}}\mathrm{i}\,\mathrm{d}y,$$

$$J_3 = \int_{L_3} \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z = \int_{R}^{-R} \frac{\mathrm{e}^{a(x+\mathrm{i}2\pi)}}{1+\mathrm{e}^{(x+\mathrm{i}2\pi)}}\,\mathrm{d}x = \mathrm{e}^{\mathrm{i}2\pi a}\int_{R}^{-R} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^{x}}\,\mathrm{d}x,$$

$$J_4 = \int_{L_3} \frac{\mathrm{e}^{az}}{1+\mathrm{e}^{z}}\,\mathrm{d}z = \int_{2\pi}^{0} \frac{\mathrm{e}^{a(-R+\mathrm{i}y)}}{1+\mathrm{e}^{-R+\mathrm{i}y}}\mathrm{i}\,\mathrm{d}y.$$

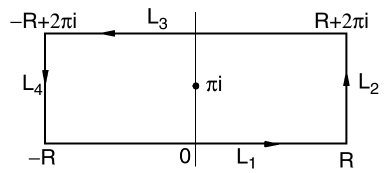

Fig. 3.11. A closed rectangular contour

As $R \to \infty$,

$$\lim_{R\to\infty} J_1 = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = I,$$

$$\lim_{R\to\infty} J_3 = \lim_{R\to\infty} e^{i2\pi a} \int_{R}^{-R} \frac{e^{ax}}{1+e^x}\,dx = -e^{i2\pi a} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = -e^{i2\pi a} I.$$

Furthermore, since $\left|e^{a(R+iy)}\right| = e^{aR}$ and the minimum value of $\left|1+e^{R+iy}\right|$ is $\left|1-e^R\right|$, hence

$$\lim_{R\to\infty} |J_2| \le \lim_{R\to\infty} \left|\frac{e^{aR}}{1-e^R}\right| 2\pi = \lim_{R\to\infty} \frac{2\pi}{e^{(1-a)R}} \to 0, \quad \text{since } a < 1.$$

Similarly,

$$\lim_{R\to\infty} |J_4| \le \lim_{R\to\infty} \left|\frac{e^{-aR}}{1-e^{-R}}\right| 2\pi = \lim_{R\to\infty} 2\pi e^{-aR} \to 0, \quad \text{since } a > 0.$$

Therefore

$$\lim_{R\to\infty} \oint \frac{e^{az}}{1+e^z}\,dz = (1-e^{i2\pi a})I.$$

Now inside the loop, there is a simple pole at $z = i\pi$, since

$$1 + e^z = 1 + e^{i\pi} = 1 - 1 = 0.$$

By the residue theorem, we have

$$\oint \frac{e^{az}}{1+e^z}\,dz = 2\pi i\,\mathrm{Res}_{z=i\pi}\left[\frac{e^{az}}{1+e^z}\right] = 2\pi i\left[\frac{e^{az}}{(1+e^z)'}\right]_{z=i\pi}$$
$$= 2\pi i \frac{e^{i\pi a}}{e^{i\pi}} = -2\pi i e^{i\pi a}.$$

Thus,

$$(1-e^{i2\pi a})I = -2\pi i e^{i\pi a},$$

so

$$I = \frac{-2\pi i e^{i\pi a}}{1-e^{i2\pi a}} = \frac{-2\pi i}{e^{-i\pi a}-e^{i\pi a}} = \frac{\pi}{\sin \pi a}.$$

Pie-shaped Contour

If the integral is from 0 to ∞, instead of from $-\infty$ to ∞ and none of the earlier methods is applicable, then a pie-shaped contour may work. In the following example, we will illustrate this method with the evaluation of the Fresnel integrals, which are important in diffraction theory and signal propagation.

Example 3.5.12. Evaluate the Fresnel integrals

$$I_c = \int_0^\infty \cos\left(x^2\right) \mathrm{d}x, \qquad I_s = \int_0^\infty \sin\left(x^2\right) \mathrm{d}x.$$

Solution 3.5.12. The two Fresnel integrals are the real and imaginary parts of the exponential integral,

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = I_c + \mathrm{i}I_s.$$

We integrate the complex function $\mathrm{e}^{\mathrm{i}z^2}$ around the pie-shaped contour shown in Fig. 3.12. Since the function is analytic within the closed contour, the loop integral must be zero,

$$\oint \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = 0.$$

This loop integral naturally divides into three parts. First from 0 to R along the real x axis, then along the path of an arc C_R from R to R'. Finally returning to 0 along the straight radial line with $\theta = \pi/4$.

$$\int_0^R \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x + \int_{C_R} \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z + \int_{R'}^0 \mathrm{e}^{\mathrm{i}z^2} \mathrm{d}z = 0.$$

In the limit of $R \to \infty$, the first integral is what we want to find,

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = \int_0^\infty \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x.$$

On the path of the third integral, with $z = r\mathrm{e}^{\mathrm{i}\theta}$ and $\theta = \pi/4$,

$$z^2 = r^2 \mathrm{e}^{\mathrm{i}2\theta} = r^2 \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}r^2,$$
$$\mathrm{d}z = \mathrm{e}^{\mathrm{i}\theta} \mathrm{d}r = \mathrm{e}^{\mathrm{i}\pi/4} \mathrm{d}r,$$

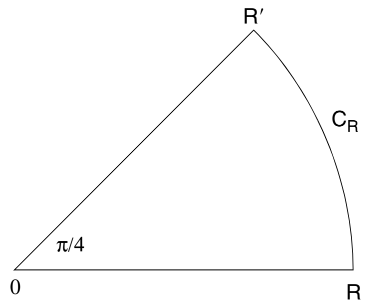

Fig. 3.12. A pie-shaped contour. In the complex plane, R' is at $z\left(R'\right) = R\mathrm{e}^{\mathrm{i}\pi/4}$

so the third integral becomes

$$\int_{R'}^{0} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = \int_{R}^{0} \mathrm{e}^{-r^2}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{d}r = -\mathrm{e}^{\mathrm{i}\pi/4}\int_{0}^{R} \mathrm{e}^{-r^2}\mathrm{d}r.$$

We will now show that the second integral along C_R is equal to zero in the limit of $R \to \infty$. On C_R

$$z = R\mathrm{e}^{\mathrm{i}\theta}, \qquad \mathrm{d}z = \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta,$$
$$z^2 = R^2\mathrm{e}^{\mathrm{i}2\theta} = R^2(\cos 2\theta + \mathrm{i}\sin 2\theta),$$

so the second integral can be written as

$$\int_{C_R} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z = \mathrm{i}R\int_{0}^{\pi/4} \mathrm{e}^{\mathrm{i}R^2(\cos 2\theta+\mathrm{i}\sin 2\theta)}\mathrm{e}^{\mathrm{i}\theta}$$
$$\mathrm{d}\theta = \mathrm{i}R\int_{0}^{\pi/4} \mathrm{e}^{\mathrm{i}(R^2\cos 2\theta+\theta)}\mathrm{e}^{-R^2\sin 2\theta}\mathrm{d}\theta.$$

Thus

$$\left|\int_{R}^{R'} \mathrm{e}^{\mathrm{i}z^2}\mathrm{d}z\right| \leq R\int_{0}^{\pi/4} \mathrm{e}^{-R^2\sin 2\theta}\mathrm{d}\theta = \frac{R}{2}\int_{0}^{\pi/2} \mathrm{e}^{-R^2\sin\phi}\mathrm{d}\phi,$$

where $\phi = 2\theta$. According to (3.23) of Jordan's lemma,

$$\int_{0}^{\pi/2} \mathrm{e}^{-R^2\sin\phi}\mathrm{d}\phi \leq \frac{\pi}{2R^2}(1-\mathrm{e}^{-R^2}).$$

Therefore, it goes to zero as $1/R^2$ for $R \to \infty$. With the second integral equal to zero, we are left with

$$\int_{0}^{\infty} \mathrm{e}^{\mathrm{i}x^2}\mathrm{d}x - \mathrm{e}^{\mathrm{i}\pi/4}\int_{0}^{\infty} \mathrm{e}^{-r^2}\mathrm{d}r = 0. \tag{3.28}$$

Now it is well known that

$$\int_{0}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

To verify this expression, define

$$I = \int_{0}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \int_{0}^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y,$$

so

$$I^2 = \int_{0}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x\int_{0}^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y = \int_{0}^{\infty}\int_{0}^{\infty} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y.$$

In polar coordinates

$$I^2 = \int_0^\infty \int_0^{\pi/2} \mathrm{e}^{-\rho^2} \rho \, \mathrm{d}\varphi \, \mathrm{d}\rho = \frac{\pi}{2} \int_0^\infty \mathrm{e}^{-r^2} \rho \, \mathrm{d}\rho = \frac{\pi}{4},$$

so $I = \sqrt{\pi}/2$.

It follows from (3.28) that

$$\int_0^\infty \mathrm{e}^{\mathrm{i}x^2} \mathrm{d}x = \mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \mathrm{e}^{-r^2} \mathrm{d}r = \mathrm{e}^{\mathrm{i}\pi/4} \frac{\sqrt{\pi}}{2} = \left(\cos \frac{\pi}{4} + \mathrm{i} \sin \frac{\pi}{4} \right) \frac{\sqrt{\pi}}{2}$$
$$= \left(\frac{1}{\sqrt{2}} + \mathrm{i} \frac{1}{\sqrt{2}} \right) \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{8}} + \mathrm{i} \sqrt{\frac{\pi}{8}}.$$

Therefore

$$\int_0^\infty \cos\left(x^2\right) \mathrm{d}x = \int_0^\infty \sin\left(x^2\right) \mathrm{d}x = \sqrt{\frac{\pi}{8}}.$$

3.5.5 Integration Along a Branch Cut

Some integrals of multivalued functions can also be evaluated by Cauchy's residue theorem. For example, the integrand of the integral

$$I = \int_0^\infty x^{-\alpha} f(x) \, \mathrm{d}x$$

is multivalued if α is not an integer. In the complex plane, $z^{-\alpha}$ is multivalued because with z expressed as

$$z = r\mathrm{e}^{\mathrm{i}(\theta + n2\pi)},$$

where n is an integer, $z^{-\alpha}$ becomes

$$z^{-\alpha} = \mathrm{e}^{-\alpha \ln z} = \mathrm{e}^{-\alpha(\ln r + \mathrm{i}\theta + \mathrm{i}n2\pi)}.$$

It is seen that $z^{-\alpha}$ is a multivalued function. For instance, with $\alpha = 1/3$,

$$z^{-\frac{1}{3}} = \begin{cases} \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} & n = 0 \\ \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} \mathrm{e}^{\mathrm{i}2\pi/3} = \left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} & n = 1 \\ \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} \mathrm{e}^{\mathrm{i}4\pi/3} = \left(-\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right) \mathrm{e}^{-\frac{1}{3}(\ln r + \mathrm{i}\theta)} & n = 2. \end{cases}$$

To define $z^{-\alpha}$ as a single valued function, the angle θ must be restricted in an interval of 2π by a branch cut. If we choose the branch cut along the positive x axis, then our real integral is an integral along the top of the branch cut. Very often the problem can be solved with a closed contour as shown in Fig. 3.13, in which the entire branch cut is excluded from the interior of the contour. Again let us illustrate the method with an example.

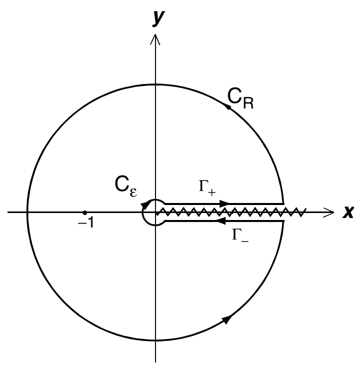

Fig. 3.13. A contour that excludes the branch cut along the positive x axis

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Example 3.5.13. Evaluate the integral

$$I = \int_0^\infty \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x, \qquad 0 < \alpha < 1.$$

Solution 3.5.13. Consider the contour integral

$$\oint \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z$$

around the closed contour shown in Fig. 3.13. Since the branch point at $z = 0$ and the entire branch cut are excluded, the only singular point inside this contour is at $z = -1$. Therefore

$$\oint \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}\left[\frac{z^{-\alpha}}{1+z}\right] = 2\pi\mathrm{i}\,(-1)^{-\alpha} = 2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi(-\alpha)} = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}}.$$

This integral consists of four parts

$$\oint \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = \int_{\Gamma_+} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z + \int_{C_R} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z + \int_{\Gamma_-} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z + \int_{C_\epsilon} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z.$$

The first integral is along the top of the branch cut with $\theta = 0$, the second integral is along the outer large circle with radius R, the third integral is along the bottom of the branch cut with $\theta = 2\pi$, and the fourth integral is along the inner small circle with radius ϵ.

With $z = r\mathrm{e}^{\mathrm{i}\theta}$, it is clear that when $\theta = 0$ and $\theta = 2\pi$, r is the same as x. Therefore

$$\int_{\Gamma_+} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = \int_\epsilon^R \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x,$$

$$\int_{\Gamma_-} \frac{z^{-\alpha}}{1+z}\,\mathrm{d}z = \int_R^\epsilon \frac{x^{-\alpha}\mathrm{e}^{\mathrm{i}2\pi(-\alpha)}}{1+x\mathrm{e}^{i2\pi}}\mathrm{e}^{\mathrm{i}2\pi}\,\mathrm{d}x = -\mathrm{e}^{-\mathrm{i}2\pi\alpha}\int_\epsilon^R \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x.$$

On C_R, $z = R\mathrm{e}^{\mathrm{i}\theta}$,

$$\left| \int_{C_R} \frac{z^{-\alpha}}{1+z} \, \mathrm{d}z \right| \leq \left| \frac{R^{-\alpha}}{1-R} 2\pi R \right|,$$

where $R^{-\alpha}$ is maximum of the numerator, $1 - R$ is the minimum of denominator, and $2\pi R$ is the length of C_R. As $R \to \infty$,

$$\left| \frac{R^{-\alpha}}{1-R} 2\pi R \right| \sim R^{-\alpha} \to 0, \quad \text{since} \quad \alpha > 0.$$

Similarly, on $C_\epsilon, \; z = \epsilon \mathrm{e}^{\mathrm{i}\theta}$,

$$\left| \int_{C_\epsilon} \frac{z^{-\alpha}}{1+z} \, \mathrm{d}z \right| \leq \frac{\epsilon^{-\alpha}}{1-\epsilon} 2\pi\epsilon.$$

As $\epsilon \to 0$,

$$\frac{\epsilon^{-\alpha}}{1-\epsilon} 2\pi\epsilon \sim \epsilon^{1-\alpha} \to 0, \quad \text{since} \quad \alpha < 1.$$

On taking the limit $R \to \infty$ and $\epsilon \to 0$, we are left with

$$\oint \frac{z^{-\alpha}}{1+z} \, \mathrm{d}z = \int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x - \mathrm{e}^{-\mathrm{i}2\pi\alpha} \int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}}.$$

Thus

$$\left(1 - \mathrm{e}^{-\mathrm{i}2\pi\alpha}\right) \int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}},$$

and

$$\int_0^\infty \frac{x^{-\alpha}}{1+x} \, \mathrm{d}x = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha}\left(1 - \mathrm{e}^{-\mathrm{i}2\pi\alpha}\right)} = \frac{2\pi\mathrm{i}}{\mathrm{e}^{\mathrm{i}\pi\alpha} - \mathrm{e}^{-\mathrm{i}\pi\alpha}} = \frac{\pi}{\sin \pi\alpha}.$$

3.5.6 Principal Value and Indented Path Integrals

Sometimes we have to deal with integrals $\int f(x) \, \mathrm{d}x$ whose integrand becomes infinite at a point $x = x_0$ in the range of integration

$$\lim_{x \to x_0} f(x) = \infty.$$

In order to make sense of this kind of integral, we define the principal value integral as

$$P \int_{-R}^{R} f(x) \, \mathrm{d}x = \lim_{\varepsilon \to 0} \left[\int_{-R}^{x_0 - \varepsilon} f(x) \, \mathrm{d}x + \int_{x_0 + \varepsilon}^{R} f(x) \, \mathrm{d}x \right].$$

It is a way to avoid the singularity. One integrates to within a small distance ε of the singularity in question, skips over the singularity, and begins integrating again at a distance ε beyond the singularity.

When evaluating the integral using the residue theorem, we are not allowed to have a singularity on the contour, however, with principal value integrals we can accommodate simple poles on the contour by deforming the contour so as to avoid the poles.

The principal value integral

$$P \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

can be evaluated by the theorem of residue for a function $f(z)$ that satisfies the asymptotic conditions that we have discussed. That is, either $zf(z) \to 0$ as $z \to \infty$, or $f(z) = \mathrm{e}^{\mathrm{i}mz} g(z)$ and $g(z) \to 0$ as $z \to \infty$. Let us first assume that $f(z)$ has one simple pole on the real axis at $z = x_0$ and is analytic everywhere else. In this case, it is clear that the closed contour integral around the indented path shown in Fig. 3.14 is equal to zero

$$\oint f(z)\,\mathrm{d}z = 0,$$

since the only singular point is outside the contour.

The integral can be written as

$$\oint f(z)\,\mathrm{d}z = \int_{-R}^{x_0-\varepsilon} f(x)\,\mathrm{d}x + \int_{C_\varepsilon} f(z)\,\mathrm{d}z + \int_{x_0+\varepsilon}^{R} f(x)\,\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z.$$

In the limit of $R \to \infty$, with $f(z)$ satisfying the specified conditions, we have shown

$$\int_{C_R} f(z)\,\mathrm{d}z = 0.$$

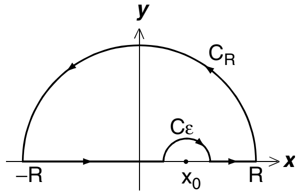

Fig. 3.14. The closed contour consists of a large semicircle C_R in the upper half-plane of radius R, the line segments from $-R$ to $x_0 - \varepsilon$ and from $x_0 + \varepsilon$ to R along the real axis, and a small semicircle C_ε of radius ϵ above the singular point x_0

Furthermore, the two line integrals along the x axis become the principal value integral as $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\left[\int_{-\infty}^{x_0-\varepsilon} f(x)\,\mathrm{d}x + \int_{x_0+\varepsilon}^{\infty} f(x)\,\mathrm{d}x\right] = P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

Since $f(z)$ has a simple pole at $z = x_0$, so in the immediate neighborhood of x_0, the Laurent series of $f(z)$ has the form

$$f(z) = \frac{a_{-1}}{z - x_0} + \sum_{n=0}^{\infty} a_n (z - x_0)^n.$$

On the semicircle C_ε around x_0,

$$z - x_0 = \varepsilon \mathrm{e}^{\mathrm{i}\theta}, \qquad \mathrm{d}z = \mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta,$$

where ε is the radius of the semicircle. The integral around C_ε can thus be written as

$$\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \int_{\pi}^{0}\left(\frac{a_{-1}}{\varepsilon \mathrm{e}^{\mathrm{i}\theta}} + \sum_{n=0}^{\infty} a_n\left(\varepsilon \mathrm{e}^{\mathrm{i}\theta}\right)^n\right)\mathrm{i}\varepsilon \mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta.$$

On taking the limit $\varepsilon \to 0$, every term vanishes except the first. Therefore

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \int_{\pi}^{0} a_{-1}\mathrm{i}\mathrm{d}\theta = -\mathrm{i}\pi a_{-1} = -\mathrm{i}\pi\mathrm{Res}_{z=x_0}[f(z)].$$

It follows that in the limit $R \to \infty$ and $\varepsilon \to 0$

$$\oint f(z)\,\mathrm{d}z = P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - \mathrm{i}\pi\mathrm{Res}[f(z)] = 0.$$

Therefore

$$P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \mathrm{i}\pi\mathrm{Res}[f(z)].$$

Note that to avoid the singular point, we can just as well go below it instead above. For the semicircle below the x axis, the direction of integration is counterclockwise

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon} f(z)\,\mathrm{d}z = \int_{\pi}^{2\pi} a_{-1}\mathrm{i}\mathrm{d}\theta = \mathrm{i}\pi a_{-1} = \mathrm{i}\pi\mathrm{Res}_{z=x_0}[f(z)].$$

However, in that case, the singular point is inside the closed contour, and the loop integral is equal to $2\pi\mathrm{i}$ times the residue at $z = x_0$. So we have

$$\oint f(z)\,\mathrm{d}z = P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x + \mathrm{i}\pi\mathrm{Res}_{z=x_0}[f(z)] = 2\pi\mathrm{i}\,\mathrm{Res}_{z=x_0}[f(z)].$$

Not surprisingly we get the same result

$$P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi \mathrm{i}\,\mathrm{Res}_{z=x_0}[f(z)] - \mathrm{i}\pi \mathrm{Res}_{z=x_0}[f(z)] = \mathrm{i}\pi \mathrm{Res}_{z=x_0}[f(z)].$$

Now if $f(z)$ has more than one pole on the real axis, (all of them first-order), furthermore, it has other singularities in the upper half-plane, (not necessary first-order), then by the same argument one can show that

$$P\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \pi i\left(\sum \text{residues on } x \text{ axis}\right) \\ +2\pi \mathrm{i}\left(\sum \text{residues in upper half-plane}\right).$$

Example 3.5.14. Find the principal value of

$$P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x$$

and use the result to show

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

Solution 3.5.14. The only singular point is at $x = 0$, therefore

$$P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = \pi \mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{\mathrm{e}^{\mathrm{i}z}}{z}\right] = \pi \mathrm{i}\left[\frac{\mathrm{e}^{\mathrm{i}z}}{z'}\right]_{z=0} = \pi \mathrm{i}.$$

Since

$$P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = P\left[\int_{-\infty}^{\infty} \frac{\cos x}{x}\,\mathrm{d}x + \mathrm{i}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x\right],$$

therefore,

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \mathrm{Im}\left(P\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x\right) = \pi.$$

We note that $x = 0$ is actually a removable singularity of $\sin x/x$, since as $x \to 0$, $\sin x/x = 1$. This means that ε, instead of approaching zero, can be set equal to exactly zero. Therefore the principal value of the integral is the integral itself

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

It is instructive to check this result in the following way. Since $\sin x/x$ is continuous at $x = 0$, if we move the path of integration an infinitesimal amount at $x = 0$, the value of the integral will not be changed. So let the path go through an infinitesimally small semicircle C_ε on top of the point at $x = 0$. Let us call the indented path as the path from $-\infty$ to ε along x axis, followed by C_ε and then continue from ε to ∞ along the x axis. Then

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \int_{\text{Indented}} \frac{\sin z}{z}\, \mathrm{d}z.$$

Now let us make use of the identity

$$\sin z = \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z} \right),$$

so

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \frac{1}{2\mathrm{i}} \int_{\text{Indented}} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\, \mathrm{d}z - \frac{1}{2\mathrm{i}} \int_{\text{Indented}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\, \mathrm{d}z.$$

For the first integral in the right-hand side, we can close the contour with an infinitely large semicircle C_R^+ in the upper half-plane, as shown in Fig. 3.15a. Since the singular point is outside the closed contour, so the contour integral vanishes

$$\int_{\text{Indented}} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\, \mathrm{d}z = \oint_{u,h.p} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\, \mathrm{d}z = 0.$$

For the second indented path integral, we cannot close the contour in the upper half-plane because of $\mathrm{e}^{-\mathrm{i}z}$, so we have to close the contour in the lower half-plane with C_R^-, as shown in Fig. 3.15b. In this case, the singular point at $z = 0$ is inside the contour, therefore

$$\int_{\text{Indented}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\, \mathrm{d}z = \oint_{\text{l.h.p}} \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\, \mathrm{d}z = -2\pi\mathrm{i}\,\mathrm{Res}_{z=0} \left[\frac{\mathrm{e}^{-\mathrm{i}z}}{z} \right] = -2\pi\mathrm{i}.$$

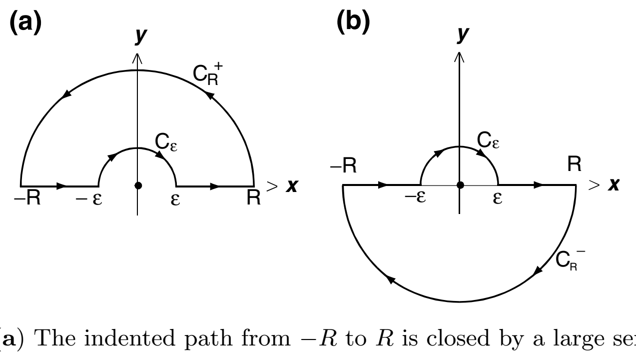

Fig. 3.15. (**a**) The indented path from $-R$ to R is closed by a large semicircle C_R^+ in the upper half-plane. (**b**) The same indented path from $-R$ to R is closed by a large semicircle C_R^- in the lower half-plane

The minus sign accounts for the clockwise direction. It follows that:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{1}{2\mathrm{i}}0 - \frac{1}{2\mathrm{i}}\left(-2\pi\mathrm{i}\right) = \pi,$$

which is the same as we obtained before.

Exercises

1. Expand $f(z) = \frac{z-1}{z+1}$ in a Taylor's series (a) around $z = 0$ and (b) around the point $z = 1$. Determine the radius of convergence of each series.

   Ans:

   $$(\mathrm{a}) f(z) = -1 + 2z - 2z^2 + 2z^2 - \cdots \quad |z| < 1$$
   $$(\mathrm{b}) f(z) = \frac{1}{2}(z-1) - \frac{1}{4}(z-1)^2 + \frac{1}{8}(z-1)^3 - \frac{1}{16}(z-1)^4 + \cdots |z-1| < 2.$$

2. Find the Taylor series expansion about the origin and the radius of convergence for

   $$(\mathrm{a}) \quad f(z) = \sin z, \qquad (\mathrm{b}) \quad f(z) = \cos z,$$
   $$(\mathrm{c}) \quad f(z) = \mathrm{e}^z, \qquad (\mathrm{d}) \quad f(z) = \frac{1}{(1-z)^m}.$$

   Ans:

   $$(\mathrm{a}) \quad \sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots \quad \text{all} \quad z,$$
   $$(\mathrm{b}) \quad \cos z = 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 + \cdots \quad \text{all} \quad z,$$
   $$(\mathrm{c}) \quad \mathrm{e}^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots \quad \text{all} \quad z,$$
   $$(\mathrm{d}) \quad \frac{1}{(1-z)^m} = 1 + mz + \frac{m(m+1)}{2}z^2 + \frac{m(m+1)(m+2)}{3!}z^3 + \cdots |z| < 1.$$

3. Find the Taylor series expansion of

   $$f(z) = \ln z$$

   around $z = 1$ by noting that

   $$\frac{\mathrm{d}}{\mathrm{d}z}\ln z = \frac{1}{z}.$$

Ans:
$\ln z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \cdots \qquad |z-1| < 1.$

4. Expand

$$f(z) = \frac{1}{(z+1)(z+2)}$$

in a Taylor's series (a) around $z = 0$ and (b) around the point $z = 2$. Determine the radius of convergence of each series.

Ans:

$$\text{(a)}\quad f(z) = \frac{1}{2} - \frac{3}{4}z + \frac{7}{8}z^2 - \frac{15}{16}z^3 + \cdots \quad |z| < 1.$$

$$\text{(b)}\quad f(z) = \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{3^2} - \frac{1}{4^2}\right)(z-2)$$

$$+ \left(\frac{1}{3^3} - \frac{1}{4^3}\right)(z-2)^2 - \cdots \quad |z-2| < 3.$$

5. Without obtaining the series, determine the radius of convergence of each of the following expansions:

$$\text{(a)}\ \tan^{-1} z \quad \text{around } z = 1,$$

$$\text{(b)}\ \frac{1}{e^z - 1} \quad \text{around } z = 4i,$$

$$\text{(c)}\ \frac{x}{x^2 + 2x + 10} \quad \text{around } x = 0.$$

Ans: (a) $\sqrt{2}$, (b) $2\pi - 4$, (c) $\sqrt{10}$.

6. Find the Laurent series for

$$f(z) = \frac{1}{z^2 - 3z + 2}$$

in the region of

$$\text{(a)}\ |z| < 1, \quad \text{(b)}\ 1 < |z| < 2, \quad \text{(c)}\ 0 < |z-1| < 1,$$

$$\text{(d)}\ 2 < |z|, \quad \text{(e)}\ |z-1| > 1, \quad \text{(f)}\ 0 < |z-2| < 1.$$

Ans:

$$\text{(a)}\ f(z) = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \cdots$$

$$\text{(b)}\ f(z) = \cdots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3.$$

$$\text{(c)}\ f(z) = -\frac{1}{(z-1)} - 1 - (z-1) - (z-1)^2 - \cdots$$

(d) $f(z) = \cdots + \dfrac{15}{z^5} + \dfrac{7}{z^4} + \dfrac{3}{z^3} + \dfrac{1}{z^2}.$

(e) $f(z) = \cdots + \dfrac{1}{(z-1)^4} + \dfrac{1}{(z-1)^3} + \dfrac{1}{(z-1)^2}.$

(f) $f(z) = \dfrac{1}{z-2} - 1 + (z-2) - (z-2)^2 + (z-2)^3 - \cdots$

7. Expand

$$f(z) = \frac{1}{z^2(z-\mathrm{i})}$$

in two different Laurent expansions around $z = \mathrm{i}$ and tell where each converges.

Ans:
$f(z) = -\frac{1}{z-\mathrm{i}} - 2\mathrm{i} + 3(z-\mathrm{i}) - 4\mathrm{i}(z-\mathrm{i})^2 + \cdots \qquad 0 < |z-\mathrm{i}| < 1$
$f(z) = \cdots \frac{4\mathrm{i}}{(z-\mathrm{i})^6} - \frac{3}{(z-\mathrm{i})^5} - \frac{2\mathrm{i}}{(z-\mathrm{i})^4} + \frac{1}{(z-\mathrm{i})^3} \qquad |z-\mathrm{i}| > 1.$

8. Find the values of $\oint_C f(z)\,\mathrm{d}z$, where C is the circle $|z| = 3$, for the following functions:

(a) $f(z) = \dfrac{1}{z(z+2)}$, (b) $f(z) = \dfrac{z+2}{z(z+1)}$, (c) $f(z) = \dfrac{z}{(z+1)(z+2)}$,

(d) $f(z) = \dfrac{1}{z(z+1)^2}$, (e) $f(z) = \dfrac{1}{(z+1)^2}$, (f) $f(z) = \dfrac{1}{z(z+1)(z+4)}$

by expanding them in an appropriate Laurent series $f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n$ and using $a_{-1} = \frac{1}{2\pi\mathrm{i}} \oint_C f(z)\mathrm{d}z$.

Ans: (a) 0, (b) $2\pi\mathrm{i}$, (c) $2\pi\mathrm{i}$, (d) 0, (e) 0, (f) $-\mathrm{i}\pi/6$.

9. Find the residue of

$$f(z) = \frac{z}{z^2+1}$$

(a) at $z = \mathrm{i}$ and (b) at $z = -\mathrm{i}$.

Ans: (a) 1/2; (b) 1/2.

10. Find the residue of

$$f(z) = \frac{z+1}{z^2(z-2)}$$

(a) at $z = 0$ and (b) at $z = 2$.

Ans: (a) $-3/4$; (b) $3/4$.

11. Find the residue of

$$f(z) = \frac{z}{z^2 + 2z + 5}$$

at each of its poles.
Ans: $r(-1 + 2\mathrm{i}) = (2 + \mathrm{i})/4$; $r(-1 - 2\mathrm{i}) = (2 - \mathrm{i})/4$

12. What is the residue of

$$f(z) = \frac{1}{(z+1)^3}$$

at $z = -1$?
Ans: 0.

13. What is the residue of

$$f(z) = \tan z$$

at $z = \pi/2$?
Ans: -1

14. What is the residue of

$$f(z) = \frac{1}{z - \sin z}$$

at $z = 0$?
Ans: 3/10

15. Use the theory of residue to evaluate $\oint_C f(z)\mathrm{d}z$ if C is the circle $|z| = 4$ for each of the following functions:

$$\text{(a)} \quad \frac{z}{z^2 - 1}, \qquad \text{(b)} \quad \frac{z+1}{z^2(z+2)}, \qquad \text{(c)} \quad \frac{1}{z(z-2)^3},$$
$$\text{(d)} \quad \frac{1}{z^2 + z + 1} \qquad \text{(e)} \quad \frac{1}{z(z^2 + 6z + 4)}.$$

Ans. (a) $2\pi\mathrm{i}$, (b) 0, (c) 0, (d) 0, (e) $(5 - 3\sqrt{5})\mathrm{i}\pi/20$.

16. Show that

$$\oint_C \frac{1}{(z^{100} + 1)(z - 4)} \mathrm{d}z = \frac{-2\pi\mathrm{i}}{4^{100} + 1},$$

if C is the circle $|z| = 3$.
Hint: First find the value of the integral along $|z| = 5$, then do the integration along $|z| = 5$ and $|z| = 3$ with a cut between them.

17. Use the theory of residue to evaluate the following definite integrals

$$\text{(a)} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2+\cos\theta},$$

$$\text{(b)} \int_0^{2\pi} \frac{\cos 3\theta \mathrm{d}\theta}{5-4\cos\theta},$$

$$\text{(c)} \int_0^{\pi} \frac{\cos 2\theta \mathrm{d}\theta}{1-2a\cos\theta+a^2} \quad \text{where} \quad (-1<a<1),$$

$$\text{(d)} \int_0^{\pi} \sin^{2n}\theta \mathrm{d}\theta \quad \text{where} \quad n=1,2,\ldots.$$

Ans. (a) $2\pi/\sqrt{3}$, (b) $\pi/12$, (c) $\pi a^2/\left(1-a^2\right)$, (d) $\pi\,(2n)!/\left(2^{2n}(n!)^2\right)$.

18. Show that

$$\text{(a)} \int_0^{\infty} \frac{\mathrm{d}x}{x^2+1} = \frac{\pi}{2},$$

$$\text{(b)} \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}\,\mathrm{d}x = \sqrt{2}\pi$$

$$\text{(c)} \int_0^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{4},$$

$$\text{(d)} \int_0^{\infty} \frac{ab}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{2(a+b)}.$$

19. Evaluate

$$\text{(a)} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}3x}}{x-2\mathrm{i}}\,\mathrm{d}x,$$

$$\text{(b)} \int_0^{\infty} \frac{\cos kx}{x^2+1}\,\mathrm{d}x,$$

$$\text{(c)} \int_{-\infty}^{\infty} \frac{\cos mx}{(x-a)^2+b^2}\,\mathrm{d}x,$$

$$\text{(d)} \int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x.$$

Ans. (a) $2\pi\mathrm{i}/\mathrm{e}^6$, (b) $\frac{\pi}{2}\,\mathrm{e}^{-|k|}$, (c) $\frac{\pi}{b}\,\mathrm{e}^{-mb}\cos ma$, (d) $\frac{\pi}{a^2-b^2}\left(\frac{\mathrm{e}^{-bm}}{b}-\frac{\mathrm{e}^{-am}}{a}\right)$.

20. Use a rectangular contour to show that

$$\int_{-\infty}^{\infty} \frac{\cos mx}{\mathrm{e}^{-x}+\mathrm{e}^{x}}\,\mathrm{d}x = \frac{\pi}{\mathrm{e}^{m\pi/2}+\mathrm{e}^{-m\pi/2}}.$$

21. Use the "integration along the branch cut" method to show that

$$\int_0^{\infty} \frac{x^{1/3}}{(1+x)^2}\,\mathrm{d}x = \frac{2\pi}{3\sqrt{3}}.$$

22. Use a pie-shaped contour with $\theta = 2\pi/3$ to show that

$$\int_0^{\infty} \frac{1}{x^3+1}\,\mathrm{d}x = \frac{2\sqrt{3}\pi}{9}.$$

23. Find the principal value of the following

$$P\int_{-\infty}^{\infty} \frac{1}{(x+1)\left(x^2+2\right)}\,\mathrm{d}x.$$

Ans. $\sqrt{2}\pi/6$.

24. Show that

$$\frac{1-\mathrm{e}^{2\mathrm{i}z}}{z^2}$$

has a simple pole at $z=0$. Find the principal value of

$$P\int_{-\infty}^{\infty} \frac{1-\mathrm{e}^{2\mathrm{i}x}}{x^2}\,\mathrm{d}x.$$

Use the result to show that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

Ans. 2π.