4

Determinants

Determinants are powerful tools for solving systems of linear equations and they are indispensable in the development of matrix theory. Most readers probably already possess the knowledge of evaluating second- and third-order determinants. After a systematic review, we introduce the formal definition of a nth order determinant through the Levi-Civita symbol. All properties of determinants can be derived from this definition.

4.1 Systems of Linear Equations

4.1.1 Solution of Two Linear Equations

Suppose we wish to solve for x and y from the system of 2×2 linear equations (2 equations and 2 unknowns)

$$a_1 x + b_1 y = d_1, \tag{4.1}$$

$$a_2 x + b_2 y = d_2, \tag{4.2}$$

where a_1, a_2, b_1, b_2, d_1, and d_2 are known constants. We can multiply (4.1) by b_2 and (4.2) by b_1, and then take the difference. In so doing, y is eliminated, and we are left with

$$(b_2 a_1 - b_1 a_2) x = b_2 d_1 - b_1 d_2,$$

therefore

$$x = \frac{d_1 b_2 - d_2 b_1}{a_1 b_2 - a_2 b_1}, \tag{4.3}$$

where we have written $b_2 a_1$ as $a_1 b_2$, since the order is immaterial in the product of two numbers. It turns out that if we use the following notation, it is much easier to generalize this process to larger systems of $n \times n$ equations

$$a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{4.4}$$

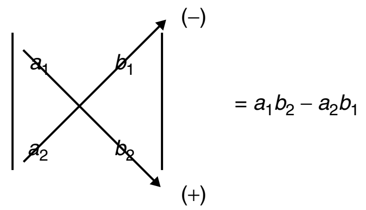


Fig. 4.1. A schematic diagram for a second-order determinant

The 2×2 square array of the four elements on the right-hand side of this equation is called a second-order determinant. Its meaning is just that its value is equal to the left-hand side of this equation. Explicitly, the value of a second-order determinant is defined as the difference between the two products of the diagonal elements as shown in the schematic diagram (Fig. 4.1).

With determinants, (4.3) can be written as

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \tag{4.5}$$

and with a similar procedure one can easily show that

$$y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}. \tag{4.6}$$

Example 4.1.1. Find the solution of

$$2x - 3y = -4,$$
$$6x - 2y = 2.$$

Solution 4.1.1.

$$x = \frac{\begin{vmatrix} -4 & -3 \\ 2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix}} = \frac{8+6}{-4+18} = 1,$$

$$y = \frac{\begin{vmatrix} 2 & -4 \\ 6 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix}} = \frac{4+24}{-4+18} = 2.$$

4.1.2 Properties of Second-Order Determinants

There are many general properties of determinants that will be discussed in later sections. At this moment we want to list a few which we need in the following discussion of third-order determinant. For a second-order determinant, these properties are almost self-evident from its definition. Although they are generally valid for nth order determinant, at this point we only need them to be valid for second-order determinant to continue our discussion:

1. If the rows and columns are interchanged, the determinant is unaltered,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{4.7}$$

2. If two columns (or two rows) are interchanged, the determinant changes sign,

$$\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1) = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \tag{4.8}$$

3. If each element in a column (or in a row) is multiplied by m, the determinant is multiplied by m,

$$\begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix} = ma_1b_2 - ma_2b_1 = m(a_1b_2 - a_2b_1) = m\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

4. If each element of a column (or of a row) is sum of two terms, the determinant equals the sum of the two corresponding determinants,

$$\begin{vmatrix} (a_1+c_1) & b_1 \\ (a_2+c_2) & b_2 \end{vmatrix} = (a_1+c_1)b_2 - (a_2+c_2)b_1 = a_1b_2 - a_2b_1 + c_1b_2 - c_2b_1$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}.$$

4.1.3 Solution of Three Linear Equations

Now suppose we want to solve a system of three equations

$$a_1x + b_1y + c_1z = d_1, \tag{4.9}$$

$$a_2x + b_2y + c_2z = d_2, \tag{4.10}$$

$$a_3x + b_3y + c_3z = d_3. \tag{4.11}$$

First we can solve for y and z in terms of x. Writing (4.10) and (4.11) as

$$b_2y + c_2z = d_2 - a_2x,$$

$$b_3y + c_3z = d_3 - a_3x,$$

then in analogy to (4.5) and (4.6), we can express y and z as

$$y = \frac{\begin{vmatrix} (d_2 - a_2 x) & c_2 \\ (d_3 - a_3 x) & c_2 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}, \tag{4.12}$$

$$z = \frac{\begin{vmatrix} b_2 & (d_2 - a_2 x) \\ b_3 & (d_3 - a_3 x) \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}. \tag{4.13}$$

Substituting these two expressions into (4.9) and then multiplying the entire equation by

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$

we have

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} x + b_1 \begin{vmatrix} (d_2 - a_2 x) & c_2 \\ (d_3 - a_3 x) & c_3 \end{vmatrix} + c_1 \begin{vmatrix} b_2 & (d_2 - a_2 x) \\ b_3 & (d_3 - a_3 x) \end{vmatrix} = d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}. \tag{4.14}$$

By properties 3 and 4, this equation becomes

$$\begin{aligned} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} x &+ b_1 \left\{ \begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} x \right\} \\ &+ c_1 \left\{ \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} - \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix} x \right\} = d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}. \end{aligned} \tag{4.15}$$

It follows:

$$Dx = N_x, \tag{4.16}$$

where

$$N_x = d_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix}, \tag{4.17}$$

and

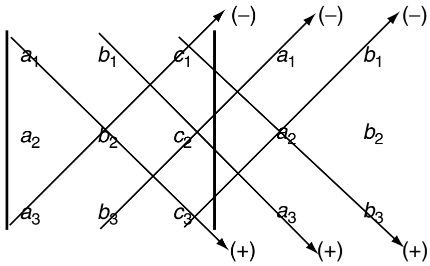
$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} b_2 & a_2 \\ b_3 & a_3 \end{vmatrix}. \tag{4.18}$$

Expanding the second-order determinants, (4.18) leads to

$$D = a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 a_3 c_2 - c_1 b_2 a_3 + c_1 b_3 a_2. \tag{4.19}$$

To express these six terms in a more systematic way, we introduce a third-order determinant as a short hand notation for (4.19)

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{4.20}$$



$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Fig. 4.2. A schematic diagram for a third-order determinant

A useful device for evaluating a third-order determinant is as follows. We write down the determinant column by column, after the third column, we repeat the first, then the second column, creating a 3×5 array of numbers. We can form a product of three elements along each of the three diagonals going from upper left to lower right. These products carry a positive sign. Similarly, three products can be formed along the diagonals from lower left to upper right. These three latter products carry a minus sign. The value of the determinant is equal to the sum of these six terms. This is shown in the diagram (Fig. 4.2).

This is seen to be exactly equal to the six terms in (4.19).

Using the determinant notation, one can easily show that N_x in (4.17) is equal to

$$N_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}. \tag{4.21}$$

Therefore

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Similarly we can define

$$N_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad N_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

and show

$$y = \frac{N_y}{D}, \qquad z = \frac{N_z}{D}.$$

The determinant in the denominator D is called the determinant of the coefficients. It is simply formed with the array of the coefficients on the left-hand sides of (4.9)–(4.11). To find the numerator determinant N_x, start with D, erase the x coefficients a_1, a_2, and a_3, and replace them with the constants d_1, d_2, and d_3 from the right-hand sides of the equations. Similarly we replace the y coefficients in D with the constant terms to find N_y, and the z coefficients in D with the constants to find N_z.

Example 4.1.2. Find the solution of

$$\begin{aligned} 3x + 2y + z &= 11, \\ 2x + 3y + z &= 13, \\ x + y + 4z &= 12. \end{aligned}$$

Solution 4.1.2.

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 36 + 2 + 2 - 3 - 3 - 16 = 18,$$

$$N_x = \begin{vmatrix} 11 & 2 & 1 \\ 13 & 3 & 1 \\ 12 & 1 & 4 \end{vmatrix} = 132 + 24 + 13 - 36 - 11 - 104 = 18,$$

$$N_y = \begin{vmatrix} 3 & 11 & 1 \\ 2 & 13 & 1 \\ 1 & 12 & 4 \end{vmatrix} = 156 + 11 + 24 - 13 - 36 - 88 = 54,$$

$$N_z = \begin{vmatrix} 3 & 2 & 11 \\ 2 & 3 & 13 \\ 1 & 1 & 12 \end{vmatrix} = 108 + 26 + 22 - 33 - 39 - 48 = 36.$$

Thus

$$x = \frac{18}{18} = 1, \quad y = \frac{54}{18} = 3, \quad z = \frac{36}{18} = 2.$$

Clearly, with determinant notation, the results can be given in a systematic way. While this procedure is still valid for systems of more than three equations, as we shall see in the section on Cramer's rule, but the diagonal scheme of expanding the determinants shown in this section is generally correct only for determinants of second- and third-orders. For determinants of higher order, we must pay attention to the formal definition of determinants.

4.2 General Definition of Determinants

4.2.1 Notations

Before we present the general definition of an arbitrary order determinant, let us write the third-order determinant in a more systematic way. Equations (4.19) and (4.20) can be written in the following form:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} a_i b_j c_k, \tag{4.22}$$

where

$$\begin{aligned} &\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \\ &\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1, \\ &\varepsilon_{ijk} = 0 \quad \text{for all others.} \end{aligned} \tag{4.23}$$

Writing out term by term the right-hand side of (4.22), one can readily verify that the six nonvanishing terms are exactly the same as in (4.19).

In order to generalize this definition for a nth order determinant, let us examine the triple sum more closely. First we note that $\varepsilon_{ijk} = 0$ if any two of the three indices i, j, k are equal, e.g., $\varepsilon_{112} = 0$, $\varepsilon_{333} = 0$. Eliminating those terms, (4.22) is a particular linear combination of six products, each product contains one and only one element from each row and from each column. Each product carries either a positive or a negative sign. The arrangements of (i, j, k) in the positive products are either in the normal order of $(1, 2, 3)$, or are the results of an even number of interchanges between two adjacent numbers of the normal order. Those in the negative products are the results of an odd number of interchanges in the normal order. For example, it takes two interchanges to get $(2, 3, 1)$ from $(1, 2, 3)$ [123 (interchange 12)→213 (interchange 13)→231], and $a_2b_3c_1$ is positive ($\varepsilon_{231} = 1$); it takes only one interchange to get $(1, 3, 2)$ from $(1, 2, 3)$ [123 (interchange 23)→132], and $a_1b_3c_2$ is negative ($\varepsilon_{132} = -1$). The diagram (Fig. 4.3) can help us to find out the value of ε_{ijk} quickly. If a set of indices goes in the clockwise direction, it gives a positive one (+1), if it goes in the counterclockwise direction, it gives a negative one (−1).

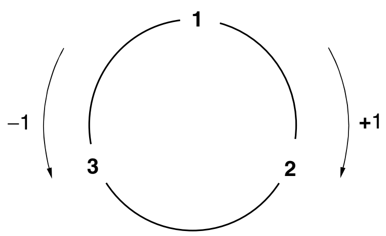


Fig. 4.3. Levi-Civita symbol ε_{ijk} where i, j, k take the value of $1, 2$, or 3. If the set of indices goes clockwise, $\varepsilon_{ijk} = +1$, if counterclockwise, $\varepsilon_{ijk} = -1$

These properties are characterized by the Levi-Civita symbol $\varepsilon_{i_1 i_2 \cdots i_n}$, which is defined as follows:

$$\varepsilon_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & \text{if} \quad (i_1, i_2, \ldots, i_n) \text{ is an even permutation} \\ & \quad \text{of the normal order } (1, 2, \ldots, n) \\ -1 & \text{if} \quad (i_1, i_2, \ldots, i_n) \text{ is an odd permutation} \\ & \quad \text{of the normal order } (1, 2, \ldots, n) \\ 0 & \quad \text{if any index is repeated.} \end{cases}$$

An even permutation means that an even number of pairwise interchanges of adjacent numbers is needed to obtain the given permutation from the normal order, and an odd permutation is associated with an odd number of pairwise interchanges. As we have shown, $(2, 3, 1)$ is an even permutation, and $(1, 3, 2)$ is an odd permutation.

An easy way to determine whether a given permutation is even or odd is to write out the normal order and write the permutation directly below it. Then connect corresponding numbers in these two arrangements with line segments, and count the number of intersections between pairs of these lines. If the number of intersections is even, then the given permutation is even. If the number of intersections is odd, then the permutation is odd. For example, to find the permutation $(2, 3, 4, 1)$, we write out the normal order and permutation in the diagram (Fig. 4.4, we call it "permutation diagram"):

There are three intersections. Therefore the permutation is odd and $\varepsilon_{2341} = -1$. The reason this scheme is valid is because of the following. Starting with the smallest number that is not directly below the same number, an exchange of this number with the number to its left will eliminate one intersection. In the earlier example, after the interchange between 1 and 4, only two intersections remain. Clearly two more interchanges will eliminate all intersections and return the permutation to the normal order. Thus three intersections indicate three interchanges are needed. Therefore the permutation is odd.

When we count the number of intersections, we are counting the intersections of pairs of lines. Therefore one should avoid to have more than two lines intersecting at a point. The lines joining the corresponding numbers need not to be straight lines.

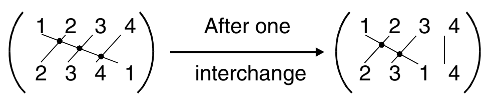


Fig. 4.4. Permutation diagram. The permutation is written directly below the normal order. The number of intersections between pairs of lines connecting the corresponding numbers is equal to the number of interchanges needed to obtain the permutation from the normal order. This diagram shows that one intersection point represents one interchange between two adjacent members

Example 4.2.1. What is the value of the the Levi-Civita symbol $\varepsilon_{1357246}$?

Solution 4.2.1. There are six intersections in Fig. 4.5, therefore the permutation is even and $\varepsilon_{1357246} = 1$.

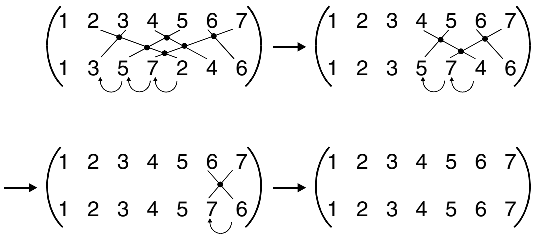


Fig. 4.5. In this diagram, six intersections represent that six interchanges are needed to obtain the permutation 1357246 from the normal order 1234567

4.2.2 Definition of a *n*th Order Determinant

In discussing a general nth order determinant, it is convenient to use the double-subscript notation. Each element of the determinant is represented by the symbol a_{ij}. The subscripts ij indicate that it is the element at ith row and jth column. With this notation, $a_1b_2c_3$ becomes $a_{11}a_{22}a_{33}$; $a_2b_3c_1$ becomes $a_{21}a_{32}a_{13}$, and $a_ib_jc_k$ becomes $a_{i1}a_{j2}a_{k3}$. The determinant itself is denoted by a variety of symbols. The following notations are all equivalent:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}| = |A| = \det|A| = D_n. \tag{4.24}$$

The value of the determinant is given by

$$D_n = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} \varepsilon_{i_1i_2\cdots i_n} a_{i_1 1}a_{i_2 2}\cdots a_{i_n n}. \tag{4.25}$$

This equation is the formal definition of a nth order determinant. Clearly, for $n = 3$, it reduces to (4.22). Note that for a nth order determinant, there are $n!$ possible products because i_1 can take one of n values, i_2 cannot repeat i_1, so it can take only one of $n-1$ values, and so on. We can think of evaluating a determinant in terms of three steps. (1) Take $n!$ products of n elements such that in each product there is one and only one element from each row and one and only one element from each column. (2) Attach a positive (+) sign to the product if the row subscripts are an even permutation of the column subscripts, and a minus sign $(-)$ if an odd perturbation. (3) Sum over $n!$ products with these signs.

Stated in this way, it is clear that the definition of a determinant is symmetrical between the rows and columns. The determinant (4.25) can just as well be written as

$$D_n = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \varepsilon_{i_1 i_2 \cdots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}. \tag{4.26}$$

It follows that any theorem about the determinant which involves the rows is also true for the columns, and vice-versa.

Another property that is clear from this definition is this. If any two rows are interchanged, the determinant changes sign. First it is easy to show that if the two rows are adjacent to each other, this is the case. This follows from the fact that an interchange of two adjacent rows corresponds to an interchange of two adjacent row indices in the Levi-Civita symbol. It changes an even permutation into an odd permutation, and vice versa. Therefore it introduces a minus sign to all the products.

Now suppose the row indices i and j are not adjacent to each other and there are n indices between them:

$$i \;\; a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_n \;\; j\,.$$

To bring j to the left requires $n+1$ adjacent interchanges leading to

$$j \;\; i \;\; a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_n\,.$$

Now bringing i to the right requires n adjacent interchanges leading to

$$j \;\; a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_n \;\; i\,.$$

Therefore all together there are $2n+1$ number of adjacent interchanges leading to the interchange of i and j. Since $2n+1$ is an odd integer, this brings in an overall minus sign.

Example 4.2.2. Let

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

use (4.25) to (a) expand this second-order determinant, (b) show explicitly that the interchange of the two rows changes its sign.

Solution 4.2.2. (a) According to (4.25)

$$D_2 = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2},$$

$$\begin{aligned}
i_1 = 1,\ i_2 = 1: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{11} a_{11} a_{12} \\
i_1 = 1,\ i_2 = 2: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{12} a_{11} a_{22} \\
i_1 = 2,\ i_2 = 1: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{21} a_{21} a_{12} \\
i_1 = 2,\ i_2 = 2: &\quad \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = \varepsilon_{22} a_{21} a_{22}.
\end{aligned}$$

Since $\varepsilon_{11} = 0,\ \varepsilon_{12} = 1,\ \varepsilon_{21} = -1,\ \varepsilon_{22} = 0$, the double sum gives the second-order determinant as

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} = a_{11} a_{22} - a_{21} a_{12}.$$

(b) To express the interchange of two rows, we can simply replace $a_{i_1 1} a_{i_2 2}$ in the double sum with $a_{i_2 1} a_{i_1 2}$ (i_1 and i_2 are interchanged), thus

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_2 1} a_{i_1 2}.$$

Since i_1 and i_2 are running indices, we can rename i_1 as j_2 and i_2 as j_1, so

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_2 1} a_{i_1 2} = \sum_{j_2=1}^{2} \sum_{j_1=1}^{2} \varepsilon_{j_2 j_1} a_{j_1 1} a_{j_2 2} = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \varepsilon_{j_2 j_1} a_{j_1 1} a_{j_2 2}.$$

The last expression is identical with that of the original determinant except the indices of the Levi-Civita symbol are interchanged.

$$\begin{aligned}
\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} &= \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \varepsilon_{j_2 j_1} a_{j_1 1} a_{j_2 2} \\
&= \varepsilon_{11} a_{11} a_{12} + \varepsilon_{21} a_{11} a_{22} + \varepsilon_{12} a_{21} a_{12} + \varepsilon_{22} a_{21} a_{22} \\
&= -a_{11} a_{22} + a_{21} a_{12} = - \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.
\end{aligned}$$

This result can, of course, be obtained by inspection. We have taken the risk of stating the obvious. Hopefully, this step by step approach will remove any uneasy feeling of working with indices.

4.2.3 Minors, Cofactors

Let us return to (4.18), written in the double-subscript notation this equation becomes

$$\begin{aligned}
D_3 &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},
\end{aligned} \tag{4.27}$$

where we have interchanged the two columns of the last second-order determinant of (4.18) and changed the sign. It is seen that

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

is the second-order determinant formed by removing the first row and first column from the original third-order determinant D_3. We call it M_{11} the minor complementary to a_{11}. In general, the minor M_{ij} complementary to a_{ij} is defined as the $(n-1)$th order determinant formed by deleting the ith row and the jth column from the original nth order determinant D_n. The cofactor C_{ij} is defined as $(-1)^{i+j}M_{ij}$.

Example 4.2.3. Find the value of the minors M_{11}, M_{23} and the cofactors C_{11}, C_{23} of the determinant

$$D_4 = \begin{vmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & 2 & 3 & 1 \end{vmatrix}.$$

Solution 4.2.3.

$$M_{11} = \begin{vmatrix} * & * & * & * \\ * & 2 & 5 & 0 \\ * & 0 & -2 & 2 \\ * & 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 0 \\ 0 & -2 & 2 \\ 2 & 3 & 1 \end{vmatrix}; \quad M_{23} = \begin{vmatrix} 2 & -1 & * & 3 \\ * & * & * & * \\ 1 & 0 & * & 2 \\ 4 & 2 & * & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & 2 & 1 \end{vmatrix}.$$

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 5 & 0 \\ 0 & -2 & 2 \\ 2 & 3 & 1 \end{vmatrix}; \quad C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & 2 & 1 \end{vmatrix}.$$

4.2.4 Laplacian Development of Determinants by a Row (or a Column)

With these notations, (4.27) becomes

$$D_3 = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = \sum_{j=1}^{3}(-1)^{1+j}a_{1j}M_{1j} \tag{4.28}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \sum_{k=1}^{3} a_{1k}C_{1k}. \tag{4.29}$$

This is known as the Laplace development of the third-order determinant on elements of the first row. It turns out this is not limited to the third-order determinant. It is a fundamental theorem that determinants of any order can be evaluated by a Laplace development on any row or column

$$D_n = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any } i, \tag{4.30}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any } j. \tag{4.31}$$

The proof may be given by induction and is based on the definition of the determinant. According to (4.25), a determinant is the sum of all the $n!$ products which are formed by taking exactly one element from each row and each column and multiplying by 1 or -1 in accordance with the Levi-Civita rule.

Now the minor M_{ij} of a nth order determinant is a $(n-1)$th determinant. It is a sum of $(n-1)!$ products. Each product has one element from each row and each column except the ith row and jth column. It is then clear that $\sum_{j=1}^{n} k_{ij} a_{ij} M_{ij}$ is a sum of $n(n-1)! = n!$ products, and each product is formed with exactly one element from each row and each column. It follows that, with the appropriate choice of k_{ij}, the determinant can be written in a row expansion

$$D_n = \sum_{j=1}^{n} k_{ij} a_{ij} M_{ij} \tag{4.32}$$

or in a column expansion

$$D_n = \sum_{i=1}^{n} k_{ij} a_{ij} M_{ij}. \tag{4.33}$$

The Laplace development will follow if we can show:

$$k_{ij} = (-1)^{i+k}.$$

First let us consider all the terms in (4.25) containing a_{11}. In these terms $i_1 = 1$. We note that if $(1, i_2, i_3, \ldots, i_n)$ is an even (or odd) permutation of $(1, 2, 3, \ldots, n)$, it means $(i_2, i_3, \ldots, i_n)$ is an even (or odd) permutation of $(2, 3, \ldots, n)$. The number of intersections in the following two "permutation diagrams" are obviously the same:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}; \qquad \begin{pmatrix} 2 & 3 & \cdots & n \\ i_2 & i_3 & \cdots & i_n \end{pmatrix},$$

therefore

$$\varepsilon_{1 i_2 \cdots i_n} = \varepsilon_{i_2 \cdots i_n}.$$

So terms containing a_{11} sum to

$$\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n}\varepsilon_{1i_2\cdots i_n}a_{11}a_{i_22}\cdots a_{i_nn}=a_{11}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n}\varepsilon_{i_2\cdots i_n}a_{i_22}\cdots a_{i_nn},$$

which is simply $a_{11}M_{11}$, where M_{11} is the minor of a_{11}. On the other hand, according to (4.32), all the terms containing a_{11} sum to $k_{11}a_{11}M_{11}$. Therefore

$$k_{11}=+1.$$

Next consider the terms in (4.25) which contain a particular element a_{ij}. If we interchange the ith row with the one above it, the determinant changes sign. If we move the row up in this way $(i-1)$ times, the ith row will have moved up into the first row, and the order of the other rows is not changed. The process will change the sign of the determinant $(i-1)$ times. In a similar way, we can move the jth column to the first column without change the order of the other columns. Then the element a_{ij} will be in the top left corner of the determinant, in the place of a_{11}, and the sign of the determinant has change $(i-1+j-1)$ times. That is

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j-2}\begin{vmatrix} a_{ij} & a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{1j} & a_{11} & a_{12} & \cdots & a_{1n} \\ a_{2j} & a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{nj} & a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}.$$

In the rearranged determinant, a_{ij} is in the place of a_{11}, thus the sum of all the terms containing a_{ij} is equal to $a_{ij}M_{ij}$. But there is a factor $(-1)^{i+j-2}$ in front of the rearranged determinant. Therefore the terms containing a_{ij} in the right-hand side of the equation sum to $(-1)^{i+j-2}a_{ij}M_{ij}$. On the other hand, according to (4.32), all the terms containing a_{ij} in the determinant of the left-hand side of the equation sum to $k_{ij}a_{ij}M_{ij}$. Therefore,

$$k_{ij}=(-1)^{i+j-2}=(-1)^{i+j}. \tag{4.34}$$

This completes the proof of the Laplace development, which is very important in both theory and computation of determinants. It is useful to keep in mind that k_{ij} forms a checkboard pattern:

$$\begin{vmatrix} +1 & -1 & +1 & & & \\ -1 & 1 & -1 & & & \\ +1 & -1 & +1 & & & \\ & & & \cdots & & \\ & & & & +1 & -1 \\ & & & & -1 & +1 \end{vmatrix}.$$

Example 4.2.4. Find the value of the determinant

$$D_3 = \begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix}$$

by (a) a Laplace development on the first row; (b) a Laplace development on the second row; (c) a Laplace development on the first column.

Solution 4.2.4.

(*a*)
$$\begin{aligned} D_3 &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 3\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\ &= 3(4+3) + 2(2+12) + 2(1-8) = 35. \end{aligned}$$

(b)

$$\begin{aligned} D_3 &= -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23} \\ &= -1\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix} \\ &= -(-4-2) + 2(6-8) + 3(3+8) = 35. \end{aligned}$$

(c)

$$\begin{aligned} D_3 &= a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \\ &= 3\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} \\ &= 3(4+3) - (-4-2) + 4(6-4) = 35. \end{aligned}$$

Example 4.2.5. Find the value of the *triangular determinant*

$$D_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix}.$$

Solution 4.2.5.

$$D_n = a_{11}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots \\ 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

4.3 Properties of Determinants

By mathematical induction, we can now show that properties 1 to 4 of second-order determinants are generally valid for nth order determinants. Based on the fact that it is true for $(n-1)$th order determinants, we will show that it must also be true for nth order determinants. All properties of the determinant can be derived directly from its definition of (4.25). However, in this section, we will demonstrate them with Laplace expansions.

1. The value of the determinant remains the same if rows and columns are interchanged.

   Let the Laplace expansion of D_n on elements of the first row be

$$D_n = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}. \tag{4.35}$$

Let D_n^{T} (known as the transpose of D_n) be the nth order determinant formed by interchanging rows and columns of the determinant D_n. The Laplace expansion of D_n^{T} on elements of the first column (which are elements of the first row of D_n) is then given by

$$D_n^{\mathrm{T}} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}^{\mathrm{T}}, \tag{4.36}$$

where M_{1j}^{T} is the minor complement to a_{1j}, and is equal to the determinant M_{1j} with rows and columns interchanged. In the case of $n = 3$, the minors are second-order determinants. By (4.7), $M_{1j}^{\mathrm{T}} = M_{1j}$. Therefore $D_3 = D_3^{\mathrm{T}}$. This process can be carried out, one step at a time, to any n. Therefore we conclude

$$D_n = D_n^{\mathrm{T}}. \tag{4.37}$$

2. The determinant changes sign if any two columns (or any two rows) are interchanged.

First we will verify this property for the third-order determinant D_3. Let E_3 be the determinant obtained by interchanging two columns of D_3. Suppose column k is not one of those exchanged. Using Laplace development to expand D_3 and E_3 by their kth column, we have

$$D_3 = \sum_{i=1}^{3} (-1)^{i+k} a_{ik} M_{ik}; \tag{4.38}$$

$$E_3 = \sum_{i=1}^{3} (-1)^{i+k} a_{ik} M'_{ik}, \tag{4.39}$$

where M'_{ik} is a second-order determinant and is equal to M_{ik} with the two columns interchanged. By (4.8), $M'_{ik} = -M_{ik}$. Hence $E_3 = -D_3$. Now by mathematical induction, we assume this property holds for $(n-1)$th order determinants. The same procedure will show that this property also holds for determinants of nth order.

This property is called antisymmetric property. It is frequently used in quantum mechanics in the construction of an antisymmetric many particle wave functions.

3. If each element in a column (or in a row) is multiplied by a constant m, the determinant is multiplied by m.

This property follows directly from the Laplacian expansion. If the ith column is multiplied by m, this property can be shown in the following way:

$$\begin{vmatrix} a_{11} & \cdots & ma_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & ma_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & ma_{ni} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} ma_{ji}C_{ji} = m\sum_{j=1}^{n} a_{ji}C_{ji}$$

$$= m\begin{vmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix}. \tag{4.40}$$

4. If each element in a column (or in a row) is a sum of two terms, the determinant equals the sum of the two corresponding determinants.

If the ith column is a sum of two terms, we can expand the determinant on elements of the ith column

$$\begin{vmatrix} a_{11} & \cdots & a_{1i}+b_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i}+b_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni}+b_{ni} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n}(a_{ji}+b_{ji})C_{ji} = \sum_{j=1}^{n} a_{ji}C_{ji} + \sum_{j=1}^{n} b_{ji}C_{ji}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & b_{1i} & \cdots & a_{1n} \\ a_{21} & \cdots & b_{2i} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & b_{ni} & \cdots & a_{nn} \end{vmatrix}. \tag{4.41}$$

From these four properties, one can derive many others. For example:

5. If two columns (or two rows) are the same, the determinant is zero.

This follows from the antisymmetric property. If we exchange the two identical columns, the determinant will obviously remain the same. Yet the antisymmetric property requires the determinant to change sign. The only number that is equal to its negative self is zero. Therefore the determinant must be zero.

6. The value of a determinant is unchanged if a multiple of one column is added to another column (or if a multiple of one row is added to another row).

   Without loss of generality, this property can be expressed as follows:

$$\begin{vmatrix} a_{11}+ma_{12} & a_{12} & \cdots & a_{1n} \\ a_{21}+ma_{22} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}+ma_{n2} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} ma_{12} & a_{12} & \cdots & a_{1n} \\ ma_{22} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ ma_{n2} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + m \begin{vmatrix} a_{12} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n2} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}. \quad (4.42)$$

   The first equal sign is by property 4, the second equal sign is because of property 3, and the last equal sign is due to property 5.

Example 4.3.1. Show that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Solution 4.3.1.

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & (bc+a^2) \\ 1 & b & (ac+ab) \\ 1 & c & (ab+ac) \end{vmatrix} = \begin{vmatrix} 1 & a & (bc+a^2+ba) \\ 1 & b & (ac+ab+b^2) \\ 1 & c & (ab+ac+bc) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & (bc+a^2+ba+ca) \\ 1 & b & (ac+ab+b^2+cb) \\ 1 & c & (ab+ac+bc+c^2) \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & (bc+ba+ca) \\ 1 & b & (ac+ab+cb) \\ 1 & c & (ab+ac+bc) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + (ab+bc+ca) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

First we multiply each element of the second column by a and add to the third column. For the second equal sign, we multiply the second column by b and add to the third column. Do the same thing except multiplying by c for the third equal sign. The fourth equal sign is due to property 4. The fifth equal sign is due to property 3. And lastly, the determinant with two identical column vanishes.

Example 4.3.2. Evaluate the determinant

$$D_n = \begin{vmatrix} 1+a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & 1+a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & 1+a_3 & \cdots & a_n \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ a_1 & a_2 & a_3 & \cdots & 1+a_n \end{vmatrix}.$$

Solution 4.3.2. Adding column 2, column 3, all the way to column n to column 1, we have

$$D_n = \begin{vmatrix} 1+a_1+a_2+a_3+\cdots+a_n & a_2 & a_3 & \cdots & a_n \\ 1+a_1+a_2+a_3+\cdots+a_n & 1+a_2 & a_3 & \cdots & a_n \\ 1+a_1+a_2+a_3+\cdots+a_n & a_2 & 1+a_3 & \cdots & a_n \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1+a_1+a_2+a_3+\cdots+a_n & a_2 & a_3 & \cdots & 1+a_n \end{vmatrix}$$

$$= (1+a_1+a_2+a_3+\cdots+a_n) \begin{vmatrix} 1 & a_2 & a_3 & \cdots & a_n \\ 1 & 1+a_2 & a_3 & \cdots & a_n \\ 1 & a_2 & 1+a_3 & \cdots & a_n \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & a_2 & a_3 & \cdots & 1+a_n \end{vmatrix}.$$

Multiplying row 1 by -1 and add it to row 2, and then add it to row 3, and so on

$$D_n = (1+a_1+a_2+a_3+\cdots+a_n) \begin{vmatrix} 1 & a_2 & a_3 & \cdots & a_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= (1+a_1+a_2+a_3+\cdots+a_n).$$

Example 4.3.3. Evaluate the following determinants (known as *Vandermonde determinant*):

$$\text{(a)} \quad D_3 = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}, \qquad \text{(b)} \quad D_n = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

Solution 4.3.3. (a) Method I.

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & (x_2 - x_1) & (x_2^2 - x_1^2) \\ 0 & (x_3 - x_1) & (x_3^2 - x_1^2) \end{vmatrix} = (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & (x_2 + x_1) \\ 1 & (x_3 + x_1) \end{vmatrix}$$
$$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

Method II. D_3 is a polynomial in x_1 and it vanishes when $x_1 = x_2$, since then the first two rows are the same. Hence it is divisible by $(x_1 - x_2)$. Similarly, it is divisible by $(x_2 - x_3)$ and $(x_3 - x_1)$. Therefore

$$D_3 = k\,(x_1 - x_2)\,(x_1 - x_3)\,(x_2 - x_3)\,.$$

Furthermore, since D_3 is of degree 3 in x_1, x_2, x_3, k must be a constant. The coefficient of the term $x_2 x_3^2$ in this expression is $k(-1)(-1)^2$. On the other hand, the diagonal product of the D_3 is $+x_2 x_3^2$. Comparing them shows that $k(-1)(-1)^2 = 1$. Therefore $k = -1$ and

$$D_3 = -\,(x_1 - x_2)\,(x_1 - x_3)\,(x_2 - x_3) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

(b) With the same reason as in Method II of (a),

$$D_n = k(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n)(x_2 - x_3)\cdots(x_2 - x_n)\cdots(x_{n-1} - x_n).$$

The coefficient of the term $x_2 x_3^2 \cdots x_n^{n-1}$ in this expression is $k(-1)(-1)^2 \cdots (-1)^{n-1}$. Compare this with the diagonal product of D_n, we have

$$1 = k(-1)(-1)^2 \cdots (-1)^{n-1} = k(-1)^{1+2+3+\cdots+(n-1)}.$$

Since

$$1 + 2 + 3 + \cdots + (n-1) = \frac{1}{2} n(n-1),$$

therefore

$$D_n = (-1)^{n(n-1)/2}(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n)(x_2 - x_3)\cdots(x_2 - x_n)\cdots (x_{n-1} - x_n).$$

Example 4.3.4. Pivotal Condensation. Show that

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}.$$

Clearly, a_{11} must be nonzero. If it is zero, then the first row (or first column) must be exchanged with another row (or another column), so that $a_{11} \neq 0$.

Solution 4.3.4.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & a_{11}a_{12} & a_{11}a_{13} \\ a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{vmatrix}$$

$$= \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & (a_{11}a_{12} - a_{11}a_{12}) & (a_{11}a_{13} - a_{11}a_{13}) \\ a_{21} & (a_{11}a_{22} - a_{21}a_{12}) & (a_{11}a_{23} - a_{21}a_{13}) \\ a_{31} & (a_{11}a_{32} - a_{31}a_{12}) & (a_{11}a_{33} - a_{31}a_{13}) \end{vmatrix}$$

$$= \frac{1}{a_{11}^2} \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & (a_{11}a_{22} - a_{21}a_{12}) & (a_{11}a_{23} - a_{21}a_{13}) \\ a_{31} & (a_{11}a_{32} - a_{31}a_{12}) & (a_{11}a_{33} - a_{31}a_{13}) \end{vmatrix}$$

$$= \frac{1}{a_{11}} \begin{vmatrix} (a_{11}a_{22} - a_{21}a_{12}) & (a_{11}a_{23} - a_{21}a_{13}) \\ (a_{11}a_{32} - a_{31}a_{12}) & (a_{11}a_{33} - a_{31}a_{13}) \end{vmatrix}$$

$$= \frac{1}{a_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}.$$

This method can be applied to reduce a nth order determinant to a $(n-1)$th order determinant and is known as pivotal condensation. It may not offer any advantage for hand calculation, but it is useful in evaluating determinants with computers.

4.4 Cramer's Rule

4.4.1 Nonhomogeneous Systems

Suppose we have a set of n equations and n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ \cdots\cdots\cdots\cdots &= \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n. \end{aligned} \tag{4.43}$$

The constants $d_1, d_2, \ldots, d_n$ on the right-hand side are known as nonhomogeneous terms. If they are not all equal to zero, the set of equations is known as a nonhomogeneous system. The problem is to find $x_1, x_2, \ldots, x_n$ to satisfy this set of equations. We will see by using the properties of determinants, this set of equations can be readily solved for any n.

Forming the determinant of the coefficients and then multiplying by x_1, with the help of property 3 we have

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1}x_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

We multiply the second column of the right-hand side determinant by x_2 and add it to the first column, and then multiply the third column by x_3 and add it to the first column and so on. According to property 6, the determinant is unchanged

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n & a_{22} & \cdots & a_{2n} \\ \cdots & \cdot & \cdots & \cdot \\ a_{n1}x_1 + a_{n2}x_2 \cdots + a_{nn}x_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Replacing the first column of the right-hand side determinant with the constants of the right-hand side of (4.43), we obtain

$$x_1 \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} d_1 & a_{12} & \cdots & a_{1n} \\ d_2 & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ d_3 & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Clearly if we multiply the determinant of the coefficients by x_2, we can analyze the second column of the determinant in the same way. In general

$$x_i D_n = N_i, \qquad 1 \le i \le n, \tag{4.44}$$

where D_n is the determinant of the coefficients

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and N_i is the determinant obtained by replacing the ith column of D_n by the nonhomogeneous terms

$$N_i = \begin{vmatrix} a_{11} & \cdots & a_{1i-1} & d_1 & a_{1i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2i-1} & d_2 & a_{2i+1} & \cdots & a_{2n} \\ \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & \cdots & a_{ni-1} & d_n & a_{ni+1} & \cdots & a_{nn} \end{vmatrix}. \tag{4.45}$$

Thus if the determinant of the coefficients is not zero, the system has a unique solution

$$x_i = \frac{N_i}{D_n}, \quad 1 \le i \le n. \tag{4.46}$$

This procedure is known as Cramer's rule. For the special cases of $n = 2$ and $n = 3$, the results are, of course, identical to what we derived in the first section. Cramer's rule is very important in the development of the theory of determinants and matrices. However, to use it for solving a set of equations with large n, it is not very practical. Either because the amount of computations is so large and/or because the demand of numerical accuracy is so high with this method, even with high speed computers it may not be possible to carry out such calculations. There are other techniques to solve that kind of problems, such as the Gauss-Jordan elimination method which we will discuss in the chapter on matrix theory.

4.4.2 Homogeneous Systems

Now if $d_1, d_2, \ldots, d_n$ in the right-hand side of (4.43) are all zero, that is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots &= \cdot \\ a_{n1}x_1 + a_{n2}x_2 \cdots + a_{nn}x_n &= 0, \end{aligned}$$

the set of equations is known as a homogeneous system. In this case, all N_i's in (4.45) are equal to zero. If $D_n \neq 0$, then the only solution by (4.46) is a trivial one, namely $x_1 = x_2 = \cdots = x_n = 0$. On the other hand, if D_n is equal to zero, then it is clear from (4.44), x_i do not have to be zero. Hence a homogeneous system can have a nontrivial solution only if the coefficient determinant is equal to zero. Conversely, one can show that if

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0, \tag{4.47}$$

then there is always a nontrivial solution of the homogeneous equations. For a 2×2 system, the existence of a solution can be shown by direct calculation. Then one can show by mathematical induction that the statement is true for any $n \times n$ system.

This simple fact has many important applications.

Example 4.4.1. For what values of λ do the equations

$$\begin{aligned} 3x + 2y &= \lambda x, \\ 4x + 5y &= \lambda y \end{aligned}$$

have a solution other than $x = y = 0$?

Solution 4.4.1. Moving the right-hand side to the left gives the homogeneous system

$$(3-\lambda)x + 2y = 0,$$
$$4x + (5-\lambda)y = 0.$$

For a nontrivial solution, the coefficient determinant must vanish:

$$\begin{vmatrix} 3-\lambda & 2 \\ 4 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 7 = (\lambda-1)(\lambda-7) = 0.$$

Thus the system has a nontrivial solution if and only if $\lambda = 1$ or $\lambda = 7$.

4.5 Block Diagonal Determinants

Frequently we encounter determinants with many zero elements and the nonzero elements which form square blocks along the diagonal. For example the following fifth-order determinant is a block diagonal determinant:

$$D_5 = |A| = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ * & * & a_{33} & a_{34} & a_{35} \\ * & * & a_{43} & a_{44} & a_{45} \\ * & * & a_{53} & a_{54} & a_{55} \end{vmatrix}.$$

In this section, we will show that

$$D_5 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix},$$

regardless the values the elements $*$ assume.

By definition

$$D_5 = \sum_{i_1=1}^{5} \sum_{i_2=1}^{5} \sum_{i_3=1}^{5} \sum_{i_4=1}^{5} \sum_{i_5=1}^{5} \varepsilon_{i_1 i_2 i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5}.$$

Since $a_{13} = a_{14} = a_{15} = a_{23} = a_{24} = a_{25} = 0$, all terms containing these elements can be excluded from the summation. Thus

$$D_5 = \sum_{i_1=1}^{5}\sum_{i_2=1}^{5}\sum_{i_3=3}^{5}\sum_{i_4=3}^{5}\sum_{i_5=3}^{5} \varepsilon_{i_1 i_2 i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5}.$$

Furthermore, the summation over i_1 and i_2 can be written as from 1 to 2, since $3, 4$, and 5 are taken up by i_3, i_4, or i_5, and the Levi-Civita symbol is equal to zero if any index is repeated. Hence

$$D_5 = \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=3}^{5}\sum_{i_4=3}^{5}\sum_{i_5=3}^{5} \varepsilon_{i_1 i_2 i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5}.$$

Under these circumstances, the permutation of i_1, i_2, i_3, i_4, i_5 can be separated into two permutations as schematically shown later:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ i_1 = 1,2 & i_2 = 1,2 & i_3 = 3,4,5 & i_4 = 3,4,5 & i_5 = 3,4,5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 \\ i_1 & i_2 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ i_3 & i_4 & i_5 \end{pmatrix}.$$

The entire permutation is even if the two separated permutations are both even or both odd. The permutation is odd if one of the separated permutations is even and the other is odd. Therefore

$$\varepsilon_{i_1 i_2 i_3 i_4 i_5} = \varepsilon_{i_1 i_2} \cdot \varepsilon_{i_3 i_4 i_5}.$$

It follows

$$\begin{aligned} D_5 &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=3}^{5}\sum_{i_4=3}^{5}\sum_{i_5=3}^{5} \varepsilon_{i_1 i_2} \cdot \varepsilon_{i_3 i_4 i_5} a_{i_1 1} a_{i_2 2} a_{i_3 3} a_{i_4 4} a_{i_5 5} \\ &= \sum_{i_1=1}^{2}\sum_{i_2=1}^{2} \varepsilon_{i_1 i_2} a_{i_1 1} a_{i_2 2} \cdot \sum_{i_3=3}^{5}\sum_{i_4=3}^{5}\sum_{i_5=3}^{5} \varepsilon_{i_3 i_4 i_5} a_{i_3 3} a_{i_4 4} a_{i_5 5} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix}. \end{aligned}$$

When the blocks are along the "antidiagonal" line, we can evaluate the determinant in a similar way, except we should be careful about its sign. For example,

$$\begin{vmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & * & * \\ a_{41} & a_{42} & * & * \end{vmatrix} = \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}, \tag{4.48}$$

and

$$\begin{vmatrix} 0 & 0 & 0 & a_{14} & a_{15} & a_{16} \\ 0 & 0 & 0 & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & * & * & * \\ a_{51} & a_{52} & a_{53} & * & * & * \\ a_{61} & a_{62} & a_{63} & * & * & * \end{vmatrix} = - \begin{vmatrix} a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{vmatrix} \cdot \begin{vmatrix} a_{14} & a_{15} & a_{16} \\ a_{24} & a_{25} & a_{26} \\ a_{34} & a_{35} & a_{36} \end{vmatrix}. \quad (4.49)$$

We can establish the result of (4.48) by changing it to a block diagonal determinant with an even number of interchanges between two rows. However, we need an odd number of interchanges between two rows to change (4.49) into a block diagonal determinant, therefore a minus sign.

Solution 4.5.1. *Example 4.5.1.* Evaluate

$$D_5 = \begin{vmatrix} 0 & 2 & 0 & 7 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}.$$

Solution 4.5.2.

$$D_5 = \begin{vmatrix} 0 & 2 & 0 & 7 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} \to (\text{Row}\,4 - \text{Row}\,2) = \begin{vmatrix} 0 & 2 & 0 & 7 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 5 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -2 \cdot 1 = -2.$$

4.6 Laplacian Developments by Complementary Minors

(This section can be skipped in the first reading.)

The Laplace expansion of D_3 by the elements of the third column is

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

The three second-order determinants are minors complementary to their respective elements. It is also useful to think that the three elements $a_{13},\ a_{23},\ a_{33}$ are complementary to their respective minors. Obviously the expansion can be written as

$$D_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} a_{33} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} a_{23} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} a_{13}. \tag{4.50}$$

In this way, it is seen that the determinant D_3 is equal to the sum of the signed products of all the second-order minors contained in the first two columns, each multiplied by its complementary element. In fact, any determinant D_n, even for $n > 3$, can be expanded in the same way, except the complementary element is of course another complementary minor. For example, for a 4th order determinant

$$D_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \tag{4.51}$$

six second-order minors can be formed from the first two columns. They are

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix}, \quad \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}.$$

Let us expand D_4 in terms of these six minors. First expanding D_4 by its first column, then expanding the four minors by their first columns, we have

$$D_4 = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}, \tag{4.52}$$

where

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{22} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - a_{32} \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} + a_{42} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}$$

$$C_{21} = - \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + a_{32} \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} - a_{42} \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix}$$

$$C_{31} = \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{12} \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} - a_{22} \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} + a_{42} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}$$

$$C_{41} = - \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} + a_{22} \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} - a_{32} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}.$$

Putting these cofactors back into (4.52) and collecting terms, we have

$$\begin{aligned}D_4 &= (a_{11}a_{22} - a_{21}a_{12}) \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - (a_{11}a_{32} - a_{31}a_{12}) \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} \\ &+(a_{11}a_{41} - a_{41}a_{12}) \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} + (a_{21}a_{32} - a_{31}a_{22}) \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} \\ &-(a_{21}a_{42} - a_{41}a_{22}) \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} + (a_{31}a_{42} - a_{41}a_{32}) \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}. \quad (4.53)\end{aligned}$$

Clearly,

$$\begin{aligned}D_4 = &\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} \\ &+ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} \\ &- \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}. \quad (4.54)\end{aligned}$$

If D_4 is a block diagonal determinant,

$$D_4 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{vmatrix},$$

then only the first term in (4.54) is nonzero, therefore

$$D_4 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix},$$

in agreement with the result derived in the last section.

If we adopt the following notation

$$A_{i_1 i_2, j_1 j_2} = \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{vmatrix}$$

and $M_{i_1 i_2, j_1 j_2}$ as the complementary minor to $A_{i_1 i_2, j_1 j_2}$, the determinant D_4 in (4.51) can be expanded in terms of the minors formed by the elements of any two columns,

$$D_4 = \sum_{i_1=1}^{3} \sum_{i_2>i_1}^{4} (-1)^{i_1+i_2+j_1+j_2} A_{i_1 i_2, j_1 j_2} M_{i_1 i_2, j_1 j_2}. \quad (4.55)$$

With $j_1 = 1,\ j_2 = 2$, it can be readily verified that (4.55) is, term by term, equal to (4.54). The proof of (4.55) goes the same way as in the Laplacian expansion by a row. First (4.55) is a linear combination of 4! products, each product has one element from each row and one from each column. The coefficients are either +1 or −1, depending on whether an even or odd number of interchange are needed to move i_1 to the first row, i_2 to the second row, and j_1 to the first column, j_2 to the second column, without changing the order of the rest of the elements. Obviously, the determinant can also be expanded in terms of the minors formed from any number of rows.

For a nth order determinant D_n, one can expand it in a similar way, not only in terms of second-order minors but also in terms of kth order minors with $k < n$. Of course, for $k = n - 1$, it reduces to the regular Laplacian development by a column. Following the same procedure of expanding D_4, one can show that

$$D_n = \sum_{(i)} (-1)^{i_1+i_2+\cdots+i_k+j_1+j_2+\cdots+j_k} A_{i_1 i_2 \cdots i_k, j_1 j_2 \cdots j_k} M_{i_1 i_2 \cdots i_k, j_1 j_2 \cdots j_k},$$

where the symbol $\sum_{(i)}$ indicates that the summation is taken over all possible permutations in the following way. The first set of subscripts $i_1 i_2 \ldots i_k$ is from n indices $12 \ldots n$ taken k at a time with the restriction $i_1 < i_2 \cdots < i_k$. The second set subscripts $j_1 j_2 \ldots j_k$ are chosen arbitrarily but remain fixed for each term of the expansion. This formula is general, but is seldom needed for the evaluation of a determinant.

Example 4.6.1. Evaluate

$$D_4 = \begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 2 & 5 \\ 2 & 1 & 1 & 3 \\ 1 & 3 & 0 & 2 \end{vmatrix}$$

by (a) expansion with minors formed from the first two columns, (b) expansion with minors formed from the second and fourth rows.

Solution 4.6.1. (a)

$$\begin{aligned} D_4 &= \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \\ &\quad + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} \\ &= -2 - 0 + 5 + 6 - 24 + 65 = 50. \end{aligned}$$

(b)

$$\begin{aligned}
D_4 &= (-1)^{2+4+1+2}\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + (-1)^{2+4+1+3}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \\
&+(-1)^{2+4+1+4}\begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + (-1)^{2+4+2+3}\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\
&+(-1)^{2+4+2+4}\begin{vmatrix} 0 & 5 \\ 3 & 2 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + (-1)^{2+4+3+4}\begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \\
&= -24 - 4 - 6 + 24 + 60 - 0 = 50.
\end{aligned}$$

4.7 Multiplication of Determinants of the Same Order

If $|A|$ and $|B|$ are determinants of order n, then the product

$$|A| \cdot |B| = |C|$$

is a determinant of the same order. Its elements are given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

(As we shall show in Chap. 5, this is the rule of multiplying two matrices.)

For second-order determinants, this relation is expressed as

$$|A| \cdot |B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{vmatrix}.$$

To prove this, we use the property of block diagonal determinants.

$$|A| \cdot |B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ -1 & 0 & b_{11} & b_{12} \\ 0 & -1 & b_{21} & b_{22} \end{vmatrix}.$$

Multiplying the elements in the first column by b_{11} and the elements in the second column by b_{21} and then add them to the corresponding elements in the third column, we obtain

$$|A| \cdot |B| = \begin{vmatrix} a_{11} & a_{12} & (a_{11}b_{11} + a_{12}b_{21}) & 0 \\ a_{21} & a_{22} & (a_{21}b_{11} + a_{22}b_{21}) & 0 \\ -1 & 0 & 0 & b_{12} \\ 0 & -1 & 0 & b_{22} \end{vmatrix}.$$

In the same way, we multiply the elements in the 1st column by b_{12} and the elements in the second column by b_{22} and then add them to the corresponding elements in the fourth column, it become

$$|A| \cdot |B| = \begin{vmatrix} a_{11} & a_{12} & (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ a_{21} & a_{22} & (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}.$$

By (4.48)

$$\begin{aligned} |A| \cdot |B| &= \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{vmatrix} \\ &= \begin{vmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{vmatrix}, \end{aligned}$$

which is the desired result. This procedure is applicable to determinants of any order. (This property is of considerable importance, we will revisit this problem for determinant of higher order in the chapter on matrices.)

Example 4.7.1. Show that

$$\begin{vmatrix} b^2 + c^2 & ab & ca \\ ab & a^2 + b^2 & bc \\ ca & bc & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2.$$

Solution 4.7.1.

$$\begin{vmatrix} b^2 + c^2 & ab & ca \\ ab & a^2 + b^2 & bc \\ ca & bc & a^2 + b^2 \end{vmatrix} = \begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix} \cdot \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = (-2abc)^2 = 4a^2b^2c^2.$$

4.8 Differentiation of Determinants

Occasionally, we require an expression for the derivative of a determinant. If the derivative is with respect to a particular element a_{ij}, then

$$\frac{\partial D_n}{\partial a_{ij}} = C_{ij},$$

where C_{ij} is the cofactor of a_{ij}, since

$$D_n = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \text{for} \;\; 1 \leq i \leq n.$$

Suppose the elements are functions of a parameter s, the derivative of D_n with respect to s is then given by

$$\frac{\mathrm{d}D_n}{\mathrm{d}s} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial D_n}{\partial a_{ij}} \frac{\mathrm{d}a_{ij}}{\mathrm{d}s} = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} \frac{da_{ij}}{\mathrm{d}s}.$$

For example

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{j=1}^{3} a_{1j}C_{1j} = \sum_{j=1}^{3} a_{2j}C_{2j} = \sum_{j=1}^{3} a_{3j}C_{3j},$$

$$\begin{aligned} \frac{\mathrm{d}D_3}{\mathrm{d}s} &= \sum_{j=1}^{3} \frac{\mathrm{d}a_{1j}}{\mathrm{d}s} C_{1j} + \sum_{j=1}^{3} \frac{\mathrm{d}a_{2j}}{\mathrm{d}s} C_{2j} + \sum_{j=1}^{3} \frac{\mathrm{d}a_{3j}}{\mathrm{d}s} C_{3j} \\ &= \begin{vmatrix} \frac{\mathrm{d}a_{11}}{\mathrm{d}s} & \frac{\mathrm{d}a_{12}}{\mathrm{d}s} & \frac{\mathrm{d}a_{13}}{\mathrm{d}s} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \frac{\mathrm{d}a_{21}}{\mathrm{d}s} & \frac{\mathrm{d}a_{22}}{\mathrm{d}s} & \frac{\mathrm{d}a_{23}}{\mathrm{d}s} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \frac{\mathrm{d}a_{31}}{\mathrm{d}s} & \frac{\mathrm{d}a_{32}}{\mathrm{d}s} & \frac{\mathrm{d}a_{33}}{\mathrm{d}s} \end{vmatrix}. \end{aligned}$$

Example 4.8.1. If $D_2 = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$, find $\frac{\mathrm{d}D_2}{\mathrm{d}x}$.

Solution 4.8.1.

$$\frac{\mathrm{d}D_2}{\mathrm{d}x} = \begin{vmatrix} -\sin x & \cos x \\ -\sin x & \cos x \end{vmatrix} + \begin{vmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{vmatrix} = 0.$$

This is an obvious result, since $D_2 = \cos^2 x + \sin^2 x = 1$.

4.9 Determinants in Geometry

It is well known in analytic geometry that a straight line in the xy-plane is represented by the equation

$$ax + by + c = 0. \tag{4.56}$$

The line is uniquely defined by two points. If the line goes through two points (x_1, y_1) and (x_2, y_2), then both of them have to satisfy the equation

$$ax_1 + by_1 + c = 0, \tag{4.57}$$

$$ax_2 + by_2 + c = 0. \tag{4.58}$$

These (4.56)–(4.58) may be regarded as a system in the unknowns a, b, c which cannot all vanish if (4.56) represents a line. Hence the coefficient determinant must vanish:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{4.59}$$

It can be easily shown that (4.59) is indeed the familiar equation of a line. Expanding (4.59) by the third column, we have

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = (x_1y_2 - x_2y_1) - (xy_2 - x_2y) + (xy_1 - x_1y) = 0.$$

This equation can be readily transformed into (4.56) with $a = y_1 - y_2,\ b = x_2 - x_1,\ c = x_1y_2 - x_2y_1$. Or it can be put in form

$$y = mx + y_0,$$

where $m = \frac{y_2 - y_1}{x_2 - x_1}$ is the slope and $y_0 = y_1 - mx_1$ is the y-axis intercept.

It follows from (4.59) that a necessary and sufficient condition for three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) to lie on a line is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{4.60}$$

Now if the three points are not on a line, then they form a triangle and the determinant (4.60) is not equal to zero. In that case it is interesting to ask what does the determinant represent. Since it has the dimension of an area, this strongly suggests that the determinant is related to the area of the triangle.

The area of the triangle formed by three points $A(x_1, y_1), B(x_2, y_2)$, and $C(x_3, y_3)$ shown in Fig. 4.6 is seen to be

$$\text{Area } ABC = \text{Area } AA'C'C + \text{Area } CC'B'B - \text{Area } AA'B'B.$$

The area of a trapezoid is equal to half of the product of its altitude and the sum of the parallel sides:

$$\text{Area } AA'C'C = \frac{1}{2}(x_3 - x_1)(y_1 + y_3),$$

$$\text{Area } CC'B'B = \frac{1}{2}(x_2 - x_3)(y_2 + y_3),$$

$$\text{Area } AA'B'B = \frac{1}{2}(x_2 - x_1)(y_1 + y_2).$$

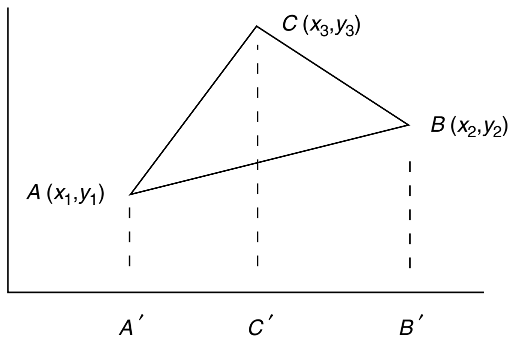


Fig. 4.6. The area of ABC is equal to the sum of the trapezoids $AA'C'C$ and $CC'B'B$ minus the trapezoid $AA'B'B$. As a consequence, the area ABC can be represented by a determinant

Hence

$$\begin{aligned}\text{Area } ABC &= \frac{1}{2}[(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) \\ &\quad -(x_2 - x_1)(y_1 + y_2)] \\ &= \frac{1}{2}\left[(x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)\right] \\ &= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \end{aligned} \tag{4.61}$$

Notice the order of the points ABC in the figure is counterclockwise. If it is clockwise, the positions of B and C are interchanged. This will result in the interchange of row 2 and row 3 in the determinant. As a consequence, a minus sign will be introduced. Thus we conclude that if the three vertices of a triangle are $A(x_1, y_1),\ B(x_2, y_2),\ C(x_3, y_3)$, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2 \times \text{Area of } ABC, \tag{4.62}$$

where the + or − sign is chosen according to the vertices being numbered consecutively in the counterclockwise or the clockwise direction.

Example 4.9.1. Use a determinant to find the circle that passes through (2,6), (6,4), (7,1).

Solution 4.9.1. The general expression of a circle is

$$a(x^2 + y^2) + bx + cy + d = 0.$$

The three points must all satisfy this equation

$$a(x_1^2+y_1^2)+bx_1+cy_1+d=0,$$
$$a(x_2^2+y_2^2)+bx_2+cy_2+d=0,$$
$$a(x_3^2+y_3^2)+bx_3+cy_3+d=0.$$

These equations may be regarded as a system of equations in the unknowns a, b, c, d which cannot all be zero. Hence the coefficient determinant must vanish

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Put in the specific values

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 40 & 2 & 6 & 1 \\ 52 & 6 & 4 & 1 \\ 50 & 7 & 1 & 1 \end{vmatrix} = 0.$$

Replacing the first row by (row 1 − row 2), and the third row by (row 3 − row 2) and the fourth row by (row 4 − row 2), we have

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 40 & 2 & 6 & 1 \\ 52 & 6 & 4 & 1 \\ 50 & 7 & 1 & 1 \end{vmatrix} = \begin{vmatrix} (x^2+y^2-40) & (x-2) & (y-6) & 0 \\ 40 & 2 & 6 & 1 \\ 12 & 4 & -2 & 0 \\ 10 & 5 & -5 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} (x^2+y^2-40) & (x-2) & (y-6) \\ 12 & 4 & -2 \\ 10 & 5 & -5 \end{vmatrix}$$

$$= -10(x^2+y^2-40)+40\,(x-2)+20\,(y-6)=0,$$

or

$$x^2+y^2-40-4(x-2)-2(y-6)=0.$$

which can be written as

$$(x-2)^2+(y-1)^2=25.$$

So the circle is centered at $x=2,\ y=1$ with a radius of 5.

Example 4.9.2. What is the area of the triangle whose vertices are $(-2,1)$, $(4,3),(0,0)$?

Solution 4.9.2.

$$\text{Area} = \begin{vmatrix} -2 & 1 & 1 \\ 4 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -10.$$

The area of the triangle is 10 and the order of the vertices is clockwise.

Exercises

1. Use determinants to solve for x, y, z from the following system of equations:

$$\begin{aligned} 3x + 6z &= 51, \\ 12y - 6z &= -6, \\ x - y - z &= 0. \end{aligned}$$

   Ans. $x = 7,\ y = 2,\ z = 5.$

2. By applying the Kirchhoff's rule to a electric circuit, the following equations are obtained for the currents i_1, i_2, i_3 in three branches

$$\begin{aligned} i_1 R_1 + i_3 R_3 &= V_A \\ i_2 R_2 + i_3 R_3 &= V_C \\ i_1 + i_2 - i_3 &= 0. \end{aligned}$$

   Express i_1, i_2, i_3 in terms of resistance R_1, R_2, R_3, and voltage source V_A, V_C.
   Ans.

$$\begin{aligned} i_1 &= \frac{(R_2 + R_3)V_A - R_3 V_C}{R_1 R_2 + R_1 R_3 + R_2 R_3}, \\ i_2 &= \frac{(R_1 + R_3)V_C - R_3 V_A}{R_1 R_2 + R_1 R_3 + R_2 R_3}, \\ i_3 &= \frac{R_2 V_A + R_1 V_C}{R_1 R_2 + R_1 R_3 + R_2 R_3}. \end{aligned}$$

3. Find the value of the following fourth-order determinant (which happens to be formed from one of the matrices appearing in Dirac's relativistic electron theory)

$$D_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}.$$

Ans. 1.

4. Without computation, show that a skew-symmetric determinant of odd order is zero

$$D_{\mathrm{ss}} = \begin{vmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ -b & -e & 0 & h & i \\ -c & -f & -h & 0 & j \\ -d & -g & -i & -j & 0 \end{vmatrix} = 0.$$

[Hint: $D^{\mathrm{T}} = D$ and $(-1)^n D_{\mathrm{ss}} = D^{\mathrm{T}}_{\mathrm{ss}}$.

5. Show that $\begin{vmatrix} a & d & 2a-3d \\ b & e & 2b-3e \\ c & f & 2c-3f \end{vmatrix} = 0.$

6. Determine x such that $\begin{vmatrix} 1 & 2 & -3 \\ -x & 1+3x & 3-x \\ 0 & -6 & 5 \end{vmatrix} = 36.$

Ans. 13.

7. The development of the determinant D_n on the ith row elements a_{ik} is $\sum_{k=1}^{n} a_{ik}C_{ik}$, where C_{ik} is the cofactor of a_{ik}. Show that

$$\sum_{k=1}^{n} a_{jk}C_{ik} = 0 \quad \text{for } j \neq i.$$

[Hint: The expansion is another determinant with two identical rows.]

8. Evaluate the following determinant by a development on (a) the first column, (b) the second row

$$D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}.$$

Ans. 1.

9. Use the properties of determinants to transform the determinant in problem 6 into a triangular form and then evaluate it as the product of the diagonal elements.

10. Evaluate the determinant in problem 6 by expanding it in terms of the 2×2 minors formed from the first two columns.

11. Evaluate the determinant

$$D_5 = \begin{vmatrix} 3 & -1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 3 & -6 & 1 \end{vmatrix}.$$

Ans. 3080.
[The quickest way to evaluate is to expand it in terms of the 2×2 minors formed from the first two columns.]

12. Without expanding, show that

$$\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

[Hint: Add row 1 and row 2, factor out $(x+y+z)$.]

13. Show that (a)

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (xy+yz+zx)\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

[Hint: Replace row 3 successively by $x \cdot \text{row}\,1 + \text{row}\,3$, then by $y \cdot \text{row}\,1 + \text{row}\,3$, then by $z \cdot \text{row}\,1 + \text{row}\,3$. Express the result as a sum of two determinants, one of them is equal to zero.]
(b) Use the result of the Vandermonde determinant to show that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (xy+yz+zx)(x-y)(y-z)(z-x).$$

14. State the reason for each step of the following identity:

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} = \begin{vmatrix} 2a & -2b & -a & b \\ 2b & 2a & -b & -a \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{vmatrix}$$

$$= \begin{vmatrix} 2a & -2b \\ 2b & 2a \end{vmatrix} \cdot \begin{vmatrix} c & -d \\ d & c \end{vmatrix} = 4(a^2+b^2)(c^2+d^2)$$

15. State the reason for each step of the following identity

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = \begin{vmatrix} (a+b) & b & (c+d) & d \\ (b+a) & a & (d+c) & c \\ (c+d) & d & (a+b) & b \\ (d+c) & c & (b+a) & a \end{vmatrix} = \begin{vmatrix} (a+b) & b & (c+d) & d \\ 0 & a-b & 0 & c-d \\ (c+d) & d & (a+b) & b \\ 0 & c-d & 0 & a-b \end{vmatrix}$$

$$= \begin{vmatrix} (a+b) & (c+d) & b & d \\ (c+d) & (a+b) & d & b \\ 0 & 0 & a-b & c-d \\ 0 & 0 & c-d & a-b \end{vmatrix}$$

$$= [(a+b)^2 - (c+d)^2][(a-b)^2 - (c-d)^2].$$

16. Show and state the reason for each step of the following identity

$$\begin{vmatrix} 0 & 1 & 2 & 3 & \cdots & n-1 \\ 1 & 0 & 1 & 2 & \cdots & n-2 \\ 2 & 1 & 0 & 1 & \cdots & n-3 \\ 3 & 2 & 1 & 0 & \cdots & n-4 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{vmatrix} = -(-2)^{n-2}(n-1).$$

[Hint: 1. Replace column 1 by column 1 + last column. 2. Factor out $(n-1)$. 3. Replace row i by row i − row$(i-1)$, starting with the last row. 3. Replace row i by row i + row 2. 4. Evaluating the triangular determinant.]

17. Evaluate the following determinant

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (n-1)n+1 & (n-1)n+2 & (n-1)n+3 & \cdots & n^2 \end{vmatrix}.$$

Ans. For $n = 1,\ D_1 = 1;\ n = 2,\ D_2 = -2;\ n \geq 3,\ D_n = 0.$
[Hint: For $n \geq 3$, replace row i by row i − row$(i-1)$.]

18. Use the rule of product of two determinants of same order to show the

$$\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & a^2+b^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = \begin{vmatrix} b^2+ac & bc & c^2 \\ ab & 2ac & bc \\ a^2 & ab & b^2+ac \end{vmatrix}.$$

[Hint:

$$\begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix} \cdot \begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix} = \begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix} \cdot \begin{vmatrix} b & c & 0 \\ a & 0 & c \\ 0 & a & b \end{vmatrix}.$$

19. If $f(s)$ is given by the following determinants, without the expansion of $f(x)$ find $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$

$$\text{(a)}\quad f(x) = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{-x} & 1 \\ \mathrm{e}^x & -\mathrm{e}^{-x} & 0 \\ \mathrm{e}^x & -\mathrm{e}^{-x} & x \end{vmatrix}; \qquad \text{(b)}\quad f(x) = \begin{vmatrix} \cos x & \sin x & \ln|x| \\ -\sin x & \cos x & \frac{1}{x} \\ -\cos x & -\sin x & -\frac{1}{x^2} \end{vmatrix}.$$

Ans. (a) -2; (b) $1/x + 2/x^3$.

20. The vertices of a triangle are $(0, t), (3t, 0), (t, 2t)$. Find a formula for the area of the triangle.

Ans. $2t$.

21. The equation representing a plane is given by $ax + by + cz + d = 0$. Find the plane that goes through $(1, 1, 1), (5, 0, 5), (3, 2, 6)$.

Ans. $3x + 4y - 2z - 5 = 0$.