5

Matrix Algebra

Matrices were introduced by British mathematician Arthur Cayley (1821–1895). The method of matrix algebra has extended far beyond mathematics into almost all disciplines of learning. In physical sciences, matrix is not only useful, but also essential in handling many complicated problems. These problems are mainly in three categories. First in the theory of transformation, second in the solution of systems of linear equations, and third in the solution of eigenvalue problems. In this chapter, we shall discuss various matrix operations and different situations in which they can be applied.

5.1 Matrix Notation

In this section, we shall define a matrix and discuss some of the simple operations by which two or more matrices can be combined.

5.1.1 Definition

Matrices

A rectangular array of elements is called a *matrix*. The array is usually enclosed within curved or square brackets. Thus, the rectangular arrays

$$\begin{pmatrix} 4 & 7 \\ 12 & 6 \\ -9 & 3 \end{pmatrix}, \quad \begin{pmatrix} x + \mathrm{i}y \\ x - \mathrm{i}y \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

are examples of a matrix. It is convenient to think of every element of a matrix as belonging to a certain row and a certain column of the matrix. If a matrix has m rows and n columns, the matrix is said of order m by n, or $m \times n$. Every element of a matrix can be uniquely characterized by a row index and a column index. It is convenient to write a $m \times n$ matrix as

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where a_{ij} is the element of ith row and jth column, it may be real or complex number or functions. The elements may even be matrices themselves, in which case the elements are called *submatrices* and the whole matrix is said to be partitioned.

Thus, if the first matrix in (5.1) is called matrix A, then A is a 3×2 matrix, it has three rows: $(4\ 7), (12\ 6), (-9\ 3)$ and two columns: $\begin{pmatrix} 4 \\ 12 \\ -9 \end{pmatrix}, \begin{pmatrix} 7 \\ 6 \\ 3 \end{pmatrix}$. Its elements are $a_{11} = 4, a_{12} = 7, a_{21} = 12, a_{22} = 6, a_{31} = -9$, and $a_{32} = 3$.

Some times it is convenient to use the notation

$$A = (a_{ij})_{m\times n}$$

to indicate that A is a $m \times n$ matrix. The elements a_{ij} can also be expressed as

$$a_{ij} = (A)_{ij}\,.$$

5.1.2 Some Special Matrices

There are some special matrices, which are named after their appearances.

Zero Matrix

A matrix of arbitrary order is said to be a zero matrix if and only if every element of the matrix equals zero. A zero matrix is sometimes called a *null matrix*.

Row Matrix

A row matrix has only one row, such as $(1\ 0\ 3)$. A row matrix is also called a *row vector*. If it is called row vector, the elements of the matrix are usually referred as components.

Column Matrix

A column matrix has only one column, such as $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$. A column matrix is also called a *column vector*. Again if it is called column vector, the elements of the matrix are usually called the components of the vector.

Square Matrix

A matrix is said to be a square matrix if the number of rows equals the number of columns. A square matrix of order n simply means it has n rows and n columns. Square matrix is of particular importance. We will be dealing mostly with square matrices together with column and row matrices.

For a square matrix A, we can calculate the determinant

$$\det(A) = |A|,$$

as defined in Chap. 4. Matrix is not a determinant. Matrix is an array of numbers, determinant is a single number. The determinant of a matrix can only be defined for a square matrix.

Let $A = (a_{ij})_n$ be a square matrix of order n. The diagonal going from the top left corner to the bottom right corner of the matrix, its elements $a_{11}, a_{22}, \ldots, a_{nn}$, are called the *diagonal elements.* All the remaining elements a_{ij} for $i \neq j$ are called the *off-diagonal elements.*

There are several special square matrices that are of interest.

Diagonal Matrix

A diagonal matrix is a square matrix whose diagonal elements are not all equal to zero, but off-diagonal elements are all zero. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

are diagonal matrices. Therefore for a diagonal matrix

$$(A)_{ij} = a_{ii}\delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

This kind of notation may seem to be redundant, as a diagonal matrix can easily be visualized. However, this notation is useful in manipulating matrices as we shall see later.

Constant Matrix

If all elements of a diagonal matrix happen to be equal to each other, it is said to be a constant matrix or a *scalar matrix.*

Unit Matrix

If the elements of a constant matrix are equal to unity, then it is a unit matrix. A unit matrix is also called the *Identity matrix,* denoted by I, that is

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Triangular Matrix

A square matrix having only zero elements on one side of the principal diagonal is a triangular matrix. Thus

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 4 & 3 & 0 \end{pmatrix}$$

are examples of a triangular matrix. A matrix for which $a_{ij} = 0$ for $i > j$ is called a *right-triangular matrix* or a *upper triangular matrix*, such as matrix A above. Whereas a matrix with $a_{ij} = 0$ for $i < j$ is called a *left-triangular matrix* or a *lower triangular matrix*, such as matrix B. If all the principal diagonal elements are zero, the matrix is a *strictly triangular matrix*, such as matrix C. Diagonal matrix, identity matrix as well as zero matrix are all triangular matrices.

5.1.3 Matrix Equation

Equality

Two matrices A and B are equal to each other if and only if, every elements of A is equal to the corresponding element of B. Clearly A and B must be of the same order, in other words they must have the same rows and columns. Thus if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix},$$

we see that

$$A \neq B, \quad B \neq C, \quad C \neq A.$$

Therefore, a matrix equation $A = B$ means that A and B are of the same order and their corresponding elements are equal, i.e., $a_{ij} = b_{ij}$. For example, the equation

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 3t & 1+2t \\ 4t^2 & 0 \end{pmatrix}$$

means $x_1 = 3t, y_1 = 4t^2, x_2 = 1 + 2t, y_2 = 0$.

With this understanding, often we can use a single matrix equation to replace a set of equations. This will not only simplify the writing but will also enable us to systematically manipulate these equations.

Addition and Subtraction

We may now define the addition and subtraction of two matrices of the same order. The sum of two matrices A and B is another matrix C. By definition

$$A + B = C$$

means

$$c_{ij} = a_{ij} + b_{ij}.$$

For example,

$$A = \begin{pmatrix} 1 & 3 & 12 \\ -2 & 4 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} -10 & 5 & -6 \\ 7 & 3 & 2 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} (1-10) & (3+5) & (12-6) \\ (-2+7) & (4+3) & (-6+2) \end{pmatrix} = \begin{pmatrix} -9 & 8 & 6 \\ 5 & 7 & -4 \end{pmatrix},$$

$$A - B = \begin{pmatrix} (1+10) & (3-5) & (12+6) \\ (-2-7) & (4-3) & (-6-2) \end{pmatrix} = \begin{pmatrix} 11 & -2 & 18 \\ -9 & 1 & -8 \end{pmatrix}.$$

The sum of several matrices is obtained by repeated addition. Since matrix addition is merely the addition of corresponding elements, it does not matter in which order we add several matrices. To be explicit, if A, B, C are three $m \times n$ matrices, then both commutative and associative laws hold

$$A + B = B + A,$$
$$A + (B + C) = (A + B) + C.$$

Multiplication by a Scalar

It is possible to combine a matrix of arbitrary order and a scalar by scalar multiplication. If A is a matrix of order $m \times n$

$$A = (a_{ij})_{m \times n}$$

and c a scalar, we define cA to be another $m \times n$ matrix such that

$$cA = (ca_{ij})_{m \times n}.$$

For example, if

$$A = \begin{pmatrix} 1 & -3 & 5 \\ -2 & 4 & -6 \end{pmatrix},$$

then

$$-2A = \begin{pmatrix} -2 & 6 & -10 \\ 4 & -8 & 12 \end{pmatrix}.$$

The scalar can be a real number, a complex number, or a function, but it cannot be a matrix quantity.

Note the difference between the scalar multiplication of a square matrix cA and the scalar multiplication of its determinant $c\,|A|$. For cA, c is multiplied to every elements of A, whereas for $c\,|A|$, c is only multiplied to the elements of a single column or a single row. Thus, if A is a square matrix of order n, then

$$\det(cA) = c^n\,|A|\,.$$

5.1.4 Transpose of a Matrix

If the rows and columns are interchanged, the resulting matrix is called the *transposed matrix*. The transposed matrix is denoted by $\widetilde{A}$, called A tilde, or by A^{T}. Usually, but not always, the transpose of a single matrix is denoted by the tilde and the transpose of the product of a number of matrices by the superscript T.

Thus, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

then

$$\widetilde{A} = A^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

By definition, if we transpose the matrix twice, we should get the original matrix, i.e.,

$$\widetilde{A}^{\mathrm{T}} = A.$$

Using index notation, this means

$$\left(\widetilde{A}\right)_{ij} = (A)_{ji}\,, \quad \left(\widetilde{A}^{\mathrm{T}}\right)_{ij} = (A)_{ij}\,.$$

It is clear that the transpose of $m \times n$ matrix is a $n \times m$ matrix. The transpose of a square matrix is another square matrix. The transpose of a column matrix is a row matrix, and the transpose of a row matrix is a column matrix.

Symmetric Matrix

A symmetric matrix is a matrix that is equal to its transpose, i.e.,

$$A = \widetilde{A},$$

which means

$$a_{ij} = a_{ji}.$$

It is symmetric with respect to its diagonal. A symmetric matrix must be a square matrix.

Antisymmetric Matrix

An antisymmetric matrix is a matrix that is equal to the negative of its transpose, i.e.,

$$A = -\widetilde{A},$$

which means

$$a_{ij} = -a_{ji}.$$

Thus the diagonal elements of an antisymmetric matrix must all be zero. An antisymmetric matrix must also be a square matrix. Antisymmetric is also known as *skew-symmetric.*

Decomposition of a Square Matrix

Any square matrix can be written as the sum of a symmetric and an antisymmetric matrix. Clearly

$$A = \frac{1}{2}\left(A + \widetilde{A}\right) + \frac{1}{2}\left(A - \widetilde{A}\right)$$

is an identity. Furthermore, let

$$A_{\mathrm{s}} = \frac{1}{2}\left(A + \widetilde{A}\right), \quad A_{\mathrm{a}} = \frac{1}{2}\left(A - \widetilde{A}\right),$$

then A_{s} is symmetric, since

$$A_{\mathrm{s}}^{\mathrm{T}} = \frac{1}{2}\left(A^{\mathrm{T}} + \widetilde{A}^{\mathrm{T}}\right) = \frac{1}{2}\left(\widetilde{A} + A\right) = A_{\mathrm{s}},$$

and A_{a} is antisymmetric, since

$$A_{\mathrm{a}}^{\mathrm{T}} = \frac{1}{2}\left(A^{\mathrm{T}} - \widetilde{A}^{\mathrm{T}}\right) = \frac{1}{2}\left(\widetilde{A} - A\right) = -A_{\mathrm{a}}.$$

Therefore

$$A = A_{\mathrm{s}} + A_{\mathrm{a}},$$
$$\widetilde{A} = A_{\mathrm{s}} - A_{\mathrm{a}}.$$

Example 5.1.1. Express the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix}$$

as the sum of a symmetric matrix and an antisymmetric matrix.

Solution 5.1.1.

$$A = A_{\mathrm{s}} + A_{\mathrm{a}},$$

$$A_{\mathrm{s}} = \frac{1}{2}\left[\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}\right] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 2 \end{pmatrix},$$

$$A_{\mathrm{a}} = \frac{1}{2}\left[\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ -1 & 4 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 4 \\ 1 & 2 & 2 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

5.2 Matrix Multiplication

5.2.1 Product of Two Matrices

The multiplication, or product, of two matrices is not a simple extension of the concept of multiplication of two numbers. The definition of matrix multiplication is motivated by the theory of linear transformation, which we will briefly discuss in Sect. 5.3.

Two matrices A and B can be multiplied together only if the number of columns of A is equal the number of rows of B. The matrix multiplication depends on the order in which the matrices occur in the product. For example, if A is of order $l \times m$, and B is of order $m \times n$, then the product matrix AB is defined but the product BA, in that order is not unless $m = l$. The multiplication is defined as follows. If

$$A = (a_{ij})_{l\times m}, \quad B = (b_{ij})_{m\times n},$$

then $AB = C$ means that C is a matrix of order $l \times n$ and

$$C = (c_{ij})_{l\times n},$$

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}.$$

So the element of the C matrix at ith row and jth column is the sum of all the products of the elements of ith row of A and the corresponding elements of jth column of B. Thus, if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}, \quad C = AB,$$

then

$$C = \begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) & (a_{11}b_{13} + a_{12}b_{23}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) & (a_{21}b_{13} + a_{22}b_{23}) \end{pmatrix}.$$

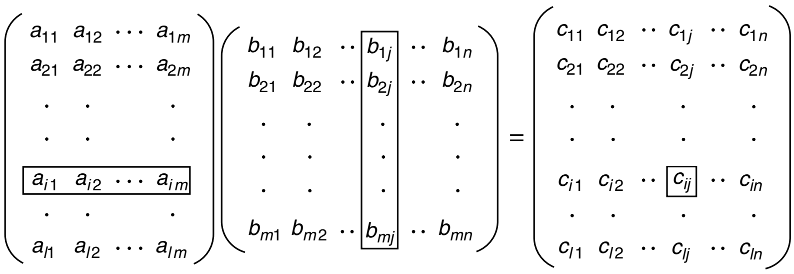

Fig. 5.1. Illustration of matrix multiplication. The number of columns of A must equal the number of rows of B for the multiplication $AB = C$ to be defined. The element at ith row and jth column of C is given by $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$

The multiplication of two matrices is illustrated in Fig. 5.1.

If the product AB is defined, A and B are said to be *comformable* (or *compatible*). If the matrix product AB is defined, the product BA is not necessarily defined. Given two matrices A and B, both the products of AB and BA will be possible if, for example, A is of order $m \times n$ and B is of order $n \times m$. AB will be of order $m \times m$, and BA of order $n \times n$. Clearly if $m \neq n$, AB cannot equal to BA, since they are of different order. Even if $n = m$, AB is still not necessarily equal to BA. The following examples will make this clear.

Example 5.2.1. Find the product AB, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix}.$$

Solution 5.2.1.

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} (1 \times 3 + 2 \times 4) & (1 \times 2 + 2 \times 5) & (1 \times 1 + 2 \times 6) \\ (3 \times 3 + 4 \times 4) & (3 \times 2 + 4 \times 5) & (3 \times 1 + 4 \times 6) \end{pmatrix} \\ &= \begin{pmatrix} 11 & 12 & 13 \\ 25 & 26 & 27 \end{pmatrix}. \end{aligned}$$

Here A is 2×2 and B is 2×3, so that AB comes out 2×3, whereas BA is not defined.

Example 5.2.2. Find the product AB, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.$$

Solution 5.2.2.

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5+12 \\ 15+24 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}.$$

Here AB is a column matrix and BA is not defined.

Example 5.2.3. Find AB and BA, if

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

Solution 5.2.3.

$$AB = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2+6+12 \end{pmatrix} = (20),$$

$$BA = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}.$$

This example dramatically shows that $AB \neq BA$.

Example 5.2.4. Find AB and BA, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

Solution 5.2.4.

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 13 & 16 \\ 29 & 36 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 15 & 22 \\ 23 & 34 \end{pmatrix}.$$

Clearly

$$AB \neq BA.$$

Example 5.2.5. Find AB and BA, if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solution 5.2.5.

$$AB = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$BA = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Not only $AB \neq BA$, but also $AB = 0$ does not necessarily imply $A = 0$ or $B = 0$ or $BA = 0$.

Example 5.2.6. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}$$

show that

$$AB = AC.$$

Solution 5.2.6.

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix},$$

$$AC = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}.$$

This example shows that $AB = AC$ can hold without $B = C$ or $A = 0$.

5.2.2 Motivation of Matrix Multiplication

Much of the usefulness of matrix algebra is due to its multiplication property. The definition of matrix multiplication, as we have seen, seems to be "unnatural" and somewhat complicated. The motivation of this definition comes from the "linear transformations." It provides a simple mechanism for changing variables. For example, suppose

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \tag{5.2a}$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \tag{5.2b}$$

and further

$$z_1 = b_{11}y_1 + b_{12}y_2, \tag{5.3a}$$
$$z_2 = b_{21}y_1 + b_{22}y_2. \tag{5.3b}$$

In these equations the x's and the y's are variables, while the a's and the b's are constants. The x's are related to the y's by the first set of equations, and the y's are related to the z's by the second set of equations. To find out how the x's are related to the z's, we must substitute the values of the y's given by the first set of equations into the second set of equations

$$z_1 = b_{11}\left(a_{11}x_1 + a_{12}x_2 + a_{13}x_3\right) + b_{12}\left(a_{21}x_1 + a_{22}x_2 + a_{23}x_3\right), \tag{5.4a}$$
$$z_2 = b_{21}\left(a_{11}x_1 + a_{12}x_2 + a_{13}x_3\right) + b_{22}\left(a_{21}x_1 + a_{22}x_2 + a_{23}x_3\right). \tag{5.4b}$$

By multiplying them out and collecting coefficients, they become

$$\begin{aligned} z_1 = {} & (b_{11}a_{11} + b_{12}a_{21})x_1 \\ & +(b_{11}a_{12} + b_{12}a_{22})x_2 + (b_{11}a_{13} + b_{12}a_{23})x_3, \end{aligned} \tag{5.5a}$$
$$\begin{aligned} z_2 = {} & (b_{21}a_{11} + b_{22}a_{21})x_1 \\ & +(b_{21}a_{12} + b_{22}a_{22})x_2 + (b_{21}a_{13} + b_{22}a_{23})x_3. \end{aligned} \tag{5.5b}$$

Now, using matrix notation, (5.2) can be written simply as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{5.6}$$

and (5.3) as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{5.7}$$

Not only the coefficients of x_1, x_2, and x_3 in (5.5) are precisely the elements of the matrix product

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

but also they are located in the proper position. In other words, (5.5) can be obtained by simply substituting $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ from (5.6) into (5.7)

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{5.8}$$

What we have shown here is essentially two things. First, matrix multiplication is defined in such a way that linear transformation can be written in compact forms. Second, if we substitute linear transformations into each other, we can obtain the composite transformation simply by multiplying coefficient matrices in the right order. This kind of transformation is not only common in mathematics, but is also extremely important in physics. We will discuss some of them in later sections.

5.2.3 Properties of Product Matrices

Transpose of a Product Matrix

A result of considerable importance in matrix algebra is that the transpose of the product of two matrices equals the product of the transposed matrices taken in reverse order,

$$(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}. \tag{5.9}$$

To prove this we must show that every element of the left-hand side is equal to the corresponding element in the right-hand side. The ijth element of the left-hand side of (5.9) is given by

$$\left((AB)^{\mathrm{T}}\right)_{ij} = (AB)_{ji} = \sum_k (A)_{jk}(B)_{ki}. \tag{5.10}$$

The ijth element of the left-hand side of (5.9) is

$$\begin{aligned}\left(\widetilde{B}\widetilde{A}\right)_{ij} &= \sum_k \left(\widetilde{B}\right)_{ik}\left(\widetilde{A}\right)_{kj} = \sum_k (B)_{ki}(A)_{jk} \\ &= \sum_k (A)_{jk}(B)_{ki},\end{aligned} \tag{5.11}$$

where in the last step we have interchanged $(B)_{kj}$ and $(A)_{jk}$ because they are just numbers. Thus (5.9) follows.

Example 5.2.7. Let

$$A = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix},$$

show that

$$(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}.$$

Solution 5.2.7.

$$AB = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 22 \\ -2 & -4 \end{pmatrix}, \quad (AB)^{\mathrm{T}} = \begin{pmatrix} 8 & -2 \\ 22 & -4 \end{pmatrix},$$

$$\widetilde{B} = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}; \quad \widetilde{B}\widetilde{A} = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ 22 & -4 \end{pmatrix}.$$

Thus, $(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}$.

Trace of a Matrix

The trace of square matrix $A = (a_{ij})$ is defined as the sum of its diagonal elements and is denoted by $\mathrm{Tr}\, A$

$$\mathrm{Tr}\, A = \sum_{i=1}^{n} a_{ii}.$$

An important theorem about trace is that the trace of the product of a finite number of matrices is invariant under any cyclic permutation of the matrices. We can first prove this theorem for the product of two matrices, and then the rest automatically follow.

Let A be a $n \times m$ matrix and B be a $m \times n$ matrix, then

$$\mathrm{Tr}\,(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji},$$
$$\mathrm{Tr}(BA) = \sum_{j=1}^{n} (BA)_{jj} = \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij}.$$

Since a_{ij} and b_{ij} are just numbers, their order can be reversed. Thus

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}(BA).$$

Notice that the trace is defined only for a square matrix, but A and B do not have to be square matrices as long as their product is a square matrix. The order of AB may be different from the order of BA, yet their traces are the same.

Now

$$\begin{aligned} \mathrm{Tr}(ABC) &= \mathrm{Tr}(A(BC)) = \mathrm{Tr}((BC)A) \\ &= \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB). \end{aligned} \tag{5.12}$$

It is important to note that the trace of the product of a number of matrices is not invariant under any permutation, but only a cyclic permutation of the matrices.

Example 5.2.8. Let

$$A = \begin{pmatrix} 4 & 0 & 6 \\ 5 & 2 & 1 \\ 7 & 8 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 9 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix},$$

show that (a) $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$, and (b) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Solution 5.2.8.

(a)

$$\begin{aligned}\mathrm{Tr}(A+B) &= \mathrm{Tr}\left\{\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix}+\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix}\right\} = \mathrm{Tr}\begin{pmatrix}5&0&7\\14&3&3\\7&12&4\end{pmatrix}\\ &= 5+3+4 = 12,\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(A)+\mathrm{Tr}(B) &= \mathrm{Tr}\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix}+\mathrm{Tr}\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix}\\ &= (4+2+3)+(1+1+1) = 12.\end{aligned}$$

(b)

$$\mathrm{Tr}(AB) = \mathrm{Tr}\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix}\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}4&24&10\\23&6&10\\79&20&26\end{pmatrix} = 36,$$

$$\mathrm{Tr}(BA) = \mathrm{Tr}\begin{pmatrix}1&0&1\\9&1&2\\0&4&1\end{pmatrix}\begin{pmatrix}4&0&6\\5&2&1\\7&8&3\end{pmatrix} = \mathrm{Tr}\begin{pmatrix}11&8&9\\55&18&61\\27&16&7\end{pmatrix} = 36.$$

Associative Law of Matrix Multiplication

If A, B, and C are three matrices such that the matrix product AB and BC are defined, then

$$(AB)\,C = A\,(BC)\,. \tag{5.13}$$

In other words, it is immaterial which two matrices are multiplied together first. To prove this, let

$$A = (a_{ij})_{m\times n}\,, \quad B = (b_{ij})_{n\times o}\,, \quad C = (c_{ij})_{o\times p}\,.$$

The ijth element of the left-hand side of (5.13) is then

$$\begin{aligned}((AB)C)_{ij} &= \sum_{k=1}^{o}(AB)_{ik}(C)_{kj} = \sum_{k=1}^{o}\left(\sum_{l=1}^{n}(A)_{il}(B)_{lk}\right)(C)_{kj}\\ &= \sum_{k=1}^{o}\sum_{l=1}^{n}a_{il}b_{lk}c_{kj},\end{aligned}$$

while the ijth element of the right-hand side (5.13) is

$$(A(BC))_{ij} = \sum_{l=1}^{n}(A)_{il}(BC)_{lj} = \sum_{l=1}^{n}(A)_{il}\sum_{k=1}^{o}(B)_{lk}(C)_{kj}$$
$$= \sum_{l=1}^{n}\sum_{k=1}^{o} a_{il}b_{lk}c_{kj}.$$

Clearly $((AB)C)_{ij} = (A(BC))_{ij}$.

Example 5.2.9. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix},$$

show that

$$A\,(BC) = (AB)\,C.$$

Solution 5.2.9.

$$A\,(BC) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ 16 & 2 \end{pmatrix},$$

$$(AB)\,C = \left[\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 2 & -1 \\ 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ 16 & 2 \end{pmatrix}.$$

Clearly $A\,(BC) = (AB)\,C$. This is one of the most important properties of matrix algebra.

Distributive Law of Matrix Multiplication

If A, B, and C are three matrices such that the addition $B + C$ and the product AB and BC are defined, then

$$A(B + C) = AB + AC. \tag{5.14}$$

To prove this, let

$$A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times p}, \quad C = (c_{ij})_{n\times p}, \tag{5.15}$$

so that the addition $B+C$ and the products AB and AC are defined. The ijth element of the left-hand side of (5.14) is then

$$\begin{aligned}(A(B+C))_{ij} &= \sum_{k=1}^{n}(A)_{ik}(B+C)_{kj} = \sum_{k=1}^{n}(A)_{ik}(B_{kj}+C_{kj}) \\ &= \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}).\end{aligned}$$

The ijth element of the right-hand side of (5.14) is

$$\begin{aligned}(AB+AC)_{ij} &= (AB)_{ij} + (AC)_{ij} \\ &= \sum_{k=1}^{n}(A)_{ik}(B)_{kj} + \sum_{k=1}^{n}(A)_{ik}(C)_{kj} \\ &= \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}).\end{aligned}$$

Thus, (5.14) follows.

Example 5.2.10. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix},$$

show that

$$C\,(A+B) = CA + CB.$$

Solution 5.2.10.

$$\begin{aligned}C\,(A+B) &= \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \left[\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \right] \\ &= \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ 21 & 13 \\ 6 & 0 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}CA + CB &= \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -2 \\ 1 & 3 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 4 \\ 10 & 2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} -2 & -10 \\ 11 & 11 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} -6 & -6 \\ 21 & 13 \\ 6 & 0 \end{pmatrix}.\end{aligned}$$

Hence, $C\,(A+B) = CA + CB$.

5.2.4 Determinant of Matrix Product

We have shown in the chapter on determinants that the value of the determinant of the product of two matrices is equal to the product of two determinants. That is, if A and B are square matrices of the same order, then

$$|AB| = |A|\,|B|\,.$$

This relation is of considerable interests. It is instructive to prove it with the properties of matrix products. We will use 2×2 matrices to illustrate the steps of the proof, but it will be obvious that the process is generally valid for all orders.

1. If D is a diagonal matrix, it is easy to show $|DA| = |D|\,|A|$.
   For example, let $D = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$, then
$$\begin{aligned} DA &= \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} d_{11}a_{11} & d_{11}a_{12} \\ d_{22}a_{21} & d_{22}a_{22} \end{pmatrix}, \\ |D| &= \begin{vmatrix} d_{11} & 0 \\ 0 & d_{22} \end{vmatrix} = d_{11}d_{22}, \\ |DA| &= \begin{vmatrix} d_{11}a_{11} & d_{11}a_{12} \\ d_{22}a_{21} & d_{22}a_{22} \end{vmatrix} = d_{11}d_{22} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = |D|\,|A|\,. \end{aligned}$$
2. Any square matrix can be diagonalized by a series of row operations which add a multiple of a row to another row.
   For example, let $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Multiply row 1 by -3 and add it to row 3, the matrix becomes $\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$. Then add row 2 to row 1, we have the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.
3. Each row operation is equivalent to premultiplying the matrix by an elementary matrix obtained from applying the same operation to the identity matrix.
   For example, multiply row 1 by -3 and add it to row 2 of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain the elementary matrix $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$. Multiply this matrix to the left of B,
$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} (B) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix},$$
   we get the same result as operating directly on B. The elementary matrix for adding row 2 to row 1 is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Multiplying this matrix to the left of $\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$, we have the diagonal matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

4. Combine the last equations

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

   Let

$$E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix},$$

   we can write the equation as

$$EB = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = D.$$

   This equation says that the matrix B is diagonalized by the matrix E, which is the product of a series of elementary matrices.

5. Because of the way E is constructed, multiplying E to the left of any matrix M is equivalent to repeatedly adding a multiple of a row to another row of M. From the theory of determinants, we know that these operations do not change the value of the determinant. For example,

$$|EB| = |D| = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2,$$
$$|B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

   Therefore the determinant of the diagonalized matrix D is equal to the determinant of the original matrix B,

$$|D| = |B|.$$

   In fact M can be any matrix, as along as it is compatible,

$$|EM| = |M|.$$

6. Now let $M = BA$,

$$|E(BA)| = |BA|.$$

   But

$$|E(BA)| = |(EB)\,A| = |DA| = |D|\,|A|,$$

   since D is diagonal. On the other hand $|D| = |B|$, therefore

$$|BA| = |B|\,|A|.$$

   Since $|B|\,|A| = |A|\,|B|$, it follows $|BA| = |AB|$, even though BA may not be equal to AB.

5.2.5 The Commutator

The difference between the two products AB and BA is known as the commutator

$$[A,B] = AB - BA.$$

If in particular, AB is equal to BA, then

$$[A,B] = 0,$$

the two matrices A and B are said to commute with each other.

It follows directly from the definition that:

- $[A,A] = 0$
- $[A,I] = [I,A] = 0$
- $[A,B] = -[B,A]$
- $[A,(B+C)] = [A,B] + [A,C]$
- $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

Example 5.2.11. Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

show that

$$[\sigma_x, \sigma_y] = 2\mathrm{i}\sigma_z, \quad [\sigma_y, \sigma_z] = 2\mathrm{i}\sigma_x, \quad [\sigma_z, \sigma_x] = 2\mathrm{i}\sigma_y.$$

Solution 5.2.11.

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_y\sigma_x = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = 2\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\mathrm{i}\sigma_z.$$

Similarly, $[\sigma_y, \sigma_z] = 2\mathrm{i}\sigma_x$ and $[\sigma_z, \sigma_x] = 2\mathrm{i}\sigma_y$.

Example 5.2.12. If a matrix B commutes with a diagonal matrix with no two elements equal to each other, then B must also be a diagonal matrix.

Solution 5.2.12. To prove this, let B commute with a diagonal matrix A of order n, whose elements are

$$\begin{gathered}(A)_{ij} = a_i \delta_{ij} \\ a_i \neq a_j \quad \text{if} \quad i \neq j.\end{gathered} \tag{5.16}$$

We are given that

$$AB = BA.$$

Let the elements of B be b_{ij}, we wish to show that $b_{ij} = 0$, unless $i = j$. Taking the ijth element of both sides, we have

$$\sum_{k=1}^{n} (A)_{ik} (B)_{kj} = \sum_{k=1}^{n} (B)_{ik} (A)_{kj}.$$

On using (5.16), this becomes

$$\sum_{k=1}^{n} a_i \delta_{ik} b_{kj} = \sum_{k=1}^{n} b_{ik} a_k \delta_{kj},$$

with the definition of delta function

$$a_i b_{ij} = b_{ij} a_j.$$

This shows

$$(a_i - a_j) b_{ij} = 0.$$

Thus b_{ij} must be all equal to zero for $i \neq j$, since for those cases $a_i \neq a_j$. The only elements of B which can be different from zero are the diagonal elements b_{ii}, proving that B must be a diagonal matrix.

5.3 Systems of Linear Equations

The method of matrix algebra is very useful in solving a system of linear equations. Let $x_1, x_2, \ldots, x_n$ be a set of n unknown variables. An equation which contains first degree of x_i and no products of two or more variables is called a *linear equation*. The most general system of m linear equations in n unknowns can be written in the form

$$\begin{gathered}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = d_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = d_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = d_m.\end{gathered} \tag{5.17}$$

Here the coefficients a_{ij} and the right-hand side terms d_i are supposed to be known constants.

We can regard the variables $x_1, x_2, \ldots, x_n$ as components of the $n \times 1$ column vector $\mathbf{x}$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and the constants $d_1, d_2, \ldots, d_m$ as components of the $m \times 1$ column vector $\mathbf{d}$

$$\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}.$$

The coefficients a_{ij} can be written as elements of the $m \times n$ matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

With the matrix multiplication defined in Sect. 5.2, (5.17) can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}.$$

If all the components of d are equal to zero, the system is called *homogeneous*. If at least one component of d is not zero, the system is called *nonhomogeneous*. If the system of linear equations is such that the equations are all satisfied simultaneously by at least one set of values of x_i, then it is said to be consistent. The system is said to be inconsistent if the equations are not satisfied simultaneously by any set of values. An inconsistent system has no solution. A consistent system may have an unique solution, or an infinite number of solutions. In the following sections, we will discuss practical ways of finding these solutions, as well as answer the question of existence and uniqueness of the solutions.

5.3.1 Gauss Elimination Method

Two linear systems are equivalent if every solution of either system is a solution of the other. There are three elementary operations that will transform a linear system into another equivalent system:

1. Interchanging two equations
2. Multiplying an equation through by a nonzero number
3. Adding to one equation a multiple of some other equation

That a system is transformed into an equivalent system by the first operation is quite apparent. The reason that the second and third kinds of operations have the same effect is that when the same operations are done on both sides of an equal sign, the equation should remain valid. In fact, these are just the techniques we learned in elementary algebra to solve a set of simultaneous equations. The goal is to transform the set of equations into a simple form so that the solution is obvious. A practical procedure is suggested by the observation that a linear system, whose coefficient matrix is either upper triangular or diagonal, is easy to solve.

For example, the system of equations

$$\begin{aligned} -2x_2 + x_3 &= 8, \\ 2x_1 - x_2 + 4x_3 &= -3, \\ x_1 - x_2 + x_3 &= -2, \end{aligned} \tag{5.18}$$

can be written as

$$\begin{pmatrix} 0 & -2 & 1 \\ 2 & -1 & 4 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -2 \end{pmatrix}.$$

Interchange equation 1 and equation 3, the system becomes

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ 2x_1 - x_2 + 4x_3 &= -3 \\ -2x_2 + x_3 &= 8 \end{aligned} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 8 \end{pmatrix},$$

where we have put the matrix equation representing the system right next to it. Multiply equation 1 of the rearranged system by -2 and add to equation 2, we have

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ x_2 + 2x_3 &= 1 \\ -2x_2 + x_3 &= 8 \end{aligned} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix}.$$

Multiply equation 2 of the last system by 2 and add to equation 3

$$\begin{aligned} x_1 - x_2 + x_3 &= -2 \\ x_2 + 2x_3 &= 1 \\ 5x_3 &= 10 \end{aligned} \qquad \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 10 \end{pmatrix}. \tag{5.19}$$

These four systems of equations are equivalent because they have the same solution. From the last set of equations, it is clear that $x_3 = 2, x_2 = 1 - 2x_3 = -3$, and $x_1 = -2 + x_2 - x_3 = -7$.

This procedure is often referred to as the *Gauss elimination method*, the *echelon method*, or *triangularization*.

Augmented Matrix

To simplify the writing further, we introduce the augmented matrix. The matrix composed of the coefficient matrix plus an additional column whose elements are the nonhomogeneous constants d_i is called the *augmented matrix* of the system. Thus

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & d_1 \\ a_{21} & a_{22} & \dots & a_{2n} & d_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & d_n \end{array}\right)$$

is the augmented matrix of (5.17). The portion in front of the vertical line is the coefficient matrix. The entire matrix, disregarding the vertical line, is the augmented matrix of the system. Clearly the augmented matrix is just a succinct expression of the linear system.

Instead of operating on the equations of the system, we can just operate on the rows of the augmented matrix with the three *elementary row operations* which consist of:

1. Interchanging of any two rows
2. Multiplying of any row by a nonzero scalar
3. Adding a multiple of a row to another row

Thus we can summarize the Gauss elimination method as using the elementary row operations to reduce the augmented matrix of the original system to an echelon form. A matrix is said to be in echelon form if:

1. The first element in the first row is nonzero.
2. The first $(n-1)$ elements of the nth row are zero, the rest elements may or may not be zero.
3. The first nonzero element of any row appears to the right of the first nonzero element in the row above.
4. As a consequence, if there are rows whose elements are all zero, then they must be at the bottom of the matrix.

Thus we can think of solving the linear system of (5.18) in the above example as reducing the augmented matrix from

$$\left(\begin{array}{ccc|c} 0 & -2 & 1 & 8 \\ 2 & -1 & 4 & -3 \\ 1 & -1 & 1 & -2 \end{array}\right)$$

to the echelon form

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

from which the solution is easily obtained.

Gauss–Jordan Elimination Method

For a large set of linear equations, it is sometimes advantageous to continue the process to reduce the coefficient matrix from the triangular form to a diagonal form. For example, multiply the third row of the last matrix by 1/5

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \tag{5.20}$$

Multiply row 3 by -2 and add to row 2

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Multiply row 3 by -1 and add to row 1:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Add row 2 to row 1:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right),$$

which corresponds to $x_1 = -7, x_2 = -3$, and $x_3 = 2$. This process is known as the *Gauss–Jordan elimination method*.

5.3.2 Existence and Uniqueness of Solutions of Linear Systems

For a linear system of m equations and n unknowns, the order of the coefficient matrix is $m \times n$ that of the augmented matrix is $m \times (n+1)$. If $m < n$, the system is underdetermined. If $m > n$, the system is overdetermined. The most interesting case is $m = n$. In all three cases, we can use Gauss elimination method to reduce the augmented matrix into an echelon form. Once in the echelon form, the problem is either solved, or else shown to be inconsistent. A few examples will make this clear.

Example 5.3.1. Solve the following system of equations:

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, \\ 2x_1 - x_2 + x_3 &= 1, \\ 3x_1 - x_2 + x_3 &= 4. \end{aligned}$$

Solution 5.3.1. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & -1 & 1 & 4 \end{array}\right).$$

Multiply row 1 by -2 and add to row 2:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 3 & -1 & 1 & 4 \end{array}\right).$$

Multiply row 1 by -3 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & -3 \\ 0 & -4 & 4 & -2 \end{array}\right).$$

Multiply row 2 by $-1/3$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & -2 \end{array}\right).$$

Multiply row 2 by 4 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right).$$

This represents the system of equations

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, \\ x_2 - x_3 &= 1, \\ 0 &= 2. \end{aligned}$$

Since no values of x_1, x_2, and x_3 can make $0 = 2$, the system is inconsistent, and has no solution.

Example 5.3.2. Solve the following system of equations:

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 6, \\ 3x_1 - 2x_2 - 8x_3 &= 7, \\ 4x_1 + 5x_2 - 3x_3 &= 17. \end{aligned}$$

Solution 5.3.2. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 3 & -2 & -8 & 7 \\ 4 & 5 & -3 & 17 \end{array}\right).$$

Multiply row 1 by -3 and add to row 2:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & -11 & -11 & -11 \\ 4 & 5 & -3 & 17 \end{array}\right).$$

Multiply row 1 by -4 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & -11 & -11 & -11 \\ 0 & -7 & -7 & -7 \end{array}\right).$$

Multiply row 2 by $-1/11$:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -7 & -7 \end{array}\right).$$

Multiply row 2 by 7 and add to row 3:

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This represents the system

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 6, \\ x_2 + x_3 &= 1, \\ 0 &= 0. \end{aligned}$$

This says $x_2 = 1 - x_3$ and $x_1 = 6 - 3x_2 - x_3 = 3 + 2x_3$. The value of x_3 may be assigned arbitrarily, therefore the system has an infinite number of solutions.

Example 5.3.3. Solve the following system of equations:

$$\begin{aligned} x_1 + x_2 &= 2, \\ x_1 + 2x_2 &= 3, \\ 2x_1 + x_2 &= 3. \end{aligned}$$

Solution 5.3.3. The augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}\right).$$

Multiply row 1 by -1 and add to row 2:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right).$$

Multiply row 1 by -2 and add to row 3:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right).$$

Add row 2 to row 3:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

The last augmented matrix says

$$\begin{aligned} x_1 + x_2 &= 2, \\ x_2 &= 1, \\ 0 &= 0. \end{aligned}$$

clearly $x_2 = 1$ and $x_1 = 1$. Therefore this system has an unique solution.

To answer questions of existence and uniqueness of solutions of linear systems, it is useful to introduce the concept of the rank of a matrix.

Rank of a Matrix

There are several equivalent definitions for the rank of a matrix. For our purpose, it is most convenient to define the rank of a matrix as the number of nonzero rows in the matrix after it has been transformed into a echelon form by elementary row operations.

In Example 5.3.1, the echelon forms of the coefficient matrix C_e and of the augmented matrix A_e are, respectively,

$$C_\mathrm{e} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\mathrm{e} = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

In C_e, there are two nonzero rows, therefore the rank of the coefficient matrix is 2. In A_e, there are three nonzero rows, therefore the rank of the augmented matrix is 3. As we have shown, this system has no solution.

In Example 5.3.2, the echelon forms of these two matrices are

$$C_\mathrm{e} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\mathrm{e} = \begin{pmatrix} 1 & 3 & 1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

They both have only two nonzero rows. Therefore the rank of the coefficient matrix equals the rank of the augmented matrix. They both equal to 2. As we have seen, this system has infinite number of solutions.

In Example 5.3.3, the two echelon forms are

$$C_\mathrm{e} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_\mathrm{e} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both of them have two nonzero rows, therefore the coefficient matrix and the augmented matrix have the same rank of 2. As we have shown, this system has an unique solution.

From the results of these examples, we can make the following observations:

1. A linear system of m equations and n unknowns has solutions if and only if the coefficient matrix and the augmented matrix have the same rank.
2. If the rank of both matrices is r, and $r < n$, the system has infinitely many solutions.
3. If $r = n$, the system has only one solution.

Actually these statements are generally valid for all linear systems regardless of whether $m < n$, $m = n$, or $m > n$.

The most interesting case is $m = n = r$. In that case, the coefficient matrix is a square matrix. The solution of such systems can be obtained from (1) the Cramer's rule discussed in the chapter of determinants, (2) the Gauss elimination method discussed in this section, and (3) the inverse matrix which we will discuss in Sect. 5.4.

5.4 Inverse Matrix

5.4.1 Nonsingular Matrix

The square matrix A is said to be nonsingular if there exists a matrix B such that

$$BA = I,$$

where I is the identity (unit) matrix. If no matrix B exists, then A is said to be singular. The matrix B is the inverse of A and vice versa. The inverse matrix is denoted by A^{-1}

$$A^{-1} = B.$$

The relationship is reciprocal. If B is the inverse of A, then A is the inverse of B. Since

$$BA = A^{-1}A = I, \tag{5.21}$$

applying B^{-1} from the left

$$B^{-1}BA = B^{-1}I.$$

It follows:

$$A = B^{-1}. \tag{5.22}$$

Existence

If A is nonsingular, then determinant $|A| \neq 0$.

Proof. If A is nonsingular, then by definition A^{-1} exists and $AA^{-1} = I$. Thus

$$\left|AA^{-1}\right| = |A| \cdot \left|A^{-1}\right| = |I| \,.$$

Since $|I| = 1$, neither $|A|$ nor $\left|A^{-1}\right|$ can be zero.

If $|A| \neq 0$, we will show in following sections that A^{-1} can always be found.
□

Uniqueness

The inverse of a matrix, if it exists, is unique. That is, if

$$AB = I,$$
$$AC = I,$$

then

$$B = C.$$

This can be seen as follows. Since $AC = I$, by definition $C = A^{-1}$. It follows that:

$$CA = AC = I.$$

Multiplying this equation from the right by B, we have

$$(CA)B = IB = B.$$

But

$$(CA)B = C(AB) = CI = C.$$

It is clear from the last two equations that $B = C$.

Inverse of Matrix Products

The inverse of the product of a number of matrices, none of which is singular, equals the product of the inverses taken in the reverse order.

Proof. Consider three nonsingular matrices A, B, and C. We will show

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

By definition

$$ABC(ABC)^{-1} = I.$$

Now

$$\begin{aligned} ABC(C^{-1}B^{-1}A^{-1}) &= AB(CC^{-1})B^{-1}A^{-1} \\ &= ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I. \end{aligned}$$

Since the inverse is unique, it follows that:

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

5.4.2 Inverse Matrix by Cramer's Rule

To find A^{-1}, let us consider the set of nonhomogeneous linear equation

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \tag{5.23}$$

written as

$$(A)(x) = (d). \tag{5.24}$$

According to Cramer's rule discussed in the chapter on determinants

$$x_i = \frac{N_i}{|A|},$$

where $|A|$ is the determinant of A and N_i is the determinant

$$N_i = \begin{vmatrix} a_{11} & \dots & a_{1i-1} & d_1 & a_{1i+1} & \dots & a_{1n} \\ a_{21} & \dots & a_{2i-1} & d_2 & a_{2i+1} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{ni-1} & d_n & a_{ni+1} & \dots & a_{nn} \end{vmatrix}.$$

Expanding N_i over the ith column, we have

$$x_i = \frac{1}{|A|} \sum_{j=1}^{n} d_j C_{ji}, \tag{5.25}$$

where C_{ji} is the cofactor of jth row and ith column of A.

Now let $A^{-1} = B$, i.e.,

$$A^{-1} = B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}.$$

Applying A^{-1} to (5.24) from the left

$$(A^{-1})(A)(x) = (A^{-1})(d),$$

so

$$(x) = (A^{-1})(d),$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Thus

$$x_i = \sum_{j=1}^{n} b_{ij} d_j. \tag{5.26}$$

Compare (5.25) and (5.26), it is clear

$$b_{ij} = \frac{1}{|A|} C_{ji} = \frac{1}{|A|} \widetilde{C}_{ij}.$$

Thus the process of obtaining the inverse of a nonsingular matrix involves the following steps:

(a) Obtain the cofactor of every element of the matrix A and write the matrix of cofactors in the form

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}.$$

(b) Transpose the matrix of cofactors to obtain

$$\widetilde{C} = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}.$$

(c) Divide this by $\det A$ to obtain the inverse of A

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}.$$

Example 5.4.1. Find the inverse of the following matrix by Cramer's rule:

$$A = \begin{pmatrix} -3 & 1 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & -2 \end{pmatrix}.$$

Solution 5.4.1. The nine cofactors of A are

$$C_{11} = \begin{vmatrix} -6 & 5 \\ 2 & -2 \end{vmatrix} = 2, \quad C_{12} = -\begin{vmatrix} 15 & 5 \\ -5 & -2 \end{vmatrix} = 5, \quad C_{13} = \begin{vmatrix} 15 & -6 \\ -5 & 2 \end{vmatrix} = 0,$$

$$C_{21} = -\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} -3 & -1 \\ -5 & -2 \end{vmatrix} = 1, \quad C_{23} = -\begin{vmatrix} -3 & 1 \\ -5 & 2 \end{vmatrix} = 1,$$

$$C_{31} = \begin{vmatrix} 1 & -1 \\ -6 & 5 \end{vmatrix} = -1, \quad C_{32} = -\begin{vmatrix} -3 & -1 \\ 15 & 5 \end{vmatrix} = 0, \quad C_{33} = \begin{vmatrix} -3 & 1 \\ 15 & -6 \end{vmatrix} = 3.$$

The value of the determinant of A can be obtained from the Laplacian expansion over any row or any column. For example, over the first column

$$|A| = -3C_{11} + 15C_{21} - 5C_{31} = -6 + 0 + 5 = -1.$$

So the inverse exists. The matrix of cofactors C is

$$C = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}.$$

The inverse of A is then obtained by transposing C and dividing it by $\det A$. Therefore

$$A^{-1} = \frac{1}{|A|}\widetilde{C} = \frac{1}{-1}\begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -2 & 0 & 1 \\ -5 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix}. \tag{5.27}$$

It can be directly verified that

$$A^{-1}A = \begin{pmatrix} -2 & 0 & 1 \\ -5 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix} \begin{pmatrix} -3 & 1 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In literature, the transpose of the cofactor matrix of A is sometimes defined as the adjoint of A, i.e., $\operatorname{adj} A = \widetilde{C}$. However, the name adjoint has another meaning, especially in quantum mechanics. It is often defined as the Hermitian conjugate $A^\dagger$, i.e., $\operatorname{adj} A = A^\dagger$. We will discuss Hermitian matrix in Chap. 6.

For a large matrix, there are more efficient techniques to find the inverse matrix. However, for a 2×2 nonsingular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

one readily obtains from this method

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

This result is simple and useful, It may even be worthwhile to memorize it.

Example 5.4.2. Find the inverse matrices for

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Solution 5.4.2.

$$A^{-1} = \frac{1}{(4-6)} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix},$$

$$R^{-1} = \frac{1}{(\cos^2\theta + \sin^2\theta)} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

One can readily verify

$$\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5.4.3 Inverse of Elementary Matrices

Elementary Matrices

An elementary matrix is a matrix that can be obtained from the identity matrix I by an elementary operation. For example, the elementary matrix E_1 obtained from interchanging row 1 and row 2 of the identity matrix of third order is

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, the elementary row operation of interchanging row 1 and row 2 of any matrix A of order $3\times n$ can be accomplished by premultiplying A by the elementary matrix E_1

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}.$$

The second elementary operation, namely multiplying a row, say row 2, by a scalar k can be accomplished as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \\ a_{31} & a_{32} \end{pmatrix}.$$

Finally, to add the third row k times to the second row, we can proceed in the following way

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} + ka_{31} & a_{22} + ka_{32} \\ a_{31} & a_{32} \end{pmatrix}.$$

Thus, to effect any elementary operation on a matrix A, one may first perform the same elementary operation on an identity matrix to obtain the corresponding elementary matrix. Then premultiply A by the elementary matrix.

Inverse of an Elementary Matrix

Since the elementary matrix is obtained from the elementary operation on the identity matrix, its inverse simply represents the reverse operation. For example, E_1 is obtained from interchanging row 1 and row 2 of the identity matrix I

$$E_1 I = E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$E_1^{-1} E_1 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

E_1^{-1} represents the operation of interchanging row 1 and row 2 of E_1. Thus E_1^{-1} is also given by

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_1.$$

The inverses of the two other kinds of elementary matrices can be obtained in a similar way, namely

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}.$$

It can be readily shown by successive elementary operations that

$$E_4 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad E_4^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$$

and

$$E_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & n & m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -n & -m \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5.4.4 Inverse Matrix by Gauss–Jordan Elimination

For a matrix of large order, Cramer's rule is of little practical use. One of the most commonly used methods for inverting a large matrix is the Gauss–Jordan method.

Equation (5.23) can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}, \tag{5.28}$$

or symbolically as

$$(A)(x) = (I)(d). \tag{5.29}$$

If the both sides of this equation is under the same operation, the equation will remain to be valid. We will operate them with the Gauss–Jordan procedure. Each step is an elementary row operation which can be thought as premultiplying (multiplying from the left) both sides by the elementary matrix representing that operation. Thus the entire Gauss–Jordan process is equivalent to multiplying (5.29) by a matrix B which is a product of all the elementary matrices representing the steps of the Gauss–Jordan procedure

$$(B)(A)(x) = (B)(I)(d). \tag{5.30}$$

Since the process reduces the coefficient matrix A to the identity matrix I, so

$$BA = I.$$

Postmultiplying both sides by A^{-1}

$$BAA^{-1} = IA^{-1},$$

we have

$$B = A^{-1}.$$

Therefore when the left-hand side of (5.30) becomes a unit matrix times the column matrix x, the right-hand side of the equation must be equal to the inverse matrix times the column matrix d.

Thus if we want to find the inverse of A, we can first augment A by the identity matrix I, and then use elementary operations to transform this matrix. When the submatrix A is in the form of I, the form assumed of the original identity matrix I must be A^{-1}.

We have found the inverse of

$$A = \begin{pmatrix} -3 & 1 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & -2 \end{pmatrix}$$

in Example 5.4.1 by Cramer's rule. Now let us do the same problem by Gauss–Jordan elimination. First we augment A by the identity matrix I

$$\left(\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 15 & -6 & 5 & 0 & 1 & 0 \\ -5 & 2 & -2 & 0 & 0 & 1 \end{array}\right).$$

Divide the first row by -3, second row by 15, and third row by -5:

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 1 & -\frac{6}{15} & \frac{5}{15} & 0 & \frac{1}{15} & 0 \\ 1 & -\frac{2}{5} & \frac{2}{5} & 0 & 0 & -\frac{1}{5} \end{array}\right),$$

leave the first row as it is, subtract the first row from the second row and put it in the second row, and subtract the first row from the third row and put it back in the third row

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{15} & 0 & \frac{1}{3} & \frac{1}{15} & 0 \\ 0 & -\frac{1}{15} & \frac{1}{15} & \frac{1}{3} & 0 & -\frac{1}{5} \end{array}\right),$$

multiply the second and third row by -15

$$\left(\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & -5 & -1 & 0 \\ 0 & 1 & -1 & -5 & 0 & 3 \end{array}\right),$$

leave the second row where it is, subtract it from the third row and put the result back to the third row, and then add 1/3 of the second row to the first row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & -2 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -5 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 3 \end{array}\right),$$

multiply the third row by -1, and then subtract 1/3 of it from the first row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -5 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -3 \end{array}\right).$$

Finally we have changed matrix A to the unit matrix I, the original unit matrix on the right side must have changed to A^{-1}, thus

$$A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ -5 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix}.$$

which is the same as (5.27) obtained in Sect. 5.3.

This technique is actually more adapted to modern computers. Computer codes and extensive literature for the Gauss–Jordan elimination method are given in W.H. Press, B.P. Flannery, S.A. Teukolsky,, and W.T. Vetterling, *Numerical Recipes,* 2nd edn. (Cambridge University Press, Cambridge 1992).

Exercises

1. Given two matrices

$$A = \begin{pmatrix} 2 & 5 \\ -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix},$$

find $B - 5A$.

Ans. $\begin{pmatrix} -8 & -25 \\ 12 & -4 \end{pmatrix}$.

2. If A and B are the 2×2 matrices

$$A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix},$$

find the products AB and BA.

Ans. $AB = \begin{pmatrix} 22 & 6 \\ -5 & 5 \end{pmatrix}, \quad BA = \begin{pmatrix} 9 & 11 \\ 2 & 18 \end{pmatrix}$.

3. If

$$A = \begin{pmatrix} 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -4 \\ 3 & -2 \\ -1 & 1 \end{pmatrix},$$

find AB and BA if they exist.

Ans. $AB = \begin{pmatrix} -5 & -2 \\ 2 & 9 \end{pmatrix}, \quad BA = \begin{pmatrix} 14 & -9 & 0 \\ 12 & -7 & 10 \\ -5 & 3 & -3 \end{pmatrix}$.

4. If

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & -2 \\ -1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 \\ 2 & 4 \\ 2 & -1 \end{pmatrix},$$

find AB and BA if they exist.

Ans. $AB = \begin{pmatrix} 2 & 7 \\ 0 & 10 \\ 1 & -7 \\ 8 & -1 \end{pmatrix}$, BA does not exist.

5. If

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix},$$

verify the associative law by showing that

$$(AB)\,C = A\,(BC)\,.$$

6. Show that if

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Hint: $A^n = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right]^n$.

7. Given

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Find all possible products of A, B, C and I, two at a time including squares.
(Note that the products of any two matrices is another matrix in this group. These four matrices form a representation of a mathematical group, known as viergruppe (vier is the German word four).)

8. If

$$A = \begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix},$$

show that $A^2 = 0$.

9. Find the value of

$$(1\;2) \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Ans. 8.

10. Explicitly verify that $(AB)^{\mathrm{T}} = \widetilde{B}\widetilde{A}$, if

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}.$$

11. Show that matrix A is symmetric, if

$$A = B\widetilde{B}.$$

Hint: $a_{ij} = \sum_k b_{ik}\widetilde{b}_{kj}$.

12. Let $A = \begin{pmatrix} 1 & 3 \\ 5 & 12 \end{pmatrix}$, find a matrix E such that EA is diagonal and $|EA| = |A|$.
Ans. $\begin{pmatrix} -4 & 1 \\ -5 & 1 \end{pmatrix}$.

13. Let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix},$$

explicitly show that

$$AB \neq BA \quad \text{but} \quad |AB| = |BA|.$$

14. Show that if

$$[A, B] \neq 0,$$

then

$$(A - B)(A + B) \neq A^2 - B^2,$$
$$(A + B)^2 \neq A^2 + 2AB + B^2.$$

15. Show that

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

16. Show that

$$\left|A^{-1}\right| = |A|^{-1}.$$

Hint: $AA^{-1} = I, \quad |AB| = |A|\,|B|$.

17. Let

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix},$$

find A^{-1}, B^{-1}, and $(AB)^{-1}$ by Cramer's rule and verify that $(AB)^{-1} = B^{-1}A^{-1}$.

18. Reduce the augmented matrix of the following system to an echelon form and show that this system has no solution

$$x_1 + x_2 + 2x_3 + x_4 = 5,$$
$$2x_1 + 3x_2 - x_3 - 2x_4 = 2,$$
$$4x_1 + 5x_2 + 3x_3 = 7.$$

Ans. $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 & -8 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right).$

19. Solve the following equations by Gauss' elimination

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= -1, \\ 3x_1 - 2x_2 + 2x_3 &= 10, \\ 4x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

Ans. $x_3 = -1, x_2 = -3, x_1 = 2$.

20. Let

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & 2 \\ 4 & 1 & 2 \end{pmatrix},$$

find A^{-1} by Gauss–Jordan elimination. Find x_1, x_2, x_3 from

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} -1 \\ 10 \\ 3 \end{pmatrix},$$

and show that

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 3 \end{pmatrix}.$$

21. Determine the rank of the following matrices:

$$\text{(a)} \begin{pmatrix} 4 & 2 & -1 & 3 \\ 0 & 5 & -1 & 2 \\ 12 & -4 & -1 & 5 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 3 & -1 & 4 & -2 \\ 0 & 2 & 4 & 6 \\ 6 & -1 & 10 & -1 \end{pmatrix}.$$

Ans. (a) 2, (b) 2.

22. Determine if the following systems are consistent. If consistent, is the solution unique?

$$\text{(a)} \quad \begin{aligned} x_1 - x_2 + 3x_3 &= -5, \\ -x_1 + 3x_3 &= 0, \\ 2x_1 + x_2 &= 1. \end{aligned} \qquad \text{(b)} \quad \begin{aligned} x_1 - 2x_2 + 3x_3 &= 0, \\ 2x_1 + 3x_2 - x_3 &= 0, \\ 4x_1 - x_2 + 5x_3 &= 0. \end{aligned}$$

Ans. (a) unique solution, (b) infinite number of solutions.

23. Find the value of λ so that the following linear system has a solution

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2, \\ 3x_1 + 2x_2 + x_3 &= 0, \\ x_1 + x_2 + x_3 &= \lambda. \end{aligned}$$

Ans. $\lambda = 0.5$.

24. Let

$$L^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$|-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |\text{null}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Show that

$$L^+ \, |-1\rangle = |0\rangle \,, \quad L^+ \, |0\rangle = |1\rangle \,, \quad L^+ \, |1\rangle = |\text{null}\rangle \,,$$
$$L^- \, |1\rangle = |0\rangle \,, \quad L^- \, |0\rangle = |-1\rangle \,, \quad L^- \, |-1\rangle = |\text{null}\rangle \,.$$