

Vector Calculus

So far we have been discussing constant vectors, but the most interesting applications of vectors involve vector functions. The simplest example is a position vector that depends on time. Such a vector can be differentiated with respect to time. The first and second derivatives are simply the velocity and acceleration of the particle whose position is given by the position vector. In this case, the coordinates of the tip of the position vector are functions of time.

Even more interesting are quantities which depend on the position in space. Such quantities are said to form fields. The word “field” has the connotation that the space has some physical properties. For example, the electrical field created by a static charge is that space surrounding the charge which has now been given a certain property, known as the electrical field. Every point in this field is associated with an electric field vector whose magnitude and direction depend on the location of the point. This electric field vector will manifest itself when another charge is brought to that point. Mathematically a vector field is simply a vector function, each of its three components depends on the coordinates of the point. The field may also depend on time, such as the electric field in the electromagnetic wave.

There are scalar fields, by which we mean that the field is characterized at each point by a single number. Of course the number may change in time, but usually we are talking about the field at a given instant. For example, temperature at different point in space is different, so the temperature is a function of the position. Thus temperature is a scalar field. It is possible to derive one kind of field from another. For example, the directional derivatives of a scalar field lead to a vector field, known as the gradient.

With vector differential calculus, we develop a set of precise terms, such as gradient, divergence, and curl, to describe the rate of change of vector functions with respect to the spatial coordinates. With vector integral calculus, we establish relationships between line, surface, and volume integrals through the theorems of Gauss and Stokes. These are important attributes of vector fields, in terms of which many fundamental laws of physics are expressed.

In this chapter, we shall assume that the vector functions are continuous and differentiable, and the region of interests is simply connected unless otherwise specified. However, this does not mean that singularities and multiple connected regions are not of our concern. They have important implications in physical problems. We will more carefully define and discuss these terms at appropriate places.

2.1 The Time Derivative

Differentiating a vector function is a simple extension of differentiating scalar quantities. If the vector \mathbf{A} depends on time t only, then the derivative of \mathbf{A} with respect to t is defined as

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{A}}{\Delta t}. \quad (2.1)$$

From this definition it follows that the sums and products involving vector quantities can be differentiated as in ordinary calculus; that is

$$\frac{d}{dt} (\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}, \quad (2.2)$$

$$\frac{d}{dt} (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \cdot \mathbf{B}, \quad (2.3)$$

$$\frac{d}{dt} (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B}. \quad (2.4)$$

Since $\Delta \mathbf{A}$ has components ΔA_x , ΔA_y , and ΔA_z ,

$$\frac{d\mathbf{A}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta A_x \mathbf{i} + \Delta A_y \mathbf{j} + \Delta A_z \mathbf{k}}{\Delta t} = \frac{dA_x}{dt} \mathbf{i} + \frac{dA_y}{dt} \mathbf{j} + \frac{dA_z}{dt} \mathbf{k}. \quad (2.5)$$

The time derivatives of a vector is thus equal to the vector sum of the time derivative of its components.

2.1.1 Velocity and Acceleration

Of particular importance is the case where \mathbf{A} is the position vector \mathbf{r} ,

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k}. \quad (2.6)$$

If t changes, the tip of \mathbf{r} traces out a space curve as shown in Fig. 2.1. If a particle is moving along this space curve, then $d\mathbf{r}/dt$ is clearly the velocity \mathbf{v} of the particle along this trajectory

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x \mathbf{i} + \Delta y \mathbf{j} + \Delta z \mathbf{k}}{\Delta t} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}. \quad (2.7)$$

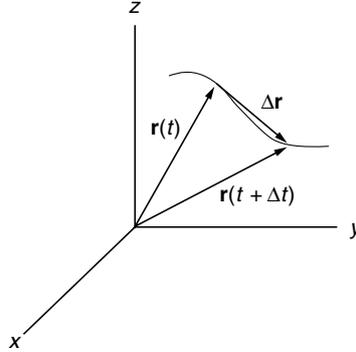


Fig. 2.1. The tip of \mathbf{r} traces out the trajectory of a particle moving in space, $\Delta\mathbf{r}$ is independent of the origin

It is important to note that the direction of $\Delta\mathbf{r}$ is unrelated to the direction of \mathbf{r} . In other words the velocity is independent of the origin chosen. Similarly, the acceleration is defined as the rate of change of velocity

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} = \frac{d^2\mathbf{r}}{dt^2}. \quad (2.8)$$

The acceleration is also independent of the origin.

Notation of differentiation with respect to time. A convenient and widely used notation (Newton's notation) is that a single dot above a symbol denotes the first time derivative and two dots denote the second time derivative, and so on. Thus

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}, \quad (2.9)$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}. \quad (2.10)$$

2.1.2 Angular Velocity Vector

For a particle moving around a circle, shown in Fig. 2.2, the rate of change of the angular position is called angular speed ω :

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \dot{\theta}. \quad (2.11)$$

The velocity \mathbf{v} of the particle is, by definition,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}, \quad (2.12)$$

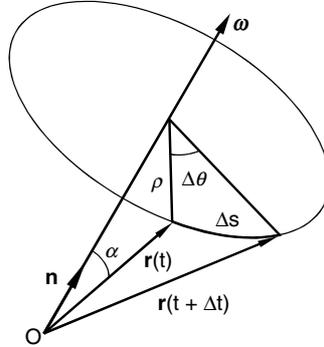


Fig. 2.2. Angular velocity vector ω . The velocity \mathbf{v} of a particle moving around a circle is given by $\mathbf{v} = \omega \times \mathbf{r}$

where \mathbf{r} is the position vector drawn from the origin to the position of the particle. The magnitude of the velocity is given by

$$v = |\mathbf{v}| = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\rho \Delta \theta}{\Delta t} = \rho \omega, \quad (2.13)$$

where ρ is the radius of the circle. The direction of the velocity is, of course, tangent to the circle.

Now, let \mathbf{n} be the unit vector drawn from the origin to the center of the circle, pointing in the positive direction of advance of a right-hand screw when turned in the same sense as the rotation of the particle. Since $|\mathbf{n} \times \mathbf{r}| = r \sin \alpha = \rho$, as shown in Fig. 2.2, the magnitude of the velocity can be written as

$$v = \rho \omega = |\mathbf{n} \times \mathbf{r}| \omega. \quad (2.14)$$

If we define the angular velocity vector ω as

$$\omega = \omega \mathbf{n}, \quad (2.15)$$

then we can write the velocity \mathbf{v} as

$$\mathbf{v} = \dot{\mathbf{r}} = \omega \times \mathbf{r}. \quad (2.16)$$

Recalling the definition of cross product of two vectors, one can easily see that both direction and magnitude of the velocity are given by this equation.

A particle moving in space, even though not in a circle, may always be considered at a given instant to be moving in a circular path. Even a straight line can be considered as a circle with infinite radius. The path, which the particle describes during an infinitesimal time interval δt , may be represented as an infinitesimal arc of a circle. Therefore, at any moment, an instantaneous angular velocity vector can be defined to describe the general motion. The instantaneous velocity is then given by (2.16).

Example 2.1.1. Show that the linear momentum, defined as $\mathbf{p} = m\dot{\mathbf{r}}$, always lies in a fixed plane in a central force field. (A central force field means that the force \mathbf{F} is in the radial direction, such as gravitational and electrostatic forces, in other words \mathbf{F} is parallel to \mathbf{r} .)

Solution 2.1.1. Let us form the angular momentum \mathbf{L}

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}}.$$

Differentiating with respect to time, we have

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}.$$

Now $\dot{\mathbf{r}} \times \mathbf{p} = \dot{\mathbf{r}} \times m\dot{\mathbf{r}} = \mathbf{0}$ and $\mathbf{r} \times \dot{\mathbf{p}} = \mathbf{r} \times m\ddot{\mathbf{r}}$. According to Newton's second law $m\ddot{\mathbf{r}} = \mathbf{F}$ and \mathbf{F} is parallel to \mathbf{r} , therefore $\mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\ddot{\mathbf{r}} = \mathbf{0}$. Thus, $\dot{\mathbf{L}} = \mathbf{0}$. In other words, \mathbf{L} is a constant vector. Furthermore, \mathbf{L} is perpendicular to \mathbf{p} , since $\mathbf{r} \times \mathbf{p}$ is perpendicular to \mathbf{p} . Therefore \mathbf{p} must always lie in the plane perpendicular to the constant vector \mathbf{L} .

Example 2.1.2. Suppose a particle is rotating around the z -axis with a constant angular velocity ω as shown in Fig. 2.3. Find the velocity and acceleration of the particle.

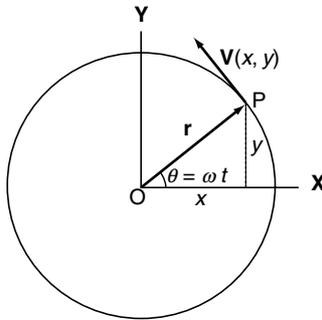


Fig. 2.3. Particle rotating around z -axis with a constant angular velocity ω

Solution 2.1.2. Method I. Since the particle is moving in a circular path and z is not changing in this motion, we will consider only the x and y components. The angular velocity vector is in the \mathbf{k} (unit vector along the z axis) direction, $\boldsymbol{\omega} = \omega\mathbf{k}$, and the position vector \mathbf{r} drawn from the origin to the point (x, y) is perpendicular to \mathbf{k} . Therefore

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

The direction of \mathbf{v} is perpendicular to \mathbf{k} and perpendicular to \mathbf{r} , that is, in the tangential direction of the circle. The magnitude of the velocity is

$$v = \omega r \sin(\pi/2) = \omega r.$$

Explicitly

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix} = -\omega y \mathbf{i} + \omega x \mathbf{j}.$$

The acceleration is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}.$$

Since $\boldsymbol{\omega}$ is a constant, $\dot{\boldsymbol{\omega}} = \mathbf{0}$. Moreover, $\dot{\mathbf{r}} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Thus

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \omega^2 \mathbf{k} \times (\mathbf{k} \times \mathbf{r}).$$

Hence, \mathbf{a} is in the $-\mathbf{r}$ direction and its magnitude is equal to $\omega^2 r$.

Method II. The position vector can be explicitly written as

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} = r \cos \omega t \mathbf{i} + r \sin \omega t \mathbf{j}. \quad (2.17)$$

The velocity and acceleration are, respectively,

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = -\omega r \sin \omega t \mathbf{i} + \omega r \cos \omega t \mathbf{j} = \omega(-y\mathbf{i} + x\mathbf{j}), \quad (2.18)$$

$$v = (\mathbf{v} \cdot \mathbf{v})^{1/2} = (\omega^2 r^2 \sin^2 \omega t + \omega^2 r^2 \cos^2 \omega t)^{1/2} = \omega r \quad (2.19)$$

$$\mathbf{a} = \dot{\mathbf{v}} = -\omega^2 \cos \omega t \mathbf{i} - \omega^2 \sin \omega t \mathbf{j} = -\omega^2(x\mathbf{i} + y\mathbf{j}) = -\omega^2 \mathbf{r} = -\frac{v^2}{r^2} \mathbf{r}. \quad (2.20)$$

We see immediately that the acceleration is toward the center with a magnitude of $\omega^2 r$. This is the familiar *centripetal acceleration*.

Also the velocity is perpendicular to the position vector since

$$\mathbf{v} \cdot \mathbf{r} = \omega(-y\mathbf{i} + x\mathbf{j}) \cdot (x\mathbf{i} + y\mathbf{j}) = 0.$$

In this example, the magnitude of \mathbf{r} is a constant, we have explicitly shown that the velocity is perpendicular to \mathbf{r} . This fact is also a consequence of the following general theorem.

If the magnitude of a vector is not changing, the vector is always orthogonal (perpendicular) to its derivative.

This follows from the fact that if $\mathbf{r} \cdot \mathbf{r} = r_0^2$ and r_0 is a constant, then

$$\begin{aligned}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) &= \frac{d}{dt}r_0^2 = 0, \\ \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) &= \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} + \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 2\frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 0.\end{aligned}$$

When the dot product of two vectors is zero, the two vectors are perpendicular to each other.

This can also be seen from geometry. In Fig. 2.1, if $\mathbf{r}(t)$ and $\mathbf{r}(t + \Delta t)$ have the same length, then $\Delta\mathbf{r}$ is the base of an isosceles triangle. When $\Delta t \rightarrow 0$, the angle between $\mathbf{r}(t)$ and $\mathbf{r}(t + \Delta t)$ also goes to zero. In that case, the two base angles approach 90° , which means $\Delta\mathbf{r}$ is perpendicular to \mathbf{r} . Since Δt is a scalar, the direction of $\Delta\mathbf{r}/\Delta t$ is determined by $\Delta\mathbf{r}$. Therefore $d\mathbf{r}/dt$ is perpendicular to \mathbf{r} .

This theorem is limited neither to the position vector, nor to the time derivative. For example, if vector \mathbf{A} is a function of the arc distance s measured from some fixed point, as long as the magnitude of \mathbf{A} is a constant, it can be shown in the same way that $d\mathbf{A}/ds$ is always perpendicular to \mathbf{A} . This theorem is of considerable importance and should always be kept in mind.

Velocity Vector Field. Sometimes the name velocity (or acceleration) vector field is used to mean that at every point (x, y, z) there is a velocity vector whose magnitude and direction depend on where the point is. In other words, the velocity is a vector function which has three components. Each component can be a function of (x, y, z) . For example, consider a rotating body. The velocity of the material of the body at any point is a vector which is a function of position. In general, a vector function may also explicitly dependent on time t . For example, in a continuum, such as a fluid, the velocity of the particles in the continuum is a vector field which is not only a function of position but may also of time. To find the acceleration, we can use the chain rule:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial\mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial\mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial\mathbf{v}}{\partial t}. \quad (2.21)$$

Since

$$\frac{dx}{dt} = v_x, \quad \frac{dy}{dt} = v_y, \quad \frac{dz}{dt} = v_z,$$

It follows that

$$\mathbf{a} = v_x \frac{\partial\mathbf{v}}{\partial x} + v_y \frac{\partial\mathbf{v}}{\partial y} + v_z \frac{\partial\mathbf{v}}{\partial z} + \frac{\partial\mathbf{v}}{\partial t}. \quad (2.22)$$

Therefore the acceleration may also be a vector field.

Example 2.1.3. A body is rotating around the z -axis with an angular velocity ω , find the velocity of the particles in the body as a function of the position, and use (2.22) to find the acceleration of these particles.

Solution 2.1.3. The angular velocity vector is $\boldsymbol{\omega} = \omega \mathbf{k}$, and the velocity of any point in the body is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y \mathbf{i} + \omega x \mathbf{j}.$$

Thus the components of the velocity vector are

$$v_x(x, y, z) = -\omega y, \quad v_y(x, y, z) = \omega x, \quad v_z(x, y, z) = 0,$$

and

$$\frac{\partial \mathbf{v}}{\partial x} = \omega \mathbf{j}, \quad \frac{\partial \mathbf{v}}{\partial y} = -\omega \mathbf{i}, \quad \frac{\partial \mathbf{v}}{\partial z} = \mathbf{0}, \quad \frac{\partial \mathbf{v}}{\partial t} = \mathbf{0}.$$

Hence according to (2.22)

$$\mathbf{a} = (-\omega y)\omega \mathbf{j} + (\omega x)(-\omega \mathbf{i}) = -\omega^2(x\mathbf{i} + y\mathbf{j}).$$

If we define $\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j}$, then the magnitude of $\boldsymbol{\rho}$ is the perpendicular distance between the particle and the rotating axis. Thus

$$\mathbf{a} = -\omega^2 \boldsymbol{\rho},$$

which shows that every particle has a centripetal acceleration $\omega^2 \rho$, as expected.

2.2 Differentiation in Noninertial Reference Systems

The acceleration \mathbf{a} in Newton's equation $\mathbf{F} = m\mathbf{a}$ is to be measured in an inertial reference system. An inertial reference system is either a coordinate system fixed in space, or a system moving with a constant velocity relative to the fixed system. A coordinate system fixed on the earth is not an inertial system because the earth is rotating.

The derivatives of a vector in a noninertial system are, of course, different from those in a fixed system. To find the relationships between them, let us first consider a moving coordinate system that has the same origin as a fixed system. Intuition tells us that, in this case, the only possible relative motion between the coordinate systems is a rotation. To transform the derivatives from one system to the other, we need to take this rotation into account.

Let us denote the quantities associated with the moving system by a prime. The position vector of a particle expressed in terms of the basis vector of the fixed system is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \quad (2.23)$$

The same position vector expressed in the moving coordinate system whose origin coincides with that of the fixed system becomes

$$\mathbf{r} = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}', \quad (2.24)$$

where \mathbf{i}' , \mathbf{j}' , \mathbf{k}' are the basis vectors of the moving system.

The velocity \mathbf{v} is by definition the time derivative of the position vector in the fixed system

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \quad (2.25)$$

If we express the time derivative of \mathbf{r} in the moving system, then with (2.24) we have

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx'}{dt}\mathbf{i}' + \frac{dy'}{dt}\mathbf{j}' + \frac{dz'}{dt}\mathbf{k}' + x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt}, \quad (2.26)$$

since \mathbf{i}' , \mathbf{j}' , \mathbf{k}' are fixed in the moving system, so they are not constant in time. Clearly, the velocity seen in the moving system is

$$\mathbf{v}' = \frac{dx'}{dt}\mathbf{i}' + \frac{dy'}{dt}\mathbf{j}' + \frac{dz'}{dt}\mathbf{k}' = \frac{D\mathbf{r}}{Dt}. \quad (2.27)$$

This equation also defines the operation D/Dt , which simply means the time derivative in the moving system. The notation

$$\frac{D\mathbf{r}}{Dt} = \dot{\mathbf{r}} \quad (2.28)$$

is also often used. As mentioned earlier, a dot on top of a symbol means the time derivative. In addition, it usually means the time derivative in the moving system. Note that while the position vector has the same appearance in both the fixed and the moving system as seen in (2.23) and (2.24), the velocity vector, or any other derivative, has more terms in the moving system than in the fixed system as seen in (2.26) and (2.25). The three derivatives (dx'/dt , dy'/dt , dz'/dt) are not the components of the velocity vector \mathbf{v} in the moving system, they only appear to be the velocity components to a stationary observer in the moving system. The velocity vector \mathbf{v} expressed in the moving system is given by (2.26), which can be written as

$$\mathbf{v} = \frac{D\mathbf{r}}{Dt} + x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt}. \quad (2.29)$$

Since \mathbf{i}' , \mathbf{j}' , \mathbf{k}' are unit vectors, their magnitudes are not changing. Therefore their derivatives must be perpendicular to themselves. For example, $d\mathbf{i}'/dt$ is perpendicular to \mathbf{i}' , and lies in the plane of \mathbf{j}' and \mathbf{k}' . Thus we can write

$$\frac{d\mathbf{i}'}{dt} = c\mathbf{j}' - b\mathbf{k}', \quad (2.30)$$

where c and $-b$ are two constants. (The reason for choosing these particular symbols for the coefficients of the linear combination is for convenience, as will be clear in a moment). Similarly

$$\frac{d\mathbf{j}'}{dt} = a\mathbf{k}' - f\mathbf{i}', \quad (2.31)$$

$$\frac{d\mathbf{k}'}{dt} = e\mathbf{i}' - d\mathbf{j}'. \quad (2.32)$$

But $\mathbf{i}' = \mathbf{j}' \times \mathbf{k}'$, so

$$\frac{d\mathbf{i}'}{dt} = \frac{d\mathbf{j}'}{dt} \times \mathbf{k}' + \mathbf{j}' \times \frac{d\mathbf{k}'}{dt} = (a\mathbf{k}' - f\mathbf{i}') \times \mathbf{k}' + \mathbf{j}' \times (e\mathbf{i}' - d\mathbf{j}') = f\mathbf{j}' - e\mathbf{k}'. \quad (2.33)$$

Comparing (2.30) and (2.33), we see that $f = c$ and $e = b$. Similarly, from $\mathbf{j}' = \mathbf{k}' \times \mathbf{i}'$ one can show that

$$\frac{d\mathbf{j}'}{dt} = d\mathbf{k}' - c\mathbf{i}'. \quad (2.34)$$

It is clear from (2.31) and (2.34) that $d = a$ and $c = f$.

It follows that

$$\begin{aligned} x' \frac{d\mathbf{i}'}{dt} + y' \frac{d\mathbf{j}'}{dt} + z' \frac{d\mathbf{k}'}{dt} &= x' (c\mathbf{j}' - b\mathbf{k}') + y' (a\mathbf{k}' - c\mathbf{i}') + z' (b\mathbf{i}' - a\mathbf{j}') \\ &= \mathbf{i}' (bz' - cy') + \mathbf{j}' (cx' - az') + \mathbf{k}' (ay' - bx') \\ &= \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ a & b & c \\ x' & y' & z' \end{vmatrix}. \end{aligned} \quad (2.35)$$

If we define

$$\boldsymbol{\omega} = a\mathbf{i}' + b\mathbf{j}' + c\mathbf{k}', \quad (2.36)$$

with \mathbf{r} given by (2.24), we can write (2.35) as

$$x' \frac{d\mathbf{i}'}{dt} + y' \frac{d\mathbf{j}'}{dt} + z' \frac{d\mathbf{k}'}{dt} = \boldsymbol{\omega} \times \mathbf{r}. \quad (2.37)$$

The meaning of $\boldsymbol{\omega} \times \mathbf{r}$ is exactly the same as in (2.16). We have thus demonstrated explicitly that the most general relative motion of two coordinate systems having a common origin is a rotation with an instantaneous angular velocity $\boldsymbol{\omega}$. Furthermore, (2.29) becomes

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}. \quad (2.38)$$

Often this equation is written in the form

$$\frac{d\mathbf{r}}{dt} = \left(\frac{D}{Dt} + \boldsymbol{\omega} \times \right) \mathbf{r} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}, \quad (2.39)$$

with the understanding that the time derivative on the left-hand side is in the fixed system, and on the right-hand side, all quantities are to be expressed in the rotating system.

This analysis is not limited to the position vector. For any vector \mathbf{A} , we can follow exactly the same procedure and show that

$$\boxed{\frac{d\mathbf{A}}{dt} = \frac{D}{Dt}\mathbf{A} + \boldsymbol{\omega} \times \mathbf{A}}, \quad (2.40)$$

where

$$\begin{aligned} \frac{D}{Dt}\mathbf{A} &= \dot{\mathbf{A}} = \dot{A}'_x \mathbf{i}' + \dot{A}'_y \mathbf{j}' + \dot{A}'_z \mathbf{k}', \\ \boldsymbol{\omega} \times \mathbf{A} &= A'_x \frac{d\mathbf{i}'}{dt} + A'_y \frac{d\mathbf{j}'}{dt} + A'_z \frac{d\mathbf{k}'}{dt}. \end{aligned}$$

Example 2.2.1. Show that the time derivative of the angular velocity vector is the same in either the fixed or the rotating system.

Solution 2.2.1. Since

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{D}{Dt}\boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\omega},$$

but $\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$, therefore the time derivative in the rotating system $\dot{\boldsymbol{\omega}}$ is the same time derivative in the fixed system.

Example 2.2.2. If the rotating system and the fixed system have the same origin, express the acceleration \mathbf{a} in the fixed system in terms of \mathbf{a}' , \mathbf{v}' , $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$ of the rotating system.

Solution 2.2.2. By definition $\mathbf{a} = d\mathbf{v}/dt$. So by (2.40)

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{D}{Dt}\mathbf{v} + \boldsymbol{\omega} \times \mathbf{v}.$$

Since

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{D}{Dt}\mathbf{r} + \boldsymbol{\omega} \times \mathbf{r} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r},$$

$$\frac{D}{Dt}\mathbf{v} = \frac{D}{Dt}(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) = \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}},$$

$$\boldsymbol{\omega} \times \mathbf{v} = \boldsymbol{\omega} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

Therefore,

$$\begin{aligned}\mathbf{a} &= \frac{D}{Dt}\mathbf{v} + \boldsymbol{\omega} \times \mathbf{v} = \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).\end{aligned}$$

Since $\ddot{\mathbf{r}} = \mathbf{a}'$ and $\dot{\mathbf{r}} = \mathbf{v}'$,

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

In general, the primed system may have both translational and rotational motion. This can be thought as a translation followed by a rotation. It is clear from Fig. 2.4 that $\mathbf{r} = \mathbf{r}' + \mathbf{r}_0$, so

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \frac{d\mathbf{r}_0}{dt} \\ &= \left(\frac{D\mathbf{r}'}{Dt} + \boldsymbol{\omega} \times \mathbf{r}' \right) + \frac{d\mathbf{r}_0}{dt}.\end{aligned}\quad (2.41)$$

The translational velocity of the coordinates is simply $\mathbf{v}_0 = d\mathbf{r}_0/dt$. Similarly the linear acceleration of the coordinates is $\mathbf{a}_0 = d^2\mathbf{r}_0/dt^2$. Therefore, the acceleration of the particle is given by

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left(\frac{D\mathbf{r}'}{Dt} + \boldsymbol{\omega} \times \mathbf{r}' \right) + \frac{d^2\mathbf{r}_0}{dt^2} \\ &= \frac{D}{Dt} \left(\frac{D\mathbf{r}'}{Dt} + \boldsymbol{\omega} \times \mathbf{r}' \right) + \boldsymbol{\omega} \times \left(\frac{D\mathbf{r}'}{Dt} + \boldsymbol{\omega} \times \mathbf{r}' \right) + \mathbf{a}_0.\end{aligned}\quad (2.42)$$

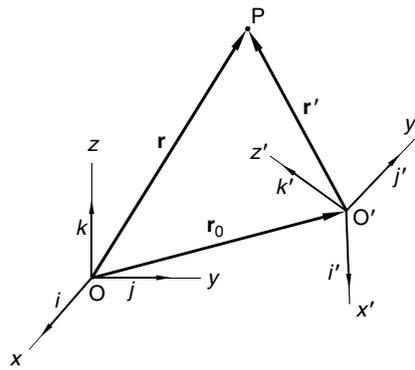


Fig. 2.4. Geometry of the coordinate systems. The primed system is a noninertial reference frame which has both translational and rotational motion relative to the fixed system

Therefore the general equations for the transformation from a fixed system to a moving system are

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{v}_0, \tag{2.43}$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{a}_0. \tag{2.44}$$

The term $\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ is known as the *transverse acceleration* and the term $2\boldsymbol{\omega} \times \mathbf{v}'$ is called the *coriolis acceleration*. The *centripetal acceleration* $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is always directed toward the center.

2.3 Theory of Space Curve

Suppose we have a particle moving on a space curve as shown in Fig. 2.5. At certain time, the particle is at some point P . In a time interval Δt the particle moves to another point P' along the path. The arc distance between P and P' is Δs . Let \mathbf{t} be the unit vector in the direction of the tangent of the curve at P . The velocity of the particle is of course in the direction of \mathbf{t} ,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v\mathbf{t}, \tag{2.45}$$

where v is the magnitude of the velocity, which is given by

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}. \tag{2.46}$$

The point P can also be specified by s , the distance along the curve measured from a fixed point to P . Then by chain rule,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{ds}{dt} \frac{d\mathbf{r}}{ds} = v \frac{d\mathbf{r}}{ds}. \tag{2.47}$$

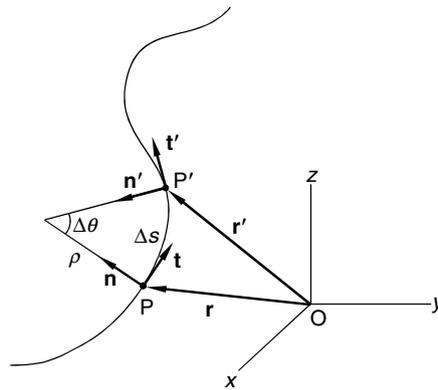


Fig. 2.5. Space curve. The tangent vector \mathbf{t} and the normal vector \mathbf{n} determine the osculating plane which may turn in space

Comparing (2.45) and (2.47), we have

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}, \quad (2.48)$$

hardly a surprising result. Clearly, as $\Delta s \rightarrow 0$, $\left| \frac{\Delta \mathbf{r}}{\Delta s} \right| = 1$.

Now, \mathbf{t} is a unit vector, which means its magnitude is not changing, therefore its derivative must be perpendicular to itself. Let \mathbf{n} be a unit vector perpendicular to \mathbf{t} , so we can write

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n}, \quad (2.49)$$

where κ is the magnitude of $d\mathbf{t}/ds$ and is called the curvature. The vector \mathbf{n} is known as the normal vector and is perpendicular to \mathbf{t} . The reciprocal of the curvature $\varrho = 1/\kappa$ is known as the radius of the curvature. Equation (2.49) defines both κ and \mathbf{n} , and tells us how fast the unit tangent vector changes direction as we move along the curve.

The acceleration of the particle is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{t}) = \dot{v}\mathbf{t} + v\dot{\mathbf{t}},$$

where

$$\dot{\mathbf{t}} = \frac{d\mathbf{t}}{dt} = \frac{ds}{dt} \frac{d\mathbf{t}}{ds} = v \frac{d\mathbf{t}}{ds} = v\kappa \mathbf{n}. \quad (2.50)$$

Therefore the acceleration can be written as

$$\mathbf{a} = \dot{v}\mathbf{t} + v^2\kappa \mathbf{n} = \dot{v}\mathbf{t} + \frac{v^2}{\varrho} \mathbf{n}. \quad (2.51)$$

The tangential component of \mathbf{a} corresponds to the change in the magnitude of v , and the normal component of \mathbf{a} corresponds to the change in direction of \mathbf{v} . The normal component is the familiar centripetal acceleration.

For the circular motion in example 2.1.2, $\mathbf{r} = r \cos \omega t \mathbf{i} + r \sin \omega t \mathbf{j}$ and $v = \omega r$,

$$\mathbf{t} = \frac{1}{v} \mathbf{v} = \frac{1}{\omega r} (-r\omega \sin \omega t \mathbf{i} + r\omega \cos \omega t \mathbf{j}) = -\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}.$$

Since by (2.50),

$$\frac{d\mathbf{t}}{ds} = \frac{1}{v} \frac{d\mathbf{t}}{dt} = \frac{1}{\omega r} (-\omega \cos \omega t \mathbf{i} - \omega \sin \omega t \mathbf{j}) = -\frac{1}{r^2} (r \cos \omega t \mathbf{i} + r \sin \omega t \mathbf{j}) = -\frac{1}{r^2} \mathbf{r}.$$

But by definition, $d\mathbf{t}/ds = \kappa \mathbf{n}$. So, in this case, $\mathbf{n} = -\mathbf{r}/r$, $\kappa = 1/r$, and $\varrho = 1/\kappa = r$. In other words, the radius of curvature in a circular motion is equal to the radius of the circle.

In a small region, we can approximate Δs by the arc of a circle. The radius of this circle is the radius of curvature of the curve as shown in Fig. 2.5.

The motion may not be confined in a plane, although both the velocity and acceleration lie in the plane of \mathbf{t} and \mathbf{n} , known as the *osculating plane*. In general, there is another degree of freedom for the motion, namely the arc as a whole may turn. In other words, the osculating plane is not necessarily fixed in space. We need another factor to compute the derivatives of the acceleration.

Let us define a third vector, known as *binormal vector*,

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}. \quad (2.52)$$

Since both \mathbf{t} and \mathbf{n} are unit vectors and they are perpendicular to each other, therefore \mathbf{b} is also a unit vector and is perpendicular to both \mathbf{t} and \mathbf{n} . It follows from the definition that, in a right-hand system,

$$\mathbf{b} \times \mathbf{t} = \mathbf{n}, \quad \mathbf{b} \times \mathbf{n} = -\mathbf{t}, \quad \mathbf{n} \times \mathbf{t} = -\mathbf{b}. \quad (2.53)$$

All vectors associated with the curve at the point P can be written as a linear combination of \mathbf{t} , \mathbf{n} , and \mathbf{b} which form a basis at P . Now we evaluate $d\mathbf{b}/ds$ and $d\mathbf{n}/ds$.

Since \mathbf{b} is perpendicular to \mathbf{t} , so $\mathbf{b} \cdot \mathbf{t} = 0$. After differentiating we have

$$\frac{d}{ds}(\mathbf{b} \cdot \mathbf{t}) = \frac{d\mathbf{b}}{ds} \cdot \mathbf{t} + \mathbf{b} \cdot \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{b}}{ds} \cdot \mathbf{t} + \mathbf{b} \cdot \kappa \mathbf{n} = 0. \quad (2.54)$$

Hence, $d\mathbf{b}/ds \cdot \mathbf{t} = -\kappa \mathbf{b} \cdot \mathbf{n}$. Since \mathbf{b} is perpendicular to \mathbf{n} , $\mathbf{b} \cdot \mathbf{n} = 0$. Thus, $d\mathbf{b}/ds \cdot \mathbf{t} = 0$, which means $d\mathbf{b}/ds$ is perpendicular to \mathbf{t} . On the other hand, since \mathbf{b} is a unit vector, so $d\mathbf{b}/ds$ is perpendicular \mathbf{b} . Therefore $d\mathbf{b}/ds$ must be in the direction of \mathbf{n} . Let

$$d\mathbf{b}/ds = \gamma \mathbf{n}, \quad (2.55)$$

where γ , by definition, is the magnitude of $d\mathbf{b}/ds$ and is called the *torsion of the curve*.

To obtain $d\mathbf{n}/ds$, we use (2.53),

$$\frac{d\mathbf{n}}{ds} = \frac{d}{ds}(\mathbf{b} \times \mathbf{t}) = \frac{d\mathbf{b}}{ds} \times \mathbf{t} + \mathbf{b} \times \frac{d\mathbf{t}}{ds} = \gamma \mathbf{n} \times \mathbf{t} + \mathbf{b} \times \kappa \mathbf{n} = -\gamma \mathbf{b} - \kappa \mathbf{t}. \quad (2.56)$$

The set of equations

$$\boxed{\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = -(\gamma \mathbf{b} + \kappa \mathbf{t}), \quad \frac{d\mathbf{b}}{ds} = \gamma \mathbf{n}} \quad (2.57)$$

are the famous *Frenet-Serret formulas*. They are fundamental equations in differential geometry.

Example 2.3.1. Find the arc length s of the curve $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$ between $t = 0$ and $t = T$. Express \mathbf{r} as a function of s .

Solution 2.3.1. Since $ds/dt = v$ and $v = (\mathbf{v} \cdot \mathbf{v})^{1/2} = (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{1/2}$,

$$ds = v dt = (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{1/2} dt.$$

This seemingly trivial formula is actually very useful in a variety of problems. In the present case

$$\begin{aligned} ds &= [(-a \sin t \mathbf{i} + a \cos t \mathbf{j}) \cdot (-a \sin t \mathbf{i} + a \cos t \mathbf{j})]^{1/2} dt = a dt, \\ s &= \int_0^T a dt = aT. \end{aligned}$$

In general $s = at$ and $t = s/a$, thus

$$\mathbf{r}(s) = a \cos \frac{s}{a} \mathbf{i} + a \sin \frac{s}{a} \mathbf{j}.$$

Example 2.3.2. A circular helix is given by $\mathbf{r} = a \cos t \mathbf{i} + a \sin t \mathbf{j} + bt \mathbf{k}$, calculate \mathbf{t} , \mathbf{n} , \mathbf{b} and κ, ρ, γ for this curve.

Solution 2.3.2.

$$\begin{aligned} \mathbf{v} = \dot{\mathbf{r}} &= -a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k}, \\ v = (\mathbf{v} \cdot \mathbf{v})^{1/2} &= [(a^2 \sin^2 t + a^2 \cos^2 t) + b^2]^{1/2} = (a^2 + b^2)^{1/2}, \\ \mathbf{t} = \frac{1}{v} \mathbf{v} &= \frac{1}{(a^2 + b^2)^{1/2}} (-a \sin t \mathbf{i} + a \cos t \mathbf{j} + b \mathbf{k}). \end{aligned}$$

Since

$$\begin{aligned} \frac{d\mathbf{t}}{dt} &= \frac{ds}{dt} \frac{d\mathbf{t}}{ds} = v \frac{d\mathbf{t}}{ds}, \\ \frac{d\mathbf{t}}{ds} &= \frac{1}{v} \frac{d\mathbf{t}}{dt} = \frac{1}{(a^2 + b^2)} (-a \cos t \mathbf{i} - a \sin t \mathbf{j}) = \kappa \mathbf{n}, \\ \kappa^2 &= (\kappa \mathbf{n} \cdot \kappa \mathbf{n}) = \frac{1}{(a^2 + b^2)^2} (a^2 \cos^2 t + a^2 \sin^2 t) = \frac{a^2}{(a^2 + b^2)^2}, \\ \kappa &= \frac{a}{(a^2 + b^2)}, \quad \rho = \frac{1}{\kappa} = \frac{a^2 + b^2}{a}, \\ \mathbf{n} &= \frac{1}{\kappa} \frac{d\mathbf{t}}{ds} = -\cos t \mathbf{i} - \sin t \mathbf{j}. \end{aligned}$$

$$\begin{aligned}
\mathbf{b} = \mathbf{t} \times \mathbf{n} &= \frac{1}{(a^2 + b^2)^{1/2}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix} \\
&= \frac{1}{(a^2 + b^2)^{1/2}} [b \sin t \mathbf{i} - b \cos t \mathbf{j} + (a \sin^2 t + a \cos^2 t) \mathbf{k}] \\
&= \frac{1}{(a^2 + b^2)^{1/2}} [b \sin t \mathbf{i} - b \cos t \mathbf{j} + a \mathbf{k}].
\end{aligned}$$

Use chain rule again,

$$\frac{d\mathbf{b}}{dt} = \frac{ds}{dt} \frac{d\mathbf{b}}{ds} = v \frac{d\mathbf{b}}{ds},$$

so

$$\begin{aligned}
\frac{d\mathbf{b}}{ds} &= \frac{1}{v} \frac{d\mathbf{b}}{dt} = \frac{b}{(a^2 + b^2)} [\cot t \mathbf{i} + \sin t \mathbf{j}] = \gamma \mathbf{n}, \\
\gamma &= -\frac{b}{(a^2 + b^2)}.
\end{aligned}$$

2.4 The Gradient Operator

The application of vector methods to physical problems most frequently takes the form of differential operations. We have discussed the rate of change with respect to time which allows us to define velocity and acceleration vectors for the motion of a particle. Now we begin a more systematic study of the rate of change with respect to the spatial coordinates. The most important differential operator is the gradient.

2.4.1 The Gradient of a Scalar Function

Before we discuss the gradient, let us briefly review the notation of derivative and partial derivative in calculus

$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}.$$

If $\Delta x \rightarrow 0$ is implicitly understood, we can multiply both sides by Δx and write

$$\Delta f = f(x + \Delta x) - f(x) = \frac{df}{dx} \Delta x. \quad (2.58)$$

This equation can also be derived from the Taylor expansion of $f(x + \Delta x)$ around x :

$$f(x + \Delta x) = f(x) + \frac{df}{dx} \Delta x + \frac{1}{2} \frac{d^2 f}{dx^2} (\Delta x)^2 + \dots$$

Equation (2.58) is obtained if one drops terms of $(\Delta x)^n$ with $n \geq 2$ as $\Delta x \rightarrow 0$.

Similarly in terms of partial derivatives

$$f(x + \Delta x, y) - f(x, y) = \frac{\partial f}{\partial x} \Delta x,$$

or

$$f(x, y + \Delta y, z + \Delta z) - f(x, y, z + \Delta z) = \frac{\partial f}{\partial y} \Delta y.$$

Let the difference of the scalar function φ between two nearby points $(x + \Delta x, y + \Delta y, z + \Delta z)$ and (x, y, z) be $\Delta\varphi$:

$$\Delta\varphi = \varphi(x + \Delta x, y + \Delta y, z + \Delta z) - \varphi(x, y, z).$$

This equation can be written as

$$\begin{aligned} \Delta\varphi &= \varphi(x + \Delta x, y + \Delta y, z + \Delta z) \\ &\quad - [\varphi(x, y + \Delta y, z + \Delta z) - \varphi(x, y + \Delta y, z + \Delta z)] \\ &\quad - [\varphi(x, y, z + \Delta z) - \varphi(x, y, z + \Delta z)] - \varphi(x, y, z), \end{aligned}$$

since the quantities in the two brackets cancel out. Removing the brackets, we have

$$\begin{aligned} \Delta\varphi &= \varphi(x + \Delta x, y + \Delta y, z + \Delta z) - \varphi(x, y + \Delta y, z + \Delta z) \\ &\quad + \varphi(x, y + \Delta y, z + \Delta z) - \varphi(x, y, z + \Delta z) \\ &\quad + \varphi(x, y, z + \Delta z) - \varphi(x, y, z). \end{aligned}$$

With the definition of partial derivative, the above equation can be written as

$$\Delta\varphi = \frac{\partial\varphi}{\partial x} \Delta x + \frac{\partial\varphi}{\partial y} \Delta y + \frac{\partial\varphi}{\partial z} \Delta z. \quad (2.59)$$

The displacement vector from (x, y, z) to $(x + \Delta x, y + \Delta y, z + \Delta z)$ is, of course,

$$\Delta\mathbf{r} = \Delta x \mathbf{i} + \Delta y \mathbf{j} + \Delta z \mathbf{k}.$$

One can readily verify that

$$\left(\mathbf{i} \frac{\partial\varphi}{\partial x} + \mathbf{j} \frac{\partial\varphi}{\partial y} + \mathbf{k} \frac{\partial\varphi}{\partial z} \right) \cdot (\Delta x \mathbf{i} + \Delta y \mathbf{j} + \Delta z \mathbf{k}) = \frac{\partial\varphi}{\partial x} \Delta x + \frac{\partial\varphi}{\partial y} \Delta y + \frac{\partial\varphi}{\partial z} \Delta z.$$

Thus,

$$\Delta\varphi = \left(\mathbf{i} \frac{\partial\varphi}{\partial x} + \mathbf{j} \frac{\partial\varphi}{\partial y} + \mathbf{k} \frac{\partial\varphi}{\partial z} \right) \cdot \Delta\mathbf{r}. \quad (2.60)$$

The vector in the parenthesis is called the *gradient* of φ , and is usually written as $\text{grad } \varphi$ or $\nabla\varphi$,

$$\nabla\varphi = \mathbf{i} \frac{\partial\varphi}{\partial x} + \mathbf{j} \frac{\partial\varphi}{\partial y} + \mathbf{k} \frac{\partial\varphi}{\partial z}. \quad (2.61)$$

Since φ is an arbitrary scalar function, it is convenient to define the differential operation in terms of the *gradient operator* ∇ (sometimes known as *del* or

del operator)

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}. \quad (2.62)$$

This is a vector operator and obeys the same convention as the derivative notation. If a function is placed on the left-hand side of it, $\varphi \nabla$ is still an operator and by itself means nothing. What is to be differentiated must be placed on the right of ∇ . When it operates on a scalar function, it turns $\nabla \varphi$ into a vector with definite magnitude and direction. It also has a definite physical meaning.

Example 2.4.1. Show that $\nabla r = \hat{\mathbf{r}}$ and $\nabla f(r) = \hat{\mathbf{r}} df/dr$, where $\hat{\mathbf{r}}$ is a unit vector along the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and r is the magnitude of \mathbf{r} .

Solution 2.4.1.

$$\begin{aligned} \nabla r &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) r, \\ \mathbf{i} \frac{\partial r}{\partial x} &= \mathbf{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{1/2} = \frac{\mathbf{i}x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{\mathbf{i}x}{r}, \quad \text{etc.} \\ \nabla r &= \frac{\mathbf{i}x}{r} + \frac{\mathbf{j}y}{r} + \frac{\mathbf{k}z}{r} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}. \end{aligned}$$

$$\begin{aligned} \nabla f(r) &= \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z}, \\ \mathbf{i} \frac{\partial f}{\partial x} &= \mathbf{i} \frac{df}{dr} \frac{\partial r}{\partial x} = \mathbf{i} \frac{df}{dr} \frac{x}{r}, \quad \text{etc.} \\ \nabla f(r) &= \mathbf{i} \frac{df}{dr} \frac{x}{r} + \mathbf{j} \frac{df}{dr} \frac{y}{r} + \mathbf{k} \frac{df}{dr} \frac{z}{r} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r} \frac{df}{dr} = \hat{\mathbf{r}} \frac{df}{dr}. \end{aligned}$$

Example 2.4.2. Show that $(\mathbf{A} \cdot \nabla) \mathbf{r} = \mathbf{A}$.

Solution 2.4.2.

$$\begin{aligned} (\mathbf{A} \cdot \nabla) \mathbf{r} &= \left[(A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}) \cdot \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \right] \mathbf{r} \\ &= \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} = \mathbf{A}. \end{aligned}$$

2.4.2 Geometrical Interpretation of Gradient

To see the physical meaning of $\nabla \varphi$, let us substitute (2.61) into (2.60)

$$\Delta \varphi = \nabla \varphi \cdot \Delta \mathbf{r}.$$

Taking the limit as $\Delta \mathbf{r}$ approaches zero yields the differential form of this equation:

$$d\varphi = \nabla\varphi \cdot d\mathbf{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz. \quad (2.63)$$

Now, $\varphi(x, y, z) = C$ represents a surface in space. For example, $\varphi(x, y, z) = x + y + z$ and $\varphi = C$ represents a family of parallel planes, as discussed in the previous chapter. Different values of C simply mean the different perpendicular distances between the plane and the origin. Another example, $\varphi(x, y, z) = x^2 + y^2 + z^2 = 4$ is the surface of a sphere of radius 2. Changing 4 to 9 simply means another sphere of radius 3.

If the two near-by points lie on the same surface $\varphi = C$, then clearly $d\varphi = 0$, since $\varphi(x + dx, y + dy, z + dz) = \varphi(x, y, z) = C$. In this case $d\mathbf{r}$ is, of course, a vector on this surface and

$$d\varphi = \nabla\varphi \cdot d\mathbf{r} = 0. \quad (2.64)$$

Since the dot product of $\nabla\varphi$ and $d\mathbf{r}$ is equal to zero, $\nabla\varphi$ must be perpendicular to $d\mathbf{r}$. Therefore $\nabla\varphi$ is normal (perpendicular) to the surface $\varphi = C$ as shown in Fig. 2.6.

We can look at it in another way. Let the unit vector in the direction $d\mathbf{r}$ be \mathbf{d} and the magnitude of $d\mathbf{r}$ be dr , then the dot product of (2.63) can be written as

$$d\varphi = \nabla\varphi \cdot \mathbf{d} dr,$$

or

$$\frac{d\varphi}{dr} = \nabla\varphi \cdot \mathbf{d}. \quad (2.65)$$

This means that the rate of change of φ in the direction of \mathbf{d} is equal to $\nabla\varphi \cdot \mathbf{d}$. Furthermore, since

$$\nabla\varphi \cdot \mathbf{d} = |\nabla\varphi| \cos\theta,$$

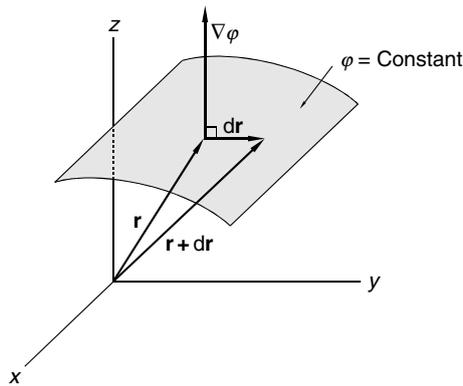


Fig. 2.6. Gradient of a scalar function. $\nabla\varphi$ is a vector normal to the surface of $\varphi = \text{constant}$

where θ is the angle between $d\mathbf{r}$ and $\nabla\varphi$, the maximum rate of change occurs at $\theta = 0$. This means that if $d\mathbf{r}$ and $\nabla\varphi$ are in the same direction, the change of φ is the largest. Therefore the meaning of $\nabla\varphi$ can be summarized as follows:

The vector $\nabla\varphi$ is in the direction of the steepest increase in φ and the magnitude of the vector $\nabla\varphi$ is equal to the rate of increase in that direction.

Example 2.4.3. Find the unit normal to the surface described by $\varphi(x, y, z) = 2x^2 + 4yz - 5z^2 = -10$ at $(3, -1, 2)$.

Solution 2.4.3. First check the point $(3, -1, 2)$ is indeed on the surface: $2(3)^2 + 4(-1)2 - 5(2)^2 = -10$. Recall the unit normal to the surface at any point is $\mathbf{n} = \nabla\varphi/|\nabla\varphi|$.

$$\begin{aligned}\nabla\varphi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(2x^2 + 4yz - 5z^2) = 4x\mathbf{i} + 4z\mathbf{j} + (4y - 10z)\mathbf{k} \\ \mathbf{n} &= \left[\frac{\nabla\varphi}{|\nabla\varphi|}\right]_{3,-1,2} = \frac{12\mathbf{i} + 8\mathbf{j} - 24\mathbf{k}}{(12^2 + 8^2 + 24^2)^{1/2}} = \frac{3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}}{\sqrt{46}}.\end{aligned}$$

Example 2.4.4. Find the maximum rate of increase for the surface $\varphi(x, y, z) = 100 + xyz$ at the point $(1, 3, 2)$. In which direction is the maximum rate of increase?

Solution 2.4.4. The maximum rate of increase is $|\nabla\varphi|_{1,3,2}$.

$$\begin{aligned}\nabla\varphi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(100 + xyz) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}, \\ |\nabla\varphi|_{1,3,2} &= |6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}| = (36 + 4 + 9)^{1/2} = 9.\end{aligned}$$

The direction of the maximum increase is given by

$$\nabla\varphi|_{1,3,2} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}.$$

Example 2.4.5. Find the rate of increase for the surface $\varphi(x, y, z) = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Solution 2.4.5.

$$\begin{aligned}\nabla\varphi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3) = y^2\mathbf{i} + (2xy + z^3)\mathbf{j} + 3yz^2\mathbf{k}, \\ \nabla\varphi_{2,-1,1} &= \mathbf{i} - 3\mathbf{j} - 3\mathbf{k}.\end{aligned}$$

The unit vector along $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is

$$\mathbf{n} = \frac{\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{1+4+4}} = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}).$$

The rate of increase is

$$\frac{d\varphi}{dr} = \nabla\varphi \cdot \mathbf{n} = (\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}) \cdot \frac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = -\frac{11}{3}.$$

Example 2.4.6. Find the equation of the tangent plane to the surface described by $\varphi(x, y, z) = 2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.

Solution 2.4.6. If \mathbf{r}_0 is a vector from the origin to the point $(1, -1, 2)$ and \mathbf{r} is a vector to any point in the tangent plane, then $\mathbf{r} - \mathbf{r}_0$ lies in the tangent plane. The tangent plane at $(1, -1, 2)$ is normal to the gradient at that point, so we have

$$\nabla\varphi|_{1,-1,2} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

$$\nabla\varphi|_{1,-1,2} = [(2z^2 - 3y - 4)\mathbf{i} - 3x\mathbf{j} - 4xz\mathbf{k}]_{1,-1,2} = 7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}.$$

Therefore the tangent plane is given by the equation

$$\begin{aligned} (7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) \cdot [(x-1)\mathbf{i} + (y+1)\mathbf{j} + (z-2)\mathbf{k}] &= 0, \\ 7(x-1) - 3(y+1) + 8(z-2) &= 0, \\ 7x - 3y + 8z &= 26. \end{aligned}$$

2.4.3 Line Integral of a Gradient Vector

Line integrals occur frequently in physical sciences. The most familiar is probably the work done by a force \mathbf{F} between A and B along some path Γ :

$$\text{Work}(A \rightarrow B) = \int_{A,\Gamma}^B \mathbf{F} \cdot d\mathbf{r},$$

where

$$d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$$

is the differential displacement vector from (x, y, z) to $(x + dx, y + dy, z + dz)$. Sometimes $d\mathbf{l}$ is used in place of $d\mathbf{r}$ to emphasize that the differential displacement vector is along a certain path for the line integral. We will not use this convention here.

For any vector field $\mathbf{A}(x, y, z)$, the line integral

$$\int_{A,\Gamma}^B \mathbf{A} \cdot d\mathbf{r} = \int_{A,\Gamma}^B (A_x dx + A_y dy + A_z dz) \quad (2.66)$$

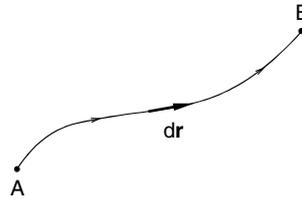


Fig. 2.7. Path of a line integral. The differential displacement $d\mathbf{r}$ is along a specified curve in space between A and B

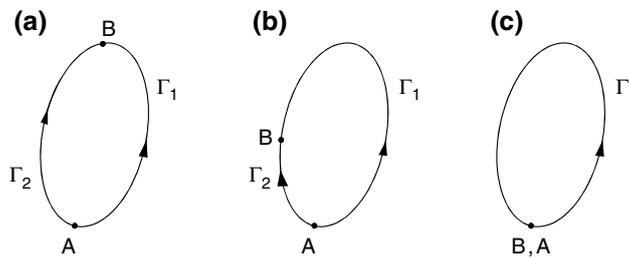


Fig. 2.8. Path independence of the line integral of $\nabla\varphi \cdot d\mathbf{r}$. (a) The value of the integral from A to B along Γ_1 is the same as along Γ_2 . (b) They are still the same even though Γ_2 is much shorter. (c) As Γ_2 shrinks to zero, the integral along Γ_2 vanishes. The integral along Γ_1 , which becomes a loop integral, must also be zero

is the sum of contributions $\mathbf{A} \cdot d\mathbf{r}$ for each differential displacement $d\mathbf{r}$ along the curve Γ in space from A to B as shown in Fig. 2.7. The line integral is also called *path integral* because it is carried out along a specific path Γ .

In general, the result depends on the path taken between A and B . However, if $\mathbf{A} = \nabla\varphi$ for some scalar function φ , the integral is independent of the path.

$$\int_{A,\Gamma}^B \mathbf{A} \cdot d\mathbf{r} = \int_{A,\Gamma}^B \nabla\varphi \cdot d\mathbf{r} = \int_A^B d\varphi = \varphi(B) - \varphi(A), \quad (2.67)$$

where we have used (2.63) to convert $\nabla\varphi \cdot d\mathbf{r}$ to the total differential $d\varphi$. Since the result depends only on the position of the two end points, the integral is independent of path. In this case, the integral from A to B in Fig. 2.8 gives the same value whether it is carried out along Γ_1 or along Γ_2 . This remains to be true as we bring the two points A and B closer, no matter how short Γ_2 becomes. When B is brought to the same place as A , the line integral over Γ_2 obviously vanishes because the length of Γ_2 is equal to zero. So the line integral over Γ_1 must also be zero. The line integral over Γ_1 is an integral around a closed loop:

$$\oint \nabla\varphi \cdot d\mathbf{r} = 0. \quad (2.68)$$

The symbol \oint means the integration is over a closed loop. The line integral around the closed loop is called the *circulation* of the vector field \mathbf{A} around the closed loop Γ . Thus we have the following result:

When a vector field \mathbf{A} is the gradient of a scalar function φ , the circulation of \mathbf{A} around any closed curve is zero.

Sometimes this is called the *fundamental theorem of gradient*. The argument can be reversed. If $\oint \mathbf{A} \cdot d\mathbf{r} = 0$, then $\int_A^B \mathbf{A} \cdot d\mathbf{r}$ is independent of path. In that case, \mathbf{A} is the gradient of some scalar function φ .

Example 2.4.7. Find the work done by the force $\mathbf{F} = 6xy\mathbf{i} + (3x^2 - 3y^2)\mathbf{j}$ in a plane along the curve $C : y = x^2 - x$ from $(0, 0)$ to $(2, 2)$.

Solution 2.4.7. The work done is by definition the line integral

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [6xy\mathbf{i} + (3x^2 - 3y^2)\mathbf{j}] \cdot (\mathbf{i} dx + \mathbf{j} dy + \mathbf{k} dz) \\ &= \int_C [6xy dx + (3x^2 - 3y^2) dy]. \end{aligned}$$

There are more than one way to carry out this integration along curve C . *Method I.* Change all variables into x .

$$y = x^2 - x, \quad dy = (2x - 1)dx,$$

$$\begin{aligned} W &= \int_C [6xy dx + (3x^2 - 3y^2) dy] \\ &= \int_0^2 \{6x(x^2 - x) dx + [3x^2 - 3(x^2 - x)^2](2x - 1)\} dx \\ &= \int_0^2 \{-6x^5 + 15x^4 - 6x^2\} dx = [-x^6 + 3x^5 - 2x^3]_0^2 = 16. \end{aligned}$$

Method II. The curve C can be considered as the trajectory described by the tip of the position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ with t as a parameter. It can be readily verified that with $x = t$ and $y = t^2 - t$, the curve $y = x^2 - x$ is traced out. Therefore the curve C is given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = t\mathbf{i} + (t^2 - t)\mathbf{j}.$$

The point $(0, 0)$ corresponds to $t = 0$, and $(2, 2)$ corresponds to $t = 2$. Now we can change all variables into t .

$$\begin{aligned}\mathbf{F}(x, y, z) \cdot d\mathbf{r} &= \mathbf{F}(x(t), y(t), z(t)) \cdot \frac{d\mathbf{r}}{dt} dt, \\ \frac{d\mathbf{r}}{dt} &= \mathbf{i} + (2t - 1)\mathbf{j}, \\ \mathbf{F} &= 6xy\mathbf{i} + (3x^2 - 3y^2)\mathbf{j} = 6t(t^2 - t)\mathbf{i} + [3t^2 - 3(x^2 - x)^2]\mathbf{j}, \\ \mathbf{F} \cdot d\mathbf{r} &= \{6t(t^2 - t) + [3t^2 - 3(x^2 - x)^2](2t - 1)\} dt, \\ W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \{6t(t^2 - t) + [3t^2 - 3(x^2 - x)^2](2t - 1)\} dt = 16.\end{aligned}$$

Example 2.4.8. Calculate the line integral of the last example from the point $(0, 0)$ to the point (x_1, y_1) along the path which runs straight from $(0, 0)$ to $(x_1, 0)$ and thence to (x_1, y_1) . Make a similar calculation for the path which runs along the other two sides of the rectangle, via the point $(0, y_1)$. If $(x_1, y_1) = (2, 2)$, what is the value of the integral?

Solution 2.4.8.

$$\begin{aligned}I_1(x_1, y_1) &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} [6xy \, dx + (3x^2 - 3y^2) \, dy], \\ C_1 &: (0, 0) \rightarrow (x_1, 0) \rightarrow (x_1, y_1).\end{aligned}$$

From $(0, 0) \rightarrow (x_1, 0)$: $y = 0$, $dy = 0$,

$$\int_{0,0}^{x_1,0} [6xy \, dx + (3x^2 - 3y^2) \, dy] = 0.$$

From $(x_1, 0) \rightarrow (x_1, y_1)$: $x = x_1$, $dx = 0$,

$$\begin{aligned}\int_{x_1,0}^{x_1,y_1} [6xy \, dx + (3x^2 - 3y^2) \, dy] &= \int_0^{y_1} (3x_1^2 - 3y^2) \, dy \\ &= [3x_1^2 y - y^3]_0^{y_1} = 3x_1^2 y_1 - y_1^3,\end{aligned}$$

$$I_1(x_1, y_1) = \left[\int_{0,0}^{x_1,0} + \int_{x_1,0}^{x_1,y_1} \right] [6xy \, dx + (3x^2 - 3y^2) \, dy] = 3x_1^2 y_1 - y_1^3,$$

$$\begin{aligned}I_2(x_1, y_1) &= \int_{C_2} [6xy \, dx + (3x^2 - 3y^2) \, dy], \\ C_2 &: (0, 0) \rightarrow (0, y_1) \rightarrow (x_1, y_1).\end{aligned}$$

From $(0, 0) \rightarrow (0, y_1)$: $x = 0$, $dx = 0$,

$$\int_{0,0}^{0,y_1} [6xy \, dx + (3x^2 - 3y^2) \, dy] = \int_0^{y_1} (-3y^2) \, dy = [-y^3]_0^{y_1} = -y_1^3.$$

From $(0, y_1) \rightarrow (x_1, y_1)$: $y = y_1$, $dy = 0$,

$$\int_{0, y_1}^{x_1, y_1} [6xy \, dx + (3x^2 - 3y^2) \, dy] = \int_0^{x_1} 6xy_1 \, dx = 3x_1^2 y_1,$$

$$I_2(x_1, y_1) = \left[\int_{0,0}^{0, y_1} + \int_{0, y_1}^{x_1, y_1} \right] [6xy \, dx + (3x^2 - 3y^2) \, dy] = -y_1^3 + 3x_1^2 y_1.$$

Clearly $I_1(x_1, y_1) = I_2(x_1, y_1)$, and $I_1(2, 2) = 3(2)^2 2 - (2)^3 = 16$.

Example 2.4.9. From the last two examples, it is clear that the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ with $\mathbf{F} = 6xy\mathbf{i} + (3x^2 - 3y^2)\mathbf{j}$ depends only on the end points and is independent of the path of integration, therefore $\mathbf{F} = \nabla\varphi$. Find $\varphi(x, y)$ and show that $\int_{0,0}^{2,2} \mathbf{F} \cdot d\mathbf{r} = \varphi(2, 2) - \varphi(0, 0)$.

Solution 2.4.9.

$$\nabla\varphi = \mathbf{i} \frac{\partial\varphi}{\partial x} + \mathbf{j} \frac{\partial\varphi}{\partial y} = 6xy\mathbf{i} + (3x^2 - 3y^2)\mathbf{j} = \mathbf{F},$$

$$\frac{\partial\varphi}{\partial x} = 6xy \implies \varphi = 3x^2 y + f(y),$$

$$\frac{\partial\varphi}{\partial y} = 3x^2 - 3y^2 = 3x^2 + \frac{df(y)}{dy},$$

$$\frac{df(y)}{dy} = -3y^2 \implies f(y) = -y^3 + k \quad (k \text{ is a constant}).$$

Thus,

$$\varphi(x, y) = 3x^2 y - y^3 + k.$$

$$\int_{0,0}^{2,2} \mathbf{F} \cdot d\mathbf{r} = \int_{0,0}^{2,2} \nabla\varphi \cdot d\mathbf{r} = \varphi(2, 2) - \varphi(0, 0) = 16 + k - k = 16.$$

Note that

$$\int_{0,0}^{x_1, y_1} \mathbf{F} \cdot d\mathbf{r} = \varphi(x_1, y_1) - \varphi(0, 0) = 3x_1^2 y_1 - y_1^3,$$

in agreement with the result of the last example.

Example 2.4.10. Find the line integral $\int_{0,0}^{2,1} \mathbf{F} \cdot d\mathbf{r}$ with $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j}$ along the path (a) $y = (1/2)x$, (b) $y = (1/4)x^2$, (c) from $(0, 0)$ straight up to $(0, 1)$ and then along a horizontal line to $(2, 1)$.

Solution 2.4.10. $\int_{0,0}^{2,1} \mathbf{F} \cdot d\mathbf{r} = \int_{0,0}^{2,1} (xy \, dx - y^2 \, dy)$ along

$$(a) \, y = \frac{1}{2}x, \quad \text{so} \quad dy = \frac{1}{2}dx,$$

$$\int_{0,0}^{2,1} (xy \, dx - y^2 \, dy) = \int_0^2 \left(\frac{1}{2}x^2 \, dx - \frac{1}{8}x^2 \, dx \right) = \left[\frac{3}{8} \cdot \frac{1}{3}x^3 \right]_0^2 = 1.$$

$$(b) \, y = \frac{1}{4}x^2, \quad \text{so} \quad dy = \frac{1}{2}x \, dx,$$

$$\int_{0,0}^{2,1} (xy \, dx - y^2 \, dy) = \int_0^2 \left(\frac{1}{4}x^3 \, dx - \frac{1}{32}x^5 \, dx \right) = \left[\frac{1}{16}x^4 - \frac{1}{32 \cdot 6}x^6 \right]_0^2 = \frac{2}{3}.$$

(c) From $(0, 0)$ straight up to $(0, 1)$: $x = 0$ so $dx = 0$;

then from $(0, 1)$ along a horizontal line to $(2, 1)$: $y = 1$ and $dy = 0$,

$$\begin{aligned} \int_{0,0}^{2,1} (xy \, dx - y^2 \, dy) &= \int_{0,0}^{0,1} (xy \, dx - y^2 \, dy) + \int_{0,1}^{2,1} (xy \, dx - y^2 \, dy) \\ &= - \int_0^1 y^2 \, dy + \int_0^2 x \, dx = -\frac{1}{3} + 2 = \frac{5}{3}. \end{aligned}$$

In general the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends on the path of integration as shown in the last example. However, if $\mathbf{F} = \nabla\varphi$, the line integral is independent of the path of integration. We are going to discuss the condition under which \mathbf{F} can be expressed as the gradient of a scalar function.

2.5 The Divergence of a Vector

Just as we can operate with ∇ on a scalar field, we can also operate with ∇ on a vector field \mathbf{A} by taking the dot product. With their components, this operation gives

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) \\ &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \end{aligned} \tag{2.69}$$

Just as the dot product of two vectors is a scalar, $\nabla \cdot \mathbf{A}$ is also a scalar. This sum, called the *divergence* of \mathbf{A} (or $\text{div } \mathbf{A}$), is a special combination of derivatives.

Example 2.5.1. Show that $\nabla \cdot \mathbf{r} = 3$ and $\nabla \cdot \mathbf{r}f(r) = 3f(r) + r(df/dr)$.

Solution 2.5.1.

$$\begin{aligned}\nabla \cdot \mathbf{r} &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) \\ &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{r}f(r) &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (\mathbf{i}xf(r) + \mathbf{j}yf(r) + \mathbf{k}zf(r)) \\ &= \frac{\partial}{\partial x}[xf(r)] + \frac{\partial}{\partial y}[yf(r)] + \frac{\partial}{\partial z}[zf(r)] \\ &= f(r) + x \frac{\partial f}{\partial x} + f(r) + y \frac{\partial f}{\partial y} + f(r) + z \frac{\partial f}{\partial z} \\ &= 3f(r) + x \frac{df}{dr} \frac{\partial r}{\partial x} + y \frac{df}{dr} \frac{\partial r}{\partial y} + z \frac{df}{dr} \frac{\partial r}{\partial z}.\end{aligned}$$

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{1/2} = \frac{1}{2} \frac{2x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r}; \\ \frac{\partial r}{\partial y} &= \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}.\end{aligned}$$

$$\begin{aligned}\nabla \cdot \mathbf{r}f(r) &= 3f(r) + \frac{x^2}{r} \frac{df}{dr} + \frac{y^2}{r} \frac{df}{dr} + \frac{z^2}{r} \frac{df}{dr} \\ &= 3f(r) + \frac{x^2 + y^2 + z^2}{r} \frac{df}{dr} = 3f(r) + r \frac{df}{dr}.\end{aligned}$$

2.5.1 The Flux of a Vector Field

To gain some physical feeling for the divergence of a vector field, it is helpful to introduce the concept of *flux* (Latin for “flow”). Consider a fluid of density ρ moving with velocity \mathbf{v} . We ask for the total mass of fluid which crosses an area Δa perpendicular to the direction of flow in a time Δt . As shown in Fig. 2.9a, all the fluid in the rectangular pipe of length $v\Delta t$ with the patch Δa as its base will cross Δa in the time interval Δt . The volume of this pipe is

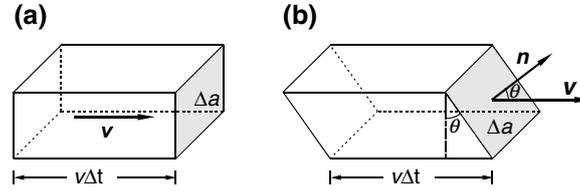


Fig. 2.9. Flux through the base. (a) Flux through $\Delta a = \rho v \Delta a$. (b) Flux through the tilted $\Delta a = \rho \mathbf{v} \cdot \mathbf{n} \Delta a$

$v\Delta t\Delta a$, and it contains a total mass $\rho v\Delta t\Delta a$. Dividing Δt will give the mass across Δa per unit time, which by definition is the rate of the flow

$$\text{Rate of flow through } \Delta a = \rho v \Delta a.$$

Now let us consider the case shown in Fig. 2.9b. In this case the area Δa is not perpendicular to the direction of the flow. The total mass which will flow through this tilted Δa in time Δt is just the density times the volume of this pipe with the slanted bases. That volume is $v\Delta t\Delta a \cos \theta$, where θ is the angle between the velocity vector \mathbf{v} and \mathbf{n} , the unit normal to Δa . But $v \cos \theta = \mathbf{v} \cdot \mathbf{n}$. So, multiplying by ρ and dividing Δt , we have

$$\text{Rate of flow through tilted } \Delta a = \rho \mathbf{v} \cdot \mathbf{n} \Delta a.$$

To get the total flow through any surface S , first we divide the whole surface into little patches which are so small that over any one patch the surface is practically flat. Then we sum up the contributions from all the patches. As the patches become smaller and more numerous without limit, the sum becomes a surface integral. Accordingly,

$$\text{Total flow through } S = \iint_S \rho \mathbf{v} \cdot \mathbf{n} \, da. \tag{2.70}$$

If we define $\mathbf{J} = \rho \mathbf{v}$, (2.70) is known as the flux of \mathbf{J} through the surface S

$$\text{Flux of } \mathbf{J} \text{ through } S = \iint_S \mathbf{J} \cdot \mathbf{n} \, da. \tag{2.71}$$

Originally it means the rate of the flow, the word flux is now generalized to mean the surface integral of the normal component of a vector. For example, the vector might be the electric field \mathbf{E} . Although electric field is not flowing in the sense in which fluid flows, we still say things like “the flux of \mathbf{E} through a closed surface is equal to the total charge inside” to help us to visualize the electric field lines “flowing” out of the electric charges.

Example 2.5.2. Let $\mathbf{J} = \rho v_0 \mathbf{k}$ where ρ is the density of the fluid and $v_0 \mathbf{k}$ is its velocity (\mathbf{k} is the unit vector in the z direction). Calculate the flux of \mathbf{J} (the flow rate of the fluid) through a hemispherical surface of radius b .

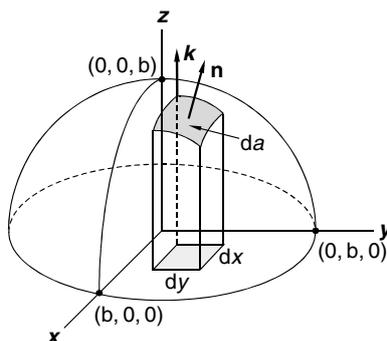


Fig. 2.10. Surface element on a hemisphere. The projected area on the xy plane is $dx dy = \cos \theta da$ where θ is the angle between the tangent plane at da and the xy plane

Solution 2.5.2. The equation of the spherical surface is $\varphi(x, y, z) = x^2 + y^2 + z^2 = b^2$. Therefore the unit normal to the surface is

$$\mathbf{n} = \frac{\nabla \varphi}{|\nabla \varphi|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{(4x^2 + 4y^2 + 4z^2)^{1/2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{b}.$$

Let the flux of \mathbf{J} through the hemisphere be Φ , which is given by

$$\Phi = \iint_S \mathbf{J} \cdot \mathbf{n} da = \iint_S \rho v_0 \mathbf{k} \cdot \mathbf{n} da,$$

where da is an element of the surface area on the hemisphere as shown in Fig. 2.10. This surface area projects onto $dx dy$ in the xy plane. Let θ be the acute angle between da (actually the tangent plane at da) and the xy plane. Then we have $dx dy = \cos \theta da$. The integral becomes

$$\Phi = \iint_S \rho v_0 \mathbf{k} \cdot \mathbf{n} da = \iint_S \rho v_0 \mathbf{k} \cdot \mathbf{n} \frac{1}{\cos \theta} dx dy,$$

where the limit on x and y must be such that we integrate over the projected area in the xy plane which is inside a circle of radius b . The angle between two planes is the same as the angle between the normals to the planes. Since \mathbf{n} is the unit normal to da and \mathbf{k} is the unit normal to xy plane, the angle θ is between \mathbf{n} and \mathbf{k} . Thus, $\cos \theta = \mathbf{n} \cdot \mathbf{k}$. Therefore,

$$\Phi = \iint_S \rho v_0 \mathbf{k} \cdot \mathbf{n} \frac{1}{\cos \theta} dx dy = \iint_S \rho v_0 dx dy = \rho v_0 \pi b^2.$$

Note that this result is the same as the flux through the circular flat area in the xy plane. In fact, it is exactly the same as the flux through any surface whose boundary is the circle of radius b in the xy plane, since we obtained the result without using the explicit expression of \mathbf{n} .

2.5.2 Divergence Theorem

The *divergence theorem* is also known as *Gauss' theorem*. It relates the flux of a vector field through a closed surface S to the divergence of the vector field in the enclosed volume

$$\boxed{\iint_{\text{closed surface } S} \mathbf{A} \cdot \mathbf{n} da = \iiint_{\text{vol. enclosed in } S} \nabla \cdot \mathbf{A} dV.} \quad (2.72)$$

The surface integral is over a closed surface as shown in Fig. 2.11. The unit normal vector \mathbf{n} is pointing outward from the enclosed volume. The right-hand side of this equation is the integration of the divergence over the volume that is enclosed in the surface.

To prove this theorem, we cut the volume V up into a very large number of tiny (differential) cubes. The volume integral is the sum of the integrals over all the cubes.

Imagine we have a parallelepiped with six surfaces enclosing a volume V . We separate the volume into two cubes by a cut as in Fig. 2.12. Note that the sum of the flux through the six surfaces of the cube on the left and the flux through the six surfaces of the cube on the right is equal to the flux through the six surfaces of the original parallelepiped before it was cut. This is because

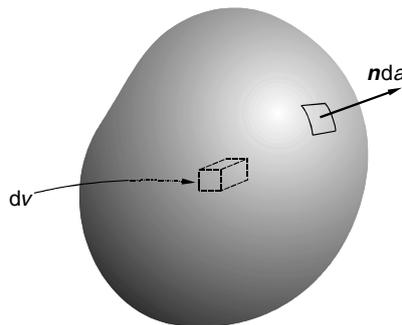


Fig. 2.11. The divergence theorem. The volume is enclosed by the surface. The integral of the divergence over the volume inside is equal to the flux through the outside surface

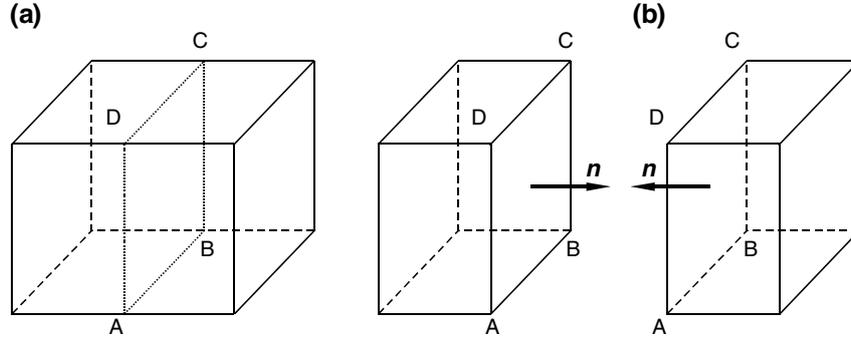


Fig. 2.12. The flux out of the touching sides of the two neighboring cubes cancel each other

the unit normal vectors on the touching sides of the two neighboring cubes are equal and opposite to each other. So the contributions to the flux for the two cubes from these two sides exactly cancel. Thus it must be generally true that the sum of the flux through the surfaces of all the cubes is equal to the integral over those surfaces that have no touching neighbors, i.e., over the original outside surface. So if we can prove the result for a small cube, we can prove it for any arbitrary volume.

Consider the flux of \mathbf{A} through the surfaces of the small cube of volume $\Delta V = \Delta x \Delta y \Delta z$ shown in Fig. 2.13. The unit vector perpendicular to the surface ABCD is clearly $-\mathbf{j}$ ($\mathbf{n} = -\mathbf{j}$). The flux through this surface is therefore given by

$$\mathbf{A}(x, y, z) \cdot (-\mathbf{j}) \Delta a = -A_y(x, y, z) \Delta x \Delta z.$$

The flux is defined as the outgoing “flow.” The minus sign simply means the flux is flowing into the volume. Similarly, the unit normal to the surface EFGH is \mathbf{j} , and the flux through this surface is

$$\mathbf{A}(x, y + \Delta y, z) \cdot \mathbf{j} \Delta a = A_y(x, y + \Delta y, z) \Delta x \Delta z.$$

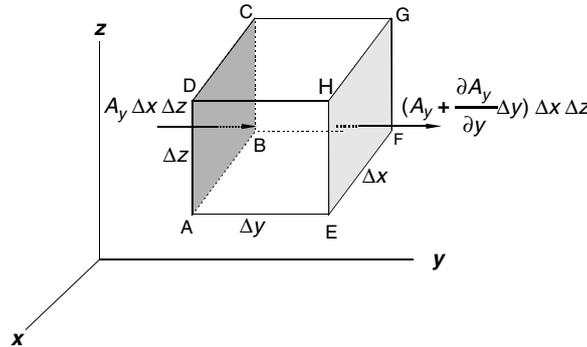


Fig. 2.13. The flux through the left and right face of a infinitesimal cube

Note that for every point (x, y, z) on ABCD, the corresponding point on EFGH is $(x, y + \Delta y, z)$. The net flux through these two surfaces is simply the sum of the two:

$$[A_y(x, y + \Delta y, z) - A_y(x, y, z)] \Delta x \Delta z = \frac{\partial A_y}{\partial y} \Delta y \Delta x \Delta z = \frac{\partial A_y}{\partial y} \Delta V. \quad (2.73)$$

By applying similar reasoning to the flux components in the two other directions, we find the total flux through all the surfaces of the cube is

$$\sum_{\text{cube}} \mathbf{A} \cdot \mathbf{n} \, da = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \Delta V = (\nabla \cdot \mathbf{A}) \Delta V. \quad (2.74)$$

This shows that the outward flux from the surface of an infinitesimal cube is equal to the divergence of the vector multiplied by the volume of the cube. Thus the divergence has the following physical meaning:

The divergence of a vector \mathbf{A} at a point is the total outward flux of \mathbf{A} per unit volume in the neighborhood of that point.

For any finite volume, the total flux of \mathbf{A} through the outside surface enclosing the volume is equal to the sum of the fluxes out of all the infinitesimal interior cubes, and the flux out of each cube is equal to the divergence of \mathbf{A} times the volume of the cube. Therefore the integral of the normal component of a vector over any closed surface is equal to the integral of the divergence of the vector over the volume enclosed by the surface,

$$\oint_S \mathbf{A} \cdot \mathbf{n} \, da = \iiint_V \nabla \cdot \mathbf{A} \, dV. \quad (2.75)$$

The small circle on the double integral sign means the surface integral is over a closed surface. The volume integral is understood to be over the entire region inside the closed surface. This is the divergent theorem of (2.72), which is sometimes called the *fundamental theorem for divergence*.

A flow field \mathbf{A} is said to be *solenoidal* if everywhere the divergence of \mathbf{A} is equal to zero ($\nabla \cdot \mathbf{A} = 0$). An incompressible fluid must flow out of a given volume as rapidly as it flows in. The divergence of such a flow field must be zero, therefore the field is solenoidal.

On the other hand if \mathbf{A} is such a field that at certain point $\nabla \cdot \mathbf{A} \neq 0$, then there is a net outward flow from a small volume surrounding that point. Fluid must be “created” or “put in” at that point. If $\nabla \cdot \mathbf{A}$ is negative, fluid must be “taken out” at that point. Therefore we come to the following conclusion.

The divergence of flow field at a point is a measure of the strength of the source (or sink) of the flow at that point.

Example 2.5.3. Verify the divergence theorem by evaluating both sides of (2.72) with $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over a cylinder described by $x^2 + y^2 = 4$ and $0 \leq z \leq 4$.

Solution 2.5.3. Since $\nabla \cdot \mathbf{A} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3$, the volume integral is

$$\iiint_V \nabla \cdot \mathbf{A} \, dV = 3 \iiint_V dV = 3(\pi 2^2)4 = 48\pi,$$

which is simply three times the volume of the cylinder. The surface of the cylinder consists of the top, bottom, and curved side surfaces. Therefore the surface integral can be divided into three parts

$$\oiint_S \mathbf{A} \cdot \mathbf{n} \, da = \iint_{\text{top}} \mathbf{A} \cdot \mathbf{n} \, da + \iint_{\text{bottom}} \mathbf{A} \cdot \mathbf{n} \, da + \iint_{\text{curved}} \mathbf{A} \cdot \mathbf{n} \, da.$$

For the top surface

$$\iint_{\text{top}} \mathbf{A} \cdot \mathbf{n} \, da = \iint_{\text{top}} (x\mathbf{i} + y\mathbf{j} + 4\mathbf{k}) \cdot \mathbf{k} \, da = 4 \iint_{\text{top}} da = 4\pi 2^2 = 16\pi.$$

For the bottom surface

$$\iint_{\text{bottom}} \mathbf{A} \cdot \mathbf{n} \, da = \iint_{\text{bottom}} (x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}) \cdot (-\mathbf{k}) \, da = 0.$$

For the curved side surface, we must first find the unit normal \mathbf{n} . Since the surface is described by $\varphi(x, y) = x^2 + y^2 = 4$,

$$\mathbf{n} = \frac{\nabla \varphi}{|\nabla \varphi|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{(4x^2 + 4y^2)^{1/2}} = \frac{x\mathbf{i} + y\mathbf{j}}{2},$$

$$\mathbf{A} \cdot \mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \frac{x\mathbf{i} + y\mathbf{j}}{2} = \frac{1}{2}(x^2 + y^2) = 2,$$

$$\iint_{\text{curved}} \mathbf{A} \cdot \mathbf{n} \, da = 2 \iint_{\text{curved}} da = 2(2\pi \cdot 2)4 = 32\pi.$$

Therefore

$$\oiint_S \mathbf{A} \cdot \mathbf{n} \, da = 16\pi + 0 + 32\pi = 48\pi,$$

which is the same as the volume integral.

2.5.3 Continuity Equation

One of the most important applications of the divergence theorem is using it to express the conservation laws in differential forms. As an example, consider a fluid of density ρ moving with velocity \mathbf{v} . According to (2.70), the rate at which the fluid flows out of a closed surface is

$$\text{Rate of outward flow through a closed surface} = \oiint_S \rho \mathbf{v} \cdot \mathbf{n} \, da. \quad (2.76)$$

Now because of the conservation of mass, this rate of out flow must be equal to the rate of decrease of the fluid inside the volume that is enclosed by the surface. Therefore

$$\oiint_S \rho \mathbf{v} \cdot \mathbf{n} \, da = - \iiint_V \frac{\partial \rho}{\partial t} \, dV. \quad (2.77)$$

The negative sign accounts for the fact that the fluid inside is decreasing if the flow is outward. Using the divergence theorem

$$\oiint_S \rho \mathbf{v} \cdot \mathbf{n} \, da = \iiint_V \nabla \cdot (\rho \mathbf{v}) \, dV, \quad (2.78)$$

we have

$$\iiint_V \left[\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} \right] \, dV = 0. \quad (2.79)$$

Since the volume V in this equation, the integrand must equal to zero, or

$$\boxed{\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0.} \quad (2.80)$$

This important equation, known as the continuity equation, relates the density and the velocity at the same point in a differential form. Many other conservation laws can be similarly expressed.

For an incompressible fluid, ρ is not changing in time. In that case, the divergence of the velocity must be zero,

$$\nabla \cdot \mathbf{v} = 0. \quad (2.81)$$

Singularities in the Field. In deriving these integral theorems, we require the scalar and vector fields to be continuous and finite at every point. Often there are points, lines, or surfaces in space at which fields become discontinuous or even infinite. Examples are the electric fields produced by point, line, or surface charges. One way of dealing with this situation is to eliminate these volume elements, by appropriate surfaces, from the domain to which the theorems are to be applied. Another scheme is to “smear out” the discontinuous quantities, such as using charge densities, so that the fields are again well behaved. Still another powerful way is to use Dirac delta function. Sect. 2.10.2 is a specific example of these procedures.

2.6 The Curl of a Vector

The cross product of the gradient operator ∇ with vector \mathbf{A} gives us another special combination of the derivatives of the components of \mathbf{A}

$$\begin{aligned}\nabla \times \mathbf{A} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \mathbf{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right). \end{aligned} \quad (2.82)$$

It is a vector known as the *curl* of \mathbf{A} . The name curl suggests that it has something to do with rotation. In fact, in European texts the word rotation (or rot) is used in place of curl. In Example 2.1.3, we have considered the motion of a body rotating around the z -axis with angular velocity $\boldsymbol{\omega}$. The velocity of the particles in the body is $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$. The circular characteristic of this velocity field is manifested in the curl of the velocity

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k}, \quad (2.83)$$

which shows that the curl of \mathbf{v} is twice the angular velocity of the rotating body.

If this velocity field describes a fluid flow, curl \mathbf{v} is called the *vorticity* vector of the fluid. It points in the direction around which a vortex motion takes place and is a measure of the the angular velocity of the flow. A small paddle wheel placed in the field will tend to rotate in regions where $\nabla \times \mathbf{v} \neq 0$. The paddle wheel will remain stationary in those regions where $\nabla \times \mathbf{v} = 0$. If the curl of a vector field is equal to zero everywhere, the field is called *irrotational*.

Example 2.6.1. Show that (a) $\nabla \times \mathbf{r} = \mathbf{0}$; (b) $\nabla \times \mathbf{r}f(r) = \mathbf{0}$ where \mathbf{r} is the position vector.

Solution 2.6.1. (a) Since $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}.$$

(b)

$$\begin{aligned}
\nabla \times \mathbf{r}f(r) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \left\{ \frac{\partial}{\partial y} [zf(r)] - \frac{\partial}{\partial z} [yf(r)] \right\} \mathbf{i} \\
&\quad + \left\{ \frac{\partial}{\partial z} [xf(r)] - \frac{\partial}{\partial x} [zf(r)] \right\} \mathbf{j} + \left\{ \frac{\partial}{\partial x} [yf(r)] - \frac{\partial}{\partial y} [xf(r)] \right\} \mathbf{k} \\
&= \left\{ z \frac{\partial}{\partial y} f(r) - y \frac{\partial}{\partial z} f(r) \right\} \mathbf{i} + \left\{ x \frac{\partial}{\partial z} f(r) - z \frac{\partial}{\partial x} f(r) \right\} \mathbf{j} \\
&\quad + \left\{ y \frac{\partial}{\partial x} f(r) - x \frac{\partial}{\partial y} f(r) \right\} \mathbf{k}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial}{\partial y} f(r) &= \frac{df}{dr} \frac{\partial r}{\partial y} \quad \text{and} \quad r = (x^2 + y^2 + z^2)^{1/2}, \\
\frac{\partial}{\partial y} f(r) &= \frac{df}{dr} \left(-\frac{y}{(x^2 + y^2 + z^2)^{1/2}} \right) = -\frac{df}{dr} \frac{y}{r}, \\
\frac{\partial}{\partial x} f(r) &= -\frac{df}{dr} \frac{x}{r}; \quad \frac{\partial}{\partial z} f(r) = -\frac{df}{dr} \frac{z}{r}.
\end{aligned}$$

Therefore

$$\nabla \times \mathbf{r}f(r) = \frac{df}{dr} \left\{ -z \frac{y}{r} + y \frac{z}{r} \right\} \mathbf{i} + \frac{df}{dr} \left\{ -x \frac{z}{r} + z \frac{x}{r} \right\} \mathbf{j} + \frac{df}{dr} \left\{ -y \frac{x}{r} + x \frac{y}{r} \right\} \mathbf{k} = \mathbf{0}.$$

2.6.1 Stokes' Theorem

Stokes' theorem relates the line integral of a vector function around a closed loop C to a surface integral of the curl of that vector over a surface S that spans the loop. The theorem states that

$$\boxed{\int_{\text{closed loop } C} \mathbf{A} \cdot d\mathbf{r} = \iint_{\text{area bounded by } C} (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da,} \quad (2.84)$$

where $d\mathbf{r}$ is a directed line element along a closed curve C and S is any surface bounded by C . At any point on the surface, the unit normal vector \mathbf{n} is perpendicular to the surface element da at that point as shown in Fig. 2.14. The sign of \mathbf{n} is determined by the convention of the "right-hand rule." Curl the fingers of your right hand in the direction $d\mathbf{r}$, then your thumb points in the positive direction of \mathbf{n} . If the curve C lies in a plane, the simplest surface spans C is a flat surface. Now imagine the flat surface is a flexible membrane

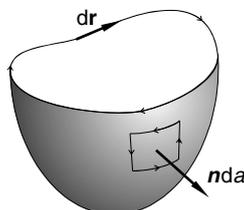


Fig. 2.14. Stokes' theorem. The integral of the curl over the surface is equal to the line integral around the closed boundary curve

which can continuously expand but remains attached to curve C . A sequence of curved surfaces is generated. Stokes theorem applies to all such surfaces. The positive direction of \mathbf{n} for the flat surface is clear from the right-hand rule. As the surface expands, \mathbf{n} moves along with it. For example, with the direction of $d\mathbf{r}$ shown in Fig. 2.14, the normal vector \mathbf{n} of the nearly flat surface bounded by C is pointing “downward” according to the right-hand rule. When the surface is expanded into the final shape, \mathbf{n} is turned to “outward” direction as shown in the figure.

The surface in Stokes' theorem must be two sided. A one-sided surface can be constructed by taking a long strip of paper, giving it a half twist, and joining the ends. If we tried to color one side of the surface we would find the whole thing colored. A belt of this shape is called a Moebius surface. Such a surface is not orientable since we cannot define the sense of the normal vector \mathbf{n} . Stokes' theorem applies only to the *orientable surface*, furthermore, the boundary curve of the surface must not cross itself.

To prove Stokes' theorem, we divide the surface into a large number of small rectangles. The surface integral on the right-hand side of (2.84) is of course just the sum of the surface integrals over all the small rectangles. If the line integrals around all the small rectangles are traced in the same direction, each interior line will be traced twice – once in each direction. Thus the line integrals of $\mathbf{A} \cdot d\mathbf{r}$ from all the interior lines will sum up to zero, since each term will appear twice with opposite sign. Therefore the sum of the line integrals around all the small rectangles will equal to the line integral around the boundary curve C , as shown in Fig. 2.15. So if we prove the result for a small rectangle, we will have proved it for any closed curve C .

Since the surface is to be composed of an infinitely large number of these infinitesimal rectangles, we may consider each to be a plane rectangle. Let us orient the coordinate axes so that one of these rectangles lies in the xy -plane, the sides will be of length Δx and Δy as shown in Fig. 2.16. Let the coordinates of the center of the loop be (x, y, z) . We designate the corners of this rectangle by A, B, C, D. So, the line integral around this rectangle is

$$\oint_{ABCD} \mathbf{A} \cdot d\mathbf{r} = \int_{AB} \mathbf{A} \cdot (\mathbf{i} dx) + \int_{BC} \mathbf{A} \cdot (\mathbf{j} dy) + \int_{CD} \mathbf{A} \cdot (-\mathbf{i} dx) + \int_{DA} \mathbf{A} \cdot (-\mathbf{j} dy). \quad (2.85)$$

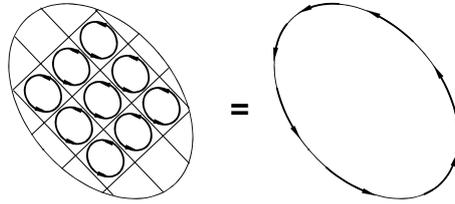


Fig. 2.15. Proof of Stokes' theorem. The surface is divided into differential surface elements. Circulation along interior lines cancel and the result is a circulation around the perimeter of the original surface

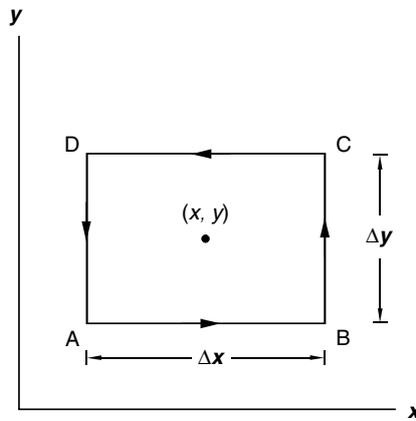


Fig. 2.16. The line integral of $\mathbf{A} \cdot d\mathbf{r}$ around the four sides of the infinitesimal square is equal to the surface integral of $\nabla \times \mathbf{A}$ over the area of this square

We use the symbol \oint to mean the line integral is over a closed loop.

Now we may approximate the integral by the average value of the integrand multiplied by the length of the integration interval. The average value of \mathbf{A} on the line AB may be taken to be the value of \mathbf{A} at the midpoint of AB. The coordinates at the midpoint of AB is $(x, y - \frac{1}{2}\Delta y, z)$. Thus

$$\int_{AB} \mathbf{A} \cdot (\mathbf{i} dx) = \int_{AB} A_x dx = A_x \left(x, y - \frac{1}{2}\Delta y, z \right) \Delta x.$$

Similarly,

$$\int_{BC} \mathbf{A} \cdot (\mathbf{j} dy) = \int_{BC} A_y dy = A_y \left(x + \frac{1}{2}\Delta x, y, z \right) \Delta y,$$

$$\int_{CD} \mathbf{A} \cdot (-\mathbf{i} dx) = - \int_{CD} A_x dx = -A_x \left(x, y + \frac{1}{2}\Delta y, z \right) \Delta x,$$

$$\int_{DA} \mathbf{A} \cdot (-\mathbf{j} dy) = - \int_{DA} A_y dy = -A_y \left(x - \frac{1}{2}\Delta x, y, z \right) \Delta y.$$

Summing all these contributions, we obtain

$$\begin{aligned} \oint_{ABCD} \mathbf{A} \cdot d\mathbf{r} &= \left(A_x \left(x, y - \frac{1}{2} \Delta y, z \right) - A_x \left(x, y + \frac{1}{2} \Delta y, z \right) \right) \Delta x \\ &\quad + \left(A_y \left(x + \frac{1}{2} \Delta x, y, z \right) - A_y \left(x - \frac{1}{2} \Delta x, y, z \right) \right) \Delta y. \end{aligned}$$

Since

$$\begin{aligned} A_y \left(x + \frac{1}{2} \Delta x, y, z \right) - A_y \left(x - \frac{1}{2} \Delta x, y, z \right) &= \frac{\partial A_y}{\partial x} \Delta x, \\ A_x \left(x, y - \frac{1}{2} \Delta y, z \right) - A_x \left(x, y + \frac{1}{2} \Delta y, z \right) &= -\frac{\partial A_x}{\partial y} \Delta y, \end{aligned}$$

so we have

$$\oint_{ABCD} \mathbf{A} \cdot d\mathbf{r} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \Delta x \Delta y. \quad (2.86)$$

Next consider the surface integral of $\nabla \times \mathbf{A}$ over ABCD. In this case the unit normal \mathbf{n} is just \mathbf{k} . Again we take the integral to be equal to the average value of the integrand over the area multiplied by the area of the integration. The average value of \mathbf{A} is simply the value of \mathbf{A} evaluated at the center. Therefore

$$\iint_{ABCD} (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da = (\nabla \times \mathbf{A}) \cdot \mathbf{k} \Delta x \Delta y = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \Delta x \Delta y, \quad (2.87)$$

which is the same as (2.86). This result can be interpreted as follows:

The component of $\nabla \times \mathbf{A}$ in the direction of \mathbf{n} at a point P is the circulation of \mathbf{A} per unit area around P in the plane normal to \mathbf{n} .

The circulation of a vector field around any closed loop can now be easily related to the curl of that field. We fill the loop with a surface S and add the circulations around a set of infinitesimal squares covering this surface. Thus we have

$$\sum \oint_{ABCD} \mathbf{A} \cdot d\mathbf{r} = \sum \iint_{ABCD} (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da, \quad (2.88)$$

which can be written as

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da. \quad (2.89)$$

This is Stokes' theorem. Sometimes this theorem is referred to as the *fundamental theorem for curls*. Note that this theorem holds for any surface S as long as the boundary of the surface is the closed loop C .

Example 2.6.2. Verify Stokes' theorem by finding the circulation of the vector field $\mathbf{A} = 4y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$ around a square of radius $2a$ in the xy plane, centered at the origin and the surface integral $\iint (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, da$ over the surface of the square.

Solution 2.6.2. We compute the circulation by calculating the line integral around each of the four sides of the square. From $(a, -a, 0)$ to $(a, a, 0)$, $x = a$, $dx = 0$, $z = 0$:

$$\begin{aligned} I_1 &= \int_{a,-a,0}^{a,a,0} \mathbf{A} \cdot d\mathbf{r} = \int_{a,-a,0}^{a,a,0} (4y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}) \cdot (\mathbf{i} \, dx + \mathbf{j} \, dy + \mathbf{k} \, dz) \\ &= \int_{a,-a,0}^{a,a,0} (4y \, dx + x \, dy + 2z \, dz) = \int_{-a}^a a \, dy = 2a^2. \end{aligned}$$

From $(a, a, 0)$ to $(-a, a, 0)$, $y = a$, $dy = 0$, $z = 0$:

$$I_2 = \int_{a,a,0}^{-a,a,0} \mathbf{A} \cdot d\mathbf{r} = \int_a^{-a} 4a \, dx = -8a^2.$$

From $(-a, a, 0)$ to $(-a, -a, 0)$, $x = -a$, $dx = 0$, $z = 0$:

$$I_3 = \int_{-a,a,0}^{-a,-a,0} \mathbf{A} \cdot d\mathbf{r} = \int_a^{-a} (-a) \, dy = 2a^2.$$

From $(-a, -a, 0)$ to $(a, -a, 0)$, $y = -a$, $dy = 0$, $z = 0$:

$$I_4 = \int_{-a,-a,0}^{a,-a,0} \mathbf{A} \cdot d\mathbf{r} = \int_{-a}^a 4(-a) \, dx = -8a^2.$$

Therefore the circulation is

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = I_1 + I_2 + I_3 + I_4 = -12a^2.$$

Now we compute the surface integral. First $\mathbf{n} \, da = \mathbf{k} \, dx \, dy$ over the square and the curl of \mathbf{A} is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & x & 2z \end{vmatrix} = -3\mathbf{k}.$$

Thus

$$\iint_S \nabla \times \mathbf{A} \cdot \mathbf{n} \, da = -3 \iint_S dx \, dy = -3(2a)^2 = -12a^2,$$

which is the same as the circulation, satisfying Stokes' theorem.

Example 2.6.3. Verify Stokes' theorem by evaluating both sides of (2.84) with $\mathbf{A} = 4y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$. This time the surface is over the hemisphere described by $\varphi(x, y, z) = x^2 + y^2 + z^2 = 4$ and the loop C is given by the circle $x^2 + y^2 = 4$.

Solution 2.6.3. Since $\nabla \times \mathbf{A} = -3\mathbf{k}$,

$$\iint_S \nabla \times \mathbf{A} \cdot \mathbf{n} \, da = -3 \iint_S \mathbf{k} \cdot \mathbf{n} \, da.$$

The surface is over a hemisphere. The geometry is shown in Fig. 2.10. The surface integral can be evaluated over the projection of the hemisphere on the xy plane using the relation

$$da = \frac{1}{\cos \theta} dx \, dy = \frac{1}{\mathbf{k} \cdot \mathbf{n}} dx \, dy.$$

The integration is simply over the disk of radius 2:

$$\iint_S \nabla \times \mathbf{A} \cdot \mathbf{n} \, da = -3 \iint_S \mathbf{k} \cdot \mathbf{n} \frac{1}{\mathbf{k} \cdot \mathbf{n}} dx \, dy = -3 \iint_S dx \, dy = -3(4\pi) = -12\pi.$$

To evaluate the line integral around the circle, it is convenient to write the circle in the parametric form

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j}, \quad x = 2 \cos \theta, \quad y = 2 \sin \theta, \quad 0 \leq \theta \leq 2\pi. \\ \frac{d\mathbf{r}}{d\theta} &= -2 \sin \theta \mathbf{i} + 2 \cos \theta \mathbf{j}, \quad \mathbf{A} = 4y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = 8 \sin \theta \mathbf{i} + 2 \cos \theta \mathbf{j} + 2z\mathbf{k}. \end{aligned}$$

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C \mathbf{A} \cdot \frac{d\mathbf{r}}{d\theta} d\theta = \int_0^{2\pi} (-16 \sin^2 \theta + 4 \cos^2 \theta) d\theta = -12\pi.$$

Thus, Stokes' theorem is verified,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{A} \cdot \mathbf{n} \, da.$$

Example 2.6.4. Use Stokes' theorem to evaluate the line integral $\oint_C \mathbf{A} \cdot d\mathbf{r}$ with $\mathbf{A} = 2yz\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ along the circle described by $x^2 + y^2 = 1$.

Solution 2.6.4. The curl of \mathbf{A} is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & x & z^2 \end{vmatrix} = 2y\mathbf{j} + (1 - 2z)\mathbf{k},$$

and according to Stokes' theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{A} \cdot \mathbf{n} \, da = \iint_S [2y\mathbf{j} + (1 - 2z)\mathbf{k}] \cdot \mathbf{n} \, da.$$

Since S can be any surface as long as it is bounded by the circle, the simplest way to do this problem is to use the flat surface inside the circle. In that case $z = 0$ and $\mathbf{n} = \mathbf{k}$. Hence,

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S da = \pi.$$

Connectivity of Space. Stokes' theorem is valid in a *simply connected region*. A region is simply connected if any closed loop in the region can be shrunk to a point without encountering any points not in the region. In a simply connected region, any two curves between two points can be distorted into each other within the region. The space inside a torus (doughnut) is multiply connected since a closed curve surrounds the hole cannot be shrunk to a point within the region. The space between two infinitely long concentric cylinders is also not simply connected. However, the region between two concentric spheres is simply connected.

If $\nabla \times \mathbf{F} = \mathbf{0}$ in a simply connected region, we can use Stokes' theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, da = 0$$

to conclude that the line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path.

If $\nabla \times \mathbf{F} = \mathbf{0}$ in a multiply connected region, then $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is not unique. In such a case, we often "cut" the region so as to make it simply connected. Then $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path inside the simply connected region, but $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ across the cut line may give a finite jump.

For example, consider the loop integral $\oint \mathbf{F} \cdot d\mathbf{r}$ with

$$\mathbf{F} = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}$$

around a unit circle centered at the origin. Since

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = \mathbf{0},$$

one might conclude that

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, da = 0.$$

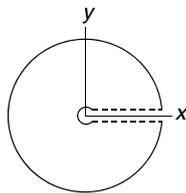


Fig. 2.17. If the function is singular on the z -axis, the region is multiply connected. A “cut” can be made to change it into a simply connected region. However, a lined integral across the cut line may give a sudden jump

This is incorrect, as one can readily see that

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint \left(-\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \right).$$

With $x = \cos \theta$, $y = \sin \theta$, (so $dx = -\sin \theta d\theta$, $dy = \cos \theta d\theta$, and $x^2 + y^2 = 1$), this integral is seen to be

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint (\sin^2 \theta + \cos^2 \theta) d\theta = \oint d\theta = 2\pi,$$

which is certainly not zero. The source of the problem is that at $x = 0$ and $y = 0$ the function blows up. Thus we can only say that the curl of the function is zero except along the z -axis. If we try to exclude the z -axis from the region of integration, the region becomes multiply connected. In a multiply connected region, Stokes' theorem does not apply.

To make it simply connected, we can cut the region, such as along the $y = 0$ plane shown in Fig. 2.17 (or along any other direction). Within the simply connected region $\int_A^B \mathbf{F} \cdot d\mathbf{r} = \theta_B - \theta_A$. It will be equal to zero if A and B are infinitesimally close. However, if the integral is across the cut line, as long as A and B are on the different side of the cut, no matter how close are A and B, there is a sudden jump of 2π .

2.7 Further Vector Differential Operations

There are several combinations of vector operations involving the del ∇ operator which appear frequently in applications. They all follow the general rules of ordinary derivatives. The distributive rules are straightforward. With the definition the del operator ∇ , one can readily verify

$$\nabla(\varphi_1 + \varphi_2) = \nabla\varphi_1 + \nabla\varphi_2; \quad (2.90)$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}; \quad (2.91)$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}. \quad (2.92)$$

However, the product rules are not so simple because there are more than one way to form a vector product.

2.7.1 Product Rules

The following is a list of useful product rules:

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi, \quad (2.93)$$

$$\nabla \cdot (\varphi\mathbf{A}) = \nabla\varphi \cdot \mathbf{A} + \varphi\nabla \cdot \mathbf{A}, \quad (2.94)$$

$$\nabla \times (\varphi\mathbf{A}) = \nabla\varphi \times \mathbf{A} + \varphi\nabla \times \mathbf{A}, \quad (2.95)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}, \quad (2.96)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}, \quad (2.97)$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}). \quad (2.98)$$

They can be verified by expanding both sides in terms of their Cartesian components. For example,

$$\begin{aligned} \nabla(\varphi\psi) &= \mathbf{i} \frac{\partial}{\partial x}(\varphi\psi) + \mathbf{j} \frac{\partial}{\partial y}(\varphi\psi) + \mathbf{k} \frac{\partial}{\partial z}(\varphi\psi) \\ &= \mathbf{i} \varphi \frac{\partial}{\partial x} \psi + \mathbf{j} \varphi \frac{\partial}{\partial y} \psi + \mathbf{k} \varphi \frac{\partial}{\partial z} \psi \\ &\quad + \mathbf{i} \psi \frac{\partial}{\partial x} \varphi + \mathbf{j} \psi \frac{\partial}{\partial y} \varphi + \mathbf{k} \psi \frac{\partial}{\partial z} \varphi \\ &= \varphi \nabla \psi + \psi \nabla \varphi. \end{aligned} \quad (2.99)$$

Similarly,

$$\begin{aligned} \nabla \cdot (\varphi\mathbf{A}) &= \frac{\partial}{\partial x}(\varphi A_x) + \frac{\partial}{\partial y}(\varphi A_y) + \frac{\partial}{\partial z}(\varphi A_z) \\ &= \left(\frac{\partial \varphi}{\partial x} A_x + \varphi \frac{\partial A_x}{\partial x} \right) + \left(\frac{\partial \varphi}{\partial y} A_y + \varphi \frac{\partial A_y}{\partial y} \right) + \left(\frac{\partial \varphi}{\partial z} A_z + \varphi \frac{\partial A_z}{\partial z} \right) \\ &= \left(\frac{\partial \varphi}{\partial x} A_x + \frac{\partial \varphi}{\partial y} A_y + \frac{\partial \varphi}{\partial z} A_z \right) + \varphi \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \\ &= \nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A}. \end{aligned} \quad (2.100)$$

Clearly it will be very tedious to explicitly prove the rest of the product rules in this way. More “elegant” proofs will be given in the chapter of tensor analysis. Here, we will use the following formal procedure to establish these relations. The procedure consists of (1) first using ∇ as a differential operator and (2) then treating ∇ as if it were a regular vector. This procedure is a mnemonic device to give correct results.

Since ∇ is a linear combination of differential operators, we require it to obey the product rule of differentiation. When ∇ operates on a product, the result is the sum of two derivatives obtained by holding one of the factors constant and allowing the other to be operated on by ∇ . As a matter of notation, we attach to ∇ a subscript indicating the one factor upon which

it is currently allowed to operate, and the other factor is kept constant. For instance,

$$\nabla \times (\varphi \mathbf{A}) = \nabla_{\varphi} \times (\varphi \mathbf{A}) + \nabla_A \times (\varphi \mathbf{A}).$$

Since $\nabla_A \times (\varphi \mathbf{A})$ means that φ is a constant, it is then clear

$$\nabla_A \times (\varphi \mathbf{A}) = \varphi \nabla_A \times \mathbf{A} = \varphi \nabla \times \mathbf{A},$$

where the subscript A is omitted from the right-hand side, since it is clear what ∇ operates on when it is followed by just one factor. Similarly, $\nabla_{\varphi} \times (\varphi \mathbf{A})$ means \mathbf{A} is constant. In this case it is easy to show

$$\nabla_{\varphi} \times (\varphi \mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi A_x & \varphi A_y & \varphi A_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \nabla \varphi \times \mathbf{A}.$$

Thus,

$$\nabla \times (\varphi \mathbf{A}) = \nabla_{\varphi} \times (\varphi \mathbf{A}) + \nabla_A \times (\varphi \mathbf{A}) = \nabla \varphi \times \mathbf{A} + \varphi \nabla \times \mathbf{A}. \quad (2.101)$$

For the divergence of a cross product, we start with

$$\begin{aligned} \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \nabla_A \cdot (\mathbf{A} \times \mathbf{B}) + \nabla_B \cdot (\mathbf{A} \times \mathbf{B}) \\ &= \nabla_A \cdot (\mathbf{A} \times \mathbf{B}) - \nabla_B \cdot (\mathbf{B} \times \mathbf{A}). \end{aligned}$$

Recall the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, the dot (\cdot) and the cross (\times) can be interchanged $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Treating ∇_A as a vector, we have

$$\nabla_A \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla_A \times \mathbf{A}) \cdot \mathbf{B} = (\nabla \times \mathbf{A}) \cdot \mathbf{B},$$

where the subscript A is dropped in the last step because the meaning is clear without it. Similarly,

$$\nabla_B \cdot (\mathbf{B} \times \mathbf{A}) = (\nabla \times \mathbf{B}) \cdot \mathbf{A}.$$

Therefore,

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}. \quad (2.102)$$

For the curl of a cross product, we will use the analogy of the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$.

$$\begin{aligned} \nabla \times (\mathbf{A} \times \mathbf{B}) &= \nabla_A \times (\mathbf{A} \times \mathbf{B}) + \nabla_B \times (\mathbf{A} \times \mathbf{B}), \\ \nabla_A \times (\mathbf{A} \times \mathbf{B}) &= (\nabla_A \cdot \mathbf{B}) \mathbf{A} - (\nabla_A \cdot \mathbf{A}) \mathbf{B} = (\mathbf{B} \cdot \nabla_A) \mathbf{A} - (\nabla_A \cdot \mathbf{A}) \mathbf{B}. \end{aligned}$$

In the last step, we have used the relation $(\nabla_A \cdot \mathbf{B}) \mathbf{A} = (\mathbf{B} \cdot \nabla_A) \mathbf{A}$, since \mathbf{B} is regarded as a constant. Similarly,

$$\nabla_B \times (\mathbf{A} \times \mathbf{B}) = (\nabla_B \cdot \mathbf{B}) \mathbf{A} - (\nabla_B \cdot \mathbf{A}) \mathbf{B} = (\nabla_B \cdot \mathbf{B}) \mathbf{A} - (\mathbf{A} \cdot \nabla_B) \mathbf{B}.$$

Therefore

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - (\nabla \cdot \mathbf{A}) \mathbf{B} + (\nabla \cdot \mathbf{B}) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B}, \quad (2.103)$$

where we have dropped the subscripts because the meaning is clear without them.

The product rule of the gradient of a dot product v is more cumbersome,

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \nabla_A(\mathbf{A} \cdot \mathbf{B}) + \nabla_B(\mathbf{A} \cdot \mathbf{B}).$$

To work out $\nabla_A(\mathbf{A} \cdot \mathbf{B})$, we use the property of the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$,

$$\mathbf{A} \times (\nabla_B \times \mathbf{B}) = \nabla_B(\mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla_B) \mathbf{B}.$$

Hence,

$$\nabla_B(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla_B) \mathbf{B} + \mathbf{A} \times (\nabla_B \times \mathbf{B}).$$

Similarly,

$$\nabla_A(\mathbf{A} \cdot \mathbf{B}) = \nabla_A(\mathbf{B} \cdot \mathbf{A}) = (\mathbf{B} \cdot \nabla_A) \mathbf{A} + \mathbf{B} \times (\nabla_A \times \mathbf{A}).$$

Dropping the subscripts when they are not necessary, we have

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A} \times (\nabla \times \mathbf{B}). \quad (2.104)$$

2.7.2 Second Derivatives

Several second derivatives can be constructed by applying ∇ twice. The following four identities of second derivatives are of great interests:

$$\nabla \times \nabla \varphi = \mathbf{0}, \quad (2.105)$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0, \quad (2.106)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \quad (2.107)$$

$$\nabla \cdot (\nabla \varphi \times \nabla \psi) = 0. \quad (2.108)$$

The first identity states that the curl of the gradient of a scalar function is identically equal to zero. This can be shown by direct expansion.

$$\begin{aligned} \nabla \times \nabla \varphi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = \mathbf{i} \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \right) \\ &+ \mathbf{j} \left(\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \right) + \mathbf{k} \left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right) = \mathbf{0}, \end{aligned} \quad (2.109)$$

provided the second cross partial derivatives of φ are continuous (which are generally satisfied by functions of interests). In such a case, the order of differentiation is immaterial.

The second identity states that the divergence of curl of a vector function is identically equal to zero. This can also be shown by direct calculation.

$$\begin{aligned}\nabla \cdot \nabla \times \mathbf{A} &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = 0.\end{aligned}\quad (2.110)$$

It is understood that the determinant is to be expanded along the first row. Again if the partial derivatives are continuous, this determinant with two identical rows is equal to zero.

The curl curl identity is equally important and is worthwhile to commit to memory. For the mnemonic purpose, we can use the analogy of the vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, with $\nabla, \nabla, \mathbf{A}$ as $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Thus, the vector triple product suggests

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}.\quad (2.111)$$

The $\nabla \cdot \nabla$ is a scalar operator. Because it appears often in physics, it has given a special name – the *Laplacian*, or just ∇^2

$$\begin{aligned}\nabla \cdot \nabla &= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2.\end{aligned}\quad (2.112)$$

Therefore, (2.111) can be written as

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.\quad (2.113)$$

Expanding both sides of this equation in rectangular coordinates, one can readily verify that this is indeed an identity.

Since ∇^2 is a scalar operator, when it operates on a vector, it means the same operation on each component of the vector

$$\nabla^2 \mathbf{A} = \mathbf{i} \nabla^2 A_x + \mathbf{j} \nabla^2 A_y + \mathbf{k} \nabla^2 A_z.\quad (2.114)$$

The identity (2.108) follows from $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \nabla \times \mathbf{A} \cdot \mathbf{B} - \nabla \times \mathbf{B} \cdot \mathbf{A}$. Since $\nabla\varphi$ and $\nabla\psi$ are two different vectors,

$$\nabla \cdot (\nabla\varphi \times \nabla\psi) = \nabla \times \nabla\varphi \cdot \nabla\psi - \nabla \times \nabla\psi \cdot \nabla\varphi.$$

Now $\nabla \times \nabla\varphi = \nabla \times \nabla\psi = \mathbf{0}$, therefore

$$\nabla \cdot (\nabla\varphi \times \nabla\psi) = 0. \quad (2.115)$$

Example 2.7.1. Show that $\nabla \times \mathbf{A} = \mathbf{B}$, if $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ and \mathbf{B} is a constant vector, first by direct expansion, then by the formula of the curl of a cross product.

Solution 2.7.1. *Method I*

$$\begin{aligned} \nabla \times \mathbf{A} &= \frac{1}{2} \nabla \times (\mathbf{B} \times \mathbf{r}) = \frac{1}{2} \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} \\ &= \frac{1}{2} \nabla \times [\mathbf{i}(B_y z - B_z y) + \mathbf{j}(B_z x - B_x z) + \mathbf{k}(B_x y - B_y x)] \\ &= \frac{1}{2} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (B_y z - B_z y) & (B_z x - B_x z) & (B_x y - B_y x) \end{vmatrix} \\ &= \frac{1}{2} [\mathbf{i}2B_x + \mathbf{j}2B_y + \mathbf{k}2B_z] = \mathbf{B}. \end{aligned}$$

Method II

$$\begin{aligned} \frac{1}{2} \nabla \times (\mathbf{B} \times \mathbf{r}) &= \frac{1}{2} [(\nabla \cdot \mathbf{r}) \mathbf{B} - (\nabla \cdot \mathbf{B}) \mathbf{r} + (\mathbf{r} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{r}] \\ &= \frac{1}{2} [(\nabla \cdot \mathbf{r}) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{r}] \quad (\text{since } \mathbf{B} \text{ is a constant}), \end{aligned}$$

$$(\nabla \cdot \mathbf{r}) \mathbf{B} = 3\mathbf{B} \quad (\text{see Example 2.5.1}),$$

$$(\mathbf{B} \cdot \nabla) \mathbf{r} = \mathbf{B} \quad (\text{see Example 2.4.2}),$$

$$\frac{1}{2} \nabla \times (\mathbf{B} \times \mathbf{r}) = \frac{1}{2} [3\mathbf{B} - \mathbf{B}] = \mathbf{B}.$$

Example 2.7.2. Show that $\nabla \times (\nabla^2 \mathbf{A}) = \nabla^2 (\nabla \times \mathbf{A})$.

Solution 2.7.2. Since $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$,

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}),$$

$$\nabla \times (\nabla^2 \mathbf{A}) = \nabla \times [\nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})].$$

Since curl gradient is equal to zero, $\nabla \times \nabla(\nabla \cdot \mathbf{A}) = \mathbf{0}$. Using curl curl formula again, we have

$$\nabla \times (\nabla^2 \mathbf{A}) = -\nabla \times \nabla \times (\nabla \times \mathbf{A}) = -\{\nabla(\nabla \cdot (\nabla \times \mathbf{A})) - \nabla^2(\nabla \times \mathbf{A})\}.$$

Since divergence of a curl is equal to zero, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, therefore

$$\nabla \times (\nabla^2 \mathbf{A}) = \nabla^2(\nabla \times \mathbf{A}).$$

Example 2.7.3. If

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{H},$$

$$\nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{H} = \frac{\partial}{\partial t} \mathbf{E},$$

show that

$$\nabla^2 \mathbf{E} = \frac{\partial^2}{\partial t^2} \mathbf{E}; \quad \nabla^2 \mathbf{H} = \frac{\partial^2}{\partial t^2} \mathbf{H}.$$

Solution 2.7.3.

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial}{\partial t} \mathbf{H} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = -\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \mathbf{E} \right) = -\frac{\partial^2}{\partial t^2} \mathbf{E},$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \quad (\text{since } \nabla \cdot \mathbf{E} = 0).$$

Therefore

$$\nabla^2 \mathbf{E} = \frac{\partial^2}{\partial t^2} \mathbf{E}.$$

Similarly,

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \left(\frac{\partial}{\partial t} \mathbf{E} \right) = \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) = \frac{\partial}{\partial t} \left(-\frac{\partial}{\partial t} \mathbf{H} \right) = -\frac{\partial^2}{\partial t^2} \mathbf{H},$$

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = -\nabla^2 \mathbf{H} \quad (\text{since } \nabla \cdot \mathbf{H} = 0).$$

It follows that

$$\nabla^2 \mathbf{H} = \frac{\partial^2}{\partial t^2} \mathbf{H}.$$

2.8 Further Integral Theorems

There are many other integral identities that are useful in physical applications. They can be derived in a variety of ways. Here we discuss some of the most useful ones and show that they all follow from the fundamental theorems of gradient, divergence, and curl.

2.8.1 Green's Theorem

The following integral identities are all named after George Green (1793–1841). To distinguish them, we adopt the following terminology.

Green's Lemma:

$$\oint_C [f(x, y) dx + g(x, y) dy] = \iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy, \quad (2.116)$$

Green's Theorem:

$$\oiint_S \varphi \nabla \psi \cdot \mathbf{n} da = \iiint_V (\nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi) dV, \quad (2.117)$$

Symmetrical form of Green's Theorem:

$$\oiint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot \mathbf{n} da = \iiint_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) dV. \quad (2.118)$$

To prove Green's Lemma, we start with Stokes' theorem

$$\oint_C \mathbf{A} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} da.$$

With the curve C lying entirely on the xy plane,

$$\mathbf{A} \cdot d\mathbf{r} = (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) \cdot (\mathbf{i} dx + \mathbf{j} dy) = A_x dx + A_y dy,$$

and \mathbf{n} is equal to \mathbf{k} , the unit vector in the z direction,

$$(\nabla \times \mathbf{A}) \cdot \mathbf{n} da = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \cdot \mathbf{k} dx dy = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx dy.$$

Thus we have

$$\oint_C (A_x dx + A_y dy) = \iint_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx dy.$$

Since \mathbf{A} in Stokes' theorem can be any vector function, Green's Lemma follows with $A_x = f(x, y)$, and $A_y = g(x, y)$.

To prove Green's theorem, we start with the divergence theorem

$$\iiint_V \nabla \cdot (\varphi \nabla \psi) dV = \iint_S \varphi \nabla \psi \cdot \mathbf{n} da.$$

Using the identity

$$\nabla \cdot (\varphi \nabla \psi) = \nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi,$$

we have

$$\iiint_V (\nabla \varphi \cdot \nabla \psi + \varphi \nabla^2 \psi) dV = \iint_S \varphi \nabla \psi \cdot \mathbf{n} da, \quad (2.119)$$

which is Green's theorem (2.117).

Clearly (2.119) is equally valid when φ and ψ are interchanged

$$\iiint_V (\nabla \psi \cdot \nabla \varphi + \psi \nabla^2 \varphi) dV = \iint_S \psi \nabla \varphi \cdot \mathbf{n} da. \quad (2.120)$$

Taking the difference of the last two equations, we obtain the symmetric form of the Green's theorem

$$\iiint_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) dV = \iint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot \mathbf{n} da.$$

2.8.2 Other Related Integrals

The divergence theorem can take some other alternative forms. Let φ be a scalar function and \mathbf{C} be an arbitrary constant vector. Then,

$$\iiint_V \nabla \cdot (\varphi \mathbf{C}) dV = \iint_S \varphi \mathbf{C} \cdot \mathbf{n} da,$$

$$\nabla \cdot (\varphi \mathbf{C}) = \nabla \varphi \cdot \mathbf{C} + \varphi \nabla \cdot \mathbf{C} = \nabla \varphi \cdot \mathbf{C},$$

since \mathbf{C} is constant and $\nabla \cdot \mathbf{C} = 0$.

$$\iiint_V \nabla \cdot (\varphi \mathbf{C}) dV = \iiint_V \nabla \varphi \cdot \mathbf{C} dV = \mathbf{C} \cdot \iiint_V \nabla \varphi dV.$$

$$\iint_S \varphi \mathbf{C} \cdot \mathbf{n} da = \mathbf{C} \cdot \iint_S \varphi \mathbf{n} da.$$

Therefore the divergence theorem can be written as

$$\mathbf{C} \cdot \left[\iiint_V \nabla \varphi dV - \iint_S \varphi \mathbf{n} da \right] = 0.$$

Since \mathbf{C} is arbitrary, the terms in the brackets must be zero. Thus we have another interesting relation between volume integral and surface integral

$$\iiint_V \nabla \varphi \, dV = \iint_S \varphi \mathbf{n} \, da. \quad (2.121)$$

Similarly, let \mathbf{A} be a vector function and \mathbf{C} , an arbitrary constant vector. $\mathbf{A} \times \mathbf{C}$ is another vector function. The divergence theorem can be written as

$$\iiint_V \nabla \cdot (\mathbf{A} \times \mathbf{C}) \, dV = \iint_S (\mathbf{A} \times \mathbf{C}) \cdot \mathbf{n} \, da.$$

Since

$$\begin{aligned} \nabla \cdot (\mathbf{A} \times \mathbf{C}) &= (\nabla \times \mathbf{A}) \cdot \mathbf{C} - (\nabla \times \mathbf{C}) \cdot \mathbf{A} = (\nabla \times \mathbf{A}) \cdot \mathbf{C}, \\ (\mathbf{A} \times \mathbf{C}) \cdot \mathbf{n} &= -(\mathbf{C} \times \mathbf{A}) \cdot \mathbf{n} = -\mathbf{C} \cdot \mathbf{A} \times \mathbf{n}, \end{aligned}$$

therefore

$$\begin{aligned} \iiint_V \nabla \cdot (\mathbf{A} \times \mathbf{C}) \, dV &= \mathbf{C} \cdot \iiint_V \nabla \times \mathbf{A} \, dV, \\ \iint_S (\mathbf{A} \times \mathbf{C}) \cdot \mathbf{n} \, da &= -\mathbf{C} \cdot \iint_S \mathbf{A} \times \mathbf{n} \, da. \end{aligned}$$

Thus we have another form of the divergence theorem

$$\iiint_V \nabla \times \mathbf{A} \, dV = -\iint_S \mathbf{A} \times \mathbf{n} \, da. \quad (2.122)$$

This exploitation of the arbitrary nature of a part of a problem is a very useful technique. In the following examples some alternative forms of Stokes' theorem will be derived using this technique.

Example 2.8.1. Show that $\oint_C \varphi \, d\mathbf{r} = -\iint_S \nabla \varphi \times \mathbf{n} \, da$.

Solution 2.8.1. Let \mathbf{C} be an arbitrary constant vector. By Stokes' theorem we have

$$\oint_C \varphi \mathbf{C} \cdot d\mathbf{r} = \iint_S \nabla \times (\varphi \mathbf{C}) \cdot \mathbf{n} \, da.$$

Since \mathbf{C} is a constant and $\nabla \times \mathbf{C} = \mathbf{0}$,

$$\begin{aligned} \nabla \times \varphi \mathbf{C} &= \nabla \varphi \times \mathbf{C} + \varphi \nabla \times \mathbf{C} = \nabla \varphi \times \mathbf{C}, \\ \iint_S \nabla \times (\varphi \mathbf{C}) \cdot \mathbf{n} \, da &= \iint_S \nabla \varphi \times \mathbf{C} \cdot \mathbf{n} \, da. \end{aligned}$$

Furthermore,

$$\nabla\varphi \times \mathbf{C} \cdot \mathbf{n} = -\mathbf{C} \times \nabla\varphi \cdot \mathbf{n} = -\mathbf{C} \cdot (\nabla\varphi \times \mathbf{n}).$$

Therefore

$$\iint_S \nabla \times (\varphi \mathbf{C}) \cdot \mathbf{n} \, da = -\mathbf{C} \cdot \iint_S \nabla\varphi \times \mathbf{n} \, da.$$

With

$$\oint_C \varphi \mathbf{C} \cdot d\mathbf{r} = \mathbf{C} \cdot \oint_C \varphi d\mathbf{r},$$

we can write Stokes' theorem as

$$\mathbf{C} \cdot \oint_C \varphi d\mathbf{r} = -\mathbf{C} \cdot \iint_S \nabla\varphi \times \mathbf{n} \, da.$$

Again since \mathbf{C} is an arbitrary constant vector, it follows that

$$\oint_C \varphi d\mathbf{r} = - \iint_S \nabla\varphi \times \mathbf{n} \, da. \quad (2.123)$$

Example 2.8.2. Show that $\oint_C \mathbf{r} \times d\mathbf{r} = 2 \iint_S \mathbf{n} \, da$ where \mathbf{r} is the position vector from an origin that can be chosen at any point in space.

Solution 2.8.2. To prove this, we use an arbitrary constant vector \mathbf{C} and start with Stokes' theorem,

$$\oint_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r} = \iint_S \nabla \times (\mathbf{C} \times \mathbf{r}) \cdot \mathbf{n} \, da.$$

Since

$$\oint_C (\mathbf{C} \times \mathbf{r}) \cdot d\mathbf{r} = \oint_C \mathbf{C} \cdot \mathbf{r} \times d\mathbf{r} = \mathbf{C} \cdot \oint_C \mathbf{r} \times d\mathbf{r},$$

and

$$\nabla \times (\mathbf{C} \times \mathbf{r}) = 2\mathbf{C} \quad (\text{see example 2.7.1}),$$

$$\iint_S \nabla \times (\mathbf{C} \times \mathbf{r}) \cdot \mathbf{n} \, da = \iint_S 2\mathbf{C} \cdot \mathbf{n} \, da = 2\mathbf{C} \cdot \iint_S \mathbf{n} \, da,$$

it follows

$$\mathbf{C} \cdot \oint_C \mathbf{r} \times d\mathbf{r} = \mathbf{C} \cdot 2 \iint_S \mathbf{n} \, da.$$

Since \mathbf{C} is an arbitrary constant vector, the integral identity

$$\oint_C \mathbf{r} \times d\mathbf{r} = 2 \iint_S \mathbf{n} \, da \quad (2.124)$$

must hold. This integral identity is of some importance in electrodynamics. This integral also shows that the area A of a flat surface S enclosed by a curve C is given by

$$A = \iint_S da = \frac{1}{2} \left| \oint_C \mathbf{r} \times d\mathbf{r} \right|. \quad (2.125)$$

2.9 Classification of Vector Fields

2.9.1 Irrotational Field and Scalar Potential

If $\nabla \times \mathbf{F} = \mathbf{0}$ in a simply connected region, we say \mathbf{F} is an *irrotational vector field*. An irrotational field is also known as a *conservative vector field*. We have seen that if $\nabla \times \mathbf{F} = \mathbf{0}$, the line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of path. This means, as shown in Sect. 2.4.3, that \mathbf{F} can be expressed as the gradient of a scalar function φ , known as the scalar potential.

Because of Stokes' theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, da,$$

an irrotational field \mathbf{F} is characterized by any of the following equivalent conditions:

- (a) $\nabla \times \mathbf{F} = \mathbf{0}$,
- (b) $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed loop,
- (c) $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of path,
- (d) $\mathbf{F} = -\nabla\varphi$.

The sign in (d) is arbitrary, since φ is yet to be specified. In hydrodynamics, often a plus sign (+) is chosen for the velocity potential. Here we have followed the convention in choosing a minus sign (−) for the convenience of establishing the principle of conservation of energy.

Conservative Force Field. To see why an irrotational field is also called a conservative vector field, consider $\mathbf{F}(x, y, z)$ as the force in Newton's equation of motion

$$\mathbf{F}(x, y, z) = m\mathbf{a} = m \frac{d\mathbf{v}}{dt}. \quad (2.126)$$

Since \mathbf{F} is irrotational, so

$$\mathbf{F}(x, y, z) = -\nabla\varphi(x, y, z). \quad (2.127)$$

Therefore

$$m \frac{d\mathbf{v}}{dt} = -\nabla\varphi. \quad (2.128)$$

Take dot product of both sides with $d\mathbf{r}$ and integrate. The left-hand side becomes

$$\begin{aligned} \int m \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} &= \int m \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{dt} dt = \int m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt \\ &= \int \frac{d}{dt} \left(\frac{1}{2} m \mathbf{v} \cdot \mathbf{v} \right) dt = \int d \left(\frac{1}{2} m v^2 \right) \\ &= \frac{1}{2} m v^2 + \text{constant}. \end{aligned} \quad (2.129)$$

The right-hand side becomes

$$\int (-\nabla\varphi) \cdot d\mathbf{r} = - \int d\varphi = -\varphi + \text{constant}. \quad (2.130)$$

Equating the results of the two sides of (2.128) gives

$$\frac{1}{2} m v^2 + \varphi = \text{constant}. \quad (2.131)$$

The expression $\frac{1}{2} m v^2$ is defined as the kinetic energy and $\varphi(x, y, z)$ is the potential energy in classical mechanics. The sum of the two is the total energy. The last equation says that no matter where and when the total energy is evaluated, it must be equal to the same constant. This is the principle of conservation of energy.

Although we have used classical mechanics to introduce the idea of conservative field, the idea can be generalized. Any vector field $\mathbf{v}(x, y, z)$ which can be expressed as the gradient of a scalar field $\varphi(x, y, z)$ is called a conservative field and the scalar function φ is called the scalar potential. Since $\nabla\varphi = \nabla(\varphi + \text{constant})$, the scalar potential is defined up to an additive constant.

Example 2.9.1. Determine which of the following is an irrotational (or conservative) field: (a) $\mathbf{F}_1 = 6xy\mathbf{i} + (3x^2 - 3y^2)\mathbf{j}$, (b) $\mathbf{F}_2 = xy\mathbf{i} - y\mathbf{j}$

Solution 2.9.1.

$$\begin{aligned} \nabla \times \mathbf{F}_1 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & 3x^2 - 3y^2 & 0 \end{vmatrix} = -\mathbf{i} \frac{\partial}{\partial z} (3x^2 - 3y^2) + \mathbf{j} \frac{\partial}{\partial z} (6xy) \\ &\quad + \mathbf{k} \left(\frac{\partial}{\partial x} (3x^2 - 3y^2) - \frac{\partial}{\partial y} (6xy) \right) = \mathbf{k} (6x - 6x) = \mathbf{0}. \end{aligned}$$

$$\begin{aligned}\nabla \times \mathbf{F}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y & 0 \end{vmatrix} = -\mathbf{i} \frac{\partial}{\partial z} y + \mathbf{j} \frac{\partial}{\partial z} (xy) \\ &\quad + \mathbf{k} \left(\frac{\partial}{\partial x} y - \frac{\partial}{\partial y} (xy) \right) = -x\mathbf{k} \neq \mathbf{0}.\end{aligned}$$

Therefore \mathbf{F}_1 is an irrotational field and \mathbf{F}_2 is not an irrotational field. We have shown explicitly, in the examples of Sect. 2.4.3, that the line integral $\int_A^B \mathbf{F}_1 \cdot d\mathbf{r}$ is independent of path and $\int_A^B \mathbf{F}_2 \cdot d\mathbf{r}$ is dependent on the path.

Example 2.9.2. Show that the force field $\mathbf{F} = -(2ax + by)\mathbf{i} - bx\mathbf{j} - c\mathbf{k}$ is conservative, and find φ such that $-\nabla\varphi = \mathbf{F}$.

Solution 2.9.2.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(2ax + by) & -bx & -c \end{vmatrix} = \mathbf{0}.$$

Therefore, \mathbf{F} is conservative, there must exist a φ such that $-\nabla\varphi = \mathbf{F}$.

$$\begin{aligned}-\frac{\partial\varphi}{\partial x} &= F_x = -(2ax + by) \implies \varphi = ax^2 + bxy + f(y, z). \\ -\frac{\partial\varphi}{\partial y} &= F_y = -bx, \text{ but } -\frac{\partial\varphi}{\partial y} = -bx - \frac{\partial}{\partial y} f(y, z) \\ \frac{\partial}{\partial y} f(y, z) &= 0 \implies f(y, z) = g(z) \\ -\frac{\partial\varphi}{\partial z} &= F_z = -c, \text{ but } -\frac{\partial\varphi}{\partial z} = -\frac{\partial}{\partial z} g(z) \\ \frac{\partial}{\partial z} g(z) &= c \implies g(z) = cz + k. \\ \varphi &= ax^2 + bxy + cz + k.\end{aligned}$$

Example 2.9.3. Suppose a particle of mass m is moving in the force field of the last example, and at $t = 0$ the particle passes through the origin with speed v_0 . What will the speed of the particle be if and when it passes through the point $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$?

Solution 2.9.3. The conservation of energy requires

$$\frac{1}{2}mv^2 + \varphi(\mathbf{r}) = \frac{1}{2}mv_0^2 + \varphi(\mathbf{0}).$$

$$v^2 = v_0^2 + \frac{2}{m} [k - (ax^2 + bxy + cz + k)].$$

At $x = 1$, $y = 2$, $z = 1$:

$$v^2 = v_0^2 + \frac{2}{m} (a + 2b + c).$$

2.9.2 Solenoidal Field and Vector Potential

If the field \mathbf{F} is divergence-less (that is $\nabla \cdot \mathbf{F} = 0$) everywhere in a simply connected region, the field is called *solenoidal*. For a solenoidal field, the surface integral of $\mathbf{F} \cdot \mathbf{n} \, da$ over any closed surface is zero, since by the divergence theorem

$$\oiint_S \mathbf{F} \cdot \mathbf{n} \, da = \iiint \nabla \cdot \mathbf{F} \, dV = 0.$$

Furthermore, \mathbf{F} can be expressed as the curl of another vector function \mathbf{A} ,

$$\mathbf{F} = \nabla \times \mathbf{A}.$$

The vector function \mathbf{A} is known as the *vector potential* of the field \mathbf{F} .

The existence of vector potentials for solenoidal fields can be shown in the following way. For any given solenoidal field \mathbf{F} (that is, $F_x(x, y, z)$, $F_y(x, y, z)$, and $F_z(x, y, z)$ are known), we shall first show that it is possible to find a vector function \mathbf{A} with one zero component to satisfy $\mathbf{F} = \nabla \times \mathbf{A}$. Then a general formula for all possible vector potentials can be found.

Let us take $A_z = 0$, and try to find A_x and A_y in $\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j}$ so that $\nabla \times \mathbf{A} = \mathbf{F}$:

$$\begin{aligned} \nabla \times \mathbf{A} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & 0 \end{vmatrix} = -\mathbf{i} \frac{\partial}{\partial z} A_y + \mathbf{j} \frac{\partial}{\partial z} A_x + \mathbf{k} \left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right) \\ &= \mathbf{i} F_x + \mathbf{j} F_y + \mathbf{k} F_z. \end{aligned}$$

For this to hold, we must have

$$\frac{\partial}{\partial z} A_y = -F_x, \quad \frac{\partial}{\partial z} A_x = F_y, \quad \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x = F_z. \quad (2.132)$$

From the first two equations we have

$$A_y = - \int F_x(x, y, z) dz + f(x, y), \quad (2.133)$$

$$A_x = \int F_y(x, y, z) dz + g(x, y). \quad (2.134)$$

With A_y and A_x so obtained, if we can show $\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = F_z$, then we would have proved that $\nabla \times \mathbf{A} = \mathbf{F}$.

Using (2.133) and (2.134), we have

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = - \int \left(\frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y \right) dz + h(x, y).$$

Since \mathbf{F} is solenoidal, $\nabla \cdot \mathbf{F} = 0$ which can be written as

$$\frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y = - \frac{\partial}{\partial z}F_z.$$

Thus,

$$\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x = \int \frac{\partial}{\partial z}F_z dz + h(x, y).$$

With proper choice of $h(x, y)$, we can certainly make

$$\int \frac{\partial}{\partial z}F_z dz + h(x, y) = F_z.$$

This proof clearly indicates that \mathbf{A} is not unique. If \mathbf{A}' is another vector potential, then both $\nabla \times \mathbf{A}$ and $\nabla \times \mathbf{A}'$ are equal to \mathbf{F} . Therefore $\nabla \times (\mathbf{A}' - \mathbf{A}) = \mathbf{0}$. Since $(\mathbf{A}' - \mathbf{A})$ is irrotational, it follows that $\mathbf{A}' - \mathbf{A} = \nabla\psi$. Thus we conclude that with one \mathbf{A} obtained from the above procedure, all other vector potentials are of the form $\mathbf{A} + \nabla\psi$ where ψ is any scalar function.

It is also possible for us to require the vector potential to be solenoidal. If we find a vector potential \mathbf{A} which is not solenoidal (that is, $\nabla \times \mathbf{A} = \mathbf{F}$ and $\nabla \cdot \mathbf{A} \neq 0$), we can construct another vector potential \mathbf{A}' which is solenoidal ($\nabla \cdot \mathbf{A}' = 0$). Let

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\psi, \\ \nabla \times \mathbf{A}' &= \nabla \times \mathbf{A} + \nabla \times \nabla\psi = \nabla \times \mathbf{A}, \\ \nabla \cdot \mathbf{A}' &= \nabla \cdot \mathbf{A} + \nabla^2\psi. \end{aligned}$$

If we choose ψ such that $\nabla^2\psi + \nabla \cdot \mathbf{A} = 0$, then we will have $\nabla \cdot \mathbf{A}' = 0$. The following example will make this clear.

Example 2.9.4. Show that $\mathbf{F} = x^2\mathbf{i} + 3xz^2\mathbf{j} - 2xz\mathbf{k}$ is solenoidal, and find a vector potential \mathbf{A} such that $\nabla \times \mathbf{A} = \mathbf{F}$ and $\nabla \cdot \mathbf{A} = 0$.

Solution 2.9.4. Since

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(3xz^2) + \frac{\partial}{\partial z}(-2xz) = 0,$$

this shows that \mathbf{F} is solenoidal. Let $\mathbf{A}_1 = A_x\mathbf{i} + A_y\mathbf{j}$ and $\nabla \times \mathbf{A}_1 = \mathbf{F}$. By (2.132)

$$\begin{aligned}\frac{\partial}{\partial z}A_y &= -F_x = -x^2, \implies A_y = -x^2z + f(x, y), \\ \frac{\partial}{\partial z}A_x &= F_y = 3xz^2, \implies A_x = xz^3 + g(x, y), \\ \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x &= F_z = -2xz, \implies -2xy + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -2xy.\end{aligned}$$

Since f and g are arbitrary, the simplest choice is to make $f = g = 0$. Thus, $\mathbf{A}_1 = xz^3\mathbf{i} - x^2z\mathbf{j}$, but $\nabla \cdot \mathbf{A}_1 = z^3 \neq 0$. Let

$$\mathbf{A} = \mathbf{A}_1 + \nabla\psi, \quad \nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_1 + \nabla^2\psi = z^3 + \nabla^2\psi.$$

If $\nabla \cdot \mathbf{A} = 0$, then $\nabla^2\psi = -z^3$. A simple solution of this equation is

$$\psi = -\frac{1}{20}z^5.$$

Since

$$\begin{aligned}\nabla\psi &= \nabla\left(-\frac{1}{20}z^5\right) = -\frac{1}{4}z^4\mathbf{k}, \\ \mathbf{A} &= \mathbf{A}_1 + \nabla\psi = xz^3\mathbf{i} - x^2z\mathbf{j} - \frac{1}{4}z^4\mathbf{k}.\end{aligned}$$

It can be readily verified that

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial}{\partial x}(xz^3) + \frac{\partial}{\partial y}(-x^2z) + \frac{\partial}{\partial z}\left(-\frac{1}{4}z^4\right) = 0, \\ \nabla \times \mathbf{A} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -x^2z & -\frac{1}{4}z^4 \end{vmatrix} = x^2\mathbf{i} + 3xz^2\mathbf{j} - 2xz\mathbf{k} = \mathbf{F}.\end{aligned}$$

This vector potential is still not unique. For example, we can assume $\mathbf{A}_2 = A_y\mathbf{j} + A_z\mathbf{k}$ and $\nabla \times \mathbf{A}_2 = \mathbf{F}$. Following the same procedure, we obtain

$$\mathbf{A}_2 = -x^2z\mathbf{j} - \frac{3}{2}x^2z^2\mathbf{k}.$$

Now, $\nabla \cdot \mathbf{A}_2 = -3x^2z \neq 0$. We can find \mathbf{A}' such that $\mathbf{A}' = \mathbf{A}_2 + \nabla\psi$ and $\nabla \cdot \mathbf{A}' = 0$. It follows that $\nabla^2\psi = -\nabla \cdot \mathbf{A}_2 = 3x^2z$. A simple solution is $\psi = \frac{1}{4}x^4z$. Therefore $\nabla\psi = x^3z\mathbf{i} + \frac{1}{4}x^4\mathbf{k}$, and

$$\mathbf{A}' = \mathbf{A}_2 + \nabla\psi = x^3z\mathbf{i} - x^2z\mathbf{j} + \left(\frac{1}{4}x^4 - \frac{3}{2}x^2z^2\right)\mathbf{k}.$$

Again, it can be readily verified that $\nabla \times \mathbf{A}' = \mathbf{F}$ and $\nabla \cdot \mathbf{A}' = 0$.

Clearly, \mathbf{A} and \mathbf{A}' are not identical. They must differ by an additive gradient

$$\mathbf{A}' = \mathbf{A} + \nabla\chi. \quad (2.135)$$

Now $\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2\chi$ and $\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} = 0$. Therefore

$$\nabla^2\chi = 0. \quad (2.136)$$

In this particular case,

$$\begin{aligned} \nabla\chi &= \mathbf{A}' - \mathbf{A} = (x^3z - xz^3)\mathbf{i} + \left(\frac{1}{4}x^4 - \frac{3}{2}x^2z^2 + \frac{1}{4}z^4\right)\mathbf{k}, \\ \chi &= \frac{1}{4}x^4z - \frac{1}{2}x^2z^3 + \frac{1}{20}z^5, \\ \nabla^2\chi &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{4}x^4z - \frac{1}{2}x^2z^3 + \frac{1}{20}z^5\right) = 0. \end{aligned}$$

Equation (2.135) is an example of what is known as a *gauge transformation*. The requirement (2.136) leads to the so-called *Coulomb gauge*. The vector potential is not as useful as the scalar potential in computation. It is in the conceptual development of time-dependent problems, especially in electrodynamics, that the vector potential is essential.

2.10 Theory of Vector Fields

2.10.1 Functions of Relative Coordinates

Very often we deal with functions that depend only on the difference of the coordinates. For example, the electric field at the point (x, y, z) due to the a point charge at (x', y', z') is a function solely of $(x - x')$, $(y - y')$, $(z - z')$. The point (x, y, z) is called the field point and the point (x', y', z') is called source point. The relative position vector \mathbf{R} shown in Fig. 2.18 can be written as

$$\mathbf{R} = \mathbf{r} - \mathbf{r}' = (x - x')\mathbf{i} + (y - y')\mathbf{j} + (z - z')\mathbf{k}. \quad (2.137)$$

The distance between these two points is

$$R = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}. \quad (2.138)$$

Let $f(\mathbf{R})$ be a function of the relative position vector. This function could be a scalar or a component of a vector. Functions of this type have some important properties. Let us define $X = (x - x')$, $Y = (y - y')$, $Z = (z - z')$. Using the chain rule of differentiation, we find

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial X} \frac{\partial X}{\partial x} = \frac{\partial f}{\partial X}; \quad \frac{\partial f}{\partial x'} = \frac{\partial f}{\partial X} \frac{\partial X}{\partial x'} = -\frac{\partial f}{\partial X}.$$

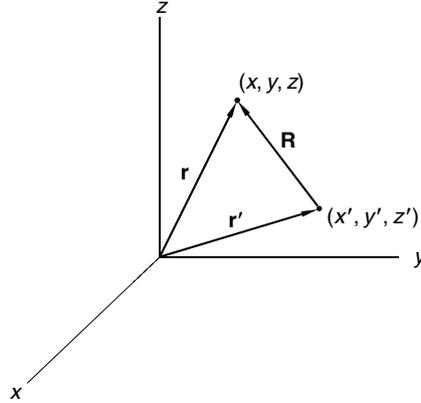


Fig. 2.18. Relative coordinates $\mathbf{R} = \mathbf{r} - \mathbf{r}'$

Similar expressions can be found for the y and z derivatives. It follows

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial x'}, \quad \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial y'}, \quad \frac{\partial f}{\partial z} = -\frac{\partial f}{\partial z'}. \quad (2.139)$$

Corresponding to the gradient ∇ with respect to the field point

$$\nabla f = \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z},$$

we define the gradient ∇' with respect to the source point

$$\nabla' f = \mathbf{i} \frac{\partial f}{\partial x'} + \mathbf{j} \frac{\partial f}{\partial y'} + \mathbf{k} \frac{\partial f}{\partial z'}.$$

It follows from (2.139) that

$$\nabla f = -\nabla' f. \quad (2.140)$$

This shows that when we deal with functions of the relative coordinates the ∇ and ∇' operator can be interchanged provided the sign is also changed. Similar calculations can be used to show

$$\nabla \cdot \mathbf{A}(\mathbf{R}) = -\nabla' \cdot \mathbf{A}(\mathbf{R}), \quad (2.141)$$

$$\nabla \times \mathbf{A}(\mathbf{R}) = -\nabla' \times \mathbf{A}(\mathbf{R}), \quad (2.142)$$

and

$$\nabla^2 f(\mathbf{R}) = \nabla'^2 f(\mathbf{R}). \quad (2.143)$$

Example 2.10.1. Show that (a) $\nabla \cdot \mathbf{R} = 3$, (b) $\nabla \times \mathbf{R} = \mathbf{0}$, (c) $\nabla \times f(R)\mathbf{R} = \mathbf{0}$, and (d) $\nabla \cdot f(R)\mathbf{R} = \frac{df(R)}{dR}R + 3f(R)$.

Solution 2.10.1.

$$\nabla \cdot \mathbf{R} = \frac{\partial}{\partial x}(x - x') + \frac{\partial}{\partial y}(y - y') + \frac{\partial}{\partial z}(z - z') = 3,$$

$$\nabla \times \mathbf{R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x - x') & (y - y') & (z - z') \end{vmatrix} = \mathbf{0},$$

$$\begin{aligned} \nabla \times f(R)\mathbf{R} &= \nabla f(R) \times \mathbf{R} + f(R)\nabla \times \mathbf{R} \\ &= \frac{df(R)}{dR}\widehat{\mathbf{R}} \times \mathbf{R} = \mathbf{0}, \end{aligned}$$

$$\begin{aligned} \nabla \cdot f(R)\mathbf{R} &= \nabla f(R) \cdot \mathbf{R} + f(R)\nabla \cdot \mathbf{R} \\ &= \frac{df(R)}{dR}\widehat{\mathbf{R}} \cdot \mathbf{R} + 3f(R) = \frac{df(R)}{dR}R + 3f(R). \end{aligned}$$

For functions that depend only on the distance between the two points, the gradient takes a simple form:

$$\nabla f(R) = \mathbf{i}\frac{\partial f(R)}{\partial x} + \mathbf{j}\frac{\partial f(R)}{\partial y} + \mathbf{k}\frac{\partial f(R)}{\partial z}.$$

By the chain rule

$$\frac{\partial f(R)}{\partial x} = \frac{df(R)}{dR} \frac{\partial R}{\partial x},$$

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x} \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \frac{x - x'}{R}.$$

With similar expressions for y and z derivatives, we have

$$\begin{aligned} \nabla f(R) &= \frac{df(R)}{dR} \left(\mathbf{i}\frac{x - x'}{R} + \mathbf{j}\frac{y - y'}{R} + \mathbf{k}\frac{z - z'}{R} \right) \\ &= \frac{df(R)}{dR} \frac{\mathbf{R}}{R} = \frac{df(R)}{dR} \widehat{\mathbf{R}}, \end{aligned} \tag{2.144}$$

where $\widehat{\mathbf{R}}$ is the unit vector in the direction of \mathbf{R} . In particular

$$\nabla R = \widehat{\mathbf{R}}, \quad (2.145)$$

$$\nabla R^n = nR^{n-1}\widehat{\mathbf{R}}. \quad (2.146)$$

For $n = -1$

$$\nabla \frac{1}{R} = -\frac{1}{R^2}\widehat{\mathbf{R}}. \quad (2.147)$$

This last expression is an especially important case because $-\widehat{\mathbf{R}}/R^2$ is the “radial inverse-square-law” field. This vector field (with appropriate multiplicative constants) describes two of the most important fundamental forces in nature, namely the gravitational force field and the Coulomb force field of a static electric charge. The divergence of this field requires our special attention.

2.10.2 Divergence of $\widehat{\mathbf{R}}/|\mathbf{R}|^2$ as a Delta Function

The divergence of $\widehat{\mathbf{R}}/R^2$ has some peculiar and important properties. Calculated directly

$$\begin{aligned} \nabla \cdot \frac{\widehat{\mathbf{R}}}{R^2} &= \nabla \cdot \frac{1}{R^3}\mathbf{R} = \left(\nabla \frac{1}{R^3}\right) \cdot \mathbf{R} + \frac{1}{R^3}\nabla \cdot \mathbf{R} \\ &= -3\frac{1}{R^4}\widehat{\mathbf{R}} \cdot \mathbf{R} + 3\frac{1}{R^3} = 0, \end{aligned}$$

we get zero. On the other hand, as we discussed earlier, the divergence is a measure of the strength of the source of the vector field. If it were zero everywhere, how could there be any gravitational and electric fields? Furthermore, if we apply the divergence theorem (2.72) to this function over a sphere of radius R around the point (x', y', z') , we will get a nonzero result,

$$\begin{aligned} \iiint_V \nabla \cdot \frac{\widehat{\mathbf{R}}}{R^2} dV &= \iint_S \frac{\widehat{\mathbf{R}}}{R^2} \cdot \mathbf{n} da = \frac{1}{R^2} \iint_S \widehat{\mathbf{R}} \cdot \widehat{\mathbf{R}} da \\ &= \frac{1}{R^2} \iint_S da = \frac{1}{R^2} 4\pi R^2 = 4\pi. \end{aligned} \quad (2.148)$$

In the integral, we have used the facts that on the surface of a sphere, the unit normal \mathbf{n} is equal to $\widehat{\mathbf{R}}$ and R is a constant. This integral would be zero if $\nabla \cdot (\widehat{\mathbf{R}}/R^2)$ were equal to zero everywhere.

The source of the problem is at the point $R = 0$ where $\widehat{\mathbf{R}}/R^2$ blows up and the derivative in the usual sense does not exist. Thus we can only say that the divergence is zero everywhere except at $R = 0$. To find out the divergence at $R = 0$, we note that the volume integral (2.148) of the divergence over a sphere is equal to 4π no matter how small R is. Evidently the entire contribution must

be coming from the point $R = 0$. A useful way to describe this behavior is through the Dirac delta function $\delta^3(\mathbf{r} - \mathbf{r}')$.

A more detailed description of the delta function is given a later chapter. Here it suffices to know that the delta function $\delta^3(\mathbf{r} - \mathbf{r}')$ is a sharply peaked function at $\mathbf{r} = \mathbf{r}'$ with the properties

$$\delta^3(\mathbf{r} - \mathbf{r}') = \begin{cases} 0 & \mathbf{r} \neq \mathbf{r}' \\ \infty & \mathbf{r} = \mathbf{r}' \end{cases} \quad (2.149)$$

and

$$\iiint_{\text{all space}} \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r} = 1, \quad (2.150)$$

where $d^3\mathbf{r}$ is a commonly used symbol for the volume element around the field point $d^3\mathbf{r} = dV = dx \, dy \, dz$. It follows that the delta function is characterized by the shifting property

$$\iiint_{\text{all space}} f(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r} = f(\mathbf{r}'), \quad (2.151)$$

because

$$\iiint_{\text{all space}} f(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r} = \iiint_{\text{all space}} f(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r},$$

since the value of $f(\mathbf{r})$ is immaterial for $\mathbf{r} \neq \mathbf{r}'$ as the integrand is going to be zero anyway. Furthermore,

$$\iiint_{\text{all space}} f(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r} = f(\mathbf{r}') \iiint_{\text{all space}} \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r} = f(\mathbf{r}'),$$

since the integration is over $d^3\mathbf{r}$. This property can also be written as

$$\iiint_{\text{all space}} f(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r}' = f(\mathbf{r}), \quad (2.152)$$

where $d^3\mathbf{r}' = dx' \, dy' \, dz'$.

With the delta function, the divergence of $\widehat{\mathbf{R}}/R^2$ can be precisely expressed as

$$\nabla \cdot \frac{\widehat{\mathbf{R}}}{R^2} = \nabla \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = 4\pi \delta^3(\mathbf{r} - \mathbf{r}'). \quad (2.153)$$

With this understanding, we see that

$$\begin{aligned} \iiint_V \nabla \cdot \frac{\widehat{\mathbf{R}}}{R^2} dV &= \iiint_V \nabla \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r} = 4\pi \iiint_V \delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r} \\ &= \begin{cases} 4\pi & \text{if the volume includes } \mathbf{r}' \\ 0 & \text{if } \mathbf{r}' \text{ is outside the body.} \end{cases} \end{aligned} \quad (2.154)$$

Since

$$\nabla \frac{1}{R} = -\frac{\widehat{\mathbf{R}}}{R^2},$$

it follows that the Laplacian of $(\frac{1}{R})$ is given by

$$\nabla^2 \frac{1}{R} = \nabla \cdot \nabla \frac{1}{R} = -\nabla \cdot \frac{\widehat{\mathbf{R}}}{R^2} = -4\pi\delta^3(\mathbf{r} - \mathbf{r}'). \quad (2.155)$$

Example 2.10.2. Evaluate the integral

$$I = \iiint_V (r^3 + 1) \nabla \cdot \frac{\widehat{\mathbf{r}}}{r^2} dV,$$

where V is a sphere of radius b centered at the origin.

Solution 2.10.2. *Method I.* Use the delta function. Since

$$\begin{aligned} \nabla \cdot \frac{\widehat{\mathbf{r}}}{r^2} &= 4\pi\delta^3(\mathbf{r}), \\ I &= \iiint_V (r^3 + 1) 4\pi\delta^3(\mathbf{r}) dV = 4\pi(0 + 1) = 4\pi. \end{aligned}$$

Method II. Use integration by parts. Since $f\nabla \cdot \mathbf{A} = \nabla \cdot (f\mathbf{A}) - \nabla f \cdot \mathbf{A}$,

$$I = \iiint_V (r^3 + 1) \nabla \cdot \frac{\widehat{\mathbf{r}}}{r^2} dV = \iiint_V \nabla \cdot \left[(r^3 + 1) \frac{\widehat{\mathbf{r}}}{r^2} \right] dV - \iiint_V \nabla(r^3 + 1) \cdot \frac{\widehat{\mathbf{r}}}{r^2} dV.$$

By the divergence theorem

$$\iiint_V \nabla \cdot \left[(r^3 + 1) \frac{\widehat{\mathbf{r}}}{r^2} \right] dV = \oiint_S (r^3 + 1) \frac{\widehat{\mathbf{r}}}{r^2} \cdot \widehat{\mathbf{r}} da = \oiint_S \left(r + \frac{1}{r^2} \right) da,$$

where S is the surface of the sphere of radius b . Since on this surface $r = b$ everywhere, therefore

$$\iiint_V \nabla \cdot \left[(r^3 + 1) \frac{\widehat{\mathbf{r}}}{r^2} \right] dV = \left(b + \frac{1}{b^2} \right) \oiint_S da = \left(b + \frac{1}{b^2} \right) 4\pi b^2 = 4\pi b^3 + 4\pi.$$

Since $\nabla(r^3 + 1) = 3r^2\widehat{\mathbf{r}}$,

$$\iiint_V \nabla(r^3 + 1) \cdot \frac{\widehat{\mathbf{r}}}{r^2} dV = \iiint_V 3r^2\widehat{\mathbf{r}} \cdot \frac{\widehat{\mathbf{r}}}{r^2} dV = 3 \iiint_V dV = 3 \frac{4}{3} \pi b^3 = 4\pi b^3.$$

Thus we have

$$I = 4\pi b^3 + 4\pi - 4\pi b^3 = 4\pi,$$

which is the same as the result of delta function method. This example illustrates the validity and power of the delta function method. If the volume is not a sphere, as long as it includes the origin, the delta function result is still valid, but the direct integration will be much more difficult to do.

2.10.3 Helmholtz's Theorem

The Helmholtz theorem deals with the question of what information we need to determine a vector field. Basically, the answer is that if the divergence and the curl of a vector field are known, with some boundary conditions the vector field can be found uniquely.

The Helmholtz theorem states that any vector field \mathbf{F} may be decomposed into the sum of two vectors, one is the gradient of a scalar potential φ and the other the curl of a vector potential \mathbf{A} ,

$$\mathbf{F} = -\nabla\varphi + \nabla \times \mathbf{A}. \quad (2.156)$$

Furthermore, if $\mathbf{F} \rightarrow \mathbf{0}$ on the surface at infinity faster than $1/R$ and $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ are known everywhere, then

$$\varphi(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{\nabla' \cdot \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}', \quad (2.157)$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{\nabla' \times \mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (2.158)$$

To prove this theorem, we first construct a vector function \mathbf{G}

$$\mathbf{G}(\mathbf{r}) = \iiint \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (2.159)$$

Let us apply the Laplacian ∇^2 to both sides of this equation. Because ∇^2 operates only on \mathbf{r} and only $|\mathbf{r} - \mathbf{r}'|^{-1}$ contains \mathbf{r} , we have

$$\nabla^2 \mathbf{G}(\mathbf{r}) = \iiint \left(\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'. \quad (2.160)$$

Since by (2.155)

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta^3(\mathbf{r} - \mathbf{r}'),$$

it follows from the definition of the delta function that

$$\nabla^2 \mathbf{G}(\mathbf{r}) = \iiint (-4\pi\delta^3(\mathbf{r} - \mathbf{r}')) \mathbf{F}(\mathbf{r}') d^3\mathbf{r}' = -4\pi\mathbf{F}(\mathbf{r}). \quad (2.161)$$

Therefore

$$\mathbf{F}(\mathbf{r}) = -\frac{1}{4\pi} \nabla^2 \mathbf{G}(\mathbf{r}). \quad (2.162)$$

Using the vector identity $\nabla \times (\nabla \times \mathbf{G}) = \nabla(\nabla \cdot \mathbf{G}) - \nabla^2 \mathbf{G}$, we have

$$\nabla^2 \mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \nabla \times (\nabla \times \mathbf{G}).$$

Thus with

$$\varphi = \frac{1}{4\pi}(\nabla \cdot \mathbf{G}), \quad \mathbf{A} = \frac{1}{4\pi}(\nabla \times \mathbf{G}),$$

the first part of the theorem follows from (2.162)

$$\mathbf{F}(\mathbf{r}) = -\frac{1}{4\pi}\nabla^2\mathbf{G}(\mathbf{r}) = -\nabla\varphi + \nabla \times \mathbf{A}.$$

To find the explicit expression for φ , we start with

$$\varphi(\mathbf{r}) = \frac{1}{4\pi}(\nabla \cdot \mathbf{G}) = \frac{1}{4\pi}\nabla \cdot \iiint \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'.$$

Since ∇ operates only on \mathbf{r} , and only $|\mathbf{r} - \mathbf{r}'|$ contains \mathbf{r} ,

$$\nabla \cdot \iiint \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \iiint \nabla \cdot \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \iiint \left(\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'.$$

Now

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

and

$$\left(\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \cdot \mathbf{F}(\mathbf{r}') = \nabla' \cdot \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{F}(\mathbf{r}'),$$

so

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi} \iiint \nabla' \cdot \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' + \frac{1}{4\pi} \iiint \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'. \quad (2.163)$$

The first integral on the right-hand side can be changed to a surface integral at infinity by the divergence theorem

$$\iiint_{\text{all space}} \nabla' \cdot \frac{\mathbf{F}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \iint_{S \rightarrow \infty} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{F}(\mathbf{r}') \cdot \mathbf{n} da'.$$

As $r' \rightarrow \infty$, $\mathbf{F}(\mathbf{r}')$ goes to zero faster than $1/r'$. Hence the surface integral is equal to zero. This follows from the fact that the surface is only proportional to r'^2 , and $\mathbf{F}(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ goes to zero faster than $1/r'^2$. Thus only the second integral on the right-hand side of (2.163) remains

$$\varphi(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'.$$

Similarly, for the vector potential we start with

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{1}{4\pi}(\nabla \times \mathbf{G}) = \frac{1}{4\pi} \iiint \nabla \times \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathbf{F}(\mathbf{r}') d^3\mathbf{r}' \\ &= \frac{1}{4\pi} \iiint \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \times \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'. \end{aligned}$$

Using the identities

$$\begin{aligned}\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{F}(\mathbf{r}') &= -\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{F}(\mathbf{r}'), \\ \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{F}(\mathbf{r}') &= \nabla' \times \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathbf{F}(\mathbf{r}') - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}'),\end{aligned}$$

we have

$$\mathbf{A}(\mathbf{r}) = -\frac{1}{4\pi} \iiint \nabla' \times \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathbf{F}(\mathbf{r}') d^3\mathbf{r}' + \frac{1}{4\pi} \iiint \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'. \quad (2.164)$$

By the integral theorem (2.122)

$$\iiint_V \nabla \times \mathbf{P} d^3\mathbf{r}' = - \iint_S \mathbf{P} \times \mathbf{n} da,$$

the first integral on the right-hand side of (2.164) can be transformed into a surface integral

$$-\frac{1}{4\pi} \iiint_{\text{all space}} \nabla' \times \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathbf{F}(\mathbf{r}') d^3\mathbf{r}' = \frac{1}{4\pi} \iint_{S \rightarrow \infty} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \mathbf{F}(\mathbf{r}') \times \mathbf{n} da',$$

which is zero because $\mathbf{F}(\mathbf{r}') \rightarrow 0$ on the surface at infinity faster than $1/r'$. Thus (2.164) becomes

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'. \quad (2.165)$$

This completes the proof. The divergence and curl of \mathbf{F} are often called the sources of the field, since \mathbf{F} can be found from the knowledge of them. The point \mathbf{r} where we evaluate \mathbf{F} is called the field point. The point \mathbf{r}' where the sources are evaluated for the purpose of integration is called the source point. The volume element $d^3\mathbf{r}'$ is at the source point. The function φ and \mathbf{A} are called scalar and vector potentials, respectively, because \mathbf{F} is obtained from them by differentiation.

It should be noted that while the field $\mathbf{F}(\mathbf{r})$ so determined is unique, the potentials φ and \mathbf{A} are not. Any constant can be added to φ , since $\nabla(\varphi + C) = \nabla\varphi$. The gradient of any scalar function can be added to \mathbf{A} , since $\nabla \times (\mathbf{A} + \nabla\psi) = \nabla \times \mathbf{A}$.

Example 2.10.3. If $\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{1}{|\mathbf{r}-\mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'$ and $\mathbf{F}(\mathbf{r}')$ goes to zero on the surface at infinity faster than $1/r'$, show that $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$.

Solution 2.10.3. Since ∇ operates only on \mathbf{r} ,

$$\begin{aligned}\nabla \cdot \mathbf{A}(\mathbf{r}) &= \frac{1}{4\pi} \iiint \nabla \cdot \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}') d^3\mathbf{r}' \\ &= \frac{1}{4\pi} \iiint \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \nabla' \times \mathbf{F}(\mathbf{r}') d^3\mathbf{r}'.\end{aligned}$$

Now

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \nabla' \times \mathbf{F}(\mathbf{r}') = -\nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \nabla' \times \mathbf{F}(\mathbf{r}')$$

and

$$\begin{aligned}\nabla' \cdot \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}') \right] &= \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \nabla' \times \mathbf{F}(\mathbf{r}') + \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \cdot \nabla' \times \mathbf{F}(\mathbf{r}') \\ &= \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \cdot \nabla' \times \mathbf{F}(\mathbf{r}'),\end{aligned}$$

because the divergence of a curl is equal to zero. Therefore we have

$$\begin{aligned}\nabla \cdot \mathbf{A}(\mathbf{r}) &= -\frac{1}{4\pi} \iiint \left[\nabla' \cdot \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}') \right] d^3\mathbf{r}' \\ &= -\frac{1}{4\pi} \oiint_S \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \times \mathbf{F}(\mathbf{r}') \cdot \mathbf{n} da.\end{aligned}$$

As $S \rightarrow \infty$, $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$.

2.10.4 Poisson's and Laplace's Equations

The Helmholtz's theorem shows that the vector field is uniquely determined by its divergence and curl. To derive the expressions for the divergence and curl from experimental observations is therefore of great importance.

One of the most important vector fields is the radial inverse square law field, which is the mathematical statement of the gravitational law and the Coulomb's law, the two fundamental laws in nature. For example, together with the principle of superposition, the electric field $\mathbf{E}(\mathbf{r})$ produced by static charges can be written as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi} \iiint \varrho(\mathbf{r}') \frac{\hat{\mathbf{R}}}{R^2} d^3\mathbf{r}' = \frac{1}{4\pi} \iiint \varrho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{(\mathbf{r} - \mathbf{r}')^3} d^3\mathbf{r}', \quad (2.166)$$

where $\varrho(\mathbf{r}')$ is the charge density (electric charge per unit volume) in the neighborhood of \mathbf{r}' . The constant $1/4\pi$ is a matter of units and need not concern us here. The divergence of $\mathbf{E}(\mathbf{r})$ is

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi} \iiint \varrho(\mathbf{r}') \left(\nabla \cdot \frac{\hat{\mathbf{R}}}{R^2} \right) d^3\mathbf{r}',$$

since ∇ operates only on \mathbf{r} . But,

$$\nabla \cdot \frac{\hat{\mathbf{R}}}{R^2} = 4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

as shown in (2.153). Thus,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi} \iiint \varrho(\mathbf{r}') 4\pi\delta^3(\mathbf{r} - \mathbf{r}') d^3\mathbf{r}' = \varrho(\mathbf{r}). \quad (2.167)$$

The fact that we can relate the divergence of \mathbf{E} at \mathbf{r} to the charge density at same point \mathbf{r} is remarkable. Coulomb's law of (2.166) is the experimental result, which says that the electric field \mathbf{E} at \mathbf{r} is due to all other charges at different places \mathbf{r}' . Yet through vector analysis, we find $\nabla \cdot \mathbf{E}$ at \mathbf{r} is equal to the charge density $\varrho(\mathbf{r})$ at the same place where \mathbf{E} is to be evaluated. This type of equation is called *field equation* which describes the property of the field at each point in space.

Since curl of $(\hat{\mathbf{R}}/R^2)$ is equal to zero, \mathbf{E} can be expressed as the gradient of scalar potential $\mathbf{E} = -\nabla\varphi$. Thus,

$$\nabla \cdot \mathbf{E} = -\nabla \cdot \nabla\varphi = \varrho.$$

Therefore,

$$\nabla^2\varphi = -\varrho. \quad (2.168)$$

This result is known as *Poisson's equation* which specifies the relationship between the source density and the scalar potential for an irrotational field.

In that part of the space where there is no charge ($\varrho = 0$), the equation reduces to

$$\nabla^2\varphi = 0, \quad (2.169)$$

which is known as *Laplace's equation*.

The equations of Poisson and Laplace are two of the most important equations in mathematical physics. They are encountered repeatedly in a variety of problems.

2.10.5 Uniqueness Theorem

In the following chapters, we shall describe various methods of solving Laplace's equation. It does not matter which method we use, as long as we can find a scalar function φ that satisfies the equation and the boundary conditions, the vector field derived from it is uniquely determined. This is known as *uniqueness theorem*.

Let the region of interests be surrounded by surface S , (if the boundary consists of many surfaces including the surface at infinity, then S represents

all of them). There are two kinds of boundary conditions (1) the values of φ are specified on S , known as Dirichlet boundary condition and (2) the normal derivatives $\partial\varphi/\partial n$ over S are specified, known as Neumann boundary condition. The theorem says:

Two solutions φ_1 and φ_2 of the Laplace equation which satisfy the first kind of boundary conditions must be identical. Two solutions φ_1 and φ_2 of the Laplace equation which satisfy the second kind of boundary conditions can differ at most by an additive constant.

To prove this theorem, we define a new function $\Phi = \varphi_1 - \varphi_2$. Obviously, $\nabla^2\Phi = \nabla^2\varphi_1 - \nabla^2\varphi_2 = 0$. Furthermore, either Φ or $\partial\Phi/\partial n = \nabla\Phi \cdot \mathbf{n}$ vanishes on S . Applying the divergence theorem to $\Phi\nabla\Phi$, we have

$$\iiint \nabla \cdot (\Phi\nabla\Phi) dV = \iint_S \Phi\nabla\Phi \cdot \mathbf{n} da = 0,$$

since the integral on the right-hand side vanishes. But

$$\nabla \cdot (\Phi\nabla\Phi) = \nabla\Phi \cdot \nabla\Phi + \Phi\nabla^2\Phi$$

and $\nabla^2\Phi = 0$ at all points, so the divergence theorem in this case becomes

$$\iiint \nabla\Phi \cdot \nabla\Phi dV = 0.$$

Now $\nabla\Phi \cdot \nabla\Phi = (\nabla\Phi)^2$ must be positive or zero, and since the integral is zero, it follows that the only possibility is $\nabla\Phi = 0$ everywhere inside the volume. A function whose gradient is zero at all points cannot change, hence Φ has the same value that it has on the boundary S . For the first kind of boundary condition, $\Phi = 0$ on S , and Φ must equal to zero at every point in the region. Therefore $\varphi_1 = \varphi_2$. For the second kind of boundary conditions, $\nabla\Phi$ equal to zero at all points in the region and $\nabla\Phi \cdot \mathbf{n} = 0$ on S , the only possible solution is Φ equal to a constant. Thus φ_1 and φ_2 can differ at most by a constant. In either case, the vector field $\nabla\varphi$ is uniquely defined.

Exercises

1. Find $\frac{d\mathbf{r}}{dt}$, $\frac{d^2\mathbf{r}}{dt^2}$, $\left|\frac{d\mathbf{r}}{dt}\right|$, $\left|\frac{d^2\mathbf{r}}{dt^2}\right|$, if $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k}$.

Ans. $\cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}$, $-\sin t\mathbf{i} - \cos t\mathbf{j}$, $\sqrt{2}$, 1.

2. Show that $\mathbf{A} \cdot \frac{d\mathbf{A}}{dt} = A \frac{dA}{dt}$.

3. A particle moves along the curve $\mathbf{r}(t) = 2t^2\mathbf{i} + (t^2 - 4t)\mathbf{j} + (3t - 5)\mathbf{k}$, where t is time. Find its velocity and acceleration at $t = 1$.
Ans. $4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $4\mathbf{i} + 2\mathbf{j}$.

4. A particle moves along the curve $\mathbf{r}(t) = (t^3 - 4t)\mathbf{i} + (t^2 + 4t)\mathbf{j} + (8t^2 - 3t^3)\mathbf{k}$, where t is time. Find the magnitudes of the tangential and normal components of its acceleration at $t = 2$.
Ans. 16, $2\sqrt{73}$.

5. A velocity field is given by $\mathbf{v} = x^2\mathbf{i} - 2xy\mathbf{j} + 4t\mathbf{k}$. Determine the acceleration at the point $(2, 1, -4)$.
Ans. $16\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$.

Hint: $\mathbf{a} = \frac{\partial \mathbf{v}}{\partial x} \frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y} \frac{dy}{dt} + \frac{\partial \mathbf{v}}{\partial z} \frac{dz}{dt} + \frac{\partial \mathbf{v}}{\partial t}$

6. A wheel of radius b rolls along the ground with a constant forward speed v_0 . Find the acceleration of any point on the rim of the wheel.
Ans. v_0^2/b toward the center of the wheel.

Hint: Let the moving origin be at the center of the wheel with x' axis passing through the point in question, thus $\mathbf{r}' = b\mathbf{i}$, $\mathbf{v}' = 0$, $\mathbf{a}' = 0$. The angular velocity vector is $\boldsymbol{\omega} = (v_0/b)\mathbf{k}'$. Then use (2.44)

7. Find the arc length of $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j} + bt\mathbf{k}$ from $t = 0$ to $t = 2\pi$.
Ans. $s = 2\pi\sqrt{a^2 + b^2}$.

Hint: $ds = v dt = (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{1/2} dt$

8. Find the arc length of $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$ from $t = 0$ to $t = \pi$.
Ans. $s = \pi^2/2$.

9. Given the space curve $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$, find (a) the curvature κ and (b) the torsion γ .
Ans. $\frac{2}{(1+2t^2)^2}$, $\frac{2}{(1+2t^2)^2}$.

10. Show that $\ddot{\mathbf{r}} = \dot{v}\mathbf{t} + v^2\kappa\mathbf{n}$.

11. Show that the curvature κ of a space curve $\mathbf{r} = \mathbf{r}(t)$ is given by

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3},$$

where dots denote differentiation with respect to time t .

Hint: first show that $\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = v\mathbf{t} \times (\dot{v}\mathbf{t} + v^2\kappa\mathbf{n})$

12. Show that the torsion γ of a space curve is given numerically by

$$\gamma = \left| \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \times \ddot{\mathbf{r}} \right| / \left| \dot{\mathbf{r}} \times \ddot{\mathbf{r}} \right|^2.$$

Hint: first show that $\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} \times \ddot{\mathbf{r}} = -v^6 \kappa^2 \gamma$, then use the result of the previous problem

13. Find the gradient of the scalar field $\phi = xyz$, and evaluate it at the point $(1, 2, 3)$, find the derivative of ϕ in the direction of $\mathbf{i} + \mathbf{j}$.

Ans. $yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$, $6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $9/\sqrt{2}$.

14. Find the unit normal to each of the following surfaces at the point indicated: (a) $x^2 + y^2 - z = 0$ at $(1, 1, 2)$, (b) $x^2 + y^2 = 5$ at $(2, 1, 0)$, and (c) $y = x^2 + z^3$ at $(1, 2, 1)$.

Ans. $(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})/3$, $(3\mathbf{i} + 4\mathbf{j})/5$, $(-\mathbf{i} - \mathbf{j} - 3\mathbf{k})/\sqrt{11}$.

15. The temperature T is given by $T = x^2 + xy + yz$. What is the unit vector that points in the direction of maximum change of temperature at $(2, 1, 4)$? What is the value of the derivative of the temperature in the x direction at that point?

Ans. $(5\mathbf{i} + 6\mathbf{j} + \mathbf{k})/\sqrt{62}$, 5.

16. Determine the equation of the plane tangent to the given surface at the point indicated: (a) $x^2 + y^2 + z^2 = 25$ $(3, 4, 0)$, and (b) $x^2 - 2xy = 0$ $(2, 2, 1)$.

Ans. $3x + 4y = 25$, $y = 2$.

17. Find the divergence of each of the following vector fields at the point $(2, 1, -1)$. (a) $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$, (b) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + y\mathbf{k}$, and (c) $\mathbf{F} = \mathbf{r}/r = (x\mathbf{i} + y\mathbf{j} + y\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$.

Ans. 3, 3, $\sqrt{6}/3$.

18. Verify the divergence theorem by calculating both the volume integral and the surface integral for the vector field $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + (z - x)\mathbf{k}$ and the volume of the unit cube $0 \leq x, y, z \leq 1$.

19. By using the divergence theorem, evaluate

$$(a) \iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n} \, da,$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 9$;

$$(b) \iint_S (x\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n} \, da,$$

where S is the surface of the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 8$;

$$(c) \iint_S (x \sin y \mathbf{i} + \cos^2 x \mathbf{j} - z \sin y \mathbf{k}) \cdot \mathbf{n} \, da,$$

where S is the surface of the sphere $x^2 + y^2 + (z - 2)^2 = 1$.
Ans. $108\pi, 288\pi, 0$.

20. Show that

$$\iint_S \mathbf{r} \cdot \mathbf{n} \, da = 3V,$$

where V is the volume bounded by the closed surface S .

21. Recognizing that $\mathbf{i} \cdot \mathbf{n} \, da = dy \, dz$; $\mathbf{j} \cdot \mathbf{n} \, da = dx \, dz$; $\mathbf{k} \cdot \mathbf{n} \, da = dx \, dy$ (see Example 2.5.2), evaluating the following integral using the divergence theorem

$$\iint_S (x \, dy \, dz + y \, dx \, dz + z \, dx \, dy),$$

where S is the surface of the cylinder $x^2 + y^2 = 9$, $0 \leq z \leq 3$.
Ans. 81π .

Hint: first show that $(x \, dy \, dz + y \, dx \, dz + z \, dx \, dy) = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n} \, da$ (see Example 2.5.2)

22. Evaluating the following integral using the divergence theorem

$$\iint_S (x \, dy \, dz + 2y \, dx \, dz + y^2 \, dx \, dy),$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$.
Ans. 32π .

23. Use the divergence theorem to evaluate the surface integral

$$\iint_S [(x + y)\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}] \cdot \mathbf{n} \, da,$$

where S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1$ with $z > 0$ and \mathbf{n} is the outward unit normal. Note that the surface is not closed.
Ans. $\frac{11}{12}\pi$.

Hint: the integral is equal to the closed surface integral over the hemisphere subtract the integral over the base.

24. Find the curl of each of the following vector fields at the point $(-2, 4, 1)$.

(a) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and (b) $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$.

Ans. $0, -\mathbf{j} + 2\mathbf{k}$.

25. Verify Stokes' theorem by evaluating both the line and surface integral for the vector field $\mathbf{A} = (2x - y)\mathbf{i} - y^2\mathbf{j} + y^2z\mathbf{k}$ and the surface S given by the disc $z = 0, x^2 + y^2 \leq 1$.
26. Ampere's law states that the total flux of electric current flowing through a loop is proportional to the line integral of the magnetic field around the loop, that is $\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \iint_S \mathbf{J} \cdot \mathbf{n} \, da$ where \mathbf{B} is the magnetic field, \mathbf{J} is the current density and μ_0 is a proportional constant. If this is true for any loop C , show that $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$.
27. Show that $\oint_C \mathbf{r} \cdot d\mathbf{r} = 0$ for any closed curve C .
28. Calculate the circulation of the vector $\mathbf{F} = y^2\mathbf{i} + xy\mathbf{j} + z^2\mathbf{k}$ ($\oint \mathbf{F} \cdot d\mathbf{r}$) around a triangle with vertices at the origin, $(2, 2, 0)$, and $(0, 2, 0)$ by (a) direct integration, and (b) using Stokes' theorem.
Ans. $8/3$.
29. Calculate the circulation of $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ around a unit circle in the xy plane with center at the origin by (a) direct integration and (b) using Stokes' theorem.
Ans. -2π .
30. Evaluate the circulation of the following vector fields around the curves specified. Use either direct integration or Stokes' theorem. (a) $\mathbf{F} = 2z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ around a triangle with vertices at the origin, $(1, 0, 0)$ and $(0, 0, 4)$. (b) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ around a unit circle in the xy plane with center at the origin.
Ans. $2, 0$.

31. Check the product rule

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}$$

by calculating each term separately for the functions $\mathbf{A} = y^2\mathbf{i} + 2xy\mathbf{j} + z^2\mathbf{k}$, $\mathbf{B} = \sin y\mathbf{i} + \sin x\mathbf{j} + z^3\mathbf{k}$.

32. Check the relation

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

by calculating each term separately for the function $\mathbf{A} = y^2\mathbf{i} + 2xy\mathbf{j} + z^2\mathbf{k}$.

33. Show that $\nabla \times (\varphi \nabla \varphi) = \mathbf{0}$.
34. Show that

$$\iiint_V (\nabla \times \mathbf{A}) \cdot \mathbf{B} \, dV = \iiint_V (\nabla \times \mathbf{B}) \cdot \mathbf{A} \, dV + \iint_S (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n} \, da,$$

where S is the surface bounding the volume V .

35. Show that for any closed surface S

$$\iint_S (\nabla \times \mathbf{B}) \cdot \mathbf{n} \, da = 0.$$

36. For what values, if any, of the constants a and b is the following vector field irrotational?

$$\mathbf{F} = (y \cos x + axz)\mathbf{i} + (b \sin x + z)\mathbf{j} + (x^2 + y)\mathbf{k}.$$

Ans. $a = 2$, $b = 1$.

37. (a) Show that $\mathbf{F} = (2xy + 3)\mathbf{i} + (x^2 - 4z)\mathbf{j} - 4y\mathbf{k}$ is a conservative field.
 (b) Find a scalar potential φ such that $\nabla\varphi = -\mathbf{F}$. (c) Evaluate the integral

$$\int_{3,-1,2}^{2,1,-1} \mathbf{F} \cdot d\mathbf{r}.$$

Ans. $\nabla \times \mathbf{F} = \mathbf{0}$, $\varphi = -x^2y - 3x + 4yz$, 6.

38. (a) Show that $\mathbf{F} = y^2z\mathbf{i} - (z^2 \sin y - 2xyz)\mathbf{j} + (2z \cos y + y^2x)\mathbf{k}$ is irrotational.

(b) find a function φ such that $\nabla\varphi = \mathbf{F}$.

(c) Evaluate the integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ where Γ is along the curve $x = \sin(\pi t/2)$, $y = t^2 - t$, $z = t^4$, $0 \leq t \leq 1$.

Ans. $\nabla \times \mathbf{F} = \mathbf{0}$, $\varphi = z^2 \cos y + xy^2z$, 1.

39. If \mathbf{A} is irrotational, show that $\mathbf{A} \times \mathbf{r}$ is solenoidal.

40. Vector \mathbf{B} is formed by the product of two gradients

$$\mathbf{B} = (\nabla u) \times (\nabla v),$$

where u and v are scalar functions. (a) Show that \mathbf{B} is solenoidal.

(b) Show that

$$\mathbf{A} = \frac{1}{2} (\mathbf{u} \nabla v - \mathbf{v} \nabla u)$$

is a vector potential for \mathbf{B} in that $\mathbf{B} = \nabla \times \mathbf{A}$.

41. Show that if $\nabla^2\varphi = 0$ in the volume V , then

$$\iint_S \nabla\varphi \cdot \mathbf{n} \, da = 0,$$

where S is the surface bounding the volume.

42. Two fields f and g are related by Poisson's equation, $\nabla^2 f = g$. Show that

$$\iiint_V g \, dV = \iint_S \nabla f \cdot \mathbf{n} \, da,$$

where S is the bounding surface of V .

43. Use Stokes' theorem to show that

$$\oint_C f \nabla g \cdot d\mathbf{r} = - \oint_C g \nabla f \cdot d\mathbf{r}$$

for any closed curve C and differentiable fields f and g .

Hint: first show $\oint_C f \nabla g \cdot d\mathbf{r} = \iint_S \nabla f \times \nabla g \cdot \mathbf{n} \, da$