3

Curved Coordinates

Up to now we have used only Cartesian (rectangular) coordinates with their constant unit vectors. Frequently, because of the geometry of the problems, other coordinate systems are much more convenient. There are many coordinate systems, each of them can be regarded as a particular case of the general curvilinear coordinate system. It would be most efficient if we first develop a theory of curvilinear coordinates and then introduce each coordinate system as a special example. However, for pedagogic reasons, we will do that after we first directly transform the vector expressions of the rectangular coordinates into the corresponding ones in the two most commonly used systems, namely cylindrical and spherical coordinates. This procedure has the advantage of emphasizing that the physical meaning of gradient, divergence, curl, and Laplacian operations remain the same in different coordinate systems. Their appearances are different only because they are expressed with different notations. Furthermore, expressions in cylindrical and spherical coordinates will serve as familiar examples to clarify the terms in the general curvilinear system. As a further example, the elliptical coordinate system is discussed in some detail because of its importance in dealing with two center problems. Within the framework of curvilinear coordinates, we introduce the Jacobian determinant for multiple integrals in Sect. 3.5.

3.1 Cylindrical Coordinates

The cylindrical coordinate system is formally known as the circular cylindrical or cylindrical polar coordinate system. In this system, the position of a point is specified by (ρ, φ, z) as shown in Fig. 3.1a: ρ is the perpendicular distance from the z-axis, φ is the angle between the x-axis and the projection of ρ on the xy-plane, and z is the same as in the rectangular coordinates. The three unit vectors, $\mathbf{e}_\rho$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$, point in the direction of increase of the corresponding coordinates. The relation to Cartesian coordinates can be easily seen from Fig. 3.1b where we have moved $\mathbf{e}_\rho, \mathbf{e}_\varphi$ to the origin:

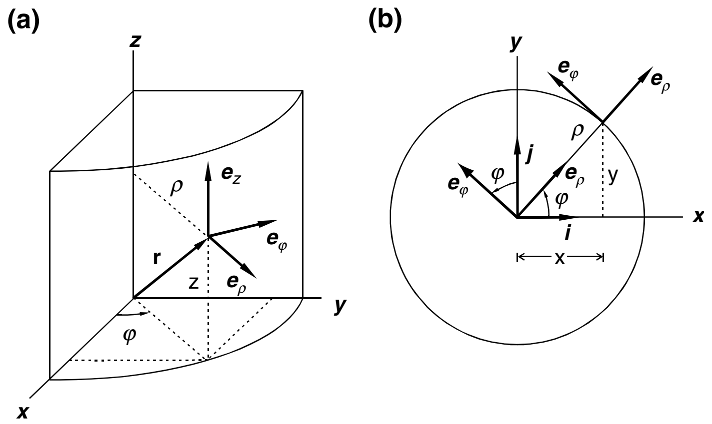

Fig. 3.1. Cylindrical coordinates. (**a**) A point is specified by (ρ, φ, z), the unit vectors $\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{e}_z$ are pointing in the direction of increase of the corresponding coordinates, (**b**) $\mathbf{e}_\rho, \mathbf{e}_\varphi$ are moved to the origin to find the relationships with $\mathbf{i}, \mathbf{j}$ of the rectangular coordinate system

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad \rho = (x^2+y^2)^{1/2}, \quad \varphi = \tan^{-1}\frac{y}{x}. \tag{3.1}$$

$$\mathbf{e}_\rho = \cos\varphi\mathbf{i} + \sin\varphi\mathbf{j} = \frac{x}{(x^2+y^2)^{1/2}}\mathbf{i} + \frac{y}{(x^2+y^2)^{1/2}}\mathbf{j}, \tag{3.2}$$

$$\mathbf{e}_\varphi = -\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j} = \frac{-y}{(x^2+y^2)^{1/2}}\mathbf{i} + \frac{x}{(x^2+y^2)^{1/2}}\mathbf{j}. \tag{3.3}$$

$$\mathbf{i} = \cos\varphi\mathbf{e}_\rho - \sin\varphi\mathbf{e}_\varphi, \tag{3.4}$$

$$\mathbf{j} = \sin\varphi\mathbf{e}_\rho + \cos\varphi\mathbf{e}_\varphi. \tag{3.5}$$

It follows

$$\frac{\partial x}{\partial\rho} = \cos\varphi, \quad \frac{\partial x}{\partial\varphi} = -\rho\sin\varphi, \quad \frac{\partial y}{\partial\rho} = \sin\varphi, \quad \frac{\partial y}{\partial\varphi} = \rho\cos\varphi, \tag{3.6}$$

$$\frac{\partial\rho}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2)^{1/2} = \frac{x}{(x^2+y^2)^{1/2}} = \frac{\rho\cos\varphi}{\rho} = \cos\varphi, \tag{3.7}$$

$$\frac{\partial\rho}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2)^{1/2} = \frac{y}{(x^2+y^2)^{1/2}} = \frac{\rho\sin\varphi}{\rho} = \sin\varphi, \tag{3.8}$$

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right) = -\frac{y}{x^2+y^2} = -\frac{\sin\varphi}{\rho}, \tag{3.9}$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right) = \frac{x}{x^2+y^2} = \frac{\cos\varphi}{\rho}. \tag{3.10}$$

The relationships between $\mathbf{e}_\rho, \mathbf{e}_\varphi, \mathbf{e}_z$ can be easily worked out, for example,

$$\begin{aligned}
\mathbf{e}_\rho \cdot \mathbf{e}_\rho &= (\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}) \cdot (\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}) = \cos^2\varphi + \sin^2\varphi = 1, \\
\mathbf{e}_\rho \cdot \mathbf{e}_\varphi &= (\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}) \cdot (-\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j}) = 0, \\
\mathbf{e}_\rho \times \mathbf{e}_\varphi &= (\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}) \times (-\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j}) = \cos^2\varphi\mathbf{k} + \sin^2\varphi\mathbf{k} = \mathbf{k} = \mathbf{e}_z.
\end{aligned}$$

Taken together, they form an orthonormal basis set

$$\begin{aligned}
\mathbf{e}_\rho \cdot \mathbf{e}_\rho &= \mathbf{e}_\varphi \cdot \mathbf{e}_\varphi = \mathbf{e}_z \cdot \mathbf{e}_z = 1, \\
\mathbf{e}_\rho \cdot \mathbf{e}_\varphi &= \mathbf{e}_\varphi \cdot \mathbf{e}_z = \mathbf{e}_z \cdot \mathbf{e}_\rho = 0, \\
\mathbf{e}_\rho \times \mathbf{e}_\varphi &= \mathbf{e}_z,\ \mathbf{e}_\varphi \times \mathbf{e}_z = \mathbf{e}_\rho,\ \mathbf{e}_z \times \mathbf{e}_\rho = \mathbf{e}_\varphi.
\end{aligned} \tag{3.11}$$

The position vector $\mathbf{r}$, from the origin to any point in space, is clearly seen in Fig. 3.1 to be

$$\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z. \tag{3.12}$$

This expression can also be obtained from directly transforming $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ into the cylindrical coordinates.

Any vector can be expressed in terms of them. If the vector is a function of the position, then

$$\mathbf{A}(\rho,\varphi,z) = A_\rho(\rho,\varphi,z)\,\mathbf{e}_\rho + A_\varphi(\rho,\varphi,z)\,\mathbf{e}_\varphi + A_z(\rho,\varphi,z)\,\mathbf{e}_z. \tag{3.13}$$

In general, each component is a function of ρ, φ, z. Unlike the constant unit vector $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the rectangular coordinate system, only $\mathbf{e}_z = \mathbf{k}$ is fixed in space, the directions of $\mathbf{e}_\rho, \mathbf{e}_\varphi$ change as the point is moved around. Note that both $\mathbf{e}_\rho$ and $\mathbf{e}_\varphi$ depend on φ. In particular,

$$\begin{aligned}
\frac{\partial}{\partial\varphi}\mathbf{e}_\rho &= \frac{\partial}{\partial\varphi}(\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}) = -\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j} = \mathbf{e}_\varphi, \\
\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi &= \frac{\partial}{\partial\varphi}(-\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j}) = -(\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j}) = -\,\mathbf{e}_\rho, \\
\frac{\partial}{\partial\rho}\mathbf{e}_\rho &= \frac{\partial}{\partial\rho}\mathbf{e}_\varphi = 0.
\end{aligned} \tag{3.14}$$

Example 3.1.1. Show that the acceleration of a particle expressed in cylindrical coordinates is given by

$$\mathbf{a} = \left(\ddot{\rho} - \rho\dot{\varphi}^2\right)\mathbf{e}_\rho + \left(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi}\right)\mathbf{e}_\varphi + \ddot{z}\mathbf{e}_z,$$

where dots denote differentiation with respect to time t.

Solution 3.1.1. Since the position vector is given by $\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$, the velocity is $\mathbf{v} = \dot{\mathbf{r}}$,

$$\dot{\mathbf{r}}=\dot{\rho}\mathbf{e}_\rho+\rho\dot{\mathbf{e}}_\rho+\dot{z}\mathbf{e}_z,$$

where $\mathbf{e}_z$ is a constant unit vector and $\mathbf{e}_\rho$ depends on φ. Since by (3.14)

$$\dot{\mathbf{e}}_\rho=\frac{\mathrm{d}\mathbf{e}_\rho}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{e}_\rho}{\mathrm{d}\varphi}=\dot{\varphi}\mathbf{e}_\varphi,$$

$$\dot{\mathbf{r}}=\dot{\rho}\mathbf{e}_\rho+\rho\dot{\varphi}\mathbf{e}_\varphi+\dot{z}\mathbf{e}_z.$$

The acceleration is the rate of change of velocity, therefore $\mathbf{a}=\dot{\mathbf{v}}=\ddot{\mathbf{r}}$,

$$\ddot{\mathbf{r}}=\ddot{\rho}\mathbf{e}_\rho+\dot{\rho}\dot{\mathbf{e}}_\rho+\dot{\rho}\dot{\varphi}\mathbf{e}_\varphi+\rho\ddot{\varphi}\mathbf{e}_\varphi+\rho\dot{\varphi}\dot{\mathbf{e}}_\varphi+\ddot{z}\mathbf{e}_z.$$

Again by (3.14),

$$\dot{\mathbf{e}}_\varphi=\frac{\mathrm{d}\mathbf{e}_\varphi}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{e}_\varphi}{\mathrm{d}\varphi}=\dot{\varphi}(-\mathbf{e}_\rho),$$

$$\ddot{\mathbf{r}}=\ddot{\rho}\mathbf{e}_\rho+\dot{\rho}\dot{\varphi}\mathbf{e}_\varphi+\dot{\rho}\dot{\varphi}\mathbf{e}_\varphi+\rho\ddot{\varphi}\mathbf{e}_\varphi-\rho\dot{\varphi}^2\mathbf{e}_\rho+\ddot{z}\mathbf{e}_z.$$

Therefore

$$\mathbf{a}=\ddot{\mathbf{r}}=\left(\ddot{\rho}-\rho\dot{\varphi}^2\right)\mathbf{e}_\rho+(2\dot{\rho}\dot{\varphi}+\rho\ddot{\varphi})\mathbf{e}_\varphi+\ddot{z}\mathbf{e}_z.$$

3.1.1 Differential Operations

Gradient. Starting from the definition of gradient in the Cartesian coordinates, we can use the coordinate transformation to express it in terms of (ρ,φ,z). Using (3.4) and (3.5),

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\Phi}&=\mathbf{i}\frac{\partial\Phi}{\partial x}+\mathbf{j}\frac{\partial\Phi}{\partial y}+\mathbf{k}\frac{\partial\Phi}{\partial z}\\&=(\cos\varphi\mathbf{e}_\rho-\sin\varphi\mathbf{e}_\varphi)\frac{\partial\Phi}{\partial x}+(\sin\varphi\mathbf{e}_\rho+\cos\varphi\mathbf{e}_\varphi)\frac{\partial\Phi}{\partial y}+\mathbf{e}_z\frac{\partial\Phi}{\partial z}\\&=\left(\cos\varphi\frac{\partial\Phi}{\partial x}+\sin\varphi\frac{\partial\Phi}{\partial y}\right)\mathbf{e}_\rho+\left(-\sin\varphi\frac{\partial\Phi}{\partial x}+\cos\varphi\frac{\partial\Phi}{\partial y}\right)\mathbf{e}_\varphi+\frac{\partial\Phi}{\partial z}\mathbf{e}_z.\end{aligned}\quad(3.15)$$

By chain rule and (3.6)

$$\frac{\partial\Phi}{\partial\rho}=\frac{\partial x}{\partial\rho}\frac{\partial\Phi}{\partial x}+\frac{\partial y}{\partial\rho}\frac{\partial\Phi}{\partial y}=\cos\varphi\frac{\partial\Phi}{\partial x}+\sin\varphi\frac{\partial\Phi}{\partial y},\quad(3.16)$$

$$\frac{\partial\Phi}{\partial\varphi}=\frac{\partial x}{\partial\varphi}\frac{\partial\Phi}{\partial x}+\frac{\partial y}{\partial\varphi}\frac{\partial\Phi}{\partial y}=-\rho\sin\varphi\frac{\partial\Phi}{\partial x}+\rho\cos\varphi\frac{\partial\Phi}{\partial y}.\quad(3.17)$$

With these expressions, (3.15) becomes

$$\boldsymbol{\nabla}\Phi=\frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho+\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\mathbf{e}_\varphi+\frac{\partial\Phi}{\partial z}\mathbf{e}_z.\quad(3.18)$$

Thus, the gradient operator in the cylindrical coordinates can be written as

$$\boldsymbol{\nabla} = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z}. \tag{3.19}$$

An immediate consequence is

$$\boldsymbol{\nabla} \rho = \mathbf{e}_\rho, \quad \boldsymbol{\nabla} \varphi = \frac{1}{\rho} \mathbf{e}_\varphi, \quad \boldsymbol{\nabla} z = \mathbf{e}_z. \tag{3.20}$$

This is not a surprising result. After all, $\boldsymbol{\nabla} u$ is a vector perpendicular to the surface $u =$ constant.

Divergence. The divergence of a vector

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \boldsymbol{\nabla} \cdot (V_\rho \mathbf{e}_\rho + V_\varphi \mathbf{e}_\varphi + V_z \mathbf{e}_z)$$

can be expanded first by the distributive law of dot product. Now,

$$\begin{aligned} \boldsymbol{\nabla} \cdot V_\rho \mathbf{e}_\rho &= \boldsymbol{\nabla} \cdot V_\rho (\mathbf{e}_\varphi \times \mathbf{e}_z) = \boldsymbol{\nabla} \cdot V_\rho (\rho \boldsymbol{\nabla} \varphi \times \boldsymbol{\nabla} z) \\ &= \boldsymbol{\nabla} (\rho V_\rho) \cdot (\boldsymbol{\nabla} \varphi \times \boldsymbol{\nabla} z) + \rho V_\rho \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \varphi \times \boldsymbol{\nabla} z). \end{aligned}$$

But $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \varphi \times \boldsymbol{\nabla} z) = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \varphi \cdot \boldsymbol{\nabla} z - \boldsymbol{\nabla} \times \boldsymbol{\nabla} z \cdot \boldsymbol{\nabla} \varphi = 0$, so

$$\begin{aligned} \boldsymbol{\nabla} \cdot V_\rho \mathbf{e}_\rho &= \boldsymbol{\nabla} (\rho V_\rho) \cdot (\boldsymbol{\nabla} \varphi \times \boldsymbol{\nabla} z) = \boldsymbol{\nabla} (\rho V_\rho) \cdot \left(\frac{1}{\rho} \mathbf{e}_\varphi \times \mathbf{e}_z \right) = \frac{1}{\rho} \boldsymbol{\nabla} (\rho V_\rho) \cdot \mathbf{e}_\rho \\ &= \frac{1}{\rho} \left(\mathbf{e}_\rho \frac{\partial \rho V_\rho}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho} \frac{\partial \rho V_\rho}{\partial \varphi} + \mathbf{e}_z \frac{\partial \rho V_\rho}{\partial z} \right) \cdot \mathbf{e}_\rho = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho V_\rho). \end{aligned} \tag{3.21}$$

$$\begin{aligned} \boldsymbol{\nabla} \cdot V_\varphi \mathbf{e}_\varphi &= \boldsymbol{\nabla} \cdot V_\varphi (\mathbf{e}_z \times \mathbf{e}_\rho) = \boldsymbol{\nabla} \cdot V_\varphi (\boldsymbol{\nabla} z \times \boldsymbol{\nabla} \rho) \\ &= \boldsymbol{\nabla} V_\varphi \cdot (\boldsymbol{\nabla} z \times \boldsymbol{\nabla} \rho) + V_\varphi \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} z \times \boldsymbol{\nabla} \rho) \\ &= \boldsymbol{\nabla} V_\varphi \cdot (\boldsymbol{\nabla} z \times \boldsymbol{\nabla} \rho) = \boldsymbol{\nabla} V_\varphi \cdot (\mathbf{e}_z \times \mathbf{e}_\rho) = \boldsymbol{\nabla} V_\varphi \cdot \mathbf{e}_\varphi \\ &= \left(\mathbf{e}_\rho \frac{\partial V_\varphi}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho} \frac{\partial V_\varphi}{\partial \varphi} + \mathbf{e}_z \frac{\partial V_\varphi}{\partial z} \right) \cdot \mathbf{e}_\varphi = \frac{1}{\rho} \frac{\partial V_\varphi}{\partial \varphi}. \end{aligned} \tag{3.22}$$

Therefore,

$$\boldsymbol{\nabla} \cdot \mathbf{V} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho V_\rho) + \frac{1}{\rho} \frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}. \tag{3.23}$$

Laplacian. By definition the Laplacian of Φ is given by

$$\nabla^2 \Phi = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \Phi = \boldsymbol{\nabla} \cdot \left(\frac{\partial \Phi}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial \Phi}{\partial \varphi} \mathbf{e}_\varphi + \frac{\partial \Phi}{\partial z} \mathbf{e}_z \right). \tag{3.24}$$

Using the expression of the divergence, we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \Phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} \left(\frac{1}{\rho} \frac{\partial \Phi}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \Phi}{\partial z} \right).$$

Therefore

$$\nabla^2 \varPhi = \frac{\partial^2 \varPhi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \varPhi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \varPhi}{\partial \varphi^2} + \frac{\partial^2 \varPhi}{\partial z^2}. \tag{3.25}$$

Since the Laplacian is a scalar operator, it is instructive to convert it directly from its definition in the rectangular coordinates

$$\nabla^2 \varPhi = \frac{\partial^2 \varPhi}{\partial x^2} + \frac{\partial^2 \varPhi}{\partial y^2} + \frac{\partial^2 \varPhi}{\partial z^2}.$$

Now with chain rule and (3.7) and (3.9), we have

$$\frac{\partial \varPhi}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial \varPhi}{\partial \rho} + \frac{\partial \varphi}{\partial x}\frac{\partial \varPhi}{\partial \varphi} = \cos\varphi \frac{\partial \varPhi}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial \varPhi}{\partial \varphi},$$

$$\begin{aligned}
\frac{\partial^2 \varPhi}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial \varPhi}{\partial x}\right] = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho}\left[\frac{\partial \varPhi}{\partial x}\right] + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\left[\frac{\partial \varPhi}{\partial x}\right] \\
&= \cos\varphi \frac{\partial}{\partial \rho}\left[\cos\varphi \frac{\partial \varPhi}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial \varPhi}{\partial \varphi}\right] - \frac{\sin\varphi}{\rho}\frac{\partial}{\partial \varphi}\left[\cos\varphi \frac{\partial \varPhi}{\partial \rho} - \frac{\sin\varphi}{\rho}\frac{\partial \varPhi}{\partial \varphi}\right] \\
&= \cos^2\varphi \frac{\partial^2 \varPhi}{\partial \rho^2} + \frac{\cos\varphi \sin\varphi}{\rho^2}\frac{\partial \varPhi}{\partial \varphi} - \frac{\cos\varphi \sin\varphi}{\rho}\frac{\partial^2 \varPhi}{\partial \rho \partial \varphi} \\
&\quad + \frac{\sin^2\varphi}{\rho}\frac{\partial \varPhi}{\partial \rho} - \frac{\sin\varphi \cos\varphi}{\rho}\frac{\partial^2 \varPhi}{\partial \varphi \partial \rho} + \frac{\sin\varphi \cos\varphi}{\rho^2}\frac{\partial \varPhi}{\partial \varphi} + \frac{\sin^2\varphi}{\rho^2}\frac{\partial^2 \varPhi}{\partial \varphi^2}.
\end{aligned}$$

Similarly,

$$\frac{\partial \varPhi}{\partial y} = \frac{\partial \rho}{\partial y}\frac{\partial \varPhi}{\partial \rho} + \frac{\partial \varphi}{\partial y}\frac{\partial \varPhi}{\partial \varphi} = \sin\varphi \frac{\partial \varPhi}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial \varPhi}{\partial \varphi},$$

$$\begin{aligned}
\frac{\partial^2 \varPhi}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial \varPhi}{\partial y}\right] = \frac{\partial \rho}{\partial y}\frac{\partial}{\partial \rho}\left[\frac{\partial \varPhi}{\partial y}\right] + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi}\left[\frac{\partial \varPhi}{\partial y}\right] \\
&= \sin\varphi \frac{\partial}{\partial \rho}\left[\sin\varphi \frac{\partial \varPhi}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial \varPhi}{\partial \varphi}\right] + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\left[\sin\varphi \frac{\partial \varPhi}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial \varPhi}{\partial \varphi}\right] \\
&= \sin^2\varphi \frac{\partial^2 \varPhi}{\partial \rho^2} - \frac{\sin\varphi \cos\varphi}{\rho^2}\frac{\partial \varPhi}{\partial \varphi} + \frac{\sin\varphi \cos\varphi}{\rho}\frac{\partial^2 \varPhi}{\partial \rho \partial \varphi} \\
&\quad + \frac{\cos^2\varphi}{\rho}\frac{\partial \varPhi}{\partial \rho} + \frac{\cos\varphi \sin\varphi}{\rho}\frac{\partial^2 \varPhi}{\partial \varphi \partial \rho} - \frac{\cos\varphi \sin\varphi}{\rho^2}\frac{\partial \varPhi}{\partial \varphi} + \frac{\cos^2\varphi}{\rho^2}\frac{\partial^2 \varPhi}{\partial \varphi^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial^2 \varPhi}{\partial x^2} + \frac{\partial^2 \varPhi}{\partial y^2} &= (\cos^2\varphi + \sin^2\varphi)\frac{\partial^2 \varPhi}{\partial \rho^2} + \frac{\sin^2\varphi + \cos^2\varphi}{\rho}\frac{\partial \varPhi}{\partial \rho} \\
&\quad + \frac{\sin^2\varphi + \cos^2\varphi}{\rho^2}\frac{\partial^2 \varPhi}{\partial \rho^2} = \frac{\partial^2 \varPhi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \varPhi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \varPhi}{\partial \varphi^2}.
\end{aligned}$$

Clearly the Laplacian obtained this way is identical to (3.25).

Curl. The curl of a vector can be written as

$$\nabla \times \mathbf{V} = \nabla \times (V_\rho \mathbf{e}_\rho + V_\varphi \mathbf{e}_\varphi + V_z \mathbf{e}_z). \tag{3.26}$$

Now

$$\nabla \times V_\rho \mathbf{e}_\rho = \nabla \times V_\rho \nabla \rho = \nabla V_\rho \times \nabla \rho + V_\rho \nabla \times \nabla \rho.$$

Since $\nabla \times \nabla \rho = \mathbf{0}$,

$$\begin{aligned}\nabla \times V_\rho \mathbf{e}_\rho &= \nabla V_\rho \times \nabla \rho = \nabla V_\rho \times \mathbf{e}_\rho \\ &= \left(\mathbf{e}_\rho \frac{\partial V_\rho}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho}\frac{\partial V_\rho}{\partial \varphi} + \mathbf{e}_z \frac{\partial V_\rho}{\partial z}\right) \times \mathbf{e}_\rho \\ &= -\frac{1}{\rho}\frac{\partial V_\rho}{\partial \varphi}\mathbf{e}_z + \frac{\partial V_\rho}{\partial z}\mathbf{e}_\varphi, \end{aligned} \tag{3.27}$$

$$\begin{aligned}\nabla \times V_\varphi \mathbf{e}_\varphi &= \nabla \times V_\varphi (\rho \nabla \boldsymbol{\varphi}) = \nabla(\rho V_\varphi) \times \nabla \varphi + \rho V_\varphi \nabla \times \nabla \varphi \\ &= \nabla(\rho V_\varphi) \times \nabla \varphi = \nabla(\rho V_\varphi) \times \frac{1}{\rho}\mathbf{e}_\varphi \\ &= \frac{1}{\rho}\left(\mathbf{e}_\rho \frac{\partial \rho V_\varphi}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho}\frac{\partial \rho V_\varphi}{\partial \varphi} + \mathbf{e}_z \frac{\partial \rho V_\varphi}{\partial z}\right) \times \mathbf{e}_\varphi \\ &= \frac{1}{\rho}\frac{\partial \rho V_\varphi}{\partial \rho}\mathbf{e}_z - \frac{1}{\rho}\frac{\partial \rho V_\varphi}{\partial z}\mathbf{e}_\rho, \end{aligned} \tag{3.28}$$

$$\begin{aligned}\nabla \times V_z \mathbf{e}_z &= \nabla V_z \times \mathbf{e}_z = \left(\mathbf{e}_\rho \frac{\partial V_z}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} + \mathbf{e}_z \frac{\partial V_z}{\partial z}\right) \times \mathbf{e}_z \\ &= -\frac{\partial V_z}{\partial \rho}\mathbf{e}_\varphi + \frac{1}{\rho}\frac{\partial V_z}{\partial \varphi}\mathbf{e}_\rho. \end{aligned}$$

Thus,

$$\begin{aligned}\nabla \times \mathbf{V} = &\left(\frac{1}{\rho}\frac{\partial V_z}{\partial \varphi} - \frac{1}{\rho}\frac{\partial \rho V_\varphi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho}\right)\mathbf{e}_\varphi \\ &+ \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\varphi) - \frac{1}{\rho}\frac{\partial V_\rho}{\partial \varphi}\right)\mathbf{e}_z. \end{aligned} \tag{3.29}$$

Example 3.1.2. (a) Show that the vector field

$$\mathbf{F} = \left(A - \frac{B}{\rho^2}\right)\cos\varphi \mathbf{e}_\rho - \left(A + \frac{B}{\rho^2}\right)\sin\varphi \mathbf{e}_\varphi$$

is irrotational $(\nabla \times \mathbf{F} = \mathbf{0})$. ($b$) Find a scalar potential $\varPhi$ such that $\nabla \varPhi = \mathbf{F}$. ($c$) Show that $\varPhi$ satisfies the Laplace's equation ($\nabla^2 \varPhi = 0$).

Solution 3.1.2. (a) All derivatives with respect to z are equal to zero, since there is no z dependence. Furthermore, $V_z = 0$. Therefore

$$\begin{aligned}\nabla\times\mathbf{F} &= \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\varphi) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial\varphi}\right)\mathbf{e}_z \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(-A-\frac{B}{\rho^2}\right)\sin\varphi\right]\mathbf{e}_z - \frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\left(A-\frac{B}{\rho^2}\right)\cos\varphi\right]\mathbf{e}_z \\ &= \frac{1}{\rho}\left[\left(-A+\frac{B}{\rho^2}\right)\sin\varphi + \left(A-\frac{B}{\rho^2}\right)\sin\varphi\right]\mathbf{e}_z = 0.\end{aligned}$$

(b)

$$\begin{aligned}\nabla\Phi &= \mathbf{e}_\rho\frac{\partial\Phi}{\partial\rho} + \mathbf{e}_\varphi\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi} + \mathbf{e}_z\frac{\partial\Phi}{\partial z} \\ &= \left(A-\frac{B}{\rho^2}\right)\cos\varphi\mathbf{e}_\rho - \left(A+\frac{B}{\rho^2}\right)\sin\varphi\mathbf{e}_\varphi,\end{aligned}$$

$$\frac{\partial\Phi}{\partial\rho} = \left(A-\frac{B}{\rho^2}\right)\cos\varphi; \quad \frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi} = -\left(A+\frac{B}{\rho^2}\right)\sin\varphi.$$

It is clear, up to an additive constant,

$$\Phi = \left(A\rho + \frac{B}{\rho}\right)\cos\varphi.$$

(c)

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\left(A\rho+\frac{B}{\rho}\right)\cos\varphi\right] + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\left(A\rho+\frac{B}{\rho}\right)\cos\varphi \\ &= \frac{1}{\rho}\left(A+\frac{B}{\rho^2}\right)\cos\varphi - \frac{1}{\rho}\left(A+\frac{B}{\rho^2}\right)\cos\varphi = 0.\end{aligned}$$

3.1.2 Infinitesimal Elements

When a point at (x, y, z) is moved to $(x+\mathrm{d}x, y+\mathrm{d}y, z+\mathrm{d}z)$, the infinitesimal displacement vector is $\mathrm{d}\mathbf{r} = \mathbf{i}\mathrm{d}x + \mathbf{j}\mathrm{d}y + \mathbf{k}\mathrm{d}z$. Similarly, when the point at (ρ, φ, z) in the cylindrical coordinates is moved to $(\rho+\mathrm{d}\rho, \varphi+\mathrm{d}\varphi, z+\mathrm{d}z)$, the infinitesimal displacement vector is

$$\mathrm{d}\mathbf{r} = \mathbf{e}_\rho\mathrm{d}\rho + \mathbf{e}_\varphi\rho\,\mathrm{d}\varphi + \mathbf{e}_z\,\mathrm{d}z. \tag{3.30}$$

Notice the distance in $\mathbf{e}_\varphi$ direction is $\rho\,\mathrm{d}\varphi$ as shown in Fig. 3.2. The infinitesimal length element is

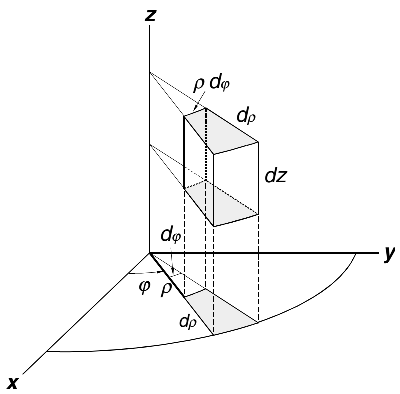

Fig. 3.2. Differential elements in cylindrical coordinates. Note that the differential length in the direction of increasing φ is $\rho\,\mathrm{d}\varphi$. The differential volume element is $\rho\ \mathrm{d}\varphi\ \mathrm{d}\rho\ \mathrm{d}z$.

$$\mathrm{d}s = (\mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r})^{1/2} = \left[(\mathrm{d}\rho)^2 + (\rho\,\mathrm{d}\varphi)^2 + (\mathrm{d}z)^2\right]^{1/2}. \tag{3.31}$$

The gradient is defined as a vector of derivatives with respect to the distances in three perpendicular directions. Thus the gradient in cylindrical coordinates should be

$$\boldsymbol{\nabla} = \mathbf{e}_\rho \frac{\partial}{\partial\rho} + \mathbf{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial\varphi} + \mathbf{e}_z \frac{\partial}{\partial z},$$

which is, of course, identical to (3.19) obtained from direct transformation.

The infinitesimal volume element $\mathrm{d}V$ is the product of the perpendicular infinitesimal displacements

$$\mathrm{d}V = (\mathrm{d}\rho)(\rho\,\mathrm{d}\varphi)(\mathrm{d}z) = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z. \tag{3.32}$$

The possible range of ρ is 0 to ∞, φ goes from 0 to 2π, and z from $-\infty$ to ∞. The infinitesimal surface element depends on the orientation of the surface. For example, on the side surface of a cylinder parallel to z-axis and of constant radius ρ, the surface element directed outward is $\mathbf{n}\ \mathrm{d}a = \rho\ \mathrm{d}\varphi\ \mathrm{d}z\mathbf{e}_\rho$.The surface element on the xy-plane directed upward is $\mathbf{n}\ \mathrm{d}a = \rho\,\mathrm{d}\varphi\ \mathrm{d}\rho\,\mathbf{e}_z$.

Example 3.1.3. Verify the divergence theorem

$$\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\ \mathrm{d}V = \oiint_S \mathbf{F}\cdot\mathbf{n}\ \mathrm{d}a$$

with a vector field

$$\mathbf{F} = \rho\left(2 + \sin^2\varphi\right)\mathbf{e}_\rho + \rho\sin\varphi\cos\varphi\,\mathbf{e}_\varphi + 3z^2\,\mathbf{e}_z$$

over a cylinder with base of radius 2 and height 5.

Solution 3.1.3.

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}\\ &= \frac{1}{\rho}2\rho\left(2+\sin^2\varphi\right) + \frac{1}{\rho}\rho\left(\cos^2\varphi - \sin^2\varphi\right) + 6z\\ &= 4+\sin^2\varphi + \cos^2\varphi + 6z = 5+6z.\end{aligned}$$

$$\begin{aligned}\iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V &= \iiint_V (5+6z)\,\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho\,\mathrm{d}z\\ &= \int_0^{2\pi}\mathrm{d}\varphi\int_0^2\rho\,\mathrm{d}\rho\int_0^5(5+6z)\,\mathrm{d}z = 400\pi.\end{aligned}$$

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a + \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a + \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a,$$

where S_1 is the side surface of the cylinder, S_2 and S_3 are, respective, the bottom and top surfaces of the cylinder.

$$\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a = \iint_{S_1}\mathbf{F}\cdot\mathbf{e}_\rho\,\mathrm{d}a = \int_0^{2\pi}\int_0^5\left[\rho\left(2+\sin^2\varphi\right)\rho\right]_{\rho=2}\mathrm{d}\varphi\,\mathrm{d}z = 100\pi.$$

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a = \iint_{S_2}\mathbf{F}\cdot(-\mathbf{e}_z)\,\mathrm{d}a = \iint_{S_2}\left[-3z^2\right]_{z=0}\mathrm{d}a = 0,$$

$$\iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a = \iint_{S_3}\mathbf{F}\cdot(\mathbf{e}_z)\,\mathrm{d}a = \int_0^{2\pi}\int_0^2\left[3z^2\right]_{z=5}\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho = 300\pi.$$

Therefore,

$$\oiint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a = 100\pi + 300\pi = 400\pi.$$

Clearly,

$$\iiint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \oiint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a.$$

3.2 Spherical Coordinates

The spherical polar coordinate system is commonly known just as the spherical coordinates. The location of a point is specified by (r,θ,φ) as shown in Fig. 3.3, where r is the distance from the origin, θ is the angle made by the position vector $\mathbf{r}$ with the positive z-axis which is often called polar angle, and φ is the angle made with the positive x-axis by the projection of $\mathbf{r}$ on the xy-plane, this angle is known as azimuthal angle. The relations between the rectangular and spherical coordinates are seen from Fig. 3.3b and c.

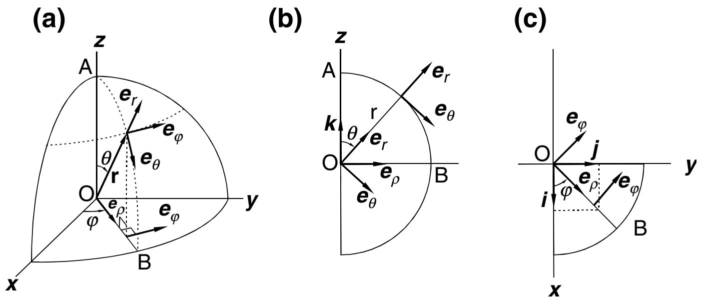

Fig. 3.3. Spherical coordinates. **(a)** A point is specified by (r, θ, φ) where r is the distance from the origin, θ is the angle made by $\mathbf{r}$ with the positive z-axis, and φ is the angle made with the positive x-axis by the projection of $\mathbf{r}$ on the xy-plane. The three unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$ are in the direction of increasing r, θ, φ, respectively. The auxiliary unit vector $\mathbf{e}_\varrho$ is in the direction of the projection of $\mathbf{r}$ on the xy-plane. **(b)** The unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are moved to the origin in the AOB plane to find their relationships with $\mathbf{k}$ and $\mathbf{e}_\varrho$. **(c)** In the xy-plane, $\mathbf{e}_\varphi$ is moved to the origin to find the relationships between $\mathbf{e}_\varphi, \mathbf{e}_\varrho$ and $\mathbf{i}, \mathbf{j}$

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta \tag{3.33}$$

and

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}, \quad \tan\theta = \frac{\left(x^2+y^2\right)^{1/2}}{z}, \quad \tan\varphi = \frac{y}{x}, \tag{3.34}$$

$$\sin\theta = \frac{\left(x^2+y^2\right)^{1/2}}{\left(x^2+y^2+z^2\right)^{1/2}}, \qquad \cos\theta = \frac{z}{\left(x^2+y^2+z^2\right)^{1/2}}, \tag{3.35}$$

$$\sin\varphi = \frac{y}{\left(x^2+y^2\right)^{1/2}}, \qquad \cos\varphi = \frac{x}{\left(x^2+y^2\right)^{1/2}}. \tag{3.36}$$

Figure 3.3 also shows a set of mutually perpendicular unit vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$ in the sense of increasing r, θ, φ, respectively. In this system, the position vector $\mathbf{r}$ is simply

$$\mathbf{r} = r\widehat{\mathbf{r}} = r\mathbf{e}_r. \tag{3.37}$$

The relations between the unit vectors in the spherical coordinates and those in the Cartesian coordinates can be seen from Fig. 3.3b. In the plane AOB, we have drawn $\mathbf{e}_r$ and $\mathbf{e}_\theta$ from the origin. It can be seen

$$\mathbf{e}_r = \sin\theta\mathbf{e}_\varrho + \cos\theta\mathbf{k},$$
$$\mathbf{e}_\theta = \cos\theta\mathbf{e}_\varrho - \sin\theta\mathbf{k},$$

where $\mathbf{e}_\varrho$ is a unit vector along OB. In Fig. 3.3c, $\mathbf{e}_\varrho$ and $\mathbf{e}_\theta$ are drawn from the origin, clearly

$$\mathbf{e}_\varrho = \cos\varphi\mathbf{i} + \sin\varphi\mathbf{j},$$
$$\mathbf{e}_\varphi = -\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j}.$$

Thus,

$$\begin{aligned}\mathbf{e}_r &= \sin\theta\cos\varphi\mathbf{i} + \sin\theta\sin\varphi\mathbf{j} + \cos\theta\mathbf{k},\\ \mathbf{e}_\theta &= \cos\theta\cos\varphi\mathbf{i} + \cos\theta\sin\varphi\mathbf{j} - \sin\theta\mathbf{k},\\ \mathbf{e}_\varphi &= -\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j}.\end{aligned} \tag{3.38}$$

The inverse relations can either be read from the same figures, or be solved for $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ from the above equations,

$$\begin{aligned}\mathbf{i} &= \sin\theta\cos\varphi\mathbf{e}_r + \cos\theta\cos\varphi\mathbf{e}_\theta - \sin\varphi\mathbf{e}_\varphi,\\ \mathbf{j} &= \sin\theta\sin\varphi\mathbf{e}_r + \cos\theta\sin\varphi\mathbf{e}_\theta + \cos\varphi\mathbf{e}_\varphi,\\ \mathbf{k} &= \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta.\end{aligned} \tag{3.39}$$

It can easily be verified that $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ form an orthonormal basis set and satisfy the following relations

$$\begin{aligned}&\mathbf{e}_r\cdot\mathbf{e}_r = \mathbf{e}_\theta\cdot\mathbf{e}_\theta = \mathbf{e}_\varphi\cdot\mathbf{e}_\varphi = 1,\\ &\mathbf{e}_r\cdot\mathbf{e}_\theta = \mathbf{e}_\theta\cdot\mathbf{e}_\varphi = \mathbf{e}_\varphi\cdot\mathbf{e}_r = 0,\\ &\mathbf{e}_r\times\mathbf{e}_\theta = \mathbf{e}_\varphi, \quad \mathbf{e}_\theta\times\mathbf{e}_\varphi = \mathbf{e}_r, \quad \mathbf{e}_\varphi\times\mathbf{e}_r = \mathbf{e}_\theta.\end{aligned} \tag{3.40}$$

Any vector can be expressed in terms of them

$$\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\varphi\mathbf{e}_\varphi, \tag{3.41}$$

where A_r, A_θ, A_φ are the radial, polar, and azimuthal components of $\mathbf{A}$. The derivatives of the unit vectors are easily obtained from (3.38):

$$\frac{\partial\mathbf{e}_r}{\partial r} = \frac{\partial\mathbf{e}_\theta}{\partial r} = \frac{\partial\mathbf{e}_\varphi}{\partial r} = \frac{\partial\mathbf{e}_\varphi}{\partial\theta} = 0, \tag{3.42}$$

$$\frac{\partial\mathbf{e}_r}{\partial\theta} = \cos\theta\cos\varphi\mathbf{i} + \cos\theta\sin\varphi\mathbf{j} - \sin\theta\mathbf{k} = \mathbf{e}_\theta, \tag{3.43}$$

$$\frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\sin\theta\cos\varphi\mathbf{i} - \sin\theta\sin\varphi\mathbf{j} - \cos\theta\mathbf{k} = -\mathbf{e}_r, \tag{3.44}$$

$$\frac{\partial\mathbf{e}_r}{\partial\varphi} = -\sin\theta\sin\varphi\mathbf{i} + \sin\theta\cos\varphi\mathbf{j} = \sin\theta\mathbf{e}_\varphi, \tag{3.45}$$

$$\frac{\partial\mathbf{e}_\theta}{\partial\varphi} = -\cos\theta\sin\varphi\mathbf{i} + \cos\theta\cos\varphi\mathbf{j} = \cos\theta\mathbf{e}_\varphi, \tag{3.46}$$

$$\frac{\partial\mathbf{e}_\varphi}{\partial\varphi} = -\cos\varphi\mathbf{i} - \sin\varphi\mathbf{j} = -(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta). \tag{3.47}$$

3.2.1 Differential Operations

Gradient. We are ready to express the gradient operator $\boldsymbol{\nabla}$ in the spherical coordinates. Using (3.39), we have

$$\begin{aligned}
\boldsymbol{\nabla} &= \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = (\sin\theta\cos\varphi\mathbf{e}_r + \cos\theta\cos\varphi\mathbf{e}_\theta - \sin\varphi\mathbf{e}_\varphi)\frac{\partial}{\partial x} \\
&\quad + (\sin\theta\sin\varphi\mathbf{e}_r + \cos\theta\sin\varphi\mathbf{e}_\theta + \cos\varphi\mathbf{e}_\varphi)\frac{\partial}{\partial y} + (\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta)\frac{\partial}{\partial z} \\
&= \mathbf{e}_r\left[\sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}\right] \\
&\quad + \mathbf{e}_\theta\left[\cos\theta\cos\varphi\frac{\partial}{\partial x} + \cos\theta\sin\varphi\frac{\partial}{\partial y} - \sin\theta\frac{\partial}{\partial z}\right] \\
&\quad + \mathbf{e}_\varphi\left[-\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y}\right].
\end{aligned} \tag{3.48}$$

The quantities in the brackets can be recognized if we use (3.33) and the chain rule of derivatives

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} \\
&= \sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z},
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} \\
&= r\cos\theta\cos\varphi\frac{\partial}{\partial x} + r\cos\theta\sin\varphi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z},
\end{aligned} \tag{3.50}$$

$$\begin{aligned}
\frac{\partial}{\partial \varphi} &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \varphi}\frac{\partial}{\partial z} \\
&= -r\sin\theta\sin\varphi\frac{\partial}{\partial x} + r\sin\theta\cos\varphi\frac{\partial}{\partial y}.
\end{aligned} \tag{3.51}$$

Thus (3.48) can be written as

$$\boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}. \tag{3.52}$$

It follows that

$$\boldsymbol{\nabla} r = \mathbf{e}_r, \quad \boldsymbol{\nabla}\theta = \frac{1}{r}\mathbf{e}_\theta, \quad \boldsymbol{\nabla}\varphi = \frac{1}{r\sin\theta}\mathbf{e}_\varphi. \tag{3.53}$$

Divergence. The divergence of a vector in the spherical coordinates is

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{V} &= \boldsymbol{\nabla}\cdot(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_\varphi\mathbf{e}_\varphi) \\ &= \boldsymbol{\nabla}V_r\cdot\mathbf{e}_r + V_r\boldsymbol{\nabla}\cdot\mathbf{e}_r + \boldsymbol{\nabla}\mathbf{V}_\theta\cdot\mathbf{e}_\theta + V_\theta\boldsymbol{\nabla}\cdot\mathbf{e}_\theta \\ &\quad +\boldsymbol{\nabla}V_\varphi\cdot\mathbf{e}_\varphi + V_\varphi\boldsymbol{\nabla}\cdot\mathbf{e}_\varphi. \end{aligned} \tag{3.54}$$

Although the divergence in the spherical coordinates can be worked out just as we did for in the cylindrical coordinates, it is instructive to find the expression by using the derivatives of (3.42)–(3.47),

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{e}_r &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\mathbf{e}_r \\ &= \mathbf{e}_r\cdot\frac{\partial\mathbf{e}_r}{\partial r} + \frac{1}{r}\mathbf{e}_\theta\cdot\frac{\partial\mathbf{e}_r}{\partial\theta} + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\cdot\frac{\partial\mathbf{e}_r}{\partial\varphi} \\ &= \frac{1}{r}\mathbf{e}_\theta\cdot\mathbf{e}_\theta + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\cdot\sin\theta\mathbf{e}_\varphi = \frac{2}{r}, \end{aligned} \tag{3.55}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{e}_\theta &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\mathbf{e}_\theta \\ &= \mathbf{e}_r\cdot\frac{\partial\mathbf{e}_\theta}{\partial r} + \frac{1}{r}\mathbf{e}_\theta\cdot\frac{\partial\mathbf{e}_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\cdot\frac{\partial\mathbf{e}_\theta}{\partial\varphi} \\ &= \frac{1}{r}\mathbf{e}_\theta\cdot(-\mathbf{e}_r) + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\cdot\cos\theta\mathbf{e}_\varphi = \frac{1}{r}\frac{\cos\theta}{\sin\theta}, \end{aligned} \tag{3.56}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{e}_\varphi &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\mathbf{e}_\varphi \\ &= \mathbf{e}_r\cdot\frac{\partial\mathbf{e}_\varphi}{\partial r} + \frac{1}{r}\mathbf{e}_\theta\cdot\frac{\partial\mathbf{e}_\varphi}{\partial\theta} + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\cdot\frac{\partial\mathbf{e}_\varphi}{\partial\varphi} \\ &= \frac{1}{r\sin\theta}\mathbf{e}_\varphi\cdot(-\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta) = 0. \end{aligned} \tag{3.57}$$

Furthermore,

$$\boldsymbol{\nabla}V_r\cdot\mathbf{e}_r = \left(\mathbf{e}_r\frac{\partial V_r}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial V_r}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\varphi}\right)\cdot\mathbf{e}_r = \frac{\partial V_r}{\partial r}. \tag{3.58}$$

Similarly,

$$\boldsymbol{\nabla}V_\theta\cdot\mathbf{e}_\theta = \frac{1}{r}\frac{\partial V_\theta}{\partial\theta}, \qquad \boldsymbol{\nabla}V_\varphi\cdot\mathbf{e}_\varphi = \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}. \tag{3.59}$$

Thus,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{V} &= \frac{\partial V_r}{\partial r} + \frac{2}{r}V_r + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{1}{r}\frac{\cos\theta}{\sin\theta}V_\theta + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, V_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}V_\varphi. \end{aligned} \tag{3.60}$$

Laplacian. The Laplacian in spherical coordinates can be written as

$$\nabla^2\Phi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Phi = \boldsymbol{\nabla}\cdot\left(\mathbf{e}_r\frac{\partial\Phi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\right).$$

Regarding $\boldsymbol{\nabla}\Phi$ as a vector and using the expression of divergence, we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\right). \tag{3.61}$$

Therefore the Laplacian operator can be written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{3.62}$$

Curl. The curl of a vector in the spherical coordinates can be written as

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{V} &= \boldsymbol{\nabla}\times(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_\varphi\mathbf{e}_\varphi)\\ &= \boldsymbol{\nabla}V_r\times\mathbf{e}_r + V_r\boldsymbol{\nabla}\times\mathbf{e}_r + \boldsymbol{\nabla}\mathbf{V}_\theta\times\mathbf{e}_\theta + V_\theta\boldsymbol{\nabla}\times\mathbf{e}_\theta\\ &\quad +\boldsymbol{\nabla}V_\varphi\times\mathbf{e}_\varphi + V_\varphi\boldsymbol{\nabla}\times\mathbf{e}_\varphi.\end{aligned} \tag{3.63}$$

Again we will derive the expression of curl in the spherical coordinates with the derivatives of (3.42)–(3.47).

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{e}_r &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\times\mathbf{e}_r\\ &= \mathbf{e}_r\times\frac{\partial\mathbf{e}_r}{\partial r} + \frac{1}{r}\mathbf{e}_\theta\times\frac{\partial\mathbf{e}_r}{\partial\theta} + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\times\frac{\partial\mathbf{e}_r}{\partial\varphi}\\ &= \frac{1}{r}\mathbf{e}_\theta\times\mathbf{e}_\theta + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\times\sin\theta\mathbf{e}_\varphi = 0,\end{aligned} \tag{3.64}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{e}_\theta &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\times\mathbf{e}_\theta\\ &= \mathbf{e}_r\times\frac{\partial\mathbf{e}_\theta}{\partial r} + \frac{1}{r}\mathbf{e}_\theta\times\frac{\partial\mathbf{e}_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\times\frac{\partial\mathbf{e}_\theta}{\partial\varphi}\\ &= \frac{1}{r}\mathbf{e}_\theta\times(-\mathbf{e}_r) + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\times\cos\theta\mathbf{e}_\varphi = \frac{1}{r}\mathbf{e}_\varphi,\end{aligned} \tag{3.65}$$

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{e}_\varphi &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\times\mathbf{e}_\varphi\\ &= \mathbf{e}_r\times\frac{\partial\mathbf{e}_\varphi}{\partial r} + \frac{1}{r}\mathbf{e}_\theta\times\frac{\partial\mathbf{e}_\varphi}{\partial\theta} + \frac{1}{r\sin\theta}\mathbf{e}_\varphi\times\frac{\partial\mathbf{e}_\varphi}{\partial\varphi}\\ &= \frac{1}{r\sin\theta}\mathbf{e}_\varphi\times(-\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta) = -\frac{1}{r}\mathbf{e}_\theta + \frac{1}{r}\frac{\cos\theta}{\sin\theta}\mathbf{e}_r.\end{aligned} \tag{3.66}$$

Furthermore

$$\begin{aligned}\nabla V_r \times \mathbf{e}_r &= \left(\mathbf{e}_r \frac{\partial V_r}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \varphi}\right) \times \mathbf{e}_r \\ &= -\frac{1}{r}\frac{\partial V_r}{\partial \theta}\mathbf{e}_\varphi + \frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \varphi}\mathbf{e}_\theta, \end{aligned} \tag{3.67}$$

$$\begin{aligned}\nabla V_\theta \times \mathbf{e}_\theta &= \left(\mathbf{e}_r \frac{\partial V_\theta}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \varphi}\right) \times \mathbf{e}_\theta \\ &= \frac{\partial V_\theta}{\partial r}\mathbf{e}_\varphi - \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \varphi}\mathbf{e}_r, \end{aligned} \tag{3.68}$$

$$\begin{aligned}\nabla V_\varphi \times \mathbf{e}_\varphi &= \left(\mathbf{e}_r \frac{\partial V_\varphi}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial V_\varphi}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial \varphi}\right) \times \mathbf{e}_\varphi \\ &= -\frac{\partial V_\varphi}{\partial r}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial V_\varphi}{\partial \theta}\mathbf{e}_r. \end{aligned} \tag{3.69}$$

Combining these six terms, we have

$$\begin{aligned}\nabla \times \mathbf{V} &= \left(\frac{1}{r}\frac{\partial V_\varphi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \varphi} + \frac{1}{r}\frac{\cos\theta}{\sin\theta}V_\varphi\right)\mathbf{e}_r \\ &+ \left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \varphi} - \frac{\partial V_\varphi}{\partial r} - \frac{1}{r}V_\varphi\right)\mathbf{e}_\theta + \left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{1}{r}V_\theta\right)\mathbf{e}_\varphi. \end{aligned} \tag{3.70}$$

3.2.2 Infinitesimal Elements

In spherical coordinates, the infinitesimal displacement vector between a point at (r, θ, φ) and at $(r + \mathrm{d}r, \theta + \mathrm{d}\theta, \varphi + \mathrm{d}\varphi)$ is

$$\mathrm{d}\mathbf{r} = \mathbf{e}_r \ \mathrm{d}r + \mathbf{e}_\theta r \ \mathrm{d}\theta + \mathbf{e}_\varphi r\sin\theta \ \mathrm{d}\varphi. \tag{3.71}$$

Note from Fig. 3.4 that only in the $\mathbf{e}_r$ direction, the increment $\mathrm{d}r$ is an element of length. Both $\mathrm{d}\theta$ and $\mathrm{d}\varphi$ are infinitesimal angles. They do not even have the units of length. The element of length in the $\mathbf{e}_\theta$ direction is $r \ \mathrm{d}\theta$ and in the $\mathbf{e}_\varphi$ direction is $r\sin\theta \ \mathrm{d}\varphi$. Thus, one would expect the gradient in the spherical coordinates to be

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial r} + \mathbf{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi},$$

which is indeed the case as shown in (3.52).

The infinitesimal volume element is the product of the three perpendicular infinitesimal displacements

$$\mathrm{d}V = (\mathrm{d}r)(r \ \mathrm{d}\theta)(r\sin\theta \ \mathrm{d}\varphi) = r^2 \sin\theta \ \mathrm{d}r \ \mathrm{d}\theta \ \mathrm{d}\varphi. \tag{3.72}$$

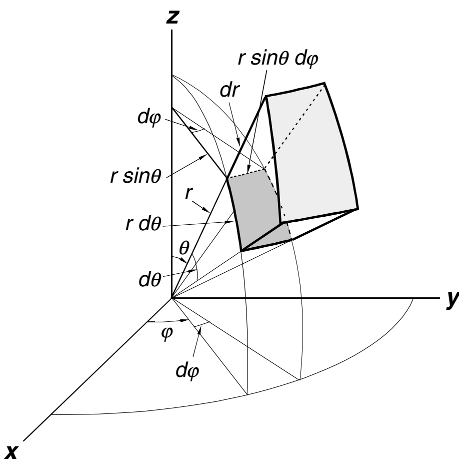

Fig. 3.4. Differential elements in the spherical coordinates. The differential length in the direction of increasing θ is $r\ \mathrm{d}\theta$. The differential length in the direction of increasing φ is $r \sin\theta\ \mathrm{d}\varphi$. The differential volume element is $r^2 \sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\varphi$

The possible range of r is 0 to ∞, θ from 0 to π, and φ from 0 to 2π. Note that θ goes from 0 to only π, and not 2π. If it goes to 2π, then every point would be counted twice.

Example 3.2.1. Use spherical coordinates to find

$$\boldsymbol{\nabla} r, \quad \boldsymbol{\nabla}\cdot\mathbf{r}, \quad \boldsymbol{\nabla} r^n, \quad \boldsymbol{\nabla}\cdot r^n\mathbf{e}_r, \quad \nabla^2 r^n, \quad \boldsymbol{\nabla}\times f(r)\,\mathbf{e}_r.$$

(We have found them in previous chapter with Cartesian coordinates. Using spherical coordinates, the results can be easily obtained, almost by inspection.)

Solution 3.2.1. Since these functions depend only on r, we need to retain only terms involving r variable:

$$\boldsymbol{\nabla} r = \mathbf{e}_r \frac{\partial}{\partial r} r = \mathbf{e}_r = \widehat{\mathbf{r}},$$

$$\boldsymbol{\nabla}\cdot\mathbf{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r\right) = 3,$$

$$\boldsymbol{\nabla} r^n = \mathbf{e}_r \frac{\partial}{\partial r} r^n = \mathbf{e}_r n r^{n-1},$$

$$\boldsymbol{\nabla}\cdot r^n\mathbf{e}_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 r^n\right) = (n+2) r^{n-1},$$

$$\nabla^2 r^n = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} r^n\right) = n\frac{1}{r^2}\frac{\partial}{\partial r} r^{n+1} = n\,(n+1)\, r^{n-2},$$

$$\nabla \times f(r)\,\mathbf{e}_r = \frac{1}{r\sin\theta}\frac{\partial f(r)}{\partial\varphi}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial f(r)}{\partial\theta}\mathbf{e}_\varphi = 0.$$

Example 3.2.2. Express $\mathbf{r} \times \nabla$ in spherical coordinates. (In quantum mechanics, the angular momentum operator $\mathbf{L}$ is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p}$ is the linear momentum operator, given by $-\mathrm{i}\hbar\nabla$.)

Solution 3.2.2.

$$\begin{aligned}\mathbf{r} \times \nabla &= r\mathbf{e}_r \times \left[\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right] \\ &= r\left[\mathbf{e}_\varphi\frac{1}{r}\frac{\partial}{\partial\theta} - \mathbf{e}_\theta\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right] \\ &= \mathbf{e}_\varphi\frac{\partial}{\partial\theta} - \mathbf{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}.\end{aligned}$$

3.3 General Curvilinear Coordinate System

3.3.1 Coordinate Surfaces and Coordinate Curves

In this section, we will develop the general theory of a curvilinear coordinate system. Suppose there is a one to one relationship between the Cartesian coordinate system (x, y, z) and another curvilinear system (u_1, u_2, u_3). This means that (x, y, z) can be written as functions of u_i,

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3), \tag{3.73}$$

and conversely,

$$u_1 = u_1(x, y, z), \quad u_2 = u_2(x, y, z), \quad u_3 = u_3(x, y, z). \tag{3.74}$$

The surfaces u_i = constant are referred to as *coordinate surfaces* and the intersections of these surfaces define the *coordinate curves*. For example, if the curvilinear system is the cylindrical coordinate system, then $u_1 = \rho$, $u_2 = \varphi$, $u_3 = z$ as shown in Fig. 3.5. Thus, u_1 = constant is the surface of the cylinder, u_2 = constant is the vertical plane, and u_3 =constant is the horizontal plane shown in the figure. The intersection of the vertical plane and the horizontal plane is the u_1 curve which is the line shown as the ρ curve. The intersection of the horizontal plane and the surface of the cylinder is the u_2 curve which is the circle shown as the φ curve. The intersection of the surface of the cylinder and the vertical plane is the u_1 curve which is the vertical line shown as the z curve.

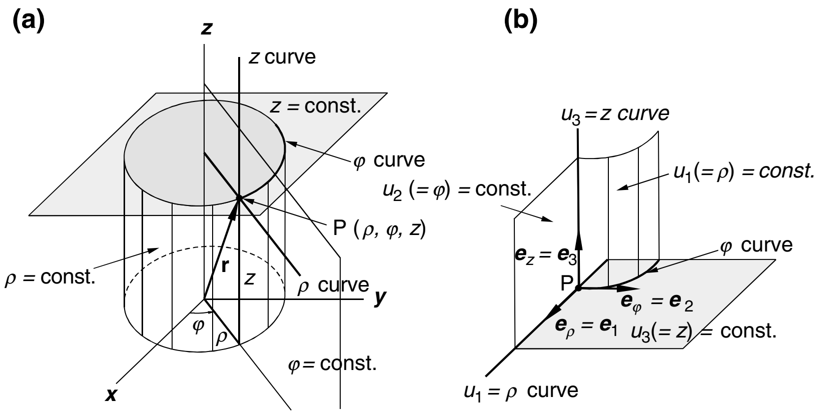

Fig. 3.5. Coordinate surfaces and coordinate curves of cylindrical coordinate system. (**a**) The side surface of the cylinder ($\rho =$ constant), the horizontal plane ($z =$ constant), and the plane containing z axis ($\varphi =$ constant) are the coordinate surfaces. The intersections of them are the coordinate curves. (**b**) Locally, the three unit vectors along the coordinate curves form an orthogonal basis set

Now the position vector $\mathbf{r}$ can be expressed as a function of u_i,

$$\mathbf{r} = \mathbf{r}\,(u_1, u_2, u_3), \tag{3.75}$$

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}\mathrm{d}u_1 + \frac{\partial \mathbf{r}}{\partial u_2}\mathrm{d}u_2 + \frac{\partial \mathbf{r}}{\partial u_3}\mathrm{d}u_3. \tag{3.76}$$

The partial derivative $\dfrac{\partial \mathbf{r}}{\partial u_1}$ means the rate of variation of $\mathbf{r}$ with u_1, while u_2 and u_3 are held fixed. So, the vector $\dfrac{\partial \mathbf{r}}{\partial u_1}$ lies in the u_2 and u_3 coordinate surfaces and is, therefore, along the u_1 coordinate curve. This enables a unit vector $\mathbf{e}_1$ to be defined in the direction of the u_1 curve,

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial u_1}/h_1 \tag{3.77}$$

where h_1 is the magnitude of $\dfrac{\partial \mathbf{r}}{\partial u_1}$

$$h_1 = \left|\frac{\partial \mathbf{r}}{\partial u_1}\right|, \tag{3.78}$$

known as the scale factor. The unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ and the corresponding scale factors h_2 and h_3 are defined in a similar way. In the case of cylindrical coordinates, $\mathbf{e}_1$ is a unit vector along the ρ curve, which is previously defined as $\mathbf{e}_\rho$, $\mathbf{e}_2$ is a unit vector tangent to the φ curve, which is previously defined as $\mathbf{e}_\varphi$, and $\mathbf{e}_3$ is a unit vector along the z curve, which is previously defined as $\mathbf{e}_z = \mathbf{k}$.

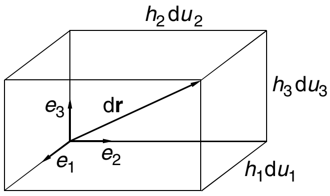

Fig. 3.6. Volume element of an orthogonal curvilinear coordinate system. A change in u_i leads to a change of distance $h_i\ \mathrm{d}u_i$ in the $\mathbf{e}_i$ direction

With the unit vectors and scale factors, the displacement vector $\mathrm{d}\mathbf{r}$ of (3.76) can be written as

$$\mathrm{d}\mathbf{r} = \mathbf{e}_1 h_1\ \mathrm{d}u_{1.} + \mathbf{e}_2 h_2\ \mathrm{d}u_2 + \mathbf{e}_3 h_3\ \mathrm{d}u_{3.} \tag{3.79}$$

If the unit vectors are orthogonal, that is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{3.80}$$

then the coordinate curves are perpendicular to each other where they intersect. Such coordinate systems are known as orthogonal curvilinear coordinates. It will also be assumed that the coordinate system is right handed, so that

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2. \tag{3.81}$$

Thus, locally $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a set of unit orthogonal basis vectors for the coordinate system (u_1, u_2, u_3), although they may change directions from point to point. In this coordinate system, a change in u_i of size $\mathrm{d}u_i$ leads to a change of distance $h_i\ \mathrm{d}u_i$ in the $\mathbf{e}_i$ direction. Schematically this is shown in Fig. 3.6.

It follows from (3.79) and Fig. 3.6 that the arc length $\mathrm{d}s$ of a line element along $\mathrm{d}\mathbf{r}$ is given by

$$\mathrm{d}s = (\mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r})^{1/2} = [(h_1\ \mathrm{d}u_1)^2 + (h_2\ \mathrm{d}u_2)^2 + (h_3\ \mathrm{d}u_3)^2]^{1/2}. \tag{3.82}$$

The directed surface element along $\mathbf{e}_1$ generated by the displacements $\mathrm{d}u_2$ and $\mathrm{d}u_3$ is

$$\mathbf{e}_1\ \mathrm{d}a = \mathbf{e}_2 h_2\ \mathrm{d}u_2 \times \mathbf{e}_3 h_3\ \mathrm{d}u, \tag{3.83}$$

and similarly for surface elements $\mathbf{e}_2\ \mathrm{d}a$ and $\mathbf{e}_3\ \mathrm{d}a$. Finally, the volume elements $\mathrm{d}V$ produced by the displacements $\mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}u_3$ are given by

$$\mathrm{d}V = |\mathbf{e}_1 h_1\ \mathrm{d}u_{1.} \cdot (\mathbf{e}_2 h_2\ \mathrm{d}u_2 \times \mathbf{e}_3 h_3\ \mathrm{d}u)| = h_1 h_2 h_3\ \mathrm{d}u_1\ \mathrm{d}u_2\ \mathrm{d}u_3, \tag{3.84}$$

since $\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = 1$.

3.3.2 Differential Operations in Curvilinear Coordinate Systems

Gradient. The gradient $\nabla\Phi$ of a scalar function is a vector perpendicular to the surface $\Phi =$ constant, defined by the equation

$$\mathrm{d}\Phi = \nabla\Phi \cdot \mathrm{d}\mathbf{r} \tag{3.85}$$

To find the expression of $\nabla\Phi$ in a curvilinear coordinate system, let us assume

$$\nabla\Phi = f_1\mathbf{e}_1 + f_2\mathbf{e}_2 + f_3\mathbf{e}_3. \tag{3.86}$$

Since

$$\mathrm{d}\mathbf{r} = h_1\ \mathrm{d}u_1\mathbf{e}_1 + h_2\ \mathrm{d}u_2\mathbf{e}_2 + h_3\ \mathrm{d}u_3\mathbf{e}_3,$$

it follows that

$$\nabla\Phi \cdot \mathrm{d}\mathbf{r} = f_1h_1\ \mathrm{d}u_1 + f_2h_2\ \mathrm{d}u_2 + f_3h_3\ \mathrm{d}u_3. \tag{3.87}$$

On the other hand

$$\mathrm{d}\Phi = \frac{\partial\Phi}{\partial u_1}\mathrm{d}u_1 + \frac{\partial\Phi}{\partial u_2}\mathrm{d}u_2 + \frac{\partial\Phi}{\partial u_3}\mathrm{d}u_3. \tag{3.88}$$

Equating the last two equations, we have

$$f_1h_1 = \frac{\partial\Phi}{\partial u_1}, \quad f_2h_2 = \frac{\partial\Phi}{\partial u_2}, \quad f_3h_3 = \frac{\partial\Phi}{\partial u_3}. \tag{3.89}$$

Thus (3.86) becomes

$$\nabla\Phi = \mathbf{e}_1\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}. \tag{3.90}$$

Therefore the del operator in curvilinear coordinates can be written as

$$\nabla = \mathbf{e}_1\frac{1}{h_1}\frac{\partial}{\partial u_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial}{\partial u_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial}{\partial u_3}. \tag{3.91}$$

In particular,

$$\nabla u_1 = \mathbf{e}_1\frac{1}{h_1}\frac{\partial u_1}{\partial u_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial u_1}{\partial u_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial u_1}{\partial u_3}.$$

Since u_1, u_2, u_3 are independent variables,

$$\frac{\partial u_1}{\partial u_1} = 1, \quad \frac{\partial u_1}{\partial u_2} = 0, \quad \frac{\partial u_1}{\partial u_3} = 0.$$

Therefore

$$\nabla u_1 = \mathbf{e}_1\frac{1}{h_1}. \tag{3.92}$$

Similarly,

$$\nabla u_2 = \mathbf{e}_2\frac{1}{h_2}, \quad \nabla u_3 = \mathbf{e}_3\frac{1}{h_3}. \tag{3.93}$$

Divergence. The expression of the divergence of a vector field $\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ in a curvilinear coordinates can be found by direct calculation using the del operator.

$$\nabla \cdot \mathbf{A} = \nabla \cdot (A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3),$$

$$\begin{aligned}\nabla \cdot A_1\mathbf{e}_1 &= \nabla \cdot A_1(\mathbf{e}_2 \times \mathbf{e}_3) = \nabla \cdot A_1 h_2 h_3 (\nabla u_2 \times \nabla u_3) \\ &= (\nabla A_1 h_2 h_3) \cdot (\nabla u_2 \times \nabla u_3) + A_1 h_2 h_3 \nabla \cdot (\nabla u_2 \times \nabla u_3).\end{aligned}$$

The term $\nabla \cdot (\nabla u_2 \times \nabla u_3) = \nabla \times \nabla u_2 \cdot \nabla u_3 - \nabla \times \nabla u_3 \cdot \nabla u_2$ vanishes because $\nabla \times \nabla f = 0$. Thus,

$$\begin{aligned}\nabla \cdot A_1\mathbf{e}_1 &= (\nabla A_1 h_2 h_3) \cdot (\nabla u_2 \times \nabla u_3) \\ &= (\nabla A_1 h_2 h_3) \cdot \frac{\mathbf{e}_2 \times \mathbf{e}_3}{h_2 h_3} = (\nabla A_1 h_2 h_3) \cdot \frac{\mathbf{e}_1}{h_2 h_3}.\end{aligned}$$

Using the del operator of (3.91), we have

$$\nabla (A_1 h_2 h_3) = \mathbf{e}_1 \frac{1}{h_1} \frac{\partial (A_1 h_2 h_3)}{\partial u_1} + \mathbf{e}_2 \frac{1}{h_2} \frac{\partial (A_1 h_2 h_3)}{\partial u_2} + \mathbf{e}_3 \frac{1}{h_3} \frac{\partial (A_1 h_2 h_3)}{\partial u_3},$$

$$(\nabla A_1 h_2 h_3) \cdot \frac{\mathbf{e}_1}{h_2 h_3} = \frac{1}{h_1 h_2 h_3} \frac{\partial (A_1 h_2 h_3)}{\partial u_1}.$$

Therefore,

$$\nabla \cdot A_1\mathbf{e}_1 = \frac{1}{h_1 h_2 h_3} \frac{\partial (A_1 h_2 h_3)}{\partial u_1}. \tag{3.94}$$

With similar expressions

$$\begin{aligned}\nabla \cdot A_2\mathbf{e}_2 &= \frac{1}{h_1 h_2 h_3} \frac{\partial (A_2 h_3 h_1)}{\partial u_2}, \\ \nabla \cdot A_3\mathbf{e}_3 &= \frac{1}{h_1 h_2 h_3} \frac{\partial (A_3 h_1 h_2)}{\partial u_3},\end{aligned}$$

we obtain

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (A_1 h_2 h_3)}{\partial u_1} + \frac{\partial (A_2 h_3 h_1)}{\partial u_2} + \frac{\partial (A_3 h_1 h_2)}{\partial u_3} \right]. \tag{3.95}$$

Laplacian. The Laplacian follows from its definition

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi.$$

Since the $\nabla \Phi$ is given by

$$\nabla \Phi = \mathbf{e}_1 \frac{1}{h_1} \frac{\partial \Phi}{\partial u_1} + \mathbf{e}_2 \frac{1}{h_2} \frac{\partial \Phi}{\partial u_2} + \mathbf{e}_3 \frac{1}{h_3} \frac{\partial \Phi}{\partial u_3},$$

the divergence of this vector is

$$\nabla \cdot \nabla \Phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{1}{h_1} \frac{\partial \Phi}{\partial u_1} h_2 h_3 \right) \right. \\ \left. + \frac{\partial}{\partial u_2} \left(\frac{1}{h_2} \frac{\partial \Phi}{\partial u_2} h_3 h_1 \right) + \frac{\partial}{\partial u_3} \left(\frac{1}{h_3} \frac{\partial \Phi}{\partial u_3} h_1 h_2 \right) \right].$$

Hence

$$\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \Phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \Phi}{\partial u_3} \right) \right]. \tag{3.96}$$

Curl. The curl of a vector field in a curvilinear coordinates can also be calculated directly,

$$\nabla \times \mathbf{A} = \nabla \times (A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3),$$

$$\nabla \times A_1 \mathbf{e}_1 = \nabla \times A_1 h_1 \nabla u_1 = \nabla (A_1 h_1) \times \nabla u_1 + A_1 h_1 \nabla \times \nabla u_1.$$

Since $\nabla \times \nabla u_1 = 0$,

$$\nabla \times A_1 \mathbf{e}_1 = \nabla (A_1 h_1) \times \nabla u_1 = \nabla (A_1 h_1) \times \frac{\mathbf{e}_1}{h_1}$$

Now

$$\nabla (A_1 h_1) = \mathbf{e}_1 \frac{1}{h_1} \frac{\partial (A_1 h_1)}{\partial u_1} + \mathbf{e}_2 \frac{1}{h_2} \frac{\partial (A_1 h_1)}{\partial u_2} + \mathbf{e}_3 \frac{1}{h_3} \frac{\partial (A_1 h_1)}{\partial u_3},$$

$$\nabla (A_1 h_1) \times \frac{\mathbf{e}_1}{h_1} = -\mathbf{e}_3 \frac{1}{h_2 h_1} \frac{\partial (A_1 h_1)}{\partial u_2} + \mathbf{e}_2 \frac{1}{h_3 h_1} \frac{\partial (A_1 h_1)}{\partial u_3}.$$

Therefore,

$$\nabla \times A_1 \mathbf{e}_1 = -\mathbf{e}_3 \frac{1}{h_2 h_1} \frac{\partial (A_1 h_1)}{\partial u_2} + \mathbf{e}_2 \frac{1}{h_3 h_1} \frac{\partial (A_1 h_1)}{\partial u_3}.$$

With similar expressions for $\nabla \times A_2 \mathbf{e}_2$ and $\nabla \times A_3 \mathbf{e}_3$, we have

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \left[\frac{1}{h_2 h_3} \frac{\partial (A_3 h_3)}{\partial u_2} - \frac{1}{h_2 h_3} \frac{\partial (A_2 h_2)}{\partial u_3} \right] \\ + \mathbf{e}_2 \left[\frac{1}{h_1 h_3} \frac{\partial (A_1 h_1)}{\partial u_3} - \frac{1}{h_1 h_3} \frac{\partial (A_3 h_3)}{\partial u_1} \right] \\ + \mathbf{e}_3 \left[\frac{1}{h_2 h_1} \frac{\partial (A_2 h_2)}{\partial u_1} - \frac{1}{h_2 h_1} \frac{\partial (A_1 h_1)}{\partial u_2} \right].$$

This expression can be put in a more symmetrical form, which is easier to remember,

$$\nabla \times \mathbf{A} = \frac{h_1\mathbf{e}_1}{h_1h_2h_3}\left[\frac{\partial (A_3h_3)}{\partial u_2} - \frac{\partial (A_2h_2)}{\partial u_3}\right] + \frac{h_2\mathbf{e}_2}{h_1h_2h_3}\left[\frac{\partial (A_1h_1)}{\partial u_3} - \frac{\partial (A_3h_3)}{\partial u_1}\right]$$
$$+\frac{h_3\mathbf{e}_3}{h_1h_2h_3}\left[\frac{\partial (A_2h_2)}{\partial u_1} - \frac{\partial (A_1h_1)}{\partial u_2}\right]$$
$$= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}. \tag{3.97}$$

Example 3.3.1. For the cylindrical coordinates, $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $z = z$. With $u_1 = \rho$, $u_2 = \varphi$, $u_3 = z$, (a) find the scale factors h_1, h_2, and h_3, (b) find the gradient, divergence, Laplacian, and curl in the cylindrical coordinates from the general formulas derived in this section.

Solution 3.3.1. (a) Since $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and x, y, z are functions of u_1, u_2, u_3, so

$$\frac{\partial \mathbf{r}}{\partial u_i} = \frac{\partial x}{\partial u_i}\mathbf{i} + \frac{\partial y}{\partial u_i}\mathbf{j} + \frac{\partial z}{\partial u_i}\mathbf{k},$$

$$h_i = \left|\frac{\partial \mathbf{r}}{\partial u_i}\right| = \left|\frac{\partial \mathbf{r}}{\partial u_i}\cdot\frac{\partial \mathbf{r}}{\partial u_i}\right|^{1/2} = \left[\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}.$$

Now

$$\frac{\partial x}{\partial u_1} = \frac{\partial x}{\partial \rho} = \cos\varphi, \quad \frac{\partial y}{\partial u_1} = \frac{\partial y}{\partial \rho} = \sin\varphi, \quad \frac{\partial z}{\partial u_1} = \frac{\partial z}{\partial \rho} = 0,$$
$$\frac{\partial x}{\partial u_2} = \frac{\partial x}{\partial \varphi} = -\rho\sin\varphi, \quad \frac{\partial y}{\partial u_2} = \frac{\partial y}{\partial \varphi} = \rho\cos\varphi, \quad \frac{\partial z}{\partial u_2} = \frac{\partial z}{\partial \varphi} = 0,$$
$$\frac{\partial x}{\partial u_3} = \frac{\partial x}{\partial z} = 0, \quad \frac{\partial y}{\partial u_3} = \frac{\partial y}{\partial z} = 0, \quad \frac{\partial z}{\partial u_3} = \frac{\partial z}{\partial z} = 1.$$

$$h_1 = \left(\cos^2\varphi + \sin^2\varphi\right)^{1/2} = 1,$$
$$h_2 = \left(\rho^2\cos^2\varphi + \rho^2\sin^2\varphi\right)^{1/2} = \rho,$$
$$h_3 = (1)^{1/2} = 1.$$

(b)

$$\nabla\Phi = \mathbf{e}_1\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}$$
$$= \mathbf{e}_\rho\frac{\partial\Phi}{\partial\rho} + \mathbf{e}_\varphi\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi} + \mathbf{e}_z\frac{\partial\Phi}{\partial z}.$$

$$\nabla\cdot\mathbf{A}=\frac{1}{h_1h_2h_3}\left[\frac{\partial(A_1h_2h_3)}{\partial u_1}+\frac{\partial(A_2h_3h_1)}{\partial u_2}+\frac{\partial(A_3h_1h_2)}{\partial u_3}\right]$$
$$=\frac{1}{\rho}\left[\frac{\partial(A_\rho\rho)}{\partial\rho}+\frac{\partial(A_\varphi)}{\partial\varphi}+\frac{\partial(A_z\rho)}{\partial z}\right]=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)+\frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi}+\frac{\partial A_z}{\partial z}.$$

$$\nabla^2\Phi=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$$
$$=\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial\Phi}{\partial z}\right)\right]$$
$$=\frac{\partial^2\Phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+\frac{\partial^2\Phi}{\partial z^2}.$$

$$\nabla\times\mathbf{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix}h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3}\\ A_1h_1 & A_2h_2 & A_3h_3\end{vmatrix}=\frac{1}{\rho}\begin{vmatrix}\mathbf{e}_\rho & \rho\mathbf{e}_\varphi & \mathbf{e}_z\\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z}\\ A_\rho & \rho A_\rho & A_z\end{vmatrix}$$
$$=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi}-\frac{1}{\rho}\frac{\partial}{\partial z}(\rho A_\varphi)\right)\mathbf{e}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\mathbf{e}_\varphi$$
$$+\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\varphi)-\frac{1}{\rho}\frac{\partial A_\rho}{\partial\varphi}\right)\mathbf{e}_z.$$

Example 3.3.2. For the spherical coordinates, $u_1=r$, $u_2=\theta$, $u_3=\varphi$, and $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$, $z=r\cos\theta$. (a) Find the scale factors h_1, h_2, and h_3, (b) find the gradient, divergence, Laplacian, and curl in the spherical coordinates from the general formulas derived in this section.

Solution 3.3.2. (a)

$$h_i=\left|\frac{\partial\mathbf{r}}{\partial u_i}\right|=\left|\frac{\partial\mathbf{r}}{\partial u_i}\cdot\frac{\partial\mathbf{r}}{\partial u_i}\right|^{1/2}=\left[\left(\frac{\partial x}{\partial u_i}\right)^2+\left(\frac{\partial y}{\partial u_i}\right)^2+\left(\frac{\partial z}{\partial u_i}\right)^2\right]^{1/2}.$$

Now

$$\frac{\partial x}{\partial u_1}=\frac{\partial x}{\partial r}=\sin\theta\cos\varphi,\qquad\frac{\partial y}{\partial u_1}=\frac{\partial y}{\partial r}=\sin\theta\sin\varphi,\qquad\frac{\partial z}{\partial u_1}=\frac{\partial z}{\partial r}=\cos\theta,$$
$$\frac{\partial x}{\partial u_2}=\frac{\partial x}{\partial\theta}=r\cos\theta\cos\varphi,\quad\frac{\partial y}{\partial u_2}=\frac{\partial y}{\partial\theta}=r\cos\theta\sin\varphi,\quad\frac{\partial z}{\partial u_2}=\frac{\partial z}{\partial\theta}=-r\sin\theta,$$
$$\frac{\partial x}{\partial u_3}=\frac{\partial x}{\partial\varphi}=-r\sin\theta\sin\varphi,\qquad\frac{\partial y}{\partial u_3}=\frac{\partial y}{\partial\varphi}=r\sin\theta\cos\varphi,\qquad\frac{\partial z}{\partial u_3}=\frac{\partial z}{\partial\varphi}=0.$$

$$h_1 = (\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta)^{1/2} = 1,$$
$$h_2 = \left(r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta\right)^{1/2} = r.$$
$$h_3 = \left(r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi\right)^{1/2} = r\sin\theta.$$

(b)

$$\begin{aligned}\boldsymbol{\nabla}\Phi &= \mathbf{e}_1\frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\\ &= \mathbf{e}_r\frac{\partial\Phi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{A} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial(A_1h_2h_3)}{\partial u_1} + \frac{\partial(A_2h_3h_1)}{\partial u_2} + \frac{\partial(A_3h_1h_2)}{\partial u_3}\right]\\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial(A_r r^2\sin\theta)}{\partial r} + \frac{\partial(A_\theta r\sin\theta)}{\partial\theta} + \frac{\partial(A_\varphi r)}{\partial\varphi}\right]\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(A_\varphi).\end{aligned}$$

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]\\ &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\varphi}\right)\right]\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.\end{aligned}$$

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3}\\ A_1h_1 & A_2h_2 & A_3h_3\end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\varphi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ A_r & rA_\theta & r\sin\theta A_\varphi\end{vmatrix}$$
$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) - \frac{\partial}{\partial\varphi}(rA_\theta)\right)\mathbf{e}_r$$
$$+ \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\varphi}(A_r) - \frac{\partial}{\partial r}(r\sin\theta A_\varphi)\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right)\mathbf{e}_\varphi.$$

3.4 Elliptical Coordinates

There are many coordinate systems. In the classical text of Morse and Feshbach, "Methods of Theoretical Physics," no less than 13 coordinate systems are discussed. Each of them is particularly convenient for certain special problems. However, because of the development of high-speed computers, the need for most of them has diminished. In this section, we will introduce only the elliptical coordinate system as one more example of special coordinate systems. The elliptical coordinate system is important in dealing with the two center problems in diatomic molecules.

3.4.1 Coordinate Surfaces

Elliptical coordinates are families of confocal ellipses and hyperbolas in two dimensions. Rotating them around the major axis of the ellipses, surfaces of prolate spheroids and hyperboloids are generated. These surfaces together with the planes containing the major axis form a three dimensional coordinate system which is commonly known as the elliptical coordinates.

The coordinate surfaces are shown in Fig. 3.7a. Let r_1 and r_2 be the distances from the two focal points which are separated by a distance $2c$ on the z axis as shown in Fig. 3.7b. A point in space is determined by r_1 and r_2, the distances from the two focal points, and the angle φ around the z axis. The coordinates of the point are λ, μ and φ with

$$\lambda = \frac{r_2 + r_1}{2c}, \tag{3.98}$$

$$\mu = \frac{r_2 - r_1}{2c}. \tag{3.99}$$

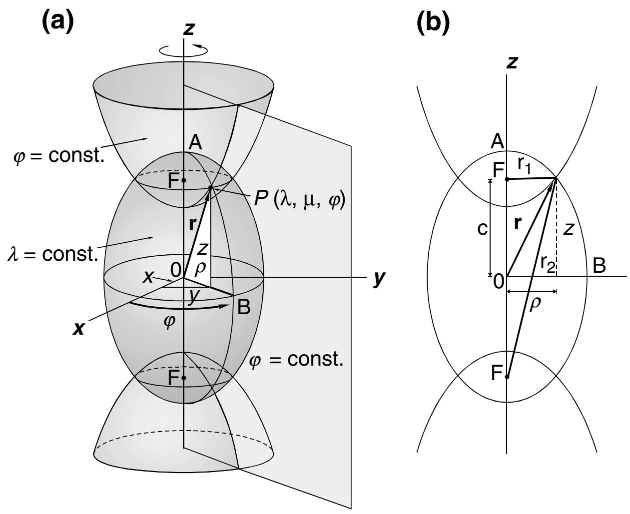

Fig. 3.7. Elliptical Coordinate System. (**a**) The coordinate surfaces generated by an ellipse, two hyperbolas and a plane containing the major axis of the ellipse. (**b**) The confocal ellipse given by $r_2 + r_1 =$ constant, and the confocal hyperbolas given by $r_2 - r_1 =$ constant

For λ = constant, (3.98) maps out a prolate spheroid in space, on any φ = constant plane, it is just an ellipse as shown in Fig. 3.7b. This can be seen as follows:

$$r_2 = \left[(z+c)^2 + \rho^2\right]^{1/2}, \tag{3.100}$$

$$r_1 = \left[(z-c)^2 + \rho^2\right]^{1/2}, \tag{3.101}$$

$$r_2 + r_1 = 2c\lambda. \tag{3.102}$$

Square both sides of $r_2 = 2c\lambda - r_1$, and collect terms, it becomes

$$z = c\lambda^2 - \lambda r_1.$$

Square both sides again, we find

$$(\lambda^2 - 1)z^2 + \lambda^2\rho^2 = c^2\lambda^2\left(\lambda^2 - 1\right).$$

This equation can be written in the standard form of an ellipse,

$$\frac{z^2}{c^2\lambda^2} + \frac{\rho^2}{c^2\left(\lambda^2 - 1\right)} = 1, \tag{3.103}$$

which cuts the z axis at $\pm c\lambda$, the ρ axis at $\pm\, c\left(\lambda^2 - 1\right)^{1/2}$. The range of λ is clearly $\infty \geq \lambda \geq 1$. When $\lambda = 1$, the ellipse reduces to the line between the two focal points.

Starting with

$$r_2 - r_1 = 2c\mu$$

and following the same procedure, we get

$$\frac{z^2}{c^2\mu^2} + \frac{\rho^2}{c^2\left(\mu^2 - 1\right)} = 1,$$

which is of the same form as the ellipse. However, in this case it is clear from Fig. 3.7b that

$$r_1 + 2c \geq r_2,$$

which simply says that the sum of two sides of a triangle must be greater than the third side. It follows that

$$2c \geq r_2 - r_1 = 2c\mu.$$

Therefore $1 \geq \mu$. Thus the equation is seen to be in the form of a hyperbola:

$$\frac{z^2}{c^2\mu^2} - \frac{\rho^2}{c^2\left(1 - \mu^2\right)} = 1, \tag{3.104}$$

which cuts the z axis at $\pm c\mu$. There are two sheets of hyperbola, one corresponds to positive value of μ and the other, negative μ. Therefore the range of μ is $1 \geq \mu \geq -1$. When $\mu = 0$, the hyperbola reduces to a straight line perpendicular to the z axis through the origin. When $\mu = 1$, it reduces to a line from $z = c$ along the z axis to ∞. When $\mu = -1$, it reduces to a line from $z = -c$ along the z axis to $-\infty$.

Surfaces of hyperboloids are generated by rotating this family of hyperbolas around the z axis. The range of the angle of rotation φ is of course, $0 \leq \varphi \leq 2\pi$.

3.4.2 Relations with Rectangular Coordinates

The transformation between (x, y, z) and (λ, u, φ) can be seen from (3.100) and (3.101):

$$r_2^2 = z^2 + 2zc + c^2 + \rho^2,$$
$$r_1^2 = z^2 - 2zc + c^2 + \rho^2.$$

It follows that

$$r_2^2 - r_1^2 = 4zc.$$

Since

$$r_2^2 - r_1^2 = (r_2 - r_1)(r_2 + r_1) = (2c\mu)(2c\lambda),$$

therefore $4zc = 4c^2\mu\lambda$ which gives

$$z = c\mu\lambda. \tag{3.105}$$

Putting this into (3.103), we have

$$\frac{c^2\mu^2\lambda^2}{c^2\lambda^2} + \frac{\rho^2}{c^2\left(\lambda^2 - 1\right)} = 1,$$

which gives

$$\rho^2 = c^2\left(\lambda^2 - 1\right)\left(1 - \mu^2\right). \tag{3.106}$$

Now, from Fig. 3.7a

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi. \tag{3.107}$$

Therefore

$$\begin{aligned} x &= c\left[\left(\lambda^2 - 1\right)\left(1 - \mu^2\right)\right]^{1/2}\cos\varphi, \\ y &= c\left[\left(\lambda^2 - 1\right)\left(1 - \mu^2\right)\right]^{1/2}\sin\varphi, \\ z &= c\mu\lambda. \end{aligned} \tag{3.108}$$

From the position vector

$$\mathbf{r} = x(\lambda, \mu, \varphi)\,\mathbf{i} + y(\lambda, \mu, \varphi)\,\mathbf{j} + z(\lambda, \mu, \varphi)\,\mathbf{k}, \tag{3.109}$$

we can find the unit vectors along the λ, μ, φ coordinate curves. The three unit vectors are defined as

$$\mathbf{e}_\lambda = \frac{\partial \mathbf{r}}{\partial \lambda}/h_\lambda, \quad \mathbf{e}_\mu = \frac{\partial \mathbf{r}}{\partial \mu}/h_\mu, \quad \mathbf{e}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi}/h_\varphi. \tag{3.110}$$

Since

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \lambda} &= \frac{\partial x}{\partial \lambda}\mathbf{i}+\frac{\partial y}{\partial \lambda}\mathbf{j}+\frac{\partial x}{\partial \lambda}\mathbf{k} = \frac{c\lambda\left(1-\mu^2\right)}{\left[\left(\lambda^2-1\right)\left(1-\mu^2\right)\right]^{1/2}}\cos\varphi\mathbf{i} \\ &+\frac{c\lambda\left(1-\mu^2\right)}{\left[\left(\lambda^2-1\right)\left(1-\mu^2\right)\right]^{1/2}}\sin\varphi\mathbf{j}+c\mu\mathbf{k},\end{aligned} \tag{3.111}$$

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \mu} &= \frac{\partial x}{\partial \mu}\mathbf{i}+\frac{\partial y}{\partial \mu}\mathbf{j}+\frac{\partial x}{\partial \mu}\mathbf{k} = \frac{-c\mu\left(\lambda^2-1\right)}{\left[\left(\lambda^2-1\right)\left(1-\mu^2\right)\right]^{1/2}}\cos\varphi\mathbf{i} \\ &+\frac{-c\mu\left(\lambda^2-1\right)}{\left[\left(\lambda^2-1\right)\left(1-\mu^2\right)\right]^{1/2}}\sin\varphi\mathbf{j}+c\lambda\mathbf{k},\end{aligned} \tag{3.112}$$

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \varphi} &= \frac{\partial x}{\partial \varphi}\mathbf{i}+\frac{\partial y}{\partial \varphi}\mathbf{j}+\frac{\partial x}{\partial \varphi}\mathbf{k} = -c\left[\left(\lambda^2-1\right)\left(1-\mu^2\right)\right]^{1/2}\sin\varphi\mathbf{i} \\ &+c\left[\left(\lambda^2-1\right)\left(1-\mu^2\right)\right]^{1/2}\cos\varphi\mathbf{j},\end{aligned} \tag{3.113}$$

the scale factors are seen to be

$$\begin{aligned}h_\lambda &= \left|\frac{\partial \mathbf{r}}{\partial \lambda}\right| = \left[\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2+\left(\frac{\partial z}{\partial \lambda}\right)^2\right]^{1/2} \\ &= \left[\frac{c^2\left(\lambda^2-\mu^2\right)}{\lambda^2-1}\right]^{1/2},\end{aligned} \tag{3.114}$$

$$h_\mu = \left[\frac{c^2\left(\lambda^2-\mu^2\right)}{1-\mu^2}\right]^{1/2}, \quad h_\varphi = \left[c^2\left(\lambda^2-1\right)\left(1-\mu^2\right)\right]^{1/2}. \tag{3.115}$$

It can be readily verified that

$$\mathbf{e}_\lambda \times \mathbf{e}_\varphi = \mathbf{e}_\mu, \quad \mathbf{e}_\varphi \times \mathbf{e}_\mu = \mathbf{e}_\lambda, \quad \mathbf{e}_\mu \times \mathbf{e}_\lambda = \mathbf{e}_\varphi. \tag{3.116}$$

Therefore $\mathbf{e}_\lambda, \mathbf{e}_\varphi, \mathbf{e}_\mu$ form an orthogonal basis set. Note that in the right-hand convention, the sequence is $(\mathbf{e}_\lambda, \mathbf{e}_\varphi, \mathbf{e}_\mu)$ and not $(\mathbf{e}_\lambda, \mathbf{e}_\mu, \mathbf{e}_\varphi)$.

The volume element in this system is

$$\mathrm{d}V = h_\lambda h\varphi h_\mu \, \mathrm{d}\lambda \, \mathrm{d}\varphi \, \mathrm{d}\mu = c^3\left(\lambda^2-\mu^2\right)\mathrm{d}\lambda \, \mathrm{d}\varphi \, \mathrm{d}\mu. \tag{3.117}$$

Example 3.4.1. Use the elliptical coordinates to find the volume of the prolate spheroid generated by rotating the ellipse

$$\frac{z^2}{a^2}+\frac{\rho^2}{b^2}=1$$

around its major axis z.

Solution 3.4.1. In terms of elliptical coordinates, the ellipse is given by

$$\frac{z^2}{c^2\lambda^2}+\frac{\rho^2}{c^2(\lambda^2-1)}=1,$$

where $2c$ is the distance between the two focal points. To find the upper limit of λ, we note $a^2=c^2\lambda^2$, or

$$\lambda=a/c$$

Furthermore,

$$b^2=c^2(\lambda^2-1)=c^2[(a/c)^2-1]=a^2-c^2.$$

The volume of the prolate spheroid is

$$\begin{aligned} V=\iiint \mathrm{d}V&=\int_0^{2\pi}\int_{-1}^{1}\int_1^{a/c} c^3\left(\lambda^2-u^2\right)\mathrm{d}\lambda\;\mathrm{d}\mu\;\mathrm{d}\varphi \\ &=2\pi c^3\left[\int_{-1}^{1}\mathrm{d}\mu\int_1^{a/c}\lambda^2\;\mathrm{d}\lambda-\int_1^{a/c}\mathrm{d}\lambda\int_{-1}^{1}\mu^2\;\mathrm{d}\mu\right], \end{aligned}$$

$$\int_{-1}^{1}\mathrm{d}\mu\int_1^{a/c}\lambda^2\;\mathrm{d}\lambda=\frac{2}{3}\left[\left(\frac{a}{c}\right)^3-1\right],\quad \int_1^{a/c}\mathrm{d}\lambda\int_{-1}^{1}\mu^2\;\mathrm{d}\mu=\frac{2}{3}\left[\frac{a}{c}-1\right].$$

$$V=\frac{4\pi}{3}c^3\left[\left(\frac{a}{c}\right)^3-\frac{a}{c}\right]=\frac{4\pi}{3}a\left(a^2-c^2\right)=\frac{4\pi}{3}ab^2.$$

Example 3.4.2. Evaluate the following integral over all space

$$I=\iiint \mathrm{e}^{-r_1}\mathrm{e}^{-r_2}\;\mathrm{d}V,$$

where r_1 and r_2 are distances from two fixed points separated by a distance R. (This happens to be the overlap integral of the H_2^+ molecular ion.)

Solution 3.4.2.

$$I=\iiint \mathrm{e}^{-r_1}\mathrm{e}^{-r_2}\;\mathrm{d}V=\iiint \mathrm{e}^{-(r_1+r_2)}\;\mathrm{d}V.$$

Using elliptical coordinates

$$r_1 + r_2 = 2c\lambda = R\lambda,$$

$$\begin{aligned} I &= \left(\frac{R}{2}\right)^3 \int_0^{2\pi} \int_{-1}^{1} \int_1^{\infty} \mathrm{e}^{-R\lambda} (\lambda^2 - \mu^2)\, \mathrm{d}\lambda\, \mathrm{d}\mu\, \mathrm{d}\varphi \\ &= \left(\frac{R}{2}\right)^3 2\pi \left[\int_{-1}^{1} \mathrm{d}\mu \int_1^{\infty} \mathrm{e}^{-R\lambda}\lambda^2\, \mathrm{d}\lambda - \int_1^{\infty} \mathrm{e}^{-R\lambda}\, \mathrm{d}\lambda \int_{-1}^{1} \mu^2\, \mathrm{d}\mu\right] \\ &= \pi(1 + R + \frac{1}{3}R^2)\mathrm{e}^{-R}. \end{aligned}$$

3.4.3 Prolate Spheroidal Coordinates

The transformation (3.108) can be expressed in a more compact form with still another change of variables. Taking advantage of the identities

$$\sin^2\theta = 1 - \cos^2\theta, \qquad \sinh^2\eta = 1 + \cosh^2\eta,$$

we can set

$$\lambda = \cosh\eta, \quad \mu = \cos\theta. \tag{3.118}$$

With this set of variables, the transformation (3.108) becomes

$$\begin{aligned} x &= c\sinh\eta\sin\theta\cos\varphi, \\ y &= c\sinh\eta\sin\theta\sin\varphi, \\ z &= c\cosh\eta\cos\theta. \end{aligned} \tag{3.119}$$

The set of coordinates (η, θ, φ) is known as the prolate spheroidal coordinate system. The range of η is $0 \le \eta < \infty$, the range of θ is $0 \le \theta \le \pi$. The scale factors for this system is

$$h_\eta = c\left(\sinh^2\eta + \sin^2\theta\right)^{1/2}, \tag{3.120}$$

$$h_\theta = h_\eta, \quad h_\varphi = c\sinh\eta\sin\theta. \tag{3.121}$$

The volume element in this system is

$$\mathrm{d}V = c^3\left(\sinh^3\eta\sin\theta + \sin^3\theta\sinh\eta\right)\mathrm{d}\eta\ \mathrm{d}\theta\ \mathrm{d}\varphi.$$

Note that $h_\eta h_\theta h_\varphi \neq h_\lambda h_\varphi h_\mu$, since $\mathrm{d}\lambda\ \mathrm{d}\mu$ is not equal to $\mathrm{d}\eta\ \mathrm{d}\theta$.

3.5 Multiple Integrals

So far we have seen how to find surface and volume elements for a multiple integral in an orthogonal coordinate system. In this section, we will show that following the same line of reasoning, this method can also be used for any change of variables in multiple integrals, regardless whether the new coordinates are orthogonal or not.

3.5.1 Jacobian for Double Integral

Consider the double integral in the Cartesian coordinates $\iint_S f(x,y)\mathrm{d}a$ where the area element $\mathrm{d}a$ is of course just $\mathrm{d}x\ \mathrm{d}y$. Very often the variables of integration (x,y) are not the most convenient for evaluating the integral. It is desirable to define double integrals in terms of a general pair of curvilinear coordinates.

Let the curvilinear coordinates be (u,v) , and there be a one-to-one transformation between (x,y) and (u,v):

$$x = x(u,v), \qquad y = y(u,v). \tag{3.122}$$

The position vector from the origin to a point inside S is

$$\mathbf{r} = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j}. \tag{3.123}$$

Therefore, $\mathbf{r}$ can also be considered as a function of the curvilinear coordinates, that is $\mathbf{r} = \mathbf{r}(u,v)$. Thus,

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v. \tag{3.124}$$

Now $(\partial\mathbf{r}/\partial u)\,\mathrm{d}u$ is an infinitesimal vector along the line where v in $\mathbf{r}(u,v)$ is kept constant and $(\partial\mathbf{r}/\partial v)\,\mathrm{d}v$ is an infinitesimal vector along the line where u is kept constant. While they may not be orthogonal, the area of the parallelogram formed by these two vectors is still given by their cross product,

$$\mathrm{d}a = \left|\frac{\partial \mathbf{r}}{\partial u}\mathrm{d}u \times \frac{\partial \mathbf{r}}{\partial v}\mathrm{d}v\right| = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\ \mathrm{d}v. \tag{3.125}$$

It follows from (3.123),

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}, \tag{3.126}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}. \tag{3.127}$$

Thus the cross product of these two vectors is

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}. \tag{3.128}$$

The last determinant is called *Jacobian determinant* (or simply as Jacobian) written as $\dfrac{\partial(x,y)}{\partial(u,v)}$,

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}. \tag{3.129}$$

It follows from (3.125) that the area element is equal to the absolute value of the Jacobian times du dv

$$\mathrm{d}a=\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\ \mathrm{d}v. \tag{3.130}$$

Therefore the double integral can be written as

$$\iint_S f(x,y)\mathrm{d}x\ \mathrm{d}y=\iint_S f(x(u,v),y(u,v))\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\ \mathrm{d}v. \tag{3.131}$$

The integrand on the right-hand side is a function of u and v. Now suppose we want to change it to an integral over x and y, we should have

$$\iint_S f(x(u,v),y(u,v)\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\ \mathrm{d}v=\iint_S f(x,y)\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}\mathrm{d}x\ \mathrm{d}y.$$

The right-hand side of this equation must be identical to the left-hand side of (3.131). Therefore

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}=1. \tag{3.132}$$

This is a useful relation. Often we need $\frac{\partial(x,y)}{\partial(u,v)}$, but $\frac{\partial(u,v)}{\partial(x,y)}$ is much easier to calculate. In that case, we simply set

$$\frac{\partial(x,y)}{\partial(u,v)}=\left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1}.$$

Now we must be careful not to assert that dx dy is equal to $\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\ \mathrm{d}v$. They are equal only in the sense that under the integral sign the area element dx dy can be changed to $\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\ \mathrm{d}v$, provided the area S covered by (x,y) is the same as covered by (u,v). Locally they cannot be equal.

From (3.122), we have

$$\mathrm{d}x=\frac{\partial x}{\partial u}\mathrm{d}u+\frac{\partial x}{\partial v}\mathrm{d}v, \tag{3.133}$$

$$\mathrm{d}y=\frac{\partial y}{\partial u}\mathrm{d}u+\frac{\partial y}{\partial v}\mathrm{d}v. \tag{3.134}$$

If we multiply dx by dy, it is certainly not equal to $\frac{\partial(x,y)}{\partial(u,v)}\mathrm{d}u\ \mathrm{d}v$.

Incidentally, the transformation of the differentials can be written as

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix} = (J) \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}, \tag{3.135}$$

where (J) is known as *Jacobian matrix*. It is the matrix associated with the Jacobian determinant $\dfrac{\partial (x, y)}{\partial (u, v)}$. Jacobian determinant and Jacobian matrix are named after the German mathematician Carl Jacobi (1804–1851). Both are very useful, but we must not get confused by the two.

3.5.2 Jacobians for Multiple Integrals

The definition of the triple integral $\iiint_V f(x, y, z)\,\mathrm{d}V$ over a given region V is entirely analogous to the definition of a double integral. If x, y, z are rectangular coordinates, then $\mathrm{d}V = \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$. Just as in double integral, often the triple integral is much easier to evaluate with a set of curvilinear coordinates u_1, u_2, u_3. Again, let

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3), \tag{3.136}$$

and the position vector be

$$\mathbf{r} = x(u_1, u_2, u_3)\,\mathbf{i} + y(u_1, u_2, u_3)\,\mathbf{j} + z(u_1, u_2, u_3)\,\mathbf{k}, \tag{3.137}$$

then

$$\mathrm{d}\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}\mathrm{d}u_1 + \frac{\partial \mathbf{r}}{\partial u_2}\mathrm{d}u_2 + \frac{\partial \mathbf{r}}{\partial u_3}\mathrm{d}u_3. \tag{3.138}$$

The partial derivative $(\partial \mathbf{r}/\partial u_1)$ is the rate of variation of $\mathbf{r}$ with u_2 and u_3 held fixed. Therefore $(\partial \mathbf{r}/\partial u_1)\,\mathrm{d}u_1$ is an infinitesimal vector along the u_1 coordinate curve. Similarly, $(\partial \mathbf{r}/\partial u_2)\,\mathrm{d}u_2$ and $(\partial \mathbf{r}/\partial u_3)\,\mathrm{d}u_3$ are, respectively, infinitesimal vectors along the u_2 and u_3 coordinate curves. Regardless whether they are orthogonal or not, the volume of parallelepiped formed by these three vectors is equal to the scalar triple product of them

$$\mathrm{d}V = \frac{\partial \mathbf{r}}{\partial u_1}\mathrm{d}u_1 \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2}\mathrm{d}u_2 \times \frac{\partial \mathbf{r}}{\partial u_3}\mathrm{d}u_3 \right). \tag{3.139}$$

It follows from (3.137) that

$$\frac{\partial \mathbf{r}}{\partial u_1} = \frac{\partial x}{\partial u_1}\mathbf{i} + \frac{\partial y}{\partial u_1}\mathbf{j} + \frac{\partial z}{\partial u_1}\mathbf{k}. \tag{3.140}$$

With similar expressions for $\partial \mathbf{r}/\partial u_2$ and $\partial \mathbf{r}/\partial u_3$, the scalar triple product can be written as

$$\mathrm{d}V = \begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} & \dfrac{\partial z}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial z}{\partial u_2} \\ \dfrac{\partial x}{\partial u_3} & \dfrac{\partial y}{\partial u_3} & \dfrac{\partial z}{\partial u_3} \end{vmatrix} \mathrm{d}u_1 \; \mathrm{d}u_2 \; \mathrm{d}u_3. \tag{3.141}$$

Again the determinant is known as the Jacobian determinant, written as

$$\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} = \begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} & \dfrac{\partial z}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial z}{\partial u_2} \\ \dfrac{\partial x}{\partial u_3} & \dfrac{\partial y}{\partial u_3} & \dfrac{\partial z}{\partial u_3} \end{vmatrix}. \tag{3.142}$$

Thus, if the region covered by x, y, z and by u_1, u_2, u_3 is the same, then

$$\iiint_V f(x,y,z)\,\mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z = \iiint_V F(u_1,u_2,u_3)\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\mathrm{d}u_1 \; \mathrm{d}u_2 \; \mathrm{d}u_3, \tag{3.143}$$

where $F(u_1,u_2,u_3) = f\left(x(u_1,u_2,u_3), y(u_1,u_2,u_3), z(u_1,u_2,u_3)\right).$

We mention in passing that it can shown by induction that a multiple integral of n variables can be similarly transformed, that is

$$\begin{aligned} &\iint \cdots \int_V f(x_1,x_2,\ldots,x_n)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n \\ &= \iint \cdots \int_V F(u_1,u_2,\ldots,u_3)\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(u_1,u_2,\ldots,u_n)}\mathrm{d}u_1\mathrm{d}u_2\cdots\mathrm{d}u_n. \end{aligned} \tag{3.144}$$

Example 3.5.1. Evaluate the integral

$$I = \iint x^2y^2\mathrm{d}x \; \mathrm{d}y$$

over the interior of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution 3.5.1. Parametrically, the coordinates of a point on the ellipse can be written as

$$x = a\cos\theta, \quad y = b\sin\theta,$$

since

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2\theta}{a^2} + \frac{b^2\sin^2\theta}{b^2} = 1.$$

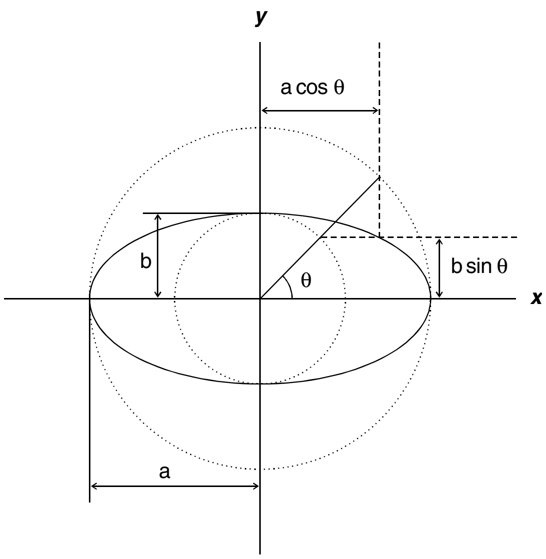

Fig. 3.8. Parametric form of an ellipse. Parametrically an ellipse can be written as $x = a\cos\theta$, $y = b\sin\theta$

This is shown in Fig. 3.8. Any point inside the ellipse can be expressed as

$$x = \gamma a \cos\theta, \quad y = \gamma b \sin\theta,$$

with $\gamma < 1$. Therefore, we can take γ and θ as curvilinear coordinates. (Note that the ellipse of $\gamma =$ constant, and the straight line of $\theta =$ constant are not orthogonal unless $a = b$.) Thus, the integral can be written as

$$I = \iint (\gamma a \cos\theta)^2 (\gamma b \sin\theta)^2 \frac{\partial(x,y)}{\partial(\gamma,\theta)} \mathrm{d}\gamma \ \mathrm{d}\theta$$

where the Jacobian is given by

$$\frac{\partial(x,y)}{\partial(\gamma,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \gamma} & \dfrac{\partial y}{\partial \gamma} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & b\sin\theta \\ -\gamma a \sin\theta & \gamma b \cos\theta \end{vmatrix} = \gamma ab.$$

Therefore

$$\begin{aligned} I &= \int_0^1 \int_0^{2\pi} (\gamma a \cos\theta)^2 (\gamma b \sin\theta)^2 \gamma ab \ \mathrm{d}\gamma \ \mathrm{d}\theta \\ &= a^3 b^3 \int_0^1 \gamma^5 \mathrm{d}\gamma \int_0^{2\pi} \cos^2\theta \sin^2\theta \ \mathrm{d}\theta = a^3 b^3 \left(\frac{1}{6}\right) \frac{\pi}{4} = \frac{\pi}{24} a^3 b^3. \end{aligned}$$

Example 3.5.2. Evaluate the integral

$$I=\int_0^\infty\int_0^\infty \frac{x^2+y^2}{1+\left(x^2-y^2\right)^2}\exp(-2xy)\mathrm{d}x\ \mathrm{d}y$$

by making a change of variable

$$u=x^2-y^2,\quad v=2xy.$$

Solution 3.5.2. First note the range of u is from $-\infty$ to ∞,

$$I=\int_0^\infty\int_{-\infty}^\infty \frac{x^2+y^2}{1+\left(x^2-y^2\right)^2}\exp(-2xy)\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}\mathrm{d}u\ \mathrm{d}v.$$

The Jacobian $\dfrac{\partial\left(x,y\right)}{\partial\left(u,v\right)}$ is not easy to calculate directly, but

$$\frac{\partial\left(u,v\right)}{\partial\left(x,y\right)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}=4\left(x^2+y^2\right).$$

Therefore,

$$\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}=\left[\frac{\partial\left(u,v\right)}{\partial\left(x,y\right)}\right]^{-1}=\frac{1}{4(x^2+y^2)}.$$

Thus

$$\begin{aligned}I&=\int_0^\infty\int_{-\infty}^\infty \frac{x^2+y^2}{1+\left(x^2-y^2\right)^2}\exp(-2xy)\frac{1}{4(x^2+y^2)}\mathrm{d}u\ \mathrm{d}v\\ &=\frac{1}{4}\int_0^\infty\int_{-\infty}^\infty \frac{1}{1+\left(x^2-y^2\right)^2}\exp(-2xy)\mathrm{d}u\ \mathrm{d}v\end{aligned}$$

$$=\frac{1}{4}\int_0^\infty\int_{-\infty}^\infty \frac{1}{1+u^2}\exp(-v)\mathrm{d}u\ \mathrm{d}v=\frac{1}{4}\int_0^\infty \exp(-v)\mathrm{d}v\times 2\int_0^\infty \frac{1}{1+u^2}\mathrm{d}u$$

$$=\frac{1}{4}\left[-\exp(-v)\right]_0^\infty 2\left[\tan^{-1}u\right]_0^\infty=\frac{\pi}{4}.$$

Exercises

1. Express the vector $\mathbf{v}=2x\mathbf{i}-z\mathbf{j}+y\mathbf{k}$ in cylindrical coordinates.
   Ans. $\mathbf{v}=\left(2\rho\cos^2\varphi-z\sin\varphi\right)\mathbf{e}_\rho-\left(2\rho\cos\varphi\sin\varphi+z\cos\varphi\right)\mathbf{e}_\varphi+\rho\sin\varphi\mathbf{e}_z$.

2. Find the curl of $\mathbf{A}$ where $\mathbf{A}=\mathbf{e}_z\ln\left(1/\rho\right)$ in cylindrical coordinates.
   Ans. $\boldsymbol{\nabla}\times\mathbf{A}=\mathbf{e}_\varphi\frac{1}{\rho}$. (The magnetic vector potentialof a long wire carrying

a current I in the z direction is $\mathbf{A} = \mathbf{e}_z \frac{\mu I}{2\pi} \ln(1/\rho)$. The magnetic field is given by $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{e}_\varphi \frac{\mu I}{2\pi} \ln(1/\rho)$.)

3. Show that $\ln \rho$ satisfies the Laplace's equation $\left(\nabla^2 \ln \rho = 0\right)$, (a) use cylindrical coordinates, (b) use spherical coordinates $(\rho = r \sin\theta)$, (c) use Cartesian coordinates $\left(\rho = \left(x^2 + y^2\right)^{1/2}\right)$.

4. Show that $1/r$ satisfies the Laplace's equation $\left(\nabla^2(1/r) = 0\right)$ for $r \neq 0$, (a) use cylindrical coordinates $\left(r = \left(\rho^2 + z^2\right)^{1/2}\right)$, (b) use spherical coordinates, (c) use Cartesian coordinates $\left(r = \left(x^2 + y^2 + z^2\right)^{1/2}\right)$.

5. (a) Show that in cylindrical coordinates

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{e}_\rho \frac{\mathrm{d}\rho}{\mathrm{d}t} + \mathbf{e}_\varphi \rho \frac{\mathrm{d}\varphi}{\mathrm{d}t} + \mathbf{e}_z \frac{\mathrm{d}z}{\mathrm{d}t},$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)^2 + \left(\rho\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]^{1/2},$$

where $\mathrm{d}s$ is the differential arc length.
(b) Find the length of the spiral described parametrically by $\rho = a$, $\varphi = t,\ z = bt$ from $t = 0$ to $t = 5$.
Ans. $5(a^2 + b^2)^{1/2}$.

6. With the vector field $\mathbf{A}$ given by $\mathbf{A} = \boldsymbol{\rho}\mathbf{e}_\rho + \mathbf{e}_z$ in cylindrical coordinates, (a) show that $\boldsymbol{\nabla} \times \mathbf{A} = 0$. (b) Find a scalar potential Φ, such that $\boldsymbol{\nabla}\Phi = \mathbf{A}$.
Ans. $\frac{1}{2}\rho^2 + z$

7. Use the infinitesimal volume element ΔV of Fig. 3.2 and the definition of the divergence

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{\Delta V} \oiint_S \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}a$$

to derive the expression of the divergence in the cylindrical coordinate system.
Hint: Find the surface elements of the six sides of ΔV, then add pairwise the surface integrals of opposite sides. For example,

$$\iint_{\text{left}} \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}a + \iint_{\text{right}} \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}a = -F_\rho(\rho, \varphi, z)\rho\ \mathrm{d}\varphi\ \mathrm{d}z$$

$$+F_\rho(\rho + \mathrm{d}\rho, \varphi, z)(\rho + \mathrm{d}\rho)\mathrm{d}\varphi\ \mathrm{d}z = \frac{\partial}{\partial\rho}(\rho F_\rho)\,\mathrm{d}\rho\ \mathrm{d}\varphi\ \mathrm{d}z.$$

With the othertwo pairs, the result is seen identical to (3.23).

8. A particle is moving in space. Show that the spherical coordinate components of its velocity and acceleration are given by

$$\begin{aligned}
v_r &= \dot{r}, \quad v_\theta = r\dot{\theta}, \quad v_\varphi = r\sin\theta\dot{\varphi}, \\
a_r &= \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2, \\
a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\cos\theta\sin\theta\dot{\varphi}^2, \\
a_\varphi &= r\sin\theta\ddot{\varphi} + 2\dot{r}\sin\theta\dot{\varphi} + 2r\cos\theta\dot{\theta}\dot{\varphi}.
\end{aligned}$$

9. Starting with the expression of $\nabla\Phi$ in spherical system, express $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$ in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, then equate it with $\nabla\Phi$ in rectangular coordinates. In this way, verify that

$$\begin{aligned}
\frac{\partial\Phi}{\partial x} &= \sin\theta\cos\varphi\frac{\partial\Phi}{\partial r} + \cos\theta\cos\varphi\frac{1}{r}\frac{\partial\Phi}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}, \\
\frac{\partial\Phi}{\partial y} &= \sin\theta\sin\varphi\frac{\partial\Phi}{\partial r} + \cos\theta\sin\varphi\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}, \\
\frac{\partial\Phi}{\partial z} &= \cos\theta\frac{\partial\Phi}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta}.
\end{aligned}$$

10. Use the infinitesimal volume element ΔV of Fig. 3.4 and the definition of the divergence

$$\nabla\cdot\mathbf{F} = \frac{1}{\Delta V}\oiint_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}a$$

to derive the expression of the divergence in the spherical coordinate system.

11. Find the expression of the Laplacian ∇^2 in spherical coordinates by directly transforming $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ into spherical coordinates using the results of the last problem.

12. Show that the following three forms of $\nabla^2\Phi(r)$ are equivalent:

$$(a)\ \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}}{\mathrm{d}r}\Phi(r)\right], \quad (b)\ \frac{\mathrm{d}^2}{\mathrm{d}r^2}\Phi(r) + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Phi(r), \quad (c)\ \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}[r\Phi(r)].$$

13. (a) Show that the vector field

$$\mathbf{F} = \left(A - \frac{B}{r^3}\right)\cos\theta\mathbf{e}_r - \left(A + \frac{B}{2r^3}\right)\sin\theta\mathbf{e}_\theta$$

is irrotational $(\nabla\times\mathbf{F} = \mathbf{0})$.

(b) Find a scalar potential Φ such that $\nabla\Phi = \mathbf{F}$.

(c) Show that Φ satisfies the Laplace equation $\nabla^2\Phi = 0$.

Ans. $\Phi = \left(Ar + \dfrac{B}{2r^2}\right)\cos\theta.$

14. Use spherical coordinates to evaluate the following integrals over a sphere of radius R centered at the origin,

$$(a) \iiint \mathrm{d}V, \quad (b) \iiint x^2 \mathrm{~d}V, \quad (c) \iiint y^2 \mathrm{~d}V, \quad (d) \iiint r^2 \mathrm{~d}V.$$

Ans. $\frac{4\pi}{3}R^3, \frac{4\pi}{15}R^3, \frac{4\pi}{15}R^3, \frac{4\pi}{5}R^3$.

15. Let

$$\mathbf{L} = -\mathrm{i}\left(\mathbf{e}_\varphi \frac{\partial}{\partial \theta} - \mathbf{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right),$$

show that
(a) $\mathbf{e}_z \cdot \mathbf{L} = -\mathrm{i}\frac{\partial}{\partial\varphi}$,
(b) $\mathbf{L}\cdot\mathbf{L} = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$.
(These are quantum mechanical L_z, L^2 angular momentum operators with $\hbar = 1$.)

16. Find the area of the Earth's surface which lies further north than the 45°N latitude. Assume the Earth is a sphere of radius R.
Ans. $\pi R^2(2-\sqrt{2})$, which is only about 15% of the total surface area of the earth ($4\pi R^2$).

17. Use spherical coordinates to verify the divergence theorem

$$\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F} \mathrm{~d}V = \oiint_S \mathbf{F}\cdot\mathbf{n} \mathrm{~d}a$$

with

$$\mathbf{F} = r^2\cos\theta\mathbf{e}_r + r^2\cos\varphi\mathbf{e}_\theta - r^2\cos\theta\sin\varphi\mathbf{e}_\varphi$$

over a sphere of radius R.
Ans. Both sides equal to 0.

18. Use elliptical coordinates to evaluate the following integral over all space

$$I = \iiint \frac{1}{r_2}\exp(-2r_1)\,\mathrm{d}V,$$

where r_1 and r_2 are the distances from two fixed points which are separated by a distance R. (This integral happens to be the so-called Coulomb integral for the H_2 molecule.)
Ans. $\frac{\pi}{R}\left[\frac{1}{R} - \exp(-2R)\left(1+\frac{1}{R}\right)\right]$.
Hint: $r_2 = \frac{1}{2}R(\lambda+\mu), \quad r_1 = \frac{1}{2}R(\lambda-\mu)$.

19. Parabolic coordinates (u, v, w) are related to Cartesian coordinates (x, y, z) by the relation $x = 2uv,\ y = u^2 - v^2,\ z = w$. (a) Find the scale factors $h_u,\ h_v,\ h_w$. (b) Show that the (u, v, w) coordinate system is orthogonal.
Ans. $2\left(u^2+v^2\right)^{1/2},\ 2\left(u^2+v^2\right)^{1/2},\ 1$.

20. Show that in terms of prolate spheroidal coordinates, the Laplace equation $\left(\nabla^2 \Phi = 0\right)$ is given by

$$\frac{1}{\left(\sinh^2\eta + \sin^2\theta\right)}\left[\frac{\partial^2}{\partial\eta^2}\Phi + \coth\eta\frac{\partial}{\partial\eta}\Phi + \frac{\partial^2}{\partial\theta^2}\Phi + \cot\theta\frac{\partial}{\partial\theta}\Phi\right]$$
$$+\frac{1}{\sinh^2\eta\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Phi = 0.$$

21. An orthogonal coordinate system (u_1, u_2, u_3) is related to Cartesian coordinates (x, y, z) by

$$x = x\left(u_1, u_2, u_3\right), \quad y = y\left(u_1, u_2, u_3\right), \quad z = z\left(u_1, u_2, u_3\right).$$

Show that

$$(a)\ \frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3} = h_1h_2h_3, \qquad (b)\ \boldsymbol{\nabla}u_1\cdot\boldsymbol{\nabla}u_2\times\boldsymbol{\nabla}u_3 = \frac{1}{h_1}\frac{1}{h_2}\frac{1}{h_3},$$

$$(c)\ \frac{\partial \mathbf{r}}{\partial u_1}\cdot\frac{\partial \mathbf{r}}{\partial u_2}\times\frac{\partial \mathbf{r}}{\partial u_3} = \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}, \quad (d)\ \boldsymbol{\nabla}u_1\cdot\boldsymbol{\nabla}u_2\times\boldsymbol{\nabla}u_3 = \frac{\partial(u_1,u_2,u_3)}{\partial(x,y,z)}.$$

22. Use the transformation $x + y = u,\ x - y = v$ to evaluate the double integral

$$I = \iint (x^2 + y^2)\mathrm{d}x\ \mathrm{d}y$$

within a square whose vertices are $(0,0),\ (1,1),\ (2,0),\ (1,-1)$.
Ans. 8/3.

Hint: Recall the Jacobian is the absolute value of the determinant $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$. Draw the square and show that the four sides of the square are $v = 0,\ u = 2,\ v = 2,\ u = 0$.