
Ordinary Differential Equations

The laws of physics that govern important and significant problems in engineering and sciences are most often expressed in the form of differential equations. A differential equation is an equation involving derivatives of an unknown function that depends upon one or more independent variables. If the unknown function depends on only one independent variable, then the equation is called an ordinary differential equation.

In this chapter, after a review of the standard methods for solving first-order differential equations, we will present a comprehensive treatment of linear differential equations with constant coefficients, in terms of which a great many physical problems are formulated. We will use mechanical vibrations and electrical circuits as illustrative examples. Then we will discuss systems of coupled differential equations and their applications.

Series solutions of differential equations will be discussed in the chapter on special functions. Another important method of solving differential equation is the Laplace transformation, which we will discuss in the next chapter.

5.1 First-Order Differential Equations

To solve a differential equation is to find a way to eliminate the derivatives in the equation so that the relation between the dependent and the independent variables can be exhibited. For a first-order differential equation, this can be achieved by carrying out an integration. The simplest type of differential equations is

$$\frac{dy}{dx} = f(x), \quad (5.1)$$

where $f(x)$ is a given function of x . We know from calculus that

$$y(x) = \int_a^x f(x')dx' \quad (5.2)$$

is a solution. Equation (5.1) contains only the first derivative of y , and is called a first-order differential equation. The order of a differential equation is equal to the order of the highest derivative in the equation. The solution (5.2) is known as a general solution, and contains an arbitrary integration constant. If the integral in (5.2) exists, then by definition there is a function $F(x)$, such that

$$\frac{d}{dx}F(x) = f(x), \quad dF(x) = f(x)dx$$

and

$$y(x) = \int_a^x dF(x') = F(x) + F(a).$$

In this sense, we often use the notation of the indefinite integral

$$y(x) = \int f(x)dx + C$$

where $C = F(a)$ is an arbitrary constant. If we know that y takes the value y_0 when $x = x_0$, then the constant is determined. This condition “ $y = y_0$ when $x = x_0$ ” is called either “initial condition” or “boundary condition.” To satisfy both the equation and the boundary condition, we can carry out the following definite integrals:

$$\int_{y_0}^y dy' = \int_{x_0}^x f(x')dx',$$

which can be written in the form of

$$y(x) = \int_{x_0}^x f(x')dx' + y_0.$$

This is known as the specific solution. (The term “particular solution” is often used, however, this may cause confusion, since “particular solution” is also used in the solution of nonhomogeneous equations, which we shall discuss a little later.) In most physical applications, it is the specific solution that is of interest. A physical problem, when formulated in the mathematical language, usually consists of a differential equation and an appropriate number of boundary and/or initial conditions. The problem is solved only after the specific solution is found.

5.1.1 Equations with Separable Variables

If an equation can be written in the form

$$f(x)dx + g(y)dy = 0$$

the solution can be immediately obtained in the form of

$$\int f(x)dx + \int g(y)dy = C.$$

This method is called solution by the separation of variables and is one of the most commonly used methods.

For example, the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

can be solved by noting that the equation can be written as

$$y dy + x dx = 0.$$

Therefore the solution is given by

$$\int y dy + \int x dx = C$$

or

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 = C.$$

This general solution can be written as

$$y(x) = (C' - x^2)^{1/2}$$

or equivalently as

$$F(x, y) = C'$$

with

$$F(x, y) = x^2 + y^2.$$

Clearly this general solution represents a family of circles with radius $\sqrt{C'}$ centered at the origin. If it is specified that $x = 5$, $y = 0$, is a point on the circle, then the specific solution is

$$x^2 + y^2 = 25.$$

This specific solution can also be obtained from the definite integral

$$\int_0^y y' dy' + \int_5^x x' dx' = 0,$$

which gives the same result by way of

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{2}5^2 = 0.$$

5.1.2 Equations Reducible to Separable Type

Certain equations of the form

$$g(x, y)dy = f(x, y)dx \quad (5.3)$$

that are not separable can be made separable by a change of variable. This can always be done, if the ratio of $f(x, y)/g(x, y)$ is a function of y/x .

Let $u = y/x$ and the function of y/x be $h(u)$, so that

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} = h(u).$$

Since y is function of x , so is u . It follows that $y(x) = xu(x)$ and

$$\frac{dy}{dx} = u + x \frac{du}{dx}.$$

Thus the differential equation can be written as

$$u + x \frac{du}{dx} = h(u),$$

or

$$x \frac{du}{dx} = h(u) - u$$

Clearly it is separable

$$\frac{du}{h(u) - u} = \frac{dx}{x}.$$

So we can solve for $u(x)$ and the solution of the original differential equation is simply

$$y = xu(x).$$

For example, if

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2} = \frac{y^2}{x^2} + \frac{y}{x},$$

then with $u = y/x$, $h(u) = u^2 + u$. Since $h(u) - u = u^2$, so

$$\frac{du}{u^2} = \frac{dx}{x}$$

and

$$\int \frac{du}{u^2} = \int \frac{dx}{x},$$

which gives

$$-\frac{1}{u} + C = \ln x.$$

Since $u = y/x$, the solution of the original differential equation is therefore given by

$$\frac{x}{y} + \ln x = C.$$

5.1.3 Exact Differential Equations

Suppose we want to find a differential equation that represents the following family of curves:

$$F(x, y) = C.$$

First let us look at two nearby points (x, y) and $(x + \Delta x, y + \Delta y)$, both on a specific curve of this family. If the curve is characterized by $C = k$, then

$$F(x, y) = k, \quad F(x + \Delta x, y + \Delta y) = k.$$

Clearly, the difference between the two is equal to zero

$$\Delta F = F(x + \Delta x, y + \Delta y) - F(x, y) = 0.$$

This difference can be written in the form of

$$\begin{aligned} F(x + \Delta x, y + \Delta y) - F(x, y) &= F(x + \Delta x, y + \Delta y) - F(x, y + \Delta y) \\ &\quad + F(x, y + \Delta y) - F(x, y). \end{aligned} \quad (5.4)$$

With the understanding that Δx and Δy are approaching zero as a limit, we can use the definition of partial derivative

$$\begin{aligned} F(x + \Delta x, y + \Delta y) - F(x, y + \Delta y) &= \frac{\partial F}{\partial x} \Delta x, \\ F(x, y + \Delta y) - F(x, y) &= \frac{\partial F}{\partial y} \Delta y \end{aligned}$$

to write (5.4) as

$$\Delta F = \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial y} \Delta y$$

or

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

This is known as the total differential. Since $\Delta F = 0$, so we have

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0. \quad (5.5)$$

This is the differential equation representing the family of curves $F(x, y) = C$. In other words, the solution of the differential equation in the form of (5.5) is given by $F(x, y) = C$.

Now let

$$\frac{\partial F}{\partial x} = f(x, y), \quad \frac{\partial F}{\partial y} = g(x, y) \quad (5.6)$$

so

$$\begin{aligned}\frac{\partial^2 F}{\partial y \partial x} &= \frac{\partial}{\partial y} \frac{\partial F}{\partial x} = \frac{\partial}{\partial y} f(x, y), \\ \frac{\partial^2 F}{\partial x \partial y} &= \frac{\partial}{\partial x} \frac{\partial F}{\partial y} = \frac{\partial}{\partial x} g(x, y).\end{aligned}$$

Since the order of differentiation can be interchanged as long as the function has continuous partial derivatives

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y},$$

one has

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial x} g(x, y). \quad (5.7)$$

Any differential equation of the form

$$f(x, y)dx + g(x, y)dy = 0$$

that satisfies (5.7) is known as an exact equation. An exact equation can be expressed as $dF = 0$, where dF is a total differential and $F(x, y) = C$ is the solution. The function $F(x, y)$ can be obtained by integrating the two equations of (5.6).

For example, the differential equation

$$\frac{dy}{dx} + \frac{xy^2}{2 + x^2y} = 0$$

can be written in the form

$$(2 + x^2y)dy + xy^2dx = 0.$$

Since

$$\frac{\partial}{\partial x}(2 + x^2y) = 2xy, \quad \frac{\partial}{\partial y}(xy^2) = 2xy$$

are equal, the differential equation is exact. Therefore, we can find the general solution in the form of

$$F(x, y) = C$$

with

$$\frac{\partial F(x, y)}{\partial y} = 2 + x^2y, \quad \frac{\partial F(x, y)}{\partial x} = xy^2.$$

The first equation yields

$$F(x, y) = 2y + \frac{1}{2}x^2y^2 + p(x).$$

The second equation requires

$$\frac{\partial F(x, y)}{\partial x} = xy^2 + \frac{d}{dx}p(x) = xy^2.$$

Therefore

$$\frac{d}{dx}p(x) = 0, \quad p(x) = k.$$

Thus the solution is

$$F(x, y) = 2y + \frac{1}{2}x^2y^2 + k = C.$$

Combining the two constants, we can write the solution as

$$2y + \frac{1}{2}x^2y^2 = C'.$$

5.1.4 Integrating Factors

A multiplying factor which will convert a differential equation that is not exact into an exact one is called an integrating factor. For example, the equation

$$y \, dx + (x^2y^3 + x)dy = 0 \tag{5.8}$$

is not exact. If, however, we multiply it by $(xy)^{-2}$, the resulting equation

$$\frac{1}{x^2y}dx + \left(y + \frac{1}{xy^2}\right)dy = 0$$

is exact, since

$$\begin{aligned} \frac{\partial}{\partial y} \left(\frac{1}{x^2y} \right) &= \frac{-1}{x^2y^2}, \\ \frac{\partial}{\partial x} \left(y + \frac{1}{xy^2} \right) &= \frac{-1}{x^2y^2}. \end{aligned}$$

Hence by definition, $(xy)^{-2}$ is an integrating factor.

By the method of exact differential equation, we have

$$\begin{aligned} \frac{\partial}{\partial x} F(x, y) &= \frac{1}{x^2y}, \\ F(x, y) &= -\frac{1}{xy} + q(y) \end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}F(x, y) &= \frac{1}{xy^2} + \frac{d}{dy}q(y) = y + \frac{1}{xy^2}, \\ \frac{d}{dy}q(y) &= y, \quad q(y) = \frac{1}{2}y^2.\end{aligned}$$

Therefore the solution of the original equation is

$$-\frac{1}{xy} + \frac{1}{2}y^2 = C.$$

It is sometimes possible to find an integrating factor by inspection. For example, one may rearrange (5.8) into

$$(y \, dx + x \, dy) + x^2 y^3 \, dy = 0$$

and recognize $y \, dx + x \, dy = d(xy)$. Then it is readily seen that the equation

$$d(xy) + x^2 y^3 \, dy = 0$$

can be solved by multiplying by a factor of $(xy)^{-2}$, since it will change the equation to

$$\frac{d(xy)}{(xy)^2} + y \, dy = 0,$$

which immediately gives the result of

$$-\frac{1}{xy} + \frac{1}{2}y^2 = C.$$

Theoretically an integrating factor exists for every differential equation of the form $f(x, y)dx + g(x, y)dy = 0$. Unfortunately no general rule is known to find it. For certain special type of differential equations, integrating factors can be found systematically.

We assume that

$$f(x, y)dx + g(x, y)dy = 0$$

is not an exact differential equation. We wish to find an integrating factor μ , so that

$$\mu f(x, y)dx + \mu g(x, y)dy = 0$$

is exact. For this equation to be exact, it must satisfy the condition

$$\frac{\partial}{\partial y}(\mu f) = \frac{\partial}{\partial x}(\mu g)$$

which gives

$$\mu \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) = \frac{\partial \mu}{\partial x} g - \frac{\partial \mu}{\partial y} f. \quad (5.9)$$

Now we consider the following possibilities.

The integrating factor μ is a function of x only. In this case, (5.9) becomes

$$\frac{1}{g} \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) = \frac{1}{\mu} \frac{\partial \mu}{\partial x}.$$

If the left-hand side of this equation is also a function of x only

$$\frac{1}{g} \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) = G(x),$$

then clearly

$$\frac{d\mu}{\mu} = G(x)dx,$$

which gives

$$\ln \mu = \int G(x)dx$$

or

$$\mu = e^{\int G(x)dx}.$$

For example, the differential equation

$$(3xy + y^2)dx + (x^2 + xy)dy = 0$$

is not exact. Written in the form of $f(x, y)dx + g(x, y)dy = 0$, we see that

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(3xy + y^2) = 3x + 2y, \\ \frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}(x^2 + xy) = 2x + y \end{aligned}$$

are not equal. But

$$\frac{1}{g} \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}$$

is a function of x only. Therefore the integrating factor is given by

$$\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

Multiply the original differential equation by x , it becomes

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0.$$

This equation is exact, since

$$\frac{\partial}{\partial y}(3x^2y + xy^2) = \frac{\partial}{\partial x}(x^3 + x^2y).$$

Integrating these two equations

$$\frac{\partial F}{\partial x} = 3x^2y + xy^2, \quad \frac{\partial F}{\partial y} = x^3 + x^2y.$$

we find

$$F(x, y) = x^3y + \frac{1}{2}x^2y^2.$$

Therefore the solution is

$$x^3y + \frac{1}{2}x^2y^2 = C.$$

The integrating factor μ is a function of y only. In this case, (5.9) becomes

$$\frac{1}{f} \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) = -\frac{1}{\mu} \frac{\partial \mu}{\partial y}.$$

If the left-hand side of this equation is also a function of y only

$$\frac{1}{f} \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) = -H(y),$$

then clearly

$$\frac{d\mu}{\mu} = H(y)dy,$$

which gives

$$\ln \mu = \int H(y)dy$$

or

$$\mu = e^{\int H(y)dy}.$$

5.2 First-Order Linear Differential Equations

A special type of first-order differential equation of some importance is of the form

$$\frac{dy}{dx} + p(x)y = q(x), \quad (5.10)$$

in which both the dependent variable y and its first derivative y' are of the first degree, and $p(x)$ and $q(x)$ are continuous functions of the independent variable x . This type of equation is called linear differential equation of the first-order. In what follows, we will derive a general solution for this equation.

First, if $q(x) = 0$, we have

$$\frac{dy}{dx} = -p(x)y.$$

In this case

$$y(x) = e^{-\int p(x)dx}.$$

For the general case, we introduce a variable coefficient

$$y(x) = f(x)e^{-\int p(x)dx}.$$

With this trial solution, (5.10) becomes

$$\frac{df(x)}{dx}e^{-\int p(x)dx} - p(x)f(x)e^{-\int p(x)dx} + p(x)f(x)e^{-\int p(x)dx} = q(x),$$

or

$$\frac{df(x)}{dx}e^{-\int p(x)dx} = q(x).$$

Thus

$$f(x) = \int q(x)e^{\int p(x)dx}dx + C.$$

Hence the solution of the first-order linear differential equation (5.10) is given by

$$\begin{aligned} y(x) &= f(x)e^{-\int p(x)dx} \\ &= e^{-\int p(x)dx} \int e^{\int p(x)dx} q(x)dx + Ce^{-\int p(x)dx}. \end{aligned} \quad (5.11)$$

To use this formula, it is important to remember to put the differential equation in the form of $y' + p(x)y = q(x)$. In other words, the coefficient of the derivative must be one.

This solution enables us to see that

$$\mu(x) = e^{\int p(x)dx}$$

is the integrating factor of the equation. In terms of $\mu(x)$, (5.11) can be written as

$$\mu(x)y = \int \mu(x)q(x)dx + C,$$

which is a solution of the differential equation

$$\frac{d}{dx}[\mu(x)y] = \mu(x)q(x). \quad (5.12)$$

Furthermore

$$\begin{aligned} \frac{d}{dx}[\mu(x)y] &= \mu(x)\frac{dy}{dx} + \left[\frac{d}{dx}\mu(x)\right]y, \\ \frac{d}{dx}\mu(x) &= \frac{d}{dx}e^{\int p(x)dx} = e^{\int p(x)dx}[p(x)] = \mu(x)p(x). \end{aligned}$$

Hence (5.12) becomes

$$\mu(x) \frac{dy}{dx} + \mu(x) p(x)y = \mu(x) q(x),$$

which clearly shows that $\mu(x)$ is an integrating factor of the original equation.

Thus, an easier way to make use of the complicated formula of (5.11) is to write it in terms of the integrating factor

$$y(x) = \frac{1}{\mu(x)} \left[\int \mu(x) q(x) dx + C \right]$$

with

$$\mu(x) = e^{\int p(x) dx}.$$

Example 5.2.1. Find the general solution of the following differential equation:

$$x \frac{dy}{dx} + (1+x)y = e^x.$$

Solution 5.2.1. This is a linear differential equation of first-order

$$\frac{dy}{dx} + \frac{1+x}{x}y = \frac{e^x}{x}.$$

The integrating factor is given by

$$\mu(x) = e^{\int \frac{1+x}{x} dx}.$$

Since

$$\begin{aligned} \int^x \frac{1+x}{x} dx &= \int \left(\frac{1}{x} + 1 \right) dx = \ln x + x, \\ \mu(x) &= e^{\ln x + x} = xe^x. \end{aligned}$$

It follows that:

$$\begin{aligned} y &= \frac{1}{xe^x} \left[\int xe^x \frac{e^x}{x} dx + C \right] \\ &= \frac{1}{xe^x} \left[\int e^{2x} dx + C \right] = \frac{1}{xe^x} \left[\frac{e^{2x}}{2} + C \right]. \end{aligned}$$

Therefore the solution is given by

$$y = \frac{e^x}{2x} + C \frac{e^{-x}}{x}.$$

5.2.1 Bernoulli Equation

The type of differential equations

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

is known as Bernoulli equations, named after Swiss mathematician James Bernoulli (1654–1705). This is a nonlinear differential equation if $n \neq 0$ or 1. However, it can be transformed into a linear equation by multiplying both sides with a factor $(1 - n)y^{-n}$

$$(1 - n)y^{-n} \frac{dy}{dx} + (1 - n)p(x)y^{1-n} = (1 - n)q(x).$$

Since

$$(1 - n)y^{-n} \frac{dy}{dx} = \frac{d}{dx}(y^{1-n}),$$

the last equation can be written as

$$\frac{d}{dx}(y^{1-n}) + (1 - n)p(x)y^{1-n} = (1 - n)q(x),$$

which is a first-order linear equation in terms of y^{1-n} . This equation can be solved for y^{1-n} , from which the solution of the original equation can be obtained.

Example 5.2.2. Find the solution of

$$\frac{dy}{dx} + \frac{1}{x}y = x^2y^3$$

with the condition $y(1) = 1$.

Solution 5.2.2. This a Bernoulli equation of $n = 3$. Multiplying this equation by $(1 - 3)y^{-3}$, we have

$$-2y^{-3} \frac{dy}{dx} - 2\frac{1}{x}y^{-2} = -2x^2,$$

which can be written as

$$\frac{d}{dx}y^{-2} - \frac{2}{x}y^{-2} = -2x^2.$$

This equation is first-order in y^{-2} and can be solved by multiplying it with an integrating factor μ ,

$$\mu = e^{\int (-\frac{2}{x})dx} = e^{-2 \ln x} = \frac{1}{x^2}.$$

Thus

$$\frac{1}{x^2}y^{-2} = \int \frac{1}{x^2}(-2x^2)dx + C = -2x + C.$$

At $x = 1$, $y = 1$, therefore

$$1 = -2 + C, \quad C = 3.$$

Hence the specific solution of the original nonlinear linear differential equation is

$$y^{-2} = -2x^3 + 3x^2$$

or written as

$$y(x) = (3x^2 - 2x^3)^{-1/2}.$$

5.3 Linear Differential Equations of Higher Order

A great many physical problems can be formulated in terms of linear differential equations. A second-order differential equation is called linear if it can be written

$$\frac{d^2}{dx^2}y(x) + p(x)\frac{d}{dx}y(x) + q(x)y(x) = h(x) \quad (5.13)$$

and nonlinear if it cannot be written in this form. To simplify the notation, this equation is also written as

$$y'' + p(x)y' + q(x)y = h(x).$$

The characteristic feature of this equation is that it is linear in the unknown function y and its derivatives. For example: $y'^2 = x$ is not linear because of the term y'^2 . The equation $yy' = 1$ is also not linear because of the product yy' . The functions p and q are called coefficients of the equation.

If $h(x) = 0$ for all x considered, the equation becomes

$$y'' + p(x)y' + q(x)y = 0$$

and is called homogeneous. If $h(x) \neq 0$, it is called nonhomogeneous.

Another convenient way of writing a differential equation is based on the so-called operator notation. The symbol of differentiation $\frac{d}{dx}$ is replaced by D :

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D^2y,$$

and so on. Therefore (5.13) can be written as

$$D^2y + p(x)Dy + q(x)y = h(x),$$

or

$$[D^2 + p(x)D + q(x)]y = h(x).$$

If we define

$$f(D) = D^2 + p(x)D + q(x),$$

then the equation is simply

$$f(D)y = h(x).$$

A fundamental theorem about homogeneous linear differential equation is the following. If $f(D)$ is second-order, then there are two linearly independent solutions y_1 and y_2 . Furthermore, any linear combination of y_1 and y_2 is also a solution. This means that if

$$f(D)y_1 = 0, \quad f(D)y_2 = 0, \quad (5.14)$$

then with any two arbitrary constants c_1 and c_2

$$f(D)(c_1y_1 + c_2y_2) = 0.$$

This is very easy to show,

$$\begin{aligned} f(D)(c_1y_1 + c_2y_2) &= f(D)c_1y_1 + f(D)c_2y_2 \\ &= c_1f(D)y_1 + c_2f(D)y_2 = 0. \end{aligned}$$

For the last step we have used (5.14). It is important to remember that this theorem does not hold for nonlinear or nonhomogeneous linear differential equations.

To discuss the general solution of the nonhomogeneous differential equation $f(D)y = h(x)$, we define a complementary function y_c and a particular solution y_p . The complementary function is the solution of the corresponding homogeneous equation, that is

$$f(D)y_c = 0.$$

If this is an n th order equation, then y_c will contain n arbitrary constants.

The particular solution is a function when it is substituted into the original nonhomogeneous equation, the result is an identity

$$f(D)y_p(x) = h(x).$$

The particular solution can be found by various methods as we shall discuss in later sections. There is no arbitrary constant in the particular solution.

The most general solution of the nonhomogeneous differential equation is the sum of the complementary function and the particular solution

$$y(x) = y_c(x) + y_p(x). \quad (5.15)$$

It is a solution since

$$f(D)[y_c(x) + y_p(x)] = f(D)y_c(x) + f(D)y_p(x) = h(x).$$

It is a general solution since the arbitrary constants, necessary for satisfying boundary or initial conditions, are contained in the complementary function.

These general statements are also true for first-order linear equations. For example, we have found that

$$y = \frac{e^x}{2x} + C \frac{e^{-x}}{x}$$

is the general solution of

$$x \frac{dy}{dx} + (1+x)y = e^x.$$

It can be readily verified that

$$\begin{aligned} \left[x \frac{d}{dx} + (1+x) \right] \frac{e^{-x}}{x} &= 0, \\ \left[x \frac{d}{dx} + (1+x) \right] \frac{e^x}{2x} &= e^x. \end{aligned}$$

Therefore e^{-x}/x is the complementary function and $e^x/(2x)$ is the particular solution.

5.4 Homogeneous Linear Differential Equations with Constant Coefficients

We will now focus our attention on linear homogeneous differential equations with constant coefficients. In searching for the solution of a homogeneous differential equation such as

$$y'' - 5y' + 6y = 0, \tag{5.16}$$

it is natural to try

$$y = e^{mx}$$

where m is a constant, because all its derivatives have the same functional form. Substituting into (5.16) and using the fact that $y' = me^{mx}$ and $y'' = m^2e^{mx}$, we have

$$e^{mx}(m^2 - 5m + 6) = 0.$$

This is the condition to be satisfied if $y = e^{mx}$ is to be a solution. Since e^{mx} can never be zero, it is thus necessary that

$$m^2 - 5m + 6 = 0.$$

This purely algebraic equation is known as the characteristic or auxiliary equation of the differential equation. The roots of this equation are $m = 2$ and $m = 3$. Therefore $y_1 = \exp(2x)$ and $y_2 = \exp(3x)$ are two solutions of (5.16). The general solution is then given by a linear combination of these two functions

$$y = c_1 e^{2x} + c_2 e^{3x}. \quad (5.17)$$

In other words, all solutions of (5.16) can be written in this form. For a second-order linear differential equation, the general solution contains two arbitrary constants c_1 and c_2 . These constants can be used to satisfy the initial conditions. For example, suppose it is given that at $x = 0$, $y = 0$ and $y' = 2$, then

$$\begin{aligned} y(0) &= c_1 + c_2 = 0, \\ y'(0) &= 2c_1 + 3c_2 = 2. \end{aligned}$$

Thus $c_1 = -2$, $c_2 = 2$. So the specific solution for the differential equation together with the given initial conditions is

$$y(x) = -2e^{2x} + 2e^{3x}.$$

To facilitate further discussion, we will repeat this process in the operator notation. If we define

$$f(D) = D^2 - 5D + 6, \quad (5.18)$$

then (5.16) can be written as $f(D)y = 0$. Substituting $y = e^{mx}$ into this equation, we obtain $f(m)e^{mx} = 0$, where

$$f(m) = m^2 - 5m + 6.$$

The characteristic equation $f(m) = 0$ has two roots; $m = 2, 3$. So the solution is given by (5.17). Although obvious, it is useful to remember that to get the characteristic equation, we need only to change D in the operator function of (5.18) to m and set it to zero.

5.4.1 Characteristic Equation with Distinct Roots

Clearly this line of reasoning can be applied to any homogeneous linear differential equation with constant coefficients, regardless of its order. If $f(D)y = 0$ is a n th-order differential equation, then the characteristic equation $f(m) = 0$ is a n th-order algebraic equation. It has n roots, $m = m_1, m_2, \dots, m_n$. If they are all distinct (different from each other), exactly n independent solutions $\exp(m_1x), \exp(m_2x), \dots, \exp(m_nx)$ of the differential equation are so obtained and the general solution is

$$y = c_1 e^{m_1x} + c_2 e^{m_2x} + \dots + c_n e^{m_nx}.$$

However, if one or more of the roots are repeated, less than n independent solutions are obtained in this way. Fortunately, it is not difficult to find the missing solutions.

5.4.2 Characteristic Equation with Equal Roots

To find the solutions when two or more roots of the characteristic equation are the same, we first consider the following identities:

$$\begin{aligned} (D - a)x^n e^{ax} &= Dx^n e^{ax} - ax^n e^{ax} \\ &= (nx^{n-1}e^{ax} + ax^n e^{ax}) - ax^n e^{ax} \\ &= nx^{n-1}e^{ax}. \end{aligned}$$

If we apply $(D - a)$ once more to both side of this equation, we have

$$\begin{aligned} (D - a)^2 x^n e^{ax} &= (D - a)nx^{n-1}e^{ax} \\ &= n(n-1)x^{n-2}e^{ax}. \end{aligned}$$

It follows that:

$$(D - a)^n x^n e^{ax} = n!e^{ax}.$$

Applying $(D - a)$ once more

$$(D - a)^{n+1} x^n e^{ax} = n!(D - a)e^{ax} = 0.$$

Clearly, if we continue to apply $(D - a)$ to both the sides of the last equation, they will all be equal to zero. Therefore

$$(D - a)^l x^n e^{ax} = 0 \quad \text{for } l > n.$$

This means that $\exp(ax)$, $x \exp(ax)$, \dots , $x^{n-1} \exp(ax)$ are solutions of the differential equation $(D - a)^n y = 0$. In other words, if the roots of the characteristic equation are repeated n times, and the common root is a , then the general solution of the differential equation is

$$y = c_1 e^{ax} + c_2 x e^{ax} + \dots + c_n x^{n-1} e^{ax}.$$

5.4.3 Characteristic Equation with Complex Roots

If the coefficients of the differential equation are real and the roots of the characteristic equation have an imaginary part, then from the theory of algebraic equations, we know that the roots must come in conjugate pairs such as $a \pm ib$. So the general solution corresponding to these two roots is

$$y = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}. \quad (5.19)$$

There are two other very useful equivalent forms of (5.19). Since

$$e^{(a \pm bi)x} = e^{ax} e^{\pm ibx} = e^{ax} (\cos bx \pm i \sin bx),$$

we can write (5.19) as

$$y = e^{ax}[c_1 \cos bx + ic_1 \sin bx + c_2 \cos bx - ic_2 \sin bx] \\ = e^{ax}[(c_1 + c_2) \cos bx + (ic_1 - ic_2) \sin bx].$$

Since c_1 and c_2 are arbitrary constants, we can replace $c_1 + c_2$ and $ic_1 - ic_2$ by two new arbitrary constants A and B . Therefore

$$y = e^{ax}(A \cos bx + B \sin bx). \tag{5.20}$$

We can write (5.20) in still another form. Recall

$$C \cos(bx - \phi) = C \cos bx \cos \phi + C \sin bx \sin \phi.$$

If we put

$$C \cos \phi = A, \quad C \sin \phi = B,$$

then $C = (A^2 + B^2)^{1/2}$ and $\phi = \tan^{-1}(B/A)$, and (5.20) becomes

$$y = Ce^{ax} \cos(bx - \phi). \tag{5.21}$$

Therefore (5.19–5.21) are all equivalent. They all contain two arbitrary constants. One set of constants can be easily transformed into another set. However, there is seldom any need to do this. In solving actual problems we simply use the form that seems best for the problem at hand, and determine the arbitrary constants in that form from the given conditions.

We summarize in Table 5.1 the relationships between the roots of the characteristic equation and the solution of the differential equation.

Table 5.1. Relationship between the roots of the characteristic equation and the general solution of the differential equation

\mathbf{m}	$\mathbf{y}(x)$
0	c_1
0, 0	$c_1 + c_2x$
0, 0, 0	$c_1 + c_2x + c_3x^2$
...	...
a	c_1e^{ax}
a, a	$c_1e^{ax} + c_2xe^{ax}$
...	...
$\pm ib$	$c_1 \cos bx + c_2 \sin bx$
$\pm ib, \pm ib$	$(c_1 + c_2x) \cos bx + (c_3 + c_4x) \sin bx$
...	...
$a \pm ib$	$e^{ax}(c_1 \cos bx + c_2 \sin bx)$
$a \pm ib, a \pm ib$	$e^{ax}[(c_1 + c_2x) \cos bx + (c_3 + c_4x) \sin bx]$
...	...

Example 5.4.1. Find the general solution of the differential equation

$$y''' = 0.$$

Solution 5.4.1. We can write the equation as

$$D^3y = 0.$$

The characteristic equation is

$$m^3 = 0.$$

The three roots are 0, 0, 0. Therefore the general solution is given by

$$\begin{aligned}y &= c_1e^{0x} + c_2xe^{0x} + c_3x^2e^{0x} \\ &= c_1 + c_2x + c_3x^2.\end{aligned}$$

This seems to be a trivial example. Obviously the result can be obtained by inspection. Here, we have demonstrated that by using the general method, we can find all the linear independent terms.

Example 5.4.2. Find the general solution of the differential equation

$$y''' - 6y'' + 9y' = 0.$$

Solution 5.4.2. We can write the equation as

$$(D^3 - 6D^2 + 9D)y = 0.$$

The characteristic equation is

$$\begin{aligned}m^3 - 6m^2 + 9m &= m(m^2 - 6m + 9) \\ &= m(m - 3)^2 = 0.\end{aligned}$$

The three roots are 0, 3, 3. Therefore the general solution is

$$y = c_1 + c_2e^{3x} + c_3xe^{3x}.$$

Again, for this third-order differential equation, the general solution has three arbitrary constants. To determine these constants, we need three conditions.

Example 5.4.3. Find the solution of

$$y'' + 9y = 0,$$

satisfying the boundary conditions $y(\pi/2) = 1$ and $y'(\pi/2) = 2$.

Solution 5.4.3. We can write the equation as

$$(D^2 + 9)y = 0.$$

The characteristic equation is

$$m^2 + 9 = 0.$$

The two roots of this equation are $m = \pm 3i$. Therefore the general solution according to (5.20) is

$$y(x) = A \cos 3x + B \sin 3x.$$

We also need y' to determine A and B

$$y'(x) = -3A \sin 3x + 3B \cos 3x.$$

The initial conditions require that

$$\begin{aligned} y\left(\frac{\pi}{2}\right) &= A \cos \frac{3\pi}{2} + B \sin \frac{3\pi}{2} \\ &= -B = 1, \end{aligned}$$

$$\begin{aligned} y'\left(\frac{\pi}{2}\right) &= -3A \sin \frac{3\pi}{2} + 3B \cos \frac{3\pi}{2} \\ &= 3A = 2. \end{aligned}$$

Thus $A = 2/3$, and $B = -1$. Therefore the solution is

$$y = \frac{2}{3} \cos 3x - \sin 3x.$$

We will get the same solution if we use either (5.19) or (5.21) instead of (5.20).

Example 5.4.4. Find the general solution of

$$y'' + y' + y = 0.$$

Solution 5.4.4. The characteristic equation is

$$m^2 + m + 1 = 0.$$

The roots of this equation are

$$m = \frac{1}{2}(-1 \pm \sqrt{1-4}) = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}.$$

Therefore the general solution is

$$y = e^{-x/2} \left(A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x \right).$$

Example 5.4.5. Find the general solution of the differential equation

$$(D^4 + 8D^2 + 16)y = 0.$$

Solution 5.4.5. The characteristic equation is

$$m^4 + 8m^2 + 16 = (m^2 + 4)^2 = 0.$$

The four roots of this equation are $m = \pm 2i, \pm 2i$. The two independent solutions associated with the roots $\pm 2i$ are $\cos 2x, \sin 2x$. The other two independent solutions corresponding to the repeated roots $\pm 2i$ are $x \cos 2x, x \sin 2x$. Therefore the general solution is given by

$$y = A \cos 2x + B \sin 2x + x(C \cos 2x + D \sin 2x).$$

5.5 Nonhomogeneous Linear Differential Equations with Constant Coefficients

5.5.1 Method of Undetermined Coefficients

We will use an example to illustrate that the general solution of nonhomogeneous differential equation is given by (5.15), namely the sum of the complementary function and the particular solution. For an equation, such as

$$(D^2 + 5D + 6)y = e^{3x} \quad (5.22)$$

it is not difficult to find the particular function $y_p(x)$. Because of e^{3x} in the right-hand side, we try

$$y_p(x) = ce^{3x}. \quad (5.23)$$

Replace y by $y_p(x)$, (5.22) becomes

$$(D^2 + 5D + 6)ce^{3x} = e^{3x}.$$

Since

$$(D^2 + 5D + 6)ce^{3x} = (9 + 5 \times 3 + 6)ce^{3x} = 30ce^{3x},$$

thus $c = 1/30$. The function $y_p(x)$ of (5.23) with $c = 1/30$ is therefore the particular solution y_p of the nonhomogeneous differential equation, that is

$$y_p = \frac{1}{30}e^{3x}.$$

Other than this particular solution, the nonhomogeneous equation has many more solutions. In fact the general solution of (5.22) should also have two

arbitrary constants. To find the general solution, let us first solve the corresponding homogeneous differential equation

$$(D^2 + 5D + 6)y_c = 0.$$

The general solution of this homogeneous equation is known as the complementary function y_c of the nonhomogeneous equation. Following the rules of solving homogeneous equation, we find

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}.$$

Thus

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{30} e^{3x} \end{aligned} \quad (5.24)$$

is the general solution of the nonhomogeneous equation (5.22).

First it is certainly a solution, since

$$\begin{aligned} (D^2 + 5D + 6)(y_c + y_p) &= (D^2 + 5D + 6)y_c + (D^2 + 5D + 6)y_p \\ &= 0 + e^{3x} = e^{3x}. \end{aligned}$$

Furthermore, it has two arbitrary constants c_1 and c_2 . This means that every possible solution of the nonhomogeneous equation (5.22) can be obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in (5.24). Clearly this principle is not limited to this particular problem.

As we have already learnt how to solve homogeneous equations, we shall now discuss some systematical ways of finding particular solutions.

Let us consider another nonhomogeneous equation

$$(D^2 - 5D + 6)y = e^{3x}. \quad (5.25)$$

Since this equation is only slightly different from (5.22), to find the particular solution y_p of this equation, we may again try

$$y_p = ce^{3x}.$$

Putting it into (5.25), we find

$$(D^2 - 5D + 6)ce^{3x} = (9 - 15 + 6)ce^{3x} = 0.$$

Obviously no value of c can make it equal to e^{3x} . Clearly, we need a more general method.

The idea is that if we can transform the nonhomogeneous equation into a homogeneous equation, then we know what to do. First we note that

$$(D - 3)e^{3x} = 0.$$

Applying $(D - 3)$ to both side of (5.25), we have

$$(D - 3)(D^2 - 5D + 6)y = (D - 3)e^{3x} = 0. \quad (5.26)$$

Then we note that the general solution $y_c + y_p$ of the nonhomogeneous equation (5.25) must also satisfy the newly formed homogeneous equation (5.26), since

$$(D - 3)(D^2 - 5D + 6)(y_c + y_p) = (D - 3)(0 + e^{3x}) = 0, \quad (5.27)$$

where we have used

$$(D^2 - 5D + 6)y_c = 0 \quad (5.28)$$

and

$$(D^2 - 5D + 6)y_p = e^{3x}. \quad (5.29)$$

This means we can obtain the general solution $y_c + y_p$ of the original nonhomogeneous equation (5.25) by assigning certain specific values to some constants in the general solution of the newly formed homogeneous equation (5.26). For example, since the roots of the characteristic equation $(m-3)(m^2-5m+6) = 0$ of the newly formed homogeneous equation are

$$m = 2, 3, 3,$$

the general solution is

$$y = c_1e^{2x} + c_2e^{3x} + c_3xe^{3x}. \quad (5.30)$$

But the complementary function given by (5.28) is

$$y_c = c_1e^{2x} + c_2e^{3x},$$

we see that the particular solution y_p can be obtained by assigning an appropriate value to c_3 , since the general solution of the original nonhomogeneous equation $y_c + y_p$ is a special solution of the newly formed homogeneous equation. To determine c_3 , we substitute c_3xe^{3x} into (5.29)

$$(D^2 - 5D + 6)c_3xe^{3x} = e^{3x}.$$

Since

$$\begin{aligned} (D^2 - 5D + 6)c_3xe^{3x} &= c_3[D(e^{3x} + 3xe^{3x}) - 5(e^{3x} + 3xe^{3x}) + 6xe^{3x}] \\ &= c_3e^{3x}, \end{aligned}$$

clearly $c_3 = 1$. Thus the particular solution is

$$y_p = xe^{3x}.$$

Hence the general solution of the nonhomogeneous differential equation (5.25) is

$$y = c_1 e^{2x} + c_2 e^{3x} + x e^{3x}.$$

Now we summarize the general procedure of solving a nonhomogeneous differential equation

$$f(D)y(x) = h(x).$$

1. The general solution is

$$y = y_c + y_p,$$

where y_c is the complementary function and y_p is the particular solution, and

$$f(D)y_c = 0, \quad f(D)y_p = h(x).$$

2. Let the roots of $f(m) = 0$ be

$$m = m_1, m_2, \dots$$

Complementary function y_c is given by the linear combination of all the linear independent functions arising from m .

3. To find y_p , we first find another equation such that

$$g(D)h(x) = 0.$$

Applying $g(D)$ to both sides of $f(D)y(x) = h(x)$, we have

$$g(D)f(D)y = 0.$$

4. The general solution $y_c + y_p$ of the original nonhomogeneous differential equation $f(D)y(x) = h(x)$ is a special solution of the newly formed homogeneous differential equation $g(D)f(D)y = 0$, since

$$\begin{aligned} g(D)f(D)(y_c + y_p) &= g(D)[f(D)y_c + f(D)y_p] \\ &= g(D)[0 + h(x)] = 0. \end{aligned}$$

5. The general solution of $g(D)f(D)y = 0$ is associated with the roots of the characteristic equation $g(m)f(m) = 0$.

Since the roots of $f(m) = 0$ lead to y_c , the roots of $g(m) = 0$ must be associated with y_p .

6. Let the roots of $g(m) = 0$ be

$$m = m'_1, m'_2, \dots$$

If there is no duplication between m'_1, m'_2, \dots and m_1, m_2, \dots , then the particular solution y_p is given by the linear combination of all the linear independent functions arising from m'_1, m'_2, \dots .

If there is duplication between m'_1, m'_2, \dots and m_1, m_2, \dots , then the functions arising from m'_1, m'_2, \dots must be multiplied by the lowest positive integer powers of x which will eliminate all such duplications.

7. Determine the arbitrary constants in the functions arising from $m'_1, m'_2 \dots$ from

$$f(D)y_p = h(x).$$

This procedure may seem to be complicated. Once understood, the implementation is actually relative simple. This we illustrate with following examples.

Example 5.5.1. Find the general solution of

$$(D^2 + 5D + 6)y = 3e^{-2x} + e^{3x}.$$

Solution 5.5.1. The characteristic equation is

$$f(m) = m^2 + 5m + 6 = 0$$

its roots are

$$m = -2, -3. \quad (m_1 = -2, m_2 = -3).$$

Therefore

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}.$$

For

$$g(D)(3e^{-2x} + e^{3x}) = 0,$$

the roots of

$$g(m) = 0$$

must be

$$m = -2, 3 \quad (m'_1 = -2, m'_2 = 3).$$

Since m'_1 repeats m_1 , the term arising from $m'_1 = -2$ must be multiplied by x . Therefore

$$y_p = c_3 x e^{-2x} + c_4 e^{3x}.$$

Since

$$(D^2 + 5D + 6)(c_3 x e^{-2x} + c_4 e^{3x}) = c_3 e^{-2x} + 30c_4 e^{3x},$$

$$(D^2 + 5D + 6)y_p = 3e^{-2x} + e^{3x},$$

we have

$$c_3 = 3; \quad c_4 = \frac{1}{30}.$$

Thus

$$y = c_1 e^{-2x} + c_2 e^{-3x} + 3x e^{-2x} + \frac{1}{30} e^{3x}.$$

Example 5.5.2. Find the general solution of

$$(D^2 + 1)y = x^2.$$

Solution 5.5.2. The roots of the characteristic equation $f(m) = m^2 + 1 = 0$ are

$$m = \pm i,$$

therefore

$$y_c = c_1 \cos x + c_2 \sin x.$$

For $g(D)x^2 = 0$, we can regard x^2 as $c_1 + c_2x + c_3x^2$ with $c_1 = 0$, $c_2 = 0$, $c_3 = 1$. The roots of $g(m) = 0$, as seen in Table 5.1, are

$$m = 0, 0, 0.$$

Thus

$$y_p = a + bx + cx^2.$$

Substituting into the original equation

$$(D^2 + 1)(a + bx + cx^2) = x^2,$$

we have

$$2c + a + bx + cx^2 = x^2.$$

Thus

$$2c + a = 0, \quad b = 0, \quad c = 1.$$

It follows that $a = -2$ and

$$y_p = -2 + x^2.$$

Finally

$$y = c_1 \cos x + c_2 \sin x - 2 + x^2.$$

Example 5.5.3. Find the general solution of

$$(D^2 + 4D + 5)y = 3e^{-2x}.$$

Solution 5.5.3. The roots of $f(m) = m^2 + 4m + 5 = 0$ are

$$m = -2 \pm i.$$

Therefore

$$y_c = e^{-2x}(c_1 \cos x + c_2 \sin x).$$

For $g(D)3e^{-2x} = 0$, $g(D) = D + 2$. Obviously, the root of $g(m) = 0$ is

$$m = -2.$$

Thus

$$y_p = Ae^{-2x}.$$

With

$$(D^2 + 4D + 5)Ae^{-2x} = 3e^{-2x}$$

we have

$$A = 3.$$

The general solution is therefore given by

$$y = e^{-2x}(c_1 \cos x + c_2 \sin x) + 3e^{-2x}.$$

Example 5.5.4. Find the general solution of

$$(D^2 - 2D + 1)y = xe^x - e^x.$$

Solution 5.5.4. The roots of $m^2 - 2m + 1 = 0$ are

$$m = 1, 1 \quad (m_1 = 1, m_2 = 1)$$

Hence

$$y_c = c_1e^x + c_2xe^x.$$

The characteristic equation of the differential equation for which $xe^x - e^x$ is the solution

$$g(D)(xe^x - e^x) = 0$$

is of course $g(m) = 0$. We can regard $xe^x - e^x$ as $ae^x + bxe^x$ (with $a = -1$, $b = 1$). We see from Table 5.1 that the roots of $g(m) = 0$ are

$$m = 1, 1 \quad (m'_1 = 1, m'_2 = 1).$$

Therefore

$$y_p = Ax^2e^x + Bx^3e^x.$$

With

$$(D^2 - 2D + 1)(Ax^2e^x + Bx^3e^x) = xe^x - e^x$$

we find

$$A = -\frac{1}{2}, \quad B = \frac{1}{6}.$$

The general solution is therefore given by

$$y = c_1e^x + c_2xe^x - \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x.$$

Example 5.5.5. Find the general solution of

$$(D^2 + 1)y = \sin x.$$

Solution 5.5.5. The roots of $m^2 + 1 = 0$ are

$$m = \pm i.$$

Therefore

$$y_c = c_1 \cos x + c_2 \sin x.$$

To find

$$g(D) \sin x = 0,$$

we can regard $\sin x$ as $a \cos x + b \sin x$ with $a = 0$ and $b = 1$. From Table 5.1, we see that the roots of $g(m) = 0$ are

$$m = \pm i \quad (m' = \pm i).$$

Thus

$$y_p = Ax \cos x + Bx \sin x.$$

With

$$(D^2 + 1)(Ax \cos x + Bx \sin x) = \sin x$$

we find

$$A = -\frac{1}{2}, \quad B = 0.$$

Therefore

$$y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x \cos x.$$

5.5.2 Use of Complex Exponentials

In applied problems, the function $h(x)$ is very often a sine or a cosine representing alternating voltage in an electric circuit or a periodic force in a vibrating system. The particular solution y_p can be found more efficiently by replacing sine or cosine by the complex exponential form.

In the last example, we can replace $\sin x$ by e^{ix} and solving the equation

$$(D^2 + 1)Y = e^{ix}. \quad (5.31)$$

The solution Y will also be complex. $Y = Y_R + iY_I$. Since $e^{ix} = \cos x + i \sin x$, the equation is equivalent to

$$(D^2 + 1)Y_R = \cos x,$$

$$(D^2 + 1)Y_I = \sin x.$$

Since the second equation is exactly the same as the original equation, we see that to find y_p , we can solve (5.31) for Y and take its imaginary part. Following the procedure of the last section, we assume

$$Y = Axe^{ix},$$

so

$$\begin{aligned}DY &= Ae^{ix} + iAxe^{ix}, \\D^2Y &= 2iAe^{ix} - Axe^{ix}, \\(D^2 + 1)Y &= 2iAe^{ix} - Axe^{ix} + Axe^{ix} = 2iAe^{ix} = e^{ix}.\end{aligned}$$

Thus

$$A = \frac{1}{2i}.$$

Taking the imaginary part of

$$Y = \frac{1}{2i}xe^{ix} = -\frac{1}{2}ix(\cos x + i\sin x),$$

we have

$$y_p = -\frac{1}{2}x \cos x,$$

which is, of course, the same as obtained in the last example.

5.5.3 Euler–Cauchy Differential Equations

An equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = h(x), \quad (5.32)$$

where the a_i are constants, is called Euler, or Cauchy, or Euler–Cauchy differential equation. By a change of variable, it can be transformed into an equation with constant coefficients which can then be solved.

If we set

$$x = e^z, \quad z = \ln x,$$

then

$$\frac{dz}{dx} = \frac{1}{x} = e^{-z}.$$

With the notation $D = \frac{d}{dz}$, we can write

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \frac{dz}{dx} = e^{-z} Dy, \\ \frac{d^2 y}{dx^2} &= \frac{d}{dx} \frac{dy}{dx} = \frac{d}{dx} (e^{-z} Dy) = \frac{d}{dz} (e^{-z} Dy) \frac{dz}{dx} \\ &= (-e^{-z} Dy + e^{-z} D^2 y) e^{-z} = e^{-2z} D(D-1)y, \\ \frac{d^3 y}{dx^3} &= \frac{d}{dz} [e^{-2z} D(D-1)y] \frac{dz}{dx} \\ &= [-2e^{-2z} D(D-1)y + e^{-2z} D^2(D-1)y] e^{-z} \\ &= e^{-3z} D(D-1)(D-2)y.\end{aligned}$$

Clearly

$$\frac{d^n y}{dx^n} = e^{-nz} D(D-1)(D-2) \cdots (D-n+1)y. \quad (5.33)$$

Substituting (5.33) into (5.32) and using $x^n = e^{nz}$, we have a differential equation with constant coefficients,

$$a_n D(D-1)(D-2) \cdots (D-n+1)y + \cdots a_1 Dy + a_0 y = h(e^z).$$

If the solution of this equation is denoted

$$y = F(z),$$

then the solution of the original equation is given by

$$y = F(\ln x).$$

The following example will make this procedure clear.

Example 5.5.6. Find the general solution of

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x \ln x.$$

Solution 5.5.6. With $x = e^z$, this equation becomes

$$[D(D-1) + D - 1]y = ze^z.$$

The complementary function comes from

$$m(m-1) + m - 1 = m^2 - 1 = 0, \quad m = 1, -1,$$

which gives

$$y_c = c_1 e^z + c_2 e^{-z}.$$

For $g(D)ze^z = 0$, $g(D)$ must be $(D-1)^2$. The characteristic equation is then

$$(m' - 1)^2 = 0, \quad m' = 1, 1.$$

Therefore the particular solution is of the form

$$y_p = c_3 z e^z + c_4 z^2 e^z.$$

Substituting it back into the differential equation

$$[D(D-1) + D - 1](c_3 z e^z + c_4 z^2 e^z) = z e^z,$$

we find

$$c_3 = -\frac{1}{4}, \quad c_4 = \frac{1}{4}.$$

Therefore

$$y = c_1 e^z + c_2 e^{-z} - \frac{1}{4} z e^z + \frac{1}{4} z^2 e^z.$$

For the general solution of the original equation, we must change z back to x .

With $z = \ln x$

$$y(x) = c_1 x + c_2 \frac{1}{x} - \frac{1}{4} x \ln x + \frac{1}{4} x (\ln x)^2,$$

which can be readily verified that this is indeed the general solution with two arbitrary constants.

For a homogeneous Euler–Cauchy equation, the following procedure is perhaps simpler. For example, to solve the equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0,$$

we can simply start with the solution

$$y(x) = x^m.$$

So

$$x^2 m(m-1)x^{m-2} + mx^{m-1} - x^m = 0$$

or

$$[m(m-1) + m - 1]x^m = 0.$$

Thus

$$\begin{aligned} m(m-1) + m - 1 &= m^2 - 1 = 0, \\ m &= 1, \quad m = -1. \end{aligned}$$

It follows that:

$$y(x) = c_1 x + c_2 \frac{1}{x}.$$

5.5.4 Variation of Parameters

The method of undetermined coefficients is simple and has important physical applications, but it applies only to constant coefficient equations with special forms of the nonhomogeneous term $h(x)$. In this section, we discuss the method of variation of parameters, which is more general. It applies to equations

$$(D^2 + p(x)D + q(x))y = h(x), \quad (5.34)$$

where p , q , and h are continuous functions of x in some interval. Let

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

be the solution of the corresponding homogeneous equation

$$(D^2 + p(x)D + q(x))y_c = 0.$$

The method of variation of parameters involves replacing the parameters c_1 and c_2 by functions u and v to be determined so that

$$y_p = u(x)y_1(x) + v(x)y_2(x)$$

is the particular solution of (5.34). Now this expression contains two unknown functions u and v , but the requirement that y_p satisfies (5.34) imposes only one condition on u and v . Therefore, we are free to impose a second arbitrary condition without loss of generality. Further calculation will show that it is convenient to require

$$u'y_1 + v'y_2 = 0. \quad (5.35)$$

Now

$$Dy_p = u'y_1 + uy_1' + v'y_2 + vy_2'.$$

With the imposed condition (5.35), we are left with

$$y_p' = uy_1' + vy_2'.$$

It follows that:

$$D^2y_p = u'y_1' + uy_1'' + v'y_2' + vy_2''.$$

Substituting them back into the equation

$$(D^2 + p(x)D + q(x))y_p = h(x)$$

and collecting terms, we have

$$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = h.$$

Since y_1 and y_2 satisfy the homogeneous equation, the quantities in the parenthesis are equal to zero. Thus

$$u'y_1' + v'y_2' = h.$$

This equation together with the imposed condition (5.35) can be solved for u' and v'

$$u' = -\frac{hy_2}{y_1y_2' - y_2y_1'}, \quad v' = \frac{hy_1}{y_1y_2' - y_2y_1'}. \quad (5.36)$$

The quantity in the denominator, known as the Wronskian W of y_1 and y_2 ,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

will not be equal to zero as long as y_1 and y_2 are linearly independent. Integration of (5.36) will enable us to determine u and v

$$u = - \int \frac{hy_2}{W} dx, \quad v = \int \frac{hy_1}{W} dx.$$

The particular solution is then

$$y_p = -y_1 \int \frac{hy_2}{W} dx + y_2 \int \frac{hy_1}{W} dx.$$

Example 5.5.7. Use the variation of parameters method to find the general solution of

$$(D^2 + 4D + 4)y = 3xe^{-2x}$$

Solution 5.5.7. This equation can be solved easily by the method of undetermined coefficients. But we want to show that the solution can also be found by the variation of parameters method. Since

$$m^2 + 4m + 4 = (m + 2)^2 = 0, \quad m = -2, -2$$

the two independent solutions of the homogeneous equation are

$$y_1 = e^{-2x}, \quad y_2 = xe^{-2x}.$$

The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x}(1 - 2x) + 2xe^{-4x} = e^{-4x}$$

and u' and v' are given by

$$u' = - \frac{3xe^{-2x}y_2}{W} = - \frac{3xe^{-2x}xe^{-2x}}{e^{-4x}} = -3x^2,$$

$$v' = \frac{3xe^{-2x}y_1}{W} = \frac{3xe^{-2x}e^{-2x}}{e^{-4x}} = 3x.$$

It follows that:

$$u = - \int 3x^2 dx = -x^3 + c_1, \quad v = \int 3x dx = \frac{3}{2}x^2 + c_2.$$

Therefore

$$y = (-x^3 + c_1)e^{-2x} + \left(\frac{3}{2}x^2 + c_2\right)xe^{-2x}$$

$$= \frac{1}{2}x^3e^{-2x} + c_1e^{-2x} + c_2xe^{-2x}.$$

This is the general solution. It is seen that this solution includes the complementary function

$$y_c = c_1e^{-2x} + c_2xe^{-2x}$$

and the particular solution

$$y_p = \frac{1}{2}x^3e^{-2x}.$$

Example 5.5.8. Find the general solution of

$$(D^2 + 2D + 1)y = h(x), \quad h(x) = \frac{2}{x^2}e^{-x}.$$

Solution 5.5.8. This equation cannot be solved by the method of undetermined coefficients, therefore we seek the solution by variation of parameters. The complementary function is obtained from

$$m^2 + 2m + 1 = (m + 1)^2 = 0, \quad m = -1, -1$$

to be

$$y_c = c_1y_1 + c_2y_2 = c_1e^{-x} + c_2xe^{-x}.$$

Let

$$y_p = uy_1 + vy_2.$$

From the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}(1 - x) + xe^{-2x} = e^{-2x},$$

we have

$$u = -\int \frac{y_2h}{W} = -\int \frac{xe^{-x}2x^{-2}e^{-x}}{e^{-2x}} dx = -2 \ln x,$$

$$v = \int \frac{y_1h}{W} = \int \frac{e^{-x}2x^{-2}e^{-x}}{e^{-2x}} dx = -\frac{2}{x}.$$

Thus

$$y_p = -2e^{-x} \ln x - 2e^{-x}.$$

Therefore the general solution is

$$y = c_1e^{-x} + c_2xe^{-x} - 2e^{-x} \ln x.$$

Note that we have dropped the term $-2e^{-x}$, since it is absorbed in c_1e^{-x} .

5.6 Mechanical Vibrations

There are countless applications of differential equations in engineering and physical sciences. As illustrative examples, we will first discuss mechanical vibrations. Any motion that repeats itself after certain time interval is called vibration or oscillation. A simple model is the spring-mass system shown in Fig. 5.1. The block of mass m is constrained to move on a frictionless table and

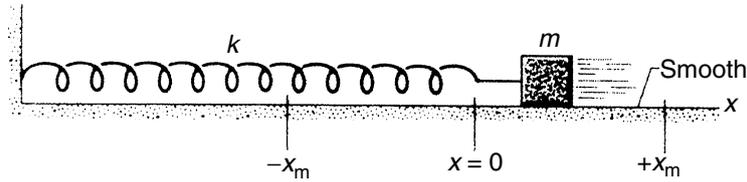


Fig. 5.1. A simple harmonic oscillator. The block moves on a frictionless table. The equilibrium position of the spring is at $x = 0$. At $t = 0$, the block is released from rest at $x = x_m$

is fastened to a spring with spring constant k . The block is pulled a distance x_m from its equilibrium position at $x = 0$ and released from rest. We want to know the subsequent motion of the block.

This system is simple enough for us to demonstrate the following steps in mathematical physics:

- Formulate the physical problem in terms of mathematical language, usually in the form of a differential equation.
- Solve the mathematical equation.
- Understand the physical meaning of the mathematical solution.

5.6.1 Free Vibration

The first step is to observe that the only horizontal force on the block is coming from the spring. According to Hooke's law, the force is proportional to the displacement but opposite in sign, that is

$$F = -kx.$$

The motion of the block is governed by the Newton's dynamic equation

$$F = ma.$$

Since the acceleration is equal to the second derivative of the displacement

$$a = \frac{d^2x}{dt^2},$$

therefore

$$m \frac{d^2x}{dt^2} = -kx.$$

This is a second-order linear homogeneous differential equation. Since the block is released from rest at $x = x_m$, the velocity of the block, which is the first derivative of the displacement, is zero at $t = 0$. Therefore the initial conditions are

$$x(0) = x_m, \quad v(0) = \left. \frac{dx}{dt} \right|_{t=0} = 0.$$

With the differential equation and the initial conditions, the mathematical problem is uniquely defined.

The second step is to solve this equation. Since the coefficients are constants, the solution of the differential equation is of the exponential form, $x = \exp(\alpha t)$ with α determined by the characteristic equation

$$m\alpha^2 = -k.$$

Clearly the roots of this equation are

$$\alpha = \pm\sqrt{-\frac{k}{m}} = \pm i\omega_0, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

Thus the general solution of the differential equation is given by

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}.$$

The initial conditions requires that

$$\begin{aligned} x(0) &= A + B = x_m, \\ \left. \frac{dx}{dt} \right|_{t=0} &= i\omega_0 A - i\omega_0 B = 0. \end{aligned}$$

Therefore $A = B = \frac{1}{2}x_m$, and

$$x(t) = x_m \cos \omega_0 t.$$

The third step is to interpret this solution. The cosine function varies between 1 and -1 , it repeats itself when its argument is increased by 2π . Therefore the block oscillates between x_m and $-x_m$. The period T of the oscillation is defined as the time required for the motion to repeat itself, this means

$$x(t + T) = x(t).$$

Thus

$$\cos(\omega_0 t + \omega_0 T) = \cos(\omega_0 t).$$

Clearly

$$\omega_0 T = 2\pi.$$

Therefore the period is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}.$$

The frequency f is defined as the number of oscillations in one second, that is

$$f = \frac{1}{T} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

Thus the block oscillates with a frequency that is prescribed by k and m . Since $\omega_0 = 2\pi f$, ω_0 is called angular frequency. Often ω_0 is referred simply as the natural frequency with the understanding that it is actually the angular frequency.

From the solution of the differential equation, we can derive all attributes of the motion, such as the velocity of the block at any given time. This type of periodic motion is called simple harmonic motion. The mass–spring system is known as a harmonic oscillator.

5.6.2 Free Vibration with Viscous Damping

In practical systems, the amplitude of the oscillation gradually decreases due to friction. This is known as damping. For example, if the system is vibrating in a fluid medium, such as air, water, oil, the resisting force offered by the viscosity of the fluid is generally proportional to the velocity of the vibrating body. Therefore with viscous damping, there is an additional force

$$F_v = -c \frac{dx}{dt}$$

where c is the coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity. Thus the equation of motion of the mass–spring system becomes

$$m \frac{d^2x}{dt^2} = -kx - c \frac{dx}{dt}$$

or

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0.$$

This equation can be written in the form

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 x = 0,$$

where $\beta = c/2m$, $\omega_0^2 = k/m$. With $x = \exp(\alpha t)$, α must satisfy the equation

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0.$$

The roots of this equation are

$$\alpha_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}, \quad \alpha_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}.$$

The solution is therefore given by

$$x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}. \quad (5.37)$$

Depending on the strength of damping, this solution takes the following three forms.

Over damping. If $\beta^2 > \omega_0^2$, then the values of α_1 and α_2 are both negative. Thus both terms in x exponentially go to zero as $t \rightarrow \infty$. In this case, the damping force represented by β overpowers the restoring force represented by ω_0 and hence prevents oscillation. The system is called overdamped.

Critical damping. In this case $\beta^2 = \omega_0^2$, the characteristic equation has a double root at $\alpha = -\beta$ twice. Hence the solution is of the form

$$x(t) = (A + Bt)e^{-\beta t}.$$

Since $\beta > 0$, both $e^{-\beta t}$ and $te^{-\beta t}$ go to zero as $t \rightarrow \infty$. The motion dies out with time and is not qualitatively different from the overdamped motion. In this case the damping force is just as strong as the restoring force, therefore the system is called critically damped.

Under damping. If $\beta^2 < \omega_0^2$, then the roots of the characteristic equation are complex

$$\alpha_1, \alpha_2 = -\beta \pm i\omega,$$

where

$$\omega = \sqrt{\omega_0^2 - \beta^2}.$$

Therefore the solution $x = e^{-\beta t}(Ae^{i\omega t} + Be^{-i\omega t})$ can be written in the form of

$$x(t) = Ce^{-\beta t} \cos(\omega t + \varphi).$$

Because of the cosine term in the solution, the motion is oscillatory. Since the maximum value of cosine is one, the displacement x must lie between the curves $x(t) = \pm Ce^{-\beta t}$. Hence it resembles a cosine curve with decreasing amplitude. In this case, the damping force represented by β is weaker than the restoring force represented by ω and thus cannot prevent oscillation. For this reason, the system is called under damped.

These three cases are illustrated in the following example with specific parameters.

Example 5.6.1. The displacement $x(t)$ of a damped harmonic oscillator satisfies the equation

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 x = 0.$$

Let the initial conditions be

$$x(0) = x_0; \quad v(0) = \left. \frac{dx}{dt} \right|_{t=0} = 0.$$

Find x as a function of time t , if $\omega_0 = 4$, and (a) $\beta = 5$, (b) $\beta = 4$, (c) $\beta = 1$. Show a sketch of the solutions of these three cases.

Solution 5.6.1. (a) Since $\beta = 5$ and $\omega_0 = 4$, the motion is overdamped. The roots of the characteristic equation are

$$\alpha_1, \alpha_2 = -5 \pm \sqrt{25 - 16} = -2, -8.$$

Thus

$$x(t) = A_1 e^{-2t} + A_2 e^{-8t}.$$

The initial conditions require A_1 and A_2 to satisfy

$$A_1 + A_2 = x_0, \quad -2A_1 - 8A_2 = 0.$$

Therefore

$$x(t) = x_0 \left(\frac{4}{3} e^{-2t} - \frac{1}{3} e^{-8t} \right).$$

The graph of this function is shown as the dotted line in Fig. 5.2.

(b) Since $\beta = 4$ and $\omega_0 = 4$, the motion is critically damped. The roots of the characteristic equation are $-\beta$ twice

$$\alpha_1, \alpha_2 = -\beta = -4.$$

Thus

$$x(t) = (A + Bt)e^{-4t}.$$

From the initial conditions, we find

$$A = x_0, \quad B = 4x_0.$$

Therefore

$$x(t) = x_0(1 + 4t)e^{-4t}.$$

The graph of this function is shown as the line of open circles in Fig. 5.2.

(c) For $\beta = 1$ and $\omega_0 = 4$, the motion is under damped. The solution can be written as

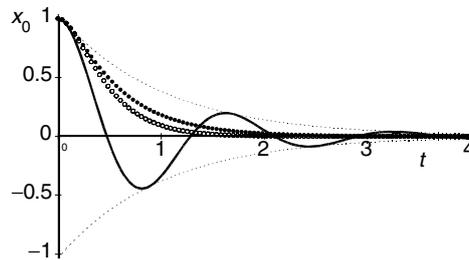


Fig. 5.2. Free vibrations with viscous damping. Initially the block is at x_0 and released from rest. The *dotted line* is for the overdamped motion, the *line of open circles* is for the critically damped motion, the *solid line* is for the underdamped motion. The damped amplitude is shown as *dashed lines*

$$x(t) = Ce^{-\beta t} \cos(\omega t - \phi),$$

where

$$\omega = \sqrt{\omega_0^2 - \beta^2} = \sqrt{15}.$$

From the initial conditions

$$\begin{aligned} x(0) &= C \cos(-\phi) = x_0, \\ \left. \frac{dx}{dt} \right|_{t=0} &= -\beta C \cos(-\phi) - \omega C \sin(-\phi) = 0, \end{aligned}$$

we find

$$\tan \phi = \frac{\beta}{\omega} = \frac{1}{\sqrt{15}}, \quad C = \frac{1}{\cos \phi} x_0.$$

Since $\cos \phi = (1 + \tan^2 \phi)^{-1/2}$,

$$C = \frac{\sqrt{\omega^2 + \beta^2}}{\omega} x_0 = \frac{4}{\sqrt{15}} x_0.$$

Therefore

$$x(t) = \frac{4}{\sqrt{15}} x_0 e^{-t} \cos \left(\sqrt{15} t - \tan^{-1} \frac{1}{\sqrt{15}} \right).$$

The graph of this function is shown as the solid line in Fig. 5.2. The damped amplitude $Ce^{-\beta t}$ is also shown as the dashed line.

In the over damped and critically damped cases, if there is a large negative initial velocity, it is possible for the block to overshoot the equilibrium position. In that case, it will come to a temporarily stop in a negative x position. After that, it will return to $x = 0$ in a monotonically decreasing way. Therefore even if it overshoots the equilibrium position, it can do that only once. Hence the motion is not oscillatory.

The under damped motion is oscillatory although its amplitude is approaching zero as time goes to ∞ . The damped frequency $\sqrt{\omega_0^2 - \beta^2}$ is always less than the natural frequency ω_0 .

5.6.3 Free Vibration with Coulomb Damping

From the first course of physics, we all learned that the friction force of a block sliding on a plane is proportional to the normal force acting on the plane of contact. This friction force acts in a direction opposite to the direction of velocity and is given by

$$F_c = \mu N$$

where N is the normal force and μ is the coefficient of friction. When the motion is damped by this friction force, it is known as Coulomb damping. Charles Augustin Coulomb (1736–1806) first proposed this relationship,

but he is much better known for his law of electrostatic force. His name is also remembered through the unit of electric charge. Coulomb damping is also known as constant damping, since the magnitude of the damping force is independent of the displacement and velocity.

However, the sign of the friction force changes with the direction of the velocity, and we need to consider the motion in two directions separately.

When the block moves from right to left, the friction force is pointing toward the right and has a positive sign. With this friction force, the equation of motion is given by

$$m \frac{d^2x}{dt^2} = -kx + \mu N.$$

This equation has a constant nonhomogeneous term. The solution is

$$x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{\mu N}{k}, \quad (5.38)$$

where $\omega_0 = \sqrt{k/m}$, which is the same as the angular frequency of the undamped oscillator.

When the block is moving from left to right, the friction force is pointing toward the left and has a negative sign, and the equation of motion becomes

$$m \frac{d^2x}{dt^2} = -kx - \mu N.$$

The solution of this equation is

$$x(t) = C \cos \omega_0 t + D \sin \omega_0 t - \frac{\mu N}{k}. \quad (5.39)$$

The constants A, B, C, D are determined by the initial conditions. For example, if the block is released from rest at a distance x_0 to the right of the equilibrium position, then $x(0) = x_0$ and the velocity v , which is the first derivative of x , at $t = 0$ is zero. In this case, the motion starts from right to left. Using (5.38), we have

$$v(t) = \frac{dx}{dt} = -\omega A \sin \omega_0 t + \omega B \cos \omega_0 t.$$

Thus the initial conditions are

$$x(0) = A + \frac{\mu N}{k} = x_0, \quad v(0) = \omega_0 B = 0.$$

Therefore $A = x_0 - \mu N/k$, $B = 0$, and

$$x(t) = \left(x_0 - \frac{\mu N}{k}\right) \cos \omega_0 t + \frac{\mu N}{k}.$$

This equation represents a simple harmonic motion with the equilibrium position shifted from zero to $\mu N/k$. However, this equation is valid only for

the first half of the first cycle. When $t = \pi/\omega_0$, the velocity of the block is equal to zero and the block is at its extreme left position x_1 which is

$$x_1 = \left(x_0 - \frac{\mu N}{k}\right) \cos \omega_0 \frac{\pi}{\omega_0} + \frac{\mu N}{k} = -x_0 + 2\frac{\mu N}{k}.$$

In the next half cycle, the block moves from left to right, so we have to use (5.39). To determine C and D , we use the fact $x(t = \pi/\omega_0) = x_1$ and $v(t = \pi/\omega_0) = 0$. With these conditions, we have

$$\begin{aligned} x\left(t = \frac{\pi}{\omega_0}\right) &= C \cos \omega_0 \frac{\pi}{\omega_0} + D \sin \omega_0 \frac{\pi}{\omega_0} - \frac{\mu N}{k} = -C - \frac{\mu N}{k} = x_1, \\ v\left(t = \frac{\pi}{\omega_0}\right) &= -\omega_0 C \sin \omega_0 \frac{\pi}{\omega_0} + \omega_0 D \cos \omega_0 \frac{\pi}{\omega_0} = -\omega_0 D = 0. \end{aligned}$$

Therefore $C = -x_1 - \frac{\mu N}{k} = x_0 - 3\frac{\mu N}{k}$, $D = 0$, and (5.39) becomes

$$x(t) = \left(x_0 - 3\frac{\mu N}{k}\right) \cos \omega_0 t - \frac{\mu N}{k}.$$

This is also a simple harmonic motion with the equilibrium position shifted to $-\mu N/k$. This equation is valid for $\pi/\omega_0 \leq t \leq 2\pi/\omega_0$. At the end of this half cycle, the velocity is again equal to zero

$$v\left(t = \frac{2\pi}{\omega_0}\right) = -\omega_0 \left(x_0 - 3\frac{\mu N}{k}\right) \sin \omega_0 \frac{2\pi}{\omega_0} = 0$$

and the block is at x_2 which is

$$x_2 = \left(x_0 - 3\frac{\mu N}{k}\right) \cos \omega_0 \frac{2\pi}{\omega_0} - \frac{\mu N}{k} = x_0 - 4\frac{\mu N}{k}.$$

These become the initial conditions for the third half cycle, and the procedure can be continued until the motion stops. The displacement x as a function of time t of this motion is shown in Fig. 5.3.

It is to be noted that the frequency of a Coulomb damped vibration is the same as that of the free vibration without damping. This is to be contrasted with the viscous damping. Furthermore, if x_n is a local maximum, then

$$x_n - x_{n-2} = -\frac{4\mu N}{k}.$$

This means that in each successive cycle, the amplitude of the motion is reduced by $4\mu N/k$ in a time interval of $2\pi/\omega_0$. Therefore the maxima of the oscillation all fall on a straight line. The slope of this straight enveloping line is

$$-\frac{4\mu N/k}{2\pi/\omega_0} = -\frac{2\mu N\omega_0}{\pi k}.$$

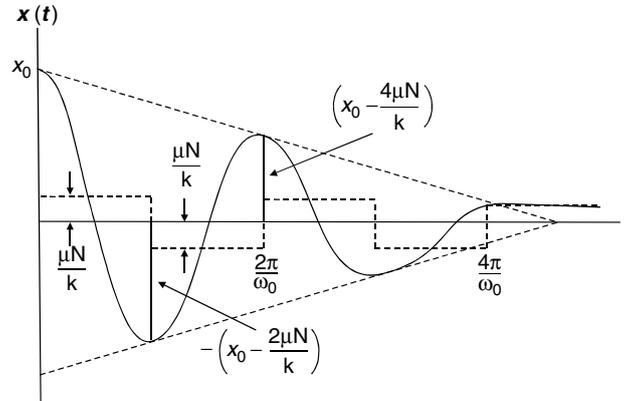


Fig. 5.3. The displacement x of the block as a function of time t in the mass-spring system with Coulomb damping. Initially the block is at $x = x_0$ and is released from rest

Similarly, all local minima must fall on a straight line with a slope of $2\mu N\omega_0/\pi k$. These characteristics are also shown in Fig. 5.3.

The motion stops when the velocity becomes zero and the restoring force kx of the spring is equal or less than the friction force μN . Thus the number of half cycles n_0 that elapse before the motion ceases can be found from the condition

$$k \left(x_0 - n_0 \frac{2\mu N}{k} \right) \leq \mu N.$$

If $x_0 \leq \mu N/k$, the motion will not even start. For $\mu N/k < x_0 < 2\mu N/k$, the block will stop before it reaches the equilibrium position. The final position of the block is usually different from the equilibrium position and represents a permanent displacement.

5.6.4 Forced Vibration without Damping

A dynamic system is often subjected to some type of external force. In this section, we shall consider the response of a spring-mass system under the external force of the form $F_0 \cos \omega t$. First suppose there is no damping, then the equation of motion is given by

$$m \frac{d^2 x}{dt^2} + kx = F_0 \cos \omega t$$

or

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t,$$

where $\omega_0 = (k/m)^{1/2}$ is the natural frequency of the system. The general solution is the sum of the complementary function x_c and the particular solution x_p . The complementary function satisfies the homogeneous equation

$$\frac{d^2x_c}{dt^2} + \omega_0^2x_c = 0$$

and is given by

$$x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

Since the nonhomogeneous term has a frequency ω , the particular solution takes the form

$$x_p(t) = A \cos \omega t + B \sin \omega t.$$

Substituting it into the equation

$$\frac{d^2x_p}{dt^2} + \omega_0^2x_p = \frac{F_0}{m} \cos \omega t$$

we find $B = 0$, and $A = F_0/[m(\omega_0^2 - \omega^2)]$. Thus

$$x_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

and the general solution, given by $x_c + x_p$

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

is the sum of two periodic motions of different frequencies.

Beats. Suppose the initial conditions are $x(0) = 0$ and $v(0) = 0$, then c_1 and c_2 are found to be

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0.$$

Thus the solution is given by

$$\begin{aligned} x(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \\ &= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{\omega_0 - \omega}{2} t \sin \frac{\omega_0 + \omega}{2} t. \end{aligned}$$

Let the forcing frequency ω be slightly less than the natural frequency ω_0 so that $\omega_0 - \omega = 2\epsilon$, where ϵ is a small positive quantity. Then $\omega_0 + \omega \approx 2\omega$ and the solution becomes

$$x(t) \approx \frac{F_0}{2m\omega\epsilon} \sin \epsilon t \sin \omega t.$$

Since ϵ is small, the function $\sin \epsilon t$ varies slowly. Thus the factor

$$\frac{F_0}{2m\omega\epsilon} \sin \epsilon t$$

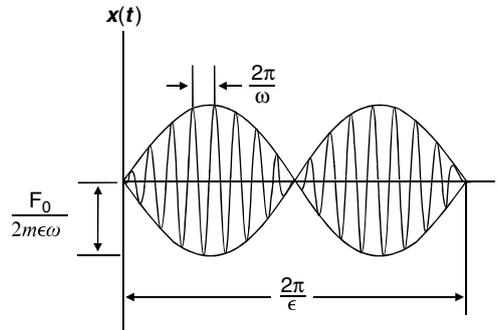


Fig. 5.4. Beats produced by the sum of two waves with approximately the same frequencies

can be regarded as the variable amplitude of the vibration whose period is $2\pi/\omega$. The oscillation of the amplitude has a large period of $2\pi/\epsilon$, which is called the period of beats. This kind of motion is shown in Fig. 5.4.

Resonance. In the case that the frequency of the forcing function is the same as the natural frequency of the system, that is $\omega = \omega_0$, then the particular solution takes the form

$$x_p(t) = At \cos \omega_0 t + Bt \sin \omega_0 t.$$

Substituting it into the original nonhomogeneous equation, we find $A = 0$, $B = F_0/(2m\omega_0)$. Thus the general solution is given by

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

Because of the presence of the term $t \sin \omega_0 t$, the motion will become unbounded as $t \rightarrow \infty$. This is known as resonance. The phenomenon of resonance is characterized by x_p which is shown in Fig. 5.5. If there is damping, the

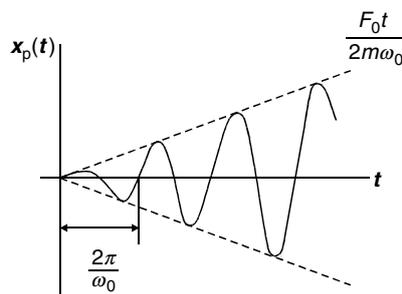


Fig. 5.5. Resonance without damping. When the forcing frequency coincide with the natural frequency, the motion will become unbounded

motion will remain bounded. However, there may still be a large response if the damping is small and ω is close to ω_0 .

5.6.5 Forced Vibration with Viscous Damping

With viscous damping, the equation of motion of the spring–mass system under a harmonic forcing function is given by

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F_0 \cos \omega t. \quad (5.40)$$

The solution of this equation is again the sum of the complementary function and the particular solution. The complementary function satisfies the homogeneous equation

$$m \frac{d^2x_c}{dt^2} + c \frac{dx_c}{dt} + kx_c = 0,$$

which represents free vibrations with damping. As discussed earlier, the free vibration dies out with time under all possible initial conditions. This part of the solution is called transient. The rate at which the transient motion decays depends on the system parameters m, k, c .

The general solution of the equation eventually reduces to the particular solution which represents the steady-state vibration.

The particular solution is expected to have the same frequency as the forcing function, we can write the solution in the following form:

$$x_p(t) = A \cos(\omega t - \phi). \quad (5.41)$$

Substituting it into the equation of motion, we find

$$A [(k - m\omega^2) \cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t.$$

Using the trigonometric identities

$$\begin{aligned} \cos(\omega t - \phi) &= \cos \omega t \cos \phi + \sin \omega t \sin \phi, \\ \sin(\omega t - \phi) &= \sin \omega t \cos \phi - \cos \omega t \sin \phi \end{aligned}$$

and equating the coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides of the resulting equation, we obtain

$$A [(k - m\omega^2) \cos \phi + c\omega \sin \phi] = F_0, \quad (5.42a)$$

$$A [(k - m\omega^2) \sin \phi - c\omega \cos \phi] = 0. \quad (5.42b)$$

It follows from (5.42b) that:

$$(k - m\omega^2) \sin \phi = c\omega \cos \phi,$$

but $\sin^2 \phi = 1 - \cos^2 \phi$, so

$$(k - m\omega^2)^2(1 - \cos^2 \phi) = (c\omega \cos \phi)^2.$$

Therefore

$$\cos \phi = \frac{k - m\omega^2}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}}.$$

It follows that:

$$\sin \phi = \frac{c\omega}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}}.$$

Substituting $\cos \phi$ and $\sin \phi$ into (5.42a), we find

$$A = \frac{F_0}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} = \frac{F_0}{[m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2]^{1/2}}, \quad (5.43)$$

where $\omega_0^2 = k/m$. The particular solution $x_p(t)$ is, therefore, given by

$$x_p(t) = \frac{F_0 \cos(\omega t - \phi)}{[m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2]^{1/2}}, \quad (5.44)$$

where

$$\phi = \tan^{-1} \frac{c\omega}{m(\omega_0^2 - \omega^2)}.$$

Notice that $m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2$ is never zero, even for $\omega = \omega_0$. Hence with damping, the motion is always bounded. However, if the damping is not strong enough, the amplitude can still get to be very large.

To find the maximum amplitude, we take the derivative of A with respect to ω , and set it to zero. This shows that the frequency that makes $\frac{dA}{d\omega} = 0$ must satisfy the equation

$$2m^2(\omega_0^2 - \omega^2) - c^2 = 0.$$

Therefore the maximum amplitude occurs at

$$\omega = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}}. \quad (5.45)$$

Note that for $c^2 > 2m^2\omega_0^2$, no real ω can satisfy this equation. In that case, there will not be any maximum for $\omega \neq 0$. The amplitude is a monotonically decreasing function of the forcing frequency.

However, if $c^2 < 2m^2\omega_0^2$, then there will be a maximum. Substituting (5.45) into the expression of A , we obtain the maximum amplitude

$$A_{\max} = \frac{2mF_0}{c(4m^2\omega_0^2 - c^2)^{1/2}}.$$

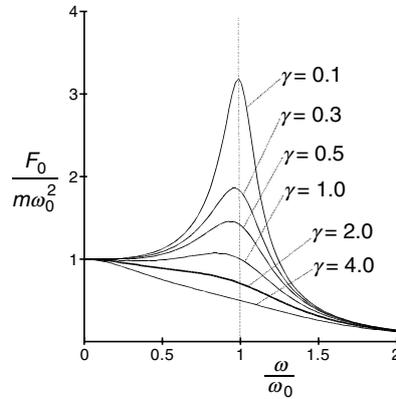


Fig. 5.6. Forced vibration with viscous damping. Amplitude of the steady-state as a function of ω/ω_0 , $\gamma = (c/m\omega_0)^2$ represents the strength of damping. For $\gamma \geq 2$, there is no maximum

To see the relation between the amplitude A and the forcing frequency ω , it is convenient to express A of (5.43) as

$$A = \frac{F_0}{m\omega_0^2 \left[(1 - (\omega/\omega_0)^2)^2 + \gamma(\omega/\omega_0)^2 \right]^{1/2}},$$

where $\gamma = c^2/m^2\omega_0^2$. The graphs of A in units of $F_0/m\omega_0^2$ as functions of ω/ω_0 are shown in Fig. 5.6 for several different values of γ . For $\gamma = 0$, it is the forced vibration without damping and the motion is unbound at $\omega = \omega_0$. For a small γ , the amplitude still has a sharp peak at a frequency slightly less than ω_0 . As γ gets larger, the peak becomes smaller and wider. When $\gamma \geq 2$, there is no longer any maximum.

In designing structures, we want to include sufficient amount of damping to avoid resonance which can lead to disaster. On the other hand, if we design a device to detect periodic force, we would want to choose m, k, c to satisfy (5.45) so that the response of the device to such a force is maximum.

5.7 Electric Circuits

As a second example of application of theory of linear second-order differential equations with constant coefficients, we consider the simple electric circuit shown in Fig. 5.7.

It consists of three kinds of circuit elements; a resistor with a resistance R measured in ohms, an inductor with an inductance L measured in henries, and a capacitor with capacitance C measured in farads. They are connected in series with a source of electromotive force (emf) that supplies at time t

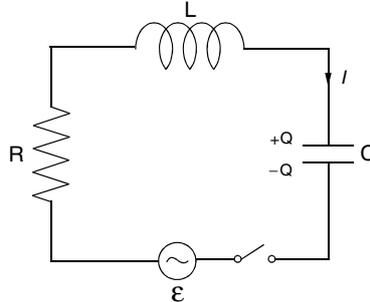


Fig. 5.7. An oscillatory electrical circuit with resistance, inductance, and capacitance

a voltage $V(t)$ measured in volts. The capacitor is a device to store electric charges Q , measured in coulombs. If the switch is closed, there will be current $I(t)$, measured in amperes, flowing in the circuit. In elementary physics, we learned that the voltage drop across the resistor is equal to IR , the voltage drop across the inductor is $L\frac{dI}{dt}$, and the voltage across the capacitor is $\frac{1}{C}Q$. The sum of these is equal to the applied voltage. Therefore

$$L\frac{dI}{dt} + RI + \frac{1}{C}Q = V(t).$$

Furthermore, the rate of increase of the charge Q on the capacitor is, by definition, equal to the current

$$\frac{dQ}{dt} = I.$$

With this relation, we obtain the following second-order linear nonhomogeneous equation for Q :

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t).$$

Suppose the circuit is driven by a generator with a pure cosine wave oscillation, $V(t) = V_0 \cos \omega t$, then the equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_0 \cos \omega t. \quad (5.46)$$

5.7.1 Analog Computation

We see that the equation describing an LRC circuit is exactly the same as (5.40), the equation describing the forced vibration of a spring-mass system with viscous damping. The fact that the same differential equation serves to describe two entirely different physical phenomena is a striking example of the

Table 5.2. The analogy between mechanical and electrical systems

Mechanical Property	Electrical Property
$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F_0 \cos \omega t$	$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C}Q = V_0 \cos \omega t$
displacement x	charge Q
velocity $v = \frac{dx}{dt}$	current $I = \frac{dQ}{dt}$
mass m	inductance L
spring constant k	inverse capacitance $1/C$
damping coefficient c	resistance R
applied force $F_0 \cos \omega t$	applied voltage $V_0 \cos \omega t$
resonant frequency $\omega_0^2 = \frac{k}{m}$	resonant frequency $\omega_0^2 = \frac{1}{LC}$

unifying role of mathematics in natural sciences. With appropriate substitutions, the solution of (5.40) can be applied to electric circuits. The correspondence between the electrical and mechanical cases are shown in Table 5.2.

The correspondence between mechanical and electrical properties can also be used to construct an electrical model of a given mechanical system. This is a very useful way to predict the performance of a mechanical system, since the electrical elements are inexpensive and electrical measurements are usually very accurate. The method of computing the motion of a mechanical system from an electrical circuit is known as analog computation.

By directly converting $x_p(t)$ of (5.44) into its electrical equivalent, the steady-state solution of (5.46) is found to be

$$Q(t) = \frac{V_0 \cos(\omega t - \phi)}{\left[\left(\frac{1}{C} - \omega^2 L\right)^2 + (\omega R)^2\right]^{1/2}},$$

$$\phi = \tan^{-1} \frac{\omega R}{\frac{1}{C} - \omega^2 L} = \tan^{-1} \frac{R}{\frac{1}{\omega C} - \omega L}.$$

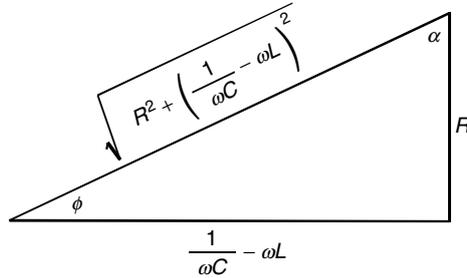
Generally, it is the current that is of primary interests, so we differentiate Q with respect to t to get the steady-state current

$$I(t) = \frac{dQ}{dt} = \frac{-\omega V_0 \sin(\omega t - \phi)}{\left[\left(\frac{1}{C} - \omega^2 L\right)^2 + (\omega R)^2\right]^{1/2}} = \frac{-V_0 \sin(\omega t - \phi)}{\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]^{1/2}}.$$

To see more clearly the phase relation between the current $I(t)$ and the applied voltage $V(t) = \cos \omega t$, we would like to express the current also in terms of a cosine function. This can be done by noting that

$$\tan \phi = \frac{R}{\frac{1}{\omega C} - \omega L}$$

can be expressed geometrically in the following triangle.



It is clear that $\phi = \frac{\pi}{2} - \alpha$ and

$$\tan \alpha = \frac{\frac{1}{\omega C} - \omega L}{R}.$$

Since $\sin(\omega t - \phi) = \sin(\omega t - \frac{\pi}{2} + \alpha) = -\cos(\omega t + \alpha)$, it follows that:

$$I(t) = \frac{V_0 \cos(\omega t + \alpha)}{\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]^{1/2}}.$$

For reasons that will soon be clear, often $I(t)$ is written in still another form:

$$I(t) = \frac{V_0 \cos(\omega t - \beta)}{\left[\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2\right]^{1/2}}, \quad (5.47)$$

where $\beta = -\alpha$, and

$$\tan \beta = \tan(-\alpha) = -\tan \alpha = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

5.7.2 Complex Solution and Impedance

The particular solution of (5.46) can be found by the complex exponential method. This method offers some computational and conceptual advantages. We can replace $V_0 \cos \omega t$ by $V_0 e^{i\omega t}$ and solve the equation

$$L \frac{d^2 Q_c}{dt^2} + R \frac{dQ_c}{dt} + \frac{1}{C} Q_c = V_0 e^{i\omega t}. \quad (5.48)$$

The real part of the solution Q_c is the charge Q , and the real part of I_c , defined as $\frac{d}{dt} Q_c$, is the current I . The time dependence of charges and currents must also be in the form of $e^{i\omega t}$

$$Q_c = \widehat{Q} e^{i\omega t}, \quad I_c = \widehat{I} e^{i\omega t},$$

where \widehat{Q} and \widehat{I} are complex but independent of t . Since

$$\frac{dQ_c}{dt} = i\omega Q_c, \quad \frac{d^2Q_c}{dt^2} = -\omega^2 Q_c,$$

the differential equation (5.48) becomes the algebraic equation

$$(-\omega^2 L + i\omega R + \frac{1}{C})Q_c = V_0 e^{i\omega t}.$$

Clearly

$$Q_c = \frac{V_0 e^{i\omega t}}{-\omega^2 L + i\omega R + \frac{1}{C}},$$

and

$$I_c = \frac{dQ_c}{dt} = \frac{i\omega V_0 e^{i\omega t}}{-\omega^2 L + i\omega R + \frac{1}{C}} = \frac{V_0 e^{i\omega t}}{R + i(\omega L - \frac{1}{\omega C})}. \quad (5.49)$$

Writing the denominator in the polar form

$$R + i(\omega L - \frac{1}{\omega C}) = \left[R^2 + (\omega L - \frac{1}{\omega C})^2 \right]^{1/2} e^{i\beta},$$

$$\beta = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R},$$

We see that

$$I_c = \frac{V_0 e^{i\omega t}}{\left[R^2 + (\omega L - \frac{1}{\omega C})^2 \right]^{1/2} e^{i\beta}} = \frac{V_0 e^{i(\omega t - \beta)}}{\left[R^2 + (\omega L - \frac{1}{\omega C})^2 \right]^{1/2}}.$$

The real part of I_c is

$$I = \frac{V_0 \cos(\omega t - \beta)}{\left[R^2 + (\omega L - \frac{1}{\omega C})^2 \right]^{1/2}},$$

which is identical to (5.47).

In electrical engineering, it is customary to define $V_0 e^{i\omega t}$ as the complex voltage V_c , and to define

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

as the complex impedance Z . With these notations, (5.49) can be written in the form

$$I_c = \frac{V_c}{Z}.$$

Note that if the circuit element had consisted of the resistance R alone, the impedance would be equal simply to R , so this relation would resemble Ohm's law for a direct current circuit: $V = RI$. Thus the role the impedance plays

in an alternating circuit with a sinusoidal voltage is exactly the same as the resistor in a direct current circuit.

It is a simple matter to show that if the circuit element consists only of the inductance L , the impedance is simply $i\omega L$. Similarly, with only capacitance C , the impedance is just $1/(i\omega C)$. Thus we see that when electrical elements are connected in series, the corresponding impedances combine just as simple resistances do.

In a similar way, we can show that when electrical elements are connected in parallel, the corresponding impedances also combine just as simple resistances do. For example, if R, L, C are connected in parallel, the complex current can be found by dividing the complex voltage by the simple impedance Z defined by the relation.

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{i\omega L} + i\omega C.$$

The real part of the result is the current in this AC circuit. This makes it very easy to determine the steady state behavior of an electrical system.

5.8 Systems of Simultaneous Linear Differential Equations

In many applications, it is necessary to simultaneously consider several dependent variables, each depending on the same independent variable, usually time t . The mathematical model is generally a system of linear differential equations. The elementary approach of solving systems of differential equations is to eliminate the dependent variables one by one through combining pairs of equations, until there is only one equation left containing one dependent variable. This equation will usually be of higher order, and can be solved by the methods we have discussed. Once this equation is solved, the other dependent variables can be found in turn. This method is similar to the solution of systems of simultaneous algebraic equations.

A closely related method is to find the eigenvalues of the matrix formed by the differential equations. This method provides a mathematical framework for the discussion of normal frequencies of the system, which are physically important.

5.8.1 The Reduction of a System to a Single Equation

Let us solve the following system of equations with two dependent variables $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = -2x + y, \tag{5.50a}$$

$$\frac{dy}{dt} = -4x + 3y + 10 \cos t \tag{5.50b}$$

with the initial conditions

$$x(0) = 0, \quad y(0) = -1.$$

From the first equation, we have

$$y = \frac{dx}{dt} + 2x, \quad \frac{dy}{dt} = \frac{d^2x}{dt^2} + 2\frac{dx}{dt}.$$

Substitute them into the second equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = -4x + 3\left(\frac{dx}{dt} + 2x\right) + 10 \cos t$$

or

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 10 \cos t. \quad (5.51)$$

This is an ordinary second-order nonhomogeneous differential equation, the complementary function $x_c(t)$ and the particular solution $x_p(t)$ are found to be, respectively, $c_1e^{-t} + c_2e^{2t}$ and $-3 \cos t - \sin t$. Therefore

$$x = x_c + x_p = c_1e^{-t} + c_2e^{2t} - 3 \cos t - \sin t.$$

The solution for $y(t)$ is then given by

$$y = \frac{dx}{dt} + 2x = c_1e^{-t} + 4c_2e^{2t} - 7 \cos t + \sin t.$$

The constants c_1, c_2 are determined by the initial conditions

$$\begin{aligned} x(0) &= c_1 + c_2 - 3 = 0, \\ y(0) &= c_1 + 4c_2 - 7 = -1, \end{aligned}$$

which gives $c_1 = 2, c_2 = 1$. Thus

$$\begin{aligned} x(t) &= 2e^{-t} + e^{2t} - 3 \cos t - \sin t, \\ y(t) &= 2e^{-t} + 4e^{2t} - 7 \cos t + \sin t. \end{aligned}$$

If the number of coupled equations is small (2 or 3), the simplest method of solving the problem is this kind of direct substitution. However, for a larger system, one may prefer the more systematic approach of Sect. 5.8.2.

5.8.2 Cramer's Rule for Simultaneous Differential Equations

We will use the same example of the last section to illustrate this method. First, use the notation D to represent $\frac{d}{dt}$, and write the set of equations (5.50) as

$$(D + 2)x - y = 0, \quad (5.52a)$$

$$4x + (D - 3)y = 10 \cos t. \quad (5.52b)$$

Recall that for a system of algebraic equations

$$\begin{aligned}a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2,\end{aligned}$$

the solution can be obtained by the Cramer's rule

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

We can use the same formalism to solve a system of differential equations. That is, $x(t)$ of (5.52) can be written as

$$x = \frac{\begin{vmatrix} 0 & -1 \\ 10 \cos t & (D+3) \end{vmatrix}}{\begin{vmatrix} (D+2) & -1 \\ 4 & (D-3) \end{vmatrix}},$$

or

$$\begin{vmatrix} (D+2) & -1 \\ 4 & (D-3) \end{vmatrix} x = \begin{vmatrix} 0 & -1 \\ 10 \cos t & (D-3) \end{vmatrix}.$$

Expanding the determinant, we have

$$[(D+2)(D-3) + 4]x = 10 \cos t.$$

This means

$$(D^2 - D - 2)x = 10 \cos t,$$

which is identical to (5.51) of the last section. Proceeding in exactly the same way as in the last section, we find

$$x(t) = x_c + x_p = c_1 e^{-t} + c_2 e^{2t} - 3 \cos t - \sin t.$$

Substituting it into the original differential equation, $y(t)$ is found to be

$$y(t) = c_1 e^{-t} + 4c_2 e^{2t} - 7 \cos t + \sin t.$$

An alternative way of finding $y(t)$ is to note that

$$y = \frac{\begin{vmatrix} (D+2) & 0 \\ 4 & 10 \cos t \end{vmatrix}}{\begin{vmatrix} (D+2) & -1 \\ 4 & (D-3) \end{vmatrix}}$$

or

$$\begin{vmatrix} (D+2) & -1 \\ 4 & (D-3) \end{vmatrix} y = \begin{vmatrix} (D+2) & 0 \\ 4 & 10 \cos t \end{vmatrix}.$$

Expanding the determinant, we have

$$(D^2 - D - 2)y = 20 \cos t - 10 \sin t.$$

The solution of this equation is

$$y(t) = y_c + y_p = k_1 e^{-t} + k_2 e^{2t} - 7 \cos t + \sin t.$$

Note that the complementary functions x_c and y_c satisfy the same homogeneous differential equation

$$(D^2 - D - 2)x_c = 0 \quad (D^2 - D - 2)y_c = 0.$$

Since we have already written $x_c = c_1 e^{-t} + c_2 e^{2t}$, we must avoid using c_1 and c_2 as the constants in y_c . That is, in $y_c = k_1 e^{-t} + k_2 e^{2t}$, k_1 and k_2 are not necessarily equal to c_1 and c_2 , because there is no reason that they should be equal. To find the relationship between them, we have to substitute $x(t)$ and $y(t)$ back into one of the original differential equations. For example, substituting them back into $(D+2)x = y$, we have

$$c_1 e^{-t} + 4c_2 e^{2t} - 7 \cos t + \sin t = k_1 e^{-t} + k_2 e^{2t} - 7 \cos t + \sin t.$$

Therefore

$$k_1 = c_1, \quad k_2 = 4c_2.$$

Thus we obtain the same result as before.

It is seen that after the first dependent variable $x(t)$ is found from Cramer's rule, it is simpler to find the second dependent variable $y(t)$ by direct substitution. If we continue to use Cramer's rule to find $y(t)$, we will introduce some additional constants which must be eliminated by substituting both $x(t)$ and $y(t)$ back into the original differential equation.

5.8.3 Simultaneous Equations as an Eigenvalue Problem

A system of simultaneous differential equations can be solved as an eigenvalue problem in matrix theory. We will continue to use the same example to illustrate the procedures of this method. First write the set of equations (5.50) in the following form:

$$\begin{aligned} -2x + y &= x', \\ -4x + 3y &= y' - 10 \cos t. \end{aligned}$$

With matrix notation, they become

$$\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' - 10 \cos t \end{pmatrix}.$$

Let

$$x = x_c + x_p, \quad y = y_c + y_p.$$

The complementary functions x_c and y_c satisfy the equation

$$\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} x'_c \\ y'_c \end{pmatrix}, \quad (5.53)$$

and the particular solutions x_p and y_p satisfy the equation

$$\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} x'_p \\ y'_p - 10 \cos t \end{pmatrix}. \quad (5.54)$$

Since these are linear equations with constant coefficients, we assume

$$x_c = c_1 e^{\lambda t}, \quad y_c = c_2 e^{\lambda t},$$

so

$$x'_c = \frac{dx_c}{dt} = \lambda c_1 e^{\lambda t}, \quad y'_c = \frac{dy_c}{dt} = \lambda c_2 e^{\lambda t}.$$

It follows that the matrix equation for the complementary functions is given by:

$$\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \lambda c_1 e^{\lambda t} \\ \lambda c_2 e^{\lambda t} \end{pmatrix}.$$

This is an eigenvalue problem

$$\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

with eigenvalue λ and eigenvector $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Therefore, λ must satisfy the secular equation

$$\begin{vmatrix} -2 - \lambda & 1 \\ -4 & 3 - \lambda \end{vmatrix} = 0$$

or

$$(-2 - \lambda)(3 - \lambda) + 4 = 0.$$

The two roots λ_1, λ_2 of this equation are easily found to be

$$\lambda_1 = -1, \quad \lambda_2 = 2.$$

Corresponding to each λ_i , there is an eigenvector $\begin{pmatrix} c_1^i \\ c_2^i \end{pmatrix}$. The coefficients c_1^i and c_2^i are not independent of each other, they must satisfy the equation

$$\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} c_1^i \\ c_2^i \end{pmatrix} = \lambda_i \begin{pmatrix} c_1^i \\ c_2^i \end{pmatrix}.$$

It follows from this equation that for $\lambda_1 = -1$, $c_2^1 = c_1^1$, and for $\lambda_2 = 2$, $c_2^2 = 4c_1^2$. Therefore, other than some multiplicative constants, the eigenvector for $\lambda = -1$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and for $\lambda = 2$ is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

The complementary functions x_c and y_c are given by the linear combinations of these two sets of solutions,

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{2t}.$$

For the particular solution, because of the nonhomogeneous term $10 \cos t$, we can assume $x_p = A \cos t + B \sin t$ and $y_p = C \cos t + D \sin t$. However, it is less cumbersome to make use of the fact that $10 \cos t$ is the real part of $10e^{it}$. We can assume x_p is the real part of $A_c e^{it}$ and y_p is the real part of $B_c e^{it}$, where A_c and B_c are complex numbers. With these assumptions, (5.54) becomes

$$\begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} A_c e^{it} \\ B_c e^{it} \end{pmatrix} = \begin{pmatrix} iA_c e^{it} \\ iB_c e^{it} - 10e^{it} \end{pmatrix}.$$

Thus

$$\begin{aligned} -2A_c + B_c &= iA_c, \\ -4A_c + 3B_c &= iB_c - 10, \end{aligned}$$

which yields

$$A_c = -3 + i, \quad B_c = -7 - i.$$

Therefore

$$\begin{aligned} x_p &= \operatorname{Re}(A_c e^{it}) = -3 \cos t - \sin t, \\ y_p &= \operatorname{Re}(B_c e^{it}) = -7 \cos t + \sin t. \end{aligned}$$

Finally, we have the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} - 3 \cos t - \sin t \\ c_1 e^{-t} + 4c_2 e^{2t} - 7 \cos t + \sin t \end{pmatrix},$$

which is what we had before.

5.8.4 Transformation of an n th Order Equation into a System of n First-Order Equations

We have seen that a system of equations can be reduced to a single equation of higher order. The reverse is also true. Any n th-order differential equation can always be transformed into a simultaneous n first-order equations. Let us

use this method to solve the second-order differential equation for the damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \frac{k}{m}x = 0.$$

Let

$$x_1 = x, \quad x_2 = \frac{dx_1}{dt}.$$

It follows:

$$\frac{dx_2}{dt} = \frac{d^2x_1}{dt^2} = -\frac{c}{m} \frac{dx_1}{dt} - \frac{k}{m}x_1.$$

Thus the second-order equation can be written as a set of two first-order equations:

$$\begin{aligned} x_2 &= \frac{dx_1}{dt}, \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 &= \frac{dx_2}{dt}. \end{aligned}$$

With matrix notation, we have

$$\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}.$$

Since the coefficients are constants, we can assume

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t},$$

thus we have the eigenvalue problem

$$\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The eigenvalues λ can be found from the characteristic equation

$$\begin{vmatrix} 0 - \lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{vmatrix} = 0.$$

The two roots of this equation are

$$\lambda_1, \lambda_2 = \frac{1}{2m}(-c \pm \sqrt{c^2 - 4km}).$$

Thus the general solution of the original problem is given by

$$x(t) = x_1 = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

This result is identical to (5.37).

The fact that a linear differential equation of n th-order can be transformed into a system of n coupled first-order equations is of some importance because mathematically one can show that there is a unique solution for a linear first-order system, provided the initial conditions are specified. However, we are not too concerned with uniqueness and existence because in physical applications, the mathematical model, if formulated correctly, must have a solution.

5.8.5 Coupled Oscillators and Normal Modes

The motion of a harmonic oscillator is described by a second-order differential equation. Its solution shows that the motion is characterized by a single natural frequency. A real physical system usually has many different characteristic frequencies. A vibration with any of these frequencies is called a normal mode of the system. The motion of the system is generally a linear combination of these normal modes.

A simple example is the system of two coupled oscillators shown in Fig. 5.8.

The system consists of two identical mass-spring oscillators of mass m and spring constant k . The two masses rest on a frictionless table and are connected by a spring with spring constant K . When the displacements x_A and x_B are zero, the springs are neither stretched or compressed. At any moment, the connecting spring is stretched an amount $x_A - x_B$ and therefore pulls or pushes on A and B with a force whose magnitude is $K(x_A - x_B)$. Thus the magnitude of the restoring force on A is

$$-kx_A - K(x_A - x_B).$$

The force on B must be

$$-kx_B + K(x_A - x_B).$$

Therefore the equations of motion for A and B are

$$\begin{aligned} -kx_A - K(x_A - x_B) &= m \frac{d^2x_A}{dt^2}, \\ -kx_B + K(x_A - x_B) &= m \frac{d^2x_B}{dt^2}. \end{aligned}$$

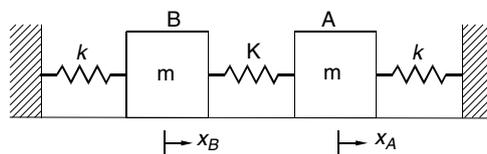


Fig. 5.8. Two coupled oscillators

With matrix notation, these equations can be written as

$$\begin{pmatrix} -\frac{k}{m} - \frac{K}{m} & \frac{K}{m} \\ \frac{K}{m} & -\frac{k}{m} - \frac{K}{m} \end{pmatrix} \begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} \frac{d^2 x_A}{dt^2} \\ \frac{d^2 x_B}{dt^2} \end{pmatrix}.$$

To simplify the writing, let $\omega_0^2 = k/m$ and $\omega_1^2 = K/m$. With the assumption $x_A = ae^{\lambda t}$ and $x_B = be^{\lambda t}$, the last equation becomes

$$\begin{pmatrix} -\omega_0^2 - \omega_1^2 & \omega_1^2 \\ \omega_1^2 & -\omega_0^2 - \omega_1^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} e^{\lambda t} = \lambda^2 \begin{pmatrix} a \\ b \end{pmatrix} e^{\lambda t}.$$

This can be regarded as an eigenvalue problem. The secular equation

$$\begin{vmatrix} -\omega_0^2 - \omega_1^2 - \lambda^2 & \omega_1^2 \\ \omega_1^2 & -\omega_0^2 - \omega_1^2 - \lambda^2 \end{vmatrix} = 0$$

shows that λ^2 satisfies the equation

$$(-\omega_0^2 - \omega_1^2 - \lambda^2)^2 - \omega_1^4 = 0$$

or

$$\lambda^2 = -\omega_0^2 - \omega_1^2 \pm \omega_1^2.$$

Thus

$$\lambda^2 = -\omega_0^2, \quad \lambda^2 = -(\omega_0^2 + 2\omega_1^2).$$

The four roots of λ are

$$\lambda_1, \lambda_2 = \pm i\omega_0, \quad \lambda_3, \lambda_4 = \pm i\omega_c,$$

where

$$\omega_c = \sqrt{\omega_0^2 + 2\omega_1^2}.$$

These frequencies, ω_0 and ω_c are known as the normal frequencies of the system. The amplitudes a and b are not independent of each other, since they must satisfy the equation

$$\begin{pmatrix} -\omega_0^2 - \omega_1^2 - \lambda^2 & \omega_1^2 \\ \omega_1^2 & -\omega_0^2 - \omega_1^2 - \lambda^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

Thus for $\lambda = \lambda_1, \lambda_2 = \pm i\omega_0$, so $\lambda^2 = -\omega_0^2$, the amplitudes a and b must satisfy

$$\begin{pmatrix} -\omega_0^2 - \omega_1^2 + \omega_0^2 & \omega_1^2 \\ \omega_1^2 & -\omega_0^2 - \omega_1^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

It follows that $a = b$.

Similarly, for $\lambda = \lambda_3, \lambda_4 = \pm i\omega_c$, the relation between a and b is given by

$$\begin{pmatrix} -\omega_0^2 - \omega_1^2 + \omega_0^2 + 2\omega_1^2 & \omega_1^2 \\ \omega_1^2 & -\omega_0^2 - \omega_1^2 + \omega_0^2 + 2\omega_1^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0,$$

which gives $b = -a$.

The displacements x_A and x_B are given by a linear combinations of these four solutions,

$$\begin{aligned} x_A &= a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t} + a_4 e^{\lambda_4 t}, \\ x_B &= a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} - a_3 e^{\lambda_3 t} - a_4 e^{\lambda_4 t}, \end{aligned}$$

where we have substituted a_1, a_2 for b_1, b_2 , and $-a_3, -a_4$ for b_3, b_4 . Since $\lambda_1 = i\omega_0$, $\lambda_2 = -i\omega_0$,

$$a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} = a_1 e^{i\omega_0 t} + a_2 e^{-i\omega_0 t} = C \cos(\omega_0 t + \alpha).$$

Similarly

$$a_3 e^{\lambda_3 t} + a_4 e^{\lambda_4 t} = D \cos(\omega_c t + \beta).$$

Thus x_A and x_B can be written as

$$\begin{aligned} x_A &= C \cos(\omega_0 t + \alpha) + D \cos(\omega_c t + \beta), \\ x_B &= C \cos(\omega_0 t + \alpha) - D \cos(\omega_c t + \beta). \end{aligned}$$

The four constants a_1, a_2, a_3, a_4 (or C, α, D, β) depend on the initial conditions. It is seen that both x_A and x_B are given by some combination of the vibrations of two normal frequencies ω_0 and ω_c .

Suppose the motion is started when we pull both A and B toward the same direction by an equal amount x_0 and then release them from rest. The distance between A and B equals the relaxed length of the coupling spring and therefore it exerts no force on each mass. Thus A and B will oscillate in phase with the same natural frequency ω_0 as if they were not coupled. Mathematically, we see that is indeed the case. With the initial conditions

$$x_A(0) = x_B(0) = x_0 \quad \text{and} \quad \left. \frac{dx_A}{dt} \right|_{t=0} = 0, \quad \left. \frac{dx_B}{dt} \right|_{t=0} = 0,$$

one can easily show that $C = x_0$, $D = 0$, $\alpha = \beta = 0$. Therefore

$$x_A = x_0 \cos \omega_0 t, \quad x_B = x_0 \cos \omega_0 t.$$

This represents a normal mode of the coupled system. Once the system is vibrating with a normal frequency, it will continue to vibrate with that frequency.

Suppose initially we pull A and B in opposite direction by the same amount x_m and then release them. The symmetry of the arrangement tells us that A and B will be mirror images of each other. They will vibrate with certain frequency, which we might expect it to be ω_c , since ω_c is the only

other normal frequency of the system. This is indeed the case. Since with the initial conditions

$$x_A(0) = -x_B(0) = x_m, \quad \text{and} \quad \left. \frac{dx_A}{dt} \right|_{t=0} = 0, \quad \left. \frac{dx_B}{dt} \right|_{t=0} = 0,$$

one can show that $C = 0$, $D = x_m$, $\alpha = \beta = 0$. Therefore

$$x_A = x_m \cos \omega_c t, \quad x_B = -x_m \cos \omega_c t.$$

They oscillate with the same frequency ω_c but they are always 180° out of phase. This constitutes the second normal mode of the system. The general motion is a linear combination of these two modes.

For a real molecule or crystal, there will be many normal modes. Each normal mode corresponds to a certain symmetry of the structure. The fact that these modes can be excited by their corresponding normal frequencies is widely used in scientific applications.

5.9 Other Methods and Resources for Differential Equations

Many readers probably had previously taken a course in ordinary differential equations. Here we just give a review so that even those who did not have previous exposure can gain enough background to continue. The literature of the theory and applications of differential equations is vast. Our discussion is far from complete.

Among the methods we have not yet discussed are the Laplace transform, Fourier analysis and power series solutions.

The Laplace transform is especially useful in solving problems with nonhomogeneous terms of a discontinuous or impulsive nature. In Chap. 6 we will study these problems in detail.

If the nonhomogeneous term is periodic but not sinusoidal, Fourier series method is particularly convenient. We will discuss this method after we study the Fourier series.

In general, a differential equation with variable coefficients cannot be solved by the methods of this chapter. The usual procedure for such equations is to obtain solutions in the form of infinite series. This is known as series method. Some most important equations in physics and engineering lead us to this type equations. The series so obtained can be taken as definitions of new functions. Some important ones are named and tabulated. We shall study this method in the chapter on special functions.

In addition, differential equations can be solved numerically. Sometimes this is the only way to solve the equation. Digital computers have made numerical solutions readily available. There are several computer programs for the integration of ordinary differential equations in "Numerical Recipes"

by William H. Press, Brian P. Flannery, Saul A. Teukolsky and William T. Vetterling (Cambridge University Press, 1986). For a discussion of the numerical methods, see R.J. Rice, “Numerical Methods, Software and Analysis” (McGraw-Hill, New York, 1983).

Finally it should be mentioned that a number of commercial computer packages are available to perform algebraic manipulations, including solving differential equations. They are called computer algebraic systems, some prominent ones are Matlab, Maple, Mathematica, MathCad and MuPAD.

This book is written with the software “Scientific WorkPlace”, which also provides an interface to MuPAD. (Before version 5, it also came with Maple). Instead of requiring the user to adhere to a rigid syntax, the user can use natural mathematical notations. For example, to solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x + y$$

all you have to do is (1) type this equation in the math-mode, and (2) click on the “Compute” button, and (3) click on the “Solve ODE” button in the pull-down menu, and (4) click on the “Exact” button in the submenu. The program will return with

$$\text{Exact solution is: } C_1 e^{x(\frac{1}{2}\sqrt{5}-\frac{1}{2})} - x + C_2 e^{x(-\frac{1}{2}\sqrt{5}-\frac{1}{2})} - 1.$$

Unfortunately, not every problem can be solved by a computer algebraic system. Sometimes it fails to find the solution. Even worse, for a variety of reasons, the intention of the user is sometimes misinterpreted, and the computer returns with an answer to a wrong problem without the user knowing it. Therefore these systems must be used with caution.

Exercises

1. Find the general solutions of the following separable differential equations:

(a) $xy' + y + 3 = 0$,

(b) $2yy' + 4x = 0$.

Ans. (a) $x(y + 3) = c$, (b) $2x^2 + y^2 = c$.

2. Find the specific solutions of the following separable differential equations:

(a) $\frac{dy}{dx} = \frac{x(1+y^2)}{y(1+x^2)}$, $y(0) = 1$,

(b) $ye^{x+y}dy = dx$, $y(0) = 0$.

Ans. (a) $1 + y^2 = 2(1 + x^2)$, (b) $(1 - y)e^y = e^{-x}$.

3. Change the following equations into separable differential equations and find the general solutions:

(a) $xyy' = y^2 - x^2$,

(b) $\frac{dy}{dx} = \frac{x-y}{x+y}$.

Ans. (a) $\ln x + \frac{y^2}{2x^2} = c$, (b) $y^2 + 2xy - x^2 = c$.

4. Show that the following differential equations are exact and find the general solutions:

(a) $(2xy - \cos x)dx + (x^2 - 1)dy = 0$,

(b) $(2x + e^y)dx + xe^y dy = 0$.

Ans. (a) $x^2y - \sin x - y = c$, (b) $x^2 + xe^y = c$.

5. Solve the following differential equations by first finding an integrating factor:

(a) $2(y^3 - 2)dx + 3xy^2dy = 0$,

(b) $(y + x^4)dx - xdy = 0$.

Ans. (a) $\mu = x$, $x^2y^3 - 2x^2 = c$, (b) $\mu = 1/x^2$, $\frac{x^3}{3} - \frac{y}{x} = c$.

6. Find the general solutions of the following first-order linear differential equations:

(a) $y' + y = x$,

(b) $xy' + (1+x)y = e^{-x}$.

Ans. (a) $y = x - 1 + ce^{-x}$, (b) $y = e^{-x} + ce^{-x}/x$.

7. Find the specific solutions of the following first-order linear differential equations:

(a) $y' - y = 1 - x$, $y(0) = 1$.

(b) $y' + \frac{1}{x}y = 3x^2$, $y(1) = 5$.

Ans. (a) $y = x + e^x$, (b) $y = \frac{3}{4}x^3 + \frac{17}{4}x^{-1}$.

8. The RL circuit is described by the equation

$$L \frac{di}{dt} + Ri = A \cos t, \quad i(0) = 0$$

where i is current. Find the current i as a function of time t .

Ans. $i(t) = \frac{AR}{R^2 + L^2} \left[\cos t + \frac{L}{R} \sin t - e^{-Rt/L} \right]$.

9. Find the general solutions of the following homogeneous second-order differential equations:

(a) $y'' - k^2y = 0$,

(b) $y'' - (a+b)y' + aby = 0$,

(c) $y'' + 2ky' + k^2y = 0$.

Ans. (a) $y(x) = c_1 e^{kx} + c_2 e^{-kx}$, (b) $y(x) = c_1 e^{ax} + c_2 e^{bx}$, (c) $y(x) = c_1 e^{-kx} + c_2 x e^{-kx}$.

10. Find the specific solutions of the following homogeneous second-order differential equations:

(a) $y'' - 2ay' + (a^2 + b^2)y = 0$, $y(0) = 0$, $y'(0) = 1$.

(b) $y'' + 4y = \sin x$, $y(0) = 0$, $y'(0) = 0$.

(c) $y''' + y' = e^{2z}$, $y(0) = y'(0) = y''(0) = 0$.

Ans. (a) $y(x) = \frac{1}{b} e^{ax} \sin bx$, (b) $y(x) = \frac{1}{3} \sin x - \frac{1}{6} \sin 2x$,

(c) $y(x) = \frac{1}{10} e^{2x} - \frac{1}{5} \sin x + \frac{2}{5} \cos x - \frac{1}{2}$.

11. Find the general solutions of the following nonhomogeneous differential equations:

(a) $y'' + k^2 y = a$,

(b) $y'' - 4y = x$,

(c) $y'' - 2y' + y = 3x^2 - 12x + 7$.

Ans. (a) $y(x) = c_1 \sin kx + c_2 \cos kx + a/k^2$, (b) $y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}x$,

(c) $y(x) = (c_1 + c_2 x)e^x + 3x^2 + 1$.

12. Find the general solutions of the following nonhomogeneous differential equations:

(a) $y'' - 3y' + 2y = e^{2x}$,

(b) $y'' - 6y' + 9y = 4e^{3x}$,

(c) $y'' + 9y = \cos(3x)$.

Ans. (a) $y = c_1 e^x + c_2 e^{2x} + x e^{2x}$, (b) $y = c_1 e^{3x} + c_2 x e^{3x} + 2x^2 e^{3x}$,

(c) $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{6}x \sin(3x)$.

13. Find the specific solutions of the following nonhomogeneous differential equations:

(a) $y'' + y' = x^2 + 2x$, $y(0) = 4$, $y'(0) = -2$.

(b) $y'' - 4y' + 4y = 6 \sin x - 8 \cos x$, $y(0) = 3$, $y'(0) = 4$.

(c) $y'' - 4y = 8e^{2x}$, $y(0) = 4$, $y'(0) = 6$.

Ans. (a) $y = \frac{1}{3}x^3 + 2e^{-x} + 2$, (b) $y = (3 - 4x)e^{2x} + 2 \sin x$, (c) $y = 3e^{2x} + e^{-2x} + 2xe^{2x}$.

14. Solve the following differential equations with the method of variation of parameters:

(a) $y'' + y = \sec x$,

(b) $y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}$.

Ans. (a) $y = c_1 \cos x + c_2 \sin x + \cos x \ln |\cos x| + x \sin x$,

(b) $y = c_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x} \ln x$.

15. Solve the following set of simultaneous linear differential equation:

$$\begin{aligned}y'(x) - z'(x) - 2y(x) + 2z(x) &= 1 - 2x, \\y''(x) + 2z'(x) + y(x) &= 0, \\y(0) = z(0) = y'(0) &= 0.\end{aligned}$$

Ans. $y(x) = -2e^{-x} - 2xe^{-x} + 2$; $z(x) = -2e^{-x} - 2xe^{-x} + 2 - x$.

16. Solve the following set of simultaneous linear differential equation:

$$\begin{aligned}y'(x) + z'(x) + y(x) + z(x) &= 1, \\y'(x) - y(x) - 2z(x) &= 0, \\y(0) = 1, z(0) &= 0.\end{aligned}$$

Ans. $y(x) = 2 - e^{-x}$; $z(x) = e^{-x} - 1$.

17. In strength of materials, you will encounter the equation

$$\frac{d^4y}{dx^4} = -4a^4y$$

where a is a positive constant. Find the general solution of this equation. ($y(x)$ with four constants).

Ans. $y(x) = e^{ax}(c_1 \cos ax + c_2 \sin ax) + e^{-ax}(c_3 \cos ax + c_4 \sin ax)$

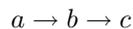
18. Solve

$$\begin{aligned}y''(t) + y(t) &= \begin{cases} 1 - \frac{t^2}{\pi^2} & \text{if } 0 \leq t \leq \pi, \\ 0 & \text{if } t > \pi, \end{cases} \\y(0) = y'(0) &= 0.\end{aligned}$$

This may be interpreted as an undamped system on which a force acts during some interval of time, for instance, the force acting on a gun barrel when a shell is fired, the barrel being braked by heavy springs (and then damped by a dashpot which we disregard for simplicity). Hint: at $t = \pi$ both y and y' must be continuous.

Ans. $y(t) = \begin{cases} -(1 + \frac{2}{\pi^2}) \cos t + (1 + \frac{2}{\pi^2}) - \frac{1}{\pi^2} t^2 & \text{for } 0 \leq t \leq \pi \\ [1 - 2(1 + \frac{2}{\pi^2})] \cos t + \frac{2}{\pi} \sin t & \text{for } t \geq \pi. \end{cases}$

19. If $N_a(t)$, $N_b(t)$, $N_c(t)$ represent the number of nuclei of three radioactive substances which decay according to the scheme



with decay constants λ_a and λ_b , the substance c is considered stable. Then the functions are known to obey the system of differential equations

$$\begin{aligned}\frac{dN_a}{dt} &= -\lambda_a N_a, \\ \frac{dN_b}{dt} &= -\lambda_b N_b + \lambda_a N_a, \\ \frac{dN_c}{dt} &= \lambda_b N_b.\end{aligned}$$

Assuming that $N_a(0) = N_0$, and $N_b(0) = N_c(0) = 0$, find $N_a(t)$, $N_b(t)$ and $N_c(t)$ as functions of time t .

$$\text{Ans. } N_a = N_0 e^{-\lambda_a t}; \quad N_b = N_0 \left[\frac{\lambda_a}{\lambda_b - \lambda_a} e^{-\lambda_a t} - \frac{\lambda_a}{\lambda_b - \lambda_a} e^{-\lambda_b t} \right]; \quad N_c = N_0 \left[1 - \frac{\lambda_b}{\lambda_b - \lambda_a} e^{-\lambda_a t} + \frac{\lambda_a}{\lambda_b - \lambda_a} e^{-\lambda_b t} \right].$$

20. Show that the particular solution of

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = F_0 \cos \omega t$$

can be written in the form of

$$x_p(t) = C_1 \cos \omega t + C_2 \sin \omega t,$$

where

$$C_1 = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad C_2 = \frac{(c\omega)F_0}{(k - m\omega^2)^2 + (c\omega)^2}.$$

21. Show that the result of previous problem can be put in the form of

$$x_p(t) = A \cos(\omega t - \phi)$$

where

$$A = \frac{F_0}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}}, \quad \phi = \tan^{-1} \frac{c\omega}{m(\omega_0^2 - \omega^2)}.$$

22. For two identical undamped oscillators, A and B, each of mass m , and natural frequency ω_0 , show that each of them is governed by the differential equation

$$m \frac{d^2 x}{dt^2} + m\omega_0^2 x = 0.$$

They are coupled in such a way that the coupling force exerted on A is $\alpha m(d^2 x_B/dt^2)$, and the coupling force on B is $\alpha m(d^2 x_A/dt^2)$, where α is the coupling constant with a magnitude less than one. Find the normal frequencies of the system.

$$\text{Ans. } \omega = \omega_0(1 \pm \alpha)^{-1/2}.$$