6

Laplace Transforms

Among the tools that are very useful in solving linear differential equations is the Laplace transform method. The idea is to use an integral to transform the differential equation into an algebraic equation, from the solution of this algebraic equation we get the desired function through the inverse transform. The Laplace transform is named after the eminent French mathematician Pierre Simon Laplace (1749–1827), who is also remembered for the Laplace equation which is one of the most important equations in mathematical physics.

Laplace first studied this method in 1782. However, the power and usefulness of this method was not recognized until 100 years later. The techniques described in this chapter are mainly due to Oliver Heaviside (1850–1925), an innovative British electrical engineer, who also made significant contributions to electromagnetic theory.

The Laplace transform is especially useful in solving problems with nonhomogeneous terms of a discontinuous or impulsive nature. Such problems are common in physical sciences but are relatively awkward to handle by the methods previously discussed.

In this chapter certain properties of Laplace transforms are investigated and relevant formulas are tabulated in such a way that the solution of initial value problems involving linear differential equations can be conveniently obtained.

6.1 Definition and Properties of Laplace Transforms

6.1.1 Laplace Transform – A Linear Operator

The Laplace transform $\mathfrak{L}[f]$ of the function $f(t)$ is defined as

$$\mathfrak{L}[f] = \int_0^\infty \mathrm{e}^{-st} f(t) \, \mathrm{d}t = F(s), \tag{6.1}$$

we assume that this integral exists. One of the reasons that Laplace transform is useful is that s can be chosen large enough that (6.1) converges even if $f(t)$ does not go to zero as $t \to \infty$. Of course, there are functions that diverge faster than e^{st}. For such functions, the Laplace transform does not exit. Fortunately such functions are of little physical interests.

Note that the transform is a function of s. The transforms of the functions of our concern not only exist, but also go to zero $(F(s) \to 0)$ as $s \to \infty$.

It follows immediately from the definition that the Laplace transform is a linear operator, that is

$$\mathfrak{L}[af(t) + bg(t)] = \int_0^\infty e^{-st}[af(t) + bg(t)]dt$$

$$= a \int_0^\infty e^{-st} f(t) \, dt + b \int_0^\infty e^{-st} g(t) \, dt$$

$$= a\mathfrak{L}[f] + b\mathfrak{L}[g]. \tag{6.2}$$

For simple functions, the integral of the Laplace transform can be readily carried out. For example:

$$\mathfrak{L}[1] = \int_0^\infty e^{-st} \, dt = \left[-\frac{1}{s} e^{-st} \right]_0^\infty = \frac{1}{s}. \tag{6.3}$$

It is also very easy to evaluate the transform of an exponential function

$$\mathfrak{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt = \left[-\frac{1}{s-a} e^{-(s-a)t} \right]_0^\infty .$$

As long as $s > a$, the upper limit vanishes and the lower limit gives $1/(s-a)$. Thus

$$\mathfrak{L}[e^{at}] = \frac{1}{s-a}. \tag{6.4}$$

Similarly

$$\mathfrak{L}[e^{-at}] = \frac{1}{s+a}. \tag{6.5}$$

With these relations, the Laplace transforms of the following hyperbolic functions

$$\cosh at = \frac{1}{2}(e^{at} + e^{-at}), \qquad \sinh at = \frac{1}{2}(e^{at} - e^{-at})$$

are easily obtained. Since Laplace transform is linear,

$$\mathfrak{L}[\cosh at] = \frac{1}{2} \left\{ \mathfrak{L}[e^{at}] + \mathfrak{L}[e^{-at}] \right\}$$

$$= \frac{1}{2} \left(\frac{1}{s-a} + \frac{1}{s+a} \right) = \frac{s}{s^2 - a^2}. \tag{6.6}$$

Similarly

$$\mathfrak{L}\left[\sinh at\right] = \frac{a}{s^2 - a^2}. \tag{6.7}$$

Now the parameter a does not have to be restricted to real numbers. If a is purely imaginary $a = i\omega$, we will have

$$\mathfrak{L}[e^{i\omega t}] = \frac{1}{s - i\omega}.$$

Since

$$\frac{1}{s - i\omega} = \frac{1}{s - i\omega} \times \frac{s + i\omega}{s + i\omega} = \frac{s}{s^2 + \omega^2} + i\frac{\omega}{s^2 + \omega^2},$$

and

$$\mathfrak{L}[e^{i\omega t}] = \mathfrak{L}[\cos \omega t + i \sin \omega t] = \mathfrak{L}[\cos \omega t] + i\mathfrak{L}[\sin \omega t],$$

equating the real part to real part and imaginary part to imaginary part we have

$$\mathfrak{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}, \tag{6.8}$$

$$\mathfrak{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}. \tag{6.9}$$

The definition of $\mathfrak{L}[\cos \omega t]$ is, of course, still

$$\mathfrak{L}[\cos \omega t] = \int_0^\infty e^{-st} \cos \omega t \, dt. \tag{6.10}$$

With integration by parts, we can evaluate this integral directly,

$$\int_0^\infty e^{-st} \cos \omega t \, dt = \left[-\frac{1}{s} e^{-st} \cos \omega t \right]_0^\infty - \int_0^\infty \frac{1}{s} e^{-st} \omega \sin \omega t \, dt$$

$$= \frac{1}{s} - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt,$$

$$\int_0^\infty e^{-st} \sin \omega t \, dt = \left[-\frac{1}{s} e^{-st} \sin \omega t \right]_0^\infty + \int_0^\infty \frac{1}{s} e^{-st} \omega \cos \omega t \, dt$$

$$= \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt.$$

Combine these two equations,

$$\int_0^\infty e^{-st} \cos \omega t \, dt = \frac{1}{s} - \frac{\omega^2}{s^2} \int_0^\infty e^{-st} \cos \omega t \, dt.$$

Move the last term to the left-hand side,

$$\left(1 + \frac{\omega^2}{s^2} \right) \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{1}{s}$$

or

$$\int_0^\infty e^{-st} \cos \omega t \ dt = \frac{s}{s^2 + \omega^2},$$

which is exactly the same as (6.8), as it should be.

In principle, the Laplace transform can be obtained directly by carrying out the integral. However, very often it is much simpler to use the properties of the Laplace transform, rather than direct integration, to obtain the transform, as shown in the last example.

The Laplace transform has many interesting properties, they are the reasons that the Laplace transform is a powerful tool of mathematical analysis. We will now discuss some of them, and use them to generate more transforms as illustrations.

6.1.2 Laplace Transforms of Derivatives

The Laplace transform of a derivative is by definition

$$\mathfrak{L}[f'] = \int_0^\infty e^{-st} \frac{df(t)}{dt} dt = \int_0^\infty e^{-st} df(t).$$

If we let $u = e^{-st}$ and $dv = df(t)$, then $du = -s \ e^{-st} dt$ and $v = f$. With integration by parts, we have $u \ dv = d(uv) - v \ du$, so

$$\mathfrak{L}[f'] = \int_0^\infty \{d[(e^{-st} f(t)] + f(t)s \ e^{-st} dt\}$$

$$= \left[e^{-st} f(t) \right]_0^\infty + s \int_0^\infty e^{-st} f(t) \ dt = -f(0) + s\mathfrak{L}[f]. \qquad (6.11)$$

Clearly

$$\mathfrak{L}[f''] = \mathfrak{L}[(f')'] = -f'(0) + s\mathfrak{L}[f']$$
$$= -f'(0) + s(-f(0) + s\mathfrak{L}[f]) = -f'(0) - sf(0) + s^2\mathfrak{L}[f]. \ (6.12)$$

Naturally this result can be extended to higher derivatives

$$\mathfrak{L}[f^{(n)}] = -f^{(n-1)}(0) - \cdots - s^{n-1}f(0) + s^n \mathfrak{L}[f]. \qquad (6.13)$$

These properties are crucial in solving differential equations. Here we will use them to generate $\mathfrak{L}[t^n]$.

First let $f(t) = t$, then $f' = 1$ and $f(0) = 0$. By (6.11)

$$\mathfrak{L}[1] = -0 + s\mathfrak{L}[t],$$

rearranging and using (6.3), we have

$$\mathfrak{L}[t] = \frac{1}{s}\mathfrak{L}[1] = \frac{1}{s^2}. \qquad (6.14)$$

If we let $f(t) = t^2$, then $f' = 2t$ and $f(0) = 0$. Again by (6.11)

$$\mathcal{L}[2t] = -0 + s\mathcal{L}[t^2].$$

Thus, with (6.14)

$$\mathcal{L}[t^2] = \frac{1}{s}\mathcal{L}[2t] = \frac{2}{s}\mathcal{L}[t] = \frac{2}{s^3}. \tag{6.15}$$

Clearly this process can be repeated

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}. \tag{6.16}$$

6.1.3 Substitution: s-Shifting

If we know the Laplace transform $F(s)$ of the function $f(t)$, we can get the transform of $e^{at}f(t)$ by replacing s with $s - a$ in $F(s)$. This can be easily shown. By definition

$$F(s) = \int_0^\infty e^{-st}f(t)\ \mathrm{d}t = \mathcal{L}\left[f(t)\right],$$

clearly

$$F(s - a) = \int_0^\infty e^{-(s-a)t}f(t)\ \mathrm{d}t = \int_0^\infty e^{-st}e^{at}f(t)\ \mathrm{d}t = \mathcal{L}\left[e^{at}f(t)\right]. \tag{6.17}$$

This simple relation is sometimes known as s-shifting (or first shifting) theorem.

With the help of the s-shifting theorem, we can derive the transforms of many more functions without carrying out the integration. For example, it follows from (6.16) and (6.17) that

$$\mathcal{L}[e^{-at}t^n] = \frac{n!}{(s + a)^{n+1}}. \tag{6.18}$$

It can also be easily shown that

$$\mathcal{L}[e^{-at}\cos\omega t] = \int_0^\infty e^{-st}e^{-at}\cos\omega t\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}\cos\omega t\ \mathrm{d}t. \tag{6.19}$$

Compare the integrals in (6.10) and (6.19), the only difference is that s is replaced by $s + a$. Therefore the last integral must equal to the right-hand side of (6.8) with s changed to $s + a$, that is

$$\mathcal{L}[e^{-at}\cos\omega t] = \frac{s + a}{(s + a)^2 + \omega^2}. \tag{6.20}$$

Similarly

$$\mathcal{L}[e^{-at}\sin\omega t] = \frac{\omega}{(s + a)^2 + \omega^2}. \tag{6.21}$$

6.1.4 Derivative of a Transform

If we differentiate the Laplace transform $F(s)$ with respect to s, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}F(s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{L}\left[f(t)\right] = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{e}^{-st}f(t)\,\mathrm{d}t$$

$$= \int_0^\infty \frac{\mathrm{d}\mathrm{e}^{-st}}{\mathrm{d}s}f(t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}(-t)f(t)\,\mathrm{d}t = \mathfrak{L}\left[-tf(t)\right]. \quad (6.22)$$

Continuing this process, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathfrak{L}\left[f(t)\right] = \mathfrak{L}\left[(-t)^n f(t)\right]. \tag{6.23}$$

Many more formulas can be derived by taking advantage of this relation. For example, differentiating both sides of (6.9) with respect to s, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{L}[\sin\omega t] = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\omega}{s^2+\omega^2}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{L}[\sin\omega t] = \frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{e}^{-st}\sin\omega t\,\mathrm{d}t = -\int_0^\infty t\,\mathrm{e}^{-st}\sin\omega t\,\mathrm{d}t = -\mathfrak{L}[t\sin\omega t],$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\omega}{s^2+\omega^2} = -\frac{2s\omega}{\left(s^2+\omega^2\right)^2},$$

therefore

$$\mathfrak{L}[t\sin\omega t] = \frac{2s\omega}{\left(s^2+\omega^2\right)^2}. \tag{6.24}$$

Similarly we can show

$$\mathfrak{L}[t\cos\omega t] = \frac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}. \tag{6.25}$$

6.1.5 A Short Table of Laplace Transforms

Since Laplace transform is a linear operator, two transforms can be combined to form a new one. For example:

$$\mathfrak{L}[1-\cos\omega t] = \frac{1}{s} - \frac{s}{s^2+\omega^2} = \frac{\omega^2}{s(s^2+\omega^2)}, \tag{6.26}$$

$$\mathfrak{L}[\omega t - \sin\omega t] = \frac{\omega}{s^2} - \frac{\omega}{s^2+\omega^2} = \frac{\omega^3}{s^2(s^2+\omega^2)}, \tag{6.27}$$

$$\mathfrak{L}[\sin\omega t - \omega t\cos\omega t] = \frac{\omega}{s^2+\omega^2} - \frac{\omega(s^2-\omega^2)}{(s^2+\omega^2)^2} = \frac{2\omega^3}{(s^2+\omega^2)^2}, \tag{6.28}$$

Table 6.1. A short table of Laplace transforms, in each case s is assumed to be sufficiently large that the transform exists

$f(t)$	$F(s) = \mathfrak{L}[f(t)]$	$f(t)$	$F(s) = \mathfrak{L}[f(t)]$
1	$\dfrac{1}{s}$	$\delta(t)$	1
t	$\dfrac{1}{s^2}$	$\delta(t-c)$	e^{-sc}
t^n	$\dfrac{n!}{s^{n+1}}$	$\delta'(t-c)$	$s\,e^{-sc}$
e^{at}	$\dfrac{1}{s-a}$	$u(t-c)$	$\dfrac{1}{s}e^{-sc}$
$t\,e^{at}$	$\dfrac{1}{(s-a)^2}$	$(t-c)^n u(t-c)$	$\dfrac{n!}{s^{n+1}}e^{-sc}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	$(t-c)^n e^{a(t-c)}u(t-c)$	$\dfrac{n!}{(s-a)^{n+1}}e^{-sc}$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	$\sin\omega(t-c)u(t-c)$	$\dfrac{\omega}{s^2+\omega^2}e^{-sc}$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$	$\cosh a(t-c)u(t-c)$	$\dfrac{s}{s^2-a^2}e^{-sc}$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$\sin\omega t$ of period $\dfrac{\pi}{\omega}$	$\dfrac{\omega}{s^2+\omega^2}\coth\dfrac{s\pi}{2\omega}$
$\cosh at$	$\dfrac{s}{s^2-a^2}$	t of period p	$\dfrac{1-(1+ps)e^{-ps}}{ps^2(1-e^{-ps})}$
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$	$\dfrac{1}{t}(e^{bt}-e^{at})$	$\ln\dfrac{s-a}{s-b}$
$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$	$\dfrac{2}{t}(1-\cosh at)$	$\ln\dfrac{s^2-a^2}{s^2}$
$t\sin\omega t$	$\dfrac{2\omega s}{(s^2+\omega^2)^2}$	$\dfrac{2}{t}(1-\cos\omega t)$	$\ln\dfrac{s^2+\omega^2}{s^2}$
$1-\cos\omega t$	$\dfrac{\omega^2}{s(s^2+\omega^2)}$	$\dfrac{\sin\omega t}{t}$	$\tan^{-1}\dfrac{\omega}{s}$
$\omega t-\sin\omega t$	$\dfrac{\omega^3}{s^2(s^2+\omega^2)}$	$t^a\ \ (a>-1)$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$
$\sin\omega t-\omega t\cos\omega t$	$\dfrac{2\omega^3}{(s^2+\omega^2)^2}$	$t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
$\sin\omega t+\omega t\cos\omega t$	$\dfrac{2\omega s^2}{(s^2+\omega^2)^2}$	$t^{1/2}$	$\dfrac{1}{2}\dfrac{\sqrt{\pi}}{s^{3/2}}$
$\cos at-\cos bt$	$\dfrac{(b^2-a^2)\,s}{(s^2+a^2)(s^2+b^2)}$	$J_0(at)$	$\dfrac{1}{(s^2+a^2)^{1/2}}$

$$\mathfrak{L}[\sin \omega t + \omega t \cos \omega t] = \frac{\omega}{s^2 + \omega^2} + \frac{\omega(s^2 - \omega^2)}{(s^2 + \omega^2)^2} = \frac{2\omega s^2}{(s^2 + \omega^2)^2}, \qquad (6.29)$$

$$\mathfrak{L}[\cos at - \cos bt] = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} = \frac{(b^2 - a^2)s}{(s^2 + a^2)(s^2 + b^2)}. \qquad (6.30)$$

There are extensive tables of Laplace transforms (For example, F. Oberherttinger and E. Badii, Tables of Laplace Transforms, Springer, New York, 1973). A short list of some simple Laplace transforms is given in Table 6.1. The items in the left-hand side of the table are the ones we have shown so far. Items in the right-hand side are relations we are going to derive in the following sections.

6.2 Solving Differential Equation with Laplace Transform

6.2.1 Inverse Laplace Transform

In solving differential equation with Laplace transform, we encounter the inverse problem of determining the unknown function $f(t)$ which has a given transform $F(s)$. The notation of $\mathfrak{L}^{-1}[F(s)]$ is conventionally used for the inverse Laplace transform of $F(s)$. That is, if

$$F(s) = \mathfrak{L}[f(t)] = \int_0^\infty e^{-st} f(t) \, dt, \qquad (6.31)$$

then

$$f(t) = \mathfrak{L}^{-1}[F(s)]. \qquad (6.32)$$

Since

$$f(t) = \mathfrak{L}^{-1}[F(s)] = \mathfrak{L}^{-1}[\mathfrak{L}[f(t)]] = I[f(t)],$$

it follows that $\mathfrak{L}^{-1}\mathfrak{L}$ is the identity operator I. The inverse transforms are of great practical importance and there are a variety of ways to get them. In this section, we will first study the transform in the form of a quotient of two polynomials

$$F(s) = \frac{p(s)}{q(s)},$$

where $p(s)$ and $q(s)$ have real coefficients and no common factors. Since

$$\lim_{s \to \infty} F(s) = \lim_{s \to \infty} \int_0^\infty e^{-st} f(t) \, dt \to 0,$$

it is clear that the degree of $p(s)$ is lower than that of $q(s)$. There are several closely related methods to get the inverse of such a transform. For the sake of clarity, we list them separately.

By Inspection. If the expression is simple enough, one can get the inverse directly from the table. This is illustrated in the following examples.

Example 6.2.1. Find (a) $\mathcal{L}^{-1}\left[\dfrac{1}{s^4}\right]$; (b) $\mathcal{L}^{-1}\left[\dfrac{4}{(s+4)^3}\right]$; (c) $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+4}\right]$.

Solution 6.2.1.
(a) Since

$$\mathcal{L}[t^3] = \frac{3!}{s^4} = \frac{6}{s^4}, \qquad t^3 = \mathcal{L}^{-1}\left[\frac{6}{s^4}\right],$$

we have

$$\mathcal{L}^{-1}\left[\frac{1}{s^4}\right] = \frac{1}{6}\mathcal{L}^{-1}\left[\frac{6}{s^4}\right] = \frac{1}{6}t^3.$$

(b) Since

$$\mathcal{L}[e^{-4t}t^2] = \frac{2}{(s+4)^3}, \qquad e^{-4t}t^2 = \mathcal{L}^{-1}\left[\frac{2}{(s+4)^3}\right],$$

so

$$\mathcal{L}^{-1}\left[\frac{4}{(s+4)^3}\right] = 2\mathcal{L}^{-1}\left[\frac{2}{(s+4)^3}\right] = 2e^{-4t}t^2.$$

(c) Since

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2+4}, \qquad \sin 2t = \mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right],$$

so

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{1}{2}\sin 2t.$$

Example 6.2.2. Find (a) $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+2s+5}\right]$; (b) $\mathcal{L}^{-1}\left[\dfrac{2s+1}{s^2+2s+5}\right]$.

Solution 6.2.2. (a) First we note that

$$\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4} = \frac{1}{2} \times \frac{2}{(s+1)^2+4}.$$

Since

$$\mathcal{L}[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+4}, \qquad e^{-t}\sin 2t = \mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right],$$

so

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+5}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right] = \frac{1}{2}e^{-t}\sin 2t.$$

(b) Recall

$$\mathfrak{L}[\mathrm{e}^{-t}\sin 2t] = \frac{2}{(s+1)^2+4},$$

$$\mathfrak{L}[\mathrm{e}^{-t}\cos 2t] = \frac{s+1}{(s+1)^2+4},$$

so we write

$$\frac{2s+1}{s^2+2s+5} = \frac{2(s+1)-1}{(s+1)^2+4} = 2\frac{(s+1)}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4}.$$

Thus

$$\mathfrak{L}^{-1}\left[\frac{2s+1}{s^2+2s+5}\right] = 2\mathfrak{L}^{-1}\left[\frac{(s+1)}{(s+1)^2+4}\right] - \frac{1}{2}\mathfrak{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right]$$

$$= 2\mathfrak{L}^{-1}[\mathfrak{L}[\mathrm{e}^{-t}\cos 2t]] - \frac{1}{2}\mathfrak{L}^{-1}[\mathfrak{L}[\mathrm{e}^{-t}\sin 2t]]$$

$$= 2\mathrm{e}^{-t}\cos 2t - \frac{1}{2}\mathrm{e}^{-t}\sin 2t.$$

Partial Fraction Decomposition. Take the partial fractions of $F(s)$ and then take the inverse of each term. Most probably you are familiar with partial fractions. We will use the following examples for review.

Example 6.2.3. Find $\mathfrak{L}^{-1}\left[\dfrac{s-1}{s^2-s-2}\right]$.

Solution 6.2.3. First we note

$$\frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}$$

$$= \frac{a}{(s-2)} + \frac{b}{(s+1)} = \frac{a(s+1)+b(s-2)}{(s-2)(s+1)}.$$

The following are three different ways to determine a and b.

- First note that
$$s-1 = a(s+1)+b(s-2)$$
must hold for all s. One way is to set $s=2$, then it follows that $a=\frac{1}{3}$. Similarly if we set $s=-1$, we see immediately that $b=\frac{2}{3}$.
- Another way is to collect the terms with the same powers in s, and require the coefficients of the corresponding terms on both sides of the equation be equal to each other. That is,
$$s-1 = (a+b)s+(a-2b).$$
This means $a+b=1$ and $a-2b=-1$. Thus $a=\frac{1}{3}$ and $b=\frac{2}{3}$.

- Still another way is to note that

$$\lim_{s \to 2} \left\{ (s-2) \frac{s-1}{(s-2)(s+1)} \right\} = \lim_{s \to 2} \left\{ (s-2) \left[\frac{a}{(s-2)} + \frac{b}{(s+1)} \right] \right\},$$

this means

$$\lim_{s \to 2} \left\{ \frac{s-1}{(s+1)} \right\} = \lim_{s \to 2} \left\{ a + (s-2) \frac{b}{(s+1)} \right\} = a.$$

We see immediately that $a = \frac{1}{3}$. Similarly

$$\lim_{s \to -1} \left\{ (s+1) \frac{s-1}{(s-2)(s+1)} \right\} = \lim_{s \to -1} \left\{ (s+1) \left[\frac{a}{(s-2)} + b \right] \right\} = b$$

gives $b = \frac{2}{3}$.

In some problems, one way is much simpler than others. Anyway, in this problem

$$\mathcal{L}^{-1} \left[\frac{s-1}{s^2 - s - 2} \right] = \mathcal{L}^{-1} \left[\frac{1}{3} \frac{1}{(s-2)} + \frac{2}{3} \frac{1}{(s+1)} \right]$$

$$= \frac{1}{3} \mathcal{L}^{-1} [\mathcal{L}[e^{2t}]] + \frac{2}{3} \mathcal{L}^{-1} [\mathcal{L}[e^{-t}]] = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}.$$

Example 6.2.4. Find $\mathcal{L}^{-1} \left[\dfrac{1}{s(s^2 + 4)} \right]$.

Solution 6.2.4. There are two ways for us to take partial fractions.

- If we use complex roots,

$$\frac{1}{s(s^2 + 4)} = \frac{a}{s} + \frac{b}{s - 2i} + \frac{c}{s + 2i}.$$

Multiplying by s and taking the limit with $s \to 0$, we have

$$a = \lim_{s \to 0} \frac{1}{s^2 + 4} = \frac{1}{4}.$$

Multiplying by $s - 2i$ and taking the limit with $s \to 2i$, we have

$$b = \lim_{s \to 2i} \frac{1}{s(s + 2i)} = -\frac{1}{8}.$$

Multiplying by $s + 2i$ and taking the limit with $s \to -2i$, we have

$$c = \lim_{s \to -2i} \frac{1}{s(s - 2i)} = -\frac{1}{8}.$$

Therefore

$$\mathfrak{L}^{-1}\left[\frac{1}{s\left(s^2+4\right)}\right] = \frac{1}{4}\mathfrak{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{8}\mathfrak{L}^{-1}\left[\frac{1}{s-2i}\right] - \frac{1}{8}\mathfrak{L}^{-1}\left[\frac{1}{s+2i}\right]$$

$$= \frac{1}{4} - \frac{1}{8}e^{2it} - \frac{1}{8}e^{-2it} = \frac{1}{4} - \frac{1}{4}\cos 2t.$$

- Another way to take partial fractions is to note that

$$\frac{b}{s-2i} + \frac{c}{s+2i} = \frac{b(s+2i)+c(s-2i)}{(s-2i)(s+2i)} = \frac{(b+c)s+2i(b-c)}{s^2+4}.$$

If we let $b+c = b'$ and $2i(b-c) = c'$, then

$$\frac{1}{s\left(s^2+4\right)} = \frac{a}{s} + \frac{b's+c'}{s^2+4}.$$

An important point we should note is that if the denominator is second order in s, the numerator must be allowed the possibility of being first order in s. In other words, it will not be possible for us to get the correct answer if b' term is missing. With this understanding, the partial fractions can be taken directly as

$$\frac{1}{s\left(s^2+4\right)} = \frac{a}{s} + \frac{bs+c}{s^2+4} = \frac{a\left(s^2+4\right)+(bs+c)s}{s\left(s^2+4\right)}$$

$$= \frac{as^2+4a+bs^2+cs}{s\left(s^2+4\right)} = \frac{(a+b)s^2+cs+4a}{s\left(s^2+4\right)}.$$

Therefore

$$1 = (a+b)s^2 + cs + 4a.$$

The coefficients of s must be equal term by term. That is, $a+b = 0$, $c = 0$, $4a = 1$. This gives $a = 1/4$, $b = -1/4$, $c = 0$. Thus

$$\mathfrak{L}^{-1}\left[\frac{1}{s\left(s^2+4\right)}\right] = \mathfrak{L}^{-1}\left[\frac{1}{4s} - \frac{1}{4}\frac{s}{s^2+4}\right] = \frac{1}{4} - \frac{1}{4}\cos 2t.$$

Example 6.2.5. Find $\mathfrak{L}^{-1}\left[\dfrac{1}{s^3(s-1)}\right]$.

Solution 6.2.5.

$$\frac{1}{s^3(s-1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s^3} + \frac{d}{(s-1)}$$

$$= \frac{as^2(s-1)+bs(s-1)+c(s-1)+ds^3}{s^3(s-1)}$$

$$= \frac{(a+d)s^3+(b-a)s^2+(c-b)s-c}{s^3(s-1)}.$$

This requires $a+d = 0$, $b-a = 0$, $c-b = 0$, $-c = 1$. Thus $c = -1$, $b = -1$, $a = -1$, $d = 1$. Hence

$$\mathfrak{L}^{-1}\left[\frac{1}{s^3(s-1)}\right] = \mathfrak{L}^{-1}\left[\frac{-1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{(s-1)}\right]$$

$$= -1 - t - \frac{1}{2}t^2 + e^t.$$

Example 6.2.6. Find $\mathfrak{L}^{-1}\left[\dfrac{2\omega}{(s^2+\omega^2)^2}\right]$.

Solution 6.2.6.

$$\frac{2\omega}{(s^2+\omega^2)^2} = \frac{2\omega}{[(s-i\omega)(s+i\omega)]^2}$$

$$= \frac{a}{(s-i\omega)} + \frac{b}{(s-i\omega)^2} + \frac{c}{(s+i\omega)} + \frac{d}{(s+i\omega)^2}.$$

Multiplying both sides by $(s-i\omega)^2$ and take the limit as $s \to i\omega$, we have

$$\lim_{s\to i\omega}\left\{\frac{2\omega}{(s+i\omega)^2}\right\} = \lim_{s\to i\omega}\left\{(s-i\omega)a + b + \frac{(s-i\omega)^2 c}{(s+i\omega)} + \frac{(s-i\omega)^2 d}{(s+i\omega)^2}\right\}.$$

Clearly

$$b = \frac{2\omega}{(2i\omega)^2} = -\frac{1}{2\omega}.$$

If after multiplying both sides by $(s-i\omega)^2$, we take the derivative first and then go to the limit $s \to i\omega$, we have

$$\lim_{s\to i\omega}\left\{\frac{d}{ds}\frac{2\omega}{(s+i\omega)^2}\right\} = \lim_{s\to i\omega}\left\{a + \frac{d}{ds}\left[\frac{(s-i\omega)^2 c}{(s+i\omega)} + \frac{(s-i\omega)^2 d}{(s+i\omega)^2}\right]\right\}.$$

This leads to

$$a = \lim_{s\to i\omega}\left\{\frac{-4\omega}{(s+i\omega)^3}\right\} = \frac{1}{2\omega^2 i}.$$

Similarly, we can show

$$d = -\frac{1}{2\omega}, \quad c = -\frac{1}{2\omega^2 i}.$$

Thus we have

$$\mathfrak{L}^{-1}\left[\frac{2\omega}{(s^2+\omega^2)^2}\right] =$$

$$\mathfrak{L}^{-1}\left[\frac{1}{2\omega^2 i}\frac{1}{(s-i\omega)} - \frac{1}{2\omega}\frac{1}{(s-i\omega)^2} - \frac{1}{2\omega^2 i}\cdot\frac{1}{(s+i\omega)} - \frac{1}{2\omega}\frac{1}{(s+i\omega)^2}\right] =$$

$$\frac{1}{2\omega^2 i}\mathfrak{L}^{-1}\left[\frac{1}{(s-i\omega)} - \frac{1}{(s+i\omega)}\right] - \frac{1}{2\omega}\mathfrak{L}^{-1}\left[\frac{1}{(s-i\omega)^2} + \frac{1}{(s+i\omega)^2}\right] =$$

$$\frac{1}{2\omega^2 i}\left(e^{i\omega t} - e^{-i\omega t}\right) - \frac{1}{2\omega}\left(t\, e^{i\omega t} + t\, e^{-i\omega t}\right) = \frac{1}{\omega^2}\sin\omega t - \frac{1}{\omega}t\cos\omega t.$$

The Heaviside Expansion. The Heaviside expansion is essentially a systematic way of taking partial fractions. In the partial fraction decomposition of $p(s)/q(s)$, an unrepeated factor $(s - a)$ of $q(s)$ gives rise to a single fraction of the form $A/(s - a)$. Thus $F(s)$ can be written as

$$F(s) = \frac{p(s)}{q(s)} = \frac{A}{s - a} + G(s), \tag{6.33}$$

where $G(s)$ is simply the rest of the expression. Multiplication by $(s-a)$ gives

$$\frac{(s - a)p(s)}{q(s)} = A + (s - a)G(s).$$

If we let s approach a, the second term in the right-hand side vanishes, since $G(s)$ has no factor that could cancel $(s - a)$. Therefore

$$A = \lim_{s \to a} \frac{(s - a)p(s)}{q(s)}. \tag{6.34}$$

Since $q(a) = 0$, because a is an unrepeated root of $q(s) = 0$, the limit in (6.34) is an indeterminant of the form of $0/0$. With the L'Hospital's rule, we have

$$A = \lim_{s \to a} \frac{p(s) + (s - a)p'(s)}{q'(s)} = \frac{p(a)}{q'(a)}. \tag{6.35}$$

Thus the constants in the partial fraction decomposition can be quickly determined.

Example 6.2.7. Use the Heaviside expansion to find $\mathfrak{L}^{-1}\left[\dfrac{s - 1}{s^2 - s - 2}\right]$.

Solution 6.2.7. The roots of $s^2 - s - 2 = 0$ are $s = 2$ and $s = -1$, and $\frac{d}{ds}(s^2 - s - 2) = 2s - 1$. Therefore

$$\frac{s - 1}{s^2 - s - 2} = \frac{a}{(s - 2)} + \frac{b}{(s + 1)},$$

$$a = \lim_{s \to 2} \frac{s - 1}{2s - 1} = \frac{1}{3}, \qquad b = \lim_{s \to -1} \frac{s - 1}{2s - 1} = \frac{2}{3}.$$

Thus

$$\mathfrak{L}^{-1}\left[\frac{s - 1}{s^2 - s - 2}\right] = \frac{1}{3}\mathfrak{L}^{-1}\left[\frac{1}{s - 2}\right] + \frac{2}{3}\mathfrak{L}^{-1}\left[\frac{1}{s + 1}\right] = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Example 6.2.8. Use the Heaviside expansion to find $\mathfrak{L}^{-1}\left[\dfrac{2s + 1}{s^2 + 2s + 5}\right]$.

Solution 6.2.8. The roots of $s^2 + 2s + 5 = 0$ are $s = -1 \pm 2i$, and $\frac{d}{ds}(s^2 + 2s + 2) = 2s + 2$. Thus

$$\frac{2s+1}{s^2+2s+5} = \frac{a}{s-(-1+2\mathrm{i})} + \frac{b}{s-(-1-2\mathrm{i})},$$

$$a = \lim_{s\to-1+2\mathrm{i}}\frac{2s+1}{2s+2} = 1+\frac{\mathrm{i}}{4}, \quad b = \lim_{s\to-1-2\mathrm{i}}\frac{2s+1}{2s+2} = 1-\frac{\mathrm{i}}{4}.$$

Therefore

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+2s+5}\right]$$

$$= \left(1+\frac{1}{4}\mathrm{i}\right)\mathcal{L}^{-1}\left[\frac{1}{s-(-1+2\mathrm{i})}\right] + \left(1-\frac{1}{4}\mathrm{i}\right)\mathcal{L}^{-1}\left[\frac{1}{s-(-1-2\mathrm{i})}\right].$$

Recall

$$\mathcal{L}^{-1}\left[\frac{1}{s-c}\right] = \mathrm{e}^{ct},$$

we have

$$\mathcal{L}^{-1}\left[\frac{1}{s-(-1+2\mathrm{i})}\right] = \mathrm{e}^{(-1+2\mathrm{i})t} = \mathrm{e}^{-t}\mathrm{e}^{\mathrm{i}2t} = \mathrm{e}^{-t}\left(\cos 2t + \mathrm{i}\sin 2t\right),$$

and

$$\mathcal{L}^{-1}\left[\frac{1}{s-(-1-2\mathrm{i})}\right] = \mathrm{e}^{-t}\mathrm{e}^{-\mathrm{i}2t} = \mathrm{e}^{-t}\left(\cos 2t - \mathrm{i}\sin 2t\right).$$

Hence

$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+2s+5}\right] = \left(1+\frac{1}{4}\mathrm{i}\right)\mathrm{e}^{-t}\left(\cos 2t + \mathrm{i}\sin 2t\right)$$

$$+ \left(1-\frac{1}{4}\mathrm{i}\right)\mathrm{e}^{-t}\left(\cos 2t - \mathrm{i}\sin 2t\right)$$

$$= 2\mathrm{e}^{-t}\cos 2t - \frac{1}{2}\mathrm{e}^{-t}\sin 2t.$$

In general, if $q(s)$ is a polynomial with unrepeated roots, the Heaviside expansion is the most efficient way in partial fraction decomposition. If $q(s)$ is already in the form of a product of factors $(s-a_1)(s-a_2)\cdots(s-a_n)$, then other methods of partial fraction may be equally or more efficient. In any case, if the complex roots are used, it is useful to keep in mind that if the original function is real, the final result must also be real. If there is an imaginary term in the final result, then there must be a mistake somewhere.

If $q(s)$ has repeated roots, we can write it as

$$\frac{p(s)}{q(s)} = \frac{A_m}{(s-a)^m} + \frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{(s-a)}.$$

With a similar argument, one can show that

$$A_k = \frac{1}{(m-k)!}\lim_{s\to a}\frac{\mathrm{d}^{m-k}}{\mathrm{d}s^{m-k}}\left[\frac{(s-a)^m p(s)}{q(s)}\right], \qquad k = 1,\ldots,m. \qquad (6.36)$$

Unfortunately, in practice this formula is not necessarily simpler than other partial fraction methods, such as the one shown in Example 6.2.6. In fact, problems of that nature are best solved by using the derivatives of a transform. *Using Derivatives of the Transform.* In Example 6.2.6, we used the partial fraction to find $\mathfrak{L}^{-1}[1/(s^2 + a^2)^2]$. A simpler way to handle such problems is to make use of the properties of derivatives. The procedures are illustrated in the following examples.

Example 6.2.9. Find   (a) $\mathfrak{L}^{-1}\left[\dfrac{1}{(s^2 + a^2)^2}\right]$,   (b) $\mathfrak{L}^{-1}\left[\dfrac{s}{(s^2 + a^2)^2}\right]$,

(c) $\mathfrak{L}^{-1}\left[\dfrac{s^2}{(s^2 + a^2)^2}\right]$, (d) $\mathfrak{L}^{-1}\left[\dfrac{s^3}{(s^2 + a^2)^2}\right]$.

Solution 6.2.9. (a) Taking the derivative

$$\frac{\mathrm{d}}{\mathrm{d}a}\frac{a}{s^2 + a^2} = \frac{1}{s^2 + a^2} - \frac{2a^2}{(s^2 + a^2)^2},$$

we can write

$$\frac{1}{(s^2 + a^2)^2} = \frac{1}{2a^2}\left(\frac{1}{s^2 + a^2} - \frac{\mathrm{d}}{\mathrm{d}a}\frac{a}{s^2 + a^2}\right).$$

Since

$$\mathfrak{L}[\sin at] = \frac{a}{s^2 + a^2},$$

we have

$$\begin{aligned}
\frac{1}{(s^2 + a^2)^2} &= \frac{1}{2a^2}\left(\frac{1}{a}\mathfrak{L}[\sin at] - \frac{\mathrm{d}}{\mathrm{d}a}\mathfrak{L}[\sin at]\right)\\
&= \frac{1}{2a^3}\mathfrak{L}[\sin at] - \frac{1}{2a^2}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty \mathrm{e}^{-st}\sin at\ \mathrm{d}t\\
&= \frac{1}{2a^3}\mathfrak{L}[\sin at] - \frac{1}{2a^2}\int_0^\infty \mathrm{e}^{-st}t\cos at\ \mathrm{d}t\\
&= \frac{1}{2a^3}\mathfrak{L}[\sin at] - \frac{1}{2a^2}\mathfrak{L}[t\cos at].
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathfrak{L}^{-1}\left[\frac{1}{(s^2 + a^2)^2}\right] &= \mathfrak{L}^{-1}\left[\frac{1}{2a^3}\mathfrak{L}[\sin at] - \frac{1}{2a^2}\mathfrak{L}[t\cos at]\right]\\
&= \frac{1}{2a^3}\sin at - \frac{1}{2a^2}t\cos at.
\end{aligned}$$

(b) Take derivative with respect to s

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{a}{s^2 + a^2} = \frac{-2as}{(s^2 + a^2)^2},$$

so

$$\frac{s}{(s^2 + a^2)^2} = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}s}\frac{a}{s^2 + a^2} = -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{L}[\sin at]$$

$$= -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty \mathrm{e}^{-st}\sin at\ \mathrm{d}t = \frac{1}{2a}\int_0^\infty \mathrm{e}^{-st}t\sin at\ \mathrm{d}t$$

$$= \frac{1}{2a}\mathfrak{L}[t\sin at].$$

Therefore

$$\mathfrak{L}^{-1}\left[\frac{s}{(s^2 + a^2)^2}\right] = \mathfrak{L}^{-1}\left[\frac{1}{2a}\mathfrak{L}[t\sin at]\right] = \frac{1}{2a}t\sin at.$$

(c) It follows from the result of (b),

$$s\mathfrak{L}\left[\frac{1}{2a}t\sin at\right] = s\frac{s}{(s^2 + a^2)^2} = \frac{s^2}{(s^2 + a^2)^2}.$$

Recall

$$\mathfrak{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] = s\mathfrak{L}[f] - f(0), \qquad s\mathfrak{L}[f] = \mathfrak{L}\left[\frac{\mathrm{d}f}{\mathrm{d}t}\right] + f(0).$$

Let $f = \frac{1}{2a}t\sin at$, so $\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{1}{2a}\sin at + \frac{1}{2}t\cos at$; $f(0) = 0$,

we have

$$s\mathfrak{L}\left[\frac{1}{2a}t\sin at\right] = \mathfrak{L}\left[\frac{1}{2a}\sin at + \frac{1}{2}t\cos at\right].$$

Therefore

$$\mathfrak{L}^{-1}\left[\frac{s^2}{(s^2 + a^2)^2}\right] = \mathfrak{L}^{-1}\left[s\mathfrak{L}\left[\frac{1}{2a}t\sin at\right]\right] = \frac{1}{2a}\sin at + \frac{1}{2}t\cos at.$$

(d) From the result of (c)

$$\frac{s^3}{(s^2 + a^2)^2} = s\frac{s^2}{(s^2 + a^2)^2} = s\mathfrak{L}\left[\frac{1}{2a}\sin at + \frac{1}{2}t\cos at\right].$$

This time, let

$$f = \frac{1}{2a}\sin at + \frac{1}{2}t\cos at, \quad so \quad \frac{\mathrm{d}f}{\mathrm{d}t} = \cos at - \frac{a}{2}t\sin at;\ f(0) = 0,$$

thus

$$s\mathfrak{L}\left[\frac{1}{2a}\sin at + \frac{1}{2}t\cos at\right] = \mathfrak{L}\left[\cos at - \frac{a}{2}t\sin at\right],$$

$$\mathfrak{L}^{-1}\left[\frac{s^3}{(s^2 + a^2)^2}\right] = \mathfrak{L}^{-1}\left[\mathfrak{L}\left[\cos at - \frac{a}{2}t\sin at\right]\right] = \cos at - \frac{a}{2}t\sin at.$$
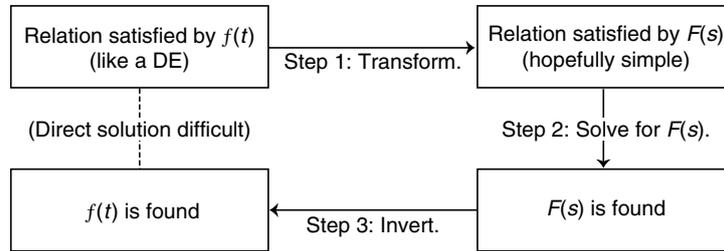
Fig. 6.1. Steps of using Laplace transform to solve differential equations

6.2.2 Solving Differential Equations

The idea of using Laplace transform to solve differential equation is expressed in Fig. 6.1. Suppose we have a differential equation in which the unknown function is $f(t)$. The first step is to apply the Laplace transform to this differential equation. The result is a relation satisfied by $F(s) = \mathfrak{L}[f]$. Generally this is an algebraic equation. The second step is to find $F(s)$ by solving this algebraic equation. The third and final step is to find the unknown function $f(t)$ by taking the inverse of the Laplace transform $F(s)$.

A few example will make this procedure clear.

Example 6.2.10. Find the solution of the differential equation

$$y'' + y = \sin 2t,$$

satisfying the initial conditions

$$y(0) = 0, \quad y'(0) = 1.$$

Solution 6.2.10. Applying Laplace transform to the equation ,

$$\mathfrak{L}[y'' + y] = \mathfrak{L}[\sin 2t],$$

we have

$$s^2 \mathfrak{L}[y] - sy(0) - y'(0) + \mathfrak{L}[y] = \frac{2}{s^2 + 4}.$$

With the initial values of $y(0)$ and $y'(0)$, this equation can be written as

$$(s^2 + 1)\mathfrak{L}[y] = 1 + \frac{2}{s^2 + 4}.$$

This algebraic equation can be easily solved to give

$$\mathfrak{L}[y] = \frac{s^2 + 6}{(s^2 + 1)\,(s^2 + 4)}.$$

Thus

$$y(t) = \mathfrak{L}^{-1}\left[\frac{s^2 + 6}{(s^2 + 1)(s^2 + 4)}\right].$$

Using methods of the last section, we find

$$y(t) = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t.$$

Example 6.2.11. Find the solution of the differential equation

$$y'' + 4y = \sin 2t, \qquad y(0) = 10, \ y'(0) = 0.$$

Solution 6.2.11. Applying the Laplace transform to both sides of the equation

$$\mathfrak{L}[y'' + 4y] = \mathfrak{L}[\sin 2t],$$

we have

$$s^2\mathfrak{L}[y] - sy(0) - y'(0) + 4\mathfrak{L}[y] = \frac{2}{s^2 + 4}.$$

With the initial values of $y(0)$ and $y'(0)$, this equation can be written as

$$(s^2 + 4)\mathfrak{L}[y] = 10s + \frac{2}{s^2 + 4}.$$

Therefore

$$y = \mathfrak{L}^{-1}\left[\frac{10s}{(s^2 + 4)} + \frac{2}{(s^2 + 4)^2}\right].$$

This leads to

$$y = 10\cos 2t + \frac{1}{8}\sin 2t - \frac{1}{4}t\cos 2t.$$

Example 6.2.12. Find the solution of the differential equation

$$y'' + 4y' + 4y = t^2\mathrm{e}^{-2t}, \qquad y(0) = 0, \ y'(0) = 0.$$

Solution 6.2.12. Applying the Laplace transform to the equation

$$\mathfrak{L}[y'' + 4y' + 4y] = \mathfrak{L}[t^2\mathrm{e}^{-2t}],$$

With the initial values of $y(0)$ and $y'(0)$, we have

$$s^2\mathfrak{L}[y] + 4s\mathfrak{L}[y] + 4\mathfrak{L}[y] = \frac{2}{(s + 2)^3}.$$

Collecting terms

$$(s^2 + 4s + 4)\mathfrak{L}[y] = (s + 2)^2\mathfrak{L}[y] = \frac{2}{(s + 2)^3},$$

or

$$\mathfrak{L}[y] = \frac{2}{(s+2)^5}.$$

The solution is the inverse transform

$$y = \frac{2}{4!}\mathfrak{L}^{-1}\left[\frac{4!}{(s+2)^5}\right] = \frac{1}{12}t^4\mathrm{e}^{-2t}.$$

Example 6.2.13. Find the solution of the set of the differential equations

$$y' - 2y + z = 0,$$
$$z' - y - 2z = 0,$$

satisfying the initial conditions

$$y(0) = 1, \quad z(0) = 0.$$

Solution 6.2.13. Applying the Laplace transform to each of the equations

$$\mathfrak{L}[y' - 2y + z] = \mathfrak{L}[0],$$
$$\mathfrak{L}[z' - y - 2z] = \mathfrak{L}[0],$$

we obtain

$$s\mathfrak{L}[y] - y(0) - 2\mathfrak{L}[y] + \mathfrak{L}[z] = 0,$$
$$s\mathfrak{L}[z] - z(0) - \mathfrak{L}[y] - 2\mathfrak{L}[z] = 0.$$

After substituting the initial conditions and collecting terms, we have

$$(s - 2)\mathfrak{L}[y] + \mathfrak{L}[z] = 1,$$
$$\mathfrak{L}[y] - (s - 2)\mathfrak{L}[z] = 0.$$

This set of algebraic equations can be easily solved to give

$$\mathfrak{L}[y] = \frac{s - 2}{(s - 2)^2 + 1},$$
$$\mathfrak{L}[z] = \frac{1}{(s - 2)^2 + 1}.$$

Thus

$$y = \mathfrak{L}^{-1}\left[\frac{s - 2}{(s - 2)^2 + 1}\right] = \mathrm{e}^{2t}\cos t,$$
$$z = \mathfrak{L}^{-1}\left[\frac{1}{(s - 2)^2 + 1}\right] = \mathrm{e}^{2t}\sin t.$$

6.3 Laplace Transform of Impulse and Step Functions

Some of the most useful and interesting applications of the Laplace transform method occur in the solution of linear differential equations with discontinuous or impulsive nonhomogeneous functions. Equations of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems, where voltages or forces of large magnitude act over very short time intervals.

To deal effectively with functions having jump discontinuities, we first introduce two functions known as delta function and step function.

6.3.1 The Dirac Delta Function

The delta function, $\delta(t)$, was first proposed in 1930 by Dirac in the development of the mathematical formalism of quantum mechanics. He required a function which is zero everywhere, except at a single point, where it is discontinuous and behaved like an infinitely high and infinitely narrow spike of unit area. Mathematicians were quick to point out that, strictly speaking, there is no function which has these properties. But Dirac supposed there was, and proceeded to use it so successfully that a new branch of mathematics was developed to justify its use. This area of mathematics is called the theory of distribution or of generalized functions. While it is nice to know that the mathematical foundation of the delta function has been established in complete details, for applications in physical sciences we need only to know its operational definition.

Definition of δ Function. The delta function is a sharply peaked function defined as

$$\delta(t - t_0) = \begin{cases} 0 & t \neq t_0 \\ \infty & t = t_0, \end{cases} \tag{6.37}$$

but such that the integral of $\delta(t - t_0)$ is normalized to unity:

$$\int_{-\infty}^{+\infty} \delta(t - t_0)\, \mathrm{d}t = 1. \tag{6.38}$$

Clearly the limits $-\infty$ and ∞ may be replaced by $t_0 - \epsilon$ and $t_0 + \epsilon$ as long as $\epsilon > 0,$ since $\delta(t - t_0)$ is equal to zero for $t \neq t_0$. We can think of it as an infinitely high and infinitely narrow function shown in Fig. 6.2, where $h \to \infty$ and $\tau \to 0$ in such a way that the area under it is equal to one.

Mathematically, the δ function is defined by how it behaves inside an integral. In fact the first operation where Dirac used the delta function is the integration

$$\int_{-\infty}^{+\infty} f(t)\delta(t - t_0)\, \mathrm{d}t,$$

where $f(t)$ is a continuous function. This integral can be evaluated by the following argument. Since $\delta(t - t_0)$ is zero for $t \neq t_0$, the limit of integration
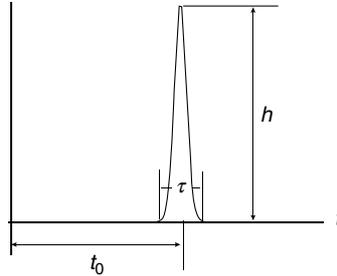
Fig. 6.2. A sharply peaked function. If $h \to \infty$ and $\tau \to 0$ in such a way that the area under it is equal to 1, then this function becomes a delta function $\delta(t - t_0)$

may be changed to $t_0 - \epsilon$ and $t_0 + \epsilon$, where ϵ is a small positive number. Moreover, since $f(x)$ is continuous at $t = t_0$, its values within the interval $(t_0 - \epsilon, t_0 + \epsilon)$ will not differ much from $f(t_0)$ and we can claim, approximately, that

$$\int_{-\infty}^{+\infty} f(t)\delta(t - t_0)\, \mathrm{d}t = \int_{t_0 - \epsilon}^{t_0 + \epsilon} f(t)\delta(t - t_0)\, \mathrm{d}t \approx f(t_0) \int_{t_0 - \epsilon}^{t_0 + \epsilon} \delta(t - t_0)\, \mathrm{d}t$$

with the approximation improving as ϵ approaches zero. However,

$$\int_{t_0 - \epsilon}^{t_0 + \epsilon} \delta(t - t_0)\, \mathrm{d}t = 1$$

for all values of ϵ. It appears then that letting $\epsilon \to 0$, we have exactly

$$\int_{-\infty}^{+\infty} f(t)\delta(t - t_0)\, \mathrm{d}t = f(t_0). \tag{6.39}$$

This integral is sometimes referred to as the shifting property of the delta function: $\delta(t - t_0)$ acts as a sieve, selecting from all possible values of $f(t)$ its value at the point $t = t_0$.

Delta Function with Complicated Arguments. In general the argument of the delta function can be any function of the independent variable. It turns out that such a function can always be rewritten as a sum of delta functions of simple argument. Here are some examples.

- $\delta(-t)$

  Let $t' = -t$, then $\mathrm{d}t = -\mathrm{d}t'$. We can write

$$\int_{-\infty}^{+\infty} f(t)\delta(-t)\, \mathrm{d}t = -\int_{+\infty}^{-\infty} f(-t')\delta(t')\, \mathrm{d}t' = \int_{-\infty}^{+\infty} f(-t')\delta(t')\, \mathrm{d}t' = f(0).$$

Since

$$\int_{-\infty}^{+\infty} f(t)\delta(t)\,\mathrm{d}t = f(0),$$

therefore

$$\delta(-t) = \delta(t). \tag{6.40}$$

This result is almost self evident.

- $\delta(at)$

    Let $t' = at$, then $\mathrm{d}t = \mathrm{d}t'/a$. Hence, if $a > 0$,

$$\int_{-\infty}^{+\infty} f(t)\delta(at)\,\mathrm{d}t = \int_{-\infty}^{+\infty} f\left(\frac{t'}{a}\right)\delta(t')\frac{1}{a}\mathrm{d}t' = \frac{1}{a}\int_{-\infty}^{+\infty} f\left(\frac{t'}{a}\right)\delta(t')\,\mathrm{d}t'$$

$$= \frac{1}{a}f\left(\frac{0}{a}\right) = \frac{1}{a}f(0).$$

Since

$$\int_{-\infty}^{+\infty} f(t)\frac{1}{a}\delta(t)\,\mathrm{d}t = \frac{1}{a}\int_{-\infty}^{+\infty} f(t)\delta(t)\,\mathrm{d}t = \frac{1}{a}f(0),$$

therefore

$$\delta(at) = \frac{1}{a}\delta(t).$$

Since

$$\delta(-at) = \delta(at),$$

we can write

$$\delta(at) = \frac{1}{|a|}\delta(t). \tag{6.41}$$

- $\delta(t^2 - a^2)$

    The argument of this function goes to zero when $t = a$ and $t = -a$, which seems to imply two δ functions. There can be contributions to the integral

$$\int_{-\infty}^{+\infty} f(t)\delta\left(t^2 - a^2\right)\,\mathrm{d}t = \int_{-\infty}^{+\infty} f(t)\delta[(t-a)(t+a)]\mathrm{d}t$$

only at the zeros of the argument of the delta function. That is

$$\int_{-\infty}^{+\infty} f(t)\delta\left(t^2 - a^2\right)\,\mathrm{d}t = \int_{-a-\epsilon}^{-a+\epsilon} f(t)\delta\left(t^2 - a^2\right)\,\mathrm{d}t + \int_{a-\epsilon}^{a+\epsilon} f(t)\delta\left(t^2 - a^2\right)\,\mathrm{d}t.$$

Near the two zeros, $t^2 - a^2$ can be approximated as

$$t^2 - a^2 = (t-a)(t+a) = \begin{cases} (-2a)(t+a) & t \to -a \\ (+2a)(t-a) & t \to +a \end{cases}.$$

In the limit as $\epsilon \to 0$, the integral becomes

$$\int_{-\infty}^{+\infty} f(t)\delta \left(t^2 - a^2\right) \mathrm{d}t = \int_{-a-\epsilon}^{-a+\epsilon} f(t)\delta \left((-2a)(t+a)\right) \mathrm{d}t$$

$$+ \int_{a-\epsilon}^{a+\epsilon} f(t)\delta \left((2a)(t-a)\right) \mathrm{d}t = \frac{1}{|2a|} \int_{-a-\epsilon}^{-a+\epsilon} f(t)\delta(t+a)\mathrm{d}t$$

$$+ \frac{1}{|2a|} \int_{a-\epsilon}^{a+\epsilon} f(t)\delta(t-a)\mathrm{d}t = \int_{-\infty}^{+\infty} f(t)\frac{1}{|2a|}[\delta(t+a) + \delta(t-a)]\mathrm{d}t.$$

Therefore

$$\delta \left(t^2 - a^2\right) = \frac{1}{|2a|}[\delta(t+a) + \delta(t-a)]. \tag{6.42}$$

The Laplace Transform of the Delta Function and Its Derivative. It follows from the definitions of the Laplace transform and the delta function that the Laplace transform of the delta function is given by

$$\mathfrak{L}[\delta(t-a)] = \int_0^\infty \mathrm{e}^{-st}\delta(t-a)\mathrm{d}t = \mathrm{e}^{-sa}. \tag{6.43}$$

The Laplace transform of the derivative of a delta function can be evaluated using integration by parts:

$$\mathfrak{L}[\delta'(t-a)] = \int_0^\infty \mathrm{e}^{-st}\frac{\mathrm{d}}{\mathrm{d}t}\delta(t-a)\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}\mathrm{d}(\delta(t-a))$$

$$= \left[\mathrm{e}^{-st}\delta(t-a)\right]_0^\infty - \int_0^\infty \delta(t-a)\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-st}\mathrm{d}t.$$

Since $\delta(t-a)$ vanishes everywhere except at $t = a$, at both upper and lower limits the integrated part is equal to zero. Therefore

$$\mathfrak{L}[\delta'(t-a)] = s \int_0^\infty \delta(t-a)\mathrm{e}^{-st}\mathrm{d}t = s\, \mathrm{e}^{-sa}. \tag{6.44}$$

In dealing with phenomena of an impulsive nature, this is a very useful expression.

6.3.2 The Heaviside Unit Step Function

Definition of the Step Function. The Heaviside unit step function $u(t-c)$ can be defined from the integration of the delta function $\delta(t'-c)$

$$u(t-c) = \int_{-\infty}^t \delta(t'-c)\mathrm{d}t'. \tag{6.45}$$

The delta function is identically equal to zero if $t' < c$. The upper limit of the integration variable t' is t. If t is less than c, then all t' will be less than c.
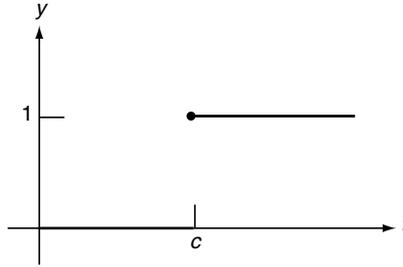
Fig. 6.3. The Heaviside unit step function $u(t-c)$

The integral is equal to zero. If t is greater than c, the integral is equal to one by the definition of the delta function. Thus

$$u(t-c) = \begin{cases} 0 & t < c \\ 1 & t > c \end{cases}.$$ (6.46)

The step function can be defined directly with (6.46) without referring to (6.45). However, with (6.45), it is immediately clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t-c) = \delta(t-c).$$ (6.47)

A plot of the Heaviside unit step function $y = u(t-c)$ is shown in Fig. 6.3. Interestingly, the function does not get its name because it is heavy on one side, but, rather from the British engineer Oliver Heaviside. Very often this function is simply called step function.

Very often we have to deal with a pulse of a finite duration. These step functions are very convenient in such situation. For example, the square pulse

$$y(t) = \begin{cases} 0 & 0 < t < \pi \\ 1 & \pi < t < 2\pi \\ 0 & 2\pi < t < \infty \end{cases}$$

can be expressed as

$$y(t) = u(t-\pi) - u(t-2\pi).$$

The sketch of this function is shown in Fig. 6.4.

Shifting Operation. In some problems a system which becomes active at $t = 0$, because of some initial disturbance, is subsequently acted upon by another disturbance beginning at a later time $t = c$. In this situation, the analytical description is greatly facilitated by the function

$$y = f(t-c)u(t-c),$$

which represent a shifting operation. First, $f(t-c)$ represents a translation of $f(t)$ by a distance c in the positive t-direction. Multiplying $u(t-c)$ has the
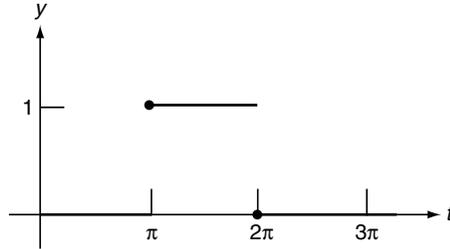
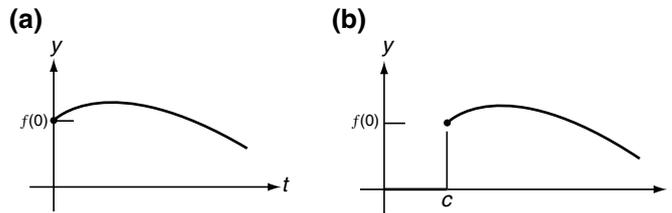**Fig. 6.4.** The square impulse $u(t - \pi) - u(t - 2\pi)$

Fig. 6.5. A translation of a given function. (**a**) $y = f(t)$; (**b**) $y = f(t - c)u(t - c)$

effect of "cutting off" or making everything vanish to the left of c. This is shown in Fig. 6.5.

Laplace Transform Involving Step Function. The Laplace transform of the step function is easily determined:

$$\mathfrak{L}[u(t - c)] = \int_0^\infty \mathrm{e}^{-st} u(t - c) \mathrm{d}t = \int_c^\infty \mathrm{e}^{-st} \mathrm{d}t$$

$$= \frac{1}{s} \mathrm{e}^{-sc}. \tag{6.48}$$

The step function is particularly important in transform theory because of the following relationship between the transform of $f(t)$ and that of its translation $f(t - c)u(t - c)$.

$$\mathfrak{L}[f(t - c)u(t - c)] = \int_0^\infty \mathrm{e}^{-st} f(t - c)u(t - c) \mathrm{d}t = \int_c^\infty \mathrm{e}^{-st} f(t - c) \mathrm{d}t.$$

Making a change of variable $t' = t - c$, we have

$$\int_c^\infty \mathrm{e}^{-st} f(t - c) \mathrm{d}t = \int_0^\infty \mathrm{e}^{-s(t' + c)} f(t') \mathrm{d}t' = \mathrm{e}^{-sc} \int_0^\infty \mathrm{e}^{-st'} f(t') \mathrm{d}t'$$

$$= \mathrm{e}^{-sc} \mathfrak{L}[f(t)].$$

Therefore

$$\mathfrak{L}[f(t - c)u(t - c)] = \mathrm{e}^{-sc} \mathfrak{L}[f(t)]. \tag{6.49}$$

Its inverse is of considerable importance.

$$f(t-c)u(t-c) = \mathfrak{L}^{-1}[e^{-sc}\mathfrak{L}[f(t)]]. \tag{6.50}$$

This relationship is sometimes referred as t-shifting (or second shifting) theorem.

6.4 Differential Equations with Discontinuous Forcing Functions

In this section we turn our attention to some examples in which the nonhomogeneous term, or forcing function, is discontinuous.

We start with the simplest case. A particle of mass m initially at rest is set into motion by a sudden blow at $t = t_0$. Assuming no friction, we wish to find the position as a function of time. Such a common every-day occurrence is rather "awkward" for "ordinary" mathematics to deal with. However, with Laplace transform and delta function, it becomes very easy.

In the Newton's dynamic equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F, \tag{6.51}$$

let us express the force of the sudden blow by the delta function

$$F = P\delta(t-t_0). \tag{6.52}$$

The initial conditions are

$$x(0) = 0, \quad x'(0) = 0. \tag{6.53}$$

Applying the Laplace transform to the differential equation

$$\mathfrak{L}[mx''] = \mathfrak{L}[P\delta(t-t_0)], \tag{6.54}$$

we obtain

$$ms^2\mathfrak{L}[x] = P\,e^{-st_0}. \tag{6.55}$$

So

$$x(t) = \frac{P}{m}\mathfrak{L}^{-1}\left[\frac{e^{-st_0}}{s^2}\right] = \frac{P}{m}\mathfrak{L}^{-1}[e^{-st_0}\mathfrak{L}[t]]$$
$$= \frac{P}{m}(t-t_0)u(t-t_0). \tag{6.56}$$

This means

$$x(t) = \begin{cases} 0 & t < t_0 \\ \frac{P}{m}(t-t_0) & t > t_0 \end{cases}. \tag{6.57}$$

The result says that the particle will stay put until t_0, after that the distance will increase linearly with time. The velocity of the particle is given by

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{P}{m}, \tag{6.58}$$

which is a constant. In fact we see that the amplitude P of the delta function is equal to mv which is the momentum. This shows that what the sudden blow did is to impart a momentum P to the particle. This momentum stays the same with the particle thereafter.

Example 6.4.1. Let us consider the damped, driven, harmonic oscillator. The mass m is driven by an applied force $F(t)$. It also experiences the spring force $-kx(t)$ and a friction force $-bx'(t)$, proportional to its velocity. The differential equation describing the motion is

$$mx'' + bx' + kx = F(t).$$

If it is at rest initially

$$x(0) = 0, \quad x'(0) = 0,$$

and the force function is an ideal impulse peaked at t_0, that is

$$F(t) = P_0\delta\left(t - t_0\right),$$

find the displacement x as a function of time t.

Solution 6.4.1. Applying the Laplace transform to both sides of the equation

$$\mathfrak{L}\left[x'' + \frac{b}{m}x' + \frac{k}{m}x\right] = \frac{P_0}{m}\mathfrak{L}[\delta\left(t - t_0\right)]$$

leads to

$$s^2\mathfrak{L}[x] + \frac{b}{m}s\mathfrak{L}[x] + \frac{k}{m}\mathfrak{L}[x] = \frac{P_0}{m}\mathrm{e}^{-st_0}.$$

Therefore

$$\mathfrak{L}[x] = \frac{P_0}{m}\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}\mathrm{e}^{-st_0}.$$

Let us write

$$s^2 + \frac{b}{m}s + \frac{k}{m} = s^2 + \frac{b}{m}s + \left(\frac{b}{2m}\right)^2 - \left(\frac{b}{2m}\right)^2 + \frac{k}{m}$$

$$= \left(s + \frac{b}{2m}\right)^2 + \frac{k}{m} - \left(\frac{b}{2m}\right)^2,$$

and simplify the notation with

$$\alpha = \frac{b}{2m}, \quad \omega^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2,$$

so we have

$$\mathfrak{L}[x] = \frac{P_0}{m\omega} \frac{\omega}{(s+\alpha)^2 + \omega^2} e^{-st_0}.$$

Three different cases arise:

(a) The oscillatory case, $\omega^2 > 0$.

$$x(t) = \mathfrak{L}^{-1}\left[\frac{P_0}{m\omega} \frac{\omega}{(s+\alpha)^2 + \omega^2} e^{-st_0}\right] = \frac{P_0}{m\omega} \mathfrak{L}^{-1}\left[e^{-st_0} \frac{\omega}{(s+\alpha)^2 + \omega^2}\right]$$

$$= \frac{P_0}{m\omega} \mathfrak{L}^{-1}\left[e^{-st_0} \mathfrak{L}[e^{-\alpha t} \sin \omega t]\right] = \frac{P_0}{m\omega} e^{-\alpha(t-t_0)} \sin \omega(t - t_0) u(t - t_0).$$

(b) The over-damped case $\omega^2 < 0$. Let $\beta^2 = -\omega^2$,

$$x(t) = \mathfrak{L}^{-1}\left[\frac{P_0}{m\beta} \frac{\beta}{(s+\alpha)^2 - \beta^2} e^{-st_0}\right] = \frac{P_0}{m\beta} \mathfrak{L}^{-1}\left[e^{-st_0} \frac{\beta}{(s+\alpha)^2 - \beta^2}\right]$$

$$= \frac{P_0}{m\beta} \mathfrak{L}^{-1}[e^{-st_0} \mathfrak{L}[e^{-\alpha t} \sinh \beta t]] = \frac{P_0}{m\beta} e^{-\alpha(t-t_0)} \sinh \beta(t - t_0) u(t - t_0).$$

(c) The critically damped case $\omega^2 = 0$.

$$x(t) = \mathfrak{L}^{-1}\left[\frac{P_0}{m} \frac{1}{(s+\alpha)^2} e^{-st_0}\right] = \frac{P_0}{m} \mathfrak{L}^{-1}\left[e^{-st_0} \frac{1}{(s+\alpha)^2}\right]$$

$$= \frac{P_0}{m} \mathfrak{L}^{-1}[e^{-st_0} \mathfrak{L}[e^{-\alpha t} t]] = \frac{P_0}{m} e^{-\alpha(t-t_0)}(t - t_0) u(t - t_0).$$

Notice in all three cases, $x(t)$ is equal to zero before $t = t_0$, as you would expect, because the system cannot respond until after the impulse has occurred. This kind of behavior is often referred to as being causal. Causality, a characteristic of solutions involving time, requires that there can be no response before the application of a drive.

It is interesting to note that Newton's equation is invariant under the transformation of $t \to -t$. Thus clearly causality is not implied by Newton's equation. The causality shown here is the result of the definition of the Laplace transform. The fact that this important physical requirement is built in the Laplace transformation is another reason that this method is so useful.

Example 6.4.2. A mass $m = 1$ is attached to spring with constant $k = 4$, and there is no friction, $b = 0$. The mass is released from rest with $x(0) = 3$. At the instant $t = 2\pi$ the mass is struck with a hammer, providing an impulse $P_0 = 8$. Determine the motion of the mass.
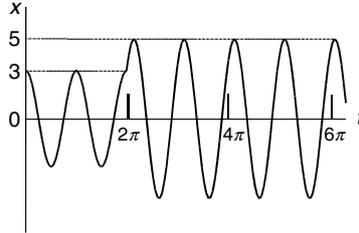
Fig. 6.6. The plot of $x(t) = 3\cos 2t + 4\sin 2(t - 2\pi)u(t - 2\pi)$

Solution 6.4.2. We need to solve the initial value problem

$$x'' + 4x = 8\delta(t - 2\pi); \quad x(0) = 3, \quad x'(0) = 0.$$

Apply the Laplace transform to get

$$s^2 \mathfrak{L}[x] - 3s + 4\mathfrak{L}[x] = 8e^{-2\pi s},$$

so

$$(s^2 + 4)\mathfrak{L}[x] = 3s + 8e^{-2\pi s}.$$

Therefore

$$\mathfrak{L}[x] = \frac{3s}{(s^2 + 4)} + \frac{8e^{-2\pi s}}{(s^2 + 4)},$$

hence

$$x(t) = \mathfrak{L}^{-1}\left[\frac{3s}{s^2 + 4}\right] + \mathfrak{L}^{-1}\left[\frac{8e^{-2\pi s}}{s^2 + 4}\right] = 3\mathfrak{L}^{-1}\left[\frac{s}{s^2 + 4}\right] + 4\mathfrak{L}^{-1}\left[\frac{2e^{-2\pi s}}{s^2 + 4}\right]$$

$$= 3\mathfrak{L}^{-1}[\mathfrak{L}[\cos 2t]] + 4\mathfrak{L}^{-1}[e^{-2\pi s}\mathfrak{L}[\sin 2t]]$$

$$= 3\cos 2t + 4\sin 2(t - 2\pi)u(t - 2\pi)$$

or

$$x(t) = \begin{cases} 3\cos 2t & t < 2\pi \\ 3\cos 2t + 4\sin 2(t - 2\pi) & t > 2\pi \end{cases}.$$

As $3\cos 2t + 4\sin 2t = 5\cos(2t - \theta)$ and $\theta = \tan^{-1}(4/3)$, we see the effect of the impulse at $t = 2\pi$. It instantaneously increases the amplitude of the oscillations from 3 to 5. Although the frequency is still the same, there is a discontinuity in velocity. The plot of $x(t)$ is shown in Fig. 6.6.

Example 6.4.3. Consider the RLC series circuit shown in Fig. 6.7 with $R = 110\ \Omega$, $L = 1$ H, $C = 0.001$ F, and a battery supplying an emf of 90 V. Initially there is no current in the circuit and no charge on the capacitor. At $t = 0$ the switch is closed and at $t = T$ ($T = 1$ s) the battery is removed from the circuit in such a way that the RLC circuit is still closed but without emf. Find the current $i(t)$ as a function of time.
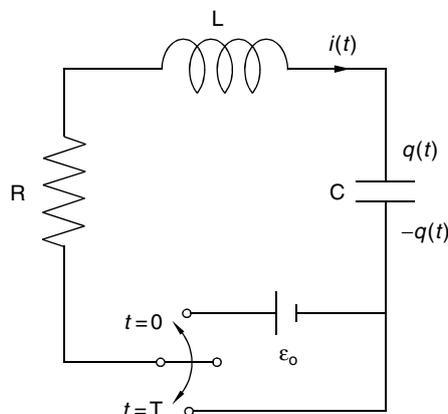
Fig. 6.7. A RLC circuit. The open circuit without charge on the capacitor is closed at $t = 0$. At $t = T$, the battery is removed from the circuit in such a way that the circuit is closed but without emf

Solution 6.4.3. The circuit equation is given by

$$Li' + Ri + \frac{1}{C}q = e(t)$$

$$i = \frac{\mathrm{d}q}{\mathrm{d}t}$$

and the initial conditions are

$$i(0) = 0, \quad q(0) = 0.$$

In this problem

$$e(t) = 90[u(t) - u(t-1)].$$

Apply the Laplace transform to the two differential equations to get

$$Ls\mathfrak{L}[i] + R\mathfrak{L}[i] + \frac{1}{C}\mathfrak{L}[q] = \mathfrak{L}[e(t)]$$

$$\mathfrak{L}[i] = s\mathfrak{L}[q].$$

Combine the two equations we obtain

$$Ls\mathfrak{L}[i] + R\mathfrak{L}[i] + \frac{1}{Cs}\mathfrak{L}[i] = \mathfrak{L}[e(t)].$$

Putting in the R, L, C, and $e(t)$ values in, we have

$$s\mathfrak{L}[i] + 110\mathfrak{L}[i] + \frac{1}{0.001s}\mathfrak{L}[i] = \mathfrak{L}[90[u(t) - u(t-1)]]$$

$$= 90\frac{1 - \mathrm{e}^{-s}}{s}.$$

Therefore

$$\mathcal{L}[i] = 90 \frac{1 - e^{-s}}{s^2 + 110s + 1000}.$$

Since

$$\frac{90}{s^2 + 110s + 1000} = \frac{1}{s + 10} - \frac{1}{s + 100},$$

so we have

$$i(t) = \mathcal{L}^{-1} \left[\frac{1}{s + 10} - \frac{1}{s + 100} - e^{-s} \left(\frac{1}{s + 10} - \frac{1}{s + 100} \right) \right]$$

$$= e^{-10t} - e^{-100t} - (e^{-10(t-1)} - e^{-100(t-1)})u(t - 1).$$

6.5 Convolution

Another important general property of the Laplace transform has to do with the products of transforms. It often happens that we are given two transforms $F(s)$ and $G(s)$ whose inverses $f(t)$ and $g(t)$ we know, and we would like to calculate the inverse of the product $F(s)G(s)$ from those known inverses $f(t)$ and $g(t)$. The inverse is called the convolution of $f(t)$ and $g(t)$. In order to understand the meaning of the mathematical formulation, we will first consider a specific example.

6.5.1 The Duhamel Integral

Let us once again consider the damped driven oscillator

$$mx'' + bx' + kx = f(t) \tag{6.59}$$

with $x(0) = 0, \ x'(0) = 0$. Applying the Laplace transform we get

$$\mathcal{L}[x] = \frac{\mathcal{L}[f(t)]}{m[(s + \alpha)^2 + \omega^2]}, \tag{6.60}$$

where $\alpha = \frac{b}{2m}, \ \omega^2 = \frac{k}{m} - (\frac{b}{2m})^2$. If $f(t)$ is a unit impulse at time τ,

$$f(t) = \delta(t - \tau), \tag{6.61}$$

then we have

$$\mathcal{L}[x] = \frac{e^{-s\tau}}{m[(s + \alpha)^2 + \omega^2]}. \tag{6.62}$$

Therefore

$$x(t) = \mathcal{L}^{-1}[e^{-s\tau}\mathcal{L}[\frac{1}{m\omega}e^{-\alpha t}\sin \omega t]]$$

$$= \frac{1}{m\omega}e^{-\alpha(t-\tau)}\sin \omega(t - \tau)u(t - \tau). \tag{6.63}$$

For $t > \tau$,

$$x(t) = \frac{1}{m\omega} e^{-\alpha(t-\tau)} \sin \omega(t - \tau) u(t - \tau). \qquad (6.64)$$

If we designate the solution as $g(t)$ in the particular case where τ is equal to zero,

$$g(t) = \frac{1}{m\omega} e^{-\alpha t} \sin \omega t, \qquad (6.65)$$

so in general if τ is not equal to zero,

$$x(t) = g(t - \tau). \qquad (6.66)$$

If the force function is

$$f(t) = P\delta(t - \tau), \qquad (6.67)$$

the solution (or the response function) is clearly

$$x(t) = Pg(t - \tau). \qquad (6.68)$$

Now we consider the response of the system under a general external force function shown in Fig. 6.8.

This force may be assumed to be made up of a series of impulses of varying magnitude. As we have discussed, the impulse is actually momentum P imparted. Since the change of momentum is equal to force $(\Delta P/\Delta t = f)$, the impulse imparted during a short time interval is equal to force multiplied by the time duration.

Assuming that at time τ, the force $f(\tau)$ acts on the system for a short period of time $\Delta\tau$, the impulse acting at $t = \tau$ is given by $f(\tau)\Delta\tau$. At any time t, the elapsed time since the impulse is $t - \tau$, so the response of the system at time t due to this impulse is

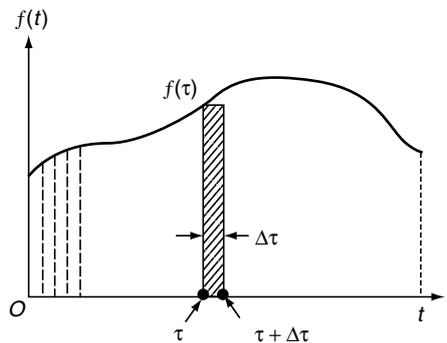$$\Delta x(t) = f(\tau)\Delta\tau g(t - \tau). \qquad (6.69)$$

Fig. 6.8. An arbitrary forcing function

The total response at time t can be found by summing all the responses due to the elementary impulses acting all times

$$x(t) = \sum f(\tau)g(t-\tau)\Delta\tau. \tag{6.70}$$

Letting $\Delta\tau \to 0$ and replacing the summation by the integration, we obtain

$$x(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau \tag{6.71}$$

or

$$x(t) = \frac{1}{m\omega}\int_0^t f(\tau)\mathrm{e}^{-\alpha(t-\tau)}\sin\omega(t-\tau)\mathrm{d}\tau. \tag{6.72}$$

This result is known as the Duhamel integral. In many cases the function $f(\tau)$ has a form that permits an explicit integration. In the case such integration is not possible, it can be evaluated numerically without much difficulty.

6.5.2 The Convolution Theorem

The Duhamel integral can also be viewed in the following way. Since

$$g(t) = \frac{1}{m\omega}\mathrm{e}^{-\alpha t}\sin\omega t = \mathfrak{L}^{-1}\left[\frac{1}{m[(s+\alpha)^2+\omega^2]}\right], \tag{6.73}$$

and by (6.60)

$$\begin{aligned}\mathfrak{L}[x(t)] &= \mathfrak{L}[f(t)]\frac{1}{m[(s+\alpha)^2+\omega^2]} \\ &= \mathfrak{L}[f(t)]\mathfrak{L}[g(t)],\end{aligned} \tag{6.74}$$

it follows that

$$x(t) = \mathfrak{L}^{-1}[\mathfrak{L}[f(t)]\mathfrak{L}[g(t)]]. \tag{6.75}$$

On the other hand

$$x(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau, \tag{6.76}$$

therefore

$$\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau = \mathfrak{L}^{-1}[\mathfrak{L}[f(t)]\mathfrak{L}[g(t)]]. \tag{6.77}$$

It turns out, as long as these transforms exist, this relationship

$$\mathfrak{L}\left[\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau\right] = \mathfrak{L}[f(t)]\mathfrak{L}[g(t)]] \tag{6.78}$$

is generally true for any arbitrary functions of f and g. It is known as the convolution theorem. If this is true, then

$$\mathfrak{L}\left[\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda\right] = \mathfrak{L}[f(t)]\mathfrak{L}[g(t)]] \tag{6.79}$$

must also be true, since the roles played by f and g in the equation are symmetric. This can be easily demonstrated directly by a change of variable. Let $\lambda = t - \tau$, then

$$\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau = \int_t^0 f(t-\lambda)g(\lambda)\mathrm{d}(-\lambda)$$

$$= \int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda. \tag{6.80}$$

The proof of the convolution theorem goes as follows.
By definition

$$\mathfrak{L}\left[\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda\right] = \int_0^\infty \mathrm{e}^{-st}\left[\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda\right]\mathrm{d}t. \tag{6.81}$$

Now with

$$u(t-\lambda) = \begin{cases} 1 & \lambda < t \\ 0 & \lambda > t \end{cases} \tag{6.82}$$

and

$$f(t-\lambda)g(\lambda)u(t-\lambda) = \begin{cases} f(t-\lambda)g(\lambda) & \lambda < t \\ 0 & \lambda > t. \end{cases} \tag{6.83}$$

We can write

$$\int_0^\infty f(t-\lambda)g(\lambda)u(t-\lambda)\mathrm{d}\lambda = \int_0^t f(t-\lambda)g(\lambda)u(t-\lambda)\mathrm{d}\lambda$$

$$+ \int_t^\infty f(t-\lambda)g(\lambda)u(t-\lambda)\mathrm{d}\lambda, \tag{6.84}$$

the second term on the right-hand side is equal to zero because the lower limit of λ is t, so $\lambda > t$. In the first term on the right-hand side, the range of λ is between 0 and t, so $\lambda < t$, using (6.83) we have

$$\int_0^\infty f(t-\lambda)g(\lambda)u(t-\lambda)\mathrm{d}\lambda = \int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda. \tag{6.85}$$

Putting (6.85) into (6.81)

$$\mathfrak{L}\left[\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda\right] = \int_0^\infty \mathrm{e}^{-st}\left[\int_0^\infty f(t-\lambda)g(\lambda)u(t-\lambda)\mathrm{d}\lambda\right]\mathrm{d}t, \tag{6.86}$$

and changing the order of integration

$$\int_0^\infty \mathrm{e}^{-st}\left[\int_0^\infty f(t-\lambda)g(\lambda)u(t-\lambda)\mathrm{d}\lambda\right]\mathrm{d}t$$

$$= \int_0^\infty g(\lambda)\left[\int_0^\infty \mathrm{e}^{-st}f(t-\lambda)u(t-\lambda)\mathrm{d}t\right]\mathrm{d}\lambda, \tag{6.87}$$

we obtain

$$\mathfrak{L}[\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda] = \int_0^\infty g(\lambda)\left[\int_0^\infty \mathrm{e}^{-st}f(t-\lambda)u(t-\lambda)\mathrm{d}t\right]\mathrm{d}\lambda. \quad (6.88)$$

Because of the presence of $u(t-\lambda)$, the integrand of the inner integral is identically zero for all $t < \lambda$. Hence the inner integration effectively starts not at $t = 0$ but at $t = \lambda$. Therefore

$$\mathfrak{L}\left[\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda\right] = \int_0^\infty g(\lambda)\left[\int_\lambda^\infty \mathrm{e}^{-st}f(t-\lambda)\mathrm{d}t\right]\mathrm{d}\lambda. \quad (6.89)$$

Now in the inner integral on the right, let $t - \lambda = \tau$ and $\mathrm{d}t = \mathrm{d}\tau$. Then

$$\begin{aligned}
\mathfrak{L}\left[\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda\right] &= \int_0^\infty g(\lambda)\left[\int_0^\infty \mathrm{e}^{-s(\tau+\lambda)}f(\tau)\mathrm{d}\tau\right]\mathrm{d}\lambda \\
&= \int_0^\infty \mathrm{e}^{-s\lambda}g(\lambda)\left[\int_0^\infty \mathrm{e}^{-s\tau}f(\tau)\mathrm{d}\tau\right]\mathrm{d}\lambda \\
&= \left[\int_0^\infty \mathrm{e}^{-s\tau}f(\tau)\mathrm{d}\tau\right]\left[\int_0^\infty \mathrm{e}^{-s\lambda}g(\lambda)\mathrm{d}\lambda\right] \\
&= \mathfrak{L}[f(t)]\mathfrak{L}[g(t)] \quad (6.90)
\end{aligned}$$

as asserted.

A common notation is to designate the convolution integral as

$$\int_0^t f(t-\lambda)g(\lambda)\mathrm{d}\lambda = f(t) * g(t). \quad (6.91)$$

So the convolution theorem is often written as

$$\mathfrak{L}[f]\mathfrak{L}[g] = \mathfrak{L}[f * g]. \quad (6.92)$$

Example 6.5.1. Use convolution to find

$$\mathfrak{L}^{-1}\left[\frac{1}{s^2(s-a)}\right].$$

Solution 6.5.1. Since

$$\mathfrak{L}[t] = \frac{1}{s^2}, \quad \mathfrak{L}[\mathrm{e}^{at}] = \frac{1}{s-a},$$

we can write

$$\mathfrak{L}^{-1}\left[\frac{1}{s^2(s-a)}\right] = \mathfrak{L}^{-1}\left[\frac{1}{s^2} \cdot \frac{1}{s-a}\right] = \mathfrak{L}^{-1}[\mathfrak{L}[t]\mathfrak{L}[\mathrm{e}^{at}]].$$

Therefore

$$\mathfrak{L}^{-1}\left[\frac{1}{s^2(s-a)}\right] = t\,\mathrm{e}^{at} = \int_0^t \tau\mathrm{e}^{a(t-\tau)}\mathrm{d}\tau = \frac{1}{a^2}\left(\mathrm{e}^{at} - at - 1\right).$$

6.6 Further Properties of Laplace Transforms

6.6.1 Transforms of Integrals

From the property of the transform of a derivative, one can derive a formula for the transform of an integral.

$$\mathfrak{L}\left[\int_0^t f(x)\mathrm{d}x\right] = \int_0^\infty \mathrm{e}^{-st}\left[\int_0^t f(x)\mathrm{d}x\right]\mathrm{d}t.$$

Let

$$g(t) = \int_0^t f(x)\mathrm{d}x,$$

then

$$g'(t) = f(t) \quad \text{and} \quad g(0) = 0.$$

Since

$$\mathfrak{L}\left[g'(t)\right] = s\mathfrak{L}\left[g(t)\right] - g(0),$$

we have

$$\mathfrak{L}\left[f(t)\right] = s\mathfrak{L}\left[\int_0^t f(x)\mathrm{d}x\right]. \tag{6.93}$$

Thus if $F(s) = \mathfrak{L}\left[f(t)\right]$,

$$\mathfrak{L}\left[\int_0^t f(x)\mathrm{d}x\right] = \frac{\mathfrak{L}\left[f(t)\right]}{s} = \frac{1}{s}F(s). \tag{6.94}$$

This formula is very useful in finding the inverse transform of a fraction that has the form of $p(s)/\left[s^n q(s)\right]$. For example:

$$\mathfrak{L}^{-1}\left[\frac{1}{s(s-a)}\right] = \mathfrak{L}^{-1}\left[\frac{1}{s}\mathfrak{L}\left[\mathrm{e}^{at}\right]\right] = \mathfrak{L}^{-1}\mathfrak{L}\left[\int_0^t \mathrm{e}^{ax}\mathrm{d}x\right]$$

$$= \int_0^t \mathrm{e}^{ax}\mathrm{d}x = \frac{1}{a}(\mathrm{e}^{at} - 1).$$

Similarly

$$\mathfrak{L}^{-1}\left[\frac{1}{s^2(s-a)}\right] = \mathfrak{L}^{-1}\left[\frac{1}{s}\mathfrak{L}\left[\frac{1}{a}(\mathrm{e}^{at} - 1)\right]\right]$$

$$= \int_0^t \frac{1}{a}(\mathrm{e}^{ax} - 1)\mathrm{d}x = \frac{1}{a^2}(\mathrm{e}^{at} - at - 1).$$

This method is often more convenient than the method of partial fractions.

6.6.2 Integration of Transforms

Differentiation of $F(s)$ corresponds to multiplication of $f(t)$ by $-t$. It is natural to expect that integration of $F(s)$ will correspond to division of $f(t)$ by t. This is indeed the case, provided the limits of integration are appropriately chosen.

If $F(s')$ is the Laplace transform of $f(t)$, then

$$\int_s^\infty F(s')\mathrm{d}s' = \int_s^\infty \left[\int_0^\infty \mathrm{e}^{-s't}f(t)\mathrm{d}t\right]\mathrm{d}s' = \int_0^\infty \left[\int_s^\infty \mathrm{e}^{-s't}f(t)\mathrm{d}s'\right]\mathrm{d}t$$

$$= \int_0^\infty f(t)\left[\int_s^\infty \mathrm{e}^{-s't}\mathrm{d}s'\right]\mathrm{d}t = \int_0^\infty f(t)\left[-\frac{1}{t}\mathrm{e}^{-s't}\right]_{s'=s}^\infty \mathrm{d}t$$

$$= \int_0^\infty f(t)\frac{1}{t}\mathrm{e}^{-st}\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}\frac{1}{t}f(t)\mathrm{d}t = \mathfrak{L}\left[\frac{f(t)}{t}\right]. \quad (6.95)$$

This relationship, namely

$$\mathfrak{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \mathfrak{L}\left[f(t)\right]\mathrm{d}s'$$

is useful if $\mathfrak{L}\left[f(t)\right]$ is known.

Example 6.6.1. Find (a) $\mathfrak{L}\left[\frac{1}{t}(\mathrm{e}^{-at} - \mathrm{e}^{-bt})\right]$,   (b) $\mathfrak{L}\left[\frac{\sin t}{t}\right]$.

Solution 6.6.1. (a)

$$\mathfrak{L}\left[\frac{1}{t}(\mathrm{e}^{-at} - \mathrm{e}^{-bt})\right] = \int_s^\infty \mathfrak{L}\left[\mathrm{e}^{-at} - \mathrm{e}^{-bt}\right]\mathrm{d}s' = \int_s^\infty \left(\frac{1}{s'+a} - \frac{1}{s'+b}\right)\mathrm{d}s'$$

$$= \left[\ln(s'+a) - \ln(s'+b)\right]_{s'=s}^\infty = \left[\ln\frac{s'+a}{s'+b}\right]_{s'=s}^\infty$$

$$= \ln 1 - \ln\frac{s+a}{s+b} = \ln\frac{s+b}{s+a}.$$

(b)

$$\mathfrak{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty \mathfrak{L}\left[\sin t\right]\mathrm{d}s' = \int_s^\infty \frac{1}{s'^2+1}\mathrm{d}s' = \left[\tan^{-1} s'\right]_{s'=s}^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = \tan^{-1}\frac{1}{s}.$$

6.6.3 Scaling

If $F(s) = \mathfrak{L}\left[f(t)\right] = \int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$ is already known, then $\mathfrak{L}\left[f(at)\right]$ can be easily obtained by a change of scale. By definition

$$\mathfrak{L}\left[f(at)\right] = \int_0^\infty \mathrm{e}^{-st}f(at)\mathrm{d}t = \frac{1}{a}\int_0^\infty \mathrm{e}^{-(s/a)at}f(at)\mathrm{d}(at). \quad (6.96)$$

Let $t' = at$, the integral becomes

$$\int_0^\infty \mathrm{e}^{-(s/a)at}f(at)\mathrm{d}(at) = \int_0^\infty \mathrm{e}^{-(s/a)t'}f(t')\mathrm{d}(t'),$$

which is just the Laplace transform of f with the parameter s replaced by s/a. Therefore

$$\mathfrak{L}\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right). \tag{6.97}$$

Example 6.6.2. If $\mathfrak{L}\left[f(t)\right]$ is known to be $\dfrac{1}{s(1+2s)}$, Find $\mathfrak{L}\left[f(2t)\right]$.

Solution 6.6.2.

$$\mathfrak{L}\left[f(2t)\right] = \frac{1}{2}\frac{1}{(s/2)[1+2(s/2)]} = \frac{1}{s(1+s)}.$$

Example 6.6.3. Find $\mathfrak{L}\left[\dfrac{\sin\omega t}{t}\right]$.

Solution 6.6.3. Since $\mathfrak{L}\left[\dfrac{\sin t}{t}\right] = \tan^{-1}\dfrac{1}{s}$, then

$$\mathfrak{L}\left[\frac{\sin\omega t}{\omega t}\right] = \frac{1}{\omega}\tan^{-1}\frac{\omega}{s}.$$

Therefore

$$\mathfrak{L}\left[\frac{\sin\omega t}{t}\right] = \tan^{-1}\frac{\omega}{s}.$$

6.6.4 Laplace Transforms of Periodic Functions

Very often the input functions in physical systems are periodic functions. A function is said to be periodic if there is a number p such that

$$f(t+p) = f(t).$$

The least value of p is called the period of f. A periodic function is one that has the characteristic

$$f(t) = f(t+p) = f(t+2p) = \cdots f(t+np)\cdots. \tag{6.98}$$

The Laplace transform of $f(t)$ is a series of integrals

$$\mathfrak{L}\left[f\right] = \int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$= \int_0^p e^{-st}f(t)\mathrm{d}t + \int_p^{2p} e^{-st}f(t)\mathrm{d}t\cdots\int_{np}^{(n+1)p} e^{-st}f(t)\mathrm{d}t\cdots. \tag{6.99}$$

It follows from a change of variable $t = \tau + np$ that

$$\int_{np}^{(n+1)p} e^{-st} f(t) dt = \int_0^p e^{-s(\tau+np)} f(\tau+np) d\tau = e^{-snp} \int_0^p e^{-s\tau} f(\tau) d\tau.$$

The dummy integration variable τ can be set equal to t, thus

$$\mathfrak{L}[f] = \int_0^p e^{-st} f(t) dt + e^{-sp} \int_0^p e^{-st} f(t) dt + \cdots + e^{-snp} \int_0^p e^{-st} f(t) dt + \cdots$$

$$= (1 + e^{-sp} + e^{-2sp} + \cdots + e^{-nsp} + \cdots) \int_0^p e^{-st} f(t) dt. \qquad (6.100)$$

With the series expansion, $1/(1-x) = 1 + x + x^2 + \cdots$, this equation becomes

$$\mathfrak{L}[f] = \frac{1}{1 - e^{-sp}} \int_0^p e^{-st} f(t) dt. \qquad (6.101)$$

Example 6.6.4. Half-wave rectifier: Find the Laplace transform of the periodic function (shown in Fig. 6.9) whose definition over one period is:

$$f(t) = \begin{cases} \sin \omega t \text{ if } 0 < t < \dfrac{\pi}{\omega} \\ 0 \quad \text{if } \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}.$$
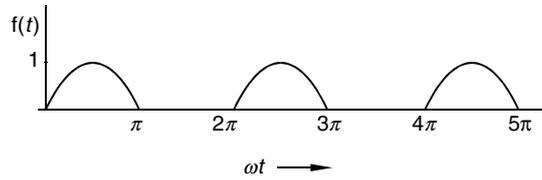

Fig. 6.9. Half-wave rectifier. The definition over one period is $f(t) = \sin \omega t$ for $0 < t < \pi/\omega$ and $f(t) = 0$ for $\pi/\omega < t < 2\pi/\omega$

Solution 6.6.4.

$$\mathfrak{L}[f] = \frac{1}{1 - e^{-s2\pi/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt = \frac{1}{1 - e^{-s2\pi/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \; dt.$$

The integral can be evaluated with integration by parts. However, it is easier to note that the integral is the imaginary part of

$$\int_0^{\pi/\omega} e^{-st} e^{i\omega t} dt = \left[\frac{1}{-s + i\omega} e^{-st+i\omega t} \right]_0^{\pi/\omega} = \frac{1}{-s + i\omega} \left(e^{-s\pi/\omega + i\pi} - 1 \right)$$

$$= \frac{-s - i\omega}{s^2 + \omega^2} \left(-e^{-s\pi/\omega} - 1 \right).$$

Thus

$$\mathcal{L}[f] = \frac{1}{1 - e^{-s2\pi/\omega}} \frac{\omega(1 + e^{-s\pi/\omega})}{s^2 + \omega^2} = \frac{\omega}{(s^2 + \omega^2)(1 - e^{-s\pi/\omega})}.$$

Example 6.6.5. Full-wave rectifier: Find the Laplace transform of the periodic function (shown in Fig. 6.10) whose definition over one period is:

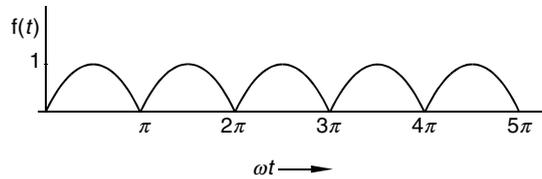$$f(t) = |\sin \omega t| \qquad 0 < t < \frac{\pi}{\omega}.$$

Fig. 6.10. Full-wave rectifier. The definition over one period is $f(t) = |\sin \omega t|$, for $0 < t < \pi/\omega$

Solution 6.6.5. In this case, the period is just π/ω. Therefore

$$\mathcal{L}[f] = \frac{1}{1 - e^{-s\pi/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t \; dt = \frac{\omega(1 + e^{-s\pi/\omega})}{(s^2 + \omega^2)(1 - e^{-s\pi/\omega})}.$$

This is a perfectly good result. It can be simplified somewhat by multiplying both numerator and denominator $\exp(\frac{s\pi}{2\omega})$

$$\mathcal{L}[f] = \frac{\omega(e^{s\pi/(2\omega)} + e^{-s\pi/(2\omega)})}{(s^2 + \omega^2)(e^{s\pi/(2\omega)} - e^{-s\pi/(2\omega)})} = \frac{\omega}{(s^2 + \omega^2)} \coth \frac{s\pi}{2\omega}.$$

6.6.5 Inverse Laplace Transforms Involving Periodic Functions

Any Laplace transform $F(s)$ that either has a factor $(1 - e^{-sp})^{-1}$, or can be written in a form with such a factor as in the last example, indicates that its inverse transform is a periodic function. However, its period may be a multiple of p. This is illustrated in the following example.

Example 6.6.6. Find the inverse and its period of the Laplace transform

$$F(s) = \frac{s}{(s^2 + 1)(1 - e^{-s\pi})}.$$

Solution 6.6.6.

$$f(t) = \mathcal{L}^{-1} \left[\frac{s}{(s^2 + 1)(1 - \mathrm{e}^{-s\pi})} \right]$$

$$= \mathcal{L}^{-1} \left[\frac{s}{(s^2 + 1)} (1 + \mathrm{e}^{-s\pi} + \mathrm{e}^{-2s\pi} + \mathrm{e}^{-3s\pi} + \cdots) \right]$$

$$= \cos t + u(t - \pi) \cos(t - \pi) + u(t - 2\pi) \cos(t - 2\pi)$$

$$+ u(t - 3\pi) \cos(t - 3\pi) + u(t - 4\pi) \cos(t - 4\pi) \cdots$$

$$= \cos t - u(t - \pi) \cos t + u(t - 2\pi) \cos t - u(t - 3\pi) \cos t + \cdots$$

$$= [1 - u(t - \pi)] \cos t + [u(t - 2\pi) - u(t - 3\pi)] \cos t + \cdots.$$

Therefore $f(t)$ is a periodic function with period 2π, whose definition over one period is

$$f(t) = \begin{cases} \cos t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \end{cases}.$$

6.6.6 Laplace Transforms and Gamma Functions

The Laplace transform of t^n is defined as

$$\mathcal{L}[t^n] = \int_0^\infty \mathrm{e}^{-st} t^n \, \mathrm{d}t. \tag{6.102}$$

If we make a change of variable and let $st = x$, the integral becomes

$$\mathcal{L}[t^n] = \int_0^\infty \mathrm{e}^{-x} \left(\frac{x}{s} \right)^n \mathrm{d}\left(\frac{x}{s} \right) = \frac{1}{s^{n+1}} \int_0^\infty \mathrm{e}^{-x} x^n \mathrm{d}x. \tag{6.103}$$

The last integral is known as the *gamma function* of $n+1$, written as $\Gamma(n+1)$. Gamma function occurs frequently in practice. It is given by

$$\Gamma(n) = \int_0^\infty \mathrm{e}^{-x} x^{n-1} \mathrm{d}x, \tag{6.104}$$

this is well defined as long as n is not zero or a negative integer. For $n = 1$,

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-x} \mathrm{d}x = \left[\mathrm{e}^{-x} \right]_0^\infty = 1. \tag{6.105}$$

With integration by parts, one can easily show

$$\Gamma(n + 1) = \int_0^\infty \mathrm{e}^{-x} x^n \mathrm{d}x = \left[-x^n \mathrm{e}^{-x} \right]_0^\infty + n \int_0^\infty \mathrm{e}^{-x} x^{n-1} \mathrm{d}x = n\Gamma(n). \tag{6.106}$$

Thus if n is a positive integer,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\cdots 1\Gamma(1) = n!, \tag{6.107}$$

and according to (6.103)

$$\mathfrak{L}[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \tag{6.108}$$

in agreement with the result we obtained before. Since $\Gamma(n)$ is a tabulated function, as long as $n > -1$, $\mathfrak{L}[t^n]$ can still be evaluated even if n is not an integer. For example

$$\mathfrak{L}\left[\frac{1}{\sqrt{t}}\right] = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{s^{1/2}}, \tag{6.109}$$

$$\mathfrak{L}\left[\sqrt{t}\right] = \frac{\Gamma(1+\frac{1}{2})}{s^{1+\frac{1}{2}}} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{\frac{3}{2}}} = \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}, \tag{6.110}$$

where we have used the well-known result $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Example 6.6.7. Find the Laplace transform of $\mathrm{e}^{at}(1+2at)/\sqrt{\pi t}$.

Solution 6.6.7. Use s-shifting property

$$\mathfrak{L}\left[\frac{\mathrm{e}^{at}}{\sqrt{\pi t}}\right] = \frac{1}{(s-a)^{1/2}},$$

$$\mathfrak{L}\left[\frac{\mathrm{e}^{at}2at}{\sqrt{\pi t}}\right] = \frac{2a}{\sqrt{\pi}}\mathfrak{L}\left[\mathrm{e}^{at}\sqrt{t}\right] = \frac{a}{(s-a)^{3/2}}.$$

Thus

$$\mathfrak{L}\left[\frac{\mathrm{e}^{at}(1+2at)}{\sqrt{\pi t}}\right] = \frac{1}{(s-a)^{1/2}} + \frac{a}{(s-a)^{3/2}} = \frac{s}{(s-a)^{3/2}}.$$

6.7 Summary of Operations of Laplace Transforms

The properties of the Laplace transform are not difficult to understand. However, because there are so many of them, it is not easy to decide which one to use for a specific problem. In Table 6.2 we summarize these operations. In the last column we give a simple example and in the first column we give a name to characterize the operation. This classification is helpful in remembering the details of each operation.

In Sect. 6.2.1, we discussed the inverse of the transform $F(s)$ in the form of a quotient of two polynomials. If $F(s)$ is not in that form, sometimes we can use the properties of the Laplace transforms to obtain the inverse.

Table 6.2. Summary of Laplace transform operations

Name	$h(t)$	$\mathfrak{L}[h(t)]$	Example: Let $f(t) = t$
Definition	$f(t)$	$F(s)$	$\mathfrak{L}[t] = \int_0^\infty e^{-st} t\, dt = \dfrac{1}{s^2} = F(s)$
Multiply t	$tf(t)$	$-\dfrac{d}{ds} F(s)$	$\mathfrak{L}[t \cdot t] = -\dfrac{d}{ds}\dfrac{1}{s^2} = \dfrac{2}{s^3}$
Divide t	$\dfrac{f(t)}{t}$	$\int_s^\infty F(\varsigma) d\varsigma$	$\mathfrak{L}\left[\dfrac{t}{t}\right] = \int_s^\infty \dfrac{1}{\varsigma^2} d\varsigma = \dfrac{1}{s}$
Derivative	$f'(t)$	$sF(s) - f(0)$	$\mathfrak{L}\left[\dfrac{dt}{dt}\right] = s\dfrac{1}{s^2} - 0 = \dfrac{1}{s}$
Integral	$\int_0^t f(\tau) d\tau$	$\dfrac{F(s)}{s}$	$\mathfrak{L}\left[\int_0^t \tau\, d\tau\right] = \dfrac{1/s^2}{s} = \dfrac{1}{s^3}$
Shifting – s	$e^{at} f(t)$	$F(s-a)$	$\mathfrak{L}\left[e^{at}t\right] = \dfrac{1}{(s-a)^2}$
Shifting – t	$u(t-a)f(t-a)$	$e^{-sa} F(s)$	$\mathfrak{L}[u(t-a)(t-a)] = e^{-sa}\dfrac{1}{s^2}$
Scaling	$f(at)$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$	$\mathfrak{L}[at] = \dfrac{1}{a}\dfrac{1}{(s/a)^2} = a\dfrac{1}{s^2}$
Period – p	$periodic\ f(t)$	$\dfrac{\int_0^p e^{-st} f(t)\, dt}{1 - e^{-ps}}$	$\mathfrak{L}[f] = \dfrac{1-(1+ps)e^{-ps}}{s^2(1-e^{-ps})}$
Convolution	$\int_0^t f(\tau)g(t-\tau) d\tau$	$F(s) \cdot G(s)$	Let $g(t) = f(t), \ G(s) = F(s)$ $\mathfrak{L}\left[\int_0^t \tau(t-\tau)d\tau\right] = \dfrac{1}{s^2}\dfrac{1}{s^2} = \dfrac{1}{s^4}$

Example 6.7.1. Find $\mathfrak{L}^{-1}\left[\ln\dfrac{s+a}{s-b}\right]$.

Solution 6.7.1. The transform is not in the form of a quotient of two polynomials, but its derivative is. Let

$$\mathfrak{L}[f(t)] = \ln\frac{s+a}{s-b}, \qquad f(t) = \mathfrak{L}^{-1}\left[\ln\frac{s+a}{s-b}\right].$$

Since

$$\mathfrak{L}[tf(t)] = -\frac{d}{ds}\mathfrak{L}[f(t)] = -\frac{d}{ds}\ln\frac{s+a}{s-b}$$

$$= -\frac{d}{ds}\ln(s+a) + \frac{d}{ds}\ln(s-b) = \frac{1}{s-b} - \frac{1}{s+a},$$

and

$$tf(t) = \mathcal{L}^{-1}\left[-\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\left[f(t)\right]\right] = \mathcal{L}^{-1}\left[\frac{1}{s-b} - \frac{1}{s+a}\right],$$

therefore

$$\mathcal{L}^{-1}\left[\ln\frac{s+a}{s-b}\right] = f(t) = \frac{1}{t}\mathcal{L}^{-1}\left[\frac{1}{s-b} - \frac{1}{s+a}\right]$$

$$= \frac{1}{t}\left(\mathrm{e}^{bt} - \mathrm{e}^{-at}\right).$$

Example 6.7.2. Find $\mathcal{L}^{-1}\left[\ln\frac{s^2 - a^2}{s^2}\right]$.

Solution 6.7.2.

$$\mathcal{L}^{-1}\left[\ln\frac{s^2 - a^2}{s^2}\right] = \frac{1}{t}\mathcal{L}^{-1}\left[-\frac{\mathrm{d}}{\mathrm{d}s}\ln\frac{s^2 - a^2}{s^2}\right] = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{2s}{s^2 - a^2} - \frac{2s}{s^2}\right]$$

$$= \frac{2}{t}\left(1 - \cosh at\right).$$

Example 6.7.3. Find $\mathcal{L}^{-1}\left[\ln\frac{s^2 + \omega^2}{s^2}\right]$.

Solution 6.7.3.

$$\mathcal{L}^{-1}\left[\ln\frac{s^2 + \omega^2}{s^2}\right] = \frac{1}{t}\mathcal{L}^{-1}\left[-\frac{\mathrm{d}}{\mathrm{d}s}\ln\frac{s^2 + \omega^2}{s^2}\right] = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{2s}{s^2 + \omega^2} - \frac{2s}{s^2}\right]$$

$$= \frac{2}{t}\left(1 - \cos\omega t\right).$$

Example 6.7.4. Find $\mathcal{L}^{-1}\left[\tan^{-1}\frac{1}{s}\right]$.

Solution 6.7.4.

$$\mathcal{L}^{-1}\left[\tan^{-1}\frac{1}{s}\right] = \frac{1}{t}\mathcal{L}^{-1}\left[-\frac{\mathrm{d}}{\mathrm{d}s}\tan^{-1}\frac{1}{s}\right] = -\frac{1}{t}\mathcal{L}^{-1}\left[\frac{1}{(1/s)^2 + 1}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s}\right]$$

$$= \frac{1}{t}\mathcal{L}^{-1}\left[\frac{1}{1 + s^2}\right] = \frac{1}{t}\sin t.$$

6.8 Additional Applications of Laplace Transforms

6.8.1 Evaluating Integrals

Many integrals from 0 *to* ∞ can be evaluated by the Laplace transform method.

By direct substitution. Integrals involving e^{-at} can be obtained from the Laplace transformation with a simple substitution.

$$\int_0^\infty e^{-at} f(t) \mathrm{d}t = \left\{ \int_0^\infty e^{-st} f(t) \mathrm{d}t \right\}_{s=a} = \{\mathfrak{L}\left[f(t)\right]\}_{s=a}. \qquad (6.111)$$

Example 6.8.1. Find $\int_0^\infty e^{-3t} \sin t \, \mathrm{d}t$.

Solution 6.8.1.

$$\int_0^\infty e^{-3t} \sin t \, \mathrm{d}t = \{\mathfrak{L}\left[\sin t\right]\}_{s=3} = \left\{ \frac{1}{s^2+1} \right\}_{s=3} = \frac{1}{10}$$

Example 6.8.2. Find $\displaystyle\int_0^\infty e^{-2t} t \cos t \, \mathrm{d}t$.

Solution 6.8.2. Since

$$\int_0^\infty e^{-2t} t \cos t \, \mathrm{d}t = \{\mathfrak{L}[t \cos t]\}_{s=2},$$

$$\{\mathfrak{L}[t \cos t]\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{L}[\cos t] = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2},$$

therefore

$$\int_0^\infty e^{-2t} t \cos t \, \mathrm{d}t = \left\{ \frac{s^2-1}{(s^2+1)^2} \right\}_{s=2} = \frac{3}{25}.$$

Use integral of the transform. In Sect. 6.6.2 we have shown

$$\mathfrak{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s') \mathrm{d}s',$$

where

$$\mathfrak{L}\left[\frac{f(t)}{t}\right] = \int_0^\infty e^{-st}\frac{f(t)}{t}\mathrm{d}t, \qquad F(s) = \mathfrak{L}\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\mathrm{d}t.$$

Setting $s = 0$, we obtain an equally important formula

$$\int_0^\infty \frac{f(t)}{t}\mathrm{d}t = \int_0^\infty \mathfrak{L}\left[f(t)\right] \mathrm{d}s. \qquad (6.112)$$

This formula can be used if the integral on left side is difficult to do directly.

Example 6.8.3. Find $\displaystyle\int_0^\infty \left[\frac{e^{-t} - e^{-3t}}{t}\right] dt$.

Solution 6.8.3.

$$\int_0^\infty \left[\frac{e^{-t} - e^{-3t}}{t}\right] dt = \int_0^\infty \mathcal{L}\left[e^{-t} - e^{-3t}\right] ds = \int_0^\infty \left(\frac{1}{s+1} - \frac{1}{s+3}\right) ds$$

$$= \left[\ln(s+1) - \ln(s+3)\right]_0^\infty = \left[\ln\frac{s+1}{s+3}\right]_0^\infty$$

$$= \ln 1 - \ln\frac{1}{3} = \ln 3.$$

Example 6.8.4. Find $\displaystyle\int_0^\infty \frac{\sin t}{t} dt$.

Solution 6.8.4.

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \mathcal{L}\left[\sin t\right] ds = \int_0^\infty \frac{1}{s^2 + 1} ds$$

$$= \left[\tan^{-1} s\right]_0^\infty = \frac{\pi}{2}. \tag{6.113}$$

Use double integrals. We can solve the problem of the last example by a double integral. Starting with

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \, dt = \frac{1}{s}, \tag{6.114}$$

if we rename t as x, and s as t, we have

$$\int_0^\infty e^{-tx} dx = \frac{1}{t}. \tag{6.115}$$

So

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \sin t \left[\frac{1}{t}\right] dt = \int_0^\infty \sin t \left[\int_0^\infty e^{-tx} dx\right] dt. \tag{6.116}$$

Interchanging the order of integration, we have

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \left[\int_0^\infty e^{-tx} \sin t \, dt\right] dx. \tag{6.117}$$

The integral in the bracket is recognized as the Laplace transform of $\sin t$ with the parameter s replaced by x, thus

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \frac{1}{1+x^2} dx = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2}. \tag{6.118}$$

This method can be applied to more complicated cases.

Example 6.8.5. Find $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\mathrm{d}t.$

Solution 6.8.5. First we note

$$\sin^2 t = \frac{1}{2}(1 - \cos 2t),$$

then write

$$\int_0^\infty \frac{\sin^2 t}{t^2}\mathrm{d}t = \frac{1}{2}\int_0^\infty (1 - \cos 2t)\left[\frac{1}{t^2}\right]\mathrm{d}t.$$

With

$$\frac{1}{t^2} = \int_0^\infty \mathrm{e}^{-tx}x\ \mathrm{d}x,$$

we have

$$\int_0^\infty \frac{\sin^2 t}{t^2}\mathrm{d}t = \frac{1}{2}\int_0^\infty (1 - \cos 2t)\left[\int_0^\infty \mathrm{e}^{-tx}x\ \mathrm{d}x\right]\mathrm{d}t$$
$$= \frac{1}{2}\int_0^\infty \left[\int_0^\infty \mathrm{e}^{-tx}(1 - \cos 2t)\mathrm{d}t\right]x\ \mathrm{d}x.$$

Since

$$\int_0^\infty \mathrm{e}^{-tx}(1 - \cos 2t)\mathrm{d}t = \left[\mathfrak{L}(1 - \cos 2t)\right]_{s=x}$$
$$= \frac{1}{x} - \frac{x}{x^2 + 4} = \frac{4}{x(x^2 + 4)},$$

$$\int_0^\infty \frac{\sin^2 t}{t^2}\mathrm{d}t = \frac{1}{2}\int_0^\infty \left[\frac{4}{x(x^2 + 4)}\right]x\ \mathrm{d}x$$
$$= \int_0^\infty \frac{2}{(x^2 + 4)}\mathrm{d}x = \left[\tan^{-1}\frac{x}{2}\right]_0^\infty = \frac{\pi}{2}.$$

Use inverse Laplace transform. If the integral is difficult to do, we can first find its Laplace transform and then take the inverse.

Example 6.8.6. Find $\displaystyle\int_0^\infty \frac{\cos x}{x^2 + b^2}\mathrm{d}x$

Solution 6.8.6. In order to use Laplace transform to evaluate this integral, we change $\cos x$ to $\cos tx$, and at the end we set $t = 1$. Let

$$I(t) = \int_0^\infty \frac{\cos tx}{x^2 + b^2}\mathrm{d}x.$$

$$\mathfrak{L}\left[I(t)\right] = \int_0^\infty \frac{1}{x^2 + b^2} \mathfrak{L}\left[\cos tx\right] \mathrm{d}x = \int_0^\infty \frac{1}{x^2 + b^2} \frac{s}{s^2 + x^2} \mathrm{d}x$$

$$= \frac{s}{s^2 - b^2} \int_0^\infty \left[\frac{1}{x^2 + b^2} - \frac{1}{s^2 + x^2}\right] \mathrm{d}x$$

$$= \frac{s}{s^2 - b^2} \left\{\left[\frac{1}{b} \tan^{-1} \frac{x}{b}\right]_0^\infty - \left[\frac{1}{s} \tan^{-1} \frac{x}{s}\right]_0^\infty\right\}$$

$$= \frac{s}{s^2 - b^2} \left\{\frac{\pi}{2b} - \frac{\pi}{2s}\right\} = \frac{\pi}{2b} \frac{1}{s + b}.$$

$$I(t) = \mathfrak{L}^{-1}\left[\frac{\pi}{2b} \frac{1}{s + b}\right] = \frac{\pi}{2b} \mathrm{e}^{-bt}.$$

Thus

$$\int_0^\infty \frac{\cos x}{x^2 + b^2} \mathrm{d}x = I(1) = \frac{\pi}{2b} \mathrm{e}^{-b}.$$

6.8.2 Differential Equation with Variable Coefficients

If $f(t)$ in the formula

$$\mathfrak{L}\left[tf(t)\right] = -\frac{\mathrm{d}}{\mathrm{d}s} \mathfrak{L}\left[f(t)\right]$$

is taken to be the nth derivative of $y(t)$, then

$$\mathfrak{L}\left[ty^{(n)}(t)\right] = -\frac{\mathrm{d}}{\mathrm{d}s} \mathfrak{L}\left[y^{(n)}(t)\right]$$

$$= -\frac{\mathrm{d}}{\mathrm{d}s} \left\{s^n \mathfrak{L}\left[y(t)\right] - s^{n-1}y(0) \cdots -y^{n-1}(0)\right\}. \quad (6.119)$$

This equation can be used to transform a linear differential equation with variable coefficients into a differential equation involving the transform. This procedure is useful if the new equation can be readily solved.

Example 6.8.7. Find the solution of

$$ty''(t) - ty'(t) - y(t) = 0, \quad y(0) = 0, \ y'(0) = 2.$$

Solution 6.8.7.

$$\mathfrak{L}\left[ty''(t)\right] = -\frac{\mathrm{d}}{\mathrm{d}s} \left\{s^2 \mathfrak{L}\left[y(t)\right] - sy(0) - y'(0)\right\}.$$

Let $\mathfrak{L}\left[y(t)\right] = F(s)$, with $y(0) = 0$ we have

$$\mathfrak{L}\left[ty''(t)\right] = -2sF(s) - s^2 F'(s),$$
$$\mathfrak{L}\left[ty'(t)\right] = -F(s) - sF'(s).$$

Therefore

$$\mathcal{L}[ty''(t) - ty'(t) - y(t)] = -s(s-1)F'(s) - 2sF(s) = 0.$$

It follows

$$\frac{\mathrm{d}F(s)}{F(s)} = -2\frac{\mathrm{d}s}{s-1},$$

$$\ln F(s) = \ln(s-1)^{-2} + \ln C$$

$$F(s) = \frac{C}{(s-1)^2}$$

$$y(t) = \mathcal{L}^{-1}\left[F(s)\right] = \mathcal{L}^{-1}\left[\frac{C}{(s-1)^2}\right] = C\,\mathrm{e}^t t.$$

Since

$$y'(t) = C\,\mathrm{e}^t t + C\,\mathrm{e}^t,$$

$$y'(0) = C = 2,$$

therefore

$$y(t) = 2\mathrm{e}^t t.$$

It can be easily verified that this is indeed the solution, since it satisfies both the equation and the initial conditions.

Example 6.8.8. Zeroth Order Bessel Function: Find the solution of

$$ty''(t) + y'(t) + ty(t) = 0, \quad y(0) = 1,\ y'(0) = 0.$$

Solution 6.8.8. With $\mathcal{L}[y(t)] = F(s)$ and $y(0) = 1, y'(0) = 0$,

$$\mathcal{L}[ty''(t) + y'(t) + ty(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}\left\{s^2 F(s) - s\right\} + sF(s) - 1 - \frac{\mathrm{d}}{\mathrm{d}s}F(s) = 0.$$

Collecting terms

$$(s^2 + 1)\frac{\mathrm{d}}{\mathrm{d}s}F(s) + sF(s) = 0,$$

or

$$\frac{\mathrm{d}F(s)}{F(s)} = -\frac{s\,\mathrm{d}s}{s^2 + 1} = -\frac{1}{2}\frac{\mathrm{d}s^2}{s^2 + 1}.$$

It follows

$$\ln F(s) = -\frac{1}{2}\ln(s^2 + 1) + \ln C,$$

$$F(s) = \frac{C}{(s^2 + 1)^{1/2}}.$$

To find the inverse of this Laplace transform, we expand it in a series in the case $s > 1$,

$$F(s) = \frac{C}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2}$$

$$= \frac{C}{s}\left[1 - \frac{1}{2s^2} + \frac{1 \cdot 3}{2^2 \cdot 2!}\frac{1}{s^4} \cdots + \frac{(-1)^n(2n)!}{(2^n n!)^2}\frac{1}{s^{2n}} + \cdots\right].$$

Inverting term by term, we have

$$y(t) = \mathfrak{L}^{-1}\left[F(s)\right] = C\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2^n n!)^2}.$$

Since $y(0) = 1$, therefore $C = 1$. It turned out this series with $C = 1$ is known as the Bessel function of zeroth order $J_0(t)$, that is,

$$J_0(t) = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2^n n!)^2},$$

which we will discuss in more detail in the chapter on Bessel functions. Thus the solution of the equation is

$$y(t) = J_0(t).$$

Furthermore, this development shows that

$$\mathfrak{L}\left[J_0(t)\right] = \frac{1}{(s^2+1)^{1/2}}.$$

With the scaling property of the Laplace transform, we also have (for $a > 0$)

$$\mathfrak{L}\left[J_0(at)\right] = \frac{1}{a}\frac{1}{[(s/a)^2 + 1]^{1/2}} = \frac{1}{(s^2 + a^2)^{1/2}}.$$

6.8.3 Integral and Integrodifferential Equations

Equations in which the unknown function appears under the integral are called integral equations. If the derivatives are also in the equation, then they are called integrodifferential equations. They are often difficult to solve. But if the integrals are in the form of a convolution, then Laplace transform can be used to solve them. The following example will make the procedure clear.

Example 6.8.9. Solve the integral equation

$$y(t) = t + \int_0^t y(\tau)\sin(t - \tau)\mathrm{d}\tau.$$

Solution 6.8.9.

$$\mathcal{L}[y(t)] = \mathcal{L}[t] + \mathcal{L}\left[\int_0^t y(\tau)\sin(t-\tau)d\tau\right]$$

$$= \mathcal{L}[t] + \mathcal{L}[y(t)]\mathcal{L}[\sin t] = \frac{1}{s^2} + \mathcal{L}[y(t)]\frac{1}{s^2+1}.$$

Solving for $\mathcal{L}[y(t)]$

$$\left(1 - \frac{1}{s^2+1}\right)\mathcal{L}[y(t)] = \frac{1}{s^2}.$$

So

$$\mathcal{L}[y(t)] = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2} + \frac{1}{s^4}\right] = t + \frac{1}{6}t^3.$$

Example 6.8.10. Find the solution of

$$y'(t) - 3\int_0^t e^{-2(t-\tau)}y(\tau)d\tau = z(t), \qquad\qquad y(0) = 4,$$

$$z(t) = \begin{cases} 4e^{-2t} & 1 < t \\ 0 & 0 < t < 1 \end{cases}.$$

Solution 6.8.10. First note

$$\mathcal{L}[z(t)] = \mathcal{L}\left[4e^{-2t}u(t-1)\right] = \mathcal{L}\left[4e^{-2}e^{-2(t-1)}u(t-1)\right] = 4e^{-2}\frac{1}{s+2}e^{-s},$$

$$\mathcal{L}\left[3\int_0^t e^{-2(t-\tau)}y(\tau)d\tau\right] = 3\frac{1}{s+2}\mathcal{L}[y].$$

Applying the Laplace transform to both sides of the equation leads to

$$s\mathcal{L}[y] - 4 - 3\frac{1}{s+2}\mathcal{L}[y] = 4e^{-2}\frac{1}{s+2}e^{-s},$$

collecting terms

$$\frac{s^2+2s-3}{s+2}\mathcal{L}[y] = 4 + 4e^{-2}\frac{1}{s+2}e^{-s},$$

or

$$\mathcal{L}[y] = \frac{4(s+2) + 4e^{-2}e^{-s}}{s^2+2s-3}$$

$$= \frac{3}{s-1} + \frac{1}{s+3} + e^{-2}\left(\frac{1}{s-1} - \frac{1}{s+3}\right)e^{-s}.$$

Thus

$$y(t) = 3e^t + e^{-3t} + e^{-2}\left(e^{(t-1)} - e^{-3(t-1)}\right)u(t-1).$$

6.9 Inversion by Contour Integration

As we have seen that to be able to invert a given transform $F(s)$ to find the function $f(t)$ is the key of solving differential equations with the Laplace Transform. For those familiar with the complex contour integration, we present in this section a universal technique of finding the inverse of the Laplace Transform. The derivation is highly imaginative, but the result is elegantly simple.

First let us extend the Laplace transform to the complex domain. The function $F(z)$ is the same function as $F(s)$ except with s replaced by z. In the complex plane, $F(z)$ will have some singular points. Let us choose a line $x = b$ in the complex plane such that all singular points of $F(z)$ are in the left-hand side of this line. Then $F(z)$ is analytic on the line $x = b$ and in the entire half plane to the right of this line. If s is any point in this half plane, we can choose a semicircular contour $C = C_1 + C_2$, as shown in Fig. 6.11, and apply the Cauchy's integral formula,

$$
\begin{aligned}
F(s) &= \frac{1}{2\pi i} \oint_C \frac{F(z)}{z - s} dz \\
&= \frac{1}{2\pi i} \int_{b+iR}^{b-iR} \frac{F(z)}{z - s} dz + \frac{1}{2\pi i} \int_{C_1} \frac{F(z)}{z - s} dz.
\end{aligned}
\tag{6.120}
$$

Now if we let R go to infinity, (6.120) is still valid, but all values of z on the semicircle C_1 are infinitely large. Since $F(z) \to 0$ as $z \to \infty$,

$$
\lim_{R \to \infty} \int_{C_1} \frac{F(z)}{z - s} dz = 0.
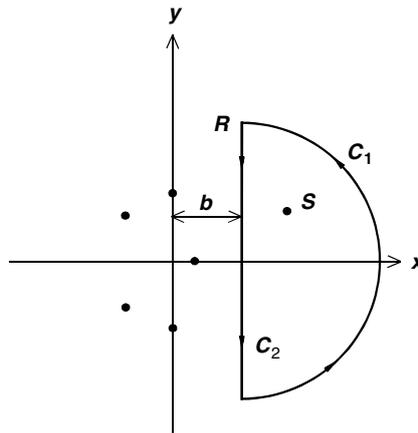$$

Fig. 6.11. The first contour used to obtain the complex inversion of the Laplace transform

Therefore in this limit (6.120) becomes

$$F(s) = \frac{1}{2\pi i} \int_{b+i\infty}^{b-i\infty} \frac{F(z)}{z-s} dz = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{F(z)}{s-z} dz.$$

In the last step we have changed the sign of the integrand and interchanged the upper and lower limits of the integral.

Taking the inverse Laplace transform, we have

$$\mathfrak{L}^{-1}[F(s)] = \mathfrak{L}^{-1}\left[\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{F(z)}{s-z} dz\right].$$

Since the inverse Laplace operator $\mathfrak{L}^{-1}$ refers only to the variable s, we can write

$$\mathfrak{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(z)\mathfrak{L}^{-1}\left[\frac{1}{s-z}\right] dz.$$

Since

$$\mathfrak{L}^{-1}\left[\frac{1}{s-z}\right] = e^{zt}$$

we have

$$\mathfrak{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(z)e^{zt} dz. \tag{6.121}$$

This procedure is called the Mellin inversion. This integral is from $b - i\infty$ to $b + i\infty$ along C_2. Usually the evaluation of this integral is accomplished by the residue theorem. To use the residue theorem, we must have a closed contour. To close the contour, we have to add a returning line integral from $b + i\infty$ to $b - i\infty$ in such a way that the value of the integral is not changed. This can be done with the semicircular contour C_3 in the left half-plane as shown in Fig. 6.12, since

$$\lim_{R\to\infty} \int_{C_3} F(z)e^{zt} dz = 0. \tag{6.122}$$

This can be understood from the fact that with a positive t, the integrand

$$F(z)e^{zt} = F(z)e^{xt+iyt}$$

goes to zero as z goes to infinity. The factor e^{iyt} is oscillatory with a maximum value of 1. For x to change from b to $-\infty$ as on C_3, the factor e^{xt} is always less than e^{bt}. Therefore $F(z)e^{zt}$ will go to zero as long as $F(z)$ is going to zero. Note that this will not be the case in the right half plane where x will go to positive infinite and e^{zt} will blow up. Thus with C_3, we have

$$\mathfrak{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} F(z)e^{zt} dz.$$

$$= \frac{1}{2\pi i} \lim_{R\to\infty} \left[\int_{C_2} F(z)e^{zt} dz + \int_{C_3} F(z)e^{zt} dz\right]$$
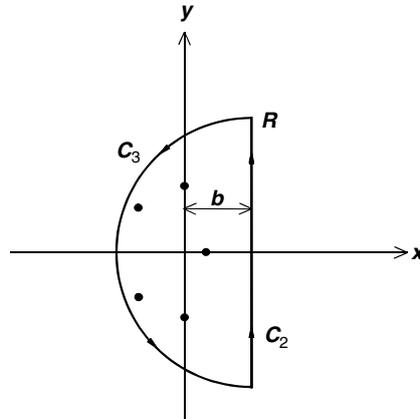
$$= \frac{1}{2\pi i} \oint_C F(z)e^{zt} dz, \tag{6.123}$$

Fig. 6.12. The contour used in evaluating the complex inversion integral

where $C = C_2 + C_3$ as shown in Fig. 6.12 with $R \to \infty$. This contour is also called the Bromwich contour. Since b is on the right of all singular points of $F(z)$, the contour C encloses all singular points of $\mathrm{e}^{zt}F(z)$. Therefore by the residue theorem

$$\mathfrak{L}^{-1}[F(s)] = \frac{1}{2\pi\mathrm{i}} \oint_C F(z)\mathrm{e}^{zt}\mathrm{d}z$$
$$= \sum \text{all residues of } F(z)\mathrm{e}^{zt}. \qquad (6.124)$$

Example 6.9.1. Using the complex inversion integral to find

$$\mathfrak{L}^{-1}\left[\frac{1}{(s+a)^2 + b^2}\right].$$

Solution 6.9.1. Since

$$\frac{1}{(s+a)^2 + b^2} = \frac{1}{[s - (-a + \mathrm{i}b)]\,[s - (-a - \mathrm{i}b)]},$$

the residues of

$$\frac{\mathrm{e}^{zt}}{(z+a)^2 + b^2}$$

at the two singular points are

$$r_1 = \lim_{z \to -a + \mathrm{i}b} [z - (-a + \mathrm{i}b)]\, \frac{\mathrm{e}^{zt}}{[z - (-a + \mathrm{i}b)]\,[z - (-a - \mathrm{i}b)]}$$
$$= \frac{\mathrm{e}^{(-a + \mathrm{i}b)t}}{2\mathrm{i}b}$$

and

$$r_2 = \lim_{z \to -a-\mathrm{i}b} \left[z - (-a - \mathrm{i}b) \right] \frac{\mathrm{e}^{zt}}{\left[z - (-a + \mathrm{i}b) \right] \left[z - (-a - \mathrm{i}b) \right]}$$
$$= \frac{\mathrm{e}^{(-a-\mathrm{i}b)t}}{-2\mathrm{i}b}.$$

Therefore

$$\mathfrak{L}^{-1} \left[\frac{1}{(s+a)^2 + b^2} \right] = \frac{\mathrm{e}^{(-a+\mathrm{i}b)t}}{2\mathrm{i}b} + \frac{\mathrm{e}^{(-a-\mathrm{i}b)t}}{-2\mathrm{i}b}$$
$$= \frac{1}{b} \mathrm{e}^{-at} \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{\mathrm{i}bt} - \mathrm{e}^{-\mathrm{i}bt} \right)$$
$$= \frac{1}{b} \mathrm{e}^{-at} \sin bt.$$

This is a familiar result. This example shows that the complex inversion integral is indeed another way of finding the inverse Laplace transform. In more difficult applications, the use of the complex inversion integral and the contour integration is either the only way or the simplest way of finding the inverse Laplace transform.

6.10 Computer Algebraic Systems for Laplace Transforms

Before closing this chapter, we should mention that a number of commercial computer packages are available to perform algebraic manipulations, including Laplace transforms. They are called computer algebraic systems, some prominent ones are Matlab, Maple, Mathematica, MathCad, and MuPAD.

This book is written with the software "Scientific WorkPlace," which also provides an interface to MuPAD. (Before version 5, it also came with Maple). Instead of requiring the user to adhere to a rigid syntax, the user can use natural mathematical notations. For example, to compute $\mathfrak{L}\left(\cos t - 2\sin t\right)$, all you have to do is (1) type $\cos t - 2\sin t$ in the math-mode, (2) click on the "Compute" button, (3) click on the "Transforms" button in the pull-down menu, and (4) click on the "Laplace" button in the submenu. The program will return with

$$\cos t - 2\sin t, \ \text{Laplace transform is}: \ \frac{s}{s^2 + 1} - \frac{2}{s^2 + 1}.$$

The program also recognizes the Laplace transform symbol. An alternative way to do the same problem is to choose, from the "Miscellaneous Symbols" panel, the Laplace transform symbol and type

$$\mathcal{L}(\cos t - 2\sin t)$$

in the math-mode, and click on "Compute" and then choose "Evaluate." The program will return with

$$\mathcal{L}(\cos t - 2\sin t) = \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

Similarly, one can compute the inverse of the Laplace transform. For example, type

$$\mathcal{L}^{-1}\left(\frac{s-2}{s^2+1}\right)$$

and click on "Compute" and then on "Evaluate," the program will return with

$$\mathcal{L}^{-1}\left(\frac{s-2}{s^2+1}\right) = \cos t - 2\sin t$$

To compute the Laplace transform of a derivative, one has to define the function first. For example, to find $\mathfrak{L}\left(y'''\right)$, first type

$$y\left(t\right)$$

and click on "Compute", then on "Definition" in the menu, then on "New Definition" in the submenu. Then type

$$\mathcal{L}(y''')$$

and click on "Compute" and then on "Evaluate". The program will return with

$$\mathcal{L}(y''') = s^3\mathcal{L}(y) - sy'(0) - s^2 y(0) - y''(0).$$

The program can also use Laplace transform to solve ordinary differential equations. For example, to solve the equation

$$y' + y = x + \sin x,$$

first type this equation in math-mode, then click on "Compute." In the pull-down menu, click on "Solve ODE," then click on "Laplace" in the submenu. The program returns with a question asking which is the independent variable. Type x and click on "OK." The program will return with

$$\text{Laplace solution is}: x - \frac{1}{2}\cos x + \frac{1}{2}\sin x + \mathrm{e}^{-x}\left(y\left(0\right) + \frac{3}{2}\right) - 1.$$

Unfortunately, not every problem can be solved by a computer algebraic system. Sometimes it fails to find the solution. Even worse, for a variety of reasons, the intention of the user is sometimes misinterpreted, and the computer returns with an answer to a wrong problem without the user knowing it. One should be aware of these pitfalls.

Computer algebraic systems are no substitute for the knowledge of the subject matter, but they are useful supplements.

Exercises

1. Find the Laplace transformation of each of the following functions by direct integration.
   (a) $\frac{1}{2}t^2$,   (b) e^{3t},   (c) $3\sin(3t)$.
   Ans. (a) $\frac{1}{s^3}$,   (b) $\frac{1}{s-3}$,   (c) $\frac{9}{s^2+9}$.

2. Find the Laplace transformation of each of the following functions by using the "Multiply t" operation (see Table 6.2).
   (a) te^t,   (b) $t\cos t$,   (c) $t^2\cos t$.
   Ans. (a) $\frac{1}{(s-1)^2}$,   (b) $\frac{s^2-1}{(s^2+1)^2}$,   (c) $\frac{2s(s^2-3)}{(s^2+1)^3}$.

3. Find the Laplace transformation of each of the following functions by using the "Divide t" operation (see Table 6.2).
   (a) $\frac{1}{t}\left(e^{2t}-e^{-2t}\right)$,   (b) $\frac{2}{t}\left(1-\cos(2t)\right)$,   (c) $\frac{1}{t}\sin(4t)$.
   Ans. (a) $\ln\left(\frac{s+2}{s-2}\right)$,   (b) $\ln\left(\frac{s^2+4}{s^2}\right)$,   (c) $\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{4}\right)$.

4. Find the Laplace transformation of each of the following functions by using the "Shifting $-$ s" operation (see Table 6.2).
   (a) $e^{at}\sin 3t$,   (b) $e^{-2t}t\sin at$,   (c) $\sinh t\cos t$.
   Ans. (a) $\frac{3}{(s-a)^2+9}$,   (b) $\frac{2a(s+2)}{[(s+2)^2+a^2]^2}$,   (c) $\frac{s^2-2}{s^4+4}$.

5. Use the definition of Laplace transformation to show
   (a) $\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$;

   use (a) to show

   (b) $\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0) - f'(0)$.

6. Use the results of previous problem and the fact that $\frac{d^2}{dt^2}\cos at = -a^2\cos at$ and $\frac{d^2}{dt^2}\sin at = -a^2\sin at$ to show
   (a) $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$,   (b)$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$.

7. Differentiate both sides of part (b) of the previous problem with respect to a, and show that
   (a) $\mathcal{L}[t\cos at] = \frac{1}{s^2+a^2} - \frac{2a^2}{(s^2+a^2)^2}$,
   Differentiate both sides of part (a) of the previous problem with respect to s, and show that
   (b) $\mathcal{L}[-t\cos at] = \frac{1}{s^2+a^2} - \frac{2s^2}{(s^2+a^2)^2}$.

8. Use the results of problems 6 and 7 to show

(a) $\mathcal{L}^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right] = \dfrac{1}{2a^3}(\sin at - at\cos at)$,

(b) $\mathcal{L}^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right] = \dfrac{1}{2}\left(t\cos at + \dfrac{1}{a}\sin at\right)$.

9. Do problem 8 with convolution theorem.
   Hint: you may need the following integral

$$\int_0^t \sin a\tau \cos a\tau\, d\tau = \frac{1}{4a}(1-\cos 2at);$$

$$\int_0^t \sin^2 a\tau\, d\tau = \frac{1}{4a}(2at - \sin 2at).$$

10. If $f(t) = t^n$, $g(t) = t^m$, $\quad n > -1$, $m > -1$,
    (a) show that

$$\int_0^t \tau^n(t-\tau)^m d\tau = t^{n+m+1}\int_0^1 y^n(1-y)^m dy.$$

   (b) By using the convolution theorem, show that

$$\int_0^1 y^n(1-y)^m dy = \frac{n!m!}{(n+m+1)!}.$$

   Hint: (a) let $\tau = yt$, (b) use convolution theorem to evaluate $\int_0^t \tau^n (t-\tau)^m d\tau$.

11. Find the Laplace transformation of each of the following functions by direct integration.
    (a) $\sin(t-a)\, u(t-a)$,

    (b) $f(t) = \begin{cases} \cos(t-\pi) & t > \pi \\ 0 & t < \pi \end{cases}$,

    (c) $f(t) = \begin{cases} 0 & 0 \le t < 5 \\ 1 & 5 \le t < 10 \\ 0 & 10 \le t \end{cases}$.

    Ans. (a) $e^{-as}\dfrac{1}{s^2+1}$, (b) $e^{-\pi s}\dfrac{s}{s^2+1}$, (c) $\dfrac{1}{s}\left(e^{-5s} - e^{-10s}\right)$.

12. Do the previous problem by using the "Shifting $-t$" operation (see Table 6.2).

13. Use the partial fraction to find the inverse Laplace transform of the following expressions.
    (a) $\mathcal{L}^{-1}\left[\dfrac{4}{s^2-4s}\right]$, (b) $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+1)}\right]$, (c) $\mathcal{L}^{-1}\left[\dfrac{1}{s^2(s^2+1)}\right]$.
    Ans. (a) $e^{4t}-1$, (b) $1-\cos t$, (c) $t-\sin t$.

14. Do the previous problem by using the formula

$$\mathscr{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}\mathscr{L}\left[f(t)\right].$$

15. Use the Heaviside expansion to solve the previous problem.

16. Use the Laplace transform to solve the following differential equations
    (a) $y'' + 2y' + y = 1$, $y(0) = 2$, $y'(0) = -2$,

    (b) (b) $y'' + y = \sin(3t)$, $y(0) = y'(0) = 0$.

    Ans. (a) $y(t) = 1 + (1-t)\,e^{-t}$,   (b) $y(t) = \frac{3}{8}\sin t - \frac{1}{8}\sin 3t$.

17. Use the Laplace transform to solve the following set of equations

$$\frac{dy}{dt} = 2y - 3z,$$
$$\frac{dz}{dt} = -2y + z,$$
$$y(0) = 8, \quad z(0) = 3.$$

    Ans. $y(t) = 3e^{4t} + 5e^{-t}$, $z(t) = 5e^{-t} - 2e^{4t}$.

18. Find the solution of the integrodifferential equation

$$y'(t) - \int_0^t y(\tau)\cos(t-\tau)\,d\tau = 0, \quad y(0) = 1.$$

    Ans. $y(t) = 1 + \frac{1}{2}t^2$.

19. Solve the following equations with the initial conditions at $t = 0$, both y and all its derivatives are equal to zero.
    (a) $y'' + 2y' + y = A\delta(t-t_0)$,
    (b) $y'''' - y = A\delta(t-t_0)$.

    Ans. (a) $y(t) = A(t-t_0)\,e^{-(t-t_0)}u(t-t_0)$,

    (b) $y(t) = \frac{1}{2}A[\sinh(t-t_0) - \sin(t-t_0)]\,u(t-t_0)$.

20. Consider a resistance R and an inductance L connected in series with a voltage $V(t)$. The equation governs the current is

$$L\frac{di}{dt} + Ri = V(t).$$

    Suppose $i(0) = 0$ and $V(t)$ is a voltage impulse at $t = t_0$ given by

$$V(t) = A\delta(t-t_0).$$

Find the current by the Laplace transform method.

Ans. $i(t) = \dfrac{A}{L} e^{-R(t-t_0)/L} u(t - t_0)$.

21. The damped harmonic oscillator is governed by

$$mx'' + bx' + kx = f(t); \quad \text{with} \ \ x(0) = x'(0) = 0.$$

(a) Find the solution by convolution. (Express $x(t)$ as an integral).
(b) If $f(t) = P\delta(t - t_0)$, find the solution by evaluating the convolution integral.
(c) If $b = 0$, and $f(t) = F_0 \sin \omega_0 t$ where $\omega_0 = \sqrt{k/m}$, solve the problem by Laplace transformation.
(d) If $b = 0$, and $f(t) = F_0 u(t - t_0)$ where $u(t - t_0)$ is the step function, solve the problem.

Ans. (a) $x(t) = \frac{1}{m\omega} \int_0^t f(\tau) e^{-\alpha(t-\tau)} \sin \omega (t - \tau) d\tau$

where $\alpha = \frac{b}{2m}$, $\omega^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2$,

(b) $x(t) = \frac{P}{m\omega} e^{-\alpha(t-t_0)} \sin \omega (t - t_0) u(t - t_0)$,

(c) $x(t) = \frac{F_0}{2m\omega_0^2} (\sin \omega_0 t - \omega_0 t \cos \omega_0 t)$,

(d) $x(t) = \frac{F_0}{m\omega_0^2} [1 - \cos \omega_0 (t - t_0)] u(t - t_0)$.

22. Using the complex inversion integral, find the inverses of the following Laplace transforms

(a) $\frac{1}{(s+1)(s+3)}$,   (b) $\frac{1}{(s+2)^2}$,   (c) $\frac{1}{(s^2+9)(s^2+4)}$.

Ans. (a) $\frac{1}{2}(e^{-t} - e^{-3t})$,   (b) $t e^{-2t}$,   (c) $\frac{1}{30}(3 \sin 2t - 2 \sin 3t)$.