2

Fourier Transforms

Fourier transform is a generalization of Fourier series. It provides representations, in terms of a superposition of sinusoidal waves, for functions defined over an infinite interval with no particular periodicity. It is an indispensable mathematical tool in the study of waves, which in one form or another, consist of most of physics and modern technology.

Like Laplace transform, Fourier transform is a member of a class of representations known as integral transforms. As such, it is useful in solving differential equations. But the importance of Fourier transforms far exceeds just being able to solve differential equations. In quantum mechanics, it enables us to look at the wave functions either in the coordinate space or in the momentum space. In information theory, it allows one to examine a wave form from the perspective of both the time and frequency domains. For these reasons, Fourier transform has become a cornerstone of diverse fields ranging from signal processing technology to quantum description of matter waves.

2.1 Fourier Integral as a Limit of a Fourier Series

As we have seen, Fourier series is useful in representing either periodic functions or functions confined in limited range of interest. However, in many problems, the function of interests, such as a single unrepeated pulse of force or voltage, is nonperiodic over an infinite range. In such a case, we can still imagine that the function is periodic with the period approaching infinity. In this limit, the Fourier series becomes the Fourier integral.

To extend the concept of Fourier series to nonperiodic functions, let us first consider a function which repeats itself after an interval of $2p$

$$f(t) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{n\pi}{p} t + b_n \sin \frac{n\pi}{p} t \right),$$

where

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{d}t,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,\mathrm{d}t, \quad n = 1, 2, \ldots,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,\mathrm{d}t, \quad n = 0, 1, 2, \ldots.$$

Note that each individual term $\cos\frac{n\pi}{p}t$ or $\sin\frac{n\pi}{p}t$ is a periodic function. Its period T_n is determined by the relation that when t is increased by T_n, the function returns to its previous value,

$$\cos\frac{n\pi}{p}(t+T_n) = \cos\left(\frac{n\pi}{p}t + \frac{n\pi}{p}T_n\right) = \cos\frac{n\pi}{p}t.$$

Thus,

$$\frac{n\pi}{p}T_n = 2\pi \text{ and } T_n = \frac{2p}{n}.$$

The frequency ν is defined as the number of oscillations in one second. Therefore, each term is associated with a frequency ν_n,

$$\nu_n = \frac{1}{T_n} = \frac{n}{2p}.$$

Now if t stands for time, then ν_n is just the usual temporal frequency. If the variable is x, standing for distance, ν_n is simply the spatial frequency. The distribution of the set of all of the frequencies $\left\{\frac{n}{2p}\right\}$ is called the frequency spectrum. To see what happens to the frequency spectra as p increases, consider the cases where $p = 1, 2$, and 10. The corresponding frequencies of the spectra are as follows:

$$p = 1, \quad \nu_n = 0,\ 0.50,\ 1.0,\ 1.50,\ 2.0, \ldots$$

$$p = 2, \quad \nu_n = 0,\ 0.25,\ 0.5,\ 0.75,\ 1.0, \ldots$$

$$p = 10, \quad \nu_n = 0,\ 0.05,\ 0.1,\ 0.15,\ 0.2, \ldots.$$

It is seen that as p increases, the discrete spectrum becomes more and more dense. It will approach a continuous spectrum as $p \to \infty$, and the Fourier series appears to be an integral. This is indeed the case, if $f(t)$ is absolutely integrable over the infinite range.

Often the angular frequency, defined as $\omega_n = 2\pi\nu_n$, is used to simplify the writing. Since

$$\omega_n = 2\pi\nu_n = 2\pi\frac{n}{2p} = \frac{n\pi}{p},$$

the Fourier series can be written as

$$f(t) = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{d}t + \sum_{n=1}^{\infty}(a_n \cos\omega_n t + b_n \sin\omega_n t).$$

As $f(t)$ is absolutely integrable over the infinite range, this means that the integral $\int_{-p}^{p} |f(t)|\,\mathrm{d}t$ exists even when $p \to \infty$. Therefore

$$\lim_{p\to\infty} \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{d}t = 0.$$

Hence,

$$f(t) = \sum_{n=1}^{\infty}(a_n \cos\omega_n t + b_n \sin\omega_n t),$$

where

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\omega_n t\,\mathrm{d}t,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\omega_n t\,\mathrm{d}t.$$

Furthermore, we can define

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\,\pi}{p} - \frac{n\pi}{p} = \frac{\pi}{p}.$$

Therefore

$$f(t) = \sum_{n=1}^{\infty}\left[\frac{\Delta\omega}{\pi}\int_{-p}^{p} f(t)\cos\omega_n t\,\mathrm{d}t\right]\cos\omega_n t + \sum_{n=1}^{\infty}\left[\frac{\Delta\omega}{\pi}\int_{-p}^{p} f(t)\sin\omega_n t\,\mathrm{d}t\right]\sin\omega_n t.$$

If we write the series as

$$f(t) = \sum_{n=1}^{\infty}\left[A_p(\omega_n)\cos\omega_n t + B_p(\omega_n)\sin\omega_n t\right]\Delta\omega,$$

then

$$A_p(\omega_n) = \frac{1}{\pi}\int_{-p}^{p} f(t)\cos\omega_n t\,\mathrm{d}t,$$

$$B_p(\omega_n) = \frac{1}{\pi}\int_{-p}^{p} f(t)\sin\omega_n t\,\mathrm{d}t.$$

Now if we let $p \to \infty$, then $\Delta\omega \to 0$ and ω_n becomes a continuous variable. Furthermore, let

$$A(\omega) = \lim_{p\to\infty} A_p(\omega_n) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t,$$

$$B(\omega) = \lim_{p\to\infty} B_p(\omega_n) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t.$$

Then the infinite series becomes a Riemann sum of an integral

$$f(t) = \int_0^{\infty} [A(\omega)\cos\omega t + B(\omega)\sin\omega t]\,\mathrm{d}\omega.$$

This integral is known as Fourier integral. This development is purely formal. However, it can be made rigorous provided (1) $f(t)$ is piecewise continuous and differentiable and (2) it is absolutely integrable in the infinite range, as we have assumed.

This integral will converge to $f(t)$ where $f(t)$ is continuous, and it converges to the average of the left- and right-hand limits of $f(t)$ at points of discontinuity, just like a Fourier series.

Example 2.1.1. (a) Find the Fourier integral of

$$f(t) = \begin{cases} 1 & \text{if} \quad -1 < t < 1, \\ 0 & \qquad \text{otherwise.} \end{cases}$$

(b) Show that

$$\int_0^{\infty} \frac{\cos\omega t \sin\omega}{\omega}\mathrm{d}\omega = \begin{cases} \frac{\pi}{2} & if \quad -1 < t < 1, \\ \frac{\pi}{4} & if \quad |t| = 1, \\ 0 & if \quad |t| > 1. \end{cases}$$

(c) Show that

$$\int_0^{\infty} \frac{\sin\omega}{\omega}\mathrm{d}\omega = \frac{\pi}{2}.$$

Solution 2.1.1. (a)

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t = \frac{1}{\pi}\int_{-1}^{1}\cos\omega t\,\mathrm{d}t = \frac{2\sin\omega}{\pi\omega}.$$

Since $f(t)$ is an even function

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t = 0.$$

Therefore the Fourier integral is given by

$$f(t) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega t\,\mathrm{d}\omega.$$

(b) In the range of $-1 < t < 1$, $f(t) = 1$, therefore

$$\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega t\,\mathrm{d}\omega = \frac{\pi}{2}, \quad \text{for} \quad -1 < t < 1.$$

At $|t| = 1$, it is a point of discontinuity, the Fourier integral converges to the average of 1 and 0, which is $\frac{1}{2}$. Therefore

$$\frac{1}{2} = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega\,\mathrm{d}\omega$$

or

$$\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega\,\mathrm{d}\omega = \frac{\pi}{4}.$$

For $|x| > 1$, $f(t) = 0$. Thus

$$\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega t\,\mathrm{d}\omega = 0, \quad for \quad |t| > 1.$$

(c) In particular at $t = 0$,

$$\int_0^\infty \frac{\sin\omega}{\omega}\cos\omega t\,\mathrm{d}\omega = \int_0^\infty \frac{\sin\omega}{\omega}\mathrm{d}\omega.$$

At $t = 0$, $f(0) = 1$, therefore

$$\int_0^\infty \frac{\sin\omega}{\omega}\mathrm{d}\omega = \frac{\pi}{2}.$$

2.1.1 Fourier Cosine and Sine Integrals

If $f(t)$ is a even function, then

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos\omega t\,\mathrm{d}t = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t,$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin\omega t\,\mathrm{d}t = 0$$

and

$$f(t) = \int_0^\infty A(\omega)\cos\omega t\,\mathrm{d}\omega.$$

This is known as Fourier cosine integral.

If $f(t)$ is an odd function, then

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t \,\mathrm{d}t = 0,$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t \,\mathrm{d}t = \frac{2}{\pi}\int_{0}^{\infty} f(t)\sin\omega t \,\mathrm{d}t$$

and

$$f(t) = \int_{0}^{\infty} B(\omega)\sin\omega t \,\mathrm{d}\omega.$$

This is known as Fourier sine integral.

Note that the function is supposed to be defined from $-\infty$ to $+\infty$, but because of the parity of the function, to define the transform, we only need the function from 0 to ∞. This also means that if we are only interested in the range of 0 to ∞, we can define the function from $-\infty$ to 0 any way we want, then we can have either cosine integral or sine integral by extending the function into the negative range either in an even or odd form. In this sense, Fourier cosine and sine integrals are equivalent to the half-range expansion of Fourier series.

Example 2.1.2. Find the Fourier cosine and sine integrals of

$$f(t) = \mathrm{e}^{-st}, \quad t > 0, \quad s > 0.$$

Solution 2.1.2. For the Fourier cosine integral, we can imagine $f(t)$ is an even function with respect to $t = 0$. Thus

$$A(\omega) = \frac{2}{\pi}\int_{0}^{\infty} \mathrm{e}^{-st}\cos\omega t \,\mathrm{d}t.$$

This integral can be evaluated with integration by parts twice. Better still, we recognize that the integral is just the Laplace transform of $\cos\omega t$. So

$$A(\omega) = \frac{2}{\pi}\frac{s}{s^2+\omega^2}\ .$$

It follows that the Fourier cosine integral is given by:

$$f(t) = \int_{0}^{\infty} A(\omega)\cos\omega t \,\mathrm{d}t = \frac{2s}{\pi}\int_{0}^{\infty} \frac{\cos\omega t}{s^2+\omega^2}\mathrm{d}t.$$

Since $f(t) = \mathrm{e}^{-st}$, a byproduct of this cosine integral is

$$\int_{0}^{\infty} \frac{\cos\omega t}{s^2+\omega^2}\mathrm{d}\omega = \frac{\pi}{2s}\mathrm{e}^{-st}$$

a formula we have obtained before by contour integration. In particular, for $t = 0$, we have

$$\int_0^\infty \frac{1}{s^2+\omega^2}\mathrm{d}\omega = \frac{\pi}{2s}.$$

Similarly, for Fourier sine integral, we can imagine $f(t)$ is an odd function. In this case

$$B(\omega) = \frac{2}{\pi}\int_0^\infty \mathrm{e}^{-st}\sin\omega t\,\mathrm{d}t = \frac{2}{\pi}\frac{\omega}{s^2+\omega^2}$$

as the integral is just a Laplace transform of $\sin\omega t$. Thus, the Fourier sine integral is given by

$$f(t) = \mathrm{e}^{-st} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{s^2+\omega^2}\sin\omega t\,\mathrm{d}\omega.$$

From this, we can obtain another integration formula

$$\int_0^\infty \frac{\omega\sin\omega t}{s^2+\omega^2}\mathrm{d}\omega = \frac{\pi}{2}\mathrm{e}^{-st}.$$

Example 2.1.3. Find $f(t)$, if $f(t)$ is an even function and

$$\int_0^\infty f(t)\cos at\,\mathrm{d}t = \begin{cases} 1-a & \text{if } \; 0 \le a \le 1, \\ 0 & \text{if } \quad a > 1. \end{cases}$$

Solution 2.1.3. We can use Fourier cosine integral to solve this integral equation. Let

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t = \begin{cases} \frac{2}{\pi}(1-\omega) & \text{if } \; 0 \le a \le 1, \\ 0 & \text{if } \quad a > 1, \end{cases}$$

then

$$\begin{aligned} f(t) &= \int_0^\infty A(\omega)\cos\omega t\,\mathrm{d}\omega = \int_0^1 \frac{2}{\pi}(1-\omega)\cos\omega t\,\mathrm{d}\omega \\ &= \frac{2}{\pi}\frac{1}{t^2}(1-\cos t). \end{aligned}$$

2.1.2 Fourier Cosine and Sine Transforms

If $f(t)$ is an even function, we have just seen that it can be expressed as a Fourier integral

$$f(t) = \int_0^\infty A(\omega)\cos\omega t\,\mathrm{d}\omega, \tag{2.1}$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,\mathrm{d}t. \tag{2.2}$$

Now if we define a function

$$\widehat{f}_{\mathrm{c}}(\omega) = \sqrt{\frac{\pi}{2}} A(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, \mathrm{d}t, \tag{2.3}$$

then

$$A(\omega) = \sqrt{\frac{2}{\pi}} \widehat{f}_{\mathrm{c}}(\omega).$$

Putting it into (2.1), we have

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f}_{\mathrm{c}}(\omega) \cos \omega t \, \mathrm{d}\omega. \tag{2.4}$$

The symmetry between (2.3) and (2.4) is unmistakable. They form what is known as the Fourier cosine transform pair. The function $\widehat{f}_{\mathrm{c}}(\omega)$ is known as the Fourier cosine transform. Formula (2.4) gives us back $f(t)$ from $\widehat{f}_{\mathrm{c}}(\omega)$, therefore it is called the inverse Fourier cosine transform of $\widehat{f}_{\mathrm{c}}(\omega)$. The process of obtaining the transform $\widehat{f}_{\mathrm{c}}(\omega)$ from a given function $f(t)$ is also called Fourier cosine transform and is denoted by $F_{\mathrm{c}}\{f(t)\}$, that is, when F_{c} operates on $f(t)$, it gives us $\widehat{f}_{\mathrm{c}}(\omega)$,

$$F_{\mathrm{c}}\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, \mathrm{d}t = \widehat{f}_{\mathrm{c}}(\omega).$$

The inverse operation is called inverse Fourier cosine transform and is denoted as $F_{\mathrm{c}}^{-1}\left\{\widehat{f}_{\mathrm{c}}(\omega)\right\}$,

$$F_{\mathrm{c}}^{-1}\left\{\widehat{f}_{\mathrm{c}}(\omega)\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f}_{\mathrm{c}}(\omega) \cos \omega t \, \mathrm{d}\omega = f(t).$$

Similarly, if $f(t)$ is an odd function, we have the Fourier sine transform pair

$$F_{\mathrm{s}}\{f(t)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \, \mathrm{d}t = \widehat{f}_{\mathrm{s}}(\omega),$$

$$F_{\mathrm{s}}^{-1}\left\{\widehat{f}_{\mathrm{s}}(\omega)\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f}_{\mathrm{s}}(\omega) \sin \omega t \, \mathrm{d}\omega = f(t).$$

Note that Fourier integral and Fourier transform are essentially the same. The modification of the multiplicative constant is of minor significance. It can be easily shown that if we define

$$\widehat{f}_{\mathrm{c}}(\omega) = \alpha \int_0^\infty f(t) \cos \omega t \, \mathrm{d}t, \tag{2.5}$$

then

$$f(t) = \beta \int_0^{\infty} \widehat{f}_{\mathrm{c}}(\omega) \cos \omega t \, \mathrm{d}\omega, \tag{2.6}$$

where

$$\beta = \frac{2}{\pi} \frac{1}{\alpha}.$$

Therefore as long as

$$\alpha\beta = \frac{2}{\pi},$$

where α can be assigned any number, (2.5) and (2.6) are still a Fourier cosine transform pair. As a matter of fact, in the literature, there are several different conventions in defining Fourier transforms. The differences are where to put the factor $\frac{2}{\pi}$. Using a Fourier transform table, one needs to pay attention to where that factor is in the definition.

Then why should we have two different names for essentially the same thing. This is because we have two different perspectives of looking at it. In Fourier integral, $f(t)$ is being described by a continuum of cosine (or sine) waves and $A(\omega)$ is just the amplitude of the harmonic components of $f(t)$ in the time domain. Whereas in Fourier transform, $\widehat{f}_{\mathrm{c}}(\omega)$ is regarded as a function in the frequency domain. This frequency domain function describes the same entity as the time domain function $f(t)$. There are many reasons why sometimes we would like to work with the transform of the function. For example, in the frequency domain we may easily perform relatively difficult mathematical operations such as differentiation and integration via simple multiplication and division.

Example 2.1.4. Show that

$$\begin{aligned}
F_{\mathrm{c}}\{f'(t)\} &= \omega F_{\mathrm{s}}\{f(t)\} - \sqrt{\frac{2}{\pi}} f(0), \\
F_{\mathrm{s}}\{f'(t)\} &= -\omega F_{\mathrm{c}}\{f(t)\}, \\
F_{\mathrm{c}}\{f''(t)\} &= -\omega^2 F_{\mathrm{c}}\{f(t)\} - \sqrt{\frac{2}{\pi}} f'(0), \\
F_{\mathrm{s}}\{f''(t)\} &= -\omega^2 F_{\mathrm{s}}\{f(t)\} + \sqrt{\frac{2}{\pi}} \omega f(0).
\end{aligned}$$

Solution 2.1.4. Since $f(t)$ is absolutely integrable, we assume

$$f(t) \to 0 \quad \text{as} \quad t \to \infty.$$

With the integration by parts, we can evaluate the transform of derivatives

$$
\begin{aligned}
F_{\mathrm{c}}\{f'(t)\} &= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\mathrm{d}f}{\mathrm{d}t}\cos\omega t\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\left[f(t)\cos\omega t|_0^{\infty} - \int_0^{\infty} f(t)\frac{\mathrm{d}}{\mathrm{d}t}\cos\omega t\,\mathrm{d}t\right] \\
&= \sqrt{\frac{2}{\pi}}\left[-f(0) + \omega\int_0^{\infty} f(t)\sin\omega t\,\mathrm{d}t\right] = \omega F_{\mathrm{s}}\{f(t)\} - \sqrt{\frac{2}{\pi}}f(0).
\end{aligned}
$$

$$
\begin{aligned}
F_{\mathrm{s}}\{f'(t)\} &= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\mathrm{d}f}{\mathrm{d}t}\sin\omega t\,\mathrm{d}t \\
&= \sqrt{\frac{2}{\pi}}\left[f(t)\sin\omega t|_0^{\infty} - \int_0^{\infty} f(t)\frac{\mathrm{d}}{\mathrm{d}t}\sin\omega t\,\mathrm{d}t\right] \\
&= \sqrt{\frac{2}{\pi}}\left[-\omega\int_0^{\infty} f(t)\cos\omega t\,\mathrm{d}t\right] = -\omega F_{\mathrm{c}}\{f(t)\}.
\end{aligned}
$$

$$
\begin{aligned}
F_{\mathrm{c}}\{f''(t)\} &= F_{\mathrm{c}}\{[f'(t)]'\} = \omega F_{\mathrm{s}}\{f'(t)\} - \sqrt{\frac{2}{\pi}}f'(0) \\
&= \omega\left[-\omega F_{\mathrm{c}}\{f(t)\}\right] - \sqrt{\frac{2}{\pi}}f'(0) = -\omega^2 F_{\mathrm{c}}\{f(t)\} - \sqrt{\frac{2}{\pi}}f'(0).
\end{aligned}
$$

$$
\begin{aligned}
F_{\mathrm{s}}\{f''(t)\} &= F_{\mathrm{s}}\{[f'(t)]'\} = -\omega F_{\mathrm{c}}\{f'(t)\} \\
&= -\omega\left[\omega F_{\mathrm{s}}\{f(t)\} - \sqrt{\frac{2}{\pi}}f(0)\right] = -\omega^2 F_{\mathrm{s}}\{f(t)\} + \omega\sqrt{\frac{2}{\pi}}f(0).
\end{aligned}
$$

Example 2.1.5. Use the transform of derivatives to show

$$F_{\mathrm{s}}\{\mathrm{e}^{-at}\} = \sqrt{\frac{2}{\pi}}\frac{\omega}{a^2+\omega^2}.$$

Solution 2.1.5. Let $f(t) = \mathrm{e}^{-at}$, so $f(0) = 1$ and

$$f'(t) = -a\,\mathrm{e}^{-at}, \quad f''(t) = a^2\,\mathrm{e}^{-at} = a^2 f(t).$$

Thus

$$F_{\mathrm{s}}\{f''(t)\} = F_{\mathrm{s}}\{a^2 f(t)\} = a^2 F_{\mathrm{s}}\{f(t)\}.$$

But

$$F_{\mathrm{s}}\{f''(t)\} = -\omega^2 F_{\mathrm{s}}\{f(t)\} + \omega\sqrt{\frac{2}{\pi}}f(0)$$

it follows that:

$$-\omega^2 F_{\mathrm{s}}\{f(t)\} + \omega\sqrt{\frac{2}{\pi}} = a^2 F_{\mathrm{s}}\{f(t)\}$$

or

$$(a^2 + \omega^2)F_{\mathrm{s}}\{f(t)\} = \omega\sqrt{\frac{2}{\pi}}.$$

Thus,

$$F_{\mathrm{s}}\{f(t)\} = F_{\mathrm{s}}\{\mathrm{e}^{-at}\} = \sqrt{\frac{2}{\pi}}\frac{\omega}{a^2 + \omega^2}.$$

Example 2.1.6. Use the Fourier sine transform to solve the following differential equation:

$$y''(t) - 9y(t) = 50\mathrm{e}^{-2t},$$
$$y(0) = y_0.$$

Solution 2.1.6. Since we are interested in positive + region, we can take $y(t)$ to be an odd function and take Fourier sine transforms. It is clear from its definition that Fourier transform is linear

$$F_{\mathrm{s}}\{af_1(t) + bf_2(t)\} = aF_{\mathrm{s}}\{f_1(t)\} + bF_{\mathrm{s}}\{f_2(t)\}.$$

Using this property and taking Fourier transform of both sides of the differential equation, we have

$$F_{\mathrm{s}}\{y''(t)\} - 9F_{\mathrm{s}}\{y(t)\} = 50F_{\mathrm{s}}\{\mathrm{e}^{-2t}\}.$$

Since

$$F_{\mathrm{s}}\{y''(t)\} = -\omega^2 F_{\mathrm{s}}\{y(t)\} + \omega\sqrt{\frac{2}{\pi}}y(0),$$

so

$$-\omega^2 F_{\mathrm{s}}\{y(t)\} + \omega\sqrt{\frac{2}{\pi}}y_0 - 9F_{\mathrm{s}}\{y(t)\} = 50F_{\mathrm{s}}\{\mathrm{e}^{-2t}\},$$

which, after collecting terms, becomes

$$(\omega^2 + 9)F_{\mathrm{s}}\{y(t)\} = -50\sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2 + 4} + \omega\sqrt{\frac{2}{\pi}}y_0.$$

Thus

$$F_{\mathrm{s}}\{y(t)\} = -50\sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2 + 4}\frac{1}{(\omega^2 + 9)} + \sqrt{\frac{2}{\pi}}y_0\frac{\omega}{(\omega^2 + 9)}.$$

With partial fraction of

$$\frac{1}{(\omega^2 + 4)(\omega^2 + 9)} = \frac{1}{5}\frac{1}{\omega^2 + 4} - \frac{1}{5}\frac{1}{\omega^2 + 9}$$

we have

$$F_{\mathrm{s}}\{y(t)\} = \sqrt{\frac{2}{\pi}}10\frac{\omega}{\omega^2+9} - \sqrt{\frac{2}{\pi}}10\frac{\omega}{\omega^2+4} + \sqrt{\frac{2}{\pi}}y_0\frac{\omega}{(\omega^2+9)}$$

$$= (10+y_0)\sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2+9} - 10\sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2+4}$$

$$= (10+y_0)F_{\mathrm{s}}\{\mathrm{e}^{-3t}\} - 10F_{\mathrm{s}}\{\mathrm{e}^{-2t}\}.$$

Taking the inverse transform, we get the solution

$$y(t) = (10+y_0)\mathrm{e}^{-3t} - 10\mathrm{e}^{-2t}.$$

2.2 Tables of Transforms

There are extensive tables of Fourier transforms (For example, A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi: "Tables of Integral Transforms," vol. 1, McGraw-Hill Book Company, New York, 1954). A short list of some simple Fourier cosine and sine transforms is given in Tables 2.1 and 2.2, respectively. A short table of Fourier transform, which we will explain in the Sect. 2.3, is given in Table 2.3.

2.3 The Fourier Transform

As we have seen in (1.28) and (1.29) that the Fourier series of a function repeating itself in the interval of $2p$, can also be written in the complex form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}\frac{n\pi}{p}t}, \quad c_n = \frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}\frac{n\pi}{p}t}\mathrm{d}t,$$

so

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2p}\int_{-p}^{p} f(t)\mathrm{e}^{-\mathrm{i}\frac{n\pi}{p}t}\mathrm{d}t\right]\mathrm{e}^{\mathrm{i}\frac{n\pi}{p}t}.$$

Again let us define

$$\omega_n = \frac{n\pi}{p}$$

and

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{p}$$

Table 2.1. A short table of Fourier cosine transforms

$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f}_c(\omega) \cos \omega t \, d\omega$	$\widehat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, dt$
$\begin{cases} 1 & \text{if} \quad 0 < t < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$
$t^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(a)}{\omega^a} \cos \frac{a\pi}{2}$
$e^{-at} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2 + \omega^2}$
$e^{-at^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\omega^2/4a}$
$\frac{1}{t^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{1}{a} e^{-a\omega}$
$t^n e^{-at} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(a^2 + \omega^2)^{n+1}} \operatorname{Re}(a + i\omega)^{n+1}$
$\begin{cases} \cos t & \text{if} \quad 0 < t < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{1}{2\pi}} \left[\frac{\sin a(1-\omega)}{1-\omega} + \frac{\sin a(1+\omega)}{1+\omega} \right]$
$\cos at^2 \quad (a > 0)$	$\sqrt{\frac{1}{2a}} \cos \left(\frac{\omega^2}{4a} - \frac{\pi}{4} \right)$
$\sin at^2 \quad (a > 0)$	$\sqrt{\frac{1}{2a}} \cos \left(\frac{\omega^2}{4a} + \frac{\pi}{4} \right)$
$\frac{\sin at}{t} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} u(a - \omega)$
Linearity of transform and inverse:	
$\alpha f(t) + \beta g(t)$	$\alpha \widehat{f}_c(\omega) + \beta \widehat{g}_c(\omega)$
Transform of derivatives:	
$f'(t)$	$\omega \widehat{f}_s(\omega) - \sqrt{\frac{2}{\pi}} f(0)$
$f''(t)$	$-\omega^2 \widehat{f}_c(\omega) - \sqrt{\frac{2}{\pi}} f'(0)$
Convolution theorem:	
$\frac{1}{2} \int_0^\infty [f(\lvert t-x \rvert) + f(\lvert t+x \rvert)] g(x) dx$	$\widehat{f}_c(\omega) \widehat{g}_c(\omega)$

Table 2.2. A short table of Fourier sine transforms

$f(t)=\sqrt{\frac{2}{\pi}}\int_0^\infty \widehat{f}_{\mathrm{s}}(\omega)\sin\omega t\,\mathrm{d}\omega$	$\widehat{f}_{\mathrm{s}}(\omega)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t\,\mathrm{d}t$
$\begin{cases}1 & \text{if } \quad 0<t<a\\ 0 & \text{otherwise}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{1-\cos a\omega}{\omega}$
$\frac{1}{\sqrt{t}}$	$\frac{1}{\sqrt{\omega}}$
$t^{a-1} \quad (0<a<1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(a)}{\omega^a}\sin\frac{a\pi}{2}$
e^{-t}	$\sqrt{\frac{2}{\pi}}\frac{\omega}{1+\omega^2}$
$\frac{t}{t^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\mathrm{e}^{-a\omega}$
$t^n\mathrm{e}^{-at} \quad (a>0)$	$\sqrt{\frac{2}{\pi}}\frac{n!}{(a^2+\omega^2)^{n+1}}\,\mathrm{Im}(a+\mathrm{i}\omega)^{n+1}$
$t\mathrm{e}^{-at^2} \quad (a>0)$	$\frac{\omega}{(2a)^{3/2}}\mathrm{e}^{-\omega^2/4a}$
$\begin{cases}\sin t & \text{if } \quad 0<t<a\\ 0 & \text{otherwise}\end{cases}$	$\sqrt{\frac{1}{2\pi}}\left[\frac{\sin a(1-\omega)}{1-\omega}-\frac{\sin a(1+\omega)}{1+\omega}\right]$
$\frac{\cos at}{t} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}u(\omega-a)$
Linearity of transform and inverse:	
$\alpha f(t)+\beta g(t)$	$\alpha\widehat{f}_{\mathrm{s}}(\omega)+\beta\widehat{g}_{\mathrm{s}}(\omega)$
Transform of derivatives:	
$f'(t)$	$-\omega\widehat{f}_{\mathrm{c}}(\omega)$
$f''(t)$	$-\omega^2\widehat{f}_{\mathrm{s}}(\omega)-\sqrt{\frac{2}{\pi}}\omega f(0)$
Convolution theorem:	
$\frac{1}{2}\int_0^\infty[f(\lvert t-x\rvert)-f(\lvert t+x\rvert)]g(x)\mathrm{d}x$	$\widehat{f}_{\mathrm{c}}(\omega)\widehat{g}_{\mathrm{s}}(\omega)$

Table 2.3. A short table of Fourier transforms: u is the Heaviside step function

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\widehat{f}(\omega)d\omega$	$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt$
$\dfrac{1}{t^2+a^2}$ $(a>0)$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$
$u(t)e^{-at}$	$\dfrac{1}{a+i\omega}$
$u(-t)e^{at}$	$\dfrac{1}{a-i\omega}$
$e^{-a\lvert t\rvert}$ $(a>0)$	$\dfrac{2a}{a^2+\omega^2}$
e^{-t^2}	$\sqrt{\pi}e^{-\omega^2/4}$
$\dfrac{1}{2a\sqrt{\pi}}e^{-t^2/(2a)^2}$ $(a>0)$	$e^{-a^2\omega^2}$
$\dfrac{1}{\sqrt{\lvert t\rvert}}$	$\sqrt{\dfrac{2\pi}{\lvert\omega\rvert}}$
$u(t+a)-u(t-a)$	$\dfrac{2\sin\omega a}{\omega}$
$\delta(t-a)$	$e^{-i\omega a}$
$f(at+b)$ $(a>0)$	$\dfrac{1}{a}e^{ib\omega/a}\widehat{f}(\dfrac{\omega}{a})$
Linearity of transform and inverse: $\alpha f(t)+\beta g(t)$	$\alpha\widehat{f}(\omega)+\beta\widehat{g}$
Transform of derivative: $f^{(n)}(t)$	$(i\omega)^n\widehat{f}(\omega)$
Transform of integral: $f(t)=\int_{-\infty}^{t} g(x)dx$	$\widehat{f}(\omega)=\dfrac{1}{i\omega}\widehat{g}(\omega)$
Convolution theorems: $f(t)*g(t)=\int_{-\infty}^{\infty} f(t-x)g(x)dx$	$\widehat{f}(\omega)\widehat{g}(\omega)$
$f(t)g(t)$	$\dfrac{1}{2\pi}\widehat{f}(\omega)*\widehat{g}(\omega)$

and write the series as

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-p}^{p} f(t) \mathrm{e}^{-\mathrm{i}\omega_n t} \mathrm{d}t \right] \mathrm{e}^{\mathrm{i}\omega_n t} \Delta\omega \tag{2.7}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \widehat{f}_p(\omega_n) \mathrm{e}^{\mathrm{i}\omega_n t} \Delta\omega$$

with

$$\widehat{f}_p(\omega_n) = \int_{-p}^{p} f(t) \mathrm{e}^{-\mathrm{i}\omega_n t} \mathrm{d}t.$$

Now if we let $p \to \infty$, then $\Delta\omega \to 0$ and ω_n becomes a continuous variable. Furthermore

$$\widehat{f}(\omega) = \lim_{p\to\infty} \widehat{f}_p(\omega_n) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t \tag{2.8}$$

and the infinite sum of (2.7) becomes an integral

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega. \tag{2.9}$$

This integral is known as Fourier integral.

The coefficient function $\widehat{f}(\omega)$ is known as the Fourier transform of $f(t)$. The process of transforming the function $f(t)$ in the time domain into the same function $\widehat{f}(\omega)$ in the frequency domain is expressed as $\mathcal{F}\{f(t)\}$,

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t = \widehat{f}(\omega). \tag{2.10}$$

The process of getting back to $f(t)$ from $\widehat{f}(\omega)$ is known as inverse Fourier transform $\mathcal{F}^{-1}\{\widehat{f}(\omega)\}$,

$$\mathcal{F}^{-1}\{\widehat{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega = f(t). \tag{2.11}$$

We have "derived" this pair of Fourier transforms with the same heuristic arguments as we introduced the Fourier cosine transform. Comments there are also applicable here. Formulas (2.10) and (2.11) can be established rigorously provided (1) $f(t)$ is piecewise continuous and differentiable and (2) it is absolutely integrable, that is, $\int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t$ is finite.

The multiplicative factor in front of the integral is somewhat arbitrary. If $\widehat{f}(\omega)$ is defined as

$$\mathcal{F}\{f(t)\} = \alpha \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t = \widehat{f}(\omega),$$

then $\mathcal{F}^{-1}\{\widehat{f}(\omega)\}$ becomes

$$\mathcal{F}^{-1}\{\widehat{f}(\omega)\} = \beta \int_{-\infty}^{\infty} \widehat{f}(\omega) \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{d}\omega = f(t),$$

where

$$\alpha\beta = \frac{1}{2\pi}.$$

Some authors chose $\alpha = \beta = \sqrt{\dfrac{1}{2\pi}}$, so that the Fourier pair is symmetrical. Others chose $\alpha = \dfrac{1}{2\pi}$, $\beta = 1$. In (2.10) and (2.11), α is chosen to be 1 and β to be $\dfrac{1}{2\pi}$.

Another convention, that is common in spectral analysis, is to use frequency ν, instead of angular frequency ω in defining the Fourier transforms. Since $\omega = 2\pi v$, (2.10) can be written as

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}2\pi\nu t} \, \mathrm{d}t = \widehat{f}(\nu) \tag{2.12}$$

and (2.11) becomes

$$\mathcal{F}^{-1}\{\widehat{f}(\nu)\} = \int_{-\infty}^{\infty} \widehat{f}(\nu) \mathrm{e}^{\mathrm{i}2\pi\nu t} \, \mathrm{d}\nu = f(t). \tag{2.13}$$

Note that in this pair of equations, the factor 2π is no longer there. Besides, frequency is a well-defined concept and no one actually measures angular frequency. These are good reasons to use (2.12) and (2.13) as the definition of Fourier transforms. However, for historic reasons, most books in engineering and physics use ω. Therefore we will continue to use (2.10) and (2.11) as the definition of the Fourier transforms.

The function $f(t)$ in the Fourier transform may or may not have any even or odd parity. However, if it is an even function, it can be easily shown that it reduces to the Fourier cosine transform. If it is an odd function, it reduces to the Fourier sine transform.

Example 2.3.1. Find the Fourier transform of

$$f(t) = \begin{cases} \mathrm{e}^{-\alpha t} & t > 0, \\ 0 & t < 0. \end{cases}$$

Solution 2.3.1.

$$\begin{aligned} \mathcal{F}\{f(t)\} &= \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t = \int_{0}^{\infty} \mathrm{e}^{-(\alpha+\mathrm{i}\omega)t} \, \mathrm{d}t \\ &= -\frac{1}{\alpha + \mathrm{i}\omega} \mathrm{e}^{-(\alpha+\mathrm{i}\omega)t} \Big|_{0}^{\infty} = \frac{1}{\alpha + \mathrm{i}\omega}. \end{aligned}$$

This result can, of course, be expressed as a real part plus an imaginary part,

$$\frac{1}{\alpha + \mathrm{i}\omega} = \frac{1}{\alpha + \mathrm{i}\omega}\frac{\alpha - \mathrm{i}\omega}{\alpha - \mathrm{i}\omega} = \frac{\alpha}{\alpha^2 + \omega^2} - \mathrm{i}\frac{\omega}{\alpha^2 + \omega^2}.$$

Example 2.3.2. Find the inverse Fourier transform of

$$\widehat{f}(\omega) = \frac{1}{\alpha + \mathrm{i}\omega}.$$

(This problem can be skipped for those who have not yet studied the complex contour integration.)

Solution 2.3.2.

$$\mathcal{F}^{-1}\{\widehat{f}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\alpha + \mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{1}{\omega - \mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

This integrals can be evaluated with contour integration. For $t > 0$, the contour can be closed counterclockwise in the upper half plane as shown in Fig. 2.1a.

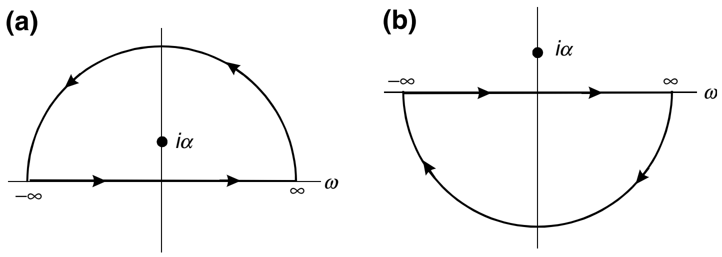

Fig. 2.1. Contour integration for inverse Fourier transform. **(a)** The contour is closed in the upper half plane. **(b)** The contour is closed in the lower half plane

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{1}{\omega - \mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi\mathrm{i}}\oint_{\mathrm{u.h.p.}}\frac{1}{\omega - \mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$
$$= \lim_{\omega \to i\alpha}\mathrm{e}^{\mathrm{i}\omega t} = \mathrm{e}^{-\alpha t}.$$

It follows that for $t > 0$:

$$\mathcal{F}^{-1}\{\widehat{f}(\omega)\} = \mathrm{e}^{-\alpha t}.$$

For $t < 0$, the contour can be closed clockwise in the lower half plane as shown in Fig. 2.1b. Since there is no singular point in the lower half plane

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{1}{\omega - \mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = \frac{1}{2\pi\mathrm{i}}\oint_{\mathrm{l.h.p.}}\frac{1}{\omega - \mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega = 0.$$

Thus, for $t < 0$,

$$\mathcal{F}^{-1}\{\widehat{f}(\omega)\} = 0.$$

With the Heaviside step function

$$u(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0, \end{cases}$$

we can combine the results for $t > 0$ and for $t < 0$ as

$$\mathcal{F}^{-1}\{\widehat{f}(\omega)\} = u(t)\mathrm{e}^{-\alpha t}.$$

It is seen that the inverse transform is indeed equal to $f(t)$ of the previous problem.

2.4 Fourier Transform and Delta Function

2.4.1 Orthogonality

If we put $\widehat{f}(\omega)$ of (2.8) back in the Fourier integral of (2.9), the Fourier representation of $f(t)$ takes the form

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t')\mathrm{e}^{-\mathrm{i}\omega t'}\,\mathrm{d}t'\right]\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$

which, after reversing the order of integration, can be written as

$$f(t) = \int_{-\infty}^{\infty} f(t')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega(t-t')}\mathrm{d}\omega\right]\mathrm{d}t'.$$

Recall that the Dirac delta function $\delta(t-t')$ is defined by the relation

$$f(t) = \int_{-\infty}^{\infty} f(t')\delta(t-t')\mathrm{d}t'.$$

Comparing the last two equations, we see that $\delta(t-t')$ can be written as

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega(t-t')}\mathrm{d}\omega. \tag{2.14}$$

Interchange the variables gives the inverted form

$$\delta(\omega-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega-\omega')t}\mathrm{d}t.$$

The last two equations are known as the orthogonality conditions. A function $e^{i\omega t}$ is orthogonal to all other functions in the form of $e^{-i\omega' t}$ when integrated over all t, as long as $\omega' \neq \omega$.

Since $\delta(x) = \delta(-x)$, (2.14) can also written as

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega.$$

These formulas are very useful representations of delta functions. The derivation of many transform pairs are greatly simplified with the use of delta functions. Although they are not proper mathematical functions, their use can be justified by the distribution theory.

2.4.2 Fourier Transforms Involving Delta Functions

Dirac Delta Function. Consider the function

$$f(t) = K\delta(t),$$

where K is a constant. The Fourier transform of $f(t)$ is easily derived using the definition of the delta function

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} K\delta(t)e^{-i\omega t}\,dt = Ke^{0} = K.$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}\{\widehat{f}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ke^{i\omega t}\,dt = K\delta(t).$$

Similarly, the Fourier transform of a constant function K is

$$\mathcal{F}\{K\} = 2\pi K\delta(\omega)$$

and its inverse is

$$\mathcal{F}^{-1}\{2\pi K\delta(\omega)\} = K.$$

These Fourier transform pairs are illustrated in Fig. 2.2.

Periodic Functions. To illustrate the Fourier transform of a periodic function, consider

$$f(t) = A\cos\omega_0 t.$$

The Fourier transform is given by

$$\mathcal{F}\{A\cos\omega_0 t\} = \int_{-\infty}^{\infty} A\cos(\omega_0 t)e^{-i\omega t}\,dt.$$

Since

$$\cos\omega_0 t = \frac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right),$$

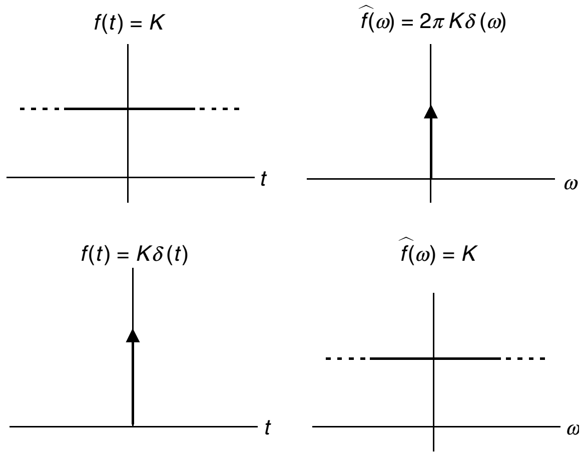

Fig. 2.2. The Fourier transform pair of constant and delta functions. The Fourier transform of constant function is a delta function. The Fourier transform of a delta function is a constant function

so

$$\mathcal{F}\{A\cos\omega_0 t\} = \frac{A}{2}\int_{-\infty}^{\infty}\left[\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)t} + \mathrm{e}^{-\mathrm{i}(\omega+\omega_0)t}\right]\mathrm{d}t.$$

Using (2.14), we have

$$\mathcal{F}\{A\cos\omega_0 t\} = \pi A\delta(\omega+\omega_0) + \pi A\delta(\omega-\omega_0). \tag{2.15}$$

Similarly,

$$\mathcal{F}\{A\sin\omega_0 t\} = \mathrm{i}\pi A\delta(\omega+\omega_0) - \mathrm{i}\pi A\delta(\omega-\omega_0). \tag{2.16}$$

Note that the Fourier transform of a sine function is imaginary.

These Fourier transform pairs are shown in Fig. 2.3, leaving out the factor of i in (2.16).

2.4.3 Three-Dimensional Fourier Transform Pair

So far we have used as variables t and ω, representing time and angular frequency, respectively. Mathematics will, of course, be exactly the same if we change the names of these variables. In describing the spatial variations of a wave, it is more natural to use either r or x, y, and z to represent distances. In a function of time, the period T is the time interval after which the function repeats itself. In a function of distance, the corresponding quantity is called wavelength λ, which is the increase in distance that the function will repeat itself. Therefore, if t is replaced by r, then the angular frequency ω, which is equal to $2\pi/T$, should be replaced by a quantity equal to $2\pi/\lambda$, which is known as the wave number k.

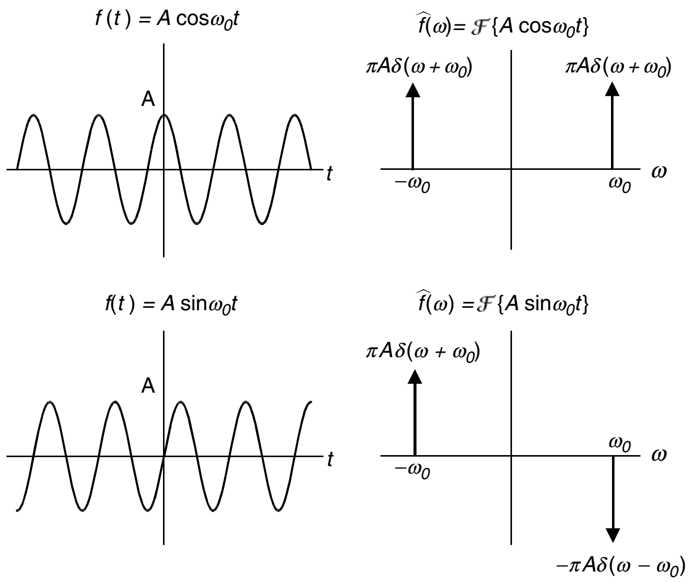

Fig. 2.3. Fourier transform pair of cosine and sine functions

Thus, corresponding to (2.14), we have

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_1(x-x')}\mathrm{d}k_1,$$

$$\delta(y-y') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_2(y-y')}\mathrm{d}k_2,$$

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_3(z-z')}\mathrm{d}k_3.$$

Therefore in three-dimensional space, the delta function is given by

$$\begin{aligned}\delta(\mathbf{r}-\mathbf{r}') &= \delta(x-x')\delta(y-y')\delta(z-z') \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_1(x-x')}\mathrm{d}k_1 \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_2(y-y')}\mathrm{d}k_2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k_3(z-z')}\mathrm{d}k_3 \\ &= \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\left[k_1(x-x')+k_2(y-y')+k_3(z-z')\right]}\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3.\end{aligned}$$

A convenient notation is to introduce a wave vector $\mathbf{k}$,

$$\mathbf{k} = k_1\widehat{\mathbf{i}} + k_2\widehat{\mathbf{j}} + k_3\widehat{\mathbf{k}}.$$

Together with

$$\mathbf{r}-\mathbf{r}' = (x-x')\widehat{\mathbf{i}} + (y-y')\widehat{\mathbf{j}} + (z-z')\widehat{\mathbf{k}}$$

the three-dimensional delta function can be written as

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3k.$$

Now by definition of the delta function

$$f(\mathbf{r}) = \iiint_{-\infty}^{\infty} f(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\mathrm{d}^3r',$$

we have

$$f(\mathbf{r})=\iiint_{-\infty}^{\infty} f(\mathbf{r}')\frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\mathrm{d}^3k\,\mathrm{d}^3r',$$

which can be written as

$$f(\mathbf{r})=\frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty}\left[\frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty} f(\mathbf{r}')\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\mathrm{d}^3r'\right]\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3k.$$

Thus, in three dimensions, we can define a Fourier transform pair

$$\widehat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty} f(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3r = \mathcal{F}\{f(\mathbf{r})\},$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty} \widehat{f}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3k = \mathcal{F}^{-1}\left\{\widehat{f}(\mathbf{k})\right\}.$$

Again, how to split $1/(2\pi)^3$ between the Fourier transform and its inverse is somewhat arbitrary. Here we split them equally to conform with most of the quantum mechanics text books.

In quantum mechanics, the momentum $\mathbf{p}$ is given by $\mathbf{p}=\hbar\mathbf{k}$. The Fourier transform pair in terms of $\mathbf{r}$ and $\mathbf{p}$ is therefore given by

$$\widehat{f}(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}}\iiint_{-\infty}^{\infty} f(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{r}/\hbar}\,\mathrm{d}^3r,$$

$$f(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{3/2}}\iiint_{-\infty}^{\infty} \widehat{f}(\mathbf{p})\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{r}/\hbar}\,\mathrm{d}^3p.$$

If $f(\mathbf{r})$ is the Schrödinger wave function, then its Fourier transform $\widehat{f}(\mathbf{p})$ is the momentum wave function. In describing a dynamic system, either space or momentum wave functions may be used, depending on which is more convenient for the particular problem.

If in three-dimensional space, a function possesses spherical symmetry, that is, $f(\mathbf{r}) = f(r)$, then its Fourier transform is reduced to a one-dimensional integral. In this case, let the wave vector $\mathbf{k}$ be along the z-axis of the coordinate space, so

$$\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$$

and

$$\mathrm{d}^3 r = r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}r \,\mathrm{d}\varphi.$$

The Fourier transform of $f(r)$ becomes

$$\begin{aligned}
\mathcal{F}\{f(\mathbf{r})\} &= \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\infty} f(r) \left[\int_0^{\pi} \mathrm{e}^{-\mathrm{i}kr\cos\theta} \sin\theta \,\mathrm{d}\theta \right] r^2 \,\mathrm{d}r \\
&= \frac{1}{(2\pi)^{3/2}} 2\pi \int_0^{\infty} f(r) \left[\frac{1}{\mathrm{i}kr} \mathrm{e}^{-\mathrm{i}kr\cos\theta} \right]_0^{\pi} r^2 \,\mathrm{d}r \\
&= \frac{1}{(2\pi)^{3/2}} 2\pi \int_0^{\infty} f(r) \frac{2\sin kr}{kr} r^2 \,\mathrm{d}r = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^{\infty} f(r) r \sin kr \,\mathrm{d}r.
\end{aligned}$$

Example 2.4.1. Find the Fourier transform of

$$f(r) = \frac{z^3}{\pi} \mathrm{e}^{-2zr},$$

where z is a constant.

Solution 2.4.1.

$$\mathcal{F}\{f(\mathbf{r})\} = \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^{\infty} \frac{z^3}{\pi} \mathrm{e}^{-2zr} r \sin kr \,\mathrm{d}r.$$

One way to evaluate this integral is to recall the Laplace transform of $\sin kr$

$$\int_0^{\infty} \mathrm{e}^{-sr} \sin kr \,\mathrm{d}r = \frac{k}{s^2 + k^2},$$

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^{\infty} \mathrm{e}^{-sr} \sin kr \,\mathrm{d}r = \int_0^{\infty} (-r) \,\mathrm{e}^{-sr} \sin kr \,\mathrm{d}r,$$

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{k}{s^2 + k^2} = \frac{-2sk}{(s^2 + k^2)^2}.$$

So

$$\int_0^{\infty} r\mathrm{e}^{-sr} \sin kr \,\mathrm{d}r = \frac{2sk}{(s^2 + k^2)^2}.$$

With $s = 2z$, we have

$$\int_0^{\infty} \mathrm{e}^{-2zr} r \sin kr \,\mathrm{d}r = \frac{4zk}{(4z^2 + k^2)^2}.$$

It follows that:

$$\mathcal{F}\{f(\mathbf{r})\} = \sqrt{\frac{2}{\pi}} \frac{1}{k} \frac{z^3}{\pi} \frac{4zk}{(4z^2 + k^2)^2} = \left(\frac{2}{\pi} \right)^{3/2} \frac{2z^4}{(4z^2 + k^2)^2}.$$

2.5 Some Important Transform Pairs

There are some prototype Fourier transform pairs that we should be familiar with. Not only they frequently occur in engineering and physics, they also form the base upon which transforms of other functions can derived.

2.5.1 Rectangular Pulse Function

The rectangular function is defined as

$$\Pi_a(t) = \begin{cases} 1 & -a \le t \le a, \\ 0 & \text{otherwise.} \end{cases}$$

This function is sometimes called box function or top-hat function. It can be expressed as

$$\Pi_a(t) = u(t+a) - u(t-a),$$

where $u(t)$ is the Heaviside step function,

$$u(t) = \begin{cases} 1 & t > 0, \\ 0 & t < 0. \end{cases}$$

The Fourier transform of this function is given by

$$\begin{aligned} \mathcal{F}\{\Pi_a'(t)\} &= \int_{-\infty}^{\infty} \Pi_a(t) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t = \int_{-a}^{a} \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t \\ &= \left. \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{-\mathrm{i}\omega} \right|_{-a}^{a} = \frac{\mathrm{e}^{-\mathrm{i}\omega a} - \mathrm{e}^{\mathrm{i}\omega a}}{-\mathrm{i}\omega} = \frac{2 \sin \omega a}{\omega} = \widehat{f}(\omega). \end{aligned}$$

In terms of "sinc function," defined as $\sin c\,(x) = \dfrac{\sin x}{x}$, we have

$$\mathcal{F}\{\Pi_a(t)\} = 2a \sin c\,(a\omega)\,.$$

This Fourier transform pair is shown in Fig. 2.4.

2.5.2 Gaussian Function

The Gaussian function is defined as

$$f(t) = \mathrm{e}^{-\alpha t^2}.$$

Its Fourier transform is given by

$$\mathcal{F}\left\{\mathrm{e}^{-\alpha t^2}\right\} = \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha t^2} \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t = \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha t^2 - \mathrm{i}\omega t}\, \mathrm{d}t = \widehat{f}(\omega).$$

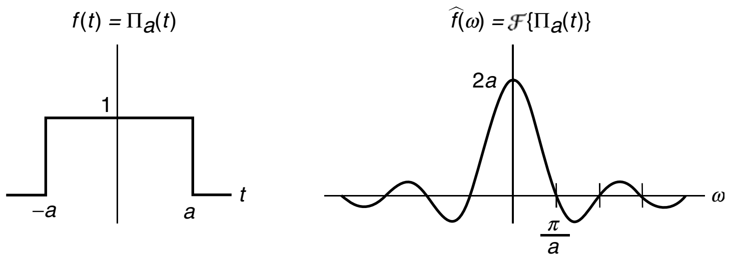

Fig. 2.4. Fourier transform pair of a rectangular function. Note that $\widehat{f}(0) = 2a$, and the zeros of $\widehat{f}(\omega)$ are at $\omega = \pi/a,\ 2\pi/a,\ 3\pi/a, \cdots$

Completing the square of the exponential

$$\alpha t^2 + \mathrm{i}\omega t = \left(\sqrt{\alpha}t + \frac{\mathrm{i}\omega}{2\sqrt{\alpha}}\right)^2 + \frac{\omega^2}{4\alpha},$$

we have

$$\int_{-\infty}^{\infty} \exp\left\{-\left[\left(\sqrt{\alpha}t + \frac{\mathrm{i}\omega}{2\sqrt{\alpha}}\right)^2 + \frac{\omega^2}{4\alpha}\right]\right\} \mathrm{d}t$$
$$= \exp\left(-\frac{\omega^2}{4\alpha}\right) \int_{-\infty}^{\infty} \exp\left\{-\left(\sqrt{\alpha}t + \frac{\mathrm{i}\omega}{2\sqrt{\alpha}}\right)^2\right\} \mathrm{d}t.$$

Let

$$u = \sqrt{\alpha}t + \frac{\mathrm{i}\omega}{2\sqrt{\alpha}}, \qquad \mathrm{d}u = \sqrt{\alpha}\,\mathrm{d}t$$

then we can write the Fourier transform as

$$\widehat{f}(\omega) = \exp\left(-\frac{\omega^2}{4\alpha}\right) \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\,\mathrm{d}u.$$

Since

$$\int_{-\infty}^{\infty} \mathrm{e}^{-u^2}\,\mathrm{d}u = \sqrt{\pi},$$

thus

$$\widehat{f}(\omega) = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\omega^2}{4\alpha}\right).$$

It is interesting to note that $\widehat{f}(\omega)$ is also of a Gaussian function with a peak at the origin, monotonically decreasing as $k \to \pm\infty$. If $f(t)$ is sharply peaked (large α), then $\widehat{f}(\omega)$ is flattened, and vice versa. This is a general feature in the theory of Fourier transforms. In quantum-mechanical applications it is related to the Heisenberg uncertainty principle. The pair of Gaussian transforms is shown in Fig. 2.5.

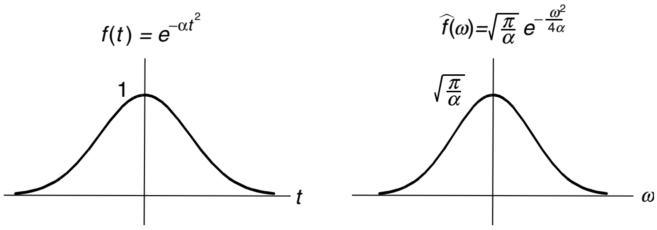

Fig. 2.5. The Fourier transform of a Gaussian function is another Gaussian function

2.5.3 Exponentially Decaying Function

The Fourier transform of the exponentially decaying function

$$f(t) = \mathrm{e}^{-a|t|}, \quad a > 0$$

is given by

$$\begin{aligned}
\mathcal{F}\left\{\mathrm{e}^{-a|t|}\right\} &= \int_{-\infty}^{\infty} \mathrm{e}^{-a|t|}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{0} \mathrm{e}^{at}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t + \int_{0}^{\infty} \mathrm{e}^{-at}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\
&= \left.\frac{\mathrm{e}^{(a-\mathrm{i}\omega)t}}{a-\mathrm{i}\omega}\right|_{-\infty}^{0} + \left.\frac{\mathrm{e}^{-(a+\mathrm{i}\omega)t}}{-(a+\mathrm{i}\omega)}\right|_{0}^{\infty} \\
&= \frac{1}{a-\mathrm{i}\omega} + \frac{1}{a+\mathrm{i}\omega} = \frac{2a}{a^2+\omega^2} = \widehat{f}(\omega).
\end{aligned}$$

This is a bell-shaped curve, similar in appearance to a Gaussian curve and is known as a Lorentz profile. This pair of transforms is shown in Fig. 2.6.

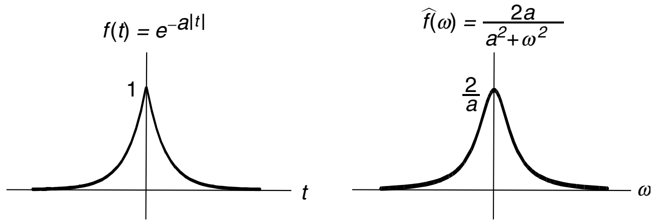

Fig. 2.6. The Fourier transform of an exponentail decaying function is Lorentz profile

2.6 Properties of Fourier Transform

2.6.1 Symmetry Property

The symmetry property of Fourier transform is of some importance.

$$\text{If} \quad \mathcal{F}\{f(t)\} = \widehat{f}(\omega), \quad \text{then} \quad \mathcal{F}\left\{\widehat{f}(t)\right\} = 2\pi f(-\omega).$$

Proof. Since

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t,$$

by definition

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega.$$

Interchanging t and ω, we have

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t.$$

Clearly,

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

Therefore

$$\mathcal{F}\left\{\widehat{f}(t)\right\} = \int_{-\infty}^{\infty} \widehat{f}(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = 2\pi f(-\omega).$$

Using this simple relation, we can avoid many complicated mathematical manipulations.

Example 2.6.1. Find

$$\mathcal{F}\left\{\frac{1}{a^2+t^2}\right\}$$

from

$$\mathcal{F}\left\{\mathrm{e}^{-a|t|}\right\} = \frac{2a}{a^2+\omega^2}.$$

Solution 2.6.1. Let

$$f(t) = \mathrm{e}^{-a|t|}, \quad \text{so} \quad f(-\omega) = \mathrm{e}^{-a|\omega|}$$

and

$$\mathcal{F}\{f(t)\} = \widehat{f}(\omega) = \frac{2a}{a^2+\omega^2}.$$

Thus

$$\widehat{f}(t) = \frac{2a}{a^2+t^2},$$

$$\mathcal{F}\left\{\widehat{f}(t)\right\} = \mathcal{F}\left\{\frac{2a}{a^2+t^2}\right\} = 2\pi f(-\omega).$$

Therefore

$$\mathcal{F}\left\{\frac{1}{a^2+t^2}\right\} = \frac{\pi}{a}\,\mathrm{e}^{-a|\omega|}.$$

This result can also be found by complex contour integration.

2.6.2 Linearity, Shifting, Scaling

Linearity of the Transform and its Inverse. If $\mathcal{F}\{f(t)\} = \widehat{f}(\omega)$ and $\mathcal{F}\{g(t)\} = \widehat{g}(\omega)$, then

$$\begin{aligned}\mathcal{F}\{af(t)+bg(t)\} &= \int_{-\infty}^{\infty}[af(t)+bg(t)]\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\ &= a\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t + b\int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\ &= a\mathcal{F}\{f(t)\} + b\mathcal{F}\{f(t)\} = a\widehat{f}(\omega) + b\widehat{g}(\omega).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{F}^{-1}\left\{a\widehat{f}(\omega)+b\widehat{g}(\omega)\right\} &= a\mathcal{F}^{-1}\left\{\widehat{f}(\omega)\right\} + b\mathcal{F}^{-1}\left\{\widehat{g}(\omega)\right\} \\ &= af(t)+bg(t).\end{aligned}$$

These simple relations are of considerable importance because it reflects the applicability of the Fourier transform to the analysis of linear systems.
Time Shifting. If time is shifted by a in the Fourier transform

$$\mathcal{F}\{f(t-a)\} = \int_{-\infty}^{\infty} f(t-a)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t,$$

then by substituting $t-a=x,\ \ \mathrm{d}t=\mathrm{d}x,\quad t=x+a$, we have

$$\begin{aligned}\mathcal{F}\{f(t-a)\} &= \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega(x+a)}\mathrm{d}x \\ &= \mathrm{e}^{-\mathrm{i}\omega a}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\omega a}\widehat{f}(\omega).\end{aligned}$$

Note that a time delay will only change the phase of the Fourier transform and not its magnitude. For example,

$$\sin\omega_0 t = \cos\left(\omega_0 t - \frac{\pi}{2}\right) = \cos\omega_0\left(t - \frac{\pi}{2}\frac{1}{\omega_0}\right).$$

Thus, if $f(t) = \cos \omega_0 t$, then $\sin \omega_0 t = f(t-a)$ with $a = \dfrac{\pi}{2}\dfrac{1}{\omega_0}$. Therefore

$$\begin{aligned}
\mathcal{F}\{A \sin \omega_0 t\} &= \mathrm{e}^{-\mathrm{i}\omega \frac{\pi}{2}\frac{1}{\omega_0}} \mathcal{F}\{A \cos \omega_0 t\} \\
&= \mathrm{e}^{-\mathrm{i}\omega \frac{\pi}{2}\frac{1}{\omega_0}} \left[A\pi\delta(\omega-\omega_0) + A\pi\delta(\omega+\omega_0)\right] \\
&= \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}} A\pi\delta(\omega-\omega_0) + \mathrm{e}^{\mathrm{i}\frac{\pi}{2}} A\pi\delta(\omega+\omega_0) \\
&= -\mathrm{i}A\pi\delta(\omega-\omega_0) + \mathrm{i}A\pi\delta(\omega+\omega_0),
\end{aligned}$$

as shown in (2.16).

Frequency Shifting. If the frequency in $\widehat{f}(\omega)$ is shifted by a constant a, its inverse is multiplied by a factor of $\mathrm{e}^{\mathrm{i}at}$. Since

$$\mathcal{F}^{-1}\left\{\widehat{f}(\omega-a)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega-a)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega,$$

substituting $\varpi = \omega - a$, we have

$$\mathcal{F}^{-1}\left\{\widehat{f}(\omega-a)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\varpi)\,\mathrm{e}^{\mathrm{i}(\varpi+a)t}\,\mathrm{d}\varpi = \mathrm{e}^{\mathrm{i}at} f(t)$$

or

$$\widehat{f}(\omega-a) = \mathcal{F}\left\{\mathrm{e}^{\mathrm{i}at} f(t)\right\}.$$

To illustrate the effect of frequency shifting, let us consider the case that $f(t)$ is multiplied by $\cos \omega_0 t$. Since $\cos \omega_0 t = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\omega_0 t} + \mathrm{e}^{-\mathrm{i}\omega_0 t}\right)$, so

$$f(t)\cos \omega_0 t = \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega_0 t} f(t) + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega_0 t} f(t)$$

and

$$\begin{aligned}
\mathcal{F}\{f(t)\cos \omega_0 t\} &= \frac{1}{2}\mathcal{F}\left\{\mathrm{e}^{\mathrm{i}\omega_0 t} f(t)\right\} + \frac{1}{2}\mathcal{F}\left\{\mathrm{e}^{-\mathrm{i}\omega_0 t} f(t)\right\} \\
&= \frac{1}{2}\widehat{f}(\omega-\omega_0) + \frac{1}{2}\widehat{f}(\omega+\omega_0).
\end{aligned}$$

This process is known as modulation. In other words, when $f(t)$ is modulated by $\cos \omega_0 t$, its frequency is symmetrically shifted up and down by ω_0.

Time Scaling. If $\mathcal{F}\{f(t)\} = \widehat{f}(\omega)$, then the Fourier transform of $f(at)$ can be determined by substituting $t' = at$ in the Fourier integral

$$\begin{aligned}
\mathcal{F}\{f(at)\} &= \int_{-\infty}^{\infty} f(at)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} f(t')\,\mathrm{e}^{-\mathrm{i}\omega t'/a}\frac{1}{a}\,\mathrm{d}t' = \frac{1}{a}\widehat{f}\left(\frac{\omega}{a}\right).
\end{aligned}$$

This is correct for $a > 0$. However, if a is negative, then $t' = at = -|a|\,t$. As a consequence, when the integration variable is changed from t to t', the integration limits should also be interchanged. That is,

$$\begin{aligned}\mathcal{F}\{f(at)\} &= \int_{-\infty}^{\infty} f(at)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \int_{\infty}^{-\infty} f(t')\,\mathrm{e}^{-\mathrm{i}\omega t'/a}\frac{1}{-|a|}\,\mathrm{d}t' \\ &= \frac{1}{|a|}\int_{-\infty}^{\infty} f(t')\,\mathrm{e}^{-\mathrm{i}\omega t'/a}\,\mathrm{d}t' = \frac{1}{|a|}\widehat{f}\left(\frac{\omega}{a}\right).\end{aligned}$$

Therefore, in general

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|}\widehat{f}\left(\frac{\omega}{a}\right).$$

This means that as the time scale expands, the frequency scale not only contracts, its amplitude will also increase. It increases in such a way as to keep the area constant.

Frequency Scaling. This is just the reverse of time scaling. If $\mathcal{F}^{-1}\left\{\widehat{f}(\omega)\right\} = f(t)$, then

$$\begin{aligned}\mathcal{F}^{-1}\left\{\widehat{f}(a\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(a\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega')\,\mathrm{e}^{\mathrm{i}\omega' t/a}\frac{1}{|a|}\,\mathrm{d}\omega' = \frac{1}{|a|}f\left(\frac{t}{a}\right).\end{aligned}$$

This means that as the frequency scale expands, the time scale will contract and the amplitude of the time function will also increase.

2.6.3 Transform of Derivatives

If the transform of nth derivative $f^n(t)$ exists, then $f^n(t)$ must be integrable over $(-\infty,\infty)$. That means $f^n(t) \to 0$, as $t \to \pm\infty$. With this assumption, the Fourier transforms of derivatives of $f(t)$ can be expressed in terms of the transform of $f(t)$. This can be shown as follows:

$$\begin{aligned}\mathcal{F}\{f'(t)\} &= \int_{-\infty}^{\infty} f'(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \int_{-\infty}^{\infty}\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \\ &= f(t)\mathrm{e}^{-\mathrm{i}\omega t}\Big|_{-\infty}^{\infty} + \mathrm{i}\omega\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.\end{aligned}$$

The integrated term is equal to zero at both limits. Thus

$$\mathcal{F}\{f'(t)\} = \mathrm{i}\omega\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{i}\omega\mathcal{F}\{f(t)\} = \mathrm{i}\omega\widehat{f}(\omega).$$

It follows that:

$$\mathcal{F}\left\{f''(t)\right\} = \mathrm{i}\omega\mathcal{F}\left\{f'(t)\right\} = (\mathrm{i}\omega)^2\,\mathcal{F}\left\{f(t)\right\} = (\mathrm{i}\omega)^2\,\widehat{f}(\omega).$$

Therefore

$$\mathcal{F}\left\{f^n(t)\right\} = (\mathrm{i}\omega)^n\,\mathcal{F}\left\{f(t)\right\} = (\mathrm{i}\omega)^n\,\widehat{f}(\omega).$$

Thus a differentiation in the time domain becomes a simple multiplication in the frequency domain.

2.6.4 Transform of Integral

The Fourier transform of the following integral:

$$I(t) = \int_{-\infty}^{t} f(x)\,\mathrm{d}x$$

can be found by using the relation for Fourier transform of derivatives. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = f(t),$$

it follows that:

$$\mathcal{F}\left\{f(t)\right\} = \mathcal{F}\left\{\frac{\mathrm{d}I(t)}{\mathrm{d}t}\right\} = \mathrm{i}\omega\mathcal{F}\left\{I\left(t\right)\right\} = \mathrm{i}\omega\mathcal{F}\left\{\int_{-\infty}^{t} f\left(x\right)\,\mathrm{d}x\right\}.$$

Therefore

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(x)\mathrm{d}x\right\} = \frac{1}{\mathrm{i}\omega}\mathcal{F}\left\{f(t)\right\}.$$

Thus an integration in the time domain becomes a division in the frequency domain.

2.6.5 Parseval's Theorem

The Parseval's theorem in Fourier series is equally valid in Fourier transform. The integral of the square of a function is related to the integral of the square of its transform in the following way:

$$\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widehat{f}\left(\omega\right)\right|^2\,\mathrm{d}\omega.$$

Since

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}\left(\omega\right)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega,$$

its complex conjugate is

$$f^*(t) = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\right]^* = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}^*(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega.$$

Thus

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} f(t) f^*(t) \mathrm{d}t = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}^*(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega \right] \mathrm{d}t.$$

Interchanging the ω and t integration,

$$\begin{aligned} \int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}^* (\omega) \left[\int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t \right] \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}^* (\omega) \, \widehat{f}(\omega) \mathrm{d}\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \widehat{f}(\omega) \right|^2 \mathrm{d}\omega. \end{aligned}$$

Written in terms of frequency ν, instead of angular frequency ω $(\omega = 2\pi\nu)$, this theorem is expressed as

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} \left| \widehat{f}(\nu) \right|^2 \mathrm{d}\nu.$$

In physics, the total energy associated with wave form $f(t)$ (electromagnetic radiation, water waves, etc.) is proportional to $\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t$. By Parseval's theorem, this energy is also given by $\int_{-\infty}^{\infty} \left| \widehat{f}(\nu) \right|^2 \mathrm{d}\nu$. Therefore $\left| \widehat{f}(\nu) \right|^2$ is the energy content per unit frequency interval, and is known as "power density." For this reason, Parseval's theorem is also known as power theorem.

Example 2.6.2. Find the value of

$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \mathrm{d}x$$

from the Parseval's theorem and the Fourier transform of

$$\Pi_1(t) = \begin{cases} 1 & |t| < 1, \\ 0 & |t| > 1. \end{cases}$$

Solution 2.6.2. Let $f(t) = \Pi_1(t)$, so

$$\begin{aligned} \mathcal{F}\{f(t)\} = \widehat{f}(\omega) &= \int_{-\infty}^{\infty} \Pi_1(t) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t = \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t \\ &= -\frac{1}{\mathrm{i}\omega} \mathrm{e}^{-\mathrm{i}\omega t} \bigg|_{-1}^{1} = \frac{1}{\mathrm{i}\omega} \left(\mathrm{e}^{\mathrm{i}\omega} - \mathrm{e}^{-\mathrm{i}\omega} \right) = \frac{2 \sin \omega}{\omega} \end{aligned}$$

and

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \int_{-1}^{1} \mathrm{d}t = 2.$$

On the other hand

$$\int_{-\infty}^{\infty} \left| \widehat{f}(\omega) \right|^2 \mathrm{d}\omega = \int_{-\infty}^{\infty} \left| \frac{2 \sin \omega}{\omega} \right|^2 \mathrm{d}\omega = 4 \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} \mathrm{d}\omega.$$

Therefore from Parseval's theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \widehat{f}(\omega) \right|^2 \mathrm{d}\omega,$$

we have

$$2 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} \mathrm{d}\omega.$$

It follows that:

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} \mathrm{d}\omega = \pi.$$

Since $\frac{\sin^2 \omega}{\omega^2}$ is an even function, so

$$\int_{0}^{\infty} \frac{\sin^2 x}{x^2} \mathrm{d}x = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \mathrm{d}x = \frac{\pi}{2}.$$

2.7 Convolution

2.7.1 Mathematical Operation of Convolution

Convolution is an important and useful concept. The convolution $c(t)$ of two functions $f(t)$ and $g(t)$ is usually written as $f(t) * g(t)$ and is defined as

$$c(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau) \mathrm{d}\tau = f(t) * g(t).$$

The mathematical operation of convolution consists of the following steps:

1. Take the mirror image of $g(\tau)$ about the coordinate axis to create $g(-\tau)$ from $g(\tau)$.
2. Shift $g(-\tau)$ by an amount t to get $g(t-\tau)$. If t is positive, the shift is to the right, if it is negative, to the left.
3. Multiply the shifted function $g(t-\tau)$ by $f(\tau)$.
4. The area under the product of $f(\tau)$ and $g(t-\tau)$ is the value of convolution at t.

Let us illustrate these steps with a simple example shown in Fig. 2.7. Suppose that $f(\tau)$ is given in (a) and $g(\tau)$ in (b). The mirror image of $g(\tau)$ is $g(-\tau)$, which is shown in (c). In (d), $g(t-\tau)$ is shown as $g(-\tau)$ shifted by an amount t.

It is clear, if $t < 0$, there is no overlap between $f(\tau)$ and $g(t-\tau)$. That means that at any value of τ, either $f(\tau)$ or $g(t-\tau)$, or both are zero. Since $f(\tau)g(t-\tau) = 0$ for $t < 0$, therefore

$$c(t) = 0, \quad \text{if} \quad t < 0.$$

Between $t = 0$ and $t = 1$, the convolution integral is simply equal to abt,

$$c(t) = abt, \quad 0 < t < 1.$$

There is full overlap at $t = 1$, so

$$c(t) = ab \quad at \quad t = 1.$$

Between $t = 1$ and $t = 2$, the overlap is steadily decreasing. The convolution integral is equal to

$$c(t) = ab[1-(t-1)] = ab(2-t), \quad \text{if} \quad 1 < t < 2.$$

For $t > 2$, there will be no overlap and the convolution integral is equal to zero. Thus the convolution of $f(t)$ and $g(t)$ is given by the triangle shown in (e) .

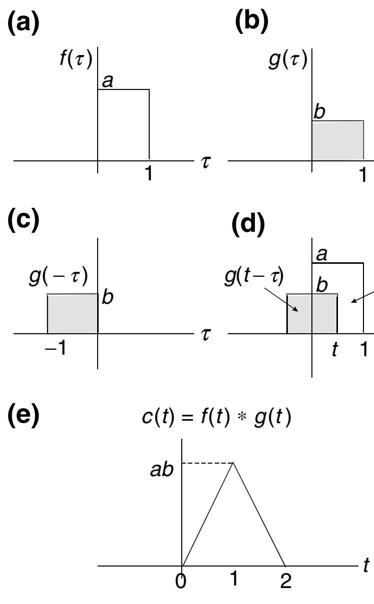

Fig. 2.7. Convolution. The convolution of $f(t)$ shown in (**a**) and $g(t)$ shown in (**b**) is given in (**e**).

2.7.2 Convolution Theorems

Time Convolution Theorem. The time convolution theorem

$$\mathcal{F}\{f(t) * g(t)\} = \widehat{f}(\omega)\,\widehat{g}\,(\omega)$$

can be proved as follows.

By definition

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f\,(\tau)\,g\,(t-\tau)\,\mathrm{d}\tau\right]\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

Interchanging the τ and t integration, we have

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty} f\,(\tau)\left[\int_{-\infty}^{\infty} g\,(t-\tau)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\right]\mathrm{d}\tau$$

Let $t-\tau = x,\ t = x+\tau$, $\mathrm{d}t = \mathrm{d}x$, then

$$\begin{aligned}\int_{-\infty}^{\infty} g\,(t-\tau)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t &= \int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{i}\omega(x+\tau)}\mathrm{d}x\\ &= \mathrm{e}^{-\mathrm{i}\omega\tau}\int_{-\infty}^{\infty} g(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\omega\tau}\widehat{g}(\omega).\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{F}\{f(t) * g(t)\} &= \int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\widehat{g}(\omega)\mathrm{d}\tau = \widehat{g}(\omega)\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\tau\\ &= \widehat{g}(\omega)\widehat{f}(\omega).\end{aligned}$$

Frequency Convolution Theorem. The frequency convolution theorem can be written as

$$\mathcal{F}^{-1}\left\{\widehat{f}(\omega) * \widehat{g}(\omega)\right\} = 2\pi f(t)g(t).$$

The proof of this theorem is also straightforward. By definition

$$\begin{aligned}\mathcal{F}^{-1}\left\{\widehat{f}(\omega) * \widehat{g}(\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\widehat{f}\,(\varpi)\,\widehat{g}\,(\omega-\varpi)\,\mathrm{d}\varpi\right]\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}\,(\varpi)\left[\int_{-\infty}^{\infty}\widehat{g}\,(\omega-\varpi)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega\right]\mathrm{d}\varpi.\end{aligned}$$

Let $\omega - \varpi = \varOmega,\quad \omega = \varOmega + \varpi,\quad \mathrm{d}\omega = \mathrm{d}\varOmega$, thus

$$\begin{aligned}\mathcal{F}^{-1}\left\{\widehat{f}(\omega) * \widehat{g}(\omega)\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}\,(\varpi)\,\mathrm{e}^{\mathrm{i}\varpi t}\mathrm{d}\varpi\int_{-\infty}^{\infty}\widehat{g}(\varOmega)\mathrm{e}^{\mathrm{i}\varOmega t}\,\mathrm{d}\varOmega\\ &= 2\pi f(t)g(t).\end{aligned}$$

Clearly this theorem can also be written as

$$\mathcal{F}\left\{f(t)g(t)\right\} = \frac{1}{2\pi}\widehat{f}(\omega) * \widehat{g}(\omega).$$

Example 2.7.1. (a) Use

$$\mathcal{F}\left\{\cos\omega_0 t\right\} = \pi\delta\left(\omega+\omega_0\right) + \pi\delta\left(\omega-\omega_0\right),$$
$$\mathcal{F}\left\{\Pi_a(t)\right\} = \frac{2\sin a\omega}{\omega},$$

and the convolution theorem to find the Fourier transform of the finite wave train $f(t)$

$$f(t) = \begin{cases} \cos\omega_0 t & |t| < a, \\ 0 & |t| > a. \end{cases}$$

(b) Use direct integration to verify the result.

Solution 2.7.1. (a) Since

$$\Pi_a(t) = \begin{cases} 1 & |t| < a, \\ 0 & |t| > a, \end{cases}$$

so we can write $f(t)$ as

$$f(t) = \cos\omega_0 t \cdot \Pi_a(t).$$

According to the convolution theorem

$$\begin{aligned}
\mathcal{F}\left\{f(t)\right\} &= \frac{1}{2\pi}\mathcal{F}\left\{\cos\omega_0 t\right\} * \mathcal{F}\left\{\Pi_a\left(t\right)\right\} \\
&= \frac{1}{2\pi}\left[\pi\delta(\omega+\omega_0) + \pi\delta\left(\omega-\omega_0\right)\right] * \frac{2\sin a\omega}{\omega} \\
&= \int_{-\infty}^{\infty}\left[\delta\left(\omega'+\omega_0\right) + \delta\left(\omega'-\omega_0\right)\right]\frac{\sin a\left(\omega-\omega'\right)}{\left(\omega-\omega'\right)}\mathrm{d}\omega' \\
&= \frac{\sin a(\omega+\omega_0)}{\omega+\omega_0} + \frac{\sin a\left(\omega-\omega_0\right)}{\omega-\omega_0}.
\end{aligned}$$

(b) By definition

$$\mathcal{F}\left\{f(t)\right\} = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \int_{-a}^{a}\cos\omega_0 t\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

Since

$$\cos\omega_0 t = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\omega_0 t} + \mathrm{e}^{-\mathrm{i}\omega_0 t}\right),$$

so

$$\mathcal{F}\{f(t)\} = \frac{1}{2}\int_{-a}^{a}(\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t} + \mathrm{e}^{-\mathrm{i}(\omega_0+\omega)t})\mathrm{d}t$$

$$= \frac{1}{2}\left[\left.\frac{\mathrm{e}^{\mathrm{i}(\omega_0-\omega)t}}{\mathrm{i}(\omega_0-\omega)}\right|_{-a}^{a} - \left.\frac{\mathrm{e}^{-\mathrm{i}(\omega_0+\omega)t}}{\mathrm{i}(\omega_0+\omega)}\right|_{-a}^{a}\right]$$

$$= \frac{\sin a\,(\omega-\omega_0)}{\omega-\omega_0} + \frac{\sin a(\omega+\omega_0)}{\omega+\omega_0}.$$

This pair of Fourier transform is shown in Fig. 2.8.

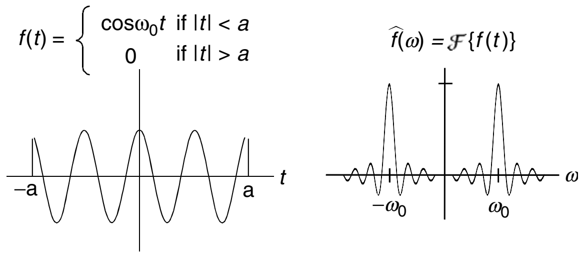

Fig. 2.8. The Fourier transform pair of a finite cosine wave

Example 2.7.2. Find the Fourier transform of the triangle function

$$f(t) = \begin{cases} t+2a & -2a < t < 0 \\ -t+2a & 0 < t < 2a \\ 0 & \text{otherwise} \end{cases}.$$

Solution 2.7.2. Following the procedure shown in Fig. 2.7, one can easily show that the triangle function is the convolution of two identical rectangle pulse function

$$f(t) = \Pi_a(t) * \Pi_a(t).$$

According to the time convolution theorem

$$\mathcal{F}\{f(t)\} = \mathcal{F}\{\Pi_a(t) * \Pi_a(t)\} = \mathcal{F}\{\Pi_a(t)\}\,\mathcal{F}\{\Pi_a\,(t)\}\,.$$

Since

$$\mathcal{F}\{\Pi_a(t)\} = \frac{2\sin a\omega}{\omega},$$

therefore

$$\mathcal{F}\{f(t)\} = \frac{2\sin a\omega}{\omega}\cdot\frac{2\sin a\omega}{\omega} = \frac{4\sin^2 a\omega}{\omega^2}.$$

This pair of transforms is shown in Fig. 2.9.

We can obtain the same result by calculating the transform directly, but that would be much more tedious.

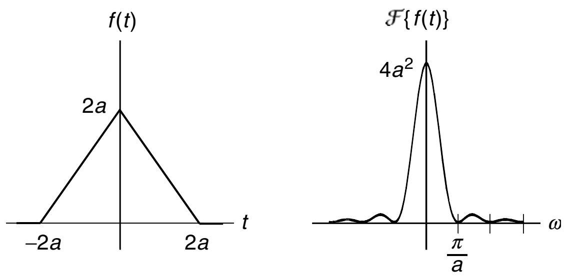

Fig. 2.9. Fourier transform of a triangular function

2.8 Fourier Transform and Differential Equations

A characteristic property of Fourier transform, like other integral transforms, is that it can be used to reduce the number of independent variables in a differential equation by one. For example, if we apply the transform to an ordinary differential equation (which has only one independent variable), then we just get an algebraic equation for the transformed function. A one-dimensional wave equation is a partial differential equation with two independent variables. It can be transformed into an ordinary differential equation in the transformed function. Usually it is easier to solve the resultant equation for the transformed function than it is to solve the original equation, since the equation for the transformed function has one less independent variable. After the transformed function is determined, we can get the solution of the original equation by an inverse transform. We will illustrate this method with the following two examples.

Example 2.8.1. Solve the following differential equation:

$$y''(t) - a^2 y(t) = f(t)$$

where a is a constant and $f(t)$ is given function. The only imposed conditions are that all functions must vanish as $t \to \pm\infty$. This ensures that their Fourier transforms exist.

Solution 2.8.1. Apply the Fourier transform to the equation, and let

$$\widehat{y}(\omega) = \mathcal{F}\left\{y(t)\right\}, \quad \widehat{f}(\omega) = \mathcal{F}\left\{f(t)\right\}.$$

Since

$$\mathcal{F}\left\{y''(t)\right\} = (\mathrm{i}\omega)^2 \mathcal{F}\{y(t)\} = -\omega^2 \widehat{y}(\omega),$$

the differential equation becomes

$$-(\omega^2 + a^2)\,\widehat{y}\,(\omega) = \widehat{f}(\omega).$$

Thus

$$\widehat{y}(\omega) = -\frac{1}{(\omega^2 + a^2)}\widehat{f}(\omega)$$

Recall

$$\mathcal{F}\left\{\mathrm{e}^{-a|t|}\right\} = \frac{2a}{(\omega^2 + a^2)},$$

therefore

$$-\frac{1}{(\omega^2 + a^2)} = \mathcal{F}\left\{-\frac{1}{2a}\mathrm{e}^{-a|t|}\right\}.$$

In other words, if we define

$$\widehat{g}(\omega) = -\frac{1}{(\omega^2 + a^2)}, \quad \text{then} \quad g(t) = -\frac{1}{2a}\mathrm{e}^{-a|t|}.$$

According to the convolution theorem,

$$\widehat{g}(\omega)\widehat{f}(\omega) = \mathcal{F}\{g(t) * f(t)\}.$$

Since

$$\widehat{y}(\omega) = -\frac{1}{(\omega^2 + a^2)}\widehat{f}(\omega) = \widehat{g}(\omega)\widehat{f}(\omega) = \mathcal{F}\{g(t) * f(t)\},$$

it follows that:

$$y(t) = \mathcal{F}^{-1}\{\widehat{y}(\omega)\} = \mathcal{F}^{-1}\mathcal{F}\{g(t) * f(t)\} = g(t) * f(t).$$

Therefore

$$y(t) = -\frac{1}{2a}\int_{-\infty}^{\infty} \mathrm{e}^{-a|t-\tau|} f(\tau)\mathrm{d}\tau.$$

This is the particular solution of the equation. With a given $f(t)$, this equation can be evaluated.

Example 2.8.2. Use the Fourier transform to solve the one-dimensional classical wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2} \tag{2.17}$$

with an initial condition

$$y(x,0) = f(x), \tag{2.18}$$

where v^2 is a constant.

Solution 2.8.2. Let us Fourier analyze $y(x,t)$ with respect to x. First express $y(x,t)$ in terms of the Fourier integral

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(k,t)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \tag{2.19}$$

so the Fourier transform is

$$\widehat{y}(k,t) = \int_{-\infty}^{\infty} y(x,t)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{2.20}$$

It follows form (2.19) and (2.18) that:

$$y(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{y}(k,0)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = f(x). \tag{2.21}$$

Since the Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \tag{2.22}$$

clearly

$$\widehat{y}(k,0) = \widehat{f}(k). \tag{2.23}$$

Taking the Fourier transform of the original equation, we have

$$\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial x^2}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{v^2}\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial t^2}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,$$

which can be written as

$$\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial x^2}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} y(x,t)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

The first term is just the Fourier transform of the second derivative of $y\,(x,t)$ with respect to x

$$\int_{-\infty}^{\infty} \frac{\partial^2 y(x,t)}{\partial x^2}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = (ik)^2\widehat{y}(k,t),$$

therefore the equation becomes

$$-k^2\widehat{y}(k,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\widehat{y}(k,t).$$

Clearly the general solution of this equation is

$$\widehat{y}(k,t) = c_1(k)\mathrm{e}^{\mathrm{i}kvt} + c_2(k)\mathrm{e}^{-\mathrm{i}kvt}.$$

where $c_1(k)$ and $c_2(k)$ are constants with respect to t. At $t = 0$, according to (2.23)

$$\widehat{y}(k,0) = c_1(k) + c_2(k) = \widehat{f}(k).$$

This equation can be satisfied by the following symmetrical and antisymmetrical forms:

$$c_1(k) = \frac{1}{2}\left[\widehat{f}(k) + \widehat{g}(k)\right],$$

$$c_2(k) = \frac{1}{2}\left[\widehat{f}(k) - \widehat{g}(k)\right],$$

where $\widehat{g}(k)$ is a yet undefined function. Thus

$$\widehat{y}(k,t) = \frac{1}{2}\widehat{f}(k)\left(\mathrm{e}^{\mathrm{i}kvt} + \mathrm{e}^{-\mathrm{i}kvt}\right) + \frac{1}{2}\widehat{g}(k)\left(\mathrm{e}^{\mathrm{i}kvt} - \mathrm{e}^{-\mathrm{i}kvt}\right).$$

Substituting it into (2.19), we have

$$\begin{aligned} y(x,t) \quad &= \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\widehat{f}(k)\left[\mathrm{e}^{\mathrm{i}k(x+vt)} + \mathrm{e}^{\mathrm{i}k(x-vt)}\right]\mathrm{d}k \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2}\widehat{g}(k)\left[\mathrm{e}^{\mathrm{i}k(x+vt)} - \mathrm{e}^{\mathrm{i}k(x-vt)}\right]\mathrm{d}k. \end{aligned}$$

Comparing the integral

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(k)\mathrm{e}^{\mathrm{i}k(x+vt)}\mathrm{d}k$$

with (2.22), we see that the integral is the same except the argument x is changed to $x + vt$. Therefore

$$I_1 = f(x + vt).$$

It follows that:

$$y(x,t) = \frac{1}{2}\left[f(x+vt) + f(x-vt)\right] + \frac{1}{2}\left[g(x+vt) - g(x-vt)\right]$$

where $g(x)$ is the Fourier inverse transform of $\widehat{g}(k)$. The function $g(x)$ is determined by additional initial, or boundary conditions.

In Chap. 5, we will have a more detailed discussion on this type of problems.

2.9 The Uncertainty of Waves

Fourier transform enables us to break a complicated, even nonperiodic wave down into simple waves. The way of doing it is to assume that the wave is a periodic function with an infinite period. Since it is not possible to observe the wave over an infinite amount of time, we have to do the analysis based on our observation over a finite period of time. Consequently we can never be 100% certain of the characteristics of a given wave.

For example, a constant function $f(t)$ has no oscillation, therefore the frequency is zero. Thus the Fourier transform $\widehat{f}$ is a delta function at $\omega = 0$, as shown in Fig. 2.2. However, this is true only if the function $f(t)$ is a constant from $-\infty$ to $+\infty$. But under no circumstances can we be sure of that. What we can say is that during certain time interval Δt, the function is a constant. This is represented by a rectangular pulse function shown in Fig. 2.4. Outside this time interval, we have no information, therefore the function is given a value of zero. The Fourier transform of this function is $2 \sin a\omega/\omega$. As we see in Fig. 2.4, now there is a spread of frequency around $\omega = 0$. In other words, there is an uncertainty of wave's frequency. We can tell how uncertain is the frequency by measuring the width $\Delta\omega$ of the central peak. In this example, $\Delta t = 2a,\ \Delta\omega = 2\pi/a$. It is interesting to note that $\Delta t\, \Delta\omega = 4\pi$, which is a constant. Since it is a constant, it can never be zero, no matter how large or small Δt may be. Therefore there is always some degree of uncertainty.

According to quantum mechanics, photons and electrons can also be thought of as waves. As waves, they are also subject to the uncertainty that applies to all waves. Therefore in the subatomic world, phenomena can only be described within a range of precision that allows for the uncertainty of waves. This is known as the Uncertainty Principle, first formulated by Werner Heisenberg.

In quantum mechanics, if $f(t)$ is normalized wave function, that is

$$\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = 1,$$

then the expectation value $\langle t^n \rangle$ is defined as

$$\langle t^n \rangle = \int_{-\infty}^{\infty} |f(t)|^2 \, t^n \, \mathrm{d}t.$$

The uncertainty Δt is given by the "root mean square" deviation, that is

$$\Delta t = \left\langle t^2 - \langle t \rangle^2 \right\rangle^{1/2}.$$

If $\widehat{f}(\omega)$ is the Fourier transform of $f(t)$, then according to Parseval's theorem

$$\int_{-\infty}^{\infty} \left| \widehat{f}(\omega) \right|^2 \mathrm{d}\omega = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = 2\pi.$$

Therefore the expectation value of $\langle \omega^n \rangle$ is given by

$$\langle \omega^n \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \widehat{f}(\omega) \right|^2 \omega^n \mathrm{d}\omega.$$

The uncertainty $\Delta\omega$ is similarly defined as

$$\Delta\omega = \left\langle \omega^2 - \langle \omega \rangle^2 \right\rangle^{1/2}.$$

If $f(t)$ is given by a normalized Gaussian function

$$f(t) = \left(\frac{2a}{\pi} \right)^{1/4} \exp\left(-at^2\right),$$

then clearly $\langle t \rangle = 0$, since the integrand of $\int_{-\infty}^{\infty} |f(t)|^2 \, t \, \mathrm{d}t$ is an odd function, and $\Delta t = \left\langle t^2 \right\rangle^{1/2}$. By definition

$$\left\langle t^2 \right\rangle = \left(\frac{2a}{\pi} \right)^{1/2} \int_{-\infty}^{\infty} \exp\left(-2at^2\right) t^2 \, \mathrm{d}t.$$

With integration by parts, it can be easily shown that

$$\int_{-\infty}^{\infty} \exp(-2at^2) t^2 \, \mathrm{d}t = -\frac{1}{4a} t \exp(-2at^2) \Big|_{-\infty}^{\infty} + \frac{1}{4a} \int_{-\infty}^{\infty} \exp(-2at^2) \mathrm{d}t$$

$$= \frac{1}{4a} \left(\frac{1}{2a} \right)^{1/2} \int_{-\infty}^{\infty} \exp(-u^2) \mathrm{d}u = \frac{1}{4a} \left(\frac{\pi}{2a} \right)^{1/2}.$$

Thus

$$\Delta t = \left\langle t^2 \right\rangle^{1/2} = \left[\left(\frac{2a}{\pi} \right)^{1/2} \frac{1}{4a} \left(\frac{\pi}{2a} \right)^{1/2} \right]^{1/2} = \left(\frac{1}{4a} \right)^{1/2}.$$

Now

$$\widehat{f}(\omega) = \mathcal{F}\{f(t)\} = \left(\frac{2a}{\pi} \right)^{1/4} \left(\frac{\pi}{a} \right)^{1/2} \exp\left(-\frac{\omega^2}{4a} \right).$$

So $\langle \omega \rangle = 0$, and

$$\left\langle \omega^2 \right\rangle = \frac{1}{2\pi} \int \left| \widehat{f}(\omega) \right|^2 \omega^2 \, \mathrm{d}\omega = \frac{1}{2\pi} \left(\frac{2a}{\pi} \right)^{1/2} \left(\frac{\pi}{a} \right) \int_{-\infty}^{\infty} \exp\left(-\frac{\omega^2}{2a} \right) \omega^2 \, \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \left(\frac{2a}{\pi} \right)^{1/2} \left(\frac{\pi}{a} \right) a (2a\pi)^{1/2} = a$$

Thus

$$\Delta\omega = \left\langle \omega^2 \right\rangle^{1/2} = (a)^{1/2}.$$

Therefore

$$\Delta t \cdot \Delta\omega = \left(\frac{1}{4a}\right)^{1/2} (a)^{1/2} = \frac{1}{2}.$$

As we have discussed, if we change the name of the variable t (representing time) to x (representing distance), the angular frequency ω is changed to the wave number k. This relation is then written as

$$\Delta x \cdot \Delta k = \frac{1}{2}.$$

The two most fundamental relations in quantum mechanics are

$$E = \hbar\omega \quad \text{and} \quad p = \hbar k,$$

where E is the energy, p the momentum, and $\hbar$ is the Planck constant, $h/2\pi$. It follows that the uncertainty in energy is $\Delta E = \hbar\,\Delta\omega$, and the uncertainty in momentum is $\Delta p = \hbar\,\Delta k$. Therefore, with a Gaussian wave, we have

$$\Delta t \cdot \Delta E = \frac{\hbar}{2}, \qquad \Delta x \cdot \Delta p = \frac{\hbar}{2}.$$

Since no other form of wave function can reduce the product of uncertainties below this value, these relations are usually presented as

$$\Delta t \cdot \Delta E \geq \frac{\hbar}{2}, \qquad \Delta x \cdot \Delta p \geq \frac{\hbar}{2},$$

which are the formal statements of uncertainty principle in quantum mechanics.

Exercises

1. Use an odd function to show that

$$\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega t\,\mathrm{d}\omega = \begin{cases} \dfrac{\pi}{2} & 0 < t < \pi \\ 0 & t > \pi \end{cases}.$$

2. Use an even function to show that

$$\int_0^\infty \frac{\cos\omega t}{1+\omega^2}\mathrm{d}\omega = \frac{\pi}{2}\mathrm{e}^{-t}.$$

3. Show that

$$\int_0^\infty \frac{\cos\omega t + \omega\sin\omega t}{1+\omega^2}\mathrm{d}\omega = \begin{cases} 0 & t < 0, \\ \dfrac{\pi}{2} & t = 0, \\ \pi\mathrm{e}^{-t} & t > 0. \end{cases}$$

4. Show that

$$\int_0^{\infty} \frac{\sin \pi\omega \sin \omega t}{1-\omega^2} \mathrm{d}\omega = \begin{cases} \dfrac{\pi \sin t}{2} & 0 \le t \le \pi. \\ 0 & t > \pi \end{cases}$$

5. Find the Fourier integral of

$$f(t) = \begin{cases} 1 & 0 < t < a, \\ 0 & t > a. \end{cases}$$

Ans. $f(t) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \frac{\sin a\omega \cos \omega t}{\omega} \mathrm{d}\omega.$

6. Find the Fourier integral of

$$f(t) = \begin{cases} t & 0 < t < a, \\ 0 & t > a. \end{cases}$$

Ans. $f(t) = \frac{2}{\pi} \displaystyle\int_0^{\infty} \left(\frac{a \sin a\omega}{\omega} + \frac{\cos a\omega - 1}{\omega^2} \right) \cos \omega t \, \mathrm{d}\omega.$

7. Find the Fourier integral of

$$f(t) = \mathrm{e}^{-t} + \mathrm{e}^{-2t}, \qquad t > 0.$$

Ans. $f(t) = \dfrac{6}{\pi} \displaystyle\int_0^{\infty} \frac{2+\omega^2}{\omega^4 + 5\omega^2 + 4} \cos \omega t \, \mathrm{d}\omega$.

8. Find the Fourier integral of

$$f(t) = \begin{cases} t^2 & 0 < t < a, \\ 0 & t > a. \end{cases}$$

Ans. $f(t) = \frac{2}{\pi} \displaystyle\int_0^{\infty} \left[\left(a^2 - \frac{2}{\omega^2} \right) \sin a\omega + \frac{2a}{\omega} \cos a\omega \right] \frac{\cos \omega t}{\omega} \mathrm{d}\omega.$

9. Find Fourier cosine and sine transform of

$$f(t) = \begin{cases} 1 & 0 < t < 1, \\ 0 & t > 1. \end{cases}$$

Ans. $\widehat{f}_{\mathrm{s}} = \dfrac{2}{\pi} \dfrac{1 - \cos \omega}{\omega}, \qquad \widehat{f}_{\mathrm{c}} = \dfrac{2}{\pi} \dfrac{\sin \omega}{\omega}.$

10. Find Fourier transform of

$$f(t) = \begin{cases} \mathrm{e}^{-t} & 0 < t, \\ 0 & t < 0. \end{cases}$$

Ans. $\dfrac{1}{(1 + \mathrm{i}\omega)}.$

11. Find Fourier transform of

$$f(t) = \begin{cases} 1-t & |t| < 1, \\ 0 & 1 < |t|. \end{cases}$$

Ans. $\left(\dfrac{2\mathrm{e}^{\mathrm{i}\omega}}{\mathrm{i}\omega} + \dfrac{\mathrm{e}^{\mathrm{i}\omega} - \mathrm{e}^{-\mathrm{i}\omega}}{\omega^2} \right).$

12. Find Fourier transform of

$$f(t) = \begin{cases} \mathrm{e}^t & |t| < 1, \\ 0 & 1 < |t|. \end{cases}$$

Ans. $\dfrac{\mathrm{e}^{1-\mathrm{i}\omega} - \mathrm{e}^{-1+\mathrm{i}\omega}}{1-\mathrm{i}\omega}.$

13. Show that if $f(t)$ is an even function, then the Fourier transform reduces to the Fourier cosine transform, and if $f(t)$ is an odd function it reduces to Fourier sine transform.

    Note that the multiplicative constants α and β may not come out the same as we have defined. But remember that as long as $\alpha \times \beta$ is equal to $2/\pi$, they are equivalent.

14. If $\widehat{f}(\omega) = \mathcal{F}\{f(t)\}$, show that

$$\mathcal{F}\{(-\mathrm{i}t)^n f(t)\} = \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} \widehat{f}(\omega).$$

Hint: First show that $\frac{\mathrm{d}\widehat{f}}{\mathrm{d}\omega} = -\mathrm{i}\mathcal{F}\{tf(t)\}$.

15. Show that

$$\mathcal{F}\{\frac{1}{t} f(t)\} = -\mathrm{i} \int_{-\infty}^{\omega} \widehat{f}(\omega')\mathrm{d}\omega'$$

16. (a) Find the normalization constant A of the Gaussian function $\exp(-at^2)$ such that

$$\int_{-\infty}^{\infty} \left| A \exp(-at^2) \right|^2 \mathrm{d}t = 1.$$

(b) Find the Fourier transform $\widehat{f}(\omega)$ of the normalized Gaussian function and verify the Parseval's theorem with explicit integration that

$$\int_{-\infty}^{\infty} \left| \widehat{f}(\omega) \right|^2 \mathrm{d}\omega = 2\pi.$$

Ans. $A = (2a/\pi)^{1/4}$.

17. Use Fourier transform of $\exp(-|t|)$ and the Parseval's theorem to show that
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{(1+\omega^2)^2} = \frac{\pi}{2}.$$

18. (a) Find the Fourier transform of
$$f(t) = \begin{cases} 1 - \left|\dfrac{t}{2}\right| & -2 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Use the result of (a) and the Parseval's theorem to evaluate the integral
$$I = \int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^4 \mathrm{d}t.$$

Ans. $I = 2\pi/3$.

19. The function $f(r)$ has a Fourier transform
$$\widehat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(r) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}^3 r = \frac{1}{(2\pi)^{3/2}} \frac{1}{k^2}.$$

Determine $f(r)$.

Ans. $f(r) = \dfrac{1}{4\pi r}$.

20. Find the Fourier transform of
$$f(t) = t\mathrm{e}^{-4t^2}.$$

Ans. $\widehat{f}(\omega) = -\mathrm{i}\frac{\sqrt{\pi}}{16}\omega \mathrm{e}^{-\omega^2/16}$.

21. Find the inverse Fourier transform of
$$\widehat{f}(\omega) = \mathrm{e}^{-2|\omega|}.$$

Ans. $f(t) = \dfrac{2}{\pi} \dfrac{1}{t^2+4}$.

22. Evaluate
$$\mathcal{F}^{-1}\left\{\frac{1}{\omega^2 + 4\omega + 13}\right\}.$$

Hint: $\omega^2 + 4\omega + 13 = (\omega+2)^2 + 9$.

Ans. $f(t) = \frac{1}{6}\mathrm{e}^{-\mathrm{i}2t}\mathrm{e}^{-3|t|}$.