4

Bessel and Legendre Functions

In the last chapter we have seen a number of named differential equations. These equations are of considerable importance in engineering and sciences because they occur in numerous applications. In Chap. 6, we will discuss a variety of physical problems which lead to these equations. Unfortunately these equations cannot be solved in terms of elementary functions. To solve them, we have to resort to power series expansions. Functions represented by these series solutions are called special functions.

An enormous amount of details are known about these special functions. Evaluations of these functions and formulas involving them can be found in many books and computer programs. We will mention some of them in the last section.

In order to be able to work with these functions and to have a feeling of understanding when results are expressed in terms of them, we need to know not only their definitions, but also some of their properties. Certain amount of familiarity with these special functions is necessary for us to deal with problems in mathematical physics.

In this chapter, we will first introduce the power series solutions of second-order differential equations, known as the Frobenius method. Next we will apply this method to finding the series solutions of Bessel's and Legendre's equations.

Undoubtedly, the most frequently encountered functions in solving second-order differential equations are trigonometric, hyperbolic, Bessel, and Legendre functions. Since the reader is certainly familiar with trigonometric and hyperbolic functions, we will not include them in our discussions. Our discussions are mostly about the characteristics and properties of Bessel and Legendre functions.

In exercises, we will present some other special functions mentioned in the last chapter. Their properties can be derived by similar methods discussed in this chapter.

4.1 Frobenius Method of Differential Equations

4.1.1 Power Series Solution of Differential Equation

A second-order linear homogeneous differential equation in the form of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0 \tag{4.1}$$

can be solved by expressing $y\,(x)$ in a power series

$$y\,(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{4.2}$$

if $p(x)$ and $q(x)$ are analytic at $x = 0$.

The idea of this method is simple. If $p(x)$ and $q(x)$ are analytic at $x = 0$, then they can be expressed in terms of Taylor series

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots$$
$$q(x) = q_0 + q_1 x + q_2 x^2 + \cdots.$$

Around the point $x = 0$, the differential equation becomes

$$y'' + p_0 y' + q_0 y = 0.$$

This is a differential equation of constant coefficients. The solution is given by either an exponential function or a power of x times an exponential function. Both of these functions can be expressed in terms of a power series around $x = 0$. Therefore it is natural for us to use (4.2) as a trial solution. After (4.2) is substituted into (4.1), we determine the coefficients a_n in such a way that the differential equation (4.1) is identically satisfied. If the series with coefficients so determined is convergent, then it is indeed a solution. The following example illustrates how this procedure works.

Example 4.1.1. Solve the differential equation

$$y'' + y = 0$$

by expanding $y(x)$ in a power series.

Solution 4.1.1. With

$$y = \sum_{n=0}^{\infty} a_n x^n,$$
$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1},$$
$$y'' = \sum_{n=0}^{\infty} a_n n\,(n-1)\,x^{n-2},$$

the differential equation can be written as

$$\sum_{n=0}^{\infty} a_n n\,(n-1)\,x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

The first two terms of the first summation $[a_0\,(0)\,(-1)\,x^{-2}$ and $a_1\,(1)\,(0)\,x^{-1}]$ are zero, so the summation is starting from $n = 2$,

$$\sum_{n=2}^{\infty} a_n n\,(n-1)\,x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0. \tag{4.3}$$

In order to collect terms, let us write the index in the first summation as

$$n = k + 2,$$

so the first summation can be written as

$$\sum_{n=2}^{\infty} a_n n\,(n-1)\,x^{n-2} = \sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\,x^k.$$

Now k is a running index, it does not matter what name it is called. So we can rename it back to n, that is

$$\sum_{k=0}^{\infty} a_{k+2}\,(k+2)\,(k+1)\,x^k = \sum_{n=0}^{\infty} a_{n+2}(n+2)\,(n+1)\,x^n.$$

Thus (4.3) can be written as

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)\,(n+1)\,x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)\,(n+1) + a_n\right] x^n = 0.$$

For this series to vanish, the coefficients of x^n have to be zero for all n. Therefore

$$a_{n+2}(n+2)\,(n+1) + a_n = 0,$$

or

$$a_{n+2} = -\frac{1}{(n+2)\,(n+1)} a_n.$$

This is known as the recurrence relation. This equation relates all even coefficients to a_0 and all odd coefficients to a_1. For

$$\begin{aligned}
n &= 0, \qquad a_2 = -\frac{1}{2 \cdot 1} a_0, \\
n &= 2, \qquad a_4 = -\frac{1}{4 \cdot 3} a_2 = -\frac{1}{4 \cdot 3}\left(-\frac{1}{2 \cdot 1} a_0\right) = \frac{1}{4!} a_0, \\
n &= 4, \qquad a_6 = -\frac{1}{6!} a_0, \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$n = 1, \qquad a_3 = -\frac{1}{3 \cdot 2} a_1,$$
$$n = 3, \qquad a_5 = -\frac{1}{5 \cdot 4} a_3 = -\frac{1}{5 \cdot 4}\left(-\frac{1}{3 \cdot 2} a_1\right) = \frac{1}{5!} a_1,$$
$$n = 5, \qquad a_7 = -\frac{1}{7!} a_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

It follows that

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)$$
$$+ a_1 \left(x - \frac{1}{3!}x^3 + \frac{1}{5!} - \frac{1}{7!} + \cdots\right).$$

These two series are readily recognized as cosine and sine functions,

$$y(x) = a_0 \cos x + a_1 \sin x.$$

4.1.2 Classifying Singular Points

Now the question is if $p(x)$ and $q(x)$ are not analytic at $x = 0$, can we still use the power series method? In other words, if $x = 0$ is a singular point for $p(x)$ and/or $q(x)$, is there a general method to solve the equation? To answer the question, we must distinguish two kinds of singular points.

Definition. Let x_0 be a singular point of $p(x)$ and/or $q(x)$. We call it a regular singular (or nonessential singular) point of the differential equation (4.1) if $(x - x_0)\, p(x)$ and $(x - x_0)^2\, q(x)$ are analytic at x_0. We call it an irregular singular (or essential singular) point of the equation if it is not a regular singular point.

By this definition, $x = 0$ is a regular singular point of the equation

$$y'' + \frac{f(x)}{x} y' + \frac{g(x)}{x^2} y = 0,$$

if $f(x)$ and $g(x)$ are analytic at $x = 0$. When we say that they are analytic, we mean that they can be expanded in terms of Taylor series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} g_n x^n,$$

including cases that $f(x)$ and $g(x)$ are polynomials of finite orders. For example, the equation

$$xy'' + 2y' + xy = 0$$

has a regular singular point at $x = 0$, since written in the form of

$$y'' + \frac{2}{x}y' + \frac{x^2}{x^2}y = 0,$$

we can see that 2 and x^2 are both analytic at $x = 0$.

If the singularity is only regular singular, we can use the following Frobenius series to solve the equation. Fortunately, almost all singular points we encounter in mathematical physics are regular singular points.

For the convenience of our discussion, we will assume that the singular point x_0 is at 0. In the case that it is not zero, all we need to do is to make a change of variable $\xi = x - x_0$, and solve the equation in ξ. At the end, ξ is changed back to x, so that the series is expanded in terms of $(x - x_0)$.

4.1.3 Frobenius Series

A differential equation with a regular singular point at $x = 0$ in the form of

$$y'' + \frac{f(x)}{x}y' + \frac{g(x)}{x^2}y = 0$$

can be solved by expression $y(x)$ in the following series

$$y(x) = x^p \sum_{n=0}^{\infty} a_n x^n. \tag{4.4}$$

if $f(x)$ and $g(x)$ are analytic at $x = 0$.

The idea of this method is simple. If $f(x)$ and $g(x)$ are analytic at $x = 0$, then they can be expressed in terms of Taylor series

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$
$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots.$$

Around the point $x = 0$, the differential equation can be written as

$$y'' + \frac{1}{x}f_0 y' + \frac{1}{x^2}g_0 y = 0.$$

or

$$x^2 y'' + f_0 x y' + g_0 y = 0 \tag{4.5}$$

This is an Euler–Cauchy differential equation which has a solution in the form of

$$y(x) = x^p.$$

This is the case because, after this function is put in (4.5)

$$p\,(p-1)\,x^p + f_0 p x^p + g_0 x^p = 0,$$

we can always find a p from the quadratic equation

$$p\,(p-1)+f_0p+g_0=0,$$

so that x^p is a solution of (4.5).

Thus it is natural for us to use (4.4) as a trial solution. In fact there is a mathematical theorem known as Fuchs' theorem which says that if $x=0$ is a regular singular point, then at least one solution can be found this way. We will be satisfied in learning how to find this solution rather than to prove this theorem.

After (4.4) is substituted into the differential equation, we determine the coefficients a_n in such a way that the equation is identically satisfied. If the series with coefficients so determined is convergent, then it is indeed a solution. In using (4.4), we can assume $a_0 \neq 0$, because if a_0 is equal to zero, we can increase p by one and rename a_1 as a_0. The following example illustrates how this procedure works.

Example 4.1.2. Solve the differential equation

$$xy''+2y'+xy=0$$

by expanding $y(x)$ in a Frobenius series.

Solution 4.1.2. With

$$y=\sum_{n=0}^{\infty}a_nx^{n+p},$$

$$y'=\sum_{n=0}^{\infty}a_n\,(n+p)\,x^{n+p-1},$$

$$y''=\sum_{n=0}^{\infty}a_n\,(n+p)\,(n+p-1)\,x^{n+p-2},$$

the differential equation becomes

$$\sum_{n=0}^{\infty}a_n\,(n+p)\,(n+p-1)\,x^{n+p-1}+2\sum_{n=0}^{\infty}a_n\,(n+p)\,x^{n+p-1}$$
$$+\sum_{n=0}^{\infty}a_nx^{n+p+1}=0,$$

or

$$x^p\left\{\sum_{n=0}^{\infty}a_n\left[(n+p)\,(n+p-1)+2\,(n+p)\right]x^{n-1}+\sum_{n=0}^{\infty}a_nx^{n+1}\right\}=0.$$

Since

$$(n+p)(n+p-1)+2(n+p)=(n+p)(n+p+1),$$

and x^p cannot be identically equal to zero for all x, so

$$\sum_{n=0}^{\infty} a_n (n+p)(n+p+1)x^{n-1}+\sum_{n=0}^{\infty} a_n x^{n+1}=0.$$

In order to collect terms, we separate out the $n=0$ and $n=1$ terms in the first summation,

$$a_0 p(p+1)x^{-1}+a_1(p+1)(p+2)+\sum_{n=2}^{\infty} a_n (n+p)(n+p+1)x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1}=0.$$

Furthermore,

$$\sum_{n=2}^{\infty} a_n (n+p)(n+p+1)x^{n-1}=\sum_{n=0}^{\infty} a_{n+2}(n+p+2)(n+p+3)x^{n+1},$$

therefore

$$a_0 p(p+1)x^{-1}+a_1(p+1)(p+2) + \sum_{n=0}^{\infty}\left[a_{n+2}(n+p+2)(n+p+3)+a_n\right]x^{n+1}=0.$$

For this to vanish, all coefficients have to be zero,

$$a_0 p(p+1)=0, \tag{4.6}$$

$$a_1(p+1)(p+2)=0, \tag{4.7}$$

$$a_{n+2}(n+p+2)(n+p+3)+a_n=0. \tag{4.8}$$

Since $a_0 \neq 0$, it follows from (4.6)

$$p(p+1)=0.$$

This equation is called the indicial equation. Clearly

$$p=-1, \quad p=0.$$

There are three possibilities that (4.7) is satisfied,

$$\begin{aligned} &\text{case 1}: \quad p=-1, \quad a_1\neq 0,\\ &\text{case 2}: \quad p=-1, \quad a_1=0,\\ &\text{case 3}: \quad p=0, \quad a_1=0. \end{aligned}$$

From here on we solve the problem in these three separate cases. In case 1, $p = -1$, it follows from (4.8) that

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n.$$

This kind of relation is known as recurrence relation. From this relation, we have

$$\begin{aligned}
n = 0: &\quad a_2 = \frac{-1}{2 \cdot 1} a_0 \\
n = 2: &\quad a_4 = \frac{-1}{4 \cdot 3} a_2 = \frac{-1}{4 \cdot 3}\left(\frac{-1}{2 \cdot 1} a_0\right) = \frac{(-1)^2}{4!} a_0 \\
n = 4: &\quad a_6 = \frac{-1}{6 \cdot 5} a_4 = \frac{-1}{6 \cdot 5}\left[\frac{(-1)^2}{4!} a_0\right] = \frac{(-1)^3}{6!} a_0 \\
&\cdots\cdot \\
n = 1: &\quad a_3 = \frac{-1}{3 \cdot 2} a_1 \\
n = 3: &\quad a_5 = \frac{-1}{5 \cdot 4} a_3 = \frac{-1}{5 \cdot 4}\left(\frac{-1}{3 \cdot 2} a_1\right) = \frac{(-1)^2}{5!} a_1 \\
n = 5: &\quad a_7 = \frac{-1}{7 \cdot 6} a_5 = \frac{-1}{7 \cdot 6}\left[\frac{(-1)^2}{5!} a_1\right] = \frac{(-1)^3}{7!} a_1 \\
&\cdots\cdot.
\end{aligned}$$

It is thus clear that the solution of the differential equation can be written as

$$\begin{aligned}
y(x) = x^{-1} a_0 &\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^5 + \cdots\right) \\
&+ x^{-1} a_1 \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right),
\end{aligned}$$

which we recognize as

$$y(x) = a_0 \frac{1}{x} \cos x + a_1 \frac{1}{x} \sin x.$$

In this we have found both linearly independent solutions of this second-order differential equation.

In case 2, $p = -1$, and $a_1 = 0$. Because of the recurrence relation, all odd coefficients are zero,

$$a_1 = a_3 = a_5 = \cdots = 0.$$

Therefore we are left with

$$y(x) = a_0 \frac{1}{x} \cos x,$$

which is one of the solutions.

In case 3, $p = 0$, and $a_1 = 0$. In this case, all odd coefficients are again equal to zero, and for the even coefficients, the recurrence relation becomes

$$a_{n+2} = \frac{-1}{(n+3)(n+2)} a_n.$$

So the solution can be written as

$$\begin{aligned} y(x) &= x^0 a_0 \left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots\right) \\ &= a_0 \frac{1}{x}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) \\ &= a_0 \frac{1}{x} \sin x, \end{aligned}$$

which is the other solution. Note that the a_0 in case 2 is not necessarily equal to the a_0 in case 3. They are arbitrary constants. We recover the general solution by a linear combination of the solutions in case 2 and in case 3,

$$y(x) = c_1 \frac{1}{x} \cos x + c_2 \frac{1}{x} \sin x.$$

The Frobenius series is a generalized power series

$$y = x^p \sum_{n=0}^{\infty} a_n x^n.$$

If the exponent p is a positive integer, it becomes a Taylor series. If p is a negative integer, it becomes a Laurent series. Any equation that can be solved by Taylor or Laurent series, it can also be solved by Frobenius series. Frobenius series is even more general than that because p may be a fraction number, in fact it may even be a complex number. Therefore if one is trying to solve a differential equation by series expansion, instead of first trying to determine if the expansion center is an ordinary point or a regular singular point, one can just try to solve it with the Frobenius series. However, before accepting the series as the solution of the equation, one must determine whether the series is convergent or divergent.

4.2 Bessel Functions

Bessel function is one of the most important special functions in mathematical physics. It occurs, mostly but not exclusively, in problems with cylindrical symmetry. It is the solution of the equation

$$x^2 y''(x) + x y'(x) + (x^2 - n^2) y(x) = 0, \tag{4.9}$$

where n is a given number. This linear homogeneous differential equation is known as the Bessel's equation, named after Wilhelm Bessel (1752–1833), a great German astronomer and mathematician.

4.2.1 Bessel Functions $J_n(x)$ of Integer Order

Although n can be any real number, but we will first limit our attention primarily to the case where n is an integer ($n = 0, 1, 2, \cdots$). We seek a solution of the Bessel's equation in the form of Frobenius series

$$y = x^p \sum_{j=0} a_j x^j = \sum_{j=0} a_j x^{j+p}, \tag{4.10}$$

where p is some constant, and

$$a_0 \neq 0.$$

Assume for the present that the function is differentiable, so

$$y' = \sum_{j=0} a_j (j+p) x^{j+p-1},$$

$$y'' = \sum_{j=0} a_j (j+p)(j+p-1) x^{j+p-2}.$$

Substituting them into (4.9) , we obtain

$$\sum_{j=0} \left[(j+p)(j+p-1) + (j+p) + (x^2 - n^2) \right] a_j x^{j+p} = 0,$$

or

$$x^p \left[\sum_{j=0} [(j+p)^2 - n^2] a_j x^j + \sum_{i=0} a_i x^{i+2} \right] = 0. \tag{4.11}$$

After $j = 0$ and $j = 1$ terms are written out explicitly, the first summation becomes

$$\sum_{j=0} [(j+p)^2 - n^2] a_j x^j = [p^2 - n^2] a_0 + [(p+1)^2 - n^2] a_1 x + \sum_{j=2} [(j+p)^2 - n^2] a_j x^j,$$

and the second summation can be written as

$$\sum_{i=0} a_i x^{i+2} = \sum_{j=2} a_{j-2} x^j,$$

The quantity in the bracket of (4.11) must be equal to zero, therefore

$$[p^2 - n^2]a_0 + [(p+1)^2 - n^2]a_1 x + \sum_{j=2}\{[(j+p)^2 - n^2]a_j + a_{j-2}\}x^j = 0.$$

For this equation to hold, the coefficient of each power of x must vanish. Thus,

$$[p^2 - n^2]a_0 = 0, \tag{4.12}$$

$$[(p+1)^2 - n^2]a_1 = 0, \tag{4.13}$$

$$[(j+p)^2 - n^2]a_j + a_{j-2} = 0. \tag{4.14}$$

Since $a_0 \neq 0$, (4.12) requires

$$p = \pm n,$$

we will first proceed with a choice of $+n$. Clearly (4.13) requires

$$a_1 = 0.$$

From (4.14), we have the recurrence relation

$$a_j = \frac{-a_{j-2}}{(j+n)^2 - n^2} = \frac{-1}{j(j+2n)}a_{j-2}. \tag{4.15}$$

Since $a_1 = 0$, this recurrence relation requires $a_3 = 0$, then $a_5 = 0$, etc.; thus

$$a_{2j-1} = 0 \quad j = 1, 2, 3, \ldots$$

Since all nonvanishing coefficients have even indices, we set

$$j = 2k, \quad k = 0, 1, 2, \ldots,$$

thus the recurrence relation (4.15) becomes

$$a_{2k} = \frac{-1}{2^2 k(k+n)}a_{2(k-1)}. \tag{4.16}$$

This relation holds for any k, specifically we have

$$a_2 = -\frac{1}{2^2 \cdot 1 \cdot (n+1)}a_0,$$

$$a_4 = -\frac{1}{2^2 \cdot 2 \cdot (n+2)}a_2 = \frac{(-1)^2}{2^4 \cdot 2! \cdot (n+2)\,(n+1)}a_0,$$

$$a_{2k} = \frac{(-1)^k}{2^{2k}k!(n+k)(n+k-1)\cdot \cdots \cdot (n+1)}a_0. \tag{4.17}$$

Thus a_0 is a common factor in all terms of the series. Therefore it is a multiplicative constant and can be set to any value. However, by convention, if a_0 is chosen to be

$$a_0 = \frac{1}{2^n n!}, \tag{4.18}$$

the resulting series for $y(x)$ is designated as $J_n(x)$, called Bessel function of the first kind of order n. With this choice, (4.17) becomes

$$a_{2k} = \frac{(-1)^k}{k!(k+n)!}\frac{1}{2^{n+2k}}, \quad k = 0, 1, 2, \ldots \tag{4.19}$$

and

$$\begin{aligned} J_n(x) &= \sum_{k=0}^{\infty} a_{2k}x^{n+2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k} \\ &= \frac{x^n}{2^n n!}\left(1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4 2!(n+1)(n+2)} - \cdots\right). \end{aligned} \tag{4.20}$$

By ratio test, this series is absolutely convergent for all x. Hence $J_n(x)$ is bounded everywhere from $x = 0$ to $x \to \infty$.

The results for J_0, J_1, J_2 are shown in Fig. 4.1. They are alternating series. The error in cutting off after n terms is less than the first term dropped. The Bessel functions oscillate but are not periodic. The amplitude of $J_n(x)$ is not constant but decreases asymptotically.

4.2.2 Zeros of the Bessel Functions

As it is seen in Fig. 4.1 for each n, there are a series of x values for which $J_n(x) = 0$. These x values are the zeros of Bessel functions. They are very important in practical applications.They can be found in tables, such as "Table of First 700 Zeros of Bessel Functions" by C.L. Beattie, Bell Tech. J. **37**, 689 (1958) and Bell Monograph 3055. The first few are listed in Table 4.1.

As an example of how to use this table, let us answer the following question. If λ_{nj} is the jth root of $J_n(\lambda c) = 0$ where $c = 2$, find $\lambda_{01},\ \lambda_{23},\ \lambda_{53}$. The answer should be

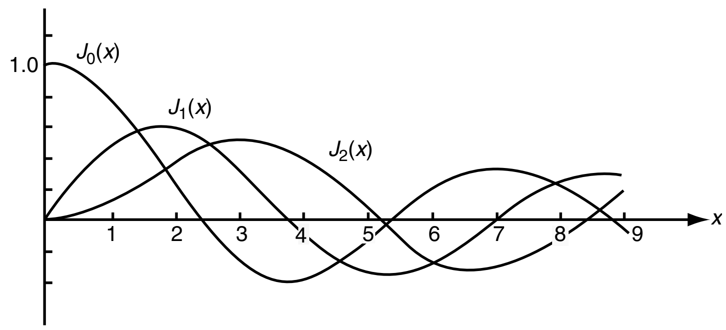

Fig. 4.1. Bessel functions, $J_0(x),\ J_1(x),\ J_2(x)$

Table 4.1. Zeros of the Bessel function

Number of zeros	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

$$\lambda_{01} = \frac{2.4048}{2} = 1.2024,$$
$$\lambda_{23} = \frac{11.6198}{2} = 5.8099,$$
$$\lambda_{53} = \frac{15.7002}{2} = 7.8501.$$

4.2.3 Gamma Function

For Bessel function of noninteger order we need an extension of the factorials. This can be done via gamma function.

The gamma function is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty \mathrm{e}^{-t} t^{\alpha-1}\,\mathrm{d}t. \tag{4.21}$$

With integration by parts, we obtain

$$\Gamma(\alpha+1) = \int_0^\infty \mathrm{e}^{-t} t^{\alpha}\,\mathrm{d}t = -\mathrm{e}^{-t} t^{\alpha}\Big|_0^\infty + \alpha \int_0^\infty \mathrm{e}^{-t} t^{\alpha-1}\,\mathrm{d}t.$$

The first expression on the right is zero, and the integral on the right is $\alpha\Gamma(\alpha)$. This gives the basic relation

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

Since

$$\Gamma(1) = \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}t = 1,$$

we conclude for integer n,

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1)\\ &= n(n-1)\cdots 1\Gamma(1) = n!\end{aligned} \tag{4.22}$$

For a noninteger α, the integral of (4.21) can be evaluated. The gamma functions $\Gamma(\alpha)$ for both positive and negative α are shown in Fig. 4.2.

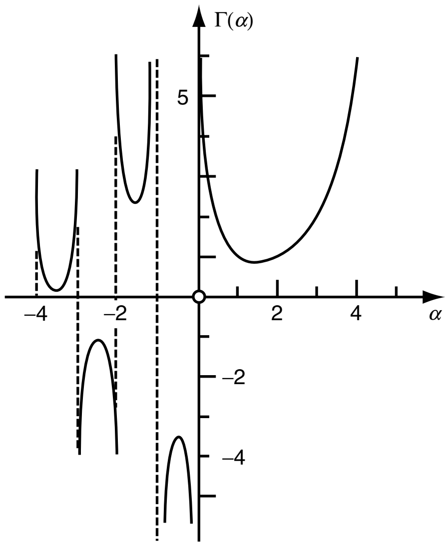

Fig. 4.2. Gamma function $\Gamma(\alpha)$

It follows from (4.22) that

$$0! = \Gamma(1) = 1. \tag{4.23}$$

Since $n\Gamma(n) = \Gamma(n+1)$, thus $\Gamma(n) = \Gamma(n+1)/n$. It follows that

$$\Gamma(0) = \frac{\Gamma(1)}{0} \to \infty,$$

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} \to \infty,$$

and for any negative integer

$$\Gamma(-n) = \frac{\Gamma(-n+1)}{-n} \to \infty.$$

The special case of $\Gamma(1/2)$ is of particular interest,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \mathrm{e}^{-t} t^{-1/2}\, \mathrm{d}t. \tag{4.24}$$

Let $t = x^2$, *so* $\mathrm{d}t = 2x\,\mathrm{d}x$ *and* $t^{-1/2} = x^{-1}$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \mathrm{e}^{-x^2} \frac{1}{x} 2x\,\mathrm{d}x = 2\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

For a definite integral, the name of the integration variable is immaterial

$$\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y,$$

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 4\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x \int_0^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y \\ &= 4\int_0^\infty \int_0^\infty \mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The double integral can be considered as a surface integral over the first quadrant of the entire plane. Change to the polar coordinates,

$$\begin{aligned}x^2 + y^2 &= \rho^2 \\ \mathrm{d}a &= \rho\,\mathrm{d}\theta\;\mathrm{d}\rho,\end{aligned}$$

we have

$$\begin{aligned}\left[\Gamma\left(\frac{1}{2}\right)\right]^2 &= 4\int_0^\infty \int_0^{\pi/2} \mathrm{e}^{-\rho^2}\rho\,\mathrm{d}\theta\,\mathrm{d}\rho \\ &= 4\frac{\pi}{2}\int_0^\infty \mathrm{e}^{-\rho^2}\rho\,\mathrm{d}\rho = \pi\left[-\mathrm{e}^{-\rho^2}\right]_0^\infty = \pi.\end{aligned}$$

Thus

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{4.25}$$

4.2.4 Bessel Function of Noninteger Order

In our development of Bessel function of integer order, we had in (4.18) $a_0 = 1/(2^n n!)$, which can be written as

$$a_0 = \frac{1}{2^n \Gamma(n+1)}.$$

This suggests that, for noninteger α, we choose

$$a_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)}.$$

Following exactly the same procedure as for the integer order, we find the noninteger order Bessel function is given by

$$J_\alpha(x) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{\alpha+2k}. \tag{4.26}$$

In fact this formula can be used for both integer and noninteger α.

Example 4.2.1. Show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

Solution 4.2.1. By definition,

$$\begin{aligned} J_{1/2}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{(1/2)+2k} \\ &= \left(\frac{x}{2}\right)^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{1}{2}+1)} \frac{x^{2k+1}}{2^{2k+1}}. \end{aligned}$$

$$\begin{aligned} \Gamma(k+\frac{1}{2}+1) &= \left(k+\frac{1}{2}\right) \Gamma\left(k+\frac{1}{2}\right) \\ &= \left(k+\frac{1}{2}\right)\left(k+\frac{1}{2}-1\right)\left(k+\frac{1}{2}-2\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\ &= \frac{(2k+1)\,(2k-1)\,(2k-3)\cdots 1}{2^{k+1}}\Gamma\left(\frac{1}{2}\right). \end{aligned}$$

It follows that

$$\begin{aligned} k!\Gamma(k+\frac{1}{2}+1)2^{2k+1} &= k!\,[(2k+1)\,(2k-1)\cdots 1]\,\Gamma\left(\frac{1}{2}\right)2^k \\ &= [2k\,(2k-2)\cdots 2]\,[(2k+1)\,(2k-1)\cdots 1]\,\Gamma\left(\frac{1}{2}\right) \\ &= (2k+1)!\Gamma\left(\frac{1}{2}\right) = (2k+1)!\sqrt{\pi}. \end{aligned}$$

Thus

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

But

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

therefore

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x. \tag{4.27}$$

Similarly,

$$\begin{aligned} J_{-1/2}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{-(1/2)+2k} \\ &= \left(\frac{x}{2}\right)^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\Gamma(\frac{1}{2})} x^{2k} = \sqrt{\frac{2}{\pi x}} \cos x. \end{aligned} \tag{4.28}$$

4.2.5 Bessel Function of Negative Order

If α is not an integer, the Bessel function of the negative order of $J_{-\alpha}(x)$ is very simple. All we have to do is to replace α by $-\alpha$ in the expression of $J_\alpha(x)$, that is

$$J_{-\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-\alpha+1)} \left(\frac{x}{2}\right)^{-\alpha+2k} . \tag{4.29}$$

Since the first term of J_α and $J_{-\alpha}$ is a finite nonzero multiple of x^α and $x^{-\alpha}$, respectively, clearly J_α and $J_{-\alpha}$ are linearly independent. Therefore the general solution of Bessel's equation of order α is

$$y(x) = c_1 J_\alpha(x) + c_2 J_{-\alpha}(x) .$$

However, if α is an integer, the negative order $J_{-n}(x)$ and the positive order Bessel function $J_n(x)$ are not linearly independent. This can be seen as follows. Starting with the definition

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-n+1)} \left(\frac{x}{2}\right)^{-n+2k} .$$

If $k < n$, then $\Gamma(k-n+1) \to \infty$ and all the corresponding terms will be zero. Therefore the series actually starts with $k = n$,

$$J_{-n}(x) = \sum_{k=n}^{\infty} \frac{(-1)^k}{k!\Gamma(k-n+1)} \left(\frac{x}{2}\right)^{-n+2k} .$$

Let us define $j = k - n$, then $k = n + j$, thus

$$\begin{aligned} J_{-n}(x) &= \sum_{j=0}^{\infty} \frac{(-1)^{n+j}}{(n+j)!\Gamma(j+1)} \left(\frac{x}{2}\right)^{-n+2(j+n)} \\ &= (-1)^n \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(n+j+1)j!} \left(\frac{x}{2}\right)^{2j+n} = (-1)^n J_n(x). \end{aligned}$$

Therefore $J_{-n}(x)$ and $J_n(x)$ are linearly dependent. So in this case, there must be another linearly independent solution of Bessel's equation of order n.

4.2.6 Neumann Functions and Hankel Functions

To determine the second linearly independent solution of the Bessel function when $\alpha = n$ and n is an integer, it is customary to form a particular linear combination of $J_\alpha(x)$ and $J_{-\alpha}(x)$ and then letting $\alpha \to n$. The combination

$$N_\alpha(x) = \frac{\cos(\alpha\pi) J_\alpha(x) - J_{-\alpha}(x)}{\sin(\alpha\pi)} \tag{4.30}$$

is called the Bessel function of the second kind of order α. It is also known as the Neumann function. In some literature, it is denoted as $Y_\alpha(x)$.

For noninteger α, $N_\alpha(x)$ is clearly a solution of the Bessel equation, since it is a linearly combination of two linearly independent solutions $J_\alpha(x)$ and $J_{-\alpha}(x)$.

For integer α, $\alpha = n$ and $n = 0, 1, 2, \ldots$, (4.30) becomes

$$N_n(x) = \frac{\cos(n\pi) J_n(x) - J_{-n}(x)}{\sin(n\pi)},$$

which gives an indeterminate form of 0/0, since $\cos(n\pi) = (-1)^n$, $\sin(n\pi) = 0$ and $J_n(x) = (-1)^n J_{-n}(x)$. We can use l'Hôspital's rule to evaluate this ratio. If we define the Neumann function $N_n(x)$ as

$$N_n(x) = \lim_{\alpha\to n} \frac{\cos(\alpha\pi) J_\alpha(x) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

Then

$$\begin{aligned} N_n(x) &= \left[\frac{\frac{\partial}{\partial\alpha}\left(\cos(\alpha\pi) J_\alpha(x) - J_{-\alpha}(x)\right)}{\frac{\partial}{\partial\alpha}\sin(\alpha\pi)}\right]_{\alpha=n} \\ &= \left[\frac{-\pi\sin(\alpha\pi) J_\alpha(x) + \cos(\alpha\pi)\frac{\partial}{\partial\alpha}J_\alpha - \frac{\partial}{\partial\alpha}J_{-\alpha}(x)}{\pi\cos(\alpha\pi)}\right]_{\alpha=n} \\ &= \frac{1}{\pi}\left[\frac{\partial}{\partial\alpha}J_\alpha(x) - (-1)^n\frac{\partial}{\partial\alpha}J_{-\alpha}(x)\right]_{\alpha=n}. \end{aligned} \tag{4.31}$$

Now we will show that $N_n(x)$ so defined is indeed a solution of the Bessel's equation. By definition, J_α and $J_{-\alpha}$, respectively, satisfy the following differential equations

$$x^2 J_\alpha''(x) + xJ_\alpha'(x) + \left(x^2 - \alpha^2\right) J_\alpha(x) = 0,$$
$$x^2 J_{-\alpha}''(x) + xJ_{-\alpha}'(x) + \left(x^2 - \alpha^2\right) J_{-\alpha}(x) = 0.$$

Differentiate with respect to α,

$$x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial J_\alpha}{\partial\alpha}\right) + x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial J_\alpha}{\partial\alpha}\right) + \left(x^2-\alpha^2\right)\frac{\partial J_\alpha}{\partial\alpha} - 2\alpha J_\alpha = 0,$$
$$x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial J_{-\alpha}}{\partial\alpha}\right) + x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial J_{-\alpha}}{\partial\alpha}\right) + \left(x^2-\alpha^2\right)\frac{\partial J_{-\alpha}}{\partial\alpha} - 2\alpha J_{-\alpha} = 0.$$

Multiplying the second equation by $(-1)^n$ and subtracting it from the first equation, we have

$$\begin{aligned} &x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial J_\alpha}{\partial\alpha} - (-1)^n\frac{\partial J_{-\alpha}}{\partial\alpha}\right) + x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial J_\alpha}{\partial\alpha} - (-1)^n\frac{\partial J_{-\alpha}}{\partial\alpha}\right) \\ &+ \left(x^2-\alpha^2\right)\left(\frac{\partial J_\alpha}{\partial\alpha} - (-1)^n\frac{\partial J_{-\alpha}}{\partial\alpha}\right) - 2\alpha\left(J_\alpha - (-1)^n J_{-\alpha}\right) = 0. \end{aligned}$$

Taking the limit $\alpha \to n$, the last term drops out because

$$J_n - (-1)^n J_{-n} = 0.$$

Clearly the Neumann function expressed in (4.31) satisfies the Bessel's equation.

Neumann function has a logarithmic term, since

$$\frac{\partial J_\alpha}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[x^\alpha \sum_{k=0}^{\infty} \frac{(-1)^n}{k!\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k}\right]$$
$$= \frac{\partial x^\alpha}{\partial \alpha} \sum_{k=0}^{\infty} \frac{(-1)^n}{k!\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k} + x^\alpha \frac{\partial}{\partial \alpha} \sum_{k=0}^{\infty} \frac{(-1)^n}{k!\Gamma(k+\alpha+1)}\left(\frac{x}{2}\right)^{2k},$$

and

$$\frac{\partial x^\alpha}{\partial \alpha} = \frac{\partial}{\partial \alpha} \mathrm{e}^{\alpha \ln x} = \mathrm{e}^{\alpha \ln x} \ln x = x^\alpha \ln x.$$

Thus $N_n(x)$ contains a term $J_n(x) \ln x$. Clearly it is linearly independent of $J_n(x)$.

Neumann functions diverge as $x \to 0$. For $\alpha \neq 0$, it diverges because the series for $J_{-\alpha}$ starts with a term $x^{-\alpha}$. For $\alpha = 0$, the term $J_0(x) \ln x$ goes to $-\infty$ as x goes to zero.

Like Bessel functions, the values and zeros of Neumann functions have been extensively tabulated. The first three orders of Neumann functions are shown in Fig. 4.3.

Since $J_\alpha(x)$ and $N_\alpha(x)$ always constitute a pair of linearly independent solutions, the general solution of the Bessel's equation can be written as

$$y(x) = c_1 J_\alpha(x) + c_2 N_\alpha(x).$$

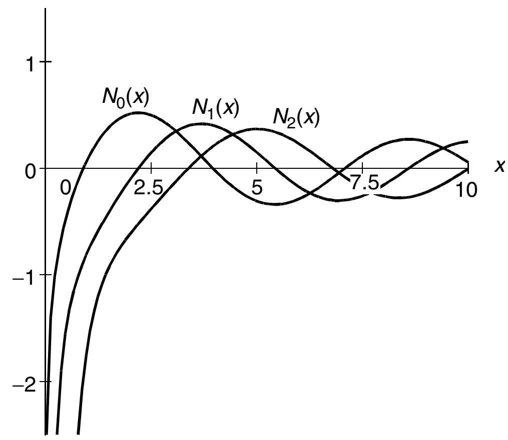

Fig. 4.3. Neumann functions $N_0(x)$, $N_1(x)$, $N_2(x)$

This expression is valid for all α.

In physical applications, we often require the solution to be bounded at the origin. Since $N_\alpha(x)$ diverges at $x = 0$, the coefficient c_2 must be set to zero, and the solution is just a constant times $J_\alpha(x)$. However, there are problems where the origin is excluded. In such cases, both $J_\alpha(x)$ and $N_\alpha(x)$ have to be used.

Hankel Functions. The following linear combinations are useful in the study of wave propagation, especially in the asymptotic region where they have pure complex exponential behavior. They are called Hankel functions of the first kind

$$H_n^{(1)}(x) = J_n(x) + \mathrm{i}N_n(x),$$

and Hankel functions of the second kind

$$H_n^{(2)}(x) = J_n(x) - \mathrm{i}N_n(x).$$

These functions are also known as Bessel functions of the third kind.

4.3 Properties of Bessel Function

4.3.1 Recurrence Relations

Starting with the series representations, we can deduce the following properties of Bessel functions

1.
$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{n+1}J_{n+1}(x)] = x^{n+1}J_n(x). \tag{4.32}$$

Proof:

$$x^{n+1}J_{n+1}(x) = x^{n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+2)}\left(\frac{x}{2}\right)^{n+2k+1}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+2)}\frac{x^{2n+2k+2}}{2^{n+2k+1}}.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{n+1}J_{n+1}(x)] = \sum_{k=0}^{\infty}\frac{(-1)^k 2(n+k+1)}{k!\Gamma(k+n+2)}\frac{x^{2n+2k+1}}{2^{n+2k+1}}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+1)}\frac{x^{2n+2k+1}}{2^{n+2k}}$$
$$= x^{n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+1)}\frac{x^{n+2k}}{2^{n+2k}} = x^{n+1}J_n(x).$$

2.
$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x). \tag{4.33}$$

Proof:

$$x^{-n}J_n(x) = x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+1)}\frac{x^{n+2k}}{2^{n+2k}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+n+1)}\frac{x^{2k}}{2^{n+2k}},$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}J_n(x)] = \sum_{k=0}^{\infty}\frac{(-1)^k 2k}{k!\Gamma(k+n+1)}\frac{x^{2k-1}}{2^{n+2k}}.$$

Since the first term with $k = 0$ is zero, it follows that the summation starts with $k = 1$.

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}J_n(x)] = \sum_{k=1}^{\infty}\frac{(-1)^k 2k}{k!\Gamma(k+n+1)}\frac{x^{2k-1}}{2^{n+2k}}$$
$$= \sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!\Gamma(k+n+1)}\frac{x^{2k-1}}{2^{n+2k-1}}.$$

Now let $j = k-1$, so $k = j+1$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}J_n(x)] = \sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{(j)!\Gamma(j+n+2)}\frac{x^{2j+1}}{2^{n+2j+1}}$$
$$= -x^{-n}\sum_{j=0}^{\infty}\frac{(-1)^{j}}{(j)!\Gamma(j+n+2)}\frac{x^{n+2j+1}}{2^{n+2j+1}} = -x^{-n}J_{n+1}(x).$$

With these properties, we can derive the following recurrence relations

3.
$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x), \tag{4.34}$$

4.
$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)], \tag{4.35}$$

5.
$$J_0'(x) = -J_1(x). \tag{4.36}$$

Starting with

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{n+1}J_{n+1}(x)] = (n+1)x^nJ_{n+1}(x) + x^{n+1}J_{n+1}'(x),$$

it follows from (4.32) that

$$(n+1)x^nJ_{n+1}(x) + x^{n+1}J_{n+1}'(x) = x^{n+1}J_n(x),$$

which can be written as

$$J'_{n+1}(x) = J_n(x) - \frac{n+1}{x} J_{n+1}(x),$$

or

$$J'_n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x). \tag{4.37}$$

From (4.33), we have

$$-nx^{-n-1} J_n(x) + x^{-n} J'_n(x) = -x^{-n} J_{n+1}(x),$$

or

$$J'_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x). \tag{4.38}$$

It follows from (4.37), (4.38) that

$$J_{n-1}(x) - \frac{n}{x} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x),$$

or

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

which is the recurrence relation 3.

Adding (4.37)–(4.38), we have

$$J'_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)].$$

which is the recurrence relation 4.

The particular case $J'_0(x)$ follows directly from property 2. Put $n = 0$ in (4.33), we have

$$J'_0(x) = -J_1(x).$$

which is the recurrence relation 5.

These recurrence relations are very useful. It means that as long as we know $J_0(x)$ and $J_1(x)$, all higher orders Bessel functions and their derivatives can be generated from these relations.

Another interesting relation follows from property 1 is

$$\int_0^r \mathrm{d}[x^{n+1} J_{n+1}(x)] = \int_0^r x^{n+1} J_n(x) \,\mathrm{d}x.$$

Therefore

6.
$$\int_0^r x^{n+1} J_n(x) \,\mathrm{d}x = r^{n+1} J_{n+1}(r). \tag{4.39}$$

An important special case is for $n = 0$,

7.
$$\int_0^r x J_0(x) \,\mathrm{d}x = r J_1(r). \tag{4.40}$$

4.3.2 Generating Function of Bessel Functions

Although Bessel functions are of interest primarily as solutions of differential equations, it is intructive and convenient to develop them from a completely different approach, that of the generating function.

Recall

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

so

$$\exp\left(\frac{xt}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{xt}{2}\right)^n, \qquad \exp\left(-\frac{x}{2t}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{x}{2t}\right)^n.$$

It follows

$$\begin{aligned} \exp\left(\frac{xt}{2} - \frac{x}{2t}\right) &= \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{xt}{2}\right)^l \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{x}{2t}\right)^m \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{l!m!}\left(\frac{x}{2}\right)^{l+m} t^{l-m}. \end{aligned} \tag{4.41}$$

Let

$$l - m = n, \text{ then } l = m + n \text{ and } l + m = 2m + n,$$

so (4.41) can be written as

$$\exp\left(\frac{xt}{2} - \frac{x}{2t}\right) = \sum_{n=-\infty}^{\infty} \left\{ \sum_{m=0}^{\infty} \frac{1}{(n+m)!} \frac{(-1)^m}{m!} \left(\frac{x}{2}\right)^{2m+n} \right\} t^n.$$

Since the Bessel function $J_n(x)$ is given by

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(n+m)!m!} \left(\frac{x}{2}\right)^{2m+n},$$

it is clear

$$\exp\left(\frac{xt}{2} - \frac{x}{2t}\right) = \sum_{n=-\infty}^{\infty} J_n(x) t^n. \tag{4.42}$$

The left-hand side of this equation is known as the generating function of the Bessel functions, sometimes designated as $G(x,t)$,

$$G(x,t) = \exp\left(\frac{xt}{2} - \frac{x}{2t}\right).$$

4.3.3 Integral Representation

A particularly useful and powerful way of treating Bessel functions employs integral representations. If we substitute

$$t = \mathrm{e}^{\mathrm{i}\theta},$$

then

$$t - \frac{1}{t} = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = 2\mathrm{i}\sin\theta.$$

Therefore (4.42) can be written as

$$\begin{aligned}
\mathrm{e}^{(\mathrm{i}x\sin\theta)} &= \sum_{n=-\infty}^{\infty} J_n(x)\mathrm{e}^{\mathrm{i}n\theta} = J_0(x) + \sum_{n=1}^{\infty}\left[J_n(x)\mathrm{e}^{\mathrm{i}n\theta} + J_{-n}(x)\mathrm{e}^{-\mathrm{i}n\theta}\right] \\
&= J_0(x) + \sum_{n=1}^{\infty} J_n(x)(\cos n\theta + \mathrm{i}\sin n\theta) \\
&\quad + \sum_{n=1}^{\infty}(-1)^n J_n(x)(\cos n\theta - \mathrm{i}\sin n\theta) \\
&= J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots] \\
&\quad + 2\mathrm{i}[J_1(x)\sin\theta + J_3(x)\sin 3\theta + J_5(x)\sin 5\theta + \cdots].
\end{aligned}$$

But

$$\mathrm{e}^{(\mathrm{i}x\sin\theta)} = \cos(x\sin\theta) + \mathrm{i}\sin(x\sin\theta),$$

thus

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots], \tag{4.43}$$

$$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + J_5(x)\sin 5\theta + \cdots]. \tag{4.44}$$

These are Fourier type representations. The coefficients $J_n(x)$ can be readily obtained. For example, multiply (4.43) by $\cos(n\theta)$ and integrate from 0 to π, we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\,\mathrm{d}\theta = \begin{cases} J_n(x), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$

Similarly from (4.44)

$$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\,\mathrm{d}\theta = \begin{cases} 0, & n \text{ even} \\ J_n(x), & n \text{ odd} \end{cases}.$$

Adding these two equations, we get

$$\begin{aligned}
J_n(x) &= \frac{1}{\pi}\int_0^{\pi}[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta - n\theta)\mathrm{d}\theta.
\end{aligned} \tag{4.45}$$

As a special case,

$$J_0(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin\theta) \mathrm{d}\theta. \tag{4.46}$$

On the other hand, cosine is an even function and sine is an odd function

$$\int_0^{\pi} \cos(x \sin\theta) \mathrm{d}\theta = \frac{1}{2} \int_0^{2\pi} \cos(x \sin\theta) \mathrm{d}\theta,$$

$$\int_0^{2\pi} \sin(x \sin\theta) \mathrm{d}\theta = 0.$$

Therefore (4.46) can be written as

$$\begin{aligned} J_0(x) &= \frac{1}{2\pi} \int_0^{2\pi} \cos(x \sin\theta) \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} [\cos(x \sin\theta) + \mathrm{i} \sin(x \sin\theta)] \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \exp(\mathrm{i}\, x \sin\theta) \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} \exp(\mathrm{i}\, x \cos\theta) \mathrm{d}\theta. \end{aligned}$$

This form is very useful in the Fraunhofer diffraction with a circular aperture.

4.4 Bessel Functions as Eigenfunctions of Sturm–Liouville Problems

4.4.1 Boundary Conditions of Bessel's Equation

As discussed in the last chapter, by itself Bessel's equation is not a Sturm–Liouville equation. There is no way for it to satisfy any given boundary condition. However, the closely related equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\lambda^2 x^2 - n^2\right) y = 0 \tag{4.47}$$

is a Sturm–Liouville equation. It can easily be shown that

$$y(x) = J_n(\lambda x)$$

is a solution of this equation. Let $z = \lambda x$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}J_n(z)}{\mathrm{d}z} \frac{\mathrm{d}z}{\mathrm{d}x} = \lambda \frac{\mathrm{d}J_n(z)}{\mathrm{d}z},$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x} \left[\lambda \frac{\mathrm{d}J_n(z)}{\mathrm{d}z} \right] = \frac{\mathrm{d}}{\mathrm{d}z} \left[\lambda \frac{\mathrm{d}J_n(z)}{\mathrm{d}z} \right] \frac{\mathrm{d}z}{\mathrm{d}x} = \lambda^2 \frac{\mathrm{d}^2 J_n(z)}{\mathrm{d}z^2}.$$

Substituting them into (4.47) , we have

$$x^2\lambda^2\frac{\mathrm{d}^2 J_n(z)}{\mathrm{d}z^2}+x\lambda\frac{\mathrm{d}J_n(z)}{\mathrm{d}z}+\left(\lambda^2x^2-n^2\right)J_n\left(z\right)=$$

$$z^2\frac{\mathrm{d}^2 J_n(z)}{\mathrm{d}z^2}+z\frac{\mathrm{d}J_n(z)}{\mathrm{d}z}+\left(z^2-n^2\right)J_n\left(z\right)=0.$$

The second line is just the regular Bessel's equation. Therefore we have established that $J_n\left(\lambda x\right)$ is the solution of (4.47) .

We have shown in the last chapter that (4.47) written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(\lambda^2x-\frac{n^2}{x}\right)y=0 \tag{4.48}$$

together with a boundary condition at $x=c$ constitute a Sturm–Liouville problem in the interval of $0\le x\le c$. The general boundary condition is of the form

$$Ay\left(c\right)+By'\left(c\right)=0,$$

where A and B are two constants. If $B=0$, it is known as the Dirichlet condition. If $A=0$, it is known as the Neumann condition.

The problem also requires that the solution be regular (bounded) at $x=0$. This precludes Neumann function as a solution.

This means that only those values of λ that satisfy the equation

$$AJ_n(\lambda c)-B\left.\frac{\mathrm{d}J_n(\lambda x)}{\mathrm{d}x}\right|_{x=c}=0 \tag{4.49}$$

are acceptable. Since Bessel functions have oscillatory character, there are infinite number of λ that satisfy this equation. These values of λ are the eigenvalues of the problem. For example, if $B=0,\ n=0,\ c=2$, then

$$J_0(2\lambda)=0.$$

The jth root of this equation, labeled λ_{0j}, can be found from the table of zeros of $J_n(x)$ (Table 4.1) as

$$\lambda_{01}=\frac{2.4048}{2}=1.2924,\quad \lambda_{02}=\frac{5.5201}{2}=2.7601,\quad \text{etc.}$$

The zeros of $J'(x)$ are also tabulated. So if $A=0$, λ_{nj} can also be read from the table. In general, if both A and B are nonzero, then λ_{nj} has to be numerically calculated.

4.4.2 Orthogonality of Bessel Functions

Corresponding to the set of eigenvalues $\{\lambda_{nj}\}$, the eigenfunctions are $\{J_n\left(\lambda_{nj}x\right)\}$. These eigenfunctions form a complete set and they are orthogonal to each other with respect to the weight function x, that is

$$\int_0^c J_n(\lambda_{ni}x)\, J_n(\lambda_{nk}x)\, x\, \mathrm{d}x = 0 \quad \text{if} \quad \lambda_{ni} \neq \lambda_{nk}.$$

Therefore any well-behaved function $f(x)$ in the interval $0 \leq x \leq c$ can be expanded into a Fourier–Bessel series

$$f(x) = \sum_{j=1}^{\infty} a_j J_n(\lambda_{nj}x),$$

where

$$a_j = \frac{1}{\int_0^c [J_n(\lambda_{nj}x)]^2\, x\, \mathrm{d}x} \int_0^c f(x) J_n(\lambda_{nj}x) x\, \mathrm{d}x.$$

In Sect. 4.4.3, we will evaluate the normalization integral

$$\beta_{nj}^2 = \int_0^c [J_n(\lambda_{nj}x)]^2\, x\, \mathrm{d}x$$

under various boundary conditions.

4.4.3 Normalization of Bessel Functions

One way to find the value of the normalization integral β_{nj}^2 is to substitute $y = J_n(\lambda x)$ into (4.48) and multiply it by $2\mathrm{x}(\mathrm{d}/\mathrm{d}\mathrm{x})J_n(\lambda x)$:

$$2x\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x)\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x)\right) + (\lambda^2 x - \frac{n^2}{x})J_n(\lambda x)\right\} = 0.$$

It is not difficult to see that this equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x)\right)^2 + (\lambda^2 x^2 - n^2)J_n(\lambda x)2\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x) = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x)\right)^2 + (\lambda^2 x^2 - n^2)\frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)^2 = 0.$$

Furthermore,

$$\lambda^2 x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(J_n(\lambda x)\right)^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left[\lambda^2 x^2 \left(J_n(\lambda x)\right)^2\right] - 2\lambda^2 x \left(J_n(\lambda x)\right)^2.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x)\right)^2 + \lambda^2 x^2\left(J_n(\lambda x)\right)^2 - n^2\left(J_n(\lambda x)\right)^2\right] = 2\lambda^2 x\left(J_n(\lambda x)\right)^2. \tag{4.50}$$

From (4.38) , we have

$$x\frac{\mathrm{d}}{\mathrm{d}x}J_n(x) = nJ_n(x) - xJ_{n+1}(x).$$

Replace x by λx leading to

$$x\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x) = nJ_n(\lambda x) - \lambda xJ_{n+1}(\lambda x). \tag{4.51}$$

Square it,

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}J_n(\lambda x)\right)^2 = n^2J_n^2(\lambda x) - 2n\lambda xJ_n(\lambda x)J_{n+1}(\lambda x) + \lambda^2x^2J_{n+1}^2(\lambda x),$$

and substitute it into (4.50)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\lambda^2x^2J_{n+1}^2(\lambda x) + \lambda^2x^2J_n^2(\lambda x) - 2n\lambda xJ_n(\lambda x)J_{n+1}(\lambda x)\right] = 2\lambda^2x\left(J_n(\lambda x)\right)^2.$$

By integrating with respect to x, we obtain

$$[\lambda^2x^2J_{n+1}^2(\lambda x)+\lambda^2x^2J_n^2(\lambda x)-2n\lambda xJ_n(\lambda x)J_{n+1}(\lambda x)]_0^c=2\lambda^2\int_0^c x(J_n(\lambda x))^2\mathrm{d}x,$$

or

$$\int_0^c x\left(J_n(\lambda x)\right)^2\mathrm{d}x = \frac{c^2}{2}\left[J_{n+1}^2(\lambda c) + J_n^2(\lambda c)\right] - \frac{nc}{\lambda}J_n(\lambda c)J_{n+1}(\lambda c). \tag{4.52}$$

Now if the boundary condition is such that $B = 0$ in (4.49) , then

$$J_n\left(\lambda_{nj}c\right) = 0.$$

In this case, the normalization constant is given by

$$\beta_{nj}^2 = \int_0^c x\left(J_n(\lambda_{nj}x)\right)^2\mathrm{d}x = \frac{1}{2}c^2J_{n+1}^2(\lambda_{nj}c), \qquad B = 0. \tag{4.53}$$

If $B \neq 0$, then (4.49) can be written as

$$\frac{A}{B}J_n\left(\lambda c\right) = \left.\frac{\mathrm{d}J_n\left(\lambda x\right)}{\mathrm{d}x}\right|_{x=c}$$

which, according to (4.51) , is given by

$$\left.\frac{\mathrm{d}J_n\left(\lambda x\right)}{\mathrm{d}x}\right|_{x=c} = \frac{n}{c}J_n\left(\lambda c\right) - \lambda J_{n+1}(\lambda c).$$

It follows that

$$J_{n+1}(\lambda c) = \frac{1}{\lambda}\left(\frac{n}{c} - \frac{A}{B}\right) J_n(\lambda c). \tag{4.54}$$

Putting it into (4.52) , we have

$$\begin{aligned}\int_0^c x\left(J_n(\lambda x)\right)^2 \mathrm{d}x &= \frac{c^2}{2} J_n^2(\lambda c)\left[1 + \frac{1}{\lambda^2}\left(\frac{n}{c} - \frac{A}{B}\right)^2\right] - \frac{nc}{\lambda^2}\left(\frac{n}{c} - \frac{A}{B}\right) J_n^2(\lambda c) \\ &= \frac{1}{2\lambda^2} J_n^2(\lambda c)\left[(\lambda c)^2 - n^2 + \left(\frac{Ac}{B}\right)^2\right].\end{aligned}$$

Therefore, in the case $\lambda = \lambda j$, where λ_j is the root of (4.54) ,

$$\beta_{nj}^2 = \int_0^c x\left(J_n(\lambda_j x)\right)^2 \mathrm{d}x = \frac{1}{2\lambda_j^2} J_n^2(\lambda_j c)\left[(\lambda_j c)^2 - n^2 + \left(\frac{Ac}{B}\right)^2\right], \quad B \neq 0. \tag{4.55}$$

4.5 Other Kinds of Bessel Functions

4.5.1 Modified Bessel Functions

Besides the Bessel equation of order n, one also encounter the modified Bessel equation

$$x^2 y'' + xy' - \left(x^2 + n^2\right) y = 0.$$

The only difference between this equation and the Bessel equation is the minus sign in front of the second x^2 term. If we change x to $\mathrm{i}x$, this equation reduces to the Bessel equation. Therefore $J_n(\mathrm{i}x)$ and $N_n(\mathrm{i}x)$ are solutions of this equation.

It is customary to define

$$I_n(x) = \mathrm{i}^{-n} J_n(\mathrm{i}x)$$

as the modified Bessel function of the first kind. Since

$$J_n(\mathrm{i}x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+n+1)}\left(\frac{\mathrm{i}x}{2}\right)^{2k+n},$$

the modified Bessel function $I_n(x)$

$$I_n(x) = \frac{1}{\mathrm{i}^n} J_n(\mathrm{i}x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+n+1)}\left(\frac{x}{2}\right)^{2k+n}$$

is a real and monotonically increasing function.

The second modified Bessel functions of the second kind are usually defined as $K_n(x)$

$$K_n(x) = \frac{\pi}{2}\mathrm{i}^{n+1}H_n^{(1)}(\mathrm{i}x) = \frac{\pi}{2}\mathrm{i}^{n+1}\left[J_n(\mathrm{i}x) + \mathrm{i}N_n(\mathrm{i}x)\right].$$

These functions do not have multiple zeros and are not orthogonal functions. They should be compared with $\sinh(x) = -i\sin(\mathrm{i}x)$ and $\cosh(x) = \cos(\mathrm{i}x)$. Because of this analogy, $I_n(x)$ and $K_n(x)$ are also called hyperbolic Bessel functions. The i factors are adjusted to make them real for real x. The first three $I_n(x)$ and $K_n(x)$ are shown in Fig. 4.4. Note that the first modified Bessel functions I_n are well behaved at the origin but diverge at infinity, the second modified Bessel functions K_n diverge at the origin but well behaved at infinity.

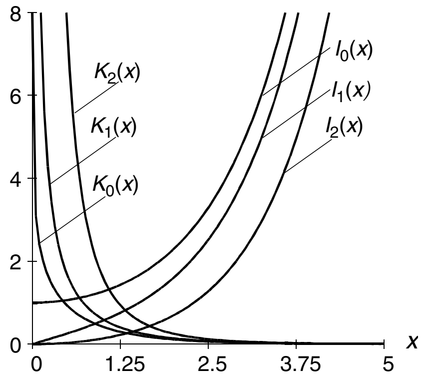

Fig. 4.4. Modified Bessel functions. The functions $I_n(x)$ diverge as $x \to \infty$, and the functions $K_n(x)$ diverge at the origin

4.5.2 Spherical Bessel Functions

The equation

$$x^2y'' + 2xy' + [x^2 - l(l+1)]y = 0 \tag{4.56}$$

with integer l, arises as the radial part of the wave equation in spherical coordinates. It is called spherical Bessel's equation because it can be transformed into a Bessel equation by a change of variables. Let

$$y = \frac{1}{(x)^{1/2}}z(x).$$

So

$$y' = \frac{z'}{x^{1/2}} - \frac{1}{2}\frac{z}{x^{3/2}},$$

$$y'' = \frac{z''}{x^{1/2}} - \frac{z'}{x^{3/2}} + \frac{3}{4}\frac{z}{x^{5/2}}.$$

Substituting them into (4.56) and multiplying it by $x^{1/2}$, we have

$$x^2[z'' - \frac{1}{x}z' + \frac{3}{4}\frac{1}{x^2}z] + 2x[z' - \frac{1}{2}\frac{1}{x}z] + [x^2 - l(l+1)]z = 0$$

or

$$x^2 z'' + xz' + [x^2 - l(l+1) - \frac{1}{4}]z = 0. \tag{4.57}$$

Since

$$l(l+1) + \frac{1}{4} = (l + \frac{1}{2})^2,$$

clearly (4.57) is a Bessel equation of the order $l + 1/2$, therefore

$$z(x) = C_1 J_{l+1/2}(x) + C_2 J_{-(l+1/2)}(x),$$

and

$$y(x) = C_1\sqrt{\frac{1}{x}}J_{l+1/2}(x) + C_2\sqrt{\frac{1}{x}}J_{-(l+1/2)}(x).$$

The two linearly independent solutions are known as spherical Bessel function $j_l(x)$ and spherical Neumann function $n_l(x)$. They are, respectively, defined as

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}(x),$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}}N_{l+1/2}(x).$$

Since

$$N_{l+1/2}(x) = \frac{\cos[(l+1/2)\pi]\,J_{l+1/2}(x) - J_{-(l+1/2)}(x)}{\sin[(l+1/2)\pi]} = (-1)^{l+1}\,J_{-(l+1/2)}(x),$$

so $n_l(x)$ can also be written as

$$n_l(x) = (-1)^{l+1}\sqrt{\frac{\pi}{2x}}J_{-(l+1/2)}(x).$$

The spherical Hankel functions are defined as

$$h_l^{(1)}(x) = j_l(x) + \mathrm{i}n_l(x),$$
$$h_l^{(2)}(x) = j_l(x) - \mathrm{i}n_l(x).$$

All these functions can be expressed in closed forms of elementary functions. In (4.27) and (4.28) , we have showed that

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-(1/2)}(x)=\sqrt{\frac{2}{\pi x}}\cos x.$$

With the recurrence relation (4.34)

$$J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x),$$

we can generate all Bessel functions of half-integer orders. For example, with $n=1/2$, we have

$$J_{3/2}(x)=\frac{1}{x}J_{1/2}(x)-J_{-1/2}(x)=\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x-\sqrt{\frac{2}{\pi x}}\cos x.$$

With $n=-1/2$, we have

$$J_{1/2}(x)=\frac{-1}{x}J_{-1/2}(x)-J_{-3/2}(x),$$

or

$$J_{-3/2}(x)=\frac{-1}{x}J_{-1/2}(x)-J_{1/2}(x)=-\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x-\sqrt{\frac{2}{\pi x}}\sin x.$$

Therefore,

$$j_0(x)=\sqrt{\frac{\pi}{2x}}J_{1/2}(x)=\frac{1}{x}\sin x,$$
$$j_1(x)=\sqrt{\frac{\pi}{2x}}J_{3/2}(x)=\frac{1}{x^2}\sin x-\frac{1}{x}\cos x,$$

and

$$n_0(x)=-\sqrt{\frac{\pi}{2x}}J_{-1/2}(x)=-\frac{1}{x}\cos x,$$
$$n_1(x)=\sqrt{\frac{\pi}{2x}}J_{-3/2}(x)=-\frac{1}{x^2}\cos x-\frac{1}{x}\sin x.$$

The higher order of spherical Bessel functions can be generated by the recurrence relation

$$f_{l+1}=\frac{2l+1}{x}f_l-f_{l-1}, \tag{4.58}$$

where f_l can be j_l, n_l, $h_l^{(1)}$, or $h_l^{(2)}$. This recurrence relation is obtained by multiplying (4.34) by $\sqrt{\pi/(2x)}$ and set $n=l+1/2$.

The asymptotic expressions of j_l, n_l, $h_l^{(1)}$, and $h_l^{(2)}$ are of considerable interests. In the asymptotic region, the $1/x$ term dominates all other terms. Since $j_0(x) = (1/x)\sin x$, and

$$\lim_{x\to\infty} j_1(x) \to -\frac{1}{x}\cos x = \frac{1}{x}\sin\left(x - \frac{\pi}{2}\right),$$

so for $l = 0$ and $l = 1$, asymptotically we can write

$$j_l(x) \to \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right), \qquad l = 0, 1.$$

For higher order of l, we see from the recurrence relation (4.58) that

$$\begin{aligned}\lim_{x\to\infty} j_{l+1}(x) &= \lim_{x\to\infty}\left[-j_{l-1}(x)\right] \to -\frac{1}{x}\sin\left(x - \frac{(l-1)\pi}{2}\right)\\ &= \frac{1}{x}\sin\left(x - \frac{(l-1)\pi}{2} - \pi\right) = \frac{1}{x}\sin\left(x - \frac{(l+1)\pi}{2}\right).\end{aligned}$$

Therefore for any integer l, asymptotically

$$j_l(x) \to \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right).$$

Similarly, we can show that asymptotically

$$n_l(x) \to -\frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right),$$

for all l. Furthermore,

$$h_l^{(1)}(x) \to \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) + \mathrm{i}\frac{-1}{x}\cos\left(x - \frac{l\pi}{2}\right) = \frac{1}{x}\mathrm{e}^{\mathrm{i}(x-((l+1)/2)\pi)},$$

$$h_l^{(2)}(x) \to \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right) - \mathrm{i}\frac{-1}{x}\cos\left(x - \frac{l\pi}{2}\right) = \frac{1}{x}\mathrm{e}^{-\mathrm{i}(x-((l+1)/2)\pi)}.$$

These asymptotic expressions are very useful for scattering problems.

We should mention that from $j_0(x)$, $j_1(x)$, and (4.58), one can show with mathematical induction that

$$j_l(x) = x^l\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\sin x}{x}.$$

Similarly,

$$n_l(x) = x^l\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l \frac{\cos x}{x}.$$

These are known as Rayleigh's formulas.

4.6 Legendre Functions

The differential equation

$$\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}y-2x\frac{\mathrm{d}}{\mathrm{d}x}y+\lambda y=0 \tag{4.59}$$

is known as Legendre's equation, after the French mathematician Adrien-Marie Legendre (1785–1833). This equation occurs in numerous physical applications. The solution of this equation is called Legendre function which is one of the most important special functions.

Legendre function ariss in the solution of partial differential equations when the Laplacian is expressed in the spherical coordinates. In that usage, the variable x is the cosine of the polar angle $(x=\cos\theta)$. Therefore x is limited in the range of $-1\leq x\leq 1$.

4.6.1 Series Solution of Legendre Equation

We start with the series solution of (4.59)

$$y(x)=x^k\sum_{n=0}a_nx^n,\quad (a_0\neq 0). \tag{4.60}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sum_{n=0}a_n(n+k)x^{n+k-1},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\sum_{n=0}a_n(n+k)(n+k-1)x^{n+k-2}.$$

Put them into (4.59) , we have

$$(1-x^2)\sum_{n=0}a_n(n+k)(n+k-1)x^{n+k-2}-2x\sum_{n=0}a_n(n+k)x^{n+k-1}$$
$$+\lambda\sum_{n=0}a_nx^{n+k}=0,$$

or

$$\sum_{n=0}a_n(n+k)(n+k-1)x^{n+k-2}-\sum_{n=0}a_n(n+k)(n+k-1)x^{n+k}$$
$$-2\sum_{n=0}a_n(n+k)x^{n+k}+\lambda\sum_{n=0}a_nx^{n+k}=0.$$

Combining the last three summations

$$\sum_{n=0}a_n(n+k)(n+k-1)x^{n+k-2}-\sum_{n=0}a_n[(n+k)(n+k-1)+2(n+k)-\lambda]x^{n+k}=0.$$

Explicitly writing out the first two terms in the first summation, amd using

$$(n+k)(n+k-1)+2(n+k)-\lambda=(n+k)(n+k+1)-\lambda,$$

we have

$$\begin{aligned} a_0k(k-1)x^{k-2}+a_1(k+1)kx^{k-1}+\sum_{n=2} a_n(n+k)(n+k-1)x^{n+k-2} \\ -\sum_{n=0} a_n[(n+k)(n+k+1)-\lambda]x^{n+k}=0. \end{aligned} \quad (4.61)$$

Shifting the index by 2 units, we can write the summation starting with $n=2$ as

$$\sum_{n=2} a_n(n+k)(n+k-1)x^{n+k-2}=\sum_{n=0} a_{n+2}(n+k+2)(n+k+1)x^{n+k}.$$

Thus (4.61) becomes

$$\begin{gathered} a_0k(k-1)x^{k-2}+a_1(k+1)kx^{k-1}+ \\ \sum_{n=0}\left\{a_{n+2}(n+k+2)(n+k+1)-a_n[(n+k)(n+k+1)-\lambda]\right\}x^{n+k}=0, \end{gathered}$$

which implies

$$a_0k(k-1)=0, \quad (4.62)$$

$$a_1(k+1)k=0, \quad (4.63)$$

$$a_{n+2}(n+k+2)(n+k+1)-a_n[(n+k)(n+k+1)-\lambda]=0. \quad (4.64)$$

Since $a_0\neq 0,$ from (4.62) we have $k=0$ or $k=1$. If $k=1$, then from (4.63) a_1 must be 0. If $k=0$, a_1 can either be 0, or not equal to 0. So we have three cases:

$$\begin{aligned} &\text{case 1}: \quad k=0 \ \text{ and } \ a_1=0, \\ &\text{case 2}: \quad k=1 \ \text{ and } \ a_1=0, \\ &\text{case 3}: \quad k=0 \ \text{ and } \ a_1\neq 0. \end{aligned}$$

We will first take up case 1. Since $k=0$, (4.64) becomes

$$a_{n+2}(n+2)(n+1)-a_n[n^2+n-\lambda]=0,$$

or

$$a_{n+2}=\frac{n(n+1)-\lambda}{(n+2)(n+1)}a_n. \quad (4.65)$$

Since $a_1=0,$ from this equation, it is seen that $a_3=a_5=\cdots=0$. Therefore we have only even terms left. Let us write this equation as

$$a_{n+2} = f(n)a_n,$$

where

$$f(n) = \frac{n(n+1) - \lambda}{(n+2)(n+1)}. \tag{4.66}$$

$$y(x) = a_0(1 + f(0)x^2 + f(2)f(0)x^4 + f(4)f(2)f(0)x^6 + \cdots), \tag{4.67}$$

which is an infinite series. We must now determine if this is a converging series. For this purpose, it is helpful to write the series in the form of

$$y(x) = \sum_{j=0} b_j (x^2)^j,$$

where

$$\begin{aligned} b_0 &= a_0, \\ b_1 &= a_0 f(0), \\ b_j &= a_0 f(2j-2) f(2j-4) \cdots f(0) \\ b_{j+1} &= a_0 f(2j) f(2j-2) \cdots f(0). \end{aligned}$$

For the ratio test, we examine

$$R = \lim_{j\to\infty} \frac{b_{j+1}(x^2)^{j+1}}{b_j (x^2)^j} = \lim_{j\to\infty} f(2j)x^2.$$

By (4.66) ,

$$f(2j) = \frac{2j(2j+1) - \lambda}{(2j+2)(2j+1)}, \tag{4.68}$$

so

$$\lim_{j\to\infty} f(2j)x^2 = x^2.$$

Therefore for $-1 < x < 1$, the series converges. However, for $x = 1$, the ordinary ratio test gives no information. We must go to the second-order ratio test (also known as Gauss's test or Raabe's test) which says with

$$R \to 1 - \frac{s}{j}, \tag{4.69}$$

if $s > 1$, the series converges, if $s \leq 1$, the series diverges. This test is based on comparison with the Riemann Zeta function $\xi(s)$, defined as

$$\xi(s) = \sum_{j=1}^{\infty} \frac{1}{j^s}.$$

Since

$$\int_1^{\infty} x^{-s}\, \mathrm{d}x = \begin{cases} \left.\dfrac{x^{-s+1}}{-s+1}\right|_1^{\infty} & s \neq 1 \\ \left.\ln x\right|_1^{\infty} & s = 1 \end{cases},$$

The integral is divergent for $s \leq 1$ and convergent for $s > 1$. Thus, according to integral test, $\xi(s)$ is divergent for $s \leq 1$ and convergent for $s > 1$. For $\xi(s)$,

$$R = \lim_{j\to\infty} \frac{1/(j+1)^s}{1/j^s} = \lim_{j\to\infty} \frac{j^s}{(j+1)^s} = \lim_{j\to\infty} \left(\frac{j+1}{j}\right)^{-s} \to 1 - \frac{s}{j},$$

which is in the form of (4.69). It can be shown that the same convergence criteria can be applied to any series that asymptotically behaves in the same way as the Riemann Zeta function $\xi(s)$ (see, for example, John M.H. Olmsted, Advance Calculus, Prentice Hall, 1961).

Now in (4.68)

$$R = \lim_{j\to\infty} f(2j) = \frac{j}{j+1} = \frac{j}{j\,(1+1/j)} = 1 - \frac{1}{j} \cdots .$$

Since $s = 1$, the series will diverge. Therefore for an arbitrary λ, the solution will not be bounded at $x = \pm 1$. However, if

$$\lambda = l(l+1), \quad l = \text{even integer},$$

then the series will terminate and become a polynomial and we will not have the convergence problem. It is clear from (4.65)

$$a_{n+2} = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)} a_n, \tag{4.70}$$

that $a_{l+2} = 0$. It follows that $a_{l+4} = a_{l+6} = \cdots = 0$. For example, if $l = 0$, then $a_2 = a_4 = \cdots = 0$. The solution, according to (4.60), is simply $y = a_0$. For any l, the solution can be systematically generated from (4.66) and (4.67). Since

$$f(n) = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)}, \tag{4.71}$$

$$y(x) = a_0(1 + f(0)x^2 + f(2)f(0)x^4 + f(4)f(2)f(0)x^6 + \cdots),$$

thus

$$l = 0: \; f(0) = 0, \quad y = a_0. \tag{4.72}$$

$$l = 2: \; f(0) = -3; \; f(2) = 0, \quad y = a_0(1 - 3x^2). \tag{4.73}$$

$$l = 4: \; f(0) = -10; \; f(2) = -\frac{7}{6}; \; f(4) = 0, \quad y = a_0(1 - 10x^2 + \frac{70}{6}x^4).$$

For case 2, $k = 1$, $a_1 = 0$, (4.64) becomes

$$a_{n+2}(n+3)(n+2) - a_n[(n+1)(n+2) - \lambda] = 0,$$

or

$$a_{n+2} = \frac{(n+1)(n+2) - \lambda}{(n+3)(n+2)} a_n. \tag{4.74}$$

Again because of $a_1 = 0$ and this recurrence relation, all a_n with odd n will be zero. The solution is then given by

$$\begin{aligned} y &= x(a_0 + a_2 x^2 + a_4 x^4 + \cdots) \\ &= a_0 x + a_2 x^3 + a_4 x^5 + \cdots , \end{aligned} \tag{4.75}$$

which is a series with odd powers of x. With a similar argument as in case 1, one can show this series diverges for $x = 1$, unless

$$\lambda = l(l+1) \quad l = \text{odd integer}.$$

In that case,

$$a_{n+2} = \frac{(n+1)(n+2) - l(l+1)}{(n+3)(n+2)} a_n = f(n+1) a_n,$$

and

$$y = a_0[x + f(1)x^3 + f(3)f(1)x^5 + f(5)f(3)f(1)x^7 + \cdots].$$

It follows

$$l = 1, \quad f(1) = 0, \quad y = a_0 x, \tag{4.76}$$

$$l = 3, \quad f(1) = -\frac{5}{3}; \; f(3) = 0, \quad y = a_0(x - \frac{5}{3}x^3). \tag{4.77}$$

For case 3, $k = 0, \; a_1 \neq 0$. Since a_0 is not equal to zero, it can be easily shown that the solution is the sum of two infinite series, one with even powers of x, and the other with odd powers of x. In this case, the series solution diverges at $x = \pm 1$.

4.6.2 Legendre Polynomials

In the last section we found that the solution of the Legendre Equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x} y(x) + l(l+1)y(x) = 0$$

is given by a polynomial of order l. Furthermore the polynomial contains only even orders of x if l is even, and only odd orders of x if l is odd. These solutions are determined up to a multiplicative constant a_0. Now, by convention, if a_0 is chosen in such a way that $y(1) = 1$, then these polynomials are known as Legendre polynomials $P_l(x)$. For example, from (4.73)

$$l = 2, \quad y_2(x) = a_0(1 - 3x^2),$$

requiring

$$y_2(1) = a_0(-2) = 1$$

we have to choose

$$a_0 = -\frac{1}{2}.$$

With this choice $y_2(x)$ is known as $P_2(x)$, that is

$$y_2(x) = -\frac{1}{2}(1 - 3x^2) = \frac{1}{2}(3x^2 - 1) = P_2(x).$$

Similarly, from (4.77)

$$l = 3, \quad y_3(x) = a_0(x - \frac{5}{3}x^3),$$

with

$$a_0 = -\frac{3}{2}, \quad y_3(1) = 1.$$

Therefore

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

The first few Legendre polynomials are shown in Fig. 4.5 and they are listed as follows

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \qquad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

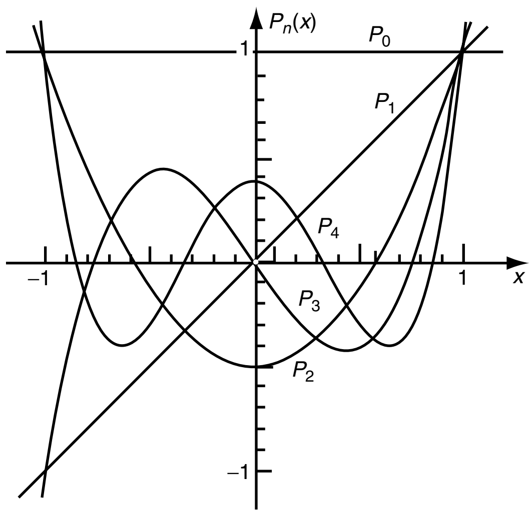

Fig. 4.5. Legendre polynomials

Clearly

$$P_l(1) = 1, \quad \text{for all } l.$$

If l is even, then $P_l(x)$ is an even function, symmetric with respect to zero, and if l is odd, then $P_l(x)$ is an odd function, antisymmetric with respect to zero. That is

$$P_l(-x) = (-1)^l P_l(x),$$

and in particular

$$P_l(-1) = (-1)^l.$$

4.6.3 Legendre Functions of the Second Kind

By choosing $\lambda = l(l+1)$, we force one of the two infinite series to become a polynomial. The second solution is still an infinite series. This infinite series can also be expressed in a closed form, although it diverges at $x = \pm 1$.

Another way to obtain the second solution is the so called "method of reduction of order." With $P_l(x)$ being a solution of the Legendre equation of order l, we write the second solution as

$$y_2(x) = u_l(x) P_l(x).$$

Requiring $y_2(x)$ to satisfy the Legendre equation of the same order, we can determine $u_l(x)$. Substituting $y_2(x)$ into the Legendre equation, we have

$$\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[u_l(x)P_l(x)\right] - 2x\frac{\mathrm{d}}{\mathrm{d}x}\left[u_l(x)P_l(x)\right] + l(l+1)\left[u_l(x)P_l(x)\right] = 0.$$

This equation can be written as

$$\begin{aligned}&\left(1-x^2\right)P_l(x)u_l''(x) - \left[2xP_l(x) - 2(1-x^2)P_l'(x)\right]u_l'(x)\\ &+\left[\left(1-x^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_l(x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}P_l(x) + l(l+1)P_l(x)\right]u_l(x) = 0.\end{aligned}$$

Since $P_l(x)$ is a solution, the term in the last bracket is zero. Therefore

$$\left(1-x^2\right)P_l(x)u_l''(x) - \left[2xP_l(x) - 2(1-x^2)P_l'(x)\right]u_l'(x) = 0,$$

or

$$u_l''(x) = \left[\frac{2x}{(1-x^2)} - \frac{2}{P_l(x)}P_l'(x)\right]u_l'(x).$$

Since $u_l''(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}u_l'(x)$, this equation can be written as

$$\frac{\mathrm{d}u_l'(x)}{u_l'(x)} = \left[\frac{2x}{(1-x^2)} - \frac{2}{P_l(x)}P_l'(x)\right]\mathrm{d}x.$$

Integrating both sides, we have

$$\ln u_l'(x) = -\ln\left(1 - x^2\right) - 2\ln P_l(x) + C,$$

or

$$u_l'(x) = \frac{c}{(1 - x^2)\, P_l^2\,(x)}.$$

Thus

$$u_l(x) = \int^x \frac{c\ \mathrm{d}x'}{(1 - x'^2)\, P_l^2\,(x')}.$$

The arbitrary constant c contributes nothing to the property of the function. With c chosen to be 1, the Legendre function of the second kind is designated as $Q_l(x)$.

With $P_0(x) = 1$,

$$u_0(x) = \int^x \frac{\mathrm{d}x'}{(1 - x'^2)} = \frac{1}{2}\int^x \left(\frac{\mathrm{d}x'}{1 + x'} + \frac{\mathrm{d}x'}{1 - x'}\right) = \frac{1}{2}\ln\frac{1 + x}{1 - x}.$$

The additive constant of this integral is chosen to be zero for convenience. With this choice

$$Q_0(x) = u_0(x)P_0(x) = \frac{1}{2}\ln\frac{1 + x}{1 - x}. \tag{4.78}$$

Similarly, with $P_1(x) = x$,

$$u_1(x) = \int^x \frac{\mathrm{d}x'}{(1 - x'^2)\, x'^2} = \int^x \left(\frac{\mathrm{d}x'}{1 - x'^2} + \frac{\mathrm{d}x'}{x'^2}\right) = \frac{1}{2}\ln\frac{1 + x}{1 - x} - \frac{1}{x},$$

and

$$Q_1(x) = u_1(x)P_1(x) = \frac{x}{2}\ln\frac{1 + x}{1 - x} - 1 = P_1(x)Q_0(x) - 1.$$

Higher order $Q_l(x)$ can be obtained with the recurrence relation

$$lQ_l(x) = (2l - 1)\, xQ_{l-1}(x) - (l - 1)\, Q_{l-2}(x),$$

which is also satisfied by P_l, as we shall show in Sect. 4.7. The first few $Q_l(x)$ are as follows:

$$\begin{aligned}
&Q_0 = \frac{1}{2}\ln\frac{1 + x}{1 - x}, \quad Q_1 = P_1Q_0 - 1, \quad Q_2 = P_2Q_0 - \frac{3}{2}x, \\
&Q_3 = P_3Q_0 - \frac{5}{2}x^2, \quad Q_4 = P_4Q_0 - \frac{35}{8}x^3 + \frac{55}{24}x, \\
&Q_5 = P_5Q_0 - \frac{63}{8}x^4 + \frac{49}{8}x^2 - \frac{8}{15}.
\end{aligned}$$

It can be shown that these expressions are the infinite series solutions obtained from the Frobenius method. For example, the coefficients in the second solution of (4.75)

$$y = a_0x + a_2x^3 + a_4x^5 + \cdots,$$

are given in (4.74)

$$a_{n+2} = \frac{(n+1)(n+2) - \lambda}{(n+3)(n+2)} a_n.$$

With $\lambda = 0$,

$$a_{n+2} = \frac{(n+1)}{(n+3)} a_n.$$

Therefore

$$y = a_0 \left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots \right) = \frac{a_0}{2} \ln \frac{1+x}{1-x}.$$

This expression is identical to $a_0 Q_0(x)$.

Therefore the general solution of the Legendre equation for integer order l is

$$y(x) = c_1 P_l(x) + c_2 Q_l(x),$$

where $P_l(x)$ is a polynomial which converges for all x, and $Q_l(x)$ diverges at $x = \pm 1$.

4.7 Properties of Legendre Polynomials

4.7.1 Rodrigues' Formula

The Legendre polynomials can be summarized by the Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} (x^2 - 1)^l. \tag{4.79}$$

Clearly this formula will give a polynomial of order l. We will prove this polynomial is the Legendre polynomial. There are two parts of the proof. First we will show that $\mathrm{d}^l/\mathrm{d}x^l (x^2-1)^l$ satisfies the Legendre equation. Then we will show that $P_l(1)$ with $P_l(x)$ given by (4.79) is equal to 1.

Before we start the proof, let us first recall the Leibniz's rule for differentiating products:

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} A(x)B(x) = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{\mathrm{d}^{m-k}}{\mathrm{d}x^{m-k}} A(x) \frac{\mathrm{d}^k}{\mathrm{d}x^k} B(x)$$

$$= \left[\frac{\mathrm{d}^m}{\mathrm{d}x^m} A(x) \right] B(x) + m \left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}} A(x) \right] \left[\frac{\mathrm{d}}{\mathrm{d}x} B(x) \right] + \cdots + A(x) \left[\frac{\mathrm{d}^m}{\mathrm{d}x^m} B(x) \right]. \tag{4.80}$$

To prove the first part of (4.79) , let

$$v = (x^2-1)^l,$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = l(x^2-1)^{l-1}2x$$

$$(x^2-1)\frac{\mathrm{d}v}{\mathrm{d}x} = l(x^2-1)^l 2x = 2lxv. \tag{4.81}$$

Differentiating the left-hand side of this equation $l+1$ times by Leibniz' rule [with $A(x) = \dfrac{\mathrm{d}v}{\mathrm{d}x}$ and $B(x) = (x^2-1)$]:

$$\frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}}(x^2-1)\frac{\mathrm{d}v}{\mathrm{d}x} = (x^2-1)\frac{\mathrm{d}^{l+2}v}{\mathrm{d}x^{l+2}} + (l+1)2x\frac{\mathrm{d}^{l+1}v}{\mathrm{d}x^{l+1}} + \frac{(l+1)l}{2!}2\frac{\mathrm{d}^l v}{\mathrm{d}x^l},$$

and differentiating the right-hand side of (4.81) $l+1$ times by Leibniz' rule [with $A(x) = v$ *and* $B(x) = 2lx$]:

$$\frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}}2lvx = 2lx\frac{\mathrm{d}^{l+1}v}{\mathrm{d}x^{l+1}} + (l+1)2l\frac{\mathrm{d}^l v}{\mathrm{d}x^l},$$

we have

$$(x^2-1)\frac{\mathrm{d}^{l+2}v}{\mathrm{d}x^{l+2}} + (l+1)2x\frac{\mathrm{d}^{l+1}v}{\mathrm{d}x^{l+1}} + \frac{(l+1)l}{2!}2\frac{\mathrm{d}^l v}{\mathrm{d}x^l} = 2lx\frac{\mathrm{d}^{l+1}v}{\mathrm{d}x^{l+1}} + (l+1)2l\frac{\mathrm{d}^l v}{\mathrm{d}x^l}.$$

Simplifying this equation, we get

$$(x^2-1)\frac{\mathrm{d}^{l+2}v}{\mathrm{d}x^{l+2}} + 2x\frac{\mathrm{d}^{l+1}v}{\mathrm{d}x^{l+1}} - l(l+1)\frac{\mathrm{d}^l v}{\mathrm{d}x^l} = 0,$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^l v}{\mathrm{d}x^l}\right]\right\} + l(l+1)\left[\frac{\mathrm{d}^l v}{\mathrm{d}x^l}\right] = 0.$$

Clearly

$$\frac{\mathrm{d}^l v}{\mathrm{d}x^l} = \frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l$$

satisfies the Legendre equation. Now if we can show

$$\left[\frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l\right]_{x=1} = 1,$$

then

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l$$

must be the Legendre polynomial.

This can be done by writing

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l = \frac{\mathrm{d}^l}{\mathrm{d}x^l}(x-1)^l(x+1)^l$$

and using the Leibniz' rule with $A(x) = (x-1)^l$ and $B(x) = (x+1)^l$

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x-1)^l(x+1)^l = \left[\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x-1)^l\right](x+1)^l + l\left[\frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}(x-1)^l\right]\left[\frac{\mathrm{d}}{\mathrm{d}x}(x+1)^l\right] + \cdots. \tag{4.82}$$

Notice

$$\frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}(x-1)^l = l!(x-1),$$

at $x = 1$, this term is equal to zero. In fact as long as the number of times $(x-1)^l$ is differentiated is less than l, the result will contain the factor $(x-1)$. At $x = 1$, the derivative is equal to zero. Therefore all terms on the right-hand side of (4.82) except the first term are equal to zero at $x = 1$, thus

$$\left[\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x-1)^l(x+1)^l\right]_{x=1} = \left\{\left[\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x-1)^l\right](x+1)^l\right\}_{x=1}.$$

Now

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x-1)^l = l!, \qquad \left[(x+1)^l\right]_{x=1} = 2^l,$$

therefore

$$\frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l\Big|_{x=1} = 1.$$

This completes our proof.

4.7.2 Generating Function of Legendre Polynomials

We will prove a very important identity

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x)z^n, \quad |z| < 1. \tag{4.83}$$

This relation has an advantage of summarizing P_n into a single function. This enables us to derive relationships between Legendre polynomials of different orders without using explicit forms. Besides its many applications in physics, it is also very useful in statistics.

To prove this relationship, we will make use of the fact that the power series

$$f = \frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} F_n(x)z^n \tag{4.84}$$

exists. Then we will show that the coefficient $F_n(x)$ satisfies the Legendre equation, and that $F_n(1) = 1$. That will enable us to identify $F_n(x)$ as $P_n(x)$. Now

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(1 - 2xz + z^2)^{-3/2}(-2z) = zf^3, \tag{4.85}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2}(1 - 2xz + z^2)^{-3/2}(-2x + 2z) = (x - z)f^3, \tag{4.86}$$

$$f = \frac{f^3}{f^2} = f^3(1 - 2xz + z^2) = (1 - x^2)f^3 + (x - z)^2 f^3.$$

It follows from the last equation

$$(1 - x^2)f^3 = f - (x - z)^2 f^3.$$

Thus

$$\begin{aligned}(1 - x^2)\frac{\partial f}{\partial x} &= (1 - x^2)zf^3 = z\left[f - (x - z)^2 f^3\right] \\ &= z\left[f - (x - z)\frac{\partial f}{\partial z}\right] = -z\frac{\partial}{\partial z}\left[(x - z)f\right].\end{aligned}$$

Differentiate both sides of the last equation with respect to x

$$\begin{aligned}\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial f}{\partial x}\right] &= -z\frac{\partial}{\partial z}\frac{\partial}{\partial x}\left[(x - z)f\right] = -z\frac{\partial}{\partial z}\left[f + (x - z)\frac{\partial f}{\partial x}\right] \\ &= -z\frac{\partial}{\partial z}\left[f + (x - z)zf^3\right] = -z\frac{\partial}{\partial z}\left[f + z\frac{\partial f}{\partial z}\right] \\ &= -z\frac{\partial}{\partial z}\frac{\partial}{\partial z}(zf) = -z\frac{\partial^2}{\partial z^2}(zf).\end{aligned}$$

With the series expansion of f, we have

$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial}{\partial x}\sum_{n=0}^{\infty} F_n(x)z^n\right] = -z\frac{\partial^2}{\partial z^2}\left[z\sum_{n=0}^{\infty} F_n(x)z^n\right],$$

or

$$\begin{aligned}\sum_{n=0}^{\infty}\left[\frac{\partial}{\partial x}(1 - x^2)\frac{\partial}{\partial x}F_n(x)\right]z^n &= -z\sum_{n=0}^{\infty} F_n(x)\frac{\partial^2}{\partial z^2}z^{n+1} \\ = -z\sum_{n=0}^{\infty} F_n(x)(n + 1)nz^{n-1} &= -\sum_{n=0}^{\infty} F_n(x)(n + 1)nz^n.\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty}\left[\frac{\partial}{\partial x}(1 - x^2)\frac{\partial}{\partial x}F_n(x) + n(n + 1)F_n(x)\right]z^n = 0.$$

Therefore $F_n(x)$ satisfies the Legendre equation

$$\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}F_n(x)+n(n+1)F_n(x)=0.$$

Furthermore, for $x=1$

$$\frac{1}{\sqrt{1-2z+z^2}}=\sum_{n=0}^{\infty}F_n(1)z^n.$$

Since

$$\frac{1}{\sqrt{1-2z+z^2}}=\frac{1}{\sqrt{(1-z)^2}}=\frac{1}{1-z},$$

and

$$\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,$$

hence

$$F_n(1)=1.$$

Thus, (4.83) is established. The left-hand side of that equation is known as the generating function $G(x,z)$ of Legendre polynomials,

$$G(x,z)=\frac{1}{\sqrt{1-2xz+z^2}}=\sum_{n=0}^{\infty}P_n(x)z^n.$$

4.7.3 Recurrence Relations

The following recurrence formulas are very useful in handling Legendre polynomials and their derivatives.

1.
$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0. \tag{4.87}$$

Proof:

$$\frac{\partial}{\partial z}\left[\left(1-2xz+z^2\right)^{-1/2}\right]=\frac{\partial}{\partial z}\sum_{n=0}^{\infty}P_n(x)z^n,$$

$$\left(1-2xz+z^2\right)^{-3/2}(x-z)=\sum_{n=0}^{\infty}P_n(x)nz^{n-1},$$

$$(x-z)\frac{\left(1-2xz+z^2\right)^{-1/2}}{1-2xz+z^2}=\sum_{n=0}^{\infty}P_n(x)nz^{n-1},$$

$$(x-z)\sum_{n=0}^{\infty}P_n(x)z^n=\left(1-2xz+z^2\right)\sum_{n=0}^{\infty}P_n(x)nz^{n-1},$$

$$\sum_{n=0}^{\infty} xP_n(x)z^n - \sum_{n=0}^{\infty} P_n(x)z^{n+1}$$

$$= \sum_{n=0}^{\infty} nP_n(x)z^{n-1} - \sum_{n=0}^{\infty} 2xnP_n(x)z^n + \sum_{n=0}^{\infty} nP_n(x)z^{n+1}.$$

Collecting the terms, we have

$$\sum_{n=0}^{\infty} nP_n(x)z^{n-1} - \sum_{n=0}^{\infty}(2n+1)xP_n(x)z^n + \sum_{n=0}^{\infty}(n+1)P_n(x)z^{n+1} = 0.$$

Since

$$\sum_{n=0}^{\infty} nP_n(x)z^{n-1} = \sum_{n=1}^{\infty} nP_n(x)z^{n-1} = \sum_{n=0}^{\infty}(n+1)P_{n+1}(x)z^n,$$

$$\sum_{n=0}^{\infty}(n+1)P_n(x)z^{n+1} = \sum_{n=1}^{\infty} nP_{n-1}(x)z^n = \sum_{n=0}^{\infty} nP_{n-1}(x)z^n,$$

thus

$$\sum_{n=0}^{\infty}\left[(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)\right] z^n = 0.$$

Hence

$$P_{n+1}(x) = \frac{1}{n+1}\left[(2n+1)xP_n(x) - nP_{n-1}(x)\right]. \qquad (4.88)$$

This means as long as we know $P_{n-1}(x),\ P_n(x)$, we can generate $P_{n+1}(x)$. For example with $P_0(x) = 1,$ and $P_1(x) = x$, this recurrence relation with $n = 1$ will give $P_2(x) = 1/2(3x^2 - 1)$. With $P_1(x)$ and $P_2(x)$, we can generate $P_3(x)$ and so on. In other words, just with the first two orders, a Do-Loop in the computer code can automatically generate all orders of the polynomials.

2.

$$nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0, \quad \text{where } P_n'(x) = \mathrm{d}/\mathrm{d}xP_n(x). \qquad (4.89)$$

Proof: Taking the derivative of the generating function with respect to z and x, we get, respectively,

$$\frac{x-z}{\left(1-2xz+z^2\right)^{3/2}} = \sum_{n=0} nP_n\left(x\right)z^{n-1},$$

$$\frac{z}{\left(1-2xz+z^2\right)^{3/2}} = \sum_{n=0} P_n'(x)z^n.$$

Taking the ratio, we get

$$\frac{x-z}{z}=\frac{\sum_{n=0} nP_n(x)z^{n-1}}{\sum_{n=0} P_n'(x)z^n}$$

It follows that

$$(x-z)\sum_{n=0}^{\infty} P_n'(x)z^n = z\sum_{n=0}^{\infty} P_n(x)nz^{n-1}$$

$$\sum_{n=0}^{\infty} xP_n'(x)z^n - \sum_{n=0}^{\infty} P_n'(x)z^{n+1} = \sum_{n=0}^{\infty} nP_n(x)z^n. \tag{4.90}$$

Now

$$\sum_{n=0}^{\infty} xP_n'(x)z^n = \sum_{n=1}^{\infty} xP_n'(x)z^n,$$

since $P_0'(x)=0$, and

$$\sum_{n=0}^{\infty} nP_n(x)z^n = \sum_{n=1}^{\infty} nP_n(x)z^n,$$

since $n=0$ term is equal to zero. It can also easily be shown that

$$\sum_{n=0}^{\infty} P_n'(x)z^{n+1} = \sum_{n=1}^{\infty} P_{n-1}'(x)z^n.$$

Thus (4.90) can be written as

$$\sum_{n=1}^{\infty}\left[xP_n'(x)-P_{n-1}'(x)-nP_n(x)\right]z^n=0.$$

It follows

$$xP_n'(x)-P_{n-1}'(x)-nP_n(x)=0. \tag{4.91}$$

This relation together with the recurrence relation of (4.87) can generate the derivative of all orders of the Legendre polynomial based on the knowledge of $P_0=1$ and $P_1=x$.

Many other relations can be derived from these two recurrence relations. For example, from (4.88) we have

$$P_{n+1}'=\frac{2n+1}{n+1}(P_n+xP_n')-\frac{n}{n+1}P_{n-1}',$$

or

$$xP_n'=\frac{n+1}{2n+1}P_{n+1}'-P_n+\frac{n}{2n+1}P_{n-1}'.$$

Putting it into (4.91)

$$\frac{n+1}{2n+1}P'_{n+1} - P_n + \frac{n}{2n+1}P'_{n-1} - P'_{n-1} - nP_n = 0,$$

and collecting terms

$$\frac{n+1}{2n+1}P'_{n+1} - \frac{n+1}{2n+1}P'_{n-1} - (n+1)P_n = 0,$$

we obtain another very important relation

$$P_n = \frac{1}{2n+1}\left[P'_{n+1} - P'_{n-1}\right]. \tag{4.92}$$

This equation enables us to do the following integral

$$\int_x^1 P_n(\zeta)\mathrm{d}\zeta = \frac{1}{2n+1}\left[P_{n-1}(x) - P_{n+1}(x)\right].$$

4.7.4 Orthogonality and Normalization of Legendre Polynomials

As we have discussed in the last chapter, the Legendre's equation by itself in the form of

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}y\right] + \lambda y = 0$$

constitutes a Sturm–Liouville problem in the interval of $-1 \leq x \leq 1$ with a unit weight function. The requirements that solutions must be regular (bounded) at $x = \pm 1$ restrict the acceptable eigenvalues λ to $l(l+1)$, where l is an integer. The corresponding eigenfunctions are the Legendre polynomials $P_l(x)$. Since eigenfunctions of a Sturm–Liouville problem must be orthogonal to each other with respect to the weight function, therefore

$$\int_{-1}^1 P_l(x)P_{l'}(x)\mathrm{d}x = 0, \quad \text{if} \quad l \neq l'.$$

An extension to this fact is that

$$\int_{-1}^1 x^n P_l(x)\mathrm{d}x = 0, \quad \text{if} \quad n < l,$$

since x^n can be expressed as a linear combination of Legendre polynomials, the highest order of which is n.

Any well-behaved function $f(x)$ in the interval $-1 \leq x \leq 1$ can be expanded as a Fourier–Legendre series

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x),$$

where

$$a_n = \frac{1}{\int_{-1}^{1} P_n^2(x)\mathrm{d}x} \int_{-1}^{1} f(x) P_n(x) \mathrm{d}x.$$

To carry out this expansion, we need to know the value of the normalization integral

$$\beta_n^2 = \int_{-1}^{1} P_n^2(x)\mathrm{d}x.$$

There are many ways to evaluate this integral. The simplest is to make use of (4.92)

$$P_n(x) = \frac{1}{2n+1} \left[P'_{n+1}(x) - P'_{n-1}(x) \right].$$

Multiply both sides by $P_n(x)$ and integrate

$$\int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{1}{2n+1} \left[\int_{-1}^{1} P_n(x) P'_{n+1}(x)\mathrm{d}x - \int_{-1}^{1} P_n(x) P'_{n-1}(x)\mathrm{d}x \right].$$

The second integral on the right-hand side is equal to zero because $P'_{n-1}(x)$ is a polynomial of order $n-2$. The first integral on the right-hand side can be written as

$$\begin{aligned} \int_{-1}^{1} P_n(x) P'_{n+1}(x)\mathrm{d}x &= \int_{-1}^{1} P_n(x) \frac{\mathrm{d}P_{n+1}(x)}{\mathrm{d}x}\mathrm{d}x = \int_{-1}^{1} P_n(x)\mathrm{d}P_{n+1}(x) \\ &= \left[P_n(x) P_{n+1}(x) \right]_{-1}^{1} - \int_{-1}^{1} P_{n+1}(x) P'_n(x)\mathrm{d}x. \end{aligned}$$

Again, the last integral is zero because $P'_n(x)$ is a polynomial of order $n-1$. The integrated part is

$$\begin{aligned} \left[P_n(x) P_{n+1}(x) \right]_{-1}^{1} &= P_n(1) P_{n+1}(1) - P_n(-1) P_{n+1}(-1) \\ &= 1 - (-1)^n (-1)^{n+1} = 2. \end{aligned}$$

Thus

$$\beta_n^2 = \int_{-1}^{1} P_n^2(x)\mathrm{d}x = \frac{2}{2n+1}.$$

4.8 Associated Legendre Functions and Spherical Harmonics

4.8.1 Associated Legendre Polynomials

The equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \left[\lambda - \frac{m^2}{1-x^2} \right] y(x) = 0. \tag{4.93}$$

is known as the associated Legendre equation. This equation is of Sturm–Liouville form. It becomes an eigenvalue equation for λ if we require that the solution be bounded (finite) at the singular points $x = \pm 1$. If $m = 0$, it reduces to the Legendre equation and $\lambda = l(l+1)$. For m equal to nonzero integer, we could solve this equation with a series expansion as we did for the Legendre equation. However, it is more efficient and interesting to relate the $m \neq 0$ case to the $m = 0$ case.

One way of doing this to start with the regular Legendre equation

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_l(x) - 2x\frac{\mathrm{d}}{\mathrm{d}x}P_l(x) + l(l+1)P_l(x) = 0 \qquad (4.94)$$

and convert it into the associated Legendre equation by multiple differentiations. With the Leibnitz's formula (4.80) , we can write

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_l\right] = (1-x^2)\frac{\mathrm{d}^{m+2}}{\mathrm{d}x^{m+2}}P_l - 2mx\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}P_l - m(m-1)\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l,$$

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[2x\frac{\mathrm{d}}{\mathrm{d}x}P_l\right] = 2x\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}P_l + 2m\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l.$$

Therefore differentiating (4.94) m times and collecting terms, we have

$$(1-x^2)\frac{\mathrm{d}^{m+2}}{\mathrm{d}x^{m+2}}P_l - 2x(m+1)\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}P_l + [l(l+1) - m(m+1)]\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l = 0. \qquad (4.95)$$

Denoting

$$u = \frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x),$$

the above equation becomes

$$(1-x^2)u'' - 2x(m+1)u' + [l(l+1) - m(m+1)]u = 0. \qquad (4.96)$$

Now we will show that with

$$u(x) = \left(1-x^2\right)^{-m/2} y(x),$$

$y(x)$ satisfies the associated Legendre equation. Inserting $u(x)$ and its derivatives

$$u' = (1-x^2)^{-m/2}y' + \frac{m}{2}(1-x^2)^{-m/2-1}2xy = \left(y' + \frac{mx}{1-x^2}y\right)(1-x^2)^{-m/2},$$

$$u'' = \left[y'' + \frac{2mx}{1-x^2}y' + \frac{m}{1-x^2}y + \frac{m(m+2)x^2}{(1-x^2)^2}y\right](1-x^2)^{-m/2},$$

into (4.96) and simplify, we see that

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y == 0,$$

which is the associated Legendre equation. Thus

$$y(x) = (1-x^2)^{m/2} u(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x)$$

must be the solution of the associated Legendre equation. A negative value for m does not change m^2 in the equation, so this solution is also a solution for the corresponding negative m.

This solution is called the associated Legendre function, customarily designated as $P_l^m(x)$. For both positive and negative m, it is defined as

$$P_l^m(x) = (1-x^2)^{|m|/2} \frac{\mathrm{d}^{|m|}}{\mathrm{d}x^{|m|}} P_l(x). \tag{4.97}$$

Using the Rodrigues formula, it can be written as

$$P_l^m(x) = \frac{1}{2^l l!} (1-x^2)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}} (x^2-1)^l. \tag{4.98}$$

Since the highest power term in $(x^2-1)^l$ is x^{2l}, clearly $-l \leq m \leq l$.

A constant times (4.97) is also a solution, this enables some authors to define P_l^m differently by multiplying a factor $(-1)^m$. Still others use (4.98) to define $P_l^{-m}(x)$ which differs from (4.97) by a constant factor (see exercise 25). Unless otherwise stated, we shall assume that the associated Legendre polynomial is defined in (4.97). The first few polynomials are listed below. In applications most often $x = \cos\theta$, therefore $P_l^m(\cos\theta)$ are also listed side by side,

$$P_1^1(x) = (1-x^2)^{1/2} \qquad P_1^1(\cos\theta) = \sin\theta$$

$$P_2^1(x) = 3x(1-x^2)^{1/2} \qquad P_2^1(\cos\theta) = 3\cos\theta\sin\theta$$

$$P_2^2(x) = 3(1-x^2) \qquad P_2^2(\cos\theta) = 3\sin^2\theta$$

$$P_3^1(x) = (3/2)(5x^2-1)(1-x^2)^{1/2} \qquad P_3^1(\cos\theta) = (3/2)(5\cos^2\theta - 1)\sin\theta$$

$$P_3^2(x) = 15x(1-x^2) \qquad P_3^2(\cos\theta) = 15\cos\theta\sin^2\theta$$

$$P_3^3(x) = 15(1-x^2)^{3/2} \qquad P_3^3(\cos\theta) = 15\sin^3\theta.$$

4.8.2 Orthogonality and Normalization of Associated Legendre Functions

The associated Legendre equation is of the Sturm–Liouville form, therefore its eigenfunctions, the associated Legendre functions, are orthogonal to each other.

Let us recall the standard Sturm–Liouville equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[r(x)\frac{\mathrm{d}}{\mathrm{d}x}y\right] + q(x)y + \lambda w(x)y = 0.$$

If $r(a) = r(b) = 0$, then this equation, by itself, is a singular Sturm–Liouville problem in the range of $a \leq x \leq b$, provided the solution is bounded at $x = a$ and $x = b$. This means that its eigenfunctions are orthogonal with respect to the weight function $w(x)$.

With the associated Legendre equation written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_l^m(x)\right] + l(l+1)P_l^m(x) - \frac{m^2}{1-x^2}P_l^m(x) = 0,$$

we can identify

$$r(x) = x^2 - 1, \quad q(x) = -l(l+1), \quad w(x) = \frac{1}{1-x^2}, \quad \lambda = m^2,$$

and conclude that

$$\int_{-1}^{1} P_l^m(x)P_l^{m'}(x)\frac{1}{1-x^2}\mathrm{d}x = 0\ , \qquad m' \neq m. \tag{4.99}$$

On the other hand, we can identify

$$r(x) = 1 - x^2, \quad q(x) = -\frac{m^2}{1-x^2}, \quad w(x) = 1, \quad \lambda = l(l+1),$$

so we see that $P_l^m(x)$ also satisfies the orthogonality condition

$$\int_{-1}^{1} P_l^m(x)P_{l'}^m(x)\mathrm{d}x = 0, \quad l \neq l'. \tag{4.100}$$

In practical applications, (4.100) is of great importance, while (4.99) is only a mathematical curiosity.

To use $\{P_l^m(x)\}$ as a basis set in the generalized Fourier series, we must evaluate the normalization integral,

$$\int_{-1}^{1} [P_l^m(x)]^2\,\mathrm{d}x = \beta_{lm}^2.$$

By definition

$$\int_{-1}^{1} [P_l^m(x)]^2\,\mathrm{d}x = \int_{-1}^{1}(1-x^2)^m\frac{\mathrm{d}^m P_l(x)}{\mathrm{d}x^m}\frac{\mathrm{d}^m P_l(x)}{\mathrm{d}x^m}\mathrm{d}x$$
$$= \int_{-1}^{1}(1-x^2)^m\frac{\mathrm{d}^m P_l(x)}{\mathrm{d}x^m}\mathrm{d}\left[\frac{\mathrm{d}^{m-1}P_l(x)}{\mathrm{d}x^{m-1}}\right].$$

Integrating by parts we have

$$\int_{-1}^{1}(1-x^2)^m\frac{\mathrm{d}^m P_l(x)}{\mathrm{d}x^m}\mathrm{d}\left[\frac{\mathrm{d}^{m-1}P_l(x)}{\mathrm{d}x^{m-1}}\right] = (1-x^2)^m\frac{\mathrm{d}^m P_l(x)}{\mathrm{d}x^m}\frac{\mathrm{d}^{m-1}P_l(x)}{\mathrm{d}x^{m-1}}\bigg|_{-1}^{1}$$
$$-\int_{-1}^{1}\frac{\mathrm{d}^{m-1}P_l(x)}{\mathrm{d}x^{m-1}}\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^m\frac{\mathrm{d}^m P_l(x)}{\mathrm{d}x^m}\right]\mathrm{d}x.$$

The integrated part vanishes at both upper and lower limits, and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)^m\frac{\mathrm{d}^mP_l(x)}{\mathrm{d}x^m}\right]=(1-x^2)^m\frac{\mathrm{d}^{m+1}P_l(x)}{\mathrm{d}x^{m+1}}-2mx(1-x^2)^{m-1}\frac{\mathrm{d}^mP_l(x)}{\mathrm{d}x^m}.$$

Replacing m with $m-1$ in (4.95) , the equation becomes

$$(1-x^2)\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}P_l-2xm\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l+[l(l+1)-(m-1)m]\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}P_l=0.$$

Multiplying this equation by $\left(1-x^2\right)^{m-1}$, we see that

$$\begin{aligned}&(1-x^2)^m\frac{\mathrm{d}^{m+1}P_l(x)}{\mathrm{d}x^{m+1}}-2mx(1-x^2)^{m-1}\frac{\mathrm{d}^mP_l(x)}{\mathrm{d}x^m}\\&\quad=-(1-x^2)^{m-1}(l+m)(l-m+1)\frac{\mathrm{d}^{m-1}P_l(x)}{\mathrm{d}x^{m-1}}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{-1}^{1}\left[P_l^m(x)\right]^2\mathrm{d}x&=(l+m)(l-m+1)\int_{-1}^{1}\frac{\mathrm{d}^{m-1}P_l(x)}{\mathrm{d}x^{m-1}}(1-x^2)^{m-1}\frac{\mathrm{d}^{m-1}P_l(x)}{\mathrm{d}x^{m-1}}\\&=(l+m)(l-m+1)\int_{-1}^{1}\left[P_l^{m-1}(x)\right]^2\mathrm{d}x.\end{aligned}$$

Clearly this process can be continued, after m times

$$\int_{-1}^{1}\left[P_l^m(x)\right]^2\mathrm{d}x=k_{lm}\int_{-1}^{1}\left[P_l(x)\right]^2\mathrm{d}x,$$

where

$$\begin{aligned}k_{lm}&=(l+m)(l-m+1)(l+m-1)(l-m+2)\cdots(l+1)l\\&=(l+m)(l+m-1)\cdots(l+1)l\cdots(l-m+2)(l-m+1)\\&=(l+m)(l+m-1)\cdots(l-m+1)\frac{(l-m)!}{(l-m)!}=\frac{(l+m)!}{(l-m)!}.\end{aligned}$$

Since

$$\int_{-1}^{1}\left[P_l(x)\right]^2\mathrm{d}x=\frac{2}{2l+1},$$

it follows that the normalization constant β_{lm}^2 is given by

$$\beta_{lm}^2=\int_{-1}^{1}\left[P_l^m(x)\right]^2\mathrm{d}x=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}.$$

4.8.3 Spherical Harmonics

The major use of the associated Legendre polynomials is in conjunction with the spherical harmonics $Y_l^m(\theta,\varphi)$ which is the angular part of the solution of the Laplace equation expressed in the spherical coordinates,

$$Y_l^m(\theta,\varphi)=(-1)^m\sqrt{\frac{1}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\varphi}. \tag{4.101}$$

where θ is the polar angle and φ is the azimuthal angle and $m\geq 0$. For $m\leq 0$,

$$Y_l^{-|m|}(\theta,\varphi)=(-1)^m\left[Y_l^{|m|}(\theta,\varphi)\right]^*. \tag{4.102}$$

Over the surface of a sphere, $\{Y_l^m\}$ forms a complete orthonormal set,

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}Y_{l_1}^{m_1*}(\theta,\varphi)Y_{l_2}^{m_2}(\theta,\varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi=\delta_{l_1,l_2}\delta_{m_1,m_2}.$$

The orthogonality with respect to (m_1,m_2) comes from the φ-dependent part $\exp(\mathrm{i}m\varphi)$

$$\int_{\varphi=0}^{2\pi}\mathrm{e}^{-\mathrm{i}m_1\varphi}\mathrm{e}^{\mathrm{i}m_2\varphi}\,\mathrm{d}\varphi=\int_{\varphi=0}^{2\pi}\mathrm{e}^{\mathrm{i}(m_2-m_1)\varphi}\,\mathrm{d}\varphi=2\pi\delta_{m_1,m_2},$$

and the orthogonality with respect to (l_1,l_2) is due to the associated Legendre function $P_l^m(\cos\theta)$,

$$\int_{\theta=0}^{\pi}P_{l_1}^m(\cos\theta)P_{l_2}^m(\cos\theta)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l_1,l_2}.$$

The first few spherical harmonics are listed as follows

$$\begin{aligned}
Y_0^0(\theta,\varphi)&=\frac{1}{\sqrt{4\pi}} & Y_2^0(\theta,\varphi)&=\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)\\
Y_1^0(\theta,\varphi)&=\sqrt{\frac{3}{4\pi}}\cos\theta & Y_2^{\pm1}(\theta,\varphi)&=\mp\sqrt{\frac{5}{24\pi}}3\sin\theta\cos\theta\mathrm{e}^{\mathrm{i}\varphi}\\
Y_1^{\pm1}(\theta,\varphi)&=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\mathrm{e}^{\mathrm{i}\varphi} & Y_2^{\pm2}(\theta,\varphi)&=\sqrt{\frac{5}{96\pi}}3\sin^2\theta\mathrm{e}^{2\mathrm{i}\varphi}.
\end{aligned}$$

The factor $(-1)^m$ in (4.101) is a phase factor. Although it is not necessary, it is most convenient in the quantum theory of angular momentum. It is called the Condon–Shortley phase, after the authors of a classic text on atomic spectroscopy. Some authors define spherical harmonics without it. Still others use $\cos\varphi$ or $\sin\,\varphi$ instead of $\mathrm{e}^{\mathrm{i}\varphi}$. So whenever spherical harmonics are used, the phase convention should be specified.

Any well–behaved function of θ and φ can be expanded as a sum of spherical harmonics,

$$f(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}c_{lm}Y_l^m(\theta,\varphi),$$

where

$$c_{lm}=\int_{-1}^{1}\int_{0}^{2\pi}\left[Y_l^m(\theta,\varphi)\right]^{*}f(\theta,\varphi)\,\mathrm{d}\varphi\,\mathrm{d}(\cos\theta)\,.$$

This is another example of generalized Fourier series where the basis set is the solutions of a Sturm–Liouville problem.

4.9 Resources on Special Functions

We have touched only a small part of the huge amount of information about special functions. For the wealth of material available, see the three volume set of "Higher Transcendental Functions" by A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Triconi (McGraw-Hill, 1981). An excellent summary is given in "Formulas and Theorems for the Special Functions of Mathematical Physics" by W. Magnus, F. Oberhettinger, and R.P. Soni, (Springer, New York, 1966).

Extensive numerical tables of many special functions are presented in "Handbook of Mathematical Function with Formulas, Graphs, and Mathematical Tables" edited by M. Abramowitz and I.A. Stegun (Dover Publications, 1972).

Digital computers have made numerical evaluations of special functions readily available. There are several computer programs for calculating special functions listed in "Numerical Recipes" by William H. Press, Brian P. Flannery, Saul A. Teukolsky and William T. Vetterling (Cambridge University Press, 1986). A more recent comprehensive compilation of special functions, including computer programs, algorithms and tables is given in "Computation of Special Functions" by S. Zhang and J. Jin (John Wiley & Sons, 1996).

It should also be mentioned that a number of commercial computer packages are available to perform algebraic manipulations, including evaluating special functions. They are called computer algebraic systems, some prominent ones are Matlab, Maple, Mathematica, MathCad, and MuPAD.

This book is written with the software "Scientific WorkPlace", which also provides an interface to MuPAD (Before version 5, it also came with Maple). Instead of requiring the user to adhere to a rigid syntax, the user can use natural mathematical notations. For example, the figures of Bessel function J_n, Neumann function N_n, and modified Bessel functions I_n and K_n shown in this chapter are all automatically plotted. To plot $J_n(x)$, all you have to do is (1) from the Insert menu, choose Math Name, (2) type BesselJ in the name box, enter a subscript, enter an argument enclosed in parentheses, (3) choose OK, and (4) click on the 2D plot button. The program will return with a graph of $J_0,\ J_1$, or J_3 depending on what subscript you put in.

Unfortunately, as powerful and convenient as these computer algebraic systems are, sometimes they fail without any apparent reason. For example, in plotting $N_n(x)$, after the program successfully returned $N_0(x)$ and $N_1(x)$, with the same scale, the program hanged up on $N_2(x)$. After the scale is changed slightly, the program worked again. Even worse, the intention of the user is sometimes misinterpreted, and the computer returns with an answer to a wrong problem without the user knowing it. Therefore these systems must be used with caution.

Exercises

1. Solve the differential equation

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} + \alpha T(t) = 0$$

by expanding $T(t)$ into a Frobenius series

$$T(t) = t^p \sum_{n=0} a_n t^n.$$

Ans. $T(t) = a_0 \left(1 - \alpha t + \frac{\alpha^2 t^2}{2!} - \frac{\alpha^3 t^3}{3!} + \cdots\right) = a_0\, \mathrm{e}^{-\alpha t}$.

2. *Laguerre Polynomials.* Use the Frobenius method to solve the Laguerre equation

$$xy'' + (1-x)y' + \lambda y = 0.$$

Show that if λ is a nonnegative integer n, then the solution is a polynomial of order n. If the polynomial is normalized such that it is equal to 1 at $x = 0$, it is known as the Laguerre polynomial $L_n(x)$. Show that

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k\, n!}{(n-k)!\,(k!)^2} x^k.$$

Find the explicit expression of $L_0(x),\ L_1(x),\ L_2(x)$ and show that they are identical to results obtained in Exercise 3 of the last chapter from the Gram–Schmidt procedure.

3. Find the coefficients c_n of the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n L_n(x)$$

in the interval of $0 \le x < \infty$. Let $f(x) = x^2$, find c_n and verify the results with the explicit expressions of $L_0(x),\ L_1(x),\ L_2(x)$.

Hint: Recall Laguerre equation is a Sturm–Liouville problem in the interval of $0 \le x < \infty$. Its eigenfunctions are orthogonal with respect to weight function e^{-x}.

4. *Rodrigues Formula for Laguerre Polynomials.* Show that

$$L_n(x) = \frac{1}{n!} \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^n \mathrm{e}^{-x} \right).$$

Hint: Use Leibnitz's rule to carry out the differentiation of $x^n \mathrm{e}^{-x}$.

5. *Associated Laguerre Polynomials.* (a) Show that the solution of the associated Laguerre equation

$$xy'' + (K + 1 - x)y' + ny = 0$$

is given by

$$y = \frac{\mathrm{d}^k}{\mathrm{d}x^k} L_{n+k}(x).$$

(b) Other than a multiplicative constant, these solutions are known as the associated Laguerre polynomials $L_n^k(x)$. Show that $L_0^1(x),\ L_1^1(x),\ L_2^1(x)$ found in Exercise 6 of the last chapter are, respectively, proportional to

$$\frac{\mathrm{d}}{\mathrm{d}x} L_l(x),\ \frac{\mathrm{d}}{\mathrm{d}x} L_2(x),\ \frac{\mathrm{d}}{\mathrm{d}x} L_3(x).$$

Hint: (a) Start with the Laguerre equation of order $n + k$. Differentiate it k times.

Warning: There are many different notations used in literature for associated Laguerre polynomials. When dealing with these polynomials, one must be careful with their definition.

6. *Hermite Polynomials.* Use the Frobenius method to show that the following polynomials

$$y = \sum_{k=0} c_k x^k,$$

where

$$\frac{c_{k+2}}{c_k} = \frac{2k - 2n}{(k+1)(k+2)},$$

are solutions to the Hermite equation

$$y'' - 2xy' + 2ny = 0.$$

Here we have a terminating series with $k = 0, 2, \ldots, n$ terms for n even and with $k = 1, 3, \ldots, n$ terms for n odd. If the coefficient with the highest power of x is normalized to 2^n, then these polynomials are known as Hermite polynomials $H_n(x)$. Find the explicit expressions of $H_0(x)$, $H_1(x)$, $H_2(x)$. Show that these are the same polynomials found in Exercise 4 of the last chapter from the Gram–Schmidt procedure.

7. With the product of

$$\mathrm{e}^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} x^{2l},$$

$$\mathrm{e}^{2xt} = 1 + 2xt + \frac{(2x)^2}{2!} t^2 + \frac{(2x)^3}{3!} t^3 + \cdots = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} t^k,$$

written as

$$\mathrm{e}^{-t^2} \mathrm{e}^{2xt} = \sum_{n=0} A_n(x) t^n,$$

show that for $n = 0,\ n = 1,\ n = 2,$

$$A_n(x) = \frac{1}{n!} H_n(x),$$

where $H_n(x)$ are the Hermite polynomials found in the previous problem.

8. *Generating Function of Hermite Polynomials.* Show that (a) $A_n(x)$ of the previous problem can be written as

$$A_n(x) = \sum_{k=0}^{n} c_{k,n} x^k, \quad \textit{and} \quad c_{n,n} = \frac{2^n}{n!},$$

where k and n are either both even or both odd.
(b) Show that c_{kn} is given by

$$c_{k,n} = \frac{2^k}{k!} \frac{(-1)^{(n-k)/2}}{[(n-k)/2]!},$$

(c) Show that the ratio of $c_{k+2,n}$ over $c_{k,n}$ is given by

$$\frac{c_{k+2,n}}{c_{k,n}} = \frac{2k - 2n}{(k+2)(k+1)}.$$

(d) Show that in general

$$\mathrm{e}^{-t^2} \mathrm{e}^{2xt} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) t^n,$$

where $H_n(x)$ is the Hermite polynomial of order n. The left-hand side of this equation is known as the generating function $G(x,t)$ of Hermite polynomials,

$$G(x,t) = \mathrm{e}^{-t^2 + 2xt}.$$

Hint: (a) The power of x can come only from the expansion of e^{2xt}.

(b) The coefficient of t^n is the product of the coefficient of t^k in the expansion of e^{2xt} and the coefficient of t^{n-k} in the expansion of e^{-t^2}. Set $(n-k) = 2l$.
(d) The coefficients satisfy the recurrence relation of the Hermite polynomials and the coefficient of the highest power of x is normalized to $(1/n!)2^n$.

9. *Recurrence Relations of Hermite Polynomials.* Show that

$$\text{(a)}\quad 2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x), \qquad n \geq 1,$$

$$\text{(b)}\quad \frac{\mathrm{d}}{\mathrm{d}x} H_n(x) = 2nH_{n-1}(x), \quad n \geq 1.$$

Hint: (a) Consider $\dfrac{\partial}{\partial t} G(x,t)$. (b) Consider $\dfrac{\partial}{\partial x} G(x,t)$.

10. *Rodrigues Formula for Hermite Polynomials.* Show that

$$\text{(a)}\quad \left.\frac{\partial^n}{\partial t^n} G(x,t)\right|_{t=0} = H_n(x),$$

$$\text{(b)}\quad \frac{\partial^n}{\partial t^n} G(x,t) = e^{x^2} \frac{\partial^n}{\partial t^n} e^{-(t-x)^2},$$

$$\text{(c)}\quad H_n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2}.$$

Hint: (b) $G(x,t) = e^{-t^2+2xt} = e^{x^2-(t-x)^2} = e^{x^2} e^{-(t-x)^2}$
(c) $e^{x^2} \dfrac{\partial^n}{\partial t^n} e^{-(t-x)^2} = (-1)^n e^{x^2} \dfrac{\partial^n}{\partial x^n} e^{-(t-x)^2}$.

11. Use the series expression of Bessel functions

$$J_n(x) = \sum_{k=0} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

to show that

$$\text{(a)}\quad J_0(0) = 1; \quad J_n(0) = 0 \;\text{ for }\; n = 1,\ 2,\ 3, \ldots,$$

$$\text{(b)}\quad J_0'(x) = -J_1(x).$$

12. Let λ_{nj} be the jth root of $J_n(\lambda c) = 0$, where $c = 2$. Find $\lambda_{01},\ \lambda_{12},\ \lambda_{23}$ with the table of "zeros of the Bessel function."
Ans. 1.2024, 3.5078, 5.8099.

13. Show that

$$\text{(a)} \int_0^c J_0(\lambda r)\, r\, \mathrm{d}r = \frac{c}{\lambda} J_1(\lambda c).$$

$$\text{(b)} \quad \int_0^1 J_1(\lambda r)\, \mathrm{d}r = \frac{1}{\lambda}, \quad \text{if} \quad J_0(\lambda) = 0.$$

Hint: (a) Use (4.32). (b) Use (4.36).

14. Show that

$$J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x) = 1.$$

Hint: Use the generating function.

15. Solve

$$y''(x) + y(x) = 0$$

by substituting $y = \sqrt{x}u$ and solving the resulting Bessel's equation. Then show that the solution is equivalent to

$$y(x) = A\sin x + B\cos x.$$

16. (a) Show that an equation of the form

$$x^2 y''(x) + xy'(x) + \left(ax^{\beta}\right)^2 y - b^2 y = 0$$

is transformed into a Bessel equation

$$z^2 y''(z) + zy'(z) + z^2 y - \left(\frac{b}{\beta}\right)^2 y = 0$$

by a change of variable

$$z = \frac{ax^{\beta}}{\beta}.$$

(b) Solve

$$x^2 y'' + xy' + 4x^4 y - 16y = 0.$$

Ans. (b) $y(x) = c_1 J_2\left(x^2\right) + c_2 N_2\left(x^2\right)$.

17. Solve

$$x^2 y''(x) - xy'(x) + x^2 y(x) = 0.$$

Hint: Let $y(x) = x^{\alpha} u(x)$, and show that the equation becomes

$$x^2 u'' + (2\alpha - 1)\, xu' + \left[x^2 + \left(\alpha^2 - 2\alpha\right)\right] u = 0.$$

then set $\alpha = 1$.
Ans. $y(x) = x\left[c_1 J_1(x) + c_2 N_1(x)\right]$.

18. Use the Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} \left(x^2 - 1\right)^l$$

to develop the following series expressions of the Legendre polynomials

$$P_l(x) = \frac{1}{2^l l!} \sum_{k=0}^{[l/2]} \frac{(-1)^k \, l!}{k! \, (l-k)!} \frac{(2l-2k)!}{(l-2k)!} x^{l-2k},$$

where $[l/2] = l/2$ if l is even, and $[l/2] = (l-1)/2$ if l is odd. Show that

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2}, \qquad P_{2n+1}(0) = 0.$$

19. Use the generating function $G(0,t)$ to find $P_{2n}(0)$.
Hint:

$$\frac{1}{\sqrt{1+t^2}} = 1 - \frac{1}{2}t^2 + \frac{1}{2!}\frac{1}{2}\frac{3}{2}t^4 + \cdots.$$

$$= \sum_{n=0}^{2n} \frac{(-1)^n}{n!} \frac{(2n-1)!!}{2^n} t^{2n},$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 1$, and

$$(2n-1)!! = (2n-1)!! \frac{(2n)!!}{(2n)!!} = \frac{(2n)!}{2^n n!}.$$

20. Use Rodrigues' formula to prove that

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x).$$

Hence show that

$$\int_x^1 P_n(x)\mathrm{d}x = \frac{1}{2n+1} \left[P_{n-1}(x) - P_{n+1}(x)\right].$$

21. Use the fact that both $P_n(x)$ and $P_m(x)$ satisfy the Legendre equation to show that

$$\int_{-1}^{1} P_n(x) P_m(x) \mathrm{d}x = 0 \quad \text{if} \quad m \neq n.$$

Hint: Multiply the Legendre equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_m(x)\right] + m(m+1)P_m(x) = 0$$

by $P_n(x)$ and integrate by parts:

$$\int_{-1}^{1}(1-x^2)\left[\frac{\mathrm{d}}{\mathrm{d}x}P_m(x)\right]\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)\mathrm{d}x = m(m+1)\int_{-1}^{1}P_n(x)P_m(x)\mathrm{d}x.$$

Similarly,

$$\int_{-1}^{1}(1-x^2)\left[\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)\right]\frac{\mathrm{d}}{\mathrm{d}x}P_m(x)\mathrm{d}x = n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\mathrm{d}x.$$

Then get the conclusion from

$$[m(m+1)-n(n+1)]\int_{-1}^{1}P_n(x)P_m(x)\mathrm{d}x = 0.$$

22. Use the Rodrigues formula to show

$$\int_{-1}^{1}[P_n(x)]^2\,\mathrm{d}x = \frac{2}{2n+1}.$$

Hint: First show that

$$\int_{-1}^{1}[P_n(x)]^2\,\mathrm{d}x = \left[\frac{1}{2^n n!}\right]^2\int_{-1}^{1}\mathrm{d}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(x^2-1\right)^n\right]\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n,$$

then use integration by parts repeatedly to show

$$\int_{-1}^{1}[P_n(x)]^2\,\mathrm{d}x = (-1)^n\left[\frac{1}{2^n n!}\right]^2\int_{-1}^{1}\left(x^2-1\right)^n\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(x^2-1\right)^n\mathrm{d}x.$$

Note that $(\mathrm{d}^{2n}/\mathrm{d}x^{2n})\left(x^2-1\right)^n = (2n)!$, so

$$\int_{-1}^{1}[P_n(x)]^2\,\mathrm{d}x = \left[\frac{1}{2^n n!}\right]^2(2n)!\int_{-1}^{1}\left(1-x^2\right)^n\mathrm{d}x.$$

Evaluate the integral on the right-hand side with a change a variable $x=\cos\theta$. First note

$$\int_{-1}^{1}\left(1-x^2\right)^n\mathrm{d}x = \int_{-1}^{1}\left(1-\cos^2\theta\right)^n\mathrm{d}\cos\theta = \int_{0}^{\pi}(\sin\theta)^{2n+1}\mathrm{d}\theta,$$

then show that

$$(2n+1)\int_{0}^{\pi}(\sin\theta)^{2n+1}\mathrm{d}\theta = 2n\int_{0}^{\pi}(\sin\theta)^{2n-1}\,\mathrm{d}\theta,$$

by repeated integration by parts of

$$\int_{-1}^{1}\left(1-\cos^2\theta\right)^n\mathrm{d}\cos\theta = \int_{-1}^{1}(\sin\theta)^{2n}\,\mathrm{d}\cos\theta = -\int_{\pi}^{0}\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)^{2n}\,\mathrm{d}\theta$$
$$= 2n\int_{0}^{\pi}(\sin\theta)^{2n-1}\,\mathrm{d}\theta - 2n\int_{0}^{\pi}(\sin\theta)^{2n+1}\,\mathrm{d}\theta.$$

Finally get the result from

$$\int_0^{\pi} (\sin\theta)^{2n+1} \mathrm{d}\theta = \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} \int_0^{\pi} \sin\theta \, \mathrm{d}\theta = 2\frac{[2^n n!]^2}{(2n+1)!}.$$

23. If

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & -1 < x < 0 \end{cases},$$

show that $f(x)$ can be expressed as

$$f(x) = \frac{1}{2} + \frac{3}{4}x + \sum_{n=1}^{\infty} (-1)^n \frac{4n+3}{4n+4} \frac{(2n)!}{2^{2n}(n!)^2} P_{2n+1}(x).$$

Hint: Show that

$$f(x) = \frac{1}{2}P_0(x) + \frac{1}{2}\sum_{n=0}^{\infty} \left[P_{2n}(0) - P_{2n+2}(0)\right] P_{2n+1}(x).$$

24. Show that

(a) $$\int_{-1}^{1} xP_n(x)P_m(x)\mathrm{d}x = \begin{cases} \dfrac{2(n+1)}{(2n+1)(2n+3)} & \text{if } m = n+1 \\ \dfrac{2n}{(2n-1)(2n+1)} & \text{if } m = n-1 \\ 0 & \text{otherwise} \end{cases},$$

(b) $$\int_{-1}^{1} x^2 P_n(x)P_m(x)\mathrm{d}x = \begin{cases} \dfrac{2(n+1)(n+2)}{(2n+1)(2n+3)(2n+5)} & m = n+2 \\ \dfrac{2(2n^2+2n-1)}{(2n-1)(2n+1)(2n+3)} & m = n \\ \dfrac{2(n-1)n}{(2n-3)(2n-1)(2n+1)} & m = n-2 \\ 0 & \text{otherwise} \end{cases}.$$

Hint: Use (4.88).

25. Show that if $P_l^{-m}(x)$ and $Y_l^m(\theta,\varphi)$ are, respectively, defined as

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x),$$

and

$$Y_l^m(\theta,\varphi) = (-1)^m \sqrt{\frac{1}{2\pi}\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\varphi}, \quad \text{for } -l \le m \le l,$$

then $Y_l^{-m}(\theta,\varphi)$ obtained from the last expression is the same as

$$Y_l^{-|m|}(\theta,\varphi) = (-1)^m \left[Y_l^{|m|}(\theta,\varphi)\right]^*,$$

defined in (4.102).