5

Partial Differential Equations in Cartesian Coordinates

An equation involving one or more partial derivatives of an unknown function of two or more independent variables is called a partial differential equation. Compared with ordinary differential equations, far more problems in physical sciences lead to partial differential equations. In fact, most of mathematical physics deals with partial differential equations.

In general, the totality of solutions of a partial differential equation is very large. However, a unique solution of a partial differential equation corresponding to a given physical problem can usually be obtained by the use of either boundary and/or initial conditions. In practice, the boundary conditions frequently serve as a guide in choosing a particular form of the solution, which satisfies the partial differential equation as well as the boundary conditions.

The field of partial differential equation is very wide. We are going to focus our attention on the equations that arise most often in physics, namely

$$\nabla^2 u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}, \quad \text{Wave equation}$$

$$\nabla^2 u = \frac{1}{a^2}\frac{\partial u}{\partial t}, \quad \text{Diffusion equation}$$

$$\nabla^2 u = 0, \quad \text{Laplace's equation,}$$

where ∇^2 is the operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The Schrödinger equation in quantum mechanics also has a similar form except that an imaginary number is attached to the time derivative.

In this chapter we only consider problems expressible in cartesian coordinates. Problems with curved boundaries will be considered in Chap. 6.

The amazing thing is that there are many problems which are physically unrelated, but they can be described by the same or very similar partial differential equation.

5.1 One-Dimensional Wave Equations

5.1.1 The Governing Equation of a Vibrating String

As an example, we will derive the equation that governs the small vibrations of an elastic string of length L, fixed at both endpoints. The dependent variable $u(x,t)$ represents, at time t, the displacement of the point of the string that is at distance x away from the first end point 0.

We shall assume that the string is homogeneous, that is, the mass of the string per unit length, denoted as ρ, is a constant. We also shall assume that the string undergoes only small vertical displacements from its equilibrium position. (The displacements do not have to be in vertical direction, but for the sake of discussion, we assume it is.)

Let us consider the segment of the string between x and $x + \Delta x$, where Δx is a small increment, as shown in Fig. 5.1. The quantities T_1 and T_2 in the figure are the tensions at the points P and Q of the string. Both T_1 and T_2 are tangential to the curve of the spring. Because there is no horizontal motion of the string, the net horizontal force exerted on the segment must be zero. In other words, the horizontal components of the tensions at P and Q must be equal and opposite. That is

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T, \tag{5.1}$$

where T is a constant equal to the horizontal force with which the string is stretched. If the amplitude is small, we can regard T as the tension of the string.

There is a net force in the vertical direction, F_u, that causes the vertical motion of the string. Clearly

$$F_u = T_2 \sin\theta_2 - T_1 \sin\theta_1 .$$

By Newton's second law, this force is equal to the mass of the segment, $\rho\,\Delta x$, times the acceleration which is the second derivative of the displacement with respect to time. That is

$$T_2 \sin\beta - T_1 \sin\alpha = \rho\,\Delta x \frac{\partial^2 u}{\partial t^2}.$$

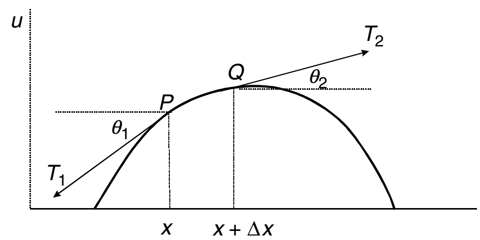

Fig. 5.1. A vibrating string at time t

Dividing this equation by T and using (5.1), we have

$$\frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} = \frac{\rho\,\Delta x}{T}\frac{\partial^2 u}{\partial t^2}.$$

Thus

$$\tan\theta_2 - \tan\theta_1 = \frac{\rho\,\Delta x}{T}\frac{\partial^2 u}{\partial t^2}. \tag{5.2}$$

But $\tan\theta_2$ and $\tan\theta_1$ are the slopes of the curve of the string at $x + \Delta x$ and x, respectively,

$$\tan\theta_2 = \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x},$$
$$\tan\theta_1 = \left(\frac{\partial u}{\partial x}\right)_{x}.$$

Hence (5.2) can be written as

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\rho\,\Delta x}{T}\frac{\partial^2 u}{\partial t^2}. \tag{5.3}$$

Recall the definition of a derivative

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \lim_{\Delta x\to 0}\frac{F(x+\Delta x) - F(x)}{\Delta x}.$$

If it is understood that Δx is approaching zero even without the limit sign, then we can write

$$F(x+\Delta x) - F(x) = \frac{\mathrm{d}F}{\mathrm{d}x}\Delta x.$$

Thus it is clear that

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\Delta x = \frac{\partial^2 u}{\partial x^2}\Delta x.$$

Therefore (5.3) becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}. \tag{5.4}$$

This is the so called one-dimensional wave equation. We see that it is linear, homogeneous, and of second-order.

If the string is fixed at both ends, we have two boundary conditions

$$\text{B.C.:}\ u(0,t) = 0;\ u(L,t) = 0.$$

Furthermore, if the string is initially displaced into a position $u = f(x)$ and released at rest from that position, then we have the following initial conditions:

$$\text{I.C.:}\ \ u(x,0) = f(x);\ u_t(x,0) = 0,$$

where $u_t(x,0)$ denotes the first partial derivative of $u(x,t)$ with respect to t and then evaluated at $t = 0$

$$u_t(x,0) = \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}.$$

The first condition says that the initial shape of the string is $f(x)$, the second condition simply says that at $t = 0$, the velocity everywhere in the string is zero.

Of course, it is possible that the string also has initial velocity. In that case, the initial conditions become

$$u(x,0) = f(x);\ u_t(x,0) = g(x).$$

5.1.2 Separation of Variables

To describe the motion of the string, one must solve the differential equation and the solution must satisfy the boundary and initial conditions. Specifically let us find a formula for the transverse displacement $u(x,t)$ of the stretched string which satisfies (5.4). For simplicity of writing, let us first define

$$a^2 = \frac{T}{\rho}. \tag{5.5}$$

It turns out a has a physical meaning which will become clear later.

A powerful and classical method of solving linear boundary value problems in partial differential equations is the method of separation of variables which reduces a partial differential equation into ordinary differential equations. Although not all problems can be solved by this method and there are other methods, generally separation of variables is the first method one should try.

Let us solve the mathematical problem consisting of the following:

$$\text{D.E.:}\ \ \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{5.6}$$

$$\text{B.C.:}\ \ u(0,t) = 0; \qquad u(L,t) = 0, \tag{5.7}$$

$$\text{I.C.:}\ \ u(x,0) = f(x); \quad u_t(x,0) = 0. \tag{5.8}$$

The assumption of separation of variables is that we can write $u(x,t)$ as

$$u(x,t) = X(x)T(t),$$

where X is a function of x alone and T is a function of t alone. The justification of the assumption of this method is that it works. It follows from this assumption, that:

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} X(x)\right) T(t) = X''(x)T(t),$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} T(t)\right) = X(x)T^{''}(t).$$

Thus (5.6) becomes

$$X''(x)T(t) = \frac{1}{a^2} X(x)T''(t).$$

Dividing both sides of the equation by $X(x)T(t)$

$$\frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{a^2}\frac{X(x)T''(t)}{X(x)T(t)},$$

we obtain

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)}.$$

The left-hand side of this equation is a function of x alone, it cannot vary with t. However, it is equal to a function of t which cannot vary with x. This is possible if and only if both sides are equal to the same common constant α. This leads to

$$\frac{X''(x)}{X(x)} = \alpha,$$

$$\frac{1}{a^2}\frac{T''(t)}{T(t)} = \alpha.$$

It follows that:

$$X''(x) = \alpha X(x) \tag{5.9}$$

$$T''(t) = \alpha a^2 T(t). \tag{5.10}$$

The partial differential equation is now decomposed into two ordinary differential equations.

Eigenvalues and Eigenfunctions. If $u(x,t)$ is to satisfy the first boundary condition, then

$$u(0,t) = X(0)T(t) = 0$$

for all t. Since $T(t)$ is changing with t, the only possibility that this can be true is that

$$X(0) = 0.$$

Similarly, the condition $u(L,t) = 0$ leads to

$$X(L) = 0.$$

So far we have not specified the value of the separation constant α, it could be less than zero, equal to zero, or greater than zero. It is easy to show that if $\alpha \geqq 0$, there is no solution that can satisfy these boundary conditions.

First, if $\alpha = 0$, the solution of (5.9) is $X(x) = Ax+B$. In this case $X(0) = 0$ requires $B = 0$. Thus, $X(L) = AL$. Since $X(L) = 0$, therefore $A = 0$. This leads to $X(x) = 0$ which is a trivial solution for the case that u is identically equal to zero for all x and t.

When $\alpha > 0$, let us write $\alpha = \mu^2$ with μ being real. Then the solution of $X''(x) = \mu^2 X(x)$ is $X(x) = C\cosh\mu x + D\sinh\mu x$. With $X(0) = 0, C$ must be equal to zero. Thus $X(x) = D\sinh\mu x$. Since $\sinh\mu L \neq 0$, $X(L) = 0$ requires $D = 0$. Again this gives only the trivial solution.

Therefore α must be less than zero. Let us write $\alpha = -\mu^2$, so (5.9) becomes

$$X''(x) = -\mu^2 X(x).$$

The general solution of this equation is

$$X(x) = A\cos\mu x + B\sin\mu x.$$

Thus $X(0) = A$ and the condition $X(0) = 0$ means $A = 0$. Hence we are left with

$$X(x) = B\sin\mu x.$$

To satisfy the condition $X(L) = 0$, μ must be chosen to be

$$\mu = \frac{n\pi}{L}, \quad n = 1,\ 2,\ 3, \ldots .$$

Therefore, for each n, there is a solution $X_n(x)$

$$X_n(x) = B_n \sin\frac{n\pi}{L}x, \quad n = 1, 2, \ldots \tag{5.11}$$

where B_n is an arbitrary constant. The numbers $\alpha = -n^2\pi^2/L^2$ for which this problem has nontrivial solutions are called eigenvalues, and the corresponding functions (5.11) are called eigenfunctions.

Solution of the Problem. It is important to keep in mind that α in (5.9) and in (5.10) must be the same. When $\alpha = -n^2\pi^2/L^2$, (5.9) is a distinct problem for each different positive integer n. For a fixed integer n, (5.10) becomes

$$T_n''(t) = -\frac{n^2\pi^2}{L^2}a^2 T_n(t).$$

The solution of this equation is

$$T_n(t) = C_n\cos\frac{n\pi a}{L}t + D_n\sin\frac{n\pi a}{L}t.$$

Thus, each

$$u_n(x,t) = X_n(x)T_n(t)$$

is a solution of the differential equation. An important theorem of the linear homogeneous partial differential equation is the principle of superposition. If u_1 and u_2 are solutions of a linear homogeneous differential equation, then

$$u = c_1 u_1 + c_2 u_2,$$

where c_1 and c_2 are arbitrary constants, is also a solution of that equation. This theorem can be easily proved by showing the equation is satisfied with this combination as the solution.

Therefore the general solution is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi a}{L} t + b_n \sin \frac{n\pi a}{L} t \right) \sin \frac{n\pi}{L} x. \end{aligned} \tag{5.12}$$

where we have combined three arbitrary constants $c_n C_n B_n$ into a single constant a_n, and $c_n D_n B_n$ into b_n. Now the coefficients a_n and b_n can be chosen to satisfy the initial conditions.

One of the initial condition is

$$u_t(x,0) = \sum_{n=1}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(a_n \cos \frac{n\pi a}{L} t + b_n \sin \frac{n\pi a}{L} t \right) \right]_{t=0} \sin \frac{n\pi}{L} x = 0,$$

which leads to

$$\sum_{n=1}^{\infty} b_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x = 0.$$

Since $\sin \frac{n\pi}{L} x$ is a complete set in the interval of $0 \leq x \leq L$, all coefficients must be zero. Another way to see that all b_n are zero is the following. This equation is a Fourier sine series of zero. The coefficients are integrals of zero times some sine function. Obviously they are zero. Therefore

$$b_n = 0.$$

Thus we are left with

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t, \tag{5.13}$$

where the coefficients a_n can be chosen to satisfy the other initial condition.

Since $u(x,0) = f(x)$, it follows from the last equation

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x = f(x).$$

This is the half-range Fourier sine series of $f(x)$ between 0 and L. Therefore a_n is given by the Fourier coefficient

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x.$$

Thus, the solution of this problem is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x')\sin\frac{n\pi x'}{L}\mathrm{d}x'\right]\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t. \tag{5.14}$$

Example 5.1.1. A guitar string of length L is pulled upward at the middle so it reaches height h. What is the subsequent motion of the string if it is released from the rest?

Solution 5.1.1. To find the subsequent motion means to find the displacement of the string as a function of t. That is, we have to solve for $u(x,t)$ from the equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}.$$

Since the two ends of the guitar string are fixed, so we have to satisfy the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

Furthermore, it can be readily shown that the initial shape of the string is given by

$$f(x) = \begin{cases} \dfrac{2h}{L}x & \text{for } 0 \le x \le \dfrac{L}{2}, \\ \dfrac{2h}{L}(L-x) & \text{for } \dfrac{L}{2} \le x \le L. \end{cases}$$

Since it is released from rest, so the initial velocity everywhere in the string is zero. This means the derivative of $u(x,t)$ with respect to time evaluated at $t=0$ is zero. Thus the initial conditions are

$$u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

According to (5.14), $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t,$$

where

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \\ &= \frac{2}{L}\int_0^{L/2}\frac{2h}{L}x\sin\frac{n\pi x}{L}\mathrm{d}x + \frac{2}{L}\int_{L/2}^{L}\frac{2h}{L}(L-x)\sin\frac{n\pi x}{L}\mathrm{d}x \\ &= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned}$$

Therefore

$$
\begin{aligned}
u(x,t) &= \frac{8h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t \\
&= \frac{8h}{\pi^2}\left(\sin\frac{\pi}{L}x\cos\frac{\pi a}{L}t-\frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi a}{L}t\right. \\
&\qquad \left.+\frac{1}{5^2}\sin\frac{5\pi}{L}x\cos\frac{5\pi a}{2}t-\frac{1}{7^2}\sin\frac{7\pi}{L}x\cos\frac{7\pi a}{L}t+\cdots\right). \qquad (5.15)
\end{aligned}
$$

It is interesting to see the time development of the displacements. The shapes of the string at various times are shown in the left-hand side column of Fig. 5.2. The individual components are shown in the right-hand side column of the same figure. The string oscillates up and down as expected. We have shown the positions of the string within half of a cycle. After that it will go back

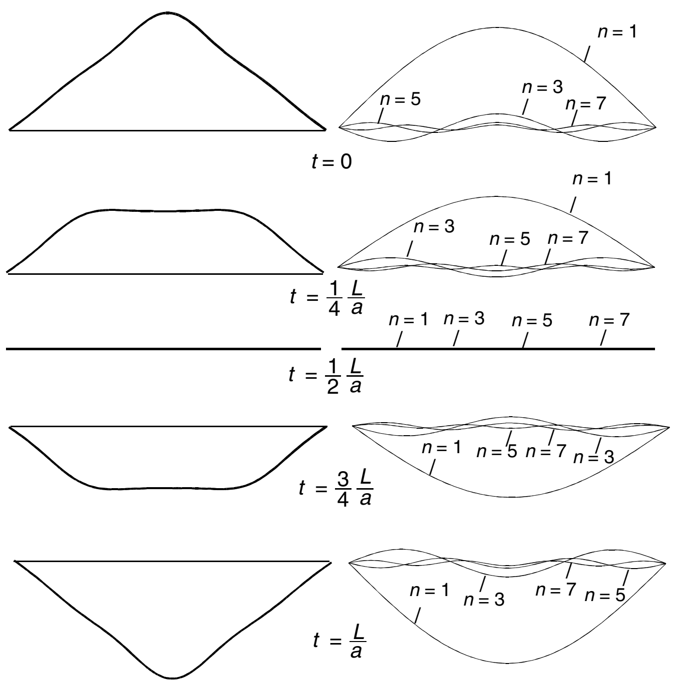

Fig. 5.2. The time development of the displacements of a string after its middle point is pulled up a distance h and released from that position. The left-hand side column is the shape of the string at various times obtained by summing up the first four nonzero terms of the series (5.15). The right-hand side column are the positions of the four individual terms of the series at the correponding times. Although different components oscillate at different frequencies, they sum up to a string going up and down as expected. It is seen that the fundamental (the first term) dominates the motion

to its original position and then repeat the motion. During this time interval of half a cycle, the fundamental (the first term in the series, $\sin\frac{\pi}{L}x\cos\frac{\pi a}{L}t$) also completes half of its cycle. The third harmonic (the second nonzero term, $\sin\frac{3\pi}{L}x\cos\frac{3\pi a}{L}t$) actually completes one and half of its cycle. Various components oscillate at various different frequencies, yet together they sum up to the oscillation shown in the left-hand side column. In fact we have summed up only four nonzero terms, so the lines for the shape of the string are somewhat curved and the corners are somewhat rounded. If we use the computer to plot

$$\frac{8h}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t,$$

with $N = 50$, then all the lines in the left-hand side column will be straight and all corners sharply pointed. The amplitudes of higher components are very small, but they do make the sum converge to the exact value. It is seen that in this case the fundamental dominates the motion.

5.1.3 Standing Wave

For the physical interpretation of the series (5.14), let us suppose that the string is suddenly released from the position $u(x,0) = \sin\frac{2\pi}{L}x$. In this case, the coefficient a_n is given by

$$a_n = \frac{2}{L}\int_0^L \sin\frac{2\pi x}{L}\sin\frac{n\pi x}{L}\mathrm{d}x = \begin{cases} 1 & n = 2, \\ 0 & n \neq 2. \end{cases}$$

Therefore the subsequent displacement of the string is

$$u(x,t) = \sin\frac{2\pi}{L}x\cos\frac{2\pi a}{L}t.$$

This motion is shown in Fig. 5.3. At any instant of time, $u(x,t)$ is a pure sine curve

$$u(x,t) = A_2(t)\sin\frac{2\pi}{L}x,$$

where $A_2(t)$ is the amplitude of the sine wave and $A_2(t) = \cos\dfrac{2\pi a}{L}t$. Note that the points at $x = 0$, $x = L/2$, and $x = L$ are fixed in time. They are called nodes. Between the nodes, the string oscillates up and down. This kind of motion is known as standing wave.

In general (5.13) can be regarded as

$$u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t), \tag{5.16}$$

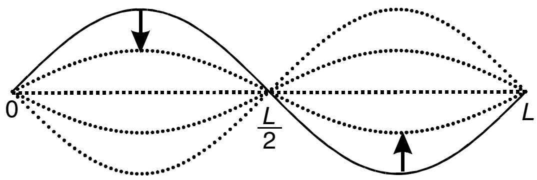

Fig. 5.3. The standing wave of $\sin \frac{2\pi}{L} x \cos \frac{2\pi a}{L} t$

where

$$u_n(x,t) = \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t$$

is known as the nth normal mode. One characteristic of a normal mode is that once the string is vibrating in the standing wave of that mode, it will continue to vibrate in that mode forever. Of course, if there is damping, its amplitude will eventually die down.

The time dependence of each normal mode is given by $\cos \frac{n\pi a t}{L}$ which is a periodic function. The period is defined as the time interval after which the function will return to its original value. Let P_n be the period, so that

$$\cos \frac{n\pi a}{L}(t + P_n) = \cos \frac{n\pi a}{L} t. \tag{5.17}$$

Since

$$\cos \frac{n\pi a}{L}(t + P_n) = \cos \left(\frac{n\pi a}{L} t + \frac{n\pi a}{L} P_n \right)$$

clearly

$$\frac{n\pi a}{L} P_n = 2\pi.$$

Therefore

$$P_n = \frac{2L}{na}.$$

Frequency ν_n is defined as the number of oscillations in one second (the unit of frequency is called Hertz, Hz), that is

$$\nu_n = \frac{1}{P_n} = \frac{na}{2L}.$$

Therefore the series (5.16) represents the motion of a string (of violin or guitar) as a superposition of infinitely many normal modes, each vibrating with a different frequency. The lowest of these frequencies

$$\nu_1 = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

is called the fundamental frequency. Here we have used the definition of a given in (5.5). The fundamental frequency usually predominates in the sound we hear. The frequency $\nu_n = n\nu_1$, of the nth overtone or harmonic is an integral multiple of ν_1.

Note that once L, T, ρ are chosen, the fundamental frequency is fixed. The initial conditions do not affect ν_1; instead, they determine the coefficients in (5.14) and hence the extent to which the higher harmonics contribute to the sound produced. Therefore the initial conditions affect the overall frequency mixture (known as timbre), rather than the fundamental frequency. For example, if the string of a violin is bowed at some other point than its center, the amplitudes of higher harmonics would have been different than shown in Fig. 5.2. By choosing the point properly any desired harmonic may be emphasized or diminished, a fact well known to musicians.

Once a musical instrument is constructed, the length of string L and the density ρ cannot be changed. Therefore tuning is usually done by changing the tension T.

The spatial dependence of the first few normal modes is shown in Fig. 5.4. The first mode ($n = 1$) is called the fundamental mode, represents a harmonic time dependence of frequency $a/2L$. The second harmonic or first overtone ($n = 2$) vibrates harmonically with frequency a/L, twice as fast as the fundamental mode. Its motion is also shown in Fig. 5.4. Note that, in addition to the two end points, the midpoint of this harmonic is a node. Similarly, the third ($n = 3$) and fourth ($n = 4$) harmonics have two and three nodes, respectively, in addition to the two end points.

In describing the frequency of the oscillation, the angular frequency ω_n (radians per second) in often used

$$\omega_n = 2\pi\nu_n = \frac{\pi n a}{L}.$$

Another quantity associated with wave motion is the wavelength. The wavelength λ_n is defined such that $u_n(x,t)$ will return to its original value if x is increased by λ_n, that is

$$u_n(x+\lambda_n, t_0) = u_n(x, t_0). \tag{5.18}$$

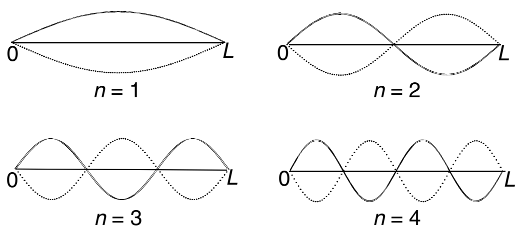

Fig. 5.4. The first four normal modes of a vibrating string. Each normal mode is a standing wave. The nth normal mode has $n-1$ nodes, excluding the nodes at the two end points

Since

$$u_n(x+\lambda_n, t_0) = \sin\left(\frac{n\pi}{L}x + \frac{n\pi}{L}\lambda_n\right)\cos\frac{n\pi a}{L}t_0;$$
$$u_n(x, t_0) = \sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t_0$$

it is clear that (5.18) will be satisfied if

$$\frac{n\pi}{L}\lambda_n = 2\pi.$$

Therefore

$$\lambda_n = \frac{2L}{n}. \tag{5.19}$$

Thus, for $n = 1,\ L = \frac{1}{2}\lambda;\ n = 2,\ L = \lambda;\ n = 3,\ L = \frac{3}{2}\lambda,\ n = 4,\ L = 2\lambda.$ These relations are clearly demonstrated in Fig. 5.4. Therefore the distance between two nodes of a standing wave is half of a wavelength.

Often a quantity known as wave number k_n (number of wavelengths in the interval of 2π)

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L} \tag{5.20}$$

is used to describe the wave form. In this notation, the normal mode $u_n(x,t)$ is written as

$$u_n(x,t) = \sin k_n x \cos \omega_n t. \tag{5.21}$$

A very important relationship between the frequency and the wavelength is

$$\nu_n \lambda_n = \frac{na}{2L}\frac{2L}{n} = a = \sqrt{\frac{T}{\rho}}. \tag{5.22}$$

This relation says that the frequency is proportional to the inverse of the wavelength and the proportionality constant is equal to the square root of tension over density. A standard physics experiment is shown in Fig. 5.5. A string of density ρ and tension T is connected to a vibrator, the frequency of which can be varied. Standing wave patterns will occur for certain discrete values of the frequency. The wavelength of each standing wave can then be measured. After several standing waves of different wavelength are measured,

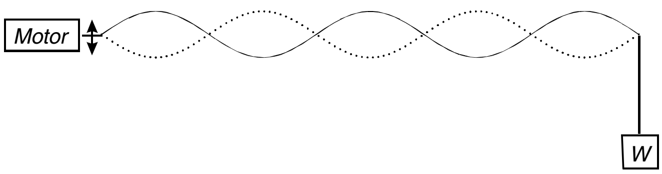

Fig. 5.5. A standing wave experiment to verify the relationship between the frequency and the wavelength

we can plot the frequency against the inverse of the wavelength. The curve is indeed a straight line and the slope of the line is indeed equal to $\sqrt{T/\rho}$.

This is not only a demonstration of a physical principle, but also a demonstration of the power of analysis. We have applied the Newton's law, which relates the force to the acceleration of the particle, to the motion of a string through the use of calculus and concluded that the frequency and the wavelength must satisfy the relation shown in (5.22). This can then be verified in the laboratory.

If the wave is traveling down on an infinite line, one may think that the frequency is the number of wave cycles generated per second and each extends a distance of one wavelength, therefore $\nu_n \lambda_n = a$ is the distance the wave travels in one second. In other words, a is the velocity of a traveling wave. This is indeed the case, as we will clearly see in Sect. 5.1.4.

5.1.4 Traveling Wave

In Sect. 5.1.3, we have shown that each normal mode is a standing wave. Now we wish to show that the same normal mode can also be regarded as a superposition of two traveling waves in the opposite direction.

Using the trigonometric identity

$$\sin a \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right],$$

we can write (5.21) as

$$\begin{aligned} u_n(x,t) &= \frac{1}{2}\sin(k_n x + \omega_n t) + \frac{1}{2}\sin(k_n x - \omega_n t) \\ &= \frac{1}{2}\sin[k_n(x+at)] + \frac{1}{2}\sin[k_n(x-at)], \end{aligned} \tag{5.23}$$

where we have used the fact

$$\frac{\omega_n}{k_n} = \nu_n \lambda_n = a.$$

Before we discuss the interpretation of (5.23), let us first examine the behavior of the function $f(x-ct)$. In this function, the variables x and t are combined in the particular way of $x - ct$. Suppose at $t = 0$, the function $f(x)$ looks like the solid curve in Fig. 5.6. If the maximum value of the function $f(x_{\rm m})$ is at $x = x_{\rm m}$, then at a later time t, the function $f(x-ct)$ will have the same maximum value $f(x_{\rm m})$ at $x = x_{\rm m} + ct$. This means that the maximum point has moved a distance ct in the time interval of t. In fact, it is not difficult to see that the whole function has moved a distance ct to the right in the time interval t, as shown by the dotted curve in Fig. 5.6. Therefore $f(x-ct)$ represents the function bodily moving (without changing the shape of the function) to the right with velocity c. Similarly, $f(x+ct)$ represents the function traveling to the left with velocity c.

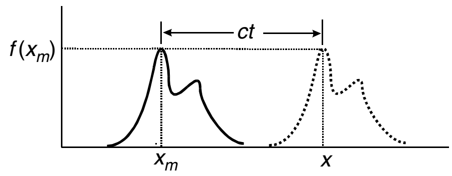

Fig. 5.6. Traveling wave. The *solid curve* shows what the function $f(x-ct)$ might be like at $t=0$, the *dashed curve* shows what the function is at a later time t

It is now clear that $\sin[k_n(x+at)]$ and $\sin[k_n(x-at)]$ in the normal mode of (5.23) are two sine waves traveling in opposite directions with the same speed a. It is interesting to write (5.13) in terms of traveling waves

$$u(x,t)=\frac{1}{2}\sum_{n=1}^{\infty}a_n\left[\sin k_n(x+at)+\sin k_n(x-at)\right]. \tag{5.24}$$

Since initially at $t=0$ the string is displaced into the form $f(x)$

$$f(x)=u(x,0)=\sum_{n=1}^{\infty}a_n\sin k_n x \tag{5.25}$$

clearly

$$f(x+at)=\sum_{n=1}^{\infty}a_n\sin k_n(x+at),$$

$$f(x-at)=\sum_{n=1}^{\infty}a_n\sin k_n(x-at).$$

Thus

$$u(x,t)=\frac{1}{2}f(x+at)+\frac{1}{2}f(x-at). \tag{5.26}$$

In other words, when the string is released at $t=0$ from the displaced position $f(x)$, it will split into two equal parts, one traveling to the right and the other to the left with the same speed a.

However, there is a question about the range over which $f(x)$ is defined. The initial displacement $f(x)$ is defined between 0 and L. But now the argument is $x+at$ or $x-at$. Since t can take any value, the argument certainly exceeds the range between 0 and L. In order to have (5.26) valid for all t, we must extend the argument of the function beyond this range. Since (5.26) is obtained from (5.25) and $\sin k_n x=\sin\frac{n\pi}{L}x$, which is an odd periodic function with period $2L$, the functions in (5.26) must also have this property. So if we denote f^* to be the odd periodic extension of f with period $2L$, then

$$u(x,t)=\frac{1}{2}f^*(x+at)+\frac{1}{2}f^*(x-at) \tag{5.27}$$

is valid for all t.

Example 5.1.2. With the traveling wave interpretation, solve the problem of the previous example of a string pulled at the middle.

Solution 5.1.2. With the initial displacement of the string

$$u(x,0) = f(x) = \begin{cases} \dfrac{2h}{L}x & \text{if} \quad 0 < x < \dfrac{L}{2} \\ \dfrac{2h}{L}(L-x) & \text{if} \quad \dfrac{L}{2} < x < L \end{cases},$$

the subsequent displacement $u(x,t)$ is given by

$$u(x,t) = \frac{1}{2}f^*(x+at) + \frac{1}{2}f^*(x-at).$$

To interpret this expression, first we imagine the function $f(x)$ is antisymmetrically extended from 0 to $-L$, and then periodically extended from $-\infty$ to ∞ with a period $2L$. Then half of this extended function is moving to the right with velocity a and the other half moving to the left with the same velocity as shown in Fig. 5.7. The sum of these two traveling waves in the region $0 \le x \le L$ is the displacement $u(x,t)$ of the string.

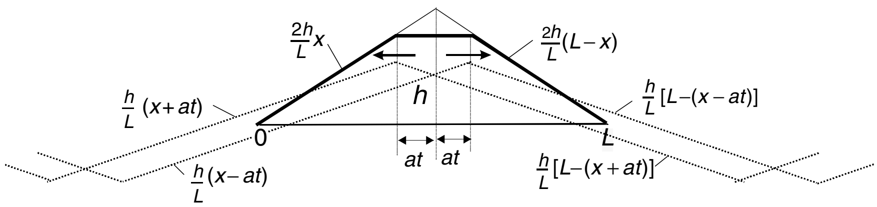

Fig. 5.7. The traveling wave interpretation of the solution of the wave equation with initial and boundary conditions. The displacement $u(x,t)$ is the sum of half of the extended initial function traveling to the left and half traveling to the right with the same velocity a

As a consequence, we see that at any time $t = T$, for $T \le \frac{L}{2a}$, the displacements are

$$\begin{aligned} u(x,T) &= \frac{1}{2}\left\{\frac{2h}{L}(x+aT) + \frac{2h}{L}(x-aT)\right\} \\ &= \frac{2h}{L}x \qquad \text{if } 0 \le x \le \left(\frac{L}{2} - aT\right), \end{aligned}$$

$$u(x,T) = \frac{1}{2}\left\{\frac{2h}{L}[L-(x+aT)] + \frac{2h}{L}(x-aT)\right\}$$
$$= \frac{2h}{L}\left(\frac{L}{2}-aT\right) \quad \text{if} \quad \left(\frac{L}{2}-aT\right) \leq x \leq \left(\frac{L}{2}+aT\right),$$

$$u(x,T) = \frac{1}{2}\left\{\frac{2h}{L}[L-(x+aT)] + \frac{2h}{L}[L-(x-aT)]\right\}$$
$$= \frac{2h}{L}(L-x) \qquad \text{if} \ (\frac{L}{2}+aT) \leq x \leq L.$$

These results are shown as the thick line in Fig. 5.7.

The displacements $u(x,t)$ as a function of time are shown in Fig. 5.8. In the left-hand side column, the positions of the string are shown at various times t. Each case is a superposition of two traveling waves, one to the left and one to the right, shown in the right-hand side column. Both of them are traveling with the same speed a. The sum of these two traveling waves describes the exact up and down motion of the string. It is interesting to compare Fig. 5.8 with Fig. 5.2. They describe the same motion but with two different interpretations.

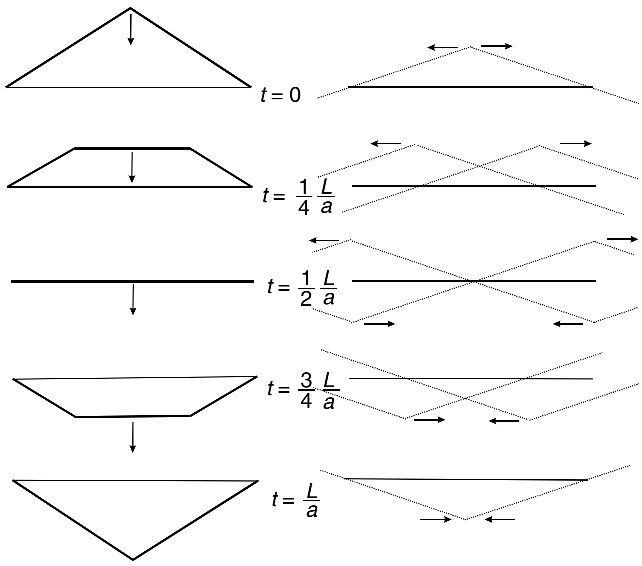

Fig. 5.8. The graph of the solution of the vibrating string with initial displacement $u(x,0)$ shown on the top of the left-hand side column. At various time t, the string will assume such positions as indicated in the left-hand side column. The positions are obtained as the superposition of a wave traveling to the right and a wave traveling to the left shown in the right-hand side column

Problems with Initial Velocity. Let us consider the case that the string is initially at rest but with a initial velocity of $g(x)$. The displacements of the string is the solution of the following problem:

$$\begin{aligned}\text{D.E.:}\quad & \frac{\partial^2 u(x,t)}{\partial x^2}=\frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2},\\ \text{B.C.:}\quad & u(0,t)=0;\ u(L,t)=0,\\ \text{I.C.:}\quad & u(x,0)=0;\ u_t(x,0)=g(x).\end{aligned}$$

With separation of variables, we will obtain (5.12) just as before, since the differential equation and the boundary conditions are the same

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi a}{L}t+b_n\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi}{L}x.$$

The initial condition $u(x,0)=0$ means that

$$u(x,0)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi}{L}x=0.$$

Therefore all a_n must be equal to zero. Thus

$$u(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi a}{L}t\sin\frac{n\pi}{L}x$$

and

$$\frac{\partial}{\partial t}u(x,t)=\sum_{n=1}^{\infty}b_n\frac{n\pi a}{L}\cos\frac{n\pi a}{L}t\ \sin\frac{n\pi}{L}x. \tag{5.28}$$

It follows from the other initial condition $u_t(x,0)=g(x)$ that:

$$u_t(x,0)=\sum_{n=1}^{\infty}b_n\frac{n\pi a}{L}\sin\frac{n\pi}{L}x=g(x). \tag{5.29}$$

This is a Fourier sine series, therefore

$$b_n\frac{n\pi a}{L}=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,\mathrm{d}x.$$

Thus the solution $u(x,t)$ is given by the infinite series

$$u(x,t)=\sum_{n=1}^{\infty}\left[\frac{2}{n\pi a}\int_0^L g\left(x'\right)\sin\frac{n\pi}{L}x'\,\mathrm{d}x'\right]\sin\frac{n\pi a}{L}t\ \sin\frac{n\pi}{L}x.$$

This solution is expressed in terms of a summation of infinite standing waves. We can also express it in terms of the sum of two traveling waves. With trigonometric identity

$$\sin a \,\cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right],$$

we can write (5.28) as

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \sum_{n=1}^{\infty} b_n \frac{n\pi a}{L}\cos\frac{n\pi a}{L}t\,\sin\frac{n\pi}{L}x \\ &= \frac{1}{2}\sum_{n=1}^{\infty} b_n \frac{n\pi a}{L}\sin\frac{n\pi}{L}(x+at) + \frac{1}{2}\sum_{n=1}^{\infty} b_n \frac{n\pi a}{L}\sin\frac{n\pi}{L}(x-at).\end{aligned}$$

Using (5.29,) we can write this expression as

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2}g^*(x+at) + \frac{1}{2}g^*(x-at),$$

where g^* is the odd periodic extension of g with period of $2L$, for the same reason as f^* is the odd periodic extension of f with a period of $2L$.

An integration of $\frac{\partial}{\partial t}u(x,t)$ will yield $u(x,t)$. The constant of integration is determined by the initial condition $u(x,0) = 0$. This condition is satisfied by the following integral:

$$u(x,t) = \int_0^t \frac{\partial u(x,t')}{\partial t'}\mathrm{d}t' = \frac{1}{2}\int_0^t g^*(x+at')\,\mathrm{d}t' + \frac{1}{2}\int_0^t g^*(x-at')\,\mathrm{d}t'.$$

With a change of variable

$$\tau = x + at', \qquad \mathrm{d}t' = \frac{1}{a}\mathrm{d}\tau,$$

the first integral on the right-hand side can be written as

$$\frac{1}{2}\int_0^t g^*(x+at')\,\mathrm{d}t' = \frac{1}{2a}\int_x^{x+at} g^*(\tau)\mathrm{d}\tau,$$

since at $t'=0,\ \tau = x$ and at $t'=t,\ \tau = x+at$.

Similarly, the second integral can be written as

$$\frac{1}{2}\int_0^t g^*(x-at')\,\mathrm{d}t' = -\frac{1}{2a}\int_x^{x-at} g^*(\tau)\mathrm{d}\tau.$$

It follows that:

$$\begin{aligned}u(x,t) &= \frac{1}{2a}\int_x^{x+at} g^*(\tau)\mathrm{d}\tau - \frac{1}{2a}\int_x^{x-at} g^*(\tau)\mathrm{d}\tau \\ &= \frac{1}{2a}\int_{x-at}^{x+at} g^*(\tau)\mathrm{d}\tau. \qquad (5.30)\end{aligned}$$

This is the solution for the case that the string has no initial displacement but is given an initial velocity $g(x)$.

Superposition of Solutions. If the string is given both an initial displacement and an initial velocity,

$$u(x,0) = f(x), \qquad u_t(x,0) = g\,(x)\,, \tag{5.31}$$

then the subsequent displacements can be written as the superposition of (5.27) and (5.30,) namely

$$u(x,t) = \frac{1}{2}\left[f^*\left(x-at\right)+f^*\left(x+at\right)\right]+\frac{1}{2a}\int_{x-at}^{x+at} g^*(\tau)\mathrm{d}\tau. \tag{5.32}$$

Note that both terms satisfy the homogeneous differential equation and the boundary conditions, while their sum clearly satisfies the initial conditions of (5.31).

In general the solution of a linear problem containing more than one nonhomogeneous conditions can be written as a sum of the solutions of problems each of which contains only one nonhomogeneous condition. The resolution of the original problem in this way, although not necessary, often simplifies the process of solving the problem.

5.1.5 Nonhomogeneous Wave Equations

Vibrating String with External Force. If there is an external force acting on the stretched string, then there will be an extra term in the governing differential equation. For example, if the weight of the string is not negligible, then in the derivation of (5.4), we must add to the equation the downward gravitational force, $-\rho\,\Delta x g$, where g is the constant gravitational acceleration. As a consequence, (5.6) becomes

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{g}{a^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{5.33}$$

Let us solve this equation with the same boundary and initial conditions as the previous problem:

$$u\,(0,t) = 0, \qquad u(L,t) = 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Since (5.33) is an nonhomogeneous equation, a straightforward application of separation of variables will not work. However, the following device will reduce this nonhomogeneous partial differential equation into a homogeneous partial differential equation plus an ordinary differential equation which we can solve. Let

$$u(x,t) = U(x,t) + \phi(x),$$

then

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2},$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 U(x,t)}{\partial t^2},$$

so the problem becomes

$$\text{D.E.:} \quad \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} - \frac{g}{a^2} = \frac{1}{a^2}\frac{\partial^2 U(x,t)}{\partial t^2},$$

$$\text{B.C.:} \quad u(0,t) = U(0,t) + \phi(0) = 0, \; u(L,t) = U(L,t) + \phi(L) = 0,$$

$$\text{I.C.:} \quad u(x,0) = U(x,0) + \phi(x) = f(x), \qquad u_t(x,0) = U_t(x,0) = 0.$$

Now we require

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} - \frac{g}{a^2} = 0,$$

$$\phi(0) = 0, \quad \phi(L) = 0.$$

This is a second-order ordinary differential equation with two boundary conditions, which can be readily solved to give

$$\phi(x) = \frac{g}{2a^2}\left(x^2 - Lx\right).$$

With $\phi(x)$ so chosen, what we are left with are the differential equation and boundary and initial conditions for $U(x,t)$

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 U(x,t)}{\partial t^2},$$

$$U(0,t) = 0, \qquad U(L,t) = 0,$$

$$U(x,0) = f(x) - \phi(x), \; U_t(x,0) = 0.$$

Note that other than the modification of one of the initial conditions, this is the same equation we solved before. Therefore we can write down its solutions immediately,

$$U(x,t) = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi a}{L}t \, \sin\frac{n\pi}{L}x,$$

$$b_n = \frac{2}{L}\int_0^L \left[f(x') - \frac{g}{2a^2}\left(x'^2 - Lx'\right)\right]\sin\frac{n\pi}{L}x' \, \mathrm{d}x'.$$

It follows that the displacements of the string, including the effect of its own weight, are given by:

$$u(x,t) = \frac{g}{2a^2}\left(x^2 - Lx\right)$$
$$+\sum_{n=1}^{\infty}\left\{\frac{2}{L}\int_0^L \left[f(x) - \frac{g}{2a^2}\left(x^2 - Lx\right)\right]\sin\frac{n\pi}{L}x\,\mathrm{d}x\right\}\cos\frac{n\pi a}{L}t\,\sin\frac{n\pi}{L}x.$$

Forced Vibration and Resonance. Now suppose that the string fixed at both ends is influenced by a periodic external force per unit length $F(t) = F_l\cos\omega t$. In this case, the string will satisfy the nonhomogeneous partial differential equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} + F_0\cos\omega t = \frac{\partial^2 u(x,t)}{\partial t^2}, \tag{5.34}$$

where $F_0 = F_l/\rho$. The boundary conditions remain the same

$$u(0,t) = 0, \quad u(L,t) = 0.$$

If the string is initially at rest in equilibrium when the external force begins to act, then the displacement $u(x,t)$ will also have to satisfy the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0.$$

Since the external force is purely sinusoidal, it is relatively easy to find a solution to satisfy the differential equation and the boundary conditions. Just like solving ordinary differential equation, we know that the particular solution will have to oscillate with $\cos\omega t$. Therefore, let us take the trial solution

$$v(x,t) = X(x)\cos\omega t.$$

Replace $u(x,t)$ with $v(x,t)$ in (5.34), we have

$$a^2X''(x)\cos\omega t + F_0\cos\omega t = -\omega^2X(x)\cos\omega t$$

or

$$X''(x) = -\frac{\omega^2}{a^2}X(t) - \frac{F_0}{a^2},$$

which yields the solution

$$X(x) = A\cos\frac{\omega x}{a} + B\sin\frac{\omega x}{a} - \frac{F_0}{\omega^2}.$$

The boundary conditions require

$$X(0) = X(L) = 0.$$

Thus

$$X(0) = A - \frac{F_0}{\omega^2} = 0, \qquad A = \frac{F_0}{\omega^2}.$$

Furthermore

$$X(L) = \frac{F_0}{\omega^2}\cos\frac{\omega L}{a} + B\sin\frac{\omega L}{a} - \frac{F_0}{\omega^2} = 0$$

or

$$B = \frac{F_0}{\omega^2}\frac{\left(1-\cos\frac{\omega L}{a}\right)}{\sin\frac{\omega L}{a}},$$

except for $\omega = n\pi a/L$ with even n, in that case $B = 0$. Therefore, in general

$$X(x) = \frac{F_0}{\omega^2}\cos\frac{\omega x}{a} + \frac{F_0}{\omega^2}\frac{\left(1-\cos\frac{\omega L}{a}\right)}{\sin\frac{\omega L}{a}}\sin\frac{\omega x}{a} - \frac{F_0}{\omega^2}, \tag{5.35}$$

$$v(x,t) = X(x)\cos\omega t.$$

But this solution does not satisfy the initial conditions. Therefore we resort to the method of splitting the solution into two parts

$$u(x,t) = v(x,t) + U(x,t).$$

In terms of v and U, the original equation and the boundary of initial conditions become

$$a^2\frac{\partial^2 v(x,t)}{\partial x^2} + a^2\frac{\partial^2 U(x,t)}{\partial x^2} + F_0\cos\omega t = \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2 U(x,t)}{\partial t^2},$$

$$\begin{aligned} u(0,t) &= v(0,t) + U(0,t) = 0, \\ u(L,t) &= v(L,t) + U(L,t) = 0, \\ u(x,0) &= v(x,0) + U(x,0) = 0, \\ u_t(x,0) &= v_t(x,0) + U_t(x,0) = 0. \end{aligned}$$

Since

$$a^2\frac{\partial^2 v(x,t)}{\partial x^2} + F_0\cos\omega t = \frac{\partial^2 v(x,t)}{\partial t^2},$$
$$v(0,t) = 0, \qquad v(L,t) = 0,$$

and

$$v(x,0) = X(x), \qquad v_t(x,0) = -\omega X(x)\sin 0 = 0.$$

Hence

$$a^2\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{\partial^2 U(x,t)}{\partial t^2},$$
$$U(0,t) = 0, \quad U(L,t) = 0,$$
$$U(x,0) = -X(x), \quad U_t(x,0) = 0.$$

This is the homogeneous differential equation we solved before

$$U(x,t) = \sum_{n=1}^{\infty} \left(-\frac{2}{L} \int_0^L X(x') \sin \frac{n\pi}{L} x' \, \mathrm{d}x' \right) \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t. \tag{5.36}$$

Therefore the solution $u(x,t)$ is given by

$$u(x,t) = X(x) \cos \omega t + U(x,t)$$

with $X(x)$ given by (5.35) and $U(x,t)$ given by (5.36).

This solution is valid for any ω. However, if ω approach $\omega_m = \frac{m\pi a}{L}$ with an odd integer m, then $X(x)$ in (5.35) approaches ∞, thus resonance occurs. But if $\omega = \frac{m\pi a}{L}$ with an even integer m, then

$$X(x) = \frac{F_0}{\omega^2} \cos \frac{m\pi x}{L} - \frac{F_0}{\omega^2}$$

and resonance does not occur in this case.

5.1.6 D'Alembert's Solution of Wave Equations

Using the separation of variables, we have solved the vibrating string problem by first finding the eigenvalues and eigenfunctions dictated by the boundary conditions. In the next step, we used the initial conditions to determine the constants in the Fourier series of the solution. Now we will introduce a method of doing just the opposite. We will first solve the initial values problem, and then find the solution to satisfy the boundary conditions.

Let us solve the following pure initial values problem:

$$\begin{aligned} \text{D.E.:} \quad & \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u(x,t)}{\partial t^2} \\ \text{I.C.:} \quad & u(x,0) = f(x), \;\; u_t(x,0) = g(x), \end{aligned}$$

for $0 < t < \infty$ and $-\infty < x < \infty$. It turns out the general solution of this equation can be found by a change of variables:

$$\begin{aligned} \zeta &= x + at \\ \eta &= x - at. \end{aligned}$$

According to the chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \zeta}{\partial x} \frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial t} &= \frac{\partial \zeta}{\partial t} \frac{\partial}{\partial \zeta} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} = a \frac{\partial}{\partial \zeta} - a \frac{\partial}{\partial \eta}, \end{aligned}$$

so the differential equation becomes

$$\left(\frac{\partial}{\partial \zeta}+\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \zeta}+\frac{\partial}{\partial \eta}\right) u=\frac{1}{a^2}\left(a \frac{\partial}{\partial \zeta}-a \frac{\partial}{\partial \eta}\right)\left(a \frac{\partial}{\partial \zeta}-a \frac{\partial}{\partial \eta}\right) u$$

or

$$\left(\frac{\partial^2}{\partial \zeta^2}+2 \frac{\partial^2}{\partial \zeta \partial \eta}+\frac{\partial^2}{\partial \eta^2}\right) u=\left(\frac{\partial^2}{\partial \zeta^2}-2 \frac{\partial^2}{\partial \zeta \partial \eta}+\frac{\partial^2}{\partial \eta^2}\right) u .$$

Clearly

$$\frac{\partial^2}{\partial \zeta \partial \eta} u=0 .$$

This new equation can be solved easily by two straightforward integrations. Integration with respect to ζ gives an arbitrary function $A\left(\eta\right)$ of η, that is

$$\frac{\partial}{\partial \eta} u=A\left(\eta\right),$$

since

$$\frac{\partial}{\partial \zeta}\left(\frac{\partial}{\partial \eta} u\right)=\frac{\partial}{\partial \zeta} A\left(\eta\right)=0 .$$

The second integration with respect to η gives

$$u=\int A\left(\eta\right) \mathrm{d} \eta+G\left(\zeta\right),$$

where $G\left(\zeta\right)$ is an arbitrary function of ζ. Since $A\left(\eta\right)$ is arbitrary, we might as well write $F\left(\eta\right)$ in place of $\int A\left(\eta\right) \mathrm{d} \eta$. Thus

$$u\left(\zeta, \eta\right)=F\left(\eta\right)+G\left(\zeta\right).$$

Substituting back the original variables, we have

$$u(x, t)=F\left(x-a t\right)+G\left(x+a t\right). \tag{5.37}$$

Thus the general solution of the wave equation is a sum of two arbitrary moving waves, each moving in opposite direction with velocity a.

It is easy to see that (5.37) is indeed the solution of the wave equation. We can use the chain rule

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} & =\frac{\mathrm{d} F(x-a t)}{\mathrm{d}(x-a t)} \frac{\partial(x-a t)}{\partial t}+\frac{\mathrm{d} G(x+a t)}{\mathrm{d}(x+a t)} \frac{\partial(x+a t)}{\partial t} \\ & =-a F^{\prime}(x-a t)+a G^{\prime}(x+a t), \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u(x, t)}{\partial t^2} & =-a \frac{\mathrm{d} F^{\prime}(x-a t)}{\mathrm{d}(x-a t)} \frac{\partial(x-a t)}{\partial t}+a \frac{\mathrm{d} G^{\prime}(x+a t)}{\mathrm{d}(x+a t)} \frac{\partial(x+a t)}{\partial t} \\ & =a^2 F^{\prime \prime}(x-a t)+a^2 G^{\prime \prime}(x+a t). \end{aligned}$$

Similarly, we obtain

$$\frac{\partial^2 u(x,t)}{\partial x^2} = F''(x-at) + G''(x+at).$$

It is readily seen that the differential equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

is satisfied.

Now if we impose the initial conditions

$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x),$$

we have

$$u(x,0) = F(x) + G(x) = f(x), \tag{5.38}$$

$$u_t(x,0) = -aF'(x) + aG'(x) = g(x). \tag{5.39}$$

Integrating (5.39) from any fixed point, say 0, to x gives

$$-aF(x) + aG(x) + aF(0) - aG(0) = \int_0^x g(x')\mathrm{d}x'$$

or

$$-F(x) + G(x) = \frac{1}{a}\int_0^x g(x')\mathrm{d}x' - F(0) + G(0). \tag{5.40}$$

Solving for $F(x)$ and $G(x)$ from (5.38) and (5.40), we get

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_0^x g(x')\mathrm{d}x' + \frac{1}{2}\left[F(0) - G(0)\right], \tag{5.41}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_0^x g(x')\mathrm{d}x' - \frac{1}{2}\left[F(0) - G(0)\right]. \tag{5.42}$$

If we replace the argument x by $x - at$ on both sides of (5.41), we can write

$$F(x-at) = \frac{1}{2}f(x-at) - \frac{1}{2a}\int_0^{x-at} g(x')\mathrm{d}x' + \frac{1}{2}\left[F(0) - G(0)\right].$$

Similarly, replacing the argument x by $x + at$ on both sides of (5.42), we have

$$G(x+at) = \frac{1}{2}f(x+at) + \frac{1}{2a}\int_0^{x+at} g(x')\mathrm{d}x' - \frac{1}{2}\left[F(0) - G(0)\right].$$

Thus

$$u(x,t) = F(x-at) + G(x+at) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] \\ +\frac{1}{2a}\int_0^{x+at} g(x')\mathrm{d}x' - \frac{1}{2a}\int_0^{x-at} g(x')\mathrm{d}x'.$$

Reversing the upper and lower limits of the last integral and combining it with the preceding one, we obtain

$$u(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(x')\mathrm{d}x'. \tag{5.43}$$

This is the solution for $-\infty < x < \infty$ without any boundary condition.

Now suppose that the string is of finite length from 0 to L, and both ends are fixed, so we have the following boundary conditions:

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

In this case, $f(x)$ and $g(x)$ are, of course, defined only in the range of $0 \le x \le L$. We want to find the solution that satisfy these additional conditions.

First, if $u(0,t) = 0$, then according to (5.43)

$$u(0,t) = \frac{1}{2}\left[f(-at) + f(at)\right] + \frac{1}{2a}\int_{-at}^{+at} g(x')\mathrm{d}x' = 0.$$

Since $f(x)$ and $g(x)$ are two unrelated functions, to satisfy this condition we must have

$$f(-at) + f(at) = 0, \tag{5.44}$$

$$\int_{-at}^{+at} g(x')\mathrm{d}x' = 0. \tag{5.45}$$

It is clear from (5.44) that

$$f(x) = -f(-x).$$

Therefore $f(x)$ must be an odd function. In other words, $f(x)$ should be antisymmetrically extended to negative x.

The integral in (5.45) can be written as

$$\int_{-at}^{0} g(x')\mathrm{d}x' + \int_0^{at} g(x)\mathrm{d}x = 0.$$

With a change of variable $x' = -x$ in the first integral, we can write it as

$$\int_{-at}^{0} g(x')\mathrm{d}x' = \int_0^{at} g(-x)\mathrm{d}x.$$

For

$$\int_0^{at} g(-x)\mathrm{d}x + \int_0^{at} g(x)\mathrm{d}x = 0,$$

we must have

$$g(-x) = -g(x).$$

Therefore, $g(x)$ also is an odd function.

Similarly, to satisfy the other boundary condition

$$u(L,t) = \frac{1}{2}\left[f(L-at) + f(L+at)\right] + \frac{1}{2a}\int_{L-at}^{L+at} g(x')\mathrm{d}x' = 0,$$

we require

$$f(L-at) + f(L+at) = 0, \tag{5.46}$$

$$\int_{L-at}^{L+at} g(x')\mathrm{d}x' = 0. \tag{5.47}$$

Thus

$$f(L-ct) = -f(L+ct).$$

Since $f(x)$ is an odd function, so

$$f(L-ct) = -f(-L+ct).$$

Therefore

$$f(-L+ct) = f(L+ct).$$

It follows that:

$$f(-L+ct+2L) = f(L+ct) = f(-L+ct)$$

This shows that $f(x)$ should be a periodic function of period $2L$.

Now (5.47) can be written as

$$\int_{L-at}^{L} g(x')\mathrm{d}x' + \int_{L}^{L+at} g(x')\mathrm{d}x' = 0.$$

With a change of variable, $x' = L - x$ in the first integral, it becomes

$$\int_{L-at}^{L} g(x')\mathrm{d}x' = \int_0^{ct} g(L-x)\mathrm{d}x.$$

Change the variable x' to $L + x$ in the second integral, we can write

$$\int_{L}^{L+at} g(x')\mathrm{d}x' = \int_0^{ct} g\left(L+x\right)\mathrm{d}x.$$

So for

$$\int_0^{ct} g(L-x)\mathrm{d}x + \int_0^{ct} g(L+x)\mathrm{d}x = 0,$$

we must have

$$g(L-x) = -g(L+x).$$

Since $g(L-x)$ is an odd function, so

$$g(L-x) = -g(-L+x).$$

Thus

$$g(-L+x) = g(L+x).$$

It follows that:

$$g(-L+x+2L) = g(L+x) = g(-L+x)$$

Therefore $g(x)$ should also be a periodic function of period $2L$.

Thus, if we define $f^*(x)$ and $g^*(x)$ as the odd periodic functions of period $2L$, whose definitions in $0 \leq x \leq L$ are, respectively, $f(x)$ and $g(x)$, then the solution of the problem is given by

$$u(x,t) = \frac{1}{2}\left[f^*(x-at) + f^*(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g^*(\tau)\mathrm{d}\tau, \tag{5.48}$$

which is exactly the same as (5.32).

This is known as D'Alembert's solution. The French mathematician Jean le Rond D'Alembert (1717–1783) first discovered it around 1750s. This method is very elegant but unfortunately it is limited to the solution of this type of equations. The method of separation of variable which, as we have seen, can also give the same solution, is more general. We shall use it to solve many other types of partial differential equations.

Example 5.1.3. A string of length L with tension T and density ρ, fixed at both ends, is initially given the displacement of a small triangular pulse at $x = L/4$, shown on the top line of Fig. 5.9, and is released from rest. Determine its subsequent motion.

Solution 5.1.3. Let the initial triangular pulse be $f(x)$, the subsequent displacement is simply given by

$$u(x,t) = \frac{1}{2}\left[f^*(x-at) + f^*(x+at)\right]$$

where f^* is equal to $f(x)$ in the range of $0 \leq x \leq L$, outside this range f^* is the antisymmetrical periodic extension of $f(x)$ with a period of $2L$. This extension is shown in Fig. 5.9 as the dotted lines. The actual displacements of

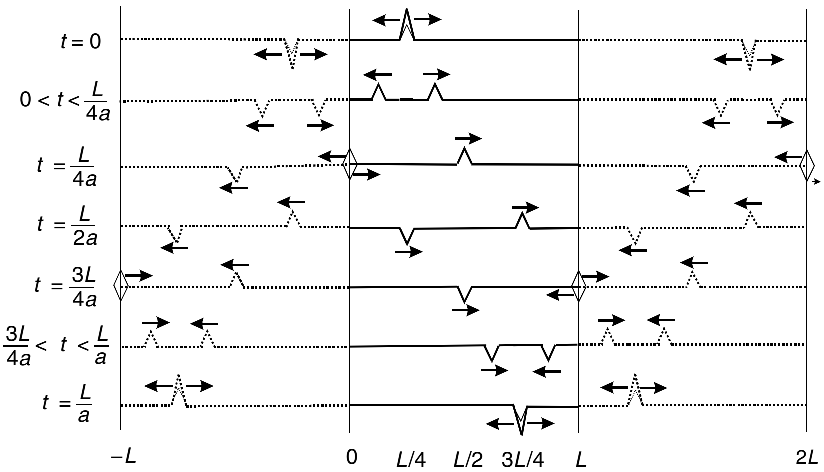

Fig. 5.9. Time development of an initial triangular pulse at $x = L/4$ on a string of length L, fixed at both ends

the string are shown in Fig. 5.9 as the thick dark lines in the "physical" space between 0 and L. However, after $t > L/4a$, what happens in the physical space is the result of some dotted pulse moving from the "mathematical" space into the "physical" space. In other words, the finite string of length L can be regarded as if it is only a segment of an infinitely long line. The pulses are moving on this infinite line. What happens in the segment between 0 and L is the displacements of the real string that we can see. Other parts of this infinitely long line are only mathematical constructs which are used to predict what will happen in the real string.

Soon after the pulse is released, it splits into two equal parts and moving in the opposite direction with the same velocity a which is equal to $\sqrt{T/\rho}$. This motion is shown in the second line of Fig. 5.9. At $t = L/4a$, the left pulse reaches the end point at $x = 0$. It will gradually disappear as shown in the third line of the figure. Then an identical pulse will reappear but upside down. In the time interval $L/4a < t < 3L/4a$, there are two pulses $L/2$ distance apart, one up and one down, both moving to the right as shown in the fourth line. At $t = 3L/4a$, the right pulse reaches the end point at $x = L$ and gradually disappears. This is shown in the fifth line. Soon after an identical pulse reappears upside down, and moves to the left. In the time interval $3L/4a < t < L/a$, two pulses, both upside down, are moving toward each other. This is shown in the sixth line. The end of the first half cycle is at $t = L/a$. At that time, the original pulse appears upside down at $x = 3L/4$. This is shown in the last line of Fig. 5.9. After that the motion will repeat itself in reverse direction until $t = 2L/a$. That is the end of the first cycle and the string will return to its original shape. These are a well-known facts which can be easily checked.

Example 5.1.4. When a piano wire is struck by a narrow hammer at x_0, a localized velocity is imparted to that point. At that moment, the wire is still at rest, but it will start to vibrate thereafter. Describe the motion assuming $a = \sqrt{T/\rho}$ and $0 < x_0 < L/4$.

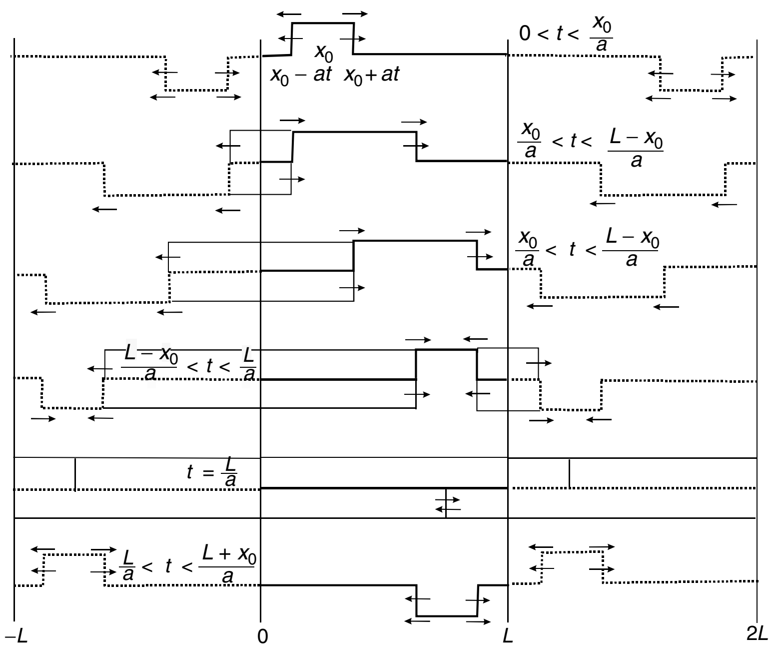

Fig. 5.10. The motion of wire of length L after a locialized velocity is imparted at x_0

Solution 5.1.4. To find the displacements of the wire, we have to solve the boundary value problem of the vibrating string with the following initial conditions

$$u(x,0) = 0 \qquad \text{and} \qquad u_t(x,0) = \delta\,(x - x_0),$$

where $\delta\,(x - x_0)$ is a delta function. According to (5.48),

$$u(x,t) = \frac{1}{2a} \int_{x-at}^{x+at} \delta^*\,(x' - x_0)\,\mathrm{d}x',$$

where $\delta^*\,(x - x_0)$ is the odd periodic function of period $2L$, whose definition between 0 and L is the delta function $\delta\,(x - x_0)$. By the definition of delta function

$$\int_{x-at}^{x+at} \delta\,(x' - x_0)\,\mathrm{d}x' = \begin{cases} 1 & \text{if} \quad x - at < x_0 < x + at \\ 0 & \qquad \text{otherwise} \end{cases}.$$

By adding at to both sides, we see that the condition $x - at < x_0$ is equivalent to $x < x_0 + at$. Similarly, $x_0 < x + at$ is equivalent to $x_0 - at < x$. That is, between $x_0 - at$ and $x_0 + at$, the integral is equal to 1, outside this range, it is equal to zero.

This result can also be obtained by using

$$\delta\left(x - x_0\right) = \frac{\mathrm{d}}{\mathrm{d}x} U\left(x - x_0\right),$$

where $U\left(x - x_0\right)$ is the step function, and

$$\begin{aligned}\int_{x-at}^{x+at} \delta\left(x' - x_0\right) \mathrm{d}x' &= U\left(x + at - x_0\right) - U(x - at - x_0) \\ &= U\left(x - \left(x_0 - at\right)\right) - U\left(x - \left(x_0 + at\right)\right).\end{aligned}$$

This gives the same result. This means that soon after the wire is struck, a rectangular pulse centered at x_0 with a height of $1/2a$ will appear as shown in Fig. 5.10.

The actual displacements of the wire between 0 and L (which we call the "physical" space) are shown as the thick dark lines in Fig. 5.10. The images due to antisymmetrical and periodic extensions in $-L < x < 0$ and $0 < x < 2L$ (which are part of the "mathematical" space) are shown as dotted lines in the figure. These rectangular pulses will continue to expand at a constant rate on an infinite line without limit. After a while, some images are coming from the mathematical space into the physical space. They will then cancel part of the expanding rectangular pulse originally in the physical space to give the actual displacements of the wire. As a result, the motion of the wire appears as follows.

First the width of the rectangular pulse will expand at a constant rate as shown in the first line of Fig. 5.10. At $t = x_0/a$, the left edge of this rectangular pulse reaches the end point at $x = 0$. Immediately it bounces back and moves toward the right. In the time interval $x_0/a < t < (L - x_0)/a$, a rectangular pulse of constant width of $2x_0$ is moving to the right with velocity of a. This motion is shown in the second and the third line of Fig. 5.10. At $t = (L - x_0)/a$, the right edge of the rectangular pulse reaches the end point at $x = L$ and bounces back. The width of the rectangular pulse starts to shrink as shown in the fourth line of Fig. 5.10. At $t = L/a$, the pulse disappears. What happened is that the two negative rectangular pulses have moved from the mathematical space so far into the physical space so that they touch each other. As a consequence, they completely cancel the positive rectangular pulse. This is shown in line five. Soon after that, the two negative rectangular pulses overlap and over compensate the positive rectangular pulse, as a result another reactangular pulse reappears up side down. This is shown in the last line of Fig. 5.10. After that the motion will repeat itself in reverse order. The end of the first cycle is at $t = 2L/a$. At that moment, the pulse will disappear, but will soon reappear to repeat the motion.

5.2 Two-Dimensional Wave Equations

5.2.1 The Governing Equation of a Vibrating Membrane

A vibrating membrane such as a drumhead is a two-dimensional version of a vibrating string. We assume that the membrane is stretched uniformly under a tension T per unit length. That is, at each point of the membrane the tension per unit length along any straight line through that point, independent of the orientation of the line, is perpendicular to that line with a magnitude T.

Let us consider the vibration of such a membrane; we shall suppose that the density ρ defined as the mass per unit area is a constant. If its equilibrium position is taken as the xy plane, then we are concerned with displacements $z\,(x, y, t)$ perpendicular to this plane. Consider a small rectangular element of sides $\Delta x, \Delta y$ shown in Fig. 5.11. We proceed as before. We assume the weight of the element is negligible compared with the tensile force. Applying Newton's second law to the element $\Delta x\,\Delta y$, we have

$$T_2\,\Delta y \sin\theta_2 - T_1\,\Delta y \sin\theta_1 + T_4\,\Delta x \sin\theta_4 - T_3\,\Delta x \sin\theta_3 = \rho\,\Delta x\,\Delta y \frac{\partial^2 z}{\partial t^2}. \quad (5.49)$$

Now there is no horizontal motion either in x- or in y-directions, therefore

$$T_2 \cos\theta_2 = T_1 \cos\theta_1, \qquad T_4 \cos\theta_4 = T_3 \cos\theta_3. \quad (5.50)$$

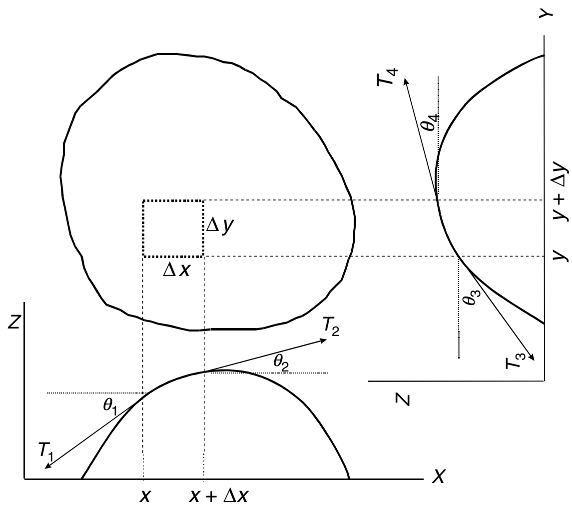

Fig. 5.11. A membrane under uniform tension

We assume the slopes $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are uniformly small over the domain, so these horizontal components in (5.50) can be regarded as the tension T of membrane. Since the tension is uniform, so

$$T_2 \cos\theta_2 = T_1 \cos\theta_1 = T_4 \cos\theta_4 = T_3 \cos\theta_3 = T.$$

Dividing both sides of (5.49) by the appropriate expression of T, we obtain

$$\Delta y \tan\theta_2 - \Delta y \tan\theta_1 + \Delta x \tan\theta_4 - \Delta x \tan\theta_3 = \frac{1}{T}\rho\,\Delta x\,\Delta y \frac{\partial^2 z}{\partial t^2}$$

or

$$\Delta y \left(\left.\frac{\partial z}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial z}{\partial x}\right|_{x} \right) + \Delta x \left(\left.\frac{\partial z}{\partial y}\right|_{y+\Delta y} - \left.\frac{\partial z}{\partial y}\right|_{y} \right) = \frac{1}{T}\rho\,\Delta x\,\Delta y \frac{\partial^2 z}{\partial t^2}.$$

In the limit of $\Delta x \to 0,\ \Delta y \to 0$, the last equation can be written as

$$\Delta y \frac{\partial^2 z}{\partial x^2}\Delta x + \Delta x \frac{\partial^2 z}{\partial y^2}\Delta y = \frac{1}{T}\rho\,\Delta x\,\Delta y \frac{\partial^2 z}{\partial t^2}.$$

It follows that:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2},$$

where

$$v = \sqrt{\frac{T}{\rho}}.$$

5.2.2 Vibration of a Rectangular Membrane

Let us consider the vibration of the rectangular membrane shown in Fig. 5.12.

The displacements $z(x, y, t)$ of the membrane out of the xy plane are given by the solution of the following problem:

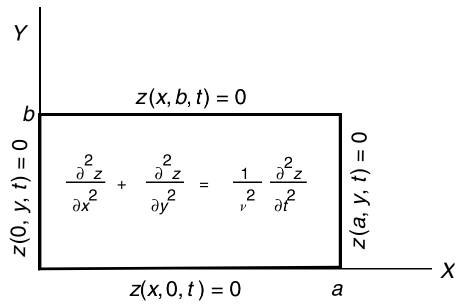

Fig. 5.12. A vibrating rectangular membrane

$$\text{D.E.:} \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2},$$

$$\text{B.C.:} \quad z(0, y, t) = 0, \quad z(a, y, t) = 0,$$

$$z(x, 0, t) = 0, \quad z(x, b, t) = 0,$$

$$\text{I.C.:} \quad z(x, y, 0) = f(x, y), \quad z_t(x, y, 0) = g(x, y).$$

We will again use the method of separation of variables,

$$z(x, y, t) = X(x)Y(y)T(t).$$

The differential equation can be written as

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = \frac{1}{v^2} X(x)Y(y)T''(t).$$

Dividing by $X(x)Y(y)T(t)$, we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{v^2} \frac{T''(t)}{T(t)}.$$

The left-hand side is a function of x and y, and the right-hand side is a function of t. Since x, y, t are independent variables, both sides must be equal to the same constant

$$\frac{1}{v^2} \frac{T''(t)}{T(t)} = \lambda, \tag{5.51}$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \lambda.$$

We can separate the x and y dependence by writing

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)}.$$

The left-hand side is a function of x alone and the right-hand side is a function of y alone, so it follows that both sides must be equal to the same constant. The constant must be a negative number for the same reason as the separation constant in the vibrating string problem is negative. Otherwise the boundary conditions in x would not be satisfied. Therefore we write

$$\lambda - \frac{Y''(y)}{Y(y)} = -\alpha^2,$$

$$\frac{X''(x)}{X(x)} = -\alpha^2.$$

Thus,

$$X''(x) = -\alpha^2 X(x), \tag{5.52}$$

$$Y''(y) = \left(\lambda + \alpha^2\right) Y(y).$$

Since λ is a constant yet to be determined, so we can combine it with α^2 as another constant. Again to satisfy the boundary conditions in y, that constant must be a negative number, therefore we write

$$\lambda + \alpha^2 = -\beta^2 \tag{5.53}$$

and

$$Y''(y) = -\beta^2 Y(y). \tag{5.54}$$

The boundary conditions in terms of $X(x)$ and $Y(y)$ are

$$X(0) = X(a) = 0; \qquad Y(0) = Y(b) = 0.$$

The solution to (5.52) and (5.54), together with these boundary conditions are

$$X(x) = \sin \alpha x, \qquad \alpha = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \ldots$$
$$Y(y) = \sin \beta y, \qquad \beta = \frac{m\pi}{b}, \qquad m = 1, 2, 3, \ldots.$$

To emphasize the fact that for each integer n, there is a separate eigenfunction solution, we write

$$X_n(x) = \sin \frac{n\pi}{a} x.$$

Similarly, for each m, there is a separate $Y_m(y)$,

$$Y_m(y) = \sin \frac{m\pi}{b} y.$$

It follows from (5.53) that for each pair of n and m, there is a constant λ

$$\lambda_{nm} = -\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right].$$

Clearly, λ_{nm} depends on two integers n and m. For each λ_{nm}, there is a time-dependent equation as seen from (5.51)

$$T''_{nm}(t) = \lambda_{nm} v^2 T_{nm}(t).$$

Therefore

$$T_{nm}(t) = a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t,$$

where

$$\omega_{nm} = \sqrt{-\lambda_{nm} v^2} = \left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right]^{1/2} \pi v.$$

Thus, for each pair of n and m, we have a solution

$$z_{nm}(x, y, t) = X_n(x) Y_m(t) T_{nm}(t).$$

We can regard this as the (n, m) normal mode. The complete solution to problem of vibrating rectangular membrane can be expressed as a superposition of these normal modes

$$\begin{aligned} z(x, y, t) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} z_{nm}(x, y, t) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}. \end{aligned} \tag{5.55}$$

The coefficients a_{nm} and b_{nm} are determined by the initial conditions. Using the condition, at $t = 0$, $z(x, y, 0) = f(x, y)$, we have

$$z(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = f(x, y).$$

This is known as a *double Fourier series*. We will assume that $f(x, y)$ can indeed be expressed in such a series. If we define $R_m(x)$ as

$$R_m(x) = \sum_{n=1}^{\infty} a_{nm} \sin \frac{n\pi x}{a}, \tag{5.56}$$

then

$$f(x, y) = \sum_{m=1}^{\infty} R_m(x) \sin \frac{m\pi y}{b}.$$

For a fixed x, this is a half-range Fourier sine expansion of $f(x, y)$ on $0 \le y \le b$. Therefore

$$R_m(x) = \frac{2}{b} \int_0^b f(x, y) \sin \frac{m\pi y}{b} \mathrm{d}y. \tag{5.57}$$

By definition, $R_m(x)$ is also given by (5.56), which is a half-range Fourier sine expansion of $R_m(x)$ on $0 \le x \le a$. So

$$a_{nm} = \frac{2}{a} \int_0^a R_m(x) \sin \frac{n\pi x}{a} \mathrm{d}x.$$

Putting $R_m(x)$ of (5.57) into this expression, we obtain

$$a_{nm} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \mathrm{d}x \, \mathrm{d}y. \tag{5.58}$$

This is the generalized Euler coefficients for the double Fourier series.

To determine b_{nm}, we differentiate (5.55) termwise with respect to t, using the condition $z_t\,(x, y, 0) = g\,(x, y)$, we obtain

$$\left.\frac{\partial z}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{nm} b_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = g(x, y).$$

Proceeding as before, we obtain

$$b_{nm} = \frac{1}{\omega_{nm}} \frac{4}{ab} \int_0^a \int_0^b g(x, y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \mathrm{d}x \,\mathrm{d}y. \tag{5.59}$$

If the initial conditions are

$$u\,(x, y, 0) = f(x, y), \qquad u_t\,(x, y, 0) = g(x, y) = 0,$$

then $b_{nm} = 0$ and

$$z\,(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \cos \omega_{nm} t \,\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

with a_{nm} given by (5.58).

In general, since

$$a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t = c_{nm} \cos\left(\omega_{nm} t + \delta_{nm}\right),$$

we can write the (n, m) normal mode as

$$z_{nm}\,(x, y, t) = c_{nm} \cos\left(\omega_{nm} t + \delta_{nm}\right) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}. \tag{5.60}$$

Its frequency is

$$\nu_{nm} = \frac{\omega_{nm}}{2\pi} = \left[\frac{n^2}{a^2} + \frac{m^2}{b^2}\right]^{1/2} \frac{\pi v}{2\pi} = \left[\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \frac{T}{4\rho}\right]^{1/2}. \tag{5.61}$$

The fundamental vibration is the $(1, 1)$ mode, for which the frequency is

$$\nu_{1,1} = \left[\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \frac{T}{4\rho}\right]^{1/2}.$$

The overtones in (5.61) are not related in any simple numerical way to the fundamental frequency, not like the vibrating string where the overtones (harmonics) are all simple integer multiples of the fundamental. For some reason, our ears find the sound more pleasing if the overtones are simply related to the fundamental. Therefore, the sound of a vibrating rectangular membrane is much less "musical" to the ear than that of a vibrating string.

According to (5.61,) the frequency of vibration depends on two integers m and n. As a result, it may happen that there are several different modes having the same frequency. For example, if $a = 3b$, then $(3, 3)$ and $(9, 1)$ modes have the same frequency. When two or more modes have the same frequency, we call it degenerate. Any combination of these degenerate modes gives another vibration with the same frequency.

In the (m, n) mode of (5.60) there are nodal lines at $x = 0, \frac{a}{n}, \frac{2a}{n}, \ldots, a$, and $y = 0, \frac{b}{m}, \frac{2b}{m}, \ldots, b$. On opposite sides of any nodal line the displacement has opposite sign. A few normal modes are shown in Fig. 5.13, in which the shaded and the unshaded parts are moving in opposite directions.

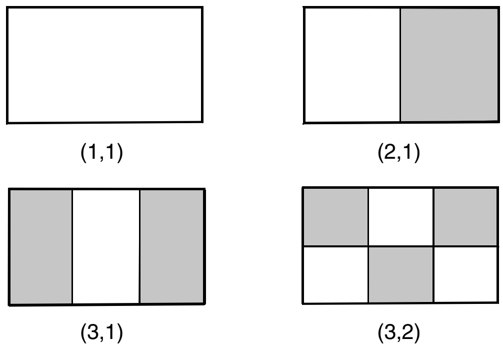

Fig. 5.13. Nodal lines and displcements of the normal modes $z_{11},\ z_{21},\ z_{31},\ z_{32}$ of a rectangular membrane

5.3 Three-Dimensional Wave Equations

A great many physical quantities satisfy the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

For example, in electrodynamics we learn that the electric field $\mathbf{E}$, the magnatic field $\mathbf{B}$, the scalar potential φ, and the vector potential $\mathbf{A}$, all together 10 quantities, satisfy this equation.

Following the separation of variables:

$$u\,(x,y,z,t)=X(x)Y(y)Z(z)T(t),$$

we obtain four ordinary differential equations

$$\frac{X''(x)}{X(x)}=-l^2,\qquad \frac{Y''(y)}{Y(y)}=-m^2,\qquad \frac{Z''(z)}{Z(z)}=-n^2,$$
$$\frac{1}{c^2}\frac{T''(t)}{T(t)}=-\alpha^2,$$

where $l,\ m,\ n,\ \alpha$ are separation constants and they must satisfy the relation

$$l^2+m^2+n^2=\alpha^2.$$

The general solutions of these equations can be easily found to be

$$\begin{aligned} X(x)&=A\cos lx+B\sin lx,\\ Y(y)&=C\cos my+D\sin my,\\ Z(z)&=E\cos nz+F\sin nz,\\ T(t)&=G\cos c\alpha t+H\sin c\alpha t, \end{aligned}\tag{5.62}$$

where $A, B, \ldots, H$ are constants. Since $\mathrm{e}^{\mathrm{i}lx} = \cos lx + \mathrm{i} \sin lx$, this set of solution can be expressed in an alternative form

$$\begin{aligned}
X(x) &= A' \exp(\mathrm{i}lx) + B' \exp(-\mathrm{i}lx), \\
Y(y) &= C' \exp(\mathrm{i}my) + D' \exp(-\mathrm{i}my), \\
Z(z) &= E' \exp(\mathrm{i}nz) + F' \exp(-\mathrm{i}nz), \\
T(t) &= G' \exp(\mathrm{i}c\alpha t) + H' \exp(-\mathrm{i}c\alpha t),
\end{aligned} \tag{5.63}$$

where $A', B', \ldots, H'$ are another set of constants. It can be readily verified by direct substitution that the expressions in (5.63) are solutions of the wave equation. Therefore we can use (5.63) and assume that we always refer to the real part, or we can just use (5.63) as it stands, without reference to its real or imaginary parts.

It is also possible that one (or two) of l^2, m^2, n^2 is negative. For example if

$$-l^2 + m^2 + n^2 = \alpha^2,$$

then

$$\begin{aligned}
X(x) &= A'' \cosh lx + B'' \sinh lx, \\
Y(y) &= C'' \cos my + D'' \sin my, \\
Z(z) &= E'' \cos nz + F'' \sin nz, \\
T(t) &= G'' \cos c\alpha t + H'' \sin c\alpha t,
\end{aligned} \tag{5.64}$$

where $A'', B'', \ldots, H''$ are still another set of constants.

Depending on the geometrical properties of any specific problem, one set of solutions is usually more convenient than others. Furthermore, the boundary conditions of the problem may restrict l, m, n to a set of allowed discrete values.

5.3.1 Plane Wave

Let us take the following set of solutions of the separated equations

$$\begin{aligned}
&X(x) = \mathrm{e}^{\mathrm{i}lx}, \quad Y(y) = \mathrm{e}^{\mathrm{i}my}, \quad Z(z) = \mathrm{e}^{\mathrm{i}nz}, \\
&T(t) = \mathrm{e}^{-\mathrm{i}c\alpha t}.
\end{aligned}$$

This gives a particular solution of the wave equation

$$u(x, y, z, t) = \mathrm{e}^{\mathrm{i}(lx+my+nz-c\alpha t)}.$$

This expression has a physical interpretation. To make it clear, we define a "wave vector" $\mathbf{k}$

$$\mathbf{k} = l\widehat{\mathbf{i}} + m\widehat{\mathbf{j}} + n\widehat{\mathbf{k}},$$

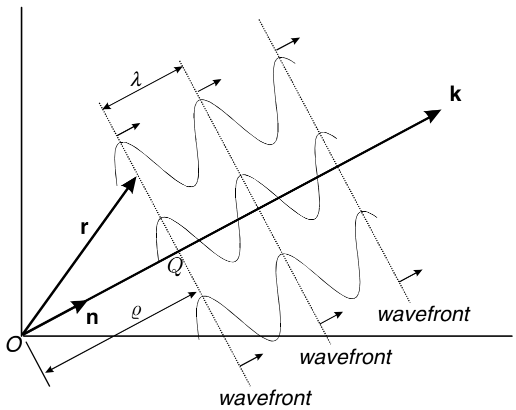

Fig. 5.14. A plane wave propagating in the direction of the vector $\mathbf{k}$

where $\widehat{\mathbf{i}}$, $\widehat{\mathbf{j}}$, $\widehat{\mathbf{k}}$ are three unit vectors along the coordinate axes. Let $\mathbf{r}$ be a position vector from origin O to a general point (x, y, z) on a plane perpendicular to $\mathbf{k}$, as shown in Fig. 5.14. Since

$$\mathbf{r} = x\widehat{\mathbf{i}} + y\widehat{\mathbf{j}} + z\widehat{\mathbf{k}},$$

so

$$\mathbf{k} \cdot \mathbf{r} = lx + my + nz.$$

Furthermore,

$$\mathbf{k} \cdot \mathbf{k} = k^2 = l^2 + m^2 + n^2 = \alpha^2.$$

Thus $u\,(x, y, z, t)$ can be written as

$$u\,(x, y, z, t) = \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - ckt)},$$

where we use the fact that $\alpha = k$. This represents a plane wave in three dimension moving in the direction of $\mathbf{k}$. A plane wave is one in which the disturbance is constant over all points of a plane perpendicular to the direction of propagation. Such a plane is often called a wavefront.

Let $\mathbf{n}$ be a unit vector in the direction of $\mathbf{k}$, then

$$\mathbf{k} \cdot \mathbf{r} = k\mathbf{n} \cdot \mathbf{r} = k\varrho$$

where ϱ is the perpendicular distance from the origin O measured along the vector $\mathbf{n}$ to the point Q at which this line meets the wavefront, as shown in Fig. 5.14. Thus

$$\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r} - ckt)} = \mathrm{e}^{\mathrm{i}(k\varrho - ckt)} = \mathrm{e}^{\mathrm{i}k(\varrho - ct)}.$$

If $\mathbf{k}$ is in the x-direction, this expression is just $\mathrm{e}^{\mathrm{i}k(x-ct)}$, which we recognize as a one-dimensional wave moving with velocity c. Furthermore, $k = 2\pi/\lambda$

and $kc = \omega$, where λ is the wavelength and ω, the angular frequency of this sinusoidal wave. Therefore

$$\mathrm{e}^{\mathrm{i}(lx+my+nz-c\alpha t)} = \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

represents a sinusoidal plane wave moving in the direction of $\mathbf{k}$, with wave length $2\pi/k$ and angular frequency $\omega = kc$.

Since $\mathbf{k}$ can be in any direction with any magnitude, the three-dimensional wave equation can have solutions which are plane waves moving in any direction with any wavelength. Since the wave equation is linear, we may have simultaneously as many plane waves, traveling in as many different directions. Thus the most general solution of the three-dimensional wave equation is a superposition of all kinds of plane waves in all kinds of directions, which is just a Fourier integral in three dimensions.

5.3.2 Particle Wave in a Rectangular Box

A free particle (a particle without force acting on it) is described in quantum mechanics by a somewhat different wave equation, known as Schrödinger equation

$$-\frac{h^2}{8\pi^2 M}\nabla^2\Psi = \mathrm{i}\frac{h}{2\pi}\frac{\partial}{\partial t}\Psi,$$

where M is the mass of the particle and h is the Planck's constant. While a discussion of quantum mechanics is outside the scope of this book, we can take it as a mathematical problem.

Using the separation of variables, we assume that

$$\Psi(x, y, z, t) = X(x)Y(y)Z(z)T(t),$$

so the equation becomes

$$-\frac{h^2}{8\pi^2 M}\left(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\right) = \mathrm{i}\frac{h}{2\pi}\frac{T'}{T}. \tag{5.65}$$

Both sides of this equation must be equal to a constant. Let

$$\mathrm{i}\frac{h}{2\pi}\frac{T'}{T} = E.$$

So

$$T(t) = \mathrm{e}^{(2\pi E/\mathrm{i}h)t}.$$

If we identify $T(t)$ as the time-dependent part of the wavefunction and write

$$\mathrm{e}^{(2\pi E/\mathrm{i}h)t} = \mathrm{e}^{-\mathrm{i}\omega t},$$

then

$$E = h\omega/2\pi = h\nu,$$

which is recognized as the energy of the particle, since according to the Planck's rule that energy is equal to h times the frequency.

The separated ordinary differential equations in $x,\ y,\ z$ are

$$\frac{X''}{X} = -l^2, \quad \frac{Y''}{Y} = -m^2, \quad \frac{Z''}{Z} = -n^2, \tag{5.66}$$

where $l,\ m,\ n$ are separation constants. Because of (5.65), they must satisfy the relation

$$E = \frac{h^2}{8\pi^2 M}\left(l^2 + m^2 + n^2\right). \tag{5.67}$$

Suppose that the particle is confined in a rectangular box of length a in x-direction, b in y-direction and c in z-direction. The fact that the wavefunction Ψ must vanish at the walls means Ψ must satisfy the following boundary conditions:

$$\begin{aligned} \Psi(0, y, z, t) &= \Psi(a, y, z, t) = 0, \\ \Psi(x, 0, z, t) &= \Psi(x, b, z, t) = 0, \\ \Psi(x, y, 0, t) &= \Psi(x, y, c, t) = 0. \end{aligned}$$

In order to satisfy these boundary conditions, the space part of the wavefunction must take the form

$$\Psi(x, y, z, t) = \sin\frac{n_1\pi}{a}x \sin\frac{n_2\pi}{b}y \sin\frac{n_3\pi}{c}y,$$

where $n_1,\ n_2,\ n_3$ independently assume the integer values of $1,\ 2,\ 3, \ldots$. This means that $l,\ m,\ n$ in (5.66) must take the values of

$$l = \frac{n_1\pi}{a}, \qquad m = \frac{n_2\pi}{b}, \qquad n = \frac{n_3\pi}{c}.$$

It follows from (5.67) that the energy is given by

$$E_{n_1,\ n_2,\ n_3} = \frac{h^2}{8M}\left[\left(\frac{n_1}{a}\right)^2 + \left(\frac{n_2}{b}\right)^2 + \left(\frac{n_3}{c}\right)^2\right].$$

Thus we see that the energy is quantized, by which we mean that the particle in the box cannot have just any energy, it has to be one of these special allowed values corresponding to $n_1,\ n_2,\ n_3$, each assuming one of the integer values of $1,\ 2,\ 3, \ldots$. This is in sharp contrast to the classical case.

Discrete energies exhibited in experiments were one of the main reasons that the quantum mechanics was developed. It is interesting to note that the quantization of energy is the consequence of the boundary conditions on the solutions of the Schrödinger equation.

5.4 Equation of Heat Conduction

To obtain the equation governing the flow of heat, we use the following experimental facts:

- Heat flows in the direction of decreasing temperature.
- The rate at which heat flows through an area is proportional to the area and to the temperature gradient normal to the area. The proportionality constant is called thermal conductivity k.
- The quantity of heat gained or lost by the body when its temperature changes is proportional to the mass of the body and to the temperature change. The proportionality constant is called the specific heat c.

The constants k, c, and density ρ (mass per unit volume) of most materials are listed in the handbooks of chemistry and physics.

Let the temperature be $u(x, y, z, t)$. The quantity of heat ΔQ required to produce a temperature change Δu in the small box with a mass Δm shown in Fig. 5.15 is

$$\Delta Q = c\, \Delta m\, \Delta u = c\rho\, \Delta x\, \Delta y\, \Delta z\, \Delta u. \tag{5.68}$$

The rate at which heat flows across the surface ABCD into the box is

$$\frac{\Delta Q_1}{\Delta t} = -k \left[\frac{\partial u}{\partial y}\right]_y \Delta x\, \Delta z.$$

Note that a positive $\dfrac{\partial u}{\partial y}$ means the temperature is increasing in the positive y-direction and the heat is flowing in the negative y-direction, so the heat is acturally flowing out of the box, thus the negative sign in the equation.

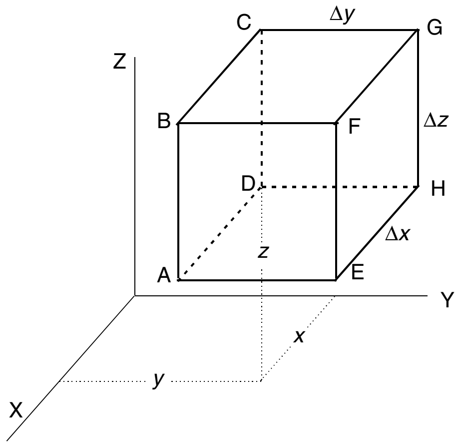

Fig. 5.15. The heat energy increased per unit time in this small element of mass is equal to the heat flux flowing into this element through its six surfaces

The subscript y in $\left[\frac{\partial u}{\partial y}\right]_y$ signifies the gradient is evaluated at the surface perpendicular to y-axis and is at a distance y units from the origin. Hence the heat flowing into the box through ABCD in the time interval Δt is

$$\Delta Q_1 = -k\left[\frac{\partial u}{\partial y}\right]_y \Delta x\,\Delta z\,\Delta t.$$

Similarly the heat flowing into the box through the surface EFGH is

$$\Delta Q_2 = k\left[\frac{\partial u}{\partial y}\right]_{y+\Delta y} \Delta x\,\Delta z\,\Delta t.$$

It follows that:

$$\begin{aligned}
\Delta Q_2 + \Delta Q_1 &= k\left[\frac{\partial u}{\partial y}\right]_{y+\Delta y} \Delta x\,\Delta z\,\Delta t - k\left[\frac{\partial u}{\partial y}\right]_y \Delta x\,\Delta z\,\Delta t \\
&= k\left\{\left[\frac{\partial u}{\partial y}\right]_{y+\Delta y} - \left[\frac{\partial u}{\partial y}\right]_y\right\}\Delta x\,\Delta z\,\Delta t \\
&= k\frac{\partial^2 u}{\partial y^2}\Delta y\,\Delta x\,\Delta z\,\Delta t.
\end{aligned}$$

It can be shown in the same way that the heat flowing into the box through the top surface BFGC and bottom surface AEHD in the time interval Δt is given by

$$\Delta Q_3 + \Delta Q_4 = k\frac{\partial^2 u}{\partial z^2}\,\Delta z\,\Delta x\,\Delta y\,\Delta t$$

and through the front and back surfaces is

$$\Delta Q_5 + \Delta Q_6 = k\frac{\partial^2 u}{\partial x^2}\Delta x\,\Delta y\,\Delta z\,\Delta t.$$

Thus the total amount heat flowing into the box through its six surfaces must be

$$\begin{aligned}
\Delta Q &= \Delta Q_1 + \Delta Q_2 + \Delta Q_3 + \Delta Q_4 + \Delta Q_5 + \Delta Q_6 \\
&= k\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2}\right)\Delta x\,\Delta y\,\Delta z\,\Delta t. \qquad (5.69)
\end{aligned}$$

This heat is responsible for raising the temperature of the box, therefore ΔQ in (5.69) must be the same ΔQ in (5.68). Thus

$$k\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2}\right)\Delta x\,\Delta y\,\Delta z\,\Delta t = c\rho\Delta x\,\Delta y\,\Delta z\,\Delta u$$

or

$$\frac{k}{c\rho}\left[\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial x^2}\right]=\frac{\Delta u}{\Delta t}.$$

In the limit of $\Delta t \to 0$, this equation becomes

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{\alpha^2}\frac{\partial u}{\partial t}, \tag{5.70}$$

where

$$\alpha^2=\frac{k}{c\rho}$$

is known as the thermal diffusivity.

This is the equation of heat conduction. It is interesting to note that nowhere in the derivation of the equation was any use made of boundary conditions. The flow of heat in a body is described by the same equation whether the surface is maintained at a constant temperature, insulated against heat loss, or allowed to cool freely by conduction to the surrounding medium. In general, as we shall soon see, the role of boundary conditions is to determine the form of those solutions of a partial differential equation which are relevant to a particular problem.

This equation differs from the one-dimensional wave equation in that the time derivative is only first order, whereas in the wave equation it is second order.

This equation is also called diffusion equation, because it not only governs the diffusion of heat, it also governs the diffusion of material, such as the diffusion of pollutants in water or the diffusion of drugs in the liver.

5.5 One-Dimensional Diffusion Equations

A number of situations involve only one coordinate. For example, if the lateral surface of a long slender rod of length L in the x-direction is insulated and no heat is generated in the rod, then the temperature distribution in the rod is governed by the one-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{\alpha^2}\frac{\partial u}{\partial t}.$$

This is because the insulation prevents heat flux in the radial direction, hence the temperature will depend on x coordinate only.

This one-dimensional equation also describes the temperature distribution of a large two-dimensional slab, infinite in y- and z-directions and bounded by planes at $x=0$ and $x=L$. If the initial and boundary conditions are known, the temperature distribution $u(x,t)$ inside the slab can be found.

In the following subsections, we will solve this one-dimensional problem with several kinds of boundary conditions.

5.5.1 Temperature Distributions with Specified Values at the Boundaries

Both Ends at the Same Temperature. A long rod is subjected to an initial temperature distribution along its axis; the rod is insulated on the lateral surfaces, and the ends of the rod are kept at the same constant temperature. As long as both ends are at the same temperature, we can assume that they are at 0°. If they are not at 0°, a simple change of scale can make them equal to 0° in the new scale. Let the length of the rod be L and the initial temperature distribution be $f(x)$. To find the temperature $u(x,t)$ anywhere in the rod at a later time is to solve the following problem:

$$\text{D.E.:} \quad \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t},$$

$$\text{B.C.:} \quad u(0,t) = 0, \quad u(L,t) = 0,$$

$$\text{I.C.:} \quad u(x,0) = f(x).$$

Following the procedure of separation of variables:

$$u(x,t) = X(x)T(t),$$

the differential equation becomes

$$X''(x)T(t) = \frac{1}{\alpha^2}X(x)T'(t).$$

Dividing both sides by $X(x)T(t)$

$$\frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{\alpha^2}\frac{X(x)T'(t)}{X(x)T(t)},$$

we have

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}.$$

This equation can hold if and only if both sides equal to the same constant

$$\frac{X''(x)}{X(x)} = -\mu^2,$$

$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\mu^2.$$

The general solution of $X''(x) = -\mu^2 X(x)$ is

$$X(x) = A\cos\mu x + B\sin\mu x.$$

Since the boundary conditions require

$$X(0) = 0, \quad X(L) = 0,$$

it follows that:

$$\begin{aligned} A &= 0, \\ \mu &= \frac{n\pi}{L}, \end{aligned}$$

where n is an integer. For each n, the solution of the space part is

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

Corresponding to this n, the equation for $T(t)$ becomes

$$T'(t) = -\left(\alpha\frac{n\pi}{L}\right)^2 T(t).$$

Therefore the time-dependent part is given by

$$T_n(t) = \exp\left[-\left(\frac{\alpha n\pi}{L}\right)^2 t\right].$$

Hence for each integer n, there is a solution $X_n(x)T_n(t)$. The general solution is the linear combination of these individual solutions

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \exp\left[-\left(\frac{\alpha n\pi}{L}\right)^2 t\right].$$

The initial condition requires

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = f(x).$$

This is a Fourier sine series, the coefficient c_n is given by

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

Thus the complete solution of this problem is

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} &\left\{\frac{2}{L}\int_0^L f(x')\sin\left(\frac{n\pi}{L}x'\right) \mathrm{d}x'\right\} \\ &\times \sin\left(\frac{n\pi}{L}x\right) \exp\left[-\left(\frac{\alpha n\pi}{L}\right)^2 t\right]. \end{aligned}$$

This solution certainly makes sense. No matter what the initial temperature is, as $t \to \infty$ the whole rod will eventually settle to 0° as its two ends.

Two Ends at Different Temperatures. A more realistic problem is that the two ends of the rod are at different temperatures. In that case our problem is changed to

$$\begin{aligned}
\text{D.E.:} \quad & \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}, \\
\text{B.C.:} \quad & u(0,t) = 0, \quad u(L,t) = K, \\
\text{I.C.:} \quad & u(x,0) = f(x).
\end{aligned}$$

One way to solve the problem is to transform it into the problem we just solved. This can be done by splitting the dependent variable $u(x,t)$ in the following way:

$$u(x,t) = v(x,t) - \psi(x).$$

So it follows:

$$\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial x^2} &= \frac{\partial^2 v(x,t)}{\partial x^2} - \psi''(x), \\
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial v(x,t)}{\partial t}.
\end{aligned}$$

Now, if we require

$$\psi''(x) = 0, \tag{5.71}$$

$$\psi(0) = 0, \quad \psi(L) = -K \tag{5.72}$$

then

$$\begin{aligned}
\frac{\partial^2 v(x,t)}{\partial x^2} &= \frac{1}{\alpha^2}\frac{\partial v(x,t)}{\partial t} \\
v(0,t) &= 0, \quad v(L,t) = 0 \\
v(x,0) &= f(x) + \psi(x).
\end{aligned}$$

Clearly we can solve for $v(x,t)$ with the same method of last problem. If we can find $\psi(x)$, then $u(x,t)$ can be obtained.

It follows from (5.71) that:

$$\psi(x) = a + bx.$$

The conditions (5.72) requires

$$a = 0 \quad \text{and} \quad b = -\frac{K}{L}.$$

Therefore

$$\psi(x) = -\frac{K}{L}x.$$

Thus

$$
\begin{aligned}
u(x,t) &= v(x,t) - \psi(x) \\
&= \frac{K}{L}x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \exp\left[-\left(\frac{\alpha n\pi}{L}\right)^2 t\right],
\end{aligned}
$$

where

$$
b_n = \frac{2}{L}\int_0^L \left[f(x) - \frac{K}{L}x\right] \sin\frac{n\pi}{L}x \, \mathrm{d}x.
$$

Again, this result makes sense, as $t \to \infty$, the temperature in the rod will increase linearly from 0 to K.

5.5.2 Problems Involving Insulated Boundaries

Both Ends Insulated. The flux of heat across the faces at $x = 0$ and $x = L$ is proportional to $\dfrac{\partial u}{\partial x}$ at those faces. To set $\dfrac{\partial u}{\partial x} = 0$ is to ensure there is no heat transfer. Thus, if both ends are insulated, to find the temperature distribution $u(x,t)$, we have to solve the following problem:

$$
\begin{aligned}
&\text{D.E.:} \quad \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}, \\
&\text{B.C.:} \quad \left[\frac{\partial u(x,t)}{\partial x}\right]_{x=0} = 0, \quad \left[\frac{\partial u(x,t)}{\partial x}\right]_{x=L} = 0. \\
&\text{I.C.:} \quad u(x,0) = f(x).
\end{aligned}
$$

Separation of variable now leads to

$$
\begin{aligned}
X''(x) &= -\mu^2 X(x), \\
X'(0) &= X'(L) = 0.
\end{aligned}
$$

It can be easily shown that

$$
\begin{aligned}
X(x) &= \cos\mu x \\
\mu &= \frac{n\pi}{L},
\end{aligned}
$$

where n is an integer starting from $n = 0$.

The corresponding $T(t)$ function is still

$$
T(t) = \exp(-\alpha^2\mu^2 t).
$$

Thus the general solution can be written as

$$
u(x,t) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L}x \exp\left[-\left(\frac{\alpha n\pi}{L}\right)^2 t\right]. \tag{5.73}
$$

The initial condition gives us the Fourier cosine series

$$u(x,0) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{L}x = f(x).$$

Thus

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,\mathrm{d}x, \quad n = 0,1,2,\ldots. \tag{5.74}$$

Therefore (5.73) with c_n given by (5.74) is our final solution.

One End at a Constant Temperature and the Other End Insulated. In this case, the problem becomes

$$\begin{aligned}
\text{D.E.:}\quad & \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t},\\
\text{B.C.:}\quad & u(0,t) = 0, \quad \left[\frac{\partial u(x,t)}{\partial x}\right]_{x=L} = 0,\\
\text{I.C.:}\quad & u(x,0) = f(x).
\end{aligned}$$

Upon separation of variables $u(x,t) = X(x)T(t)$, we find

$$\begin{aligned}
X''(x) &= -\mu^2 X(x),\\
X(0) &= 0, \quad X'(L) = 0
\end{aligned} \tag{5.75}$$

and

$$T'(t) = -\alpha^2\mu^2 T(t). \tag{5.76}$$

From (5.75) we have

$$X(x) = A\cos\mu x + B\sin\mu x.$$

The condition $X\,(0) = 0$ requires $A = 0$. Therefore we are left with

$$X(x) = B\sin\mu x$$

so

$$X'(x) = \mu B\cos\mu x.$$

The other boundary condition $X'(L) = 0$ becomes

$$X'(L) = \mu B\cos\mu L = 0.$$

This condition requires μL to be a half integer times π, that is

$$\mu = \left(\frac{2n-1}{2}\right)\frac{\pi}{L}, \quad n = 1,2,3,\ldots.$$

Thus for each n, the solution of the space part is

$$X_n(x) = \sin\left(\frac{2n-1}{2}\right)\frac{\pi}{L}x$$

and the solution of the corresponding equation for $T(t)$ is

$$T_n(t) = \exp\left[-\left(\frac{\alpha(2n-1)\pi}{2L}\right)^2 t\right].$$

Therefore the general solution is a linear combination of $X_n\,(x)\,T_n(t)$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)\pi}{2L}x\,\exp\left[-\left(\frac{\alpha(2n-1)\pi}{2L}\right)^2 t\right]. \tag{5.77}$$

From the initial condition

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{(2n-1)\pi}{2L}x = f(x),$$

the coefficients c_n can be easily determined to be

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi}{2L}x\,\mathrm{d}x, \qquad n = 1,2,3,\ldots.$$

Therefore (5.77) with c_n given by the above expression is the solution of our problem.

5.5.3 Heat Exchange at the Boundary

If heat exchange takes place, then according to Newton's law of cooling, the temperature function satisfies the relation

$$hu(x,t) + \frac{\partial}{\partial x}u(x,t) = 0,$$

where the constant h is an appropriate heat transfer coefficient.

Suppose we want to know the temperature $u(x,t)$ of a slab which initially has a uniform temperature u_0. The face of the slab at $x = 0$ is held at temperature 0, at the face $x = L$, heat exchange takes place so that

$$u_x(L,t) = -hu(L,t).$$

To find $u(x,t)$, we have to solve the following problem:

$$\begin{aligned}
&\text{D.E.:} && \frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t},\\
&\text{B.C.:} && u(0,t) = 0, \;\; u_x(L,t) = -hu(L,t),\\
&\text{I.C.:} && u(x,0) = u_0.
\end{aligned}$$

Again, we assume the variables can be separated,

$$u(x,t) = X(x)T(t),$$

so

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T'}{T} = -\lambda^2$$

and

$$T' = -a^2\lambda^2 T, \qquad X'' = -\lambda^2 X.$$

The solution for X is

$$X(x) = A\cos\lambda x + B\sin\lambda x.$$

The boundary condition $u(0,t) = 0$ means $X(0) = 0$. Thus

$$X(0) = A = 0.$$

The other boundary condition at $x = L$, $u_x(L,t) = -hu(L,t)$, becomes

$$X'(L)T(t) = -hX(x)T(t)$$

or

$$X'(L) = -hX(x).$$

Thus

$$B\lambda\cos\lambda L = -hB\sin\lambda L$$

or

$$\tan\lambda L = -\frac{\lambda}{h}. \tag{5.78}$$

The values of λ that satisfy this equation are the eigenvalues of this problem. Let $\lambda L = \beta$, so

$$\tan\beta = -\frac{\beta}{hL}. \tag{5.79}$$

The solution of this equation are the points of intersection of the graphs of $y = \tan x$ and $y = -x/hL$, as indicated in Fig. 5.16. It is apparent from the figure that there is an infinite sequence of positive roots $\beta_1, \beta_2, \beta_3, \ldots$. The eigenvalues of (5.78), are given by

$$\lambda_n = \frac{\beta_n}{L}$$

for $n = 1, 2, 3, \ldots$. In other words,

$$\tan\lambda_n L = -\frac{\lambda_n}{h}. \tag{5.80}$$

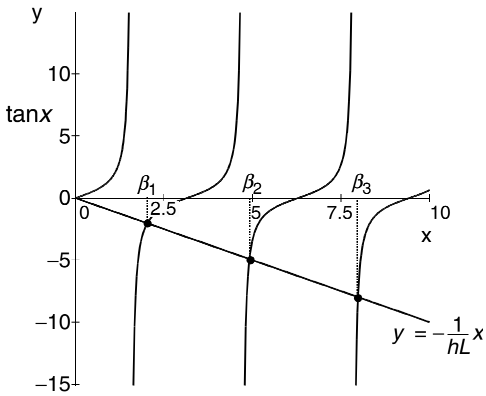

Fig. 5.16. The solutions of $\tan \beta = -\beta/hL$ are the intersections of $y = \tan x$ and $y = -x/hL$

Equation (5.79) appears frequently in many applications, its solutions for various values of hL are tabulated in Table 4.19 of Abramowitz and Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).

The eigenfunction $X_n(x)$ associate with the eigenvalue λ_n is

$$X_n(x) = \sin \lambda_n x.$$

According to Sturm–Liouville theory, these eigenfunctions are orthogonal. This can be explicitly shown. Let

$$I_{nm} = \int_0^L \sin \lambda_n x \, \sin \lambda_m x \, \mathrm{d}x.$$

By trigonometrical identity or by changing it into exponential form, one can show that the integral is equal to

$$I_{nm} = \frac{1}{2}\left[\frac{\sin(\lambda_n - \lambda_m) L}{(\lambda_n - \lambda_m)} - \frac{\sin(\lambda_n + \lambda_m) L}{(\lambda_n + \lambda_m)}\right]. \tag{5.81}$$

If $\lambda_n \neq \lambda_m$, then

$$I_{nm} = \frac{1}{\left(\lambda_n^2 - \lambda_m^2\right)}\left[\lambda_m \sin \lambda_n L \cos \lambda_m L - \lambda_n \sin \lambda_m L \cos \lambda_n L\right].$$

Using (5.80,) which can be written as

$$h \sin \lambda_i L = -\lambda_i \cos \lambda_i L, \tag{5.82}$$

We see that

$$I_{nm} = \frac{1}{\left(\lambda_n^2 - \lambda_m^2\right)}\left[-h \sin \lambda_n L \sin \lambda_m L + h \sin \lambda_m L \sin \lambda_n L\right] = 0.$$

Therefore $\{\sin \lambda_n x\}$ is an orthogonal set.

For $\lambda_n = \lambda_m$, we can either use L'Hospital's rule on the first term of (5.81) or integrate directly, the result is

$$I_{nn} = \frac{1}{2}\left[L - \frac{1}{2\lambda_n}\sin 2\lambda_n L\right] = \frac{1}{2}\left[L - \frac{1}{\lambda_n}\sin \lambda_n L \cos \lambda_n L\right].$$

Again by (5.82,)

$$I_{nn} = \frac{1}{2}\left[L + \frac{1}{h}\cos^2 \lambda_n L\right] = \frac{1}{2h}\left(Lh + \cos^2 \lambda_n L\right).$$

Corresponding to each λ_n, the solution to the equation for $T_n(t)$ is

$$T_n(t) = \mathrm{e}^{-\lambda_n^2 a^2 t}.$$

So the general solution for $u(x,t)$ can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} c_n \mathrm{e}^{-\lambda_n^2 a^2 t} \sin \lambda_n x.$$

The coefficients c_n can be determined by the initial condition, $u(x,0) = u_0$,

$$\sum_{n=1}^{\infty} c_n \sin \lambda_n x = u_0.$$

Multiplying both sides by $\sin \lambda_m x$ and integrating from 0 to L,

$$\sum_{n=1}^{\infty} c_n \int_0^L \sin \lambda_n x \sin \lambda_m x \,\mathrm{d}x = \int_0^L u_0 \sin \lambda_m x \,\mathrm{d}x,$$

we have

$$\sum_{n=1}^{\infty} c_n I_{nm} = c_m I_{mm} = \frac{1}{\lambda_m}(1 - \cos \lambda_m L) u_0.$$

Thus

$$c_n = \frac{2hu_0}{Lh + \cos^2 \lambda_n L}\frac{1}{\lambda_n}(1 - \cos \lambda_n L)$$

and

$$u(x,t) = 2hu_0 \sum_{n=1}^{\infty} \frac{1 - \cos \lambda_n L}{\lambda_n(Lh + \cos^2 \lambda_n L)} \mathrm{e}^{-\lambda_n^2 a^2 t} \sin \lambda_n x,$$

where $\lambda_1,\ \lambda_2,\ \ldots$ are positive roots of

$$\tan \lambda L = -\frac{\lambda}{h}.$$

5.6 Two-Dimensional Diffusion Equations: Heat Transfer in a Rectangular Plate

Suppose that the edges of a rectangular plate are bounded by the line $x = 0$, $x = a$, $y = 0$, and $y = b$. Its faces are insulated so that $\frac{\partial u}{\partial z} = 0$. The edges are held at temperature zero and initially the temperature distribution in the plate is $f(x, y)$. We want to find the expression for $u(x, y, t)$.

Our problem can be formulated as follows:

$$\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} = \frac{1}{\alpha^2}\frac{\partial u(x,y,t)}{\partial t}$$
$$u(0,y,t) = 0, \quad u(a,y,t) = 0,$$
$$u(x,0,t) = 0, \quad u(x,b,t) = 0,$$
$$u(x,y,0) = f(x,y).$$

Again we assume the variables can be separated

$$u(x,y,t) = X(x)Y(y)T(t).$$

Then the differential equation can be written as

$$X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = \frac{1}{\alpha^2}X(x)Y(y)T'(t).$$

Dividing by $X(x)Y(y)T(t)$, we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}$$

or

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} - \frac{Y''(y)}{Y(y)}.$$

Since the left-hand side is a function of x only, and right-hand side is a function of t and y, they can be equal if and only if both sides equal to the same constant

$$\frac{X''(x)}{X(x)} = -\lambda^2 = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} - \frac{Y''(y)}{Y(y)}.$$

For the same reason as in the one-dimensional case, we anticipate the constant to be $-\lambda^2$,

$$\frac{X''(x)}{X(x)} = -\lambda^2,$$
$$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} - \frac{Y''(y)}{Y(y)} = -\lambda^2.$$

The last equation can be written as

$$\frac{Y''(y)}{Y(y)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} + \lambda^2.$$

Again, both sides have to equal to another constant

$$\frac{Y''(y)}{Y(y)} = -\mu^2 = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} + \lambda^2.$$

It is easy to see that the boundary conditions require

$$X(0) = X(a) = 0, \quad Y(0) = Y(b) = 0.$$

With these conditions, the equation

$$X''(x) = -\lambda^2 X(x)$$

can have solution only if

$$\lambda = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots.$$

and corresponding to each n the solution X_n is

$$X_n(x) = \sin\frac{n\pi}{a}x.$$

The equation governing $Y(y)$

$$Y''(y) = -\mu^2 Y(y)$$

has solution only if

$$\mu = \frac{m\pi}{b}, \quad m = 1, 2, 3, \ldots$$

with

$$Y_m(y) = \sin\frac{m\pi}{b}y.$$

The equation for $T(t)$ is

$$T'(t) = -(\lambda^2 + \mu^2)\alpha^2 T(t).$$

For each set of (n, m), the solution of this equation is

$$T_{n,m}(t) = \exp\left\{-\left[\left(\frac{n\pi\alpha}{a}\right)^2 + \left(\frac{m\pi\alpha}{b}\right)^2\right]t\right\}.$$

The linear combination of $X_n(x)Y_m(y)T_{n,m}(t)$ is the most general solution, thus

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}\sin\frac{n\pi}{a}x\,\sin\frac{m\pi}{b}y\,\exp\left\{-\left[\left(\frac{n\pi\alpha}{a}\right)^2+\left(\frac{m\pi\alpha}{b}\right)^2\right]t\right\}. \tag{5.83}$$

Applying the initial condition, we obtain

$$u(x,y,0)=f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}\sin\frac{n\pi}{a}x\,\sin\frac{m\pi}{b}y.$$

Now if we define

$$g_m(x)=\sum_{n=1}^{\infty}c_{nm}\sin\frac{n\pi}{a}x, \tag{5.84}$$

we can write

$$f(x,y)=\sum_{m=1}^{\infty}g_m(x)\sin\frac{m\pi}{b}y.$$

We can regard this equation as a Fourier sine series in y for each fixed x, and the coefficient $g_m(x)$ is given by

$$g_m(x)=\frac{2}{b}\int_0^b f(x,y)\sin\frac{m\pi y}{b}\mathrm{d}y.$$

On the other hand (5.84) is a Fourier sine series in x, and the coefficient c_{nm} is given by

$$\begin{aligned}c_{nm}&=\frac{2}{a}\int_0^a g_m(x)\sin\frac{n\pi x}{a}\mathrm{d}x\\&=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi y}{b}\sin\frac{n\pi x}{a}\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{5.85}$$

Thus the solution of our problem is (5.83) with coefficient c_{nm} given by (5.85). This problem is another example of the two-dimensional double Fourier series.

5.7 Laplace's Equations

One of the most important partial differential equations in physics is the Laplace's equation

$$\nabla^2 u=0,$$

named after French mathematician Pierre-Simon Laplace (1740–1827) The theory of the solution of Laplace equation is called potential theory. Solutions of the equation that have second derivatives are called harmonic functions.

Laplace equation may be obtained by setting $\dfrac{\partial u}{\partial t}=0$ in the heat equation. This describes the steady-state temperature distribution in a solid in which there is no heat source or sink.

Laplace's equation also describes the electrostatic potential in a charge free region. Since the electric field is given by the gradient of a potential

$$\mathbf{E} = -\boldsymbol{\nabla} V$$

and the divergent of $\mathbf{E}$ is equal to zero in a free space, therefore

$$\boldsymbol{\nabla} \cdot \mathbf{E} = -\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} V = -\nabla^2 V = 0,$$

the potential V is the solution of the Laplace equation. Similarly, the potential of the gravitational field in a region containing no matter also satisfies the Laplace equation.

Furthermore, Laplace equation is also very important in hydrodynamics. It applies to the incompressible fluid flow with no source, sink, and vortex. In this case, the velocity is given by the gradient of the velocity potential which satisfies the Laplace equation.

5.7.1 Two-Dimensional Laplace's Equation: Steady-State Temperature in a Rectangular Plate

Suppose that the three edges $x = 0,\ x = a$, and $y = 0$ are maintained at zero temperature

$$u(0, y) = 0, \quad u\,(a, y) = 0, \quad u(x, 0) = 0, \tag{5.86}$$

and the fourth edge $y = b$ is maintained at a temperature distribution $f(x)$

$$u\,(x, b) = f(x). \tag{5.87}$$

We want to find the temperature throughout the plate after the steady-state temperature distribution is reached.

To find the answer we must determine the solution of the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions (5.86) and (5.87).

Again we use the method of separation of variables

$$u(x, y) = X(x)Y(y)$$

to write the equation as

$$X''Y + XY'' = 0.$$

Dividing by $u(x, y)$, we have

$$\frac{X''}{X} = -\frac{Y''}{Y}.$$

It follows that both sides must be equal to a constant

$$\frac{X''}{X} = -\lambda^2, \qquad -\frac{Y''}{Y} = -\lambda^2.$$

Thus

$$X''(x) = -\lambda^2 X(x),$$
$$Y''(y) = \lambda^2 Y(y).$$

Note that we choose a negative constant in order for $X(x)$ to satisfy the boundary conditions. With this choice, $X(x)$ is given by

$$X(x) = A\cos\lambda x + B\sin\lambda x.$$

The boundary conditions $u(0,y) = u(a,y) = 0$ require $X(x)$ to be one of

$$X_n(x) = \sin\lambda_n x,$$

where

$$\lambda_n = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \ldots .$$

The solutions for $Y(y)$ can be written in terms of $\mathrm{e}^{\lambda y}$ and $\mathrm{e}^{-\lambda y}$, or in terms of hyperbolic cosine and sine

$$\cosh\lambda y = \frac{1}{2}\left(\mathrm{e}^{\lambda y} + \mathrm{e}^{-\lambda y}\right), \qquad \sinh\lambda y = \frac{1}{2}\left(\mathrm{e}^{\lambda y} - \mathrm{e}^{-\lambda y}\right).$$

In the present problem, it is more convenient to express the solution in the hyperbolic functions, since at $y = 0$, $\cosh\lambda y = 1$, and $\sinh\lambda y = 0$. Thus

$$Y(y) = C\cosh\lambda y + D\sin\lambda y$$

and the boundary condition $u(x,0) = 0$ requires C to be zero

$$Y(0) = C = 0.$$

Therefore

$$Y_n(y) = \sinh\lambda_n y.$$

A linear combination of $X_n(x)Y_n(y)$ is the general solution

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}.$$

To satisfy the other boundary condition $u(x,b) = f(x)$, we must have

$$u(x,b) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a} = f(x).$$

It is clear that $c_n \sinh\frac{n\pi b}{a}$ is the Fourier coefficient of the sine series expansion of $f(x)$. Therefore

$$c_n \sinh \frac{n\pi b}{a} = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{a} \mathrm{d}x.$$

Thus, the steady-state temperature distribution is given by

$$u(x, y) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(x') \sin \frac{n\pi x'}{a} \mathrm{d}x' \right) \left[\sinh \frac{n\pi b}{a} \right]^{-1} \sinh \frac{n\pi y}{a} \sin \frac{n\pi x}{a}.$$

It is clear that the solution of the more general problem where the temperature is prescribed by nonzero functions along all four edges can be obtained by superimposing four solutions analogous to the one obtained here, each corresponding to a problem in which zero temperatures are prescribed along three of the four edges.

5.7.2 Three-Dimensional Laplace's Equation: Steady-State Temperature in a Rectangular Parallelepiped

Suppose that the temperatures of five faces of a rectangular parallelepiped are maintained at zero

$$u(0, y, z) = u(a, y, z) = u(x, 0, z) = u(x, b, z) = 0, \tag{5.88}$$

$$u(x, y, 0) = 0$$

and the sixth face is maintained at a prescribed temperature distribution

$$u(x, y, d) = f(x, y).$$

We want to know the steady-state temperature distribution in the interior.

In this case we have to solve the three-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

with the specified boundary conditions.

With the assumption

$$u(x, y, z) = X(x)Y(y)Z(z),$$

the equation can be written as

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z}.$$

Since both sides must be equal to the same constant, we have

$$-\frac{X''}{X} = \alpha^2$$

$$\frac{Y''}{Y} + \frac{Z''}{Z} = \alpha^2.$$

It follows from the second equation:

$$\frac{Y''}{Y} = \alpha^2 - \frac{Z''}{Z} = -\beta^2.$$

Thus

$$\begin{aligned} X'' &= -\alpha^2 X, \\ Y'' &= -\beta^2 Y, \\ Z'' &= (\alpha^2 + \beta^2) Z. \end{aligned}$$

The homogeneous boundary conditions (5.88) are satisfied if $X(x)$ and $Y(y)$ are one of the following eigenfunctions

$$X_n(x) = \sin \alpha_n x, \qquad \alpha_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots,$$
$$Y_m(y) = \sin \beta_m y, \quad \beta_m = \frac{m\pi}{b}, \quad m = 1, 2, 3, \ldots$$

The solution for $Z(z)$ corresponding to $X_n(x)$ and $Y_m(y)$ is

$$Z_{nm}(z) = A \cosh \gamma_{nm} z + B \sinh \gamma_{nm} z,$$

where

$$\gamma_{nm} = \left[\left(\frac{n\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 \right]^{1/2}.$$

The boundary condition $u(x, y, 0) = 0$ requires $Z(0) = 0$

$$Z_{nm}(0) = A = 0.$$

Thus

$$Z_{nm}(z) = \sinh \gamma_{nm} z.$$

Therefore, for each set of (n, m), there is a solution

$$u_{nm}(x, y, z) = X_n(x) Y_m(y) Z_{nm}(z).$$

The general solution is given by the linear combination

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \sin \alpha_n x \, \sin \beta_m y \, \sinh \gamma_{nm} z.$$

To satisfy the boundary condition on the top surface, we have

$$u(x, y, d) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \sin \alpha_n x \, \sin \beta_m y \, \sinh \gamma_{nm} d = f(x, y).$$

Clearly $c_{nm} \sinh \gamma_{nm} d$ are the coefficients of the double Fourier series of $f(x, y)$. Therefore

$$c_{nm}\sinh\gamma_{nm}d=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\mathrm{d}y\,\mathrm{d}x.$$

Thus the steady-state temperature distribution inside the parallelepiped is given by

$$u\,(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}a_{nm}\left[\sinh\gamma_{nm}d\right]^{-1}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sinh\gamma_{nm}z,$$

where

$$a_{nm}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\mathrm{d}y\,\mathrm{d}x,$$

$$\gamma_{nm}=\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]^{1/2}.$$

5.8 Helmholtz's Equations

An equivalent approach to the Laplace's equation of Sect. 5.7 consists of first seeking a product in the form of

$$u\,(x,y,z)=F(x,y)Z(z),$$

where the $Z(z)$ factor is treated in a distinct way since only along a boundary z = constant is a nonhomogeneous condition imposed. The process of separation of variables will give

$$\frac{1}{F\,(x,y)}\left(\frac{\partial^2F\,(x,y)}{\partial x^2}+\frac{\partial^2F\,(x,y)}{\partial y^2}\right)=-\frac{\partial^2Z(z)}{\partial z^2}=-k^2,$$

where k^2 is the separation constant. Thus, $F(x,y)$ is the solution of the following equation:

$$\frac{\partial^2F\,(x,y)}{\partial x^2}+\frac{\partial^2F\,(x,y)}{\partial y^2}+k^2F\,(x,y)=0.$$

This equation is known as the Helmholtz's equation named after Hermann von Helmholtz (1821–1894) who studied this equation in connection with acoustics.

A great number of important engineering and physics problems can be reduced to solving the Helmholtz equation. As we have seen, if we put a time dependence $\exp\,(\mathrm{i}\omega t)$ in the wave equation, the space part is given by the Helmholtz equation. Similarly, with a time dependence $\exp\,(-\lambda t)$, the space part of the heat equation is also given by the Helmholtz equation.

So far we have used only cartesian coordinates. The success of solving the problems in this chapter are due to the facts that the Helmholtz equation is separable in cartesian coordinates and the boundaries of straight lines and planes can be easily described by the rectangular coordinates.

Actually the Helmholtz equation is separable in 11 different curvilinear, orthogonal coordinate systems. However, most of the problems in engineering and physics can be formulated in either cartesian, cylindrical, or spherical coordinates. In Chap. 6, we will study the solutions of the Helmholtz equation in the cylindrical and spherical coordinates.

Exercises

1. Show that the solution of

$$\begin{aligned} &\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \\ &u(0,t) = 0; \quad u(L,t) = 0, \\ &u(x,0) = f(x); \quad u_t(x,0) = g(x) \end{aligned}$$

   is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi v}{L}t + B_n \sin\frac{n\pi v}{L}t\right)\sin\frac{n\pi}{L}x,$$

   where

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\mathrm{d}x, \\ B_n &= \frac{2}{n\pi v}\int_0^L g(x)\sin\frac{n\pi x}{L}\mathrm{d}x. \end{aligned}$$

2. The transverse displacements $u(x,t)$ in a string of length L stretched between the point 0 and L, and initially displaced into a position $u(x,0) = f(x)$ and released from rest is given by the solution of

$$\begin{aligned} &\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \\ &u(0,t) = 0; \quad u(L,t) = 0, \\ &u(x,0) = f(x); \quad u_t(x,0) = 0. \end{aligned}$$

   Show that the solution can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\frac{n\pi}{L}(x - vt) + B_n \sin\frac{n\pi}{L}(x + vt)\right].$$

   Express A_n and B_n in terms of quantities given in the problem.

3. Show that the motion of every point of the string of the previous problem is periodic in t with a period $2L/v$.

4. If the initial displacement of the previous problem is $f(x) = C \sin \frac{3\pi}{L} x$, find $u(x,t)$. What is the frequency of this vibration?

5. If the initial displacement of the previous problem is $f(x) = A \sin \frac{\pi}{L} x + B \sin \frac{2\pi}{L} x$, find the subsequent displacements $u(x,t)$.

6. Find the solution of the following boundary value problem

$$\text{D.E.: } \frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2},$$

$$\text{B.C.: } u(0,t) = 0, \quad u(L,t) = 0,$$

$$\text{I.C.: } u(x,0) = 0, \quad u_t(x,0) = \begin{cases} x & 0 \le x \le \frac{L}{2}, \\ L - x & \frac{L}{2} \le x \le L. \end{cases}$$

Ans. $u(x,t) = \frac{4L^2}{a\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L} \sin \frac{n\pi a}{L} t.$

7. Find $u(x,t)$ of the previous problem if the initial velocity is changed to

$$u_t(x,0) = \begin{cases} 0 & 0 \le x < \frac{L}{2} - w, \\ h & \frac{L}{2} - w < x < \frac{L}{2} + w, \\ 0 & \frac{L}{2} + w < x < L. \end{cases}$$

Ans. $u(x,t) = \frac{4Lh}{a\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi w}{L} \sin \frac{(2n-1)\pi}{L} x \sin \frac{(2n-1)\pi a}{L} t.$

8. If the vibrating string is subject to viscous damping, the governing equation can be written as

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} - 2h \frac{\partial u(x,t)}{\partial t},$$

where h is a constant. Let the boundary and initial conditions be

$$\text{B.C.:} \quad u(0,t) = 0, \qquad u(L,t) = 0,$$

$$\text{I.C.:} \quad u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Show that, assuming $h < \pi/L$, $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \, \mathrm{e}^{-ht} \left(\cos \frac{k_n}{L} t + \frac{hL}{k_n} \sin \frac{k_n}{L} t \right) \sin \frac{n\pi x}{L},$$

$$\text{where } b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \mathrm{d}x, \quad k_n = \sqrt{(n\pi)^2 - (hL)^2}.$$

9. Show explicitly that the following functions:

$$(a)\ (x+at)^2, \qquad (b)\ 2\mathrm{e}^{-(x-at)^2}, \qquad (c)\ 5\sin[3\,(x-at)]+(x+at)\,.$$

satisfy the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2\frac{\partial^2 u(x,t)}{\partial x^2}\ .$$

10. What is the solution to the initial value problem

$$\text{D.E.:}\quad \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{I.C.:}\quad \begin{cases} u(x,0) = \mathrm{e}^{-x^2} \\ u_t(x,0) = 0 \end{cases}\ .$$

Ans. $u(x,t) = \frac{1}{2}\left[\mathrm{e}^{-(x-t)^2} + \mathrm{e}^{-(x+t)^2}\right]$.

11. What is the solution to the initial value problem

$$\text{D.E.:}\quad \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad -\infty < x < \infty, \quad 0 < t < \infty$$

$$\text{I.C.:}\quad \begin{cases} u(x,0) = 0 \\ u_t(x,0) = x\,\mathrm{e}^{-x^2} \end{cases}\ .$$

Ans. $u(x,t) = \frac{1}{4}\left[\mathrm{e}^{-(x-t)^2} - \mathrm{e}^{-(x+t)^2}\right]$.

12. A stretched string between 0 and L is set to motion by a sudden blow at x_0. This problem can be modeled by the fact that the string at x_0 is given a certain velocity at $t=0$. Find the displacements of the string $u(x,t)$ by solving the following boundary problem:

$$\begin{aligned} &\text{D.E.:} && \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u(x,t)}{\partial t^2}, \\ &\text{B.C.:} && u(0,t) = 0, \qquad u\,(L,t) = 0, \\ &\text{I.C.:} && u(x,0) = 0, \qquad u_t\,(x,0) = Lv_0\delta\,(x-x_0)\,. \end{aligned}$$

Ans. $u(x,t) = \frac{2v_0L}{\pi a}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi}{L}x\sin\frac{n\pi a}{L}t$.

13. If the external force acting on the stretched string is proportional to the distance from one end, then the displacement $u(x,t)$ is governed by the differential equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} + Ax = \frac{\partial^2 u(x,t)}{\partial t^2}.$$

If the boundary and initial conditions are

$$u(0,t)=0, \qquad u(L,t)=0,$$
$$u(x,0)=f(x), \quad u_t(x,0)=0,$$

find $u(x,t)$.

Ans. $u(x,t)=\frac{A}{6a^2}x\left(L^2-x^2\right)+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t,$

$b_n=\frac{2}{L}\int_0^L\left[f(x)-\frac{A}{6a^2}x\left(L^2-x^2\right)\right]\sin\frac{n\pi}{L}x\,\mathrm{d}x.$

14. Determine the vibrational behavior of a rectangular membrane described by

$$\text{D.E.:} \quad \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$$
$$\text{B.C.:} \quad z(0,y,t)=z(a,y,t)=0,$$
$$z(x,0,t)=z(x,b,t)=0,$$

if it is initially displaced according to

$$z(x,y,0)=\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$

and then released from rest.

Ans. $z(x,y,t)=\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\cos\left[\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{1/2}v\pi t\right]$

15. Solve the equation

$$\frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$$

with the conditions

$$z(0,y,t)=z(a,y,t)=0,$$
$$z_y(x,0,t)=z_y(x,b,t)=0,$$
$$z(x,y,0)=f(x,y), \qquad z_t(x,y,0)=0.$$

Ans. $z(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}a_{nm}\cos\omega_{nm}t\,\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b},$ where

$$\omega_{nm}=v\pi\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)^{1/2}, \; a_{nm}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y.$$

16. Find the solution of the following heat conduction problem:

$$\text{D.E.:} \quad \frac{\partial^2 u(x,t)}{\partial x^2}=\frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}$$
$$\text{B.C.:} \quad u(0,t)=0, \qquad u(1,t)=0,$$
$$\text{I.C.:} \quad u(x,0)=\sin(2\pi x)+\frac{1}{3}\sin(4\pi x).$$

Ans. $u(x,t) = \mathrm{e}^{-(2\pi\alpha)^2 t} \sin(2\pi x) + \frac{1}{3}\mathrm{e}^{-(4\pi\alpha)^2 t} \sin(4\pi x)$.

17. What would be the solution to the previous problem if the initial condition is changed to

$$u(x,0) = x - x^2, \qquad 0 < x < 1.$$

Ans. $u(x,t) = \frac{8}{\pi^3}\left[\mathrm{e}^{-(\pi\alpha)^2 t}\sin(\pi x) + \frac{1}{27}\mathrm{e}^{-(3\pi\alpha)^2 t}\sin(3\pi x) + \cdots\right]$.

18. Solve the following nonhomogeneous heat equation:

$$\begin{aligned} \text{D.E.:} &\quad \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \sin(\pi x) \\ \text{B.C.:} &\quad u(0,t) = 0, \qquad u(1,t) = 0, \\ \text{I.C.:} &\quad u(x,0) = \sin(2\pi x). \end{aligned}$$

Ans. $u(x,t) = \frac{1}{\pi^2}\left(1 - \mathrm{e}^{-(\pi\alpha)^2 t}\right)\sin(\pi x) + \mathrm{e}^{-(2\pi\alpha)^2 t}\sin(2\pi x)$.

19. Solve the following problem where $u(x,t)$ is the temperature of a bar of length L with its lateral surface and one end insulated and the other end kept at a constant temperature.

$$\begin{aligned} \text{D.E.:} &\quad \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \\ \text{B.C.:} &\quad u_x(0,t) = 0, \ u(L,t) = 1, \\ \text{I.C.:} &\quad u(x,0) = 0. \end{aligned}$$

Ans. $u(x,t) = 1 - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\mathrm{e}^{-\left[\frac{(2n+1)\alpha\pi}{2L}\right]^2 t}\cos\frac{(2n+1)\pi}{2L}x$.

20. Solve the following problem:

$$\begin{aligned} \text{D.E.:} &\quad \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \\ \text{B.C.:} &\quad u(0,t) = 0, \qquad -u_x(1,t) = K, \\ \text{I.C.:} &\quad u(x,0) = 0. \end{aligned}$$

Ans. $u(x,t) = \frac{8K}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\mathrm{e}^{-\left[\frac{(2n+1)\alpha\pi}{2}\right]^2 t}\sin\frac{(2n+1)\pi}{2}x - Kx$.

21. Assume the temperature $u(x,t)$ in a bar of thermal diffusivity a^2 and length L that is perfectly insulated, including the ends at $x = 0$ and $x = L$, is initially given by $u(x,0) = f(x)$, the subsequent distribution of the temperature is given by the solution of the following problem:

$$\begin{aligned}
\text{D.E.:}&\quad \frac{1}{\alpha^2}\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}\\
\text{B.C.:}&\quad u_x(0,t)=0,\qquad u_x(L,t)=0,\\
\text{I.C.:}&\quad u(x,0)=f(x).
\end{aligned}$$

Express $u(x,t)$ in an infinite series.

Ans. $u(x,t)=\sum_{n=0}^{\infty} A_n \mathrm{e}^{-\left[\frac{n\alpha\pi}{L}\right]^2 t}\cos\frac{n\pi}{L}x$, where

$$A_0=\frac{1}{L}\int_0^L f(x)\mathrm{d}x,\qquad A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,\mathrm{d}x,\quad n=1,2,3,\ldots.$$

22. All four faces of an infinite long rectangular prism with thermal diffusivity α^2 bounded by the planes $x=0$, $x=a$, $y=0$, and $y=b$ are kept at temperature zero. If the initial temperature distribution is $f(x,y)$, derive the following formula for the temperature $u\,(x,y,t)$ in the prism:

$$u\,(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\mathrm{e}^{-\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2\alpha^2 t}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b},$$

where

$$a_{nm}=\frac{4}{ab}\int_0^a\int_0^b f\,(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\mathrm{d}x\,\mathrm{d}y.$$

If $f(x,y)=g(x)h(y)$, show that the double Fourier series reduces to product of two series

$$u\,(x,y,t)=v(x,t)w\,(y,t)$$

and note that v and w represent temperatures in the slabs $0\le x\le a$ and $0\le y\le b$ with their faces at temperature zero and with initial temperatures $g(x)$ and $h(y)$, respectively.

23. Let $u(x,y)$ be the steady-state temperature in a thin plate in the shape of a semi-infinite strip. Let the surface heat transfer takes place at the face so that

$$\frac{\partial^2 u\,(x,y)}{\partial x^2}+\frac{\partial^2 u\,(x,y)}{\partial y^2}-bu\,(x,y)=0,\quad 0\le x\le 1;\ y\ge 0.$$

If u is bounded as $y\to\infty$ and satisfies the conditions

$$u(0,y)=0,\quad u_x\,(1,y)=-hu\,(1,y)\,,\quad u(x,0)=1.$$

Show that

$$u(x,y)=2h\sum_{n=1}^{\infty}\frac{A_n}{\alpha_n}\mathrm{e}^{-\sqrt{b+\alpha_n^2}\,y}\sin\alpha_n x,$$

where

$$A_n = \frac{1 - \cos\alpha_n}{h + \cos^2\alpha_n}$$

and $\alpha_1, \alpha_2, \alpha_3, \ldots$ are positive roots of the equation

$$\tan\alpha = -\frac{\alpha}{h}.$$

Hint: With $u(x, y) = X(x)Y(y)$, show that

$$X_n(x) = \sin\alpha_n x, \qquad Y_n(y) = \mathrm{e}^{-\sqrt{b+\alpha_n^2}y},$$

$$\int_0^1 \sin^2\alpha_n x \, \mathrm{d}x = \frac{1}{2} - \frac{\sin 2\alpha_n}{4\alpha_n} = 2h\left(h + \cos^2\alpha_n\right).$$

24. Find the solution of the problem consisting of

$$\begin{aligned} &\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = N\mathrm{e}^{-kx} \\ &u(0, t) = 0, \quad u(L, t) = 0, \\ &u(x, 0) = f(x). \end{aligned}$$

Here the term on the right may represent loss of heat due to radioactive decay in the bar.

Ans. $u(x, t) = U(x, t) + \Psi(x)$,

$$\begin{aligned} U(x, t) &= \sum_{n=1}\left[\frac{2}{L}\int_0^L [f(x) - \Psi(x)]\sin\frac{n\pi}{L}x \, \mathrm{d}x\right]\mathrm{e}^{-(n\pi a/L)^2 t}\sin\frac{n\pi}{L}x \\ \Psi(x) &= -\frac{N}{a^2k^2}\left[\mathrm{e}^{-kx} + (1 - \mathrm{e}^{-kL})x/L - 1\right]. \end{aligned}$$

25. Find the electrostatic potential inside an infinitely long rectangular wave guide with conducting walls. The guide measures L by b. One of the sides of length L is held at a constant potential V_0, the other three sides are grounded (That is, $V = 0$).

Hint: The answer is given by the solution of the problem:

$$\begin{aligned} &\frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = 0, \\ &V(0, y) = V(L, y) = V(x, 0) = 0, \\ &V(x, b) = V_0. \end{aligned}$$

Ans. $V(x, y) = \frac{4V_0}{\pi}\sum_{n=1}\frac{1}{2n-1}\left[\sinh\frac{(2n-1)\pi b}{L}\right]^{-1}\sinh\frac{(2n-1)\pi}{L}y\,\sin\frac{(2n-1)\pi}{L}x.$

26. Find the solution of the problem consisting of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le \infty, \quad 0 \le y \le b.$$
$$u(x,0) = 0, \quad u(x,b) = 0,$$
$$u(0,y) = u_0, \quad u(\infty,y) = 0.$$

Here $u(x,y)$ is the steady-state temperature distribution in a semi-infinite rectangular plate of width b, the temperature at far end and along its two long sides is fixed at 0°, and temperature at $x = 0$ is fixed at a constant temperature u_0.
Ans. $u(x,y) = \sum_{n=1} \frac{4u_0}{(2n-1)\pi} \mathrm{e}^{-(2n-1)\pi x/b} \sin \frac{2n-1}{b}\pi y$.

27. Determine the solution of the Laplace's equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

in the rectangular region $0 \le x \le L$, $0 \le y \le b$, which satisfies the conditions

$$u(0,y) = u(L,y) = 0,$$
$$u(x,0) = f(x), \quad u(x,b) = g(x).$$

Hint: Show that $u_{n,1} = \sinh \frac{n\pi}{L} y \sin \frac{n\pi}{L} x$ satisfies the equation and the boundary conditions: $u(0,y) = u(L,y) = u(x,0) = 0$, and $u_{n,2} = \sinh \frac{n\pi}{L}(b-y) \sin \frac{n\pi}{L} x$ satisfies the equation and the boundary conditions: $u(0,y) = u(L,y) = u(x,b) = 0$. Thus, $u_n = u_{n,1} + u_{n,2}$.
Ans. $u(x,y) = \sum_{n=1} \left[a_n \sinh \frac{n\pi}{L} y \sin \frac{n\pi}{L} x + b_n \sinh \frac{n\pi}{L}(b-y) \sin \frac{n\pi}{L} x\right]$, where

$$a_n = \left[\sinh \frac{n\pi b}{L}\right]^{-1} \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \mathrm{d}x,$$
$$b_n = \left[\sinh \frac{n\pi b}{L}\right]^{-1} \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \mathrm{d}x.$$