6

Partial Differential Equations with Curved Boundaries

In Chap. 5, we saw that many physical problems can be formulated in terms of partial differential equations. The solutions of these equations have to satisfy certain boundary conditions. Most of these problems involve the Laplacian operator. Depending on the specific problems, either Laplace equation or Helmholtz equation has to be solved.

When the boundaries are mutually perpendicular straight lines or planes, the cartesian coordinate system is very convenient. When the partial differential equation is reduced to ordinary differential equations by the separation of variables, we are able to recognize the solutions of these ordinary differential equations. Eigenvalues and eigenfunctions dictated by the boundary conditions are readily found and the solution of the partial differential equation can be obtained by the Fourier expansion.

However, many physical problems involve boundaries in the shape of circles, cylinders, and spheres. In these cases, it is much simpler to use polar, cylindrical, or spherical coordinates. The partial differential equations expressed in these coordinates are more complicated. For most problems, the radial and angular parts can be separated, but the ordinary differential equations that govern the angular coordinates are different from those for radial coordinates. Some of them are differential equations with variable coefficients which cannot be solved in terms of elementary functions. We have solved most of these ordinary differential equations in Chap. 4 with Bessel and Legendre functions. Generally, the solutions for problems with cylindrical symmetry are expressible in terms of Fourier–Bessel series, and for those with spherical symmetry, in Fourier–Legendre series. But it is the boundary condition that determines how these solutions of ordinary differential equations should be put together to solve a particular problem. Two identical partial differential equations with slightly different boundary conditions can lead to completely different results.

It is not possible to discuss all the different types of partial differential equations encountered in engineering and physics. In this chapter, we will mainly discuss the Laplace's equation $\nabla^2 u = 0$ and the Helmholtz's equation

$\nabla^2 u + ku = 0$, because they occur in many types of practical problems. These two equations look similar, but their solutions are completely different. For example, the radial solution of the Laplace equation in spherical coordinates is given by r raised to negative or positive integer powers, whereas the solutions of the Helmholtz's equation are given by spherical Bessel functions.

Instead of following the usual procedure to classify the partial differential equations in hyperbolic, parabolic, and elliptic types, we will use enough examples to illustrate how similar methods can be used to solve these equations. It is important to know the general characteristics of different kinds of physical problems that can be solved by these methods. Otherwise details may look confusing. A careful examination of the table of contents may help to gain an overall picture of these problems.

6.1 The Laplacian

As we have seen, the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

appears in a many different partial differential equations of mathematical physics. Why should the sum of three second derivatives have so much to do with the law of nature? The answer lies in the fact that the Laplacian of a function is a comparison between the value of the function at a point and the average value of the function at points around it. This is clear in one dimension. If $u''(x) = 0$, then $u(x) = mx + b$. One can readily verify that

$$u(x) = \frac{1}{2}[u(x+\epsilon) + u(x-\epsilon)].$$

That is, the average value of two nearby points is equal to the value at the midpoint. Furthermore, if $u''(x) > o$, then the curve $u(x)$ is concaved upward, and the average value of two nearby points is greater than the value at the midpoint. If $u''(x) < 0$, then the curve is concaved downward, and the average value of two nearby points is less than the value at the midpoint.

The Laplacian may be thought of as the second derivative generalized to higher dimensions. In fact, we will soon show that in two dimensions, if

- $\nabla^2 u(x,y) = 0$, then the average value of u on a small circle is equal to the value of u at the center of the circle. This is what makes the Laplacian so useful, because
- $\nabla^2 u(x,y) > 0$ means the surface is concaved upward, and the value of u at the center of a small circle is less than the average value of u on the circle, and
- $\nabla^2 u(x,y) < 0$ means the surface is concaved downward, and $u(x,y)$ is greater than the average of u at its neighbors.

Similar statements can be made about $\nabla^2 u$ in three dimensions, if we change circle to sphere.

With these principles, we can have an intuitive understanding of some of the basic partial differential equations of physics.

The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \nabla^2 u.$$

If we use this equation to describe the vibration of a membrane, then u is the displacement (the height) of the membrane. This equation says the membrane at a point is accelerating upward (pulled up) if the membrane at that point is below the average of its neighbors.

The diffusion equation:

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u.$$

If we use this equation to describe heat transfer, then u is the temperature. This equation says that the temperature at a point will be increasing (positive rate of change) if the temperature at that point is less than the average temperature on a circle around that point.

Laplace's equation:

$$\nabla^2 u = 0.$$

This equation describes the steady-state of u, in which the rate of change with respect to time is zero. For example, if u is temperature, then this equation says that the temperature will not change if the temperature at a point is equal to the average temperature of the surrounding points.

Poisson's equation:

$$\nabla^2 u(x, y) = -g(x, y).$$

If $g(x, y)$ is positive at a point, and $u(x, y)$ is the temperature at that point, then this equation says that the temperature at that point is greater than its surroundings. In other words, heat is generated at that point, and $g(x, y)$ is the heat source.

Clearly, the meaning of the Laplacian will not change, no matter what coordinate system we choose to express it. Many problems in two and three dimensions are more naturally expressed in polar, cylindrical, or spherical coordinates, either because of the boundary conditions, or because we want to take advantage of the symmetry of the problem. These coordinates are shown in Fig. 6.1. Let us recall the expressions of the Laplacian operator in these systems. We had a detailed discussion of these expressions in Chap. 3 of volume II.

In polar coordinates:

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2}.$$

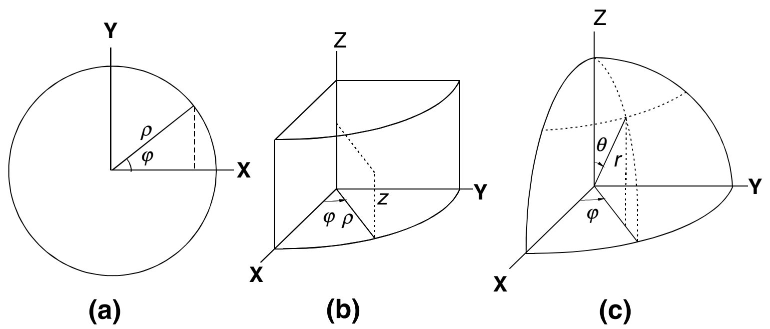


Fig. 6.1. (**a**) Polar coordinates, (**b**) cylindrical coordinates, (**c**) spherical coordinates

In cylindrical coordinates:

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}.$$

In spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

6.2 Two-Dimensional Laplace's Equations

6.2.1 Laplace's Equation in Polar Coordinates

Suppose that we need to find the solution of a two-dimensional Laplace's equation that has a prescribed behavior on a circle. Naturally we want to write the equation in the polar coordinates

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}u(\rho,\varphi)\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}u(\rho,\varphi) = 0,$$

so that we can accommodate the boundary condition by examining the solution for $\rho = a$. To solve the equation by separation of variables, we assume $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ and put it in the above equation

$$\frac{\Phi(\varphi)}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{R(\rho)}{\rho^2}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} = 0.$$

Dividing through by $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ and multiplying through by ρ^2, we have

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{1}{\Phi(\varphi)}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} = 0$$

or

$$\frac{1}{R(\rho)}\left(\rho^2\frac{\partial^2 R(\rho)}{\partial\rho^2}+\rho\frac{\partial R(\rho)}{\partial\rho}\right)=-\frac{1}{\Phi(\varphi)}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}.$$

Since the term on the left-hand side is a function of ρ only, while the term on the right-hand side depends only on φ, both must be equal to the same constant. So we obtain two ordinary differential equations

$$\frac{1}{R(\rho)}\left(\rho^2\frac{\mathrm{d}^2 R(\rho)}{\mathrm{d}\rho^2}+\rho\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho}\right)=\lambda, \tag{6.1}$$

$$\frac{1}{\Phi(\varphi)}\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2}=-\lambda, \tag{6.2}$$

where λ is the separation constant. The separation constant must be equal to n^2 with n as an integer for the following reason. Since $(\rho,\ \varphi)$ and $(\rho,\ \varphi+2\pi)$ are the same point, for the description of a real physical system, we must require $\Phi(\varphi)$ to be a periodic function with period 2π. That is, $\Phi(\varphi)$ must satisfy the condition

$$\Phi(\varphi+2\pi)=\Phi(\varphi).$$

With $\lambda=n^2$, (6.2) becomes

$$\frac{\mathrm{d}^2\Phi_n(\varphi)}{\mathrm{d}\varphi^2}=-n^2\Phi_n(\varphi).$$

Only if n is an integer ($n=0,1,2,\ldots$) can the solution of this equation be such a periodic function,

$$\Phi_n(\varphi)=A_n\cos n\varphi+B_n\sin n\varphi. \tag{6.3}$$

Note that for $n=0$, the solution $\Phi_0(\varphi)=\varphi$ is not periodic, therefore not allowed. However, the other solution for $n=0$, namely $\Phi_0(\varphi)=$ constant satisfies the periodic condition and is included in (6.3).

The radial part of the solution must come from (6.1) with the same n

$$\rho^2\frac{\mathrm{d}^2 R_n(\rho)}{\mathrm{d}\rho^2}+\rho\frac{\mathrm{d}R_n(\rho)}{\mathrm{d}\rho}=n^2R_n(\rho). \tag{6.4}$$

This is an Euler–Cauchy differential equation. The standard method is to set

$$\rho=\mathrm{e}^x,\quad \text{so}\ \ x=\ln\rho\quad\text{and}\quad \frac{\mathrm{d}x}{\mathrm{d}\rho}=\frac{1}{\rho}.$$

It follows that:

$$\frac{\mathrm{d}R_n}{\mathrm{d}\rho}=\frac{\mathrm{d}R_n}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\rho}=\frac{1}{\rho}\frac{\mathrm{d}R_n}{\mathrm{d}x},$$

$$\frac{\mathrm{d}^2R_n(\rho)}{\mathrm{d}\rho^2}=\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{\rho}\frac{\mathrm{d}R_n}{\mathrm{d}x}\right)=-\frac{1}{\rho^2}\frac{\mathrm{d}R_n}{\mathrm{d}x}+\frac{1}{\rho^2}\frac{\mathrm{d}^2R_n}{\mathrm{d}x^2},$$

so that (6.4) can be written as

$$\rho^2\left(\frac{1}{\rho^2}\frac{\mathrm{d}^2R_n}{\mathrm{d}x^2}-\frac{1}{\rho^2}\frac{\mathrm{d}R_n}{\mathrm{d}x}\right)+\rho\frac{1}{\rho}\frac{\mathrm{d}R_n}{\mathrm{d}x}-n^2R_n=0,$$

or

$$\frac{\mathrm{d}^2R_n}{\mathrm{d}x^2}-n^2R_n=0.$$

Clearly

$$R_n(x)=\begin{cases}C_n\mathrm{e}^{nx}+D_n\mathrm{e}^{-nx} & \text{for} \quad n\neq 0,\\ C_0+D_0x \quad \text{for} \quad n=0.\end{cases}$$

Thus, with $\rho=\mathrm{e}^x$, we have

$$R_n(\rho)=\begin{cases}C_n\rho^n+D_n\rho^{-n} & \text{for} \quad n\neq 0,\\ C_0+D_0\ln\rho & \text{for} \quad n=0.\end{cases}$$

For each integer n, the solution is $u_n(\rho,\varphi)=R_n(\rho)\Phi_n(\varphi)$. A linear combination of them is the general solution

$$u(\rho,\varphi)=\sum_{n=0}^{\infty}e_nu_n(\rho,\varphi).$$

Therefore we can write the solution of the Laplace's equation as

$$u(\rho,\varphi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}[(a_n\rho^n+b_n\rho^{-n})\cos n\varphi+(c_n\rho^n+d_n\rho^{-n})\sin n\varphi], \tag{6.5}$$

where we have renamed the combinations of constants for convenience.

Whenever we solve the equation in a region containing the origin, we must set b_0, b_n, and d_n to zero, because we are only interested in bounded solutions and $\ln\rho$ and ρ^{-n} go to infinity as ρ approaches zero.

Similarly if the region extends to infinity, then a_0, b_0, a_n, and c_n must be set to zero, unless there is a source at infinity. For example, a uniform electric field extending from $-\infty$ to $+\infty$ must be described by an electric potential that is not vanishing at infinity. (The electric potential satisfies the Laplace's equation, its gradient is the electric field.) In this case a_1 or c_1 may not be zero, but the rest of a_n, c_n, and b_0 must still be all zero.

The following examples will illustrate how these constants are determined by the boundary conditions.

Example 6.2.1. Show that if the temperature $f(\varphi)$ on the boundary of a circular disk of radius r_0 can be expressed as

$$f(\varphi)=\sum_{n=0}(A_n\cos n\varphi+B_n\sin n\varphi),$$

then the steady-state temperature at any point inside the disk is given by

$$u(\rho, \varphi) = \sum_{n=0} (\rho/r_0)^n (A_n \cos n\varphi + B_n \sin n\varphi).$$

Find the steady-state temperature distribution in a circular plate of radius r_0 if the upper semicircular boundary is held at 100°, and the lower at 0°.

Solution 6.2.1. The temperature function $u(\rho, \varphi)$ satisfies the Laplace's equation for steady-state distribution. So we have to solve the following boundary value problem:

$$\begin{aligned} &\text{D.E.:} \quad \nabla^2 u(\rho, \varphi) = 0, \\ &\text{B.C.:} \quad u(r_0, \varphi) = f(\varphi). \end{aligned}$$

Since the origin is inside the plate, for the solution to be bounded, we have to set b_n and d_n in (6.5) to zero. What is left in (6.5) can be written as

$$u(\rho, \varphi) = \sum_{n=0} (a_n \rho^n \cos n\varphi + c_n \rho^n \sin n\varphi).$$

Therefore, at $\rho = r_0$, we have

$$u(r_0, \varphi) = \sum_{n=0} (a_n r_0^n \cos n\varphi + c_n r_0^n \sin n\varphi).$$

But it is given that

$$u(r_0, \varphi) = f(\varphi) = \sum_{n=0} (A_n \cos n\varphi + B_n \sin n\varphi).$$

It is clear that

$$a_n = \frac{A_n}{r_0^n}, \quad c_n = \frac{B_n}{r_0^n}.$$

Substituting them into the equation for $u(\rho, \varphi)$, we obtain the required result

$$u(\rho, \varphi) = \sum_{n=0} (\rho/r_0)^n (A_n \cos n\varphi + B_n \sin n\varphi).$$

Now we can expand the given boundary condition

$$u(r_0, \varphi) = f(\varphi) = \begin{cases} 100 & \text{for } 0 < \varphi < \pi, \\ 0 & \text{for } \pi < \varphi < 2\pi \end{cases}$$

into a Fourier series

$$f(\varphi) = \sum_{n=0} (A_n \cos n\varphi + B_n \sin n\varphi),$$

where the coefficients are given by

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} u(r_0,\varphi)\mathrm{d}\varphi = \frac{1}{2\pi}\int_0^{\pi} 100\,\mathrm{d}\varphi = 50,$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} u(r_0,\varphi)\cos n\varphi\,\mathrm{d}\varphi = \frac{1}{\pi}\int_0^{\pi} 100\cos n\varphi\,\mathrm{d}\varphi = 0, \quad n \neq 0,$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} u(r_0,\varphi)\sin n\varphi\,\mathrm{d}\varphi = \frac{100}{\pi}\int_0^{\pi} \sin n\varphi\,\mathrm{d}\varphi = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{200}{n\pi} & \text{for } n \text{ odd} \end{cases}.$$

Thus the temperature distribution in the disk is given by

$$u(\rho,\varphi) = 50 + \frac{200}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\left(\frac{\rho}{r_0}\right)^n \sin n\varphi.$$

It may be noted that $u(\rho,\varphi)$ at $\varphi = 0$ and π, is 50. In general the Fourier series gives the average values at points of discontinuity. In the present case 50 is the average value of 0 and 100.

Example 6.2.2. Determine the steady-state temperature at points of the sector $0 \leq \varphi \leq \theta_0$, $0 \leq \rho \leq r_0$ of a circular plate if the temperature is maintained at zero along the straight edges and at u_0 along the curved edge.

Solution 6.2.2. To find the steady-state temperature $u(\rho,\varphi)$, we have to solve the following boundary value problem:

$$\text{D.E.:} \quad \nabla^2 u(\rho,\varphi) = 0,$$

$$\text{B.C.:} \quad u(\rho,0) = 0, \quad u(\rho,\theta_0) = 0, \quad u(r_0,\varphi) = u_0.$$

The usual separation assumption $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ leads to

$$\rho^2\frac{\mathrm{d}^2R(\rho)}{\mathrm{d}\rho^2} + \rho\frac{\mathrm{d}R(\rho)}{\mathrm{d}\rho} = \lambda R(\rho),$$

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2} = -\lambda\Phi(\varphi).$$

The boundary conditions along the straight edges require that

$$\Phi(0) = 0, \quad \Phi(\theta_0) = 0.$$

The differential equation and the boundary conditions for $\Phi(\varphi)$ actually form a Sturm–Liouville problem. So we know that the eigenvalues λ are positive ($\lambda = \alpha^2$), and the eigenfunctions are orthogonal. Thus

$$\Phi(\varphi) = A\cos\alpha\varphi + B\sin\alpha\varphi.$$

Since $\varPhi(0) = A = 0$, the boundary condition $\varPhi(\theta_0) = 0$ requires $\sin \alpha\theta_0 = 0$. Therefore
$$\alpha = \frac{n\pi}{\theta_0},$$
where n is a nonzero integer. Hence the eigenfunctions are
$$\varPhi_n(\varphi) = \sin \frac{n\pi}{\theta_0}\varphi, \quad n = 1,\ 2,\ 3,\ \ldots\ .$$
Note that $n = 0$ is not allowed in this case. The corresponding radial equation for $R_n(\rho)$ is
$$\rho^2 \frac{\mathrm{d}^2 R_n(\rho)}{\mathrm{d}\rho^2} + \rho \frac{\mathrm{d}R_n(\rho)}{\mathrm{d}\rho} = \left(\frac{n\pi}{\theta_0}\right)^2 R_n(\rho).$$
The bounded solution to this equation is
$$R_n(\rho) = c_n \rho^{n\pi/\theta_0}.$$
Therefore the general solution is
$$u(\rho, \varphi) = \sum_{n=1} c_n \rho^{n\pi/\theta_0} \sin \frac{n\pi}{\theta_0}\varphi.$$
The boundary condition $u(r_0, \varphi) = u_0$ requires c_n to satisfy the relation
$$u_0 = \sum_{n=1} c_n r_0^{n\pi/\theta_0} \sin \frac{n\pi}{\theta_0}\varphi.$$
Since $\left\{\sin \frac{n\pi}{\theta_0}\varphi\right\}$ is a complete orthogonal set in the interval $0 \le \varphi \le \theta_0$,
$$c_n r_0^{n\pi/\theta_0} = \frac{1}{\int_0^{\theta_0} \sin^2 \frac{n\pi}{\theta_0}\varphi \,\mathrm{d}\varphi} \int_0^{\theta_0} u_0 \sin \frac{n\pi}{\theta_0}\varphi \,\mathrm{d}\varphi.$$
These integrals can be easily evaluated,
$$\int_0^{\theta_0} u_0 \sin \frac{n\pi}{\theta_0}\varphi \,\mathrm{d}\varphi = \begin{cases} u_0 \frac{2\theta_0}{n\pi} & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}, \end{cases}$$
$$\int_0^{\theta_0} \sin^2 \frac{n\pi}{\theta_0}\varphi \,\mathrm{d}\varphi = \frac{1}{2}\theta_0.$$
Thus the steady-state temperature in the sector is given by
$$u(\rho, \varphi) = \frac{4u_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left(\frac{\rho}{r_0}\right)^{n\pi/\theta_0} \sin \frac{n\pi}{\theta_0}\varphi.$$

Example 6.2.3. Suppose that the temperatures along the inner circle of radius of r_1 and along the outer circle of radius r_2 of an annulus are maintained at $f_1(\varphi)$ and $f_2(\varphi)$, respectively, as shown in Fig. 6.2. Determine the steady-state temperature in the annulus, if

$$\begin{aligned} &\text{(a)} \quad f_1(\varphi) = 0, \quad f_2(\varphi) = \sin\varphi, \quad r_1 = 1, \quad r_2 = 2. \\ &\text{(b)} \quad f_1(\varphi) = 1, \quad f_2(\varphi) = 10, \quad r_1 = 1, \quad r_2 = e. \end{aligned}$$

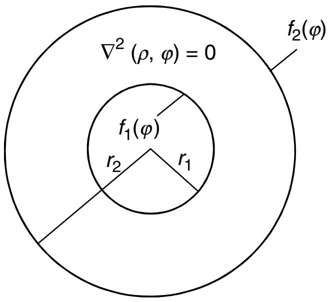


Fig. 6.2. Laplace's equation in an annulus

Solution 6.2.3. The steady-state temperature is determined by the following boundary problem:

$$\begin{aligned} &\text{D.E.:} \qquad \nabla^2 u(\rho,\varphi) = 0, \quad r_1 \le \rho \le r_2, \\ &\text{B.C.:} \qquad u(r_1,\varphi) = f_1(\varphi), \quad u(r_2,\varphi) = f_2(\varphi). \end{aligned}$$

Since the region of interest neither contains the origin nor extends to infinity, all terms in (6.5) have to be retained. Thus the boundary conditions take the form

$$f_1(\varphi) = a_0 + b_0 \ln r_1 + \sum_{n=1}^{\infty} [(a_n r_1^n + b_n r_1^{-n}) \cos n\varphi + (c_n r_1^n + d_n r_1^{-n}) \sin n\varphi],$$

$$f_2(\varphi) = a_0 + b_0 \ln r_2 + \sum_{n=1}^{\infty} [(a_n r_2^n + b_n r_2^{-n}) \cos n\varphi + (c_n r_2^n + d_n r_2^{-n}) \sin n\varphi].$$

According to the theory of Fourier series

$$\begin{aligned} a_0 + b_0 \ln r_1 &= \frac{1}{2\pi} \int_0^{2\pi} f_1(\varphi) \mathrm{d}\varphi, \\ a_0 + b_0 \ln r_2 &= \frac{1}{2\pi} \int_0^{2\pi} f_2(\varphi) \mathrm{d}\varphi \end{aligned}$$

and for $n \neq 0$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_1(\varphi) \cos n\varphi \, \mathrm{d}\varphi,$$

$$a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_2(\varphi) \cos n\varphi \, \mathrm{d}\varphi,$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_1(\varphi) \sin n\varphi \, \mathrm{d}\varphi,$$

$$c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi} \int_0^{2\pi} f_2(\varphi) \sin n\varphi \, \mathrm{d}\varphi.$$

From these equations, we can solve for the constants a_0, b_0, a_n, b_n, c_n, and d_n.

(a) For $f_1(\varphi) = 0$, $f_2(\varphi) = \sin\varphi$, $r_1 = 1$, $r_2 = 2$, all integrals are equal to zero except

$$\frac{1}{\pi} \int_0^{2\pi} f_2(\varphi) \sin\varphi \, \mathrm{d}\varphi = \frac{1}{\pi} \int_0^{2\pi} \sin^2\varphi \, \mathrm{d}\varphi = 1.$$

Therefore all constants are equal to zero except

$$c_1 + d_1 = 0,$$
$$c_1 2 + d_1 \frac{1}{2} = 1.$$

It follows that:

$$c_1 = \frac{2}{3}, \quad d_1 = -\frac{2}{3}.$$

Thus

$$u(\rho, \varphi) = \frac{2}{3} \left(\rho - \frac{1}{\rho} \right) \sin\varphi.$$

(b) For $f_1(\varphi) = 1$, $f_2(\varphi) = 10$, $r_1 = 1$, $r_2 = e$, the only nonvanishing coefficients are

$$a_0 + b_0 \ln r_1 = \frac{1}{2\pi} \int_0^{2\pi} f_1(\varphi) \mathrm{d}\varphi = 1,$$

$$a_0 + b_0 \ln r_2 = \frac{1}{2\pi} \int_0^{2\pi} f_2(\varphi) \mathrm{d}\varphi = 10.$$

Thus $a_0 = 1$, $b_0 = 10 - a_0 = 9$. Therefore

$$u(\rho, \varphi) = 1 + 9 \ln \rho.$$

Example 6.2.4. Solve the following boundary value problem:

$$\text{D.E.:} \quad \nabla^2 u(\rho,\varphi) = 0,$$
$$\text{B.C.:} \quad u(\rho \to \infty, \varphi) = E_0 \rho \cos\varphi, \quad u(r_0,\varphi) = 0.$$

(The solution is the electric potential produced by placing a long grounded conducting cylinder of radius r_0 in a previous uniform electric field $-\mathbf{E}_0$ with the axis of the cylinder perpendicular to the field $\mathbf{E}_0$.)

Solution 6.2.4. Since ρ^{-n} goes to zero as $\rho \to \infty$, asymptotically (6.5) becomes

$$u(\rho \to \infty, \varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}[a_n \rho^n \cos n\varphi + c_n \rho^n \sin n\varphi].$$

Therefore the condition $u(\rho \to \infty, \varphi) = E_0 \rho \cos\varphi$ requires that a_0, b_0, a_n, and c_n to be set to zero, except $a_1 = E_0$. What is left in (6.5) is

$$u(\rho,\varphi) = \sum_{n=1}^{\infty}[b_n \rho^{-n} \cos n\varphi + d_n \rho^{-n} \sin n\varphi] + E_0 \rho \cos\varphi.$$

At $\rho = r_0$, $u(r_0,\varphi) = 0$ becomes

$$(b_1 r_0^{-1} + E_0 r_0)\cos\varphi + d_1 r_0^{-1} \sin\varphi + \sum_{n=2}^{\infty}[b_n r_0^{-n} \cos n\varphi + d_n r_0^{-n} \sin n\varphi] = 0.$$

This requires all coefficients to vanish. That means all b_n and d_n are equal to zero except b_1, and

$$b_1 r_0^{-1} + E_0 r_0 = 0, \quad \text{or} \quad b_1 = -E_0 r_0^2.$$

Therefore the solution of this boundary value problem is given by

$$u(\rho,\varphi) = E_0 \rho \cos\varphi - E_0 r_0^2 \rho^{-1} \cos\varphi = E_0 \left(1 - \frac{r_0^2}{\rho^2}\right) \rho \cos\varphi. \tag{6.6}$$

6.2.2 Poisson's Integral Formula

Let us return to the solution $u(\rho,\varphi)$ of the Laplace's equation in the interior of a circular disk of radius r_0. If the boundary condition is $u(r_0,\varphi) = f(\varphi)$, then

$$u(\rho,\varphi) = \sum_{n=0} (\rho/r_0)^n (A_n \cos n\varphi + B_n \sin n\varphi),$$

where A_n and B_n are the Fourier coefficients of

$$f(\varphi) = \sum_{n=0} (A_n \cos n\varphi + B_n \sin n\varphi).$$

Substituting the expressions of these Fourier coefficients into $u(\rho, \varphi)$, we have

$$\begin{aligned}
u(\rho, \varphi) &= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi')\mathrm{d}\varphi' + \sum_{n=1} \left(\frac{\rho}{r_0}\right)^n \left[\frac{1}{\pi} \int_0^{2\pi} f(\varphi') \cos n\varphi' \mathrm{d}\varphi'\right] \cos n\varphi \\
&\quad + \sum_{n=1} \left(\frac{\rho}{r_0}\right)^n \left[\frac{1}{\pi} \int_0^{2\pi} f(\varphi') \sin n\varphi' \mathrm{d}\varphi'\right] \sin n\varphi \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi') \left\{1 + 2\sum_{n=1} \left(\frac{\rho}{r_0}\right)^n [\cos n\varphi' \cos n\varphi + \sin n\varphi' \sin n\varphi]\right\} \mathrm{d}\varphi'.
\end{aligned}$$

Since

$$\cos n\varphi' \cos n\varphi + \sin n\varphi' \sin n\varphi = \cos n(\varphi' - \varphi) = \mathrm{Re}\left[\mathrm{e}^{\mathrm{i}n(\varphi'-\varphi)}\right],$$

where Re stands for real part, so

$$\sum_{n=1} \left(\frac{\rho}{r_0}\right)^n [\cos n\varphi' \cos n\varphi + \sin n\varphi' \sin n\varphi] = \mathrm{Re} \sum_{n=1} \left[\frac{\rho}{r_0} \mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}\right]^n.$$

Using the facts that

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n, \quad \text{for} \quad |z| < 1$$

and $\rho/r_0 < 1$, we can write

$$\begin{aligned}
\sum_{n=1} \left[\frac{\rho}{r_0} \mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}\right]^n &= \frac{1}{1 - \frac{\rho}{r_0}\mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}} - 1 = \frac{\frac{\rho}{r_0}\mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}}{1 - \frac{\rho}{r_0}\mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}} \\
&= \frac{\frac{\rho}{r_0}\mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}}{1 - \frac{\rho}{r_0}\mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}} \cdot \frac{1 - \frac{\rho}{r_0}\mathrm{e}^{-\mathrm{i}(\varphi'-\varphi)}}{1 - \frac{\rho}{r_0}\mathrm{e}^{-\mathrm{i}(\varphi'-\varphi)}} = \frac{\frac{\rho}{r_0}\mathrm{e}^{\mathrm{i}(\varphi'-\varphi)} - \left(\frac{\rho}{r_0}\right)^2}{1 - 2\frac{\rho}{r_0}\cos(\varphi' - \varphi) + \left(\frac{\rho}{r_0}\right)^2}
\end{aligned}$$

and

$$\mathrm{Re} \sum_{n=1} \left[\frac{\rho}{r_0} \mathrm{e}^{\mathrm{i}(\varphi'-\varphi)}\right]^n = \frac{(\rho/r_0)\cos(\varphi' - \varphi) - (\rho/r_0)^2}{1 - 2(\rho/r_0)\cos(\varphi' - \varphi) + (\rho/r_0)^2}.$$

Thus

$$\begin{aligned}
u(\rho, \varphi) &= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi') \left\{1 + 2\frac{(\rho/r_0)\cos(\varphi' - \varphi) - (\rho/r_0)^2}{1 - 2(\rho/r_0)\cos(\varphi' - \varphi) + (\rho/r_0)^2}\right\} \mathrm{d}\varphi' \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi') \left\{\frac{r_0^2 - \rho^2}{r_0^2 - 2\rho r_0 \cos(\varphi' - \varphi) + \rho^2}\right\} \mathrm{d}\varphi'.
\end{aligned}$$

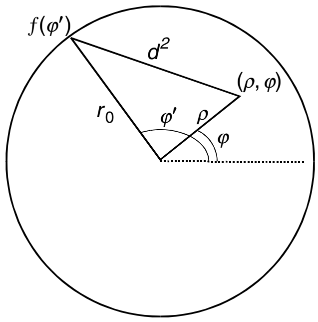


Fig. 6.3. Poisson's integral formula

This remarkable result is known as Poisson integral formula.

According to the law of cosine

$$r_0^2 - 2\rho r_0 \cos(\varphi' - \varphi) + \rho^2 = d^2,$$

where d is the distance between (ρ, φ) and the point (r_0, φ) on the boundary, as shown in Fig. 6.3.

So this formula can be written as

$$u(\rho, \varphi) = \frac{r_0^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{f(\varphi')}{d^2} \mathrm{d}\varphi'.$$

It shows that the value of $u(\rho, \varphi)$ at all points inside a circle of radius r_0 is the weighted sum of the values of the function $f(\varphi')$ on the circumference of the circle, and the weight is given by $(r_0^2 - \rho^2)/d^2$. If we were to increase the values of $f(\varphi')$ on even a small segment of the boundary, this would produce a corresponding increase in the values of u at every interior points. This is in accordance with the fact that the physical system described by the Laplace's equation is in a steady-state. An increase or decrease on the boundary requires the interior region to readjust to bring the system into equilibrium again.

At the center of the circle, $\rho = 0$ and $r_0 = d$, the formula reduces to

$$u(\text{center}) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi') \mathrm{d}\varphi'.$$

In other words, the average value of u on a circle is equal to the value of u at the center of the circle. This is known as the average value property of the Laplace equation.

From this property, we can deduce the fact that the solution of Laplace equation cannot have local maximum or minimum in the interior region. If $u(p)$ were a local maximum at the point P, then there must be a small circle around P, everywhere on it the value of u is smaller than $u(p)$. Obviously the average of the values of u on that circle could not possibly equal to $u(p)$. This violates the average value property of the Laplace equation. Similar argument will also show that there cannot be any local minimum.

6.3 Two-Dimensional Helmholtz's Equations in Polar Coordinates

As we have seen in Chap. 5, when the time dependence part is separated out, the space part of both wave equation and heat equation is given by the Helmholtz equation

$$\nabla^2 u + k^2 u = 0.$$

In polar coordinates, the Helmholtz equation takes the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}u(\rho,\varphi)\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}u(\rho,\varphi) + k^2 u(\rho,\varphi) = 0.$$

Proceeding in the same way as we solved the Laplace equation, with the assumption $u(\rho,\varphi) = R(\rho)\,\Phi(\varphi)$, we find that the angular part satisfies the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\Phi_n(\varphi) = -n^2\Phi_n(\varphi)$$

and has the familiar solution

$$\Phi_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi,$$

where n is an integer. The radial equation

$$\rho^2\frac{\mathrm{d}^2 R_n(\rho)}{\mathrm{d}\rho^2} + \rho\frac{\mathrm{d}R_n(\rho)}{\mathrm{d}\rho} + (k^2\rho^2 - n^2)R_n(\rho) = 0$$

differs from that found in the solution of Laplace equation. The solution of this equation, as we have shown in Chap. 4, is

$$R_n(\rho) = C_n J_n(k\rho) + D_n N_n(k\rho),$$

where $J_n(k\rho)$ and $N_n(k\rho)$ are, respectively, Bessel and Neumann functions of order n. Recall that $N_n(k\rho)$ goes to $-\infty$ as ρ approaches zero. Therefore for solutions that are required to be finite at the origin, we must set $D_n = 0$. Thus, for most applications, the solution of the Helmholtz equation is given by a linear combination of $u_n(\rho,\varphi)$ with

$$u_n(\rho,\varphi) = J_n(k\rho)(A_n \cos n\varphi + B_n \sin n\varphi). \tag{6.7}$$

In what follows, we shall show that not only the two-dimensional wave equation and heat equation reduce to the Helmholtz equation when the time-dependent parts are separated out, but also the three-dimensional Laplace equation in cylindrical coordinates reduce to the Helmholtz equation when the z dependence part is separated out. In that case, the separation constant can be either $+k^2$ or $-k^2$.

6.3.1 Vibration of a Drumhead: Two-Dimensional Wave Equation in Polar Coordinates

Whenever a membrane is a plane and its material is elastic, its vibrations are governed by the wave equation

$$\nabla^2 z = \frac{1}{a^2}\frac{\partial^2}{\partial t^2} z,$$

where z is the displacement (the height of the membrane from the plane). For a circular drumhead, naturally we want to use the polar coordinates.

Let the membrane be stretched over a fixed circular frame of radius c in the plane of $z = 0$. Initially the membrane is deformed in the shape of $z(\rho, \varphi, 0) = f(\rho, \varphi)$ and released from rest in that position. Hence to find the displacements $z(\rho, \varphi, t)$ is to solve the following problem:

$$\begin{aligned}
&\text{D.E.:} \quad && \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\right] z(\rho, \varphi, t) = \frac{1}{a^2}\frac{\partial^2}{\partial t^2} z(\rho, \varphi, t), \\
&\text{B.C.:} \quad && z(c, \varphi, t) = 0, \\
&\text{I.C.:} \quad && z(\rho, \varphi, 0) = f(\rho, \varphi), \qquad \left.\frac{\partial z}{\partial t}\right|_{t=0} = 0.
\end{aligned}$$

To separate the time-dependent part and the space-dependent part with the assumption $z(\rho, \varphi, t) = u(\rho, \varphi)T(t)$, we find two ordinary differential equations

$$\frac{T''(t)}{a^2 T(t)} = \lambda$$

and

$$\frac{1}{u(\rho, \varphi)}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho} u(\rho, \varphi)\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} u(\rho, \varphi)\right] = \lambda.$$

For the same reason as the vibration of a string, the separation constant must be

$$\lambda = -k^2,$$

so the time-dependent part is governed by

$$T''(t) = -k^2 a^2 T(t) \tag{6.8}$$

and the space-dependent part by the Helmholtz equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho} u(\rho, \varphi)\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} u(\rho, \varphi) + k^2 u(\rho, \varphi) = 0$$

with (6.7) as its solution

$$u(\rho, \varphi) = J_n(k\rho)(A_n \cos n\varphi + B_n \sin n\varphi).$$

In order to satisfy the boundary condition $z(c, \varphi, t) = 0$, we must have $u(c, \varphi) = 0$. This means that k cannot be any constant, it must satisfy the condition

$$J_n(kc) = 0.$$

Let $k_{nj}c$ be the jth zero of nth order Bessel function $J_n(x)$, then k must be equal to one of the k_{nj}. The same k_{nj} must be used in (6.8), so

$$T(t) = c_1 \cos k_{nj}at + c_2 \sin k_{nj}at.$$

Furthermore, to satisfy the initial condition

$$\left.\frac{\partial}{\partial t} z(\rho, \varphi, t)\right|_{t=0} = u(\rho, \varphi) \left.\frac{\mathrm{d}T(t)}{\mathrm{d}t}\right|_{t=0} = 0$$

the derivative of $T(t)$ must be zero, therefore c_2 must be set to zero. Thus for each n and j

$$z_{nj}(\rho, \varphi, t) = J_n(k_{nj}\rho)(A_n \cos n\varphi + B_n \sin n\varphi) \cos(ak_{nj}t).$$

The general solution, which is a linear combination of all these terms, is a double sum of all possible n and j

$$z(\rho, \varphi, t) = \sum_{n=0} \sum_{j=1} J_n(k_{nj}\rho)(a_{nj} \cos n\varphi + b_{nj} \sin n\varphi) \cos(ak_{nj}t),$$

where we have consolidated the constants into the coefficients a_{nj} and b_{nj}.

These coefficients can be determined by the other initial condition $z(\rho, \varphi, 0) = f(\rho, \varphi)$

$$z(\rho, \varphi, 0) = \sum_{n=0} \sum_{j=1} J_n(k_{nj}\rho)(a_{nj} \cos n\varphi + b_{nj} \sin n\varphi) = f(\rho, \varphi).$$

Let us first define

$$F_n(\rho) = \sum_{j=1} a_{nj} J_n(k_{nj}\rho), \tag{6.9}$$

$$G_n(\rho) = \sum_{j=1} b_{nj} J_n(k_{nj}\rho) \tag{6.10}$$

and express $z(\rho, \varphi, 0)$ in terms of them

$$\sum_{n=0} \{F_n(\rho) \cos n\varphi + G_n(\rho) \sin n\varphi\} = f(\rho, \varphi).$$

Regarding ρ as a parameter, this equation is in the form of a Fourier series, therefore $F_n(\rho)$ and $G_n(\rho)$ are given by

$$F_n(\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi, \quad n = 1, 2, \dots .$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi, \quad n = 0,$$
$$G_n(\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho,\varphi)\sin n\varphi\,\mathrm{d}\varphi, \quad n = 1, 2, \dots .$$

Putting them back into (6.9) and (6.10), we have

$$\sum_{j=1} a_{nj} J_n(k_{nj}\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi, \quad n = 1, 2, \dots,$$
$$\sum_{j=1} a_{nj} J_n(k_{nj}\rho) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi, \quad n = 0,$$
$$\sum_{j=1} b_{nj} J_n(k_{nj}\rho) = \frac{1}{\pi}\int_0^{2\pi} f(\rho,\varphi)\sin n\varphi\,\mathrm{d}\varphi, \quad n = 1, 2, \dots .$$

For each fixed n, the series is recognized as a Fourier–Bessel series. The coefficients can be determined by multiplying both sides by $\rho J_n(k_{ni}\rho)$ and integrating from 0 to c

$$\int_0^c \rho J_n(k_{ni}\rho) \sum_{j=1} a_{nj} J_n(k_{nj}\rho)\mathrm{d}\rho$$
$$= \int_0^c \rho J_n(k_{ni}\rho) \frac{1}{\pi}\int_0^{2\pi} f(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi\,\mathrm{d}\rho, \quad n = 1, 2, \dots .$$

Because of the orthogonality of the Bessel functions, all terms on the left-hand side are zero except the term with $j = i$

$$a_{ni}\int_0^c \rho J_n^2(k_{ni}\rho)\mathrm{d}\rho = \frac{1}{\pi}\int_0^c\int_0^{2\pi} \rho J_n(k_{ni}\rho) f(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi\mathrm{d}\rho.$$

Recall
$$\int_0^c \rho J_n^2(k_{ni}\rho)\mathrm{d}\rho = \frac{1}{2}c^2 J_{n+1}^2(k_{ni}c),$$
therefore

$$a_{ni} = \frac{2}{\pi c^2 J_{n+1}^2(k_{ni}c)}\int_0^c\int_0^{2\pi} \rho J_n(k_{ni}\rho) f(\rho,\varphi)\cos n\varphi\,\mathrm{d}\varphi\mathrm{d}\rho, \quad n = 1, 2, \dots \tag{6.11}$$

For $n = 0$, we have

$$a_{0i} = \frac{2}{2\pi c^2 J_1^2(k_{0i}c)}\int_0^c\int_0^{2\pi} \rho J_0(k_{0i}\rho) f(\rho,\varphi)\mathrm{d}\varphi\mathrm{d}\rho. \tag{6.12}$$

Similarly,

$$b_{ni} = \frac{2}{\pi c^2 J_{n+1}^2(k_{ni}c)} \int_0^c \int_0^{2\pi} \rho J_n(k_{ni}\rho) f(\rho,\varphi) \sin n\varphi \, \mathrm{d}\varphi \, \mathrm{d}\rho, \quad n = 1, 2, \dots . \tag{6.13}$$

Thus the solution of problem is given by

$$z(\rho,\varphi,t) = \sum_{n=0} \sum_{i=1} z_{nj} = \sum_{n=0} \sum_{i=1} J_n(k_{ni}\rho)(a_{ni}\cos n\varphi + b_{ni}\sin n\varphi)\cos(ak_{ni}t) \tag{6.14}$$

with the coefficients given by (6.11)–(6.13).

Each $z_{nj}(\rho,\varphi,t)$ term is known as a normal mode. Each mode is vibrating in a periodic motion with a frequency of $ak_{nj}/2\pi$. If the initial displacements happen to be in the shape of a particular mode, then the system will vibrate in that mode. For example, if

$$z(\rho,\varphi,0) = f(\rho,\varphi) = AJ_2(k_{21}\rho)\cos 2\varphi,$$

then all the coefficients in (6.14) are equal to zero, except $a_{21} = A$, and

$$z(\rho,\varphi,t) = AJ_2(k_{21}\rho)\cos 2\varphi \cos(ak_{21}t).$$

Each mode has nodal lines (lines without motion), which consist of circles and radial lines. Figure 6.4 shows a few of these modes of vibration, shaded parts being displaced in the opposite direction to unshaded parts. If the drumhead is vibrating in one of these normal modes, saw-dust sprinkled on it will collect along the nodal lines, so that you can see them. It is not easy to obtain the pure normal modes with initial conditions. However, if an oscillator vibrating

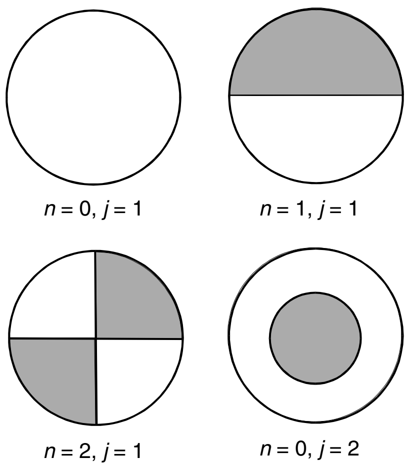


Fig. 6.4. Normal modes of drumhead vibration

in the frequency of a particular mode is placed nearby, the drumhead will eventually be vibrating in that mode, and the nodal lines become visible.

The general motion of a vibrating drumhead is a superposition of all these normal modes. If nodal lines exist at all, they will not usually be of the simple patterns.

Note that if the initial displacement is independent of φ, then $f(\rho,\varphi)$ should be replaced by $f(\rho)$. In that case, it follows from (6.11) and (6.13) that $a_{ni}=0$ and $b_{ni}=0$ for all n except for $n=0$. For $n=0$, (6.12) gives

$$a_{0i}=\frac{2}{c^2 J_1^2(k_{0i}c)}\int_0^c \rho J_0(k_{0i}\rho)f(\rho)\mathrm{d}\rho.$$

In this case the solution is given by

$$z(\rho,\varphi,t)=\sum_{i=1}\left[\frac{2}{c^2 J_1^2(k_{0i}c)}\int_0^c \rho' J_0(k_{0i}\rho')f(\rho)\mathrm{d}\rho'\right]J_0(k_{0i}\rho)\cos(ak_{0i}t). \tag{6.15}$$

This is what one would expect if both boundary and initial conditions are independent of φ, then there is no reason that the final solution is dependent on φ. Therefore in this case one can start with

$$z=R(\rho)T(t)$$

and obtain the result of (6.15) directly.

Example 6.3.1. Find the frequency of the normal modes shown in Fig. 6.4.

Solution 6.3.1. The frequency of vibration for the nj normal mode is $k_{nj}a/2\pi$. From the "Table of zeros" of Bessel functions, we find

$$\begin{aligned}
J_0(x)&=0 \quad \text{for } x=2.4048,\ 5.5201,\ \ldots\\
J_1(x)&=0 \quad \text{for } x=3.8317,\ 7.0156,\ \ldots\\
J_2(x)&=0 \quad \text{for } x=5.1356,\ 8.4172,\ \ldots
\end{aligned}$$

Therefore for the 01 mode, $k_{01}c=2.4048$, and the frequency of this mode is given by

$$\nu_{01}=\frac{k_{01}a}{2\pi}=\frac{2.4048a}{2\pi c}.$$

Similarly,

$$\nu_{11}=\frac{3.8317a}{2\pi c},\quad \nu_{21}=\frac{5.1356a}{2\pi c},\quad \nu_{02}=\frac{5.5201a}{2\pi c}.$$

Example 6.3.2. If the membrane and its frame are moving as a rigid body with unit velocity perpendicular to the membrane and the frame is suddenly brought to rest, then the membrane will start to vibrate. The vibration can be modeled with the following boundary value problem:

$$\begin{aligned} &\text{D.E.:} \quad && \nabla^2 z(\rho,t) = \frac{1}{a^2}\frac{\partial^2}{\partial t^2} z(\rho,t), \\ &\text{B.C.:} \quad && z(c,t) = 0, \\ &\text{I.C.:} \quad && z(\rho,0) = 0, \quad \left.\frac{\partial z}{\partial t}\right|_{t=0} = 1. \end{aligned}$$

Find the displacements $z(\rho,t)$.

Solution 6.3.2. With $z(\rho,t) = R(\rho)T(t)$, the time- and space-dependent parts are, respectively, given by

$$T(t) = c_1 \cos kat + c_2 \sin kat$$

and

$$R(\rho) = J_0(k\rho).$$

The boundary condition requires that $J_0(kc) = 0$. Therefore k must be equal to one of k_{0j}, where $k_{0j}c$ is the jth root of $J_0(x) = 0$. The initial condition requires $T(0) = c_1 = 0$. Thus

$$z(\rho,t) = \sum_{j=1} b_j J_0(k_{0j}\rho) \sin k_{0j}at.$$

Now the other initial condition requires that

$$\left.\frac{\partial z}{\partial t}\right|_{t=0} = \sum_{j=1} b_j k_{0j} a J_0(k_{0j}\rho) = 1.$$

Multiplying by $\rho J_0(k_{0j}\rho)$, integrating from 0 to c, we obtain

$$b_j k_{0j} a = \frac{2}{c^2 J_1^2(k_{0j}c)} \int_0^c \rho J_0(k_{0j}\rho) \mathrm{d}\rho.$$

Recall

$$\int_0^c x J_0(x) \mathrm{d}x = \int_0^c \mathrm{d}[x J_1(x)],$$

so

$$\int_0^c \rho J_0(k_{0j}\rho) \mathrm{d}\rho = \frac{1}{k_{0j}} c J_1(k_{0j}c).$$

It follows:

$$b_j k_{0j} a = \frac{2}{c^2 J_1^2(k_{0j}c)} \frac{1}{k_{0j}} c J_1(k_{0j}c),$$

$$b_j = \frac{2}{ack_{0j}^2} \frac{1}{J_1(k_{0j}c)}.$$

Therefore

$$z(\rho, t) = \frac{2}{ac} \sum_{j=1} \frac{\sin(k_{0j}at)}{k_{0j}^2 J_1(k_{0j}c)} J_0(k_{0j}\rho)$$

is the displacement, where $k_{0j}c$ are the positive roots of $J_0(x) = 0$.

6.3.2 Heat Conduction in a Disk: Two-Dimensional Diffusion Equation in Polar Coordinates

The solution of the diffusion equation

$$\nabla^2 F = \frac{1}{a^2} \frac{\partial}{\partial t} F$$

in the form of $F(\rho, \varphi, t) = u(\rho, \varphi) T(t)$ is similar to that of the wave equation. In terms of $u(\rho, \varphi)$ and $T(t)$, the equation can be written as

$$\frac{1}{u(\rho, \varphi)} \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right] u(\rho, \varphi) = \frac{1}{T(t)} \frac{1}{a^2} \frac{\partial}{\partial t} T(t).$$

Again both sides must equal to the same separation constant

$$\frac{1}{u(\rho, \varphi)} \left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} \right] u(\rho, \varphi) = \lambda,$$

$$\frac{1}{T(t)} \frac{1}{a^2} \frac{\partial}{\partial t} T(t) = \lambda.$$

The separation constant must be negative, $\lambda = -k^2$, just as in the wave equation, but for a slightly different reason. If λ were equal to $+k^2$, then $T(t) = \mathrm{e}^{k^2 a^2 t}$, which would make the temperature increase without bound as time increases. This is physically unreasonable. (Formally we say that it violates the principle of conservation of energy.) Not only that, $\lambda = k^2$ would render the space part fail to satisfy the boundary conditions. It is possible to have $\lambda = 0$, indicating that the system has already reached equilibrium. However, we should be able to solve the problem with $\lambda \neq 0$, and show that the solution reduces to that of $\lambda = 0$ as $t \to \infty$.

With $\lambda = -k^2$, $T(t) = \mathrm{e}^{-k^2 a^2 t}$ and the space part is given by the Helmholtz equation. While this equation is identical to the one obtained in solving the wave equation, it is the boundary conditions that can make a difference.

In the following examples, we will consider the heat conduction in a coin. Its circumference can either be kept at a constant temperature, or be insulated.

Example 6.3.3. Find the temperature in the disk of radius c, the flat faces of which are insulated. Initially the disk is at a prescribed temperature $f(\rho)$. Its outer edge is kept at $100°$ for all times.

Solution 6.3.3. Since neither the boundary condition nor the initial condition has angular dependence, so we know that the temperature distribution in the disk is independent of the angle. Thus the temperature in the disk $F(\rho, t)$ is the solution of the following problem:

$$\begin{aligned} \text{D.E.:} \quad & \nabla^2 F = \frac{1}{a^2}\frac{\partial}{\partial t}F, \\ \text{B.C.:} \quad & F(c,t) = 100, \\ \text{I.C.:} \quad & F(\rho, 0) = f(\rho). \end{aligned}$$

The problem is easier to solve if we make a change of variable. Let

$$u(\rho, t) = F(\rho, t) - 100.$$

The differential equation that governs $u(\rho, t)$ remains the same

$$\nabla^2 u(\rho, t) = \frac{1}{a^2}\frac{\partial}{\partial t}u(\rho, t)$$

but the boundary condition and the initial condition are changed to

$$u(c,t) = 0, \quad u(\rho, 0) = f(\rho) - 100.$$

The solution is given by

$$u(\rho, t) = J_0(k\rho)\mathrm{e}^{-k^2 a^2 t}.$$

In order to satisfy the boundary condition $u(c,t) = 0$, k must be equal to one of k_{0j}, where $k_{0j}c$ is the jth root of $J_0(x) = 0$. Therefore the general solution is the following linear combination:

$$u(\rho, t) = \sum_{j=1} c_j J_0(k_{0j}\rho)\mathrm{e}^{-k_{0j}^2 a^2 t}.$$

Since $u(\rho, 0) = f(\rho) - 100$, so

$$f(\rho) - 100 = u(\rho, 0) = \sum_{j=1} c_j J_0(k_{0j}\rho).$$

It follows that:

$$c_j = \frac{2}{c^2 J_1^2(k_{0j}c)} \int_0^c (f(\rho) - 100) J_0(k_{0j}\rho)\rho \, \mathrm{d}\rho.$$

Therefore the temperature distribution in the disk is given by

$$F(\rho, t) = 100 + \sum_{j=1} A_j J_0(k_{0j}\rho) \mathrm{e}^{-k_{0j}^2 a^2 t},$$

$$A_j = \frac{2}{c^2 J_1^2(k_{0j}c)} \int_0^c (f(\rho') - 100) J_0(k_{0j}\rho')\rho' \, \mathrm{d}\rho'.$$

Clearly, as $t \to \infty$, the temperature everywhere in the disk will settle down to $100°$, as it should be, regardless of the initial temperature.

Example 6.3.4. Replace the boundary condition of the previous problem with the condition that the edge of the disk is thermally insulated. Find the temperature distribution $u(\rho, t)$ in the disk if $u(\rho, 0) = f(\rho)$.

Solution 6.3.4. An insulated edge means that there is no heat flowing in and out of the disk. Since the heat flux is proportional to the gradient of the temperature, an insulated edge corresponds to the boundary condition

$$\left. \frac{\partial}{\partial \rho} u(\rho, t) \right|_{\rho = c} = 0.$$

The solution of the diffusion equation is still given by

$$u(\rho, t) = J_0(k\rho) \mathrm{e}^{-k^2 a^2 t}.$$

Now the boundary condition requires that

$$\left. \frac{\mathrm{d}}{\mathrm{d}\rho} J_0(k\rho) \right|_{\rho = c} = 0.$$

Recall

$$\frac{\mathrm{d}}{\mathrm{d}x} J_0(x) = -J_1(x).$$

This means

$$\left. \frac{\mathrm{d}}{\mathrm{d}\rho} J_0(k\rho) \right|_{\rho = c} = -kJ_1(kc) = 0.$$

Thus k must be equal to one of k_{1j}, where $k_{1j}c$ is the jth root of $J_1(x) = 0$. Therefore the general solution is

$$u(\rho, t) = \sum_{j=0} c_j J_0(k_{1j}\rho) \mathrm{e}^{-k_{1j}^2 a^2 t}.$$

Note that in this case $k_{10} = 0$ is also an eigenvalue, since $J_1(0) = 0$. Furthermore, since $J_0(0) = 1$, we can write this expansion as

$$u(\rho, t) = c_0 + \sum_{j=1} c_j J_0(k_{1j}\rho)\mathrm{e}^{-k_{1j}^2 a^2 t}.$$

With the initial condition $u(\rho, 0) = f(\rho)$, we have

$$f(\rho) = c_0 + \sum_{j=1} c_j J_0(k_{1j}\rho). \tag{6.16}$$

The coefficient c_j can be determined with the help of the orthogonality relations of the Bessel functions

$$\int_0^c \rho J_0(k_{1j}\rho) J_0(k_{1i}\rho)\mathrm{d}\rho = \delta_{ij}\beta_{0j}^2,$$

where β_{0j}^2 is given by

$$\beta_{0j}^2 = \frac{1}{2}c^2 J_0^2(k_{1j}c),$$

which follows from (4.55) with $n = 0$ and $A = 0$. Furthermore

$$\int_0^c \rho J_0(k_{1j}\rho)\mathrm{d}\rho = \frac{c}{k_{1j}} J_1(k_{1j}c) = 0,$$

since $k_{1j}c$ is one of the roots of $J_1(x) = 0$. Multiplying (6.16) by ρ and integrating from 0 to c, we have

$$\int_0^c \rho f(\rho)\mathrm{d}\rho = \int_0^c \rho c_0 \mathrm{d}\rho = \frac{1}{2}c^2 c_0$$

or

$$c_0 = \frac{2}{c^2}\int_0^c \rho f(\rho)\,\mathrm{d}\rho.$$

Multiplying (6.16) by $\rho J_0(k_{1i}\rho)$ and integrating from 0 to c, we have

$$c_i = \frac{1}{\beta_{0i}^2}\int_0^c \rho J_0(k_{1i}\rho) f(\rho)\,\mathrm{d}\rho.$$

Thus the temperature distribution in the disk is given by

$$u(\rho, t) = \frac{2}{c^2}\int_0^c \rho f(\rho)\mathrm{d}\rho + \sum_{j=1} B_j J_0(k_{1j}\rho)\mathrm{e}^{-k_{1j}^2 a^2 t},$$

$$B_j = \frac{2}{c^2 J_0^2(k_{1j}c)}\int_0^c \rho' J_0(k_{1j}\rho') f(\rho')\mathrm{d}\rho'.$$

6.3.3 Laplace's Equations in Cylindrical Coordinates

Three-dimensional Laplace equation $\nabla^2 V = 0$ in cylindrical coordinates takes the form

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}\right] V(\rho,\varphi,z) = 0.$$

With $V(\rho,\varphi,z) = u(\rho,\varphi)Z(z)$, this equation can be written as

$$\frac{1}{u(\rho,\varphi)}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right] u(\rho,\varphi) = -\frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z).$$

Again both sides must be equal to the same separation constant,

$$\frac{1}{u(\rho,\varphi)}\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right] u(\rho,\varphi) = \lambda,$$

$$-\frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) = \lambda.$$

It is seen that after the $Z(z)$ is separated out, the remaining equation for $u(\rho,\varphi)$ is again a Helmholtz equation in polar coordinates. In fact, the equation looks exactly the same as that obtained from wave equation or diffusion equation. However, there is a subtle difference. The difference is that in this case, the separation constant λ can be either positive $+k^2$ or negative $-k^2$, depending on the boundary conditions. In the following examples, we will illustrate that under certain boundary conditions we have to use $+k^2$, while under other boundary conditions we need to use $-k^2$. In general, both $+k^2$ and $-k^2$ are needed.

Example 6.3.5. Consider a solid circular cylinder of radius c and length L. The temperatures at the bases of the cylinder are maintained at 0° while the temperature of its lateral surface is specified by a given function $f(z)$, as shown is Fig. 6.5. Find the steady-state distribution of the temperature inside the cylinder.

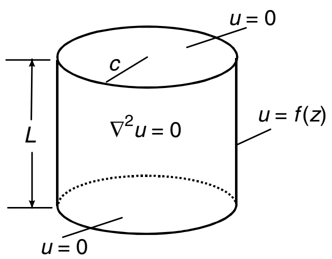


Fig. 6.5. Laplace equation in a circular cylinder. With this set of boundary conditions, the solution is given by a series of products of trigonometric and modified Bessel functions

Solution 6.3.5. The steady-state temperature u satisfies the Laplace equation $\nabla^2 u = 0$. Furthermore, since the boundary conditions do not depend on angle, the solution will also not be a function of φ. Therefore our task is to solve the following boundary value problem:

$$\begin{aligned} &\text{D.E.:} \quad \nabla^2 u(\rho, z) = 0, \\ &\text{B.C.:} \quad u(\rho, 0) = u(\rho, L) = 0, \quad u(c, z) = f(z). \end{aligned}$$

With $u(\rho, z) = R(\rho)Z(z)$, we have

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right] R(\rho) = \lambda R(\rho)$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} Z(z) = -\lambda Z(z).$$

The boundary condition $u(\rho, 0) = R(\rho)Z(0) = 0$ translates into $Z(0) = 0$. Similarly $Z(L) = 0$. The Z equation and its boundary conditions remind us of the problem of one-dimensional heat conduction with both ends at $0°$. These boundary conditions can be satisfied if the solutions are cosine and sine functions. With the separation constant λ chosen as $+k^2$,

$$Z(z) = c_1 \cos kz + c_2 \sin kz.$$

The corresponding R equation can be written as

$$\left(\rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} - k^2 \rho^2\right) R(\rho) = 0.$$

As we have seen in Chap. 4, this is the modified Bessel function of order 0. Its solution is

$$R(\rho) = c_3 I_0(k\rho) + c_4 K_0(k\rho),$$

where I_0 and K_0 are modified Bessel functions of first and second kind of order 0. Since K_0 diverges as $\rho \to 0$, to keep the temperature finite on the axis of the cylinder, c_4 must be set to zero. The condition $Z(0) = 0$ requires $c_1 = 0$. The condition $Z(L) = 0$ requires k to be one of k_n, where $k_n = n\pi/L$ with n as an integer ($n = 1,\ 2, \ldots$). Thus the general solution is in the form of

$$u(\rho, z) = \sum_{n=1} A_n I_0\left(\frac{n\pi}{L}\rho\right) \sin\left(\frac{n\pi}{L}z\right).$$

The coefficients A_n are determined by the other boundary condition $u(c, z) = f(z)$,

$$f(z)=\sum_{n=1} A_n I_0\left(\frac{n\pi}{L}c\right)\sin\left(\frac{n\pi}{L}z\right).$$

This is a Fourier sine series. It follows that:

$$A_n I_0\left(\frac{n\pi}{L}c\right)=\frac{2}{L}\int_0^L f(z)\sin\left(\frac{n\pi}{L}z\right)\mathrm{d}z.$$

Therefore the temperature distribution is given by

$$u(\rho,z)=\sum_{n=1}\left[\frac{1}{I_0\left(\frac{n\pi}{L}c\right)}\frac{2}{L}\int_0^L f(z')\sin\left(\frac{n\pi}{L}z'\right)\mathrm{d}z'\right]I_0\left(\frac{n\pi}{L}\rho\right)\sin\left(\frac{n\pi}{L}z\right).$$

Example 6.3.6. Solve the previous problem with the boundary conditions replaced by the conditions that both the curved and bottom surfaces are kept at 0° and the top surface is specified by a given function $g(\rho)$, as shown in Fig. 6.6.

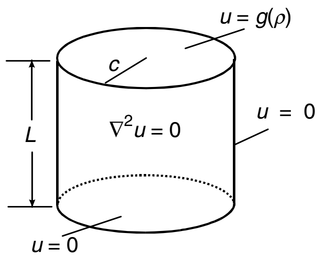


Fig. 6.6. Laplace equation in a circular cylinder. With this set of boundary conditions, the solution is given by a series of products of hyperbolic and Bessel functions

Solution 6.3.6. This time the problem we need to solve is

$$\begin{aligned}&\text{D.E.:} \quad \nabla^2 u(\rho,z)=0,\\ &\text{B.C.:} \quad u(c,z)=0, \quad u(\rho,0)=0, \quad u(\rho,L)=g(\rho).\end{aligned}$$

Again with $u(\rho,z)=R(\rho)Z(z)$ we have

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]R(\rho)=\lambda R(\rho)$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}Z(z)=-\lambda Z(z).$$

The boundary condition $u(c,z)=R(c)Z(z)=0$ requires $R(c)=0$.

Now if $\lambda = +k^2$, as we have seen in the last example

$$R(\rho) = I_0(k\rho).$$

Since I_0 is a monotonically increasing function and is never equal to zero, this choice cannot possibly satisfy the boundary condition.

If $\lambda = 0$, the R equation becomes a Euler–Cauchy differential equation, its solution is

$$R(\rho) = c_1 + c_2 \ln \rho.$$

To keep the temperature finite on the axis of the cylinder, we must set $c_2 = 0$. To make $R(c) = 0$, c_1 must also be zero. Therefore we will not get a solution for $\lambda = 0$.

Finally, if $\lambda = -k^2$, the R equation becomes

$$\left(\rho^2 \frac{\partial^2}{\partial \rho^2} + \rho \frac{\partial}{\partial \rho} + k^2 \rho^2\right) R(\rho) = 0,$$

and the solution that is finite on the axis of the cylinder is given by the Bessel function of zeroth-order

$$R(\rho) = J_0(k\rho).$$

To satisfy the boundary condition $R(c) = 0$, k must be equal to k_{0j}, where $k_{0j}c$ is the jth root of $J_0(x) = 0$.

The corresponding Z equation is

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} Z(z) = k_{0j}^2 Z(z).$$

The solution of this equation is

$$Z(z) = c_3 \cosh k_{0j} z + c_4 \sinh k_{0j} z.$$

The condition $u(\rho, 0) = R(\rho)Z(0) = 0$ requires $Z(0) = 0$, therefore c_3 must be zero. Thus the general solution is in the form of

$$u(\rho, z) = \sum_{j=1} A_j J_0(k_{0j}\rho) \sinh k_{0j} z.$$

The coefficients A_j can be determined by the condition $u(\rho, L) = g(\rho)$,

$$g(\rho) = \sum_{j=1} A_j J_0(k_{0j}\rho) \sinh k_{0j} L.$$

It follows that:

$$A_j \sinh k_{0j} L = \frac{2}{c^2 J_1^2(k_{0j}c)} \int_0^c g(\rho) J_0(k_{0j}\rho)\rho \, \mathrm{d}\rho.$$

Therefore the temperature distribution in this case is given by

$$u(\rho, z) = \sum_{j=1} \left[\frac{2}{c^2 J_1^2(k_{0j}c) \sinh k_{0j} L} \int_0^c g(\rho') J_0(k_{0j}\rho')\rho' \mathrm{d}\rho' \right] J_0(k_{0j}\rho) \sinh k_{0j} z.$$

Example 6.3.7. Find the steady-state temperature $u(\rho, z)$ in the cylinder of the previous example, subject to the boundary conditions: $u(\rho, 0) = 0,\ u(c, z) = T_1,\ u(\rho, L) = T_2$, where T_1 and T_2 are two constants.

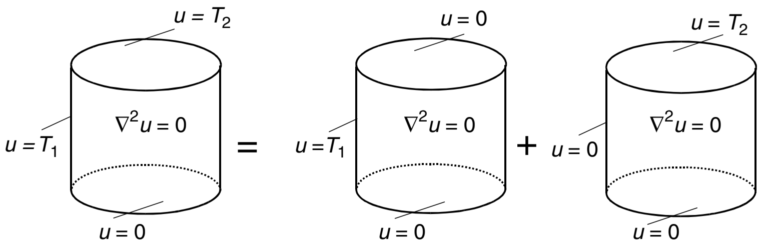


Fig. 6.7. Principle of superposition. The problem on the left-hand side can be broken apart as the sum of the two problems on the right-hand side

Solution 6.3.7. This problem can be solved by using the principle of superposition and the results of the previous examples. The problem is shown on the left-hand side of Fig. 6.7. We break it into two parts on the right-hand side of Fig. 6.7. Suppose that we have solved the following two problems:

$$\begin{aligned} &\text{D.E.:} \quad \nabla^2 u_1(\rho, z) = 0, \\ &\text{B.C.:} \quad u_1(\rho, 0) = 0, \quad u_1(\rho, L) = 0, \quad u_1(c, z) = T_1 \end{aligned}$$

and

$$\begin{aligned} &\text{D.E.:} \quad \nabla^2 u_2(\rho, z) = 0, \\ &\text{B.C.:} \quad u_2(\rho, 0) = 0, \quad u_2(\rho, L) = T_2, \quad u_2(c, z) = 0. \end{aligned}$$

Clearly the required solution is

$$u(\rho, z) = u_1(\rho, z) + u_2(\rho, z),$$

since

$$\begin{aligned} \nabla^2 u(\rho, z) &= \nabla^2 u_1(\rho, z) + \nabla^2 u_2(\rho, z), \\ u(\rho, 0) &= u_1(\rho, 0) + u_2(\rho, 0) = 0, \\ u(\rho, L) &= u_1(\rho, L) + u_2(\rho, L) = T_2, \\ u(c, z) &= u_1(c, z) + u_2(c, z) = T_1. \end{aligned}$$

From the results of previous examples

$$u_1(\rho,z)=\sum_{n=1}\left[\frac{1}{I_0\left(\frac{n\pi}{L}c\right)}\frac{2}{L}\int_0^L T_1\sin\frac{n\pi}{L}z'\mathrm{d}z'\right]I_0\left(\frac{n\pi}{L}\rho\right)\sin\left(\frac{n\pi}{L}z\right)$$

and

$$u_2(\rho,z)=\sum_{j=1}\left[\frac{2}{c^2J_1^2(k_{0j}c)\sinh k_{0j}L}\int_0^c T_2J_0(k_{0j}\rho')\rho'\mathrm{d}\rho'\right]J_0(k_{0j}\rho)\sinh k_{0j}z.$$

Since

$$\frac{2}{L}\int_0^L T_1\sin\frac{n\pi}{L}z'\mathrm{d}z'=\begin{cases}\frac{4T_1}{n\pi} & n=\text{odd}\\ 0 & n=\text{even}\end{cases}$$

and

$$\int_0^c T_2J_0(k_{0j}\rho')\rho'\mathrm{d}\rho'=T_2\frac{c}{k_{0j}}J_1(k_{0j}c),$$

it follows that the required solution is given by:

$$\begin{aligned}u(\rho,z)=&\frac{4T_1}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\frac{1}{I_0\left(\frac{n\pi}{L}c\right)}I_0\left(\frac{n\pi}{L}\rho\right)\sin\left(\frac{n\pi}{L}z\right)\\&+\frac{2T_2}{c}\sum_{j=1}\frac{1}{k_{0j}J_1(k_{0j}c)\sinh k_{0j}L}J_0(k_{0j}\rho)\sinh k_{0j}z.\end{aligned}$$

6.3.4 Helmholtz's Equations in Cylindrical Coordinates

Generalizing the above method for Helmholtz equation in cylindrical coordinates is straightforward. However, it is important to appreciate the distinction between solving Laplace equation and solving Helmholtz equation as an eigenvalue problem. For Helmholtz equation, a homogeneous boundary condition can be imposed (the function can vanish on all boundary surfaces), whereas the only solution of Laplace equation satisfying the homogeneous boundary condition is the trivial solution that the function is identically zero everywhere. This is best illustrated by an example.

Example 6.3.8. **Heat Conduction in a Finite Cylinder.** Consider a solid circular cylinder of radius c and length L, its surfaces are kept at 0° all the time. Initially the cylinder has a uniform temperature T_0, as shown in Fig. 6.8. Find the temperature distribution inside the cylinder as a function of time.

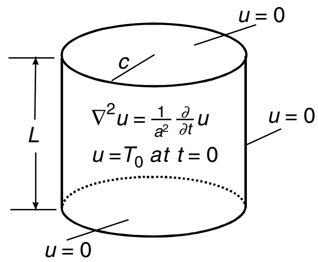


Fig. 6.8. Diffusion equation in a cylinder. Homogeneous boundary conditions can be imposed for a Helmholtz equation

Solution 6.3.8. Let u be the temperature. It satisfies the diffusion equation, which in cylindrical coordinates take the form

$$\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right] u(\rho, z, t) = \frac{1}{a^2}\frac{\partial}{\partial t} u(\rho, z, t),$$

where we have taken the advantage that this problem has no angular dependence because of the axial symmetry. The boundary and initial conditions are

$$u(c, z, t) = u(\rho, 0, t) = u(\rho, L, t) = 0,$$
$$u(\rho, z, 0) = T_0.$$

With $u(\rho, z, t) = R(\rho)Z(z)T(t)$, this equation is first separated into two equations

$$\frac{1}{R(\rho)Z(z)}\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right] R(\rho)Z(z) = -k^2,$$
$$\frac{1}{T(t)}\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}T(t) = -k^2.$$

The separation constant has to be a negative number $-k^2$, because the temperature should be in the form of $\exp(-k^2a^2t)$. The differential equation of the space part is a Helmholtz equation in cylindrical coordinates without angular dependence. It can be written as:

$$\frac{1}{R(\rho)}\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right] R(\rho) + \frac{1}{Z(z)}\frac{\partial^2}{\partial z^2} Z(z) = -k^2.$$

We can set

$$\frac{1}{R(\rho)}\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right] R(\rho) = -k_1^2,$$
$$\frac{1}{Z(z)}\frac{\partial^2}{\partial z^2} Z(z) = -k_2^2,$$

so that $k^2 = k_1^2 + k_2^2$. To satisfy the boundary conditions $Z(0) = Z(L) = 0$, k_2 has to be $k_2 = n\pi/L$, so that

$$Z(z) = \sin\left(\frac{n\pi}{L}z\right), \quad n = 1, 2, \dots \; .$$

The solution for $R(\rho)$ is the zeroth order Bessel function

$$R(\rho) = J_0(k_1\rho).$$

In order to satisfy the condition $R(c) = 0$, k_1 must be one of the k_{0j} where $k_{0j}c$ is the jth root of $J_0(x) = 0$. Thus

$$k^2 = k_{0j}^2 + \left(\frac{n\pi}{L}\right)^2$$

and

$$u(\rho, z, t) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} a_{nj} J_0(k_{0j}\rho)\sin\left(\frac{n\pi}{L}z\right)\mathrm{e}^{-\left[k_{0j}^2+(n\pi/L)^2\right]a^2t}.$$

The initial condition is such that

$$u(\rho, z, 0) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} a_{nj} J_0(k_{0j}\rho)\sin\left(\frac{n\pi}{L}z\right) = T_0.$$

It follows that:

$$a_{nj} = \frac{4}{c^2 J_1^2(k_{0j}c)L}\int_0^c\int_0^L T_0\rho J_0(k_{0j}\rho)\sin(\frac{n\pi}{L}z)\mathrm{d}z\mathrm{d}\rho.$$

Since

$$\int_0^L \sin\left(\frac{n\pi}{L}z\right)\mathrm{d}z = \begin{cases} \frac{2L}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases},$$

$$\int_0^c \rho J_0(k_{0j}\rho)\mathrm{d}\rho = \frac{c}{k_{0j}}J_1(k_{0j}c),$$

therefore

$$u(\rho, z, t) = \frac{8}{\pi c}\sum_{n=\text{odd}}^{\infty}\sum_{j=1}^{\infty}\frac{1}{nk_{0j}J_1(k_{0j}c)}J_0(k_{0j}\rho)\sin\left(\frac{n\pi}{L}z\right)\mathrm{e}^{-\left[k_{0j}^2+(n\pi/L)^2\right]a^2t}.$$

As expected $u(\rho, z, t) \to 0$ for $t \to \infty$, at all points.

6.4 Three-Dimensional Laplacian in Spherical Coordinates

6.4.1 Laplace's Equations in Spherical Coordinates

The Laplace equation $\nabla^2 F = 0$ written in the spherical coordinates is in the form of

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]F(r,\theta,\varphi)=0.$$

With the assumption $F(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$, we can multiply this equation by r^2/F and obtain

$$\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r)+\frac{1}{\Theta(\theta)\Phi(\varphi)}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Theta(\theta)\Phi(\varphi)=0$$

or

$$\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r)=-\frac{1}{\Theta(\theta)\Phi(\varphi)}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Theta(\theta)\Phi(\varphi).$$

The left-hand side is a function or r, and the right-hand side is a function of θ and φ. Since $r,\ \theta,\ \varphi$ are independent variables, they can equal to each other, if and only if both sides equal to the same constant. That is

$$\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r)=\lambda, \tag{6.17}$$

$$-\frac{1}{\Theta(\theta)\Phi(\varphi)}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Theta(\theta)\Phi(\varphi)=\lambda. \tag{6.18}$$

Multiplying (6.18) by $\sin^2\theta$, we have

$$-\frac{1}{\Theta(\theta)}\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta)-\frac{1}{\Phi(\varphi)}\frac{\partial^2}{\partial\varphi^2}\Phi(\varphi)=\lambda\sin^2\theta.$$

Rearranging

$$\frac{1}{\Theta(\theta)}\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta)+\lambda\sin^2\theta=-\frac{1}{\Phi(\varphi)}\frac{\partial^2}{\partial\varphi^2}\Phi(\varphi),$$

we see again that both sides must be equal to another constant

$$\frac{1}{\Theta(\theta)}\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta)+\lambda\sin^2\theta=\mu, \tag{6.19}$$

$$-\frac{1}{\Phi(\varphi)}\frac{\partial^2}{\partial\varphi^2}\Phi(\varphi)=\mu.$$

The $\Phi(\varphi)$ Part of the Laplace's Equation

The last equation governs $\Phi(\varphi)$

$$\frac{\partial^2}{\partial \varphi^2}\Phi(\varphi) = -\mu\Phi(\varphi).$$

So $\Phi(\varphi)$ is given by

$$\Phi(\varphi) = \exp[\pm \mathrm{i}\sqrt{\mu}\varphi]$$

or by its real value equivalent

$$\Phi(\varphi) = c_1 \cos\sqrt{\mu}\varphi + c_2 \sin\sqrt{\mu}\varphi.$$

But when φ is increased by 2π, we come back to the same point. Therefore

$$\Phi(\varphi + 2\pi) = \Phi(\varphi),$$

which means

$$\exp[\pm \mathrm{i}\sqrt{\mu}(\varphi + 2\pi)] = \exp[\pm \mathrm{i}\sqrt{\mu}\varphi].$$

This requires $\sqrt{\mu}$ to be an integer, $\sqrt{\mu} = m$ $(m = 0, 1, 2, \ldots)$. In other words

$$\mu = m^2$$

and

$$\Phi(\varphi) = \exp[\pm \mathrm{i}m\varphi].$$

The $\Theta(\theta)$ Part of the Laplace's Equation

With $\mu = m^2$, (6.19) becomes

$$\frac{1}{\Theta(\theta)}\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta) + \lambda\sin^2\theta = m^2$$

or

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta(\theta) + \left[\lambda - \frac{m^2}{\sin^2\theta}\right]\Theta(\theta) = 0. \tag{6.20}$$

Make a change of variable

$$x = \cos\theta,$$

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} = -\sin\theta\frac{\partial}{\partial x},$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - x^2,$$

and let $\Theta(\theta) = y(x)$, so (6.20) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right] + \left[\lambda - \frac{m^2}{1-x^2}\right]y(x) = 0. \tag{6.21}$$

As we have discussed in Chap. 4, the solution of this equation will diverge at $x = \pm 1$, unless λ is equal to $l(l+1)$, where l is an integer ($l = 0, 1, 2, \ldots$). In other words, in order to have a physically acceptable solution, the separation constant λ, as an eigenvalue, must be

$$\lambda = l(l+1).$$

With this λ, the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y(x) = 0$$

is known as the associated Legendre equation. An additional requirement for this equation to yield an acceptable solution is that m must be in the range of $-l \leq m \leq l$. If these requirements are met, the solution of this equation is known as the associated Legendre polynomials

$$y(x) = P_l^{|m|}(x).$$

Associated Legendre polynomials are all related to (derivable from) Legendre polynomial $P_l(x)$, which is the solution with $m = 0$. The equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right] + l(l+1)y(x) = 0$$

is known as the Legendre equation. Its solution is known as the Legendre polynomial $P_l(x)$. Thus the $\Theta(\theta)$ part of the Laplace equation is given by

$$\Theta(\theta) = P_l^{|m|}(\cos\theta).$$

Therefore, together with $\Phi(\varphi) = \mathrm{e}^{\pm \mathrm{i}m\varphi}$, the angular part of the solution of the Laplace equation in spherical coordinates can be expressed as spherical harmonics $Y_l^m(\theta, \varphi)$.

Radial Part $R(r)$ of the Laplace Equation

The radial part of the Laplace equation is given by (6.17). With $\lambda = l(l+1)$, this equation becomes

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r) + 2r\frac{\mathrm{d}}{\mathrm{d}r}R(r) = l(l+1)R(r). \tag{6.22}$$

This type of equation is known as Euler–Cauchy equation, and can be solved by a change of variable

$$r = \mathrm{e}^t, \quad t = \ln r, \quad \frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{r}.$$

$$\frac{\mathrm{d}R}{\mathrm{d}r} = \frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}t},$$

$$\begin{aligned}\frac{\mathrm{d}^2R}{\mathrm{d}r^2} &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}t}\right] = -\frac{1}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}t} \\ &= -\frac{1}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}R}{\mathrm{d}t}\right]\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{1}{r^2}\frac{\mathrm{d}R}{\mathrm{d}t} + \frac{1}{r^2}\frac{\mathrm{d}^2R}{\mathrm{d}t^2}.\end{aligned}$$

So

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}R + 2r\frac{\mathrm{d}}{\mathrm{d}r}R = \frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{\mathrm{d}R}{\mathrm{d}t}.$$

Thus (6.22) becomes

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{\mathrm{d}R}{\mathrm{d}t} = l(l+1)R.$$

This is a differential equation with constant coefficients. The standard way to solve it is to set

$$R(t) = \mathrm{e}^{mt},$$

so

$$\frac{\mathrm{d}R}{\mathrm{d}t} = m\mathrm{e}^{mt}, \quad \frac{\mathrm{d}^2R}{\mathrm{d}t^2} = m^2\mathrm{e}^{mt}.$$

Thus m can be determined from

$$m^2 + m = l(l+1)$$

or

$$(m-l)(m+l+1) = 0.$$

Clearly

$$m = \begin{cases} l, \\ -l-1. \end{cases}$$

Therefore

$$R(r) = \begin{cases} \mathrm{e}^{lt} = \mathrm{e}^{l\ln r} = r^l, \\ \mathrm{e}^{-(l+1)t} = \mathrm{e}^{-(l+1)\ln r} = \dfrac{1}{r^{l+1}}. \end{cases}$$

Thus, the general solution of the Laplace equation is given by

$$F(r,\theta) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(a_{lm}r^l + b_{lm}\frac{1}{r^{l+1}}\right)P_l^{|m|}(\cos\theta)\mathrm{e}^{\pm m\varphi}.$$

If the problem has no φ dependence ($m = 0$, a common case), the solution reduces to

$$F(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + b_l\frac{1}{r^{l+1}}\right)P_l(\cos\theta).$$

The coefficients a_{lm} and b_{lm} are to be determined by the boundary conditions.

Example 6.4.1. A hollow conducting sphere of radius c is divided into two halves at the equator by a thin insulating strip. The upper hemisphere is charged to a potential V_0 and the lower hemisphere is kept at zero potential, as shown in Fig. 6.9. Find the potential inside and outside the sphere.

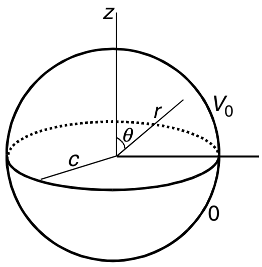


Fig. 6.9. A hollow conducting sphere

Solution 6.4.1. The electrostatic potential V satisfies the Laplace equation and the problem is axially symmetric, therefore the general form of the potential is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(a_l r^l + b_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta).$$

The boundary conditions are

$$V(c,\theta) = \begin{cases} V_0 & \text{for } 0 \le \theta \le \pi/2, \\ 0 & \text{for } \pi/2 \le \theta \le \pi. \end{cases}$$

Inside the sphere (for $r < c$) we require the solution to be finite at the origin and so $b_l = 0$ for all l. Imposing the boundary conditions at $r = c$, we have

$$\sum_{l=0}^{\infty} a_l c^l P_l(\cos\theta) = V(c,\theta).$$

The orthogonality of the Legendre polynomials enable us to determine the coefficients a_l. We multiply both sides of this equation by $P_n(\cos\theta)\sin\theta$ and integrate from $\theta = 0$ to π,

$$\sum_{l=0}^{\infty} a_l c^l \int_{\theta=0}^{\pi} P_l(\cos\theta) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta = \int_{\theta=0}^{\pi} V(c,\theta) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta.$$

Since

$$\int_{\theta=0}^{\pi} P_l(\cos\theta) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta = \int_{-1}^{1} P_l(x) P_n(x) \mathrm{d}x = \frac{2}{2n+1} \delta_{nl},$$

so

$$a_n c^n \frac{2}{2n+1} = \int_{\theta=0}^{\pi} V(c,\theta) P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta = V_0 \int_{\theta=0}^{\pi/2} P_n(\cos\theta)\sin\theta\,\mathrm{d}\theta.$$

Changing n to l and $\cos\theta$ to x, we see that a_l is given by

$$a_l = \frac{1}{c^l}\frac{2l+1}{2} V_0 \int_0^1 P_l(x)\mathrm{d}x.$$

Therefore

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(\frac{2l+1}{2} V_0 \int_0^1 P_l(x)\mathrm{d}x\right)\left(\frac{r}{c}\right)^l P_l(\cos\theta).$$

Since low order Legendre polynomials are relatively simple, the first few coefficients can be easily evaluated

$$a_0 = \frac{1}{2}V_0, \quad a_1 = \frac{3}{4c}V_0, \quad a_2 = 0, \quad a_3 = -\frac{7}{16c^3}V_0, \dots .$$

Thus

$$V(r,\theta) = \frac{1}{2}V_0 + \frac{3}{4}V_0\frac{r}{c}P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{c}\right)^3 P_3(\cos\theta) + \cdots .$$

For this problem, the general coefficients a_l can be expressed in a closed expression. Sometimes, this is quite important. For example, if we want to program the computer to sum up the potential to a high degree of accuracy, then it is useful to have an analytic expression. Recall

$$\int_0^1 P_l(x)\mathrm{d}x = \frac{1}{2l+1}[P_{l-1}(0) - P_{l+1}(0)]$$

and $P_{l-1}(0) = P_{l+1}(0) = 0$ if l is an even number (so $l-1$ and $l+1$ are odd). Thus $a_l = 0$ for all even l except a_0, which is equal to $V_0/2$. Therefore, $V(r,\theta)$ can be written as

$$V(r,\theta) = \frac{1}{2}V_0 + \sum_{n=0}^{\infty} a_{2n+1} r^{2n+1} P_{2n+1}(\cos\theta),$$

where

$$a_{2n+1} = \frac{1}{c^{2n+1}}\frac{4n+3}{2} V_0 \int_0^1 P_{2n+1}(x)\mathrm{d}x.$$

Now, since

$$\int_0^1 P_{2n+1}(x)\mathrm{d}x = \frac{1}{4n+3}[P_{2n}(0) - P_{2n+2}(0)]$$

and we have shown in Chap. 4 that

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2},$$

so

$$\int_0^1 P_{2n+1}(x)\mathrm{d}x = \frac{1}{4n+3}\left[(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\frac{4n+3}{2n+2}\right]. \tag{6.23}$$

Therefore $V(r,\theta)$ inside the sphere is given by

$$V(r,\theta) = \frac{V_0}{2}\left\{1 + \sum_{n=0}^{\infty}\left[(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\frac{4n+3}{2n+2}\right]\left(\frac{r}{c}\right)^{2n+1} P_{2n+1}(\cos\theta)\right\}.$$

Outside the sphere (for $r > c$), we require the potential to be bounded as $r \to \infty$. So we must set a_l in the general expression to zero for all l. In this case, by imposing the boundary condition at $r = c$, we have

$$\sum_{l=0}^{\infty} b_l \frac{1}{c^{l+1}} P_l(\cos\theta) = V(c,\theta).$$

Following the same argument as before, we find

$$b_l \frac{1}{c^{l+1}} = \frac{2l+1}{2} V_0 \int_0^1 P_l(x)\mathrm{d}x.$$

Therefore $V(r,\theta)$ outside the sphere is given by

$$V(r,\theta) = \frac{V_0}{2}\frac{c}{r}\left\{1 + \sum_{n=0}^{\infty}\left[(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}\frac{4n+3}{2n+2}\right]\left(\frac{c}{r}\right)^{2n+1} P_{2n+1}(\cos\theta)\right\}.$$

It may be noted that the solution for $r < c$ and the solution for $r > c$ are equal at $r = c$.

Example 6.4.2. Electrostatic Potential of a Ring of Charge. Find the electrostatic potential at a general point in space due to a total charge of Q uniformly distributed over a ring of radius c.

Solution 6.4.2. From elementary physics we learnt that the electrostatic potential V at a field point P due to a point charge Q is

$$V = \frac{Q}{d},$$

where d is the distance between the field point P and the charge. Therefore at the point P on the z-axis at a distance z above the plane of the ring (shown in Fig. 6.10), the electrostatic potential is given by

$$V = \frac{Q}{(z^2 + c^2)^{1/2}},$$

since all points of the ring are at the same distance $(z^2 + c^2)^{1/2}$ from the field point P.

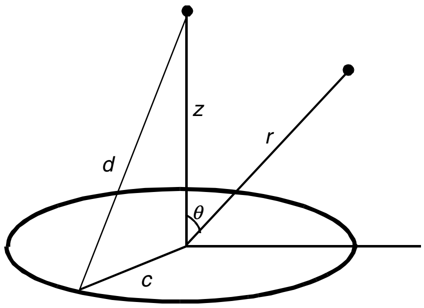


Fig. 6.10. Electrostatic potential of a ring of charge

To calculate the potential at a general point off the z-axis would be quite difficult if we are trying to sum up the potential due to the charges on the ring. However, we can find the potential by using the facts that the potential $V(r,\theta)$ satisfies the Laplace equation and it reduces to $V(z,0)$ when $r = z$ and $\theta = 0$. First if $r > c$, since we require $V(r,\theta)$ to be bounded as $r \to \infty$, all a_l in the solution of the Laplace equation must be set to zero, and what is left is

$$V(r,\theta) = \sum_{l=0}^{\infty} b_l \frac{1}{r^{l+1}} P_l(\cos\theta).$$

Furthermore, for $r = z$ and $\theta = 0$,

$$V(z,0) = \sum_{l=0}^{\infty} b_l \frac{1}{z^{l+1}} P_l(1) = \sum_{l=0}^{\infty} b_l \frac{1}{z^{l+1}}, \tag{6.24}$$

since $P_l(1) = 1$. But for $r > c$,

$$V(z,0) = \frac{Q}{(z^2+c^2)^{1/2}} = \frac{Q}{z}\left[1 + \left(\frac{c}{z}\right)^2\right]^{-1/2},$$

and

$$\left[1 + \left(\frac{c}{z}\right)^2\right]^{-1/2} = 1 - \frac{1}{2}\left(\frac{c}{z}\right)^2 + \frac{3}{8}\left(\frac{c}{z}\right)^4 + \cdots = \sum_{k=0}^{\infty} c_{2k}\left(\frac{c}{z}\right)^{2k},$$

where the binomial coefficients c_{2k} are given by

$$\begin{aligned} c_{2k} &= \frac{\left[-\frac{1}{2}\right]\left[-\frac{1}{2}-1\right]\cdots\left[-\frac{1}{2}-(k-1)\right]}{k!} = (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!} \\ &= (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!}\,\frac{(2k)(2k-2)\cdots 2}{(2k)(2k-2)\cdots 2} = (-1)^k \frac{(2k)!}{2^{2k}(k!)^2}. \end{aligned}$$

Therefore

$$V(z,0) = \frac{Q}{z}\left\{1 + \sum_{k=1}^{\infty}\left[(-1)^k \frac{(2k)!}{2^{2k}(k!)^2}\right]\left(\frac{c}{z}\right)^{2k}\right\}.$$

Comparing with (6.24), we see that $b_l = 0$ for all odd l. Thus (6.24) can be written as

$$V(z,0) = \frac{1}{z}\sum_{l=0}^{\infty}\frac{b_{2l}}{z^{2l}}$$

and

$$b_{2l} = Q(-1)^l \frac{(2l)!}{2^{2l}(l!)^2}c^{2l}.$$

Now we conclude that the potential $V(r,\theta)$ for $r > c$ must be

$$V(r,\theta) = \frac{Q}{r}\left\{1 + \sum_{l=1}(-1)^l \frac{(2l)!}{2^{2l}(l!)^2}\left(\frac{c}{r}\right)^{2l} P_{2l}(\cos\theta)\right\}.$$

This is so because this function without any arbitrary constant satisfies the Laplace equation and reduces to the correct answer on z-axis. The uniqueness theorem (shown in Chap. 2, vol. II) states that there is only one such function.

The potential for $r < c$ can be found in a similar way. Because of the axial symmetry and the requirement that the potential is bounded at $r = 0$, the solution of the Laplace equation must be in the following form:

$$V(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta).$$

For $r = z,\ \theta = 0$, this solution becomes

$$V(z,0) = \sum_{l=0}^{\infty} a_l z^l P_l(1) = \sum_{l=0}^{\infty} a_l z^l.$$

Comparison of this expression with

$$\begin{aligned} V(z,0) &= \frac{Q}{(z^2+c^2)^{1/2}} = \frac{Q}{c}\left[1 + \left(\frac{z}{c}\right)^2\right]^{-1/2} \\ &= \frac{Q}{c}\left\{1 + \sum_{k=1}^{\infty}\left[(-1)^k \frac{(2k)!}{2^{2k}(k!)^2}\right]\left(\frac{z}{c}\right)^{2k}\right\}, \quad \text{for } z < c \end{aligned}$$

clearly shows

$$a_0 = \frac{Q}{c}, \quad a_{2l+1} = 0,$$

$$a_{2l} = \frac{Q}{c^{2l+1}}(-1)^l \frac{(2l)!}{2^{2l}(l!)^2}c^{2l}.$$

Therefore the potential for $r < c$ is given by

$$V(r,\theta) = \frac{Q}{c}\left\{1 + \sum_{l=1}(-1)^l \frac{(2l)!}{2^{2l}(l!)^2}\left(\frac{r}{c}\right)^{2l} P_{2l}(\cos\theta)\right\}.$$

Example 6.4.3. Conducting sphere in a uniform electric field. A grounded conducting sphere of radius r_0 is placed in a previously uniform electric field $\mathbf{E}_0$. Find the electrostatic potential outside the sphere.

Solution 6.4.3. The electric field $\mathbf{E}$ is related to the potential V by $-\boldsymbol{\nabla}V = \mathbf{E}$. Therefore the uniform field, taken in the direction of z-axis, has an electrostatic potential

$$V = -E_0 z = -E_0 r \cos\theta,$$

since

$$-\boldsymbol{\nabla}(-E_0 z) = \mathbf{k}E_0 \frac{\partial}{\partial z} z = \mathbf{E}_0.$$

This potential satisfies the Laplace equation $\nabla^2 V = 0$, as must the potential when the sphere is present. We select the spherical coordinates (r, θ, φ) because of the spherical shape of the conductor. Furthermore, with the positive z-axis chosen to be the direction of the field, this problem is clearly axially symmetric. Therefore the general solution of the Laplace equation is given by

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(a_l r^l + b_l \frac{1}{r^{l+1}} \right) P_l(\cos\theta). \tag{6.25}$$

Since the original unperturbed electrostatic field is $\mathbf{k}E_0$, we require, as one boundary condition

$$V(r \to \infty) = -E_0 r \cos\theta = -E_0 r P_1(\cos\theta).$$

At large r, (6.25) becomes

$$V(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta),$$

since $1/r^{l+1} \to 0$, as $r \to \infty$. The last two equations clearly show that

$$a_1 = -E_0 \quad \text{and} \quad a_l = 0,\ l \neq 1.$$

The other boundary condition requires $V(r_0, \theta) = 0$, because the sphere is grounded. That is

$$V(r_0,\theta) = -E_0 r_0 P_1(\cos\theta) + \sum_{l=0}^{\infty} b_l \frac{1}{r_0^{l+1}} P_l(\cos\theta) = 0.$$

Clearly

$$b_1 = E_0 r_0^3 \quad \text{and} \quad b_l = 0,\ l \neq 1.$$

Thus the potential is given by

$$V(r,\theta) = -E_0 r \cos\theta \left[1 - \frac{r_0^3}{r^3} \right]. \tag{6.26}$$

This solution is similar to (6.6), which describes the potential produced by a conducting cylinder placed in a uniform field. The only difference is that $(r_0/r)^3$ is replaced by $(r_0/\rho)^2$.

Example 6.4.4. Sphere in a uniform stream. For an ideal incompressible fluid, the velocity $\mathbf{v}$ of the flow is derivable from a velocity potential U, such that $\mathbf{v} = \boldsymbol{\nabla} U$, where U satisfies the Laplace equation $\nabla^2 U = 0$. A sphere of radius r_0 is placed in a uniform stream, find the velocity potential.

Solution 6.4.4. At infinite distance from the sphere, the flow is uniform. Let the velocity v be in the direction of z-axis,

$$U(r \to \infty) = vz = vr\cos\theta = vrP_1(\cos\theta).$$

The velocity normal to the surface of the sphere must be zero

$$\left[\frac{\partial U}{\partial r}\right]_{r=r_0} = 0.$$

Again our solution is given by

$$U(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + b_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta).$$

To satisfy the asymptotic boundary condition

$$U(r \to \infty) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) = vrP_1(\cos\theta),$$

we must have

$$a_1 = v \quad \text{and} \quad a_l = 0,\ l \neq 1.$$

Thus

$$U(r,\theta) = vrP_1(\cos\theta) + \sum_{l=0}^{\infty} b_l \frac{1}{r^{l+1}} P_l(\cos\theta).$$

Now

$$\frac{\partial U}{\partial r} = vP_1(\cos\theta) - \sum_{l=0}^{\infty} b_l \frac{(l+1)}{r^{l+2}} P_l(\cos\theta),$$

$$\left[\frac{\partial U}{\partial r}\right]_{r=r_0} = vP_1(\cos\theta) - \sum_{l=0}^{\infty} b_l \frac{(l+1)}{r_0^{l+2}} P_l(\cos\theta) = 0.$$

Clearly

$$b_1 = \frac{1}{2} v r_0^3 \quad \text{and} \quad b_l = 0,\ l \neq 1.$$

Therefore

$$U(r,\theta) = v\cos\theta\left[r + \frac{r_0^3}{2r^2}\right]. \tag{6.27}$$

This result is similar to (6.26) but not identical. Note that the tangential component of the electric field is zero at the surface of the sphere, whereas the normal component of the velocity of the flow at the surface of the sphere is zero.

6.4.2 Helmholtz's Equations in Spherical Coordinates

In spherical coordinates, the Helmholtz equation takes the form

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + k^2\right]F(r,\theta,\varphi) = 0.$$

With $F(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$, this equation can be written as

$$\frac{1}{R}\left[\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + k^2r^2\right]R = -\frac{1}{\Theta\Phi}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Theta\Phi.$$

Both sides of this equation must be equal to the same separation constant. It is seen that the angular part of the solution $\Theta(\theta)\Phi(\varphi)$ is identical to those of Laplace equation. Therefore the separation constant must be $l(l+1)$. Therefore the radial equation takes the form

$$r^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}R(r) + 2r\frac{\mathrm{d}}{\mathrm{d}r}R(r) + [k^2r^2 - l(l+1)]R(r) = 0.$$

With a change of variable $x = kr$, this equation becomes the spherical Bessel equation. Therefore the solutions of this equation are spherical Bessel and Neumann functions $j_l(kr)$ and $n_l(kr)$. The following example is an illustration of the use of these solutions.

Example 6.4.5. A particle of mass m is contained in a sphere of radius R. The particle is described by a wavefunction satisfying the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,z) = E\psi(r,\theta,z)$$

and the conditions that the wavefunction ψ remains finite and goes to zero over the surface of the sphere. Find the lowest permitted energy E.

Solution 6.4.5. This equation can be written in the form of a Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0$$

with

$$k^2 = \frac{2m}{\hbar^2}E.$$

The radial part $R_l(r)$ of the solution $\psi = R_l(r)Y_l^m(\theta,\varphi)$ is given by

$$R_l(r) = a_l j_l(kr) + b_l n_l(kr).$$

Since the function has to remain finite, we set $b_l = 0$ as the spherical Neumann function diverges as $r \to 0$. For the function to go to zero over the surface of the sphere, kR must be one the roots of $j_l(x) = 0$. For the lowest energy, kR must be the first zero of $j_0(x)$. Since

$$j_0(x) = \frac{1}{x}\sin x$$

the zeros are at $x = n\pi,\ n$ an integer. Thus $k = n\pi/R$. Therefore the energies are

$$E_{\min} = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2}{2m}\left(\frac{n\pi}{R}\right)^2$$

with the minimum energy for $n = 1$,

$$E_{\min} = \frac{\hbar^2}{2m}\left(\frac{\pi}{R}\right)^2.$$

6.4.3 Wave Equations in Spherical Coordinates

In Chap. 5, we have seen that one of the solutions of the wave equation

$$\frac{\partial^2}{\partial x^2}u = \frac{1}{a^2}\frac{\partial^2}{\partial t^2}u$$

is the plane wave

$$u(x,t) = \mathrm{e}^{\mathrm{i}(kx-kat)} = \mathrm{e}^{\mathrm{i}(kx-\omega t)},$$

where the wavelength is $2\pi/k$ and the angular frequency is $\omega = ka$. This is a plane wave because its wave front is a plane perpendicular to the x-axis. It moves from left to right with velocity a.

By expressing the wave equation in spherical coordinates, we can study the spherical wave, the wavefront of which is a spherical.

First let us assume that the spherical wave depends only on the radial distance r. In this case, the wave equation can be written as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]u(r,t) = \frac{1}{a^2}\frac{\partial^2}{\partial t^2}u(r,t).$$

Multiplying both the sides by r, we have

$$\left[r\frac{\partial^2}{\partial r^2}+2\frac{\partial}{\partial r}\right]u(r,t)=\frac{1}{a^2}\frac{\partial^2}{\partial t^2}[ru(r,t)].$$

Since

$$\left[r\frac{\partial^2}{\partial r^2}+2\frac{\partial}{\partial r}\right]u(r,t)=\frac{\partial^2}{\partial r^2}[ru(r,t)],$$

we can write the three-dimensional wave equation as

$$\frac{\partial^2}{\partial r^2}[ru(r,t)]=\frac{1}{a^2}\frac{\partial^2}{\partial t^2}[ru(r,t)].$$

Clearly $ru(r,t)$ plays the same role as $u(x,t)$ in the one-dimensional wave equation. So one of its solutions can be written as

$$ru(r,t)=\mathrm{e}^{\mathrm{i}(kr-\omega t)}$$

or

$$u(r,t)=\frac{1}{r}\mathrm{e}^{\mathrm{i}(kr-\omega t)}.$$

We can interpret this solution as the outgoing spherical wave with a wave length $2\pi/k$ and a angular frequency ω. Note that its amplitude is decreasing by a factor of $1/r$, as r is increasing. This is what it should be, since the area of the wave front is increasing by a factor of r^2. So the intensity of the wave (energy per unit area which is proportional to $|u|^2$) summed over the wave front is a constant.

In general, spherical waves also depend on θ and φ. The angular dependence is determined by the boundary conditions, or on how it is generated.

The separated solution of the wave equation is the product of a time-dependent part and a space part. The time-dependent part is exactly the same as that of the one- and two-dimensional wave equations, namely

$$T(t)=\mathrm{e}^{\mathrm{i}kat},\ \mathrm{e}^{-\mathrm{i}kat}$$

or their real part equivalents. Furthermore, $\mathrm{e}^{\pm\mathrm{i}kat}$ can be written as $\mathrm{e}^{\pm\mathrm{i}\omega t}$, where ω is the angular frequency.

The space part is governed by the Helmholtz equation. In spherical coordinates, solutions are given by

$$F_{lm}(r,\theta,\varphi)=[a_{lm}j_l(kr)+b_{lm}n_l(kr)]Y_l^m(\theta,\varphi).$$

If we are looking for an outgoing spherical wave at large r, then we have to combine the spherical Bessel function $j_l(kr)$ and the spherical Neumann function $n_l(kr)$ into the spherical Hankel function $h_l^{(1)}(kr)$ of the first kind. That is, b_{lm} has to be equal to $\mathrm{i}a_{lm}$. Recall as $x\to\infty$

$$h_l^{(1)}(x) \to \frac{1}{x}\mathrm{e}^{\mathrm{i}[x-(l+1)\pi/2]}.$$

In this way, asymptotically $u(r,\theta,\varphi,t) = F(r,\theta,\varphi)T(t)$ will take the form of

$$u_{lm}(r,\theta,\varphi,t) \to \frac{1}{kr}\mathrm{e}^{\mathrm{i}(kr\pm\omega t)}Y_l^m(\theta,\varphi).$$

The minus sign $(-\omega t)$ is for a spherical outgoing wave, and the plus sign $(+\omega t)$ is for a spherical incoming wave. Thus, the general outgoing spherical wave is a linear combination of the these components

$$u(r,\theta,\varphi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}h_l^{(1)}(kr)Y_l^m(\theta,\varphi)\mathrm{e}^{-\mathrm{i}\omega t}.$$

Example 6.4.6. The acoustic wave $u(\mathbf{r},t)$, satisfying the wave equation

$$\nabla^2 u = \frac{1}{a^2}\frac{\partial^2}{\partial t^2}u$$

is emitted from a split spherical antenna. At $r = r_0$, it satisfies the boundary condition

$$u = \begin{cases} V_0\mathrm{e}^{-\mathrm{i}\omega t} & 0 < \theta < \frac{\pi}{2}, \\ -V_0\mathrm{e}^{-\mathrm{i}\omega t} & \frac{\pi}{2} < \theta < \pi. \end{cases}$$

Find the outgoing solution of the wave equation.

Solution 6.4.6. First the separation constant k must be equal to

$$k = \frac{\omega}{a}.$$

Then we note that the boundary conditions are axially symmetric, we need only to consider solutions with $m = 0$. Therefore

$$u(r,\theta,t) = \sum_{l=0}^{\infty} a_l h_l^{(1)}(kr)P_l(\cos\theta)\mathrm{e}^{-\mathrm{i}\omega t}.$$

At $r = r_0$

$$u(r_0,\theta,0) = \sum_{l=0}^{\infty} a_l h_l^{(1)}(kr_0)P_l(\cos\theta) = \begin{cases} V_0 & 0 < \theta < \frac{\pi}{2}, \\ -V_0 & \frac{\pi}{2} < \theta < \pi. \end{cases}$$

This is a Fourier–Legendre series, the coefficients are

$$\begin{aligned} a_l h_l^{(1)}(kr_0) &= \frac{2l+1}{2}V_0\left[\int_0^{\pi/2} P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta - \int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta\right] \\ &= \begin{cases} 0 & l \text{ even}, \\ (2l+1)V_0\int_0^1 P_l(x)\mathrm{d}x & l \text{ odd}. \end{cases} \end{aligned}$$

Therefore l has to be odd and can be written as $l = 2n+1$ with $n = 0, 1, 2, \ldots$. Thus

$$u(r, \theta, t) = \sum_{n=0}^{\infty} a_{2n+1} h_{2n+1}^{(1)}(kr) P_{2n+1}(\cos\theta) \mathrm{e}^{-\mathrm{i}\omega t}$$

with

$$a_{2n+1} = \frac{4n+3}{h_{2n+1}^{(1)}(kr_0)} V_0 \int_0^1 P_{2n+1}(x) \mathrm{d}x.$$

With (6.23)

$$\int_0^1 P_{2n+1}(x) \mathrm{d}x = \frac{1}{4n+3} \left[(-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \frac{4n+3}{2n+2} \right],$$

the outgoing acoustic wave is therefore given by

$$u(r, \theta, t) = \sum_{n=0}^{\infty} A_n V_0 \frac{1}{h_{2n+1}^{(1)}(kr_0)} h_{2n+1}^{(1)}(kr) P_{2n+1}(\cos\theta) \mathrm{e}^{-\mathrm{i}\omega t},$$
$$A_n = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \frac{4n+3}{2n+2}.$$

6.5 Poisson's Equations

Poisson's equation is a nonhomogeneous partial differential equation. The most familiar one is probably

$$\nabla^2 V(\mathbf{r}) = -\varrho(\mathbf{r}),$$

where $V(\mathbf{r})$ is the electrostatic potential and $\varrho(\mathbf{r})$ is the density of electric charge (see Chap. 2 of vol. II). This is a field equation. Although the potential at the point $\mathbf{r}$ is created by all the charges everywhere, yet $\nabla^2 V(\mathbf{r})$ is miraculously related, through the Poisson equation, to only the charge density at the same point $\mathbf{r}$. The discovery of this kind of field equation is one of the greatest achievements of analysis. At places where there is no charge, this equation reduces to the Laplace equation, $\nabla^2 V(\mathbf{r}) = 0$.

In elementary physics, we learnt that the electrostatic potential due to the charges uniformly distributed over a sphere is the same as if all the charges are concentrated at the center, provided the point where the potential is evaluated is outside the sphere. In the following example, we will illustrate how this problem is solved in the context of theory of partial differential equations.

Example 6.5.1. Find the electrostatic potential V which satisfies the differential equation

$$\nabla^2 V(\mathbf{r}) = \begin{cases} -\varrho_0 & r < r_0, \\ 0 & r > r_0 \end{cases}$$

and the boundary conditions

$$V \to 0, \text{ as } r \to \infty,$$

and that V remains finite everywhere.

Solution 6.5.1. First we observe that the Poisson's equation is a second-order differential equation. The nonhomogeneous term, although discontinuous, is finite everywhere, therefore the solution and its first derivative must be continuous. Furthermore, we note that this problem is spherically symmetric, the solution will not have angular dependence. Thus for $r < r_0$,

$$\nabla^2 V(\mathbf{r}) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2 \frac{\mathrm{d}}{\mathrm{d}r} V(r)\right] = -\varrho_0,$$

which can be written as

$$r^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} V(r) + 2r \frac{\mathrm{d}}{\mathrm{d}r} V(r) = -\varrho_0 r^2.$$

This equation is an Euler–Cauchy equation. With a substitution of $r = \exp(t)$, it can be changed into a nonhomogeneous equation of constant coefficients,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} V(t) + \frac{\mathrm{d}}{\mathrm{d}t} V(t) = -\varrho_0 \mathrm{e}^{2t}.$$

The solution of this equation is

$$V(t) = c_1 + c_2 \mathrm{e}^{-t} - \frac{1}{6}\varrho_0 \mathrm{e}^{2t}.$$

Therefore

$$V(r) = c_1 + c_2 \frac{1}{r} - \frac{1}{6}\varrho_0 r^2.$$

Because of the requirement that $V(r)$ must be finite at $r = 0$, c_2 must be set to zero. Thus, for $r < r_0$,

$$V(r) = c_1 - \frac{1}{6}\varrho_0 r^2 = V_{\mathrm{in}}(r).$$

For $r > r_0$, the part of the solution of the Laplace equation that has no angular dependence (depends on $P_0(\cos\theta) = 1$) is

$$V(r) = a_0 + b_0 \frac{1}{r}.$$

The coefficient a_0 must be set to zero to make $V(r) \to 0$ as $r \to \infty$. Therefore, for $r > r_0$,

$$V(r) = b_0 \frac{1}{r} = V_{\text{out}}(r).$$

At $r = r_0$,

$$V_{\text{in}}(r_0) = V_{\text{out}}(r_0), \qquad \left.\frac{\mathrm{d}V_{\text{in}}(r)}{\mathrm{d}r}\right|_{r=r_0} = \left.\frac{\mathrm{d}V_{\text{out}}(r)}{\mathrm{d}r}\right|_{r=r_0}.$$

Therefore

$$c_1 - \frac{1}{6}\varrho_0 r_0^2 = b_0 \frac{1}{r_0}, \quad -\frac{1}{3}\varrho_0 r_0 = -b_0 \frac{1}{r_0^2}.$$

Thus

$$V(r) = \begin{cases} \frac{1}{r}\left(\frac{1}{3}\varrho_0 r_0^3\right) & \text{for} \quad r > r_0, \\ \frac{1}{2}\varrho_0\left(r_0^2 - \frac{1}{3}r^2\right) & \text{for} \quad r < r_0. \end{cases}$$

Not surprisingly, these results are in agreement with what we obtain with Gauss' law in elementary physics. For example, the potential produced by a point charge Q at the origin is

$$V(r) = \frac{1}{4\pi}\frac{Q}{r}.$$

Our solution for $r > r_0$ can be written as

$$V(r) = \frac{1}{4\pi}\frac{1}{r}\left(\varrho_0 \frac{4\pi}{3} r_0^3\right).$$

Since $4\pi r_0^3/3$ is the volume of the sphere, we can see that this result is the same as if all the charges are put at the center.

6.5.1 Poisson's Equation and Green's Function

The Green's function approach to solving ordinary differential equations can be extended to partial differential equations.

The Green's function $G(\mathbf{r}, \mathbf{r}')$ for Poisson's equation is defined as

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \tag{6.28}$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is a three-dimensional delta function. In electrostatics, the Green's function is the potential at $\mathbf{r}$ due to a point charge of unit strength at $\mathbf{r}'$. In terms of Green's function, the solution of the Poisson's equation

$$\nabla^2 u(\mathbf{r}) = \varrho(\mathbf{r}) \tag{6.29}$$

can be expressed as

$$u(\mathbf{r}) = \iiint_V G(\mathbf{r}, \mathbf{r}')\varrho(\mathbf{r}')\mathrm{d}^3\tau',$$

where the volume integral is over the entire space. It can be easily shown that $u(\mathbf{r})$ so expressed satisfies the Poisson's equation

$$\begin{aligned}\nabla^2 u(\mathbf{r}) &= \nabla^2 \iiint_V G(\mathbf{r}, \mathbf{r}')\varrho(\mathbf{r}')\mathrm{d}^3\tau' = \iiint_V \nabla^2 G(\mathbf{r}, \mathbf{r}')\varrho(\mathbf{r}')\mathrm{d}^3\tau' \\ &= \iiint_V \delta(\mathbf{r} - \mathbf{r}')\varrho(\mathbf{r}')\mathrm{d}^3\tau' = \varrho(\mathbf{r}).\end{aligned}$$

We can bring ∇^2 inside the integral because ∇^2 is operating on $\mathbf{r}$ and the integration is respect to $\mathbf{r}'$. Physically, (6.29) is a statement of principle of superposition, the total potential is the sum of individual potentials created by all the charges everywhere.

Now let us find $G(\mathbf{r}, \mathbf{r}')$ satisfying (6.28) and the asymptotical (boundary) condition $G(\mathbf{r}, \mathbf{r}') \to 0$, as $|\mathbf{r} - \mathbf{r}'| \to \infty$. Since the problem is spherically symmetric about $\mathbf{r}'$, let us consider a sphere with center at $\mathbf{r}'$. Integrating (6.28) over this sphere, by definition of the delta function we obtain

$$\iiint_V \nabla^2 G(\mathbf{r}, \mathbf{r}')\,\mathrm{d}^3\tau = \iiint_V \delta(\mathbf{r} - \mathbf{r}')\,\mathrm{d}^3\tau = 1. \tag{6.30}$$

However,

$$\iiint_V \nabla^2 G(\mathbf{r}, \mathbf{r}')\mathrm{d}^3\tau = \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} G(\mathbf{r}, \mathbf{r}')\mathrm{d}^3\tau$$

and according to divergence theorem

$$\iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} G(\mathbf{r}, \mathbf{r}')\mathrm{d}^3\tau = \iint_S \boldsymbol{\nabla} G(\mathbf{r}, \mathbf{r}') \cdot \mathbf{n}\,\mathrm{d}S,$$

where S is the surface of the sphere and $\mathbf{n}$ is the normal to the surface. Note that in the surface of the integral, $\mathbf{r}$ is at the surface of the sphere and $\mathbf{r}'$ is at the center of the sphere. Because of the symmetry around $\mathbf{r}'$, we expect that G has the same value everywhere on the surface of the sphere, i.e.,

$$G(\mathbf{r}, \mathbf{r}') = G(|\mathbf{r} - \mathbf{r}'|) = G(r),$$

where r is radius of the sphere. Thus

$$\boldsymbol{\nabla} G(\mathbf{r}, \mathbf{r}') \cdot \mathbf{n} = \boldsymbol{\nabla} G(r) \cdot \hat{\mathbf{r}} = \frac{\partial}{\partial r} G(r)$$

and

$$\iint_S \boldsymbol{\nabla} G(\mathbf{r}, \mathbf{r}') \cdot \mathbf{n}\,\mathrm{d}S = 4\pi r^2 \frac{\partial}{\partial r} G(r).$$

Comparing with (6.30), we have

$$4\pi r^2 \frac{\partial}{\partial r} G(r) = 1 \quad \text{or} \quad \frac{\partial}{\partial r} G(r) = \frac{1}{4\pi r^2}.$$

Integrating this expression, we obtain

$$G(r) = -\frac{1}{4\pi r} + C.$$

Since we require $G(r) \to 0$ as $r \to \infty$, the constant C must be zero. Consequently the Green's function is given by

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}.$$

This Green's function is sometimes called fundamental solution to distinguish it from other Green's functions which satisfy additional boundary conditions. Before we discuss them, we will use this Green's function to solve the problem of the previous example.

Example 6.5.2. Solve the problem of the previous example with the Green's function method.

Solution 6.5.2. Let $u(\mathbf{r})$ be the solution of the Poisson's equation $\nabla^2 u(\mathbf{r}) = \varrho(\mathbf{r})$, so

$$u(\mathbf{r}) = \iiint_V G(\mathbf{r}, \mathbf{r}')\varrho(\mathbf{r}')\mathrm{d}^3\tau'. \tag{6.31}$$

Since

$$\varrho(\mathbf{r}') = \begin{cases} -\varrho_0 & r' < r_0 \\ 0 & r' > r_0 \end{cases}$$

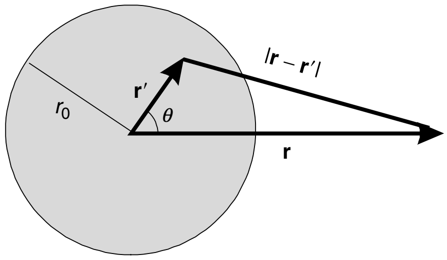
and

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|},$$

the problem is clearly spherically symmetric. Therefore $u(\mathbf{r})$ is a function of r only

$$u(r) = 2\pi \int_0^{r_0} \int_0^{\pi} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \varrho_0 r'^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}r',$$

where r, r', and θ are defined in the following figure:



Using the law of cosine and the generating function of Legendre polynomials, we have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{(r^2-2rr'\cos\theta+r'^2)^{1/2}}$$

$$=\begin{cases}\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta) & \text{for} \quad r>r' \\ \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\theta) & \text{for} \quad r<r'\end{cases}.$$

If $r>r_0$, then clearly $r>r'$, and $u(r)$ can be expressed as

$$u(r)=\frac{1}{2}\varrho_0\int_0^{r_0}\int_0^{\pi}\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)r'^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r'.$$

With $P_0(\cos\theta)=1$ and the orthogonality of the Legendre polynomials, we can carry out the θ integration

$$\int_0^{\pi}P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta=\int_0^{\pi}P_0(\cos\theta)P_l(\cos\theta)\sin\theta\,\mathrm{d}\theta=\frac{2}{2l+1}\delta_{l0}.$$

Therefore for $r>r_0$,

$$u(r)=\frac{1}{2}\varrho_0\int_0^{r_0}\frac{1}{r}2r'^2\mathrm{d}r'=\frac{1}{r}\varrho_0\frac{1}{3}r_0^3.$$

For $r<r_0$, the r' integration can be divided into two parts, first from 0 to r, then from r to r_0. That is

$$u(r)=u_1(r)+u_2(r),$$

where

$$u_1(r)=2\pi\int_0^{r}\int_0^{\pi}\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\varrho_0 r'^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r',$$

$$u_2(r)=2\pi\int_r^{r_0}\int_0^{\pi}\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\varrho_0 r'^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r'.$$

For $u_1(r),\ r > r'$ so

$$u_1(r) = \frac{1}{2}\varrho_0 \int_0^r \frac{1}{r} 2r'^2 \mathrm{d}r' = \frac{1}{r}\varrho_0 \frac{1}{3} r^3 = \frac{1}{3}\varrho_0 r^2.$$

For $u_2(r),\ r < r'$

$$u_2(r) = \frac{1}{2}\varrho_0 \int_r^{r_0} \int_0^{\pi} \frac{1}{r'} \sum_{l=0}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\theta) r'^2 \sin\theta \,\mathrm{d}\theta \mathrm{d}r'$$

$$= \varrho_0 \int_r^{r_0} r' \mathrm{d}r' = \frac{1}{2}\varrho_0 (r_0^2 - r^2).$$

Thus for $r < r_0$,

$$u(r) = \frac{1}{3}\varrho_0 r^2 + \frac{1}{2}\varrho_0 (r_0^2 - r^2) = \frac{1}{2}\varrho_0 r_0^2 - \frac{1}{6}\varrho_0 r^2.$$

6.5.2 Green's Function for Boundary Value Problems

The Green's function derived in the last section (fundamental solution) enables us to obtain the solution $u(\mathbf{r})$ of Poisson's equation which goes to zero at infinity. Often the solution of the Poisson's equation is required to satisfy additional or other boundary conditions. For example, suppose we have an electric charge distribution near a grounded conducting sphere. The electrostatic potential, in addition to satisfying the Poisson's equation, must vanish at the surface of the sphere. If we can find a Green's function that also satisfies this boundary condition, then we can still use the integral (6.31) to find the potential.

Thus, we want to find a $G(\mathbf{r}, \mathbf{r}')$ that satisfy the equation $\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$ and at the same time vanishes at certain boundaries. In general, it is difficult to find such a function directly. However, we note that if $F(\mathbf{r}, \mathbf{r}')$ satisfies the Laplace's equation $\nabla^2 F(\mathbf{r}, \mathbf{r}') = 0$, then

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}') \tag{6.32}$$

satisfies the equation that defines the Green's function

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \nabla^2 \left[-\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\right] + \nabla^2 F(\mathbf{r}, \mathbf{r}') = \nabla^2 \left[-\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}\right] = \delta(\mathbf{r} - \mathbf{r}').$$

It might be possible to adjust the constants in the solution of the Laplace's equation $F(\mathbf{r}, \mathbf{r}')$ in such a way that (6.32) is equal to zero at the boundaries. For boundaries of simple geometry, such as planes, spheres, circular cylinders, this can indeed be done. We will illustrate this procedure in the following example.

Example 6.5.3. Find the Green's function $G(\mathbf{r},\mathbf{r}')$ for solving Poisson's equation outside of a grounded sphere of radius R centered at the origin.

Solution 6.5.3. We try to find $G(\mathbf{r},\mathbf{r}')$ in the form of

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} + F(\mathbf{r},\mathbf{r}'),$$

where $F(\mathbf{r},\mathbf{r}')$ satisfies the Laplace's equation

$$\nabla^2 F(\mathbf{r},\mathbf{r}') = 0,$$

where ∇^2 is acting on $\mathbf{r}$ (not on $\mathbf{r}'$). With $\mathbf{r} = \mathbf{r}(r,\theta)$ and $\mathbf{r}'$ as the z-axis, $F(\mathbf{r},\mathbf{r}')$ is given by

$$F(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\left(a_l r^l + b_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta).$$

Since $F(\mathbf{r},\mathbf{r}')$ is required to go to zero as $r \to \infty$, all a_l must be set to zero. Therefore

$$F(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty} b_l \frac{1}{r^{l+1}} P_l(\cos\theta).$$

If $\mathbf{r}$ is a point on the surface of the sphere (or as a radial vector from the center to a point on the surface of the sphere, i.e., $|\mathbf{r}| = r = R$), then

$$F(\mathbf{r},\mathbf{r}')|_{r=R} = \sum_{l=0}^{\infty} b_l \frac{1}{R^{l+1}} P_l(\cos\theta),$$

where $\mathbf{r}$, $\mathbf{r}'$, and θ are shown in Fig. 6.11.

With this configuration,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{(R^2 - 2Rr'\cos\theta + r'^2)^{1/2}} = \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{R}{r'}\right)^l P_l(\cos\theta).$$

The condition $G(\mathbf{r},\mathbf{r}')|_{r=R} = 0$ requires

$$-\frac{1}{4\pi}\frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{R}{r'}\right)^l P_l(\cos\theta) + \sum_{l=0}^{\infty} b_l \frac{1}{R^{l+1}} P_l(\cos\theta) = 0.$$

Clearly

$$\frac{1}{4\pi}\frac{R^l}{r'^{l+1}} = \frac{b_l}{R^{l+1}}$$

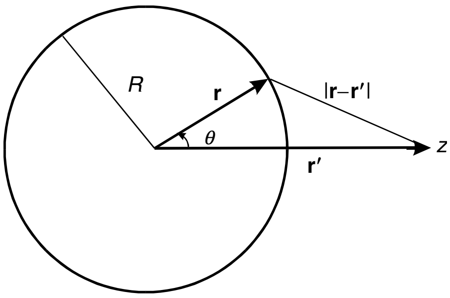


Fig. 6.11. The $\mathbf{r}, \mathbf{r}'$, and θ in the configuration of the Green's function. When $\mathbf{r}$ is a point on the surface of the sphere, then $|\mathbf{r}| = R$

or

$$b_l = \frac{R^{2l+1}}{4\pi r'^{l+1}}.$$

Therefore

$$F(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{R^{2l+1}}{r'^{l+1}} \frac{1}{r^{l+1}} P_l(\cos\theta).$$

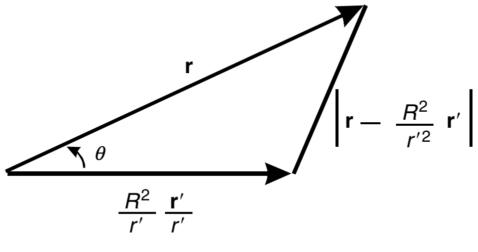
It is interesting to write the last expression in the form of

$$\sum_{l=0}^{\infty} \frac{R^{2l+1}}{r'^{l+1}} \frac{1}{r^{l+1}} P_l(\cos\theta) = \frac{R}{r'r} \sum_{l=0}^{\infty} \left(\frac{R^2}{r'r}\right)^l P_l(\cos\theta)$$

$$= \frac{R}{r'r} \frac{1}{\left[1 - 2\frac{R^2}{r'r}\cos\theta + \left(\frac{R^2}{r'r}\right)^2\right]^{1/2}} = \frac{R}{r'} \frac{1}{\left[r^2 - 2r\frac{R^2}{r'}\cos\theta + \left(\frac{R^2}{r'}\right)^2\right]^{1/2}}.$$

With the help of the following figure and the law of cosine, we see that

$$\left[r^2 - 2r\frac{R^2}{r'}\cos\theta + \left(\frac{R^2}{r'}\right)^2\right]^{1/2} = \left|\mathbf{r} - \frac{R^2}{r'^2}\mathbf{r}'\right|.$$



Therefore the Green's function in the presence of a grounded sphere is given by

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\frac{R/r'}{|\mathbf{r}-(R^2/r'^2)\mathbf{r}'|}.$$

This kind of Green's function is sometimes called Dirichlet Green's function. (When the values of the solution on the boundary are specified, it is called Dirichlet boundary condition.)

Example 6.5.4. Find the electrostatic potential outside of a grounded conducting sphere of radius R. The potential is produced by a point charge q located at a distance a from the center of the sphere, and $a > R$.

Solution 6.5.4. The potential is given by the solution of the Poisson's equation

$$\nabla^2 u(\mathbf{r}) = -\varrho(\mathbf{r})$$

with the boundary condition

$$u(\mathbf{r})|_{r=R} = 0.$$

Let the line joining the center of the sphere and the charge coincide with the z-axis and $\mathbf{k}$ be a unit vector in the z direction, as shown in Fig. 6.12. So the charge distribution can be expressed as

$$\varrho(\mathbf{r}) = q\delta(\mathbf{r}-a\mathbf{k}).$$

In terms of the Dirichlet Green's function, the potential is given by

$$\begin{aligned} u(\mathbf{r}) &= -\iiint \varrho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\mathrm{d}^3\tau' \\ &= \frac{1}{4\pi}\iiint q\delta(\mathbf{r}'-a\mathbf{k})\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-(R^2/r'^2)\mathbf{r}'|}\right]\mathrm{d}^3\tau'. \end{aligned}$$

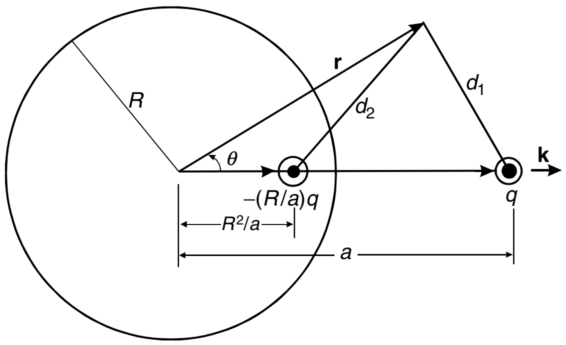


Fig. 6.12. Method of image. A point charge q is placed at a distance a from the center of a grounded sphere of radius R. The unit vector $\mathbf{k}$ is the direction of the line joining the center and the charge

The effect of the delta function is to replace $\mathbf{r}'$ by $a\mathbf{k}$ and r' by a in the Green's function. Thus

$$u(\mathbf{r}) = \frac{1}{4\pi}\left[\frac{q}{|\mathbf{r}-a\mathbf{k}|} - \frac{(R/a)q}{|\mathbf{r}-(R^2/a)\mathbf{k}|}\right] = \frac{1}{4\pi}\frac{q}{d_1} - \frac{1}{4\pi}\frac{(R/a)q}{d_2}$$
$$= \frac{1}{4\pi}\left[\frac{q}{[r^2-2ra\cos\theta+a^2]^{1/2}} - \frac{(R/a)q}{[r^2-2r(R^2/a)\cos\theta+(R^2/a)^2]^{1/2}}\right].$$

The first term is simply the Coulomb potential due to the point charge. The second term is caused by the presence of the sphere. It is interesting to note that if we replace the sphere by a charge $-(R/a)q$ placed at $(R^2/a)\mathbf{k}$, we would get the same potential for $r > R$. This is shown in Fig. 6.12. In fact, this is routinely done in electromagnetic theory. It is known as the "method of images."

Exercises

1. Find the steady-state distribution of temperature in a sector of a circular plate of radius 10 and angle $\pi/4$ if the temperature is maintained at 0° along the radii and at 100° along the curved edge.

   Ans. $u = \dfrac{400}{\pi}\sum_{n\ \text{odd}} \frac{1}{n}\left(\dfrac{\rho}{10}\right)^{4n}\sin 4n\varphi.$

2. Find the steady-state distribution of temperature in a circular annulus of inner radius 1 and outer radius 2 if the inner circle is held at 0° and the outer circle has half its circumference at 0° and half at 100°.

   Ans. $u = \dfrac{50\ln\rho}{\ln 2} + \dfrac{200}{\pi}\sum_{n\ \text{odd}} \dfrac{1}{n}\dfrac{\rho^n-\rho^{-n}}{(2^n-2^{-n})}\sin n\varphi.$

3. Find the solution the following boundary value problem inside the circular annulus

   $$\nabla^2 u(\rho,\varphi) = 0, \quad 1 \le \rho \le 2,$$
   $$u(1,\varphi) = \sin\varphi, \quad u(2,\varphi) = \sin\varphi.$$

   Ans. $u = \left(\frac{1}{3}\rho + \frac{2}{3}\dfrac{1}{\rho}\right)\sin\varphi.$

4. A particle of mass m is contained in a right circular cylindrical box of radius R and height H. The particle is described by a wavefunction satisfying the Schrödinger wave equation

   $$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,z) = E\psi(r,\theta,z)$$

and the condition that the wavefunction go to zero over the surface of the box. Find the lowest permitted energy.

Ans. $E_{\min} = \dfrac{\hbar^2}{2m}\left[\left(\dfrac{2.4048}{R}\right)^2 + \left(\dfrac{\pi}{H}\right)^2\right].$

5. A membrane is stretched over a fixed circular frame at $r = c$. Its displacement $z(r, \theta, t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \nabla^2 z$$

and the boundary condition

$$z(c, \theta, t) = 0.$$

If the initial displacement of the membrane is

$$z(r, \theta, 0) = f(r),$$

show that z has no θ dependence and

$$z(r, \theta, t) = \frac{2}{c^2} \sum_{j=1}^{\infty} \frac{J_0(\lambda_j r) \cos \lambda_j a t}{[J_1(\lambda_j c)]^2} \int_0^c r J_0(\lambda_j r) f(r) \mathrm{d}r,$$

where λ_j are the roots of $J_0(\lambda c) = 0$.

6. In previous problem, if

$$z(r, \theta, 0) = A J_0(\lambda_k r),$$

where λ_k is some root of $J_0(\lambda c) = 0$, show that

$$z(r, \theta, t) = A J_0(\lambda_k r) \cos(\lambda_k a t).$$

Observe that these displacements are periodic in t, thus the membrane gives a musical note.

7. Consider a solid circular cylinder of radius c and length L. Find the steady-state distribution of temperature inside the cylinder if the temperature at both the curved and top surfaces are kept a $0°$ and the bottom surface is specified by a given function $g(\rho)$.

Ans. $u(\rho, z) = \sum_{g=1} A_j J_0(k_{0j}\rho) \sinh k_{0j}(L - z)$,

$$A_j = \frac{2}{c^2 J_1^2(k_{0j}c) \sinh k_{0j} L} \int_0^c g(\rho') J_0(k_{0j}\rho')\rho' \mathrm{d}\rho'.$$

8. Starting with the heat conduction equation

$$\nabla^2 u = \frac{1}{a^2}\frac{\partial u}{\partial t},$$

derive the following formula for the temperature in an infinitely long right-angled cylindrical wedge bounded by a surface $r = c$ and the planes $\theta = 0$ and $\theta = \pi/2$, when these surfaces are kept at temperature zero and the initial temperature is $f(r, \theta)$

$$u(r, \theta, t) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} A_{nj} J_{2n}(\lambda_{nj} r) \sin 2n\theta \mathrm{e}^{-(\lambda_{nj} a)^2 t},$$

where λ_{nj} are the positive roots of $J_{2n}(\lambda c) = 0$ and A_{nj} are given by

$$\pi c^2 [J_{2n+1}(\lambda_{nj} c)]^2 A_{nj} = 8 \int_0^{\pi/2} \int_0^c \sin 2n\theta r J_{2n}(\lambda_{nj} r) f(r, \theta) \mathrm{d}r \mathrm{d}\theta.$$

9. Let $u(r, t)$ be the temperature in a thin circular plate whose edge, $r = 1$, is kept at temperature zero and whose initial temperature is 1. If there is surface heat transfer from the faces of the plate, the heat equation has the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - hu, \quad h > 0.$$

Show that

$$u(r, t) = 2\mathrm{e}^{-ht} \sum_{j=1}^{\infty} \frac{J_0(\lambda_j r)}{\lambda_j J_1(\lambda_j)} \mathrm{e}^{-\lambda_j^2 t},$$

where λ_j are the positive roots of $J_0(\lambda) = 0$.

10. *Shrunken fitting.* Over a long solid cylinder $r \leq 1$, initially at uniform temperature A, is tightly fitted a long hollow cylinder $1 \leq r \leq 2$ of the same material at temperature B. The outer surface $r = 2$ is then kept at temperature B. Show that the temperature in the cylinder of radius 2 so formed is

$$u(r, t) = B + \frac{A - B}{2} \sum_{j=1}^{\infty} \frac{J_1(\lambda_j)}{\lambda_j [J_1(2\lambda_j)]^2} J_0(\lambda_j r) \exp(-\lambda_j^2 a^2 t),$$

where λ_j are the positive roots of $J_0(2\lambda) = 0$ and a^2 is the constant in the heat conduction equation.

Hint: Let $u(r, t) = U(r, t) + B$, find the differential equation, boundary conditions and initial condition of $U(r, t)$. Then solve for $U(r, t)$.

11. A two-dimensional region having the shape of a quarter of a circle of radius b is initially at the uniform temperature of 100°. At $t = 0$, the temperature around the entire boundary is reduced to 0° and maintained thereafter at that value. Find the temperature at any point of the region at any subsequent time.

    Ans. $u(r,\theta,t) = \sum_m \sum_n A_{nm} J_{2n}(\lambda_{nm} r) \sin 2n\theta \exp(-\lambda_{nm}^2 a^2 t)$, where λ_{nm} are the roots of $J_{2n}(\lambda_{nm} b) = 0$, and A_{nm} is given by

    $$\frac{b^2}{2} J_{2n+1}(\lambda_{nm} b) A_{nm} = \frac{400}{n\pi} \int_0^b r J_{2n}(\lambda_{nm} r) \mathrm{d}r.$$

12. A very long cylinder of radius c is split lengthwise into two halves. The surfaces of the halves of the cylinder are kept at temperature of $+u_0$ and $-u_0$, respectively. Find the steady-state temperature everywhere inside the cylinder.

    Ans. $u(r,\theta) = \dfrac{4u_0}{\pi} \sum_{n=1} \dfrac{1}{2n-1} \left(\dfrac{r}{c}\right)^{2n-1} \sin(2n-1)\theta.$

13. A two-dimensional region having the shape of a semicircle of radius b is initially at the uniform temperature of 100°. At $t = 0$, the temperature around the bounding diameter is reduced to 0° and maintained thereafter at that temperature. The curved boundary is maintained at 100°. Find the temperature at any point of the region at any subsequent time.

    Ans. $u(r,\theta,t) = \frac{400}{\pi} \sum_{n(\text{odd})} \frac{1}{n} \left(\frac{r}{b}\right)^n \sin n\theta +$ $\sum_{n(\text{odd})} \sum_m B_{nm} J_n(\lambda_{nm} r) \sin n\theta \exp(-\lambda_{nm}^2 a^2 t)$, where B_{nm} is given by $B_{nm} = \dfrac{2}{b^2 J_{nm}^2(\lambda_{n+1} b)} \dfrac{400}{n\pi} \displaystyle\int_0^b \left[1 - \left(\frac{r}{b}\right)^n\right] J_n(\lambda_{nm} r) r \, \mathrm{d}r$, and λ_{nm} are the roots of $J_n(\lambda b) = 0$.

    Hint: Let $u(r,\theta,t) = V(r,\theta) + U(r,\theta,t)$, and choose the boundary conditions for $V(r,\theta)$ to be $V(r,0) = V(r,\pi) = 0$ and $V(b,\theta) = 100$. Find the boundary and initial conditions for $U(r,\theta,t)$. Then solve for V and U.

14. Find the steady-state temperature at any point in a spherical shell of inner radius b_1 and outer radius b_2 if the temperature distributions $u(b_1,\theta) = f_1(\theta)$ and $u(b_2,\theta) = f_2(\theta)$ are maintained over the inner and outer surfaces, respectively.

    Ans. $u(r,\theta) = \sum_{n=1} \left(A_n r^n + B_n \dfrac{1}{r^{n+1}}\right) P_n(\cos\theta)$ with A_n and B_n given by

    $$A_n b_1^n + B_n \frac{1}{b_1^{n+1}} = \frac{2n+1}{2} \int_0^\pi f_1(\theta) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta,$$

    $$A_n b_2^n + B_n \frac{1}{b_2^{n+1}} = \frac{2n+1}{2} \int_0^\pi f_2(\theta) P_n(\cos\theta) \sin\theta \, \mathrm{d}\theta.$$

15. The temperature distribution $u(r,\theta) = f(\theta)$ is maintained over the curved surface of a hemisphere of radius b. The plane boundary of the hemisphere is kept at the temperature zero. Find the steady-state temperature at any point in the hemisphere.

    Ans. $u(r,\theta) = \sum_{n \text{ odd}} \dfrac{2n+1}{b^n} r^n P_n(\cos\theta) \displaystyle\int_0^{\pi/2} f(\theta')P_n(\cos\theta')\sin\theta' \mathrm{d}\theta'$.

16. If V satisfies the Laplace's equation $\nabla^2 V = 0$ throughout the domains $r < c$ and $r > c$, and if $V \to 0$ as $r \to \infty$, and $V = 1$ on the spherical surface $r = c$, show that $V = 1$ when $r \leq c$ and $V = c/r$ when $r \geq c$.

17. Write a formula for the steady temperature $u(r,\theta)$ in a solid sphere $r \leq 1$ if, for all ϕ, $u(1,\theta) = 1$ when $0 < \theta < \frac{1}{2}\pi$ and $u(1,\theta) = 0$ when $\frac{1}{2}\pi < \theta < \pi$.
    Ans. $u(r,\theta) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}[P_{2n}(0) - P_{2n+2}(0)]r^{2n+1}P_{2n+1}(\cos\theta)$.

18. Let $u(r,\theta)$ denote steady temperatures in a hollow sphere $a \leq r \leq b$ when $u(a,\theta) = f(\cos\theta)$ and $u(b,\theta) = 0,\ 0 < \theta < \pi$. Derive the formula

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \frac{b^{2n+1} - r^{2n+1}}{b^{2n+1} - a^{2n+1}} \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta),$$
$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(\cos\theta)P_n(\cos\theta)\mathrm{d}\cos\theta.$$

19. Find the electrostatic potential u, which satisfies the Poisson's equation

$$\nabla^2 u(\mathbf{r}) = -\varrho(\mathbf{r}),$$

due to the charge distribution

$$\varrho(\mathbf{r}) = \begin{cases} Ar\cos\theta & \text{for } 0 \leq r < R, \\ 0 & \text{for } \quad r > R. \end{cases}$$

Ans. $u(r,\theta) = \begin{cases} \dfrac{A}{15}\dfrac{R^5}{r^2}\cos\theta & \text{for } r > R, \\ A\left(\dfrac{1}{6}R^2 r - \dfrac{1}{10}r^3\right)\cos\theta & \text{for } r < R. \end{cases}$

20. If electric charges are distributed over the surface of a sphere of radius R with surface charge densities

$$\varrho(\mathbf{r}) = C\cos\theta,$$

where θ is the angle between $\mathbf{r}$ and the z-axis, and C is a constant. Let z be a point on the z-axis and $z > R$. Use the Green's function to show that the electrostatic potential u is given by

$$u(z) = \frac{1}{4\pi}\int_0^{\pi} \frac{2\pi\varrho(\theta')R^2\sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}\mathrm{d}\theta'.$$

Evaluate this integral. Find the potential $u(r, \theta)$ for $r > R$.

Hint: Use the generating function of the Legendre polynomial, and note that $P_1(\cos\theta) = \cos\theta$.

Ans. $u(z) = \dfrac{1}{z^2}\left(\dfrac{1}{3}R^3 C\right), \quad u(r,\theta) = \dfrac{1}{r^2}\left(\dfrac{1}{3}R^3 C\right)\cos\theta.$