7

Calculus of Variation

A basic problem in calculus is to find the value of x that maximizes or minimizes a given function $y = f(x)$. The calculus of variation extends this problem to finding a function that maximizes or minimizes a definite integral.

Consider the integral

$$I = \int_{x_1}^{x_2} F(y, y', x)\mathrm{d}x, \tag{7.1}$$

where F depends not only on x, but also on y which is a function of x, and on y', the derivative of y with respect to x. The form of F is fixed by the physics of the problem. The only thing that can be changed in the attempt to make I larger or smaller is the form of the function $y(x)$. In this sense, the integral is a function of y. The common terminology is that I is a functional of the curve $y(x)$.

The calculus of variation provides a method for finding the function $y(x)$ that makes the integral stationary, i.e., the function that makes the value of the integral a local maximum or minimum.

Calculus of variation is one of the oldest problems in mathematical physics, developed soon after the invention of calculus. At first it was used to solve mathematically interesting problems, such as finding the shape of a hanging rope. Later it was found that many principles in classical physics, ranging from optics to mechanics, can be stated in the form that certain integrals have stationary values. In modern physics, calculus of variation was found to be equally useful in finding the eigenvalues and eigenfunctions of quantum systems.

In this chapter, y' is used to denote $\mathrm{d}y/\mathrm{d}x$, unless stated otherwise. We will also assume that all functions we need to deal with are sufficiently smooth and differentiable.

7.1 The Euler–Lagrange Equation

7.1.1 Stationary Value of a Functional

If a given form of the function $y = y(x)$ gives the integral in (7.1) a minimum value, any neighboring function must give the integral a value equal to or greater than the minimum. This is illustrated in Fig. 7.1. The solid line is the curve $y = y(x)$, along which the integral is a minimum, the broken lines represent small variations of this path. This family of curves is conveniently represented by

$$Y(x) = y(x) + \alpha\eta(x), \tag{7.2}$$

where α is a small parameter, and $\eta(x)$ is an arbitrary function of x that is bounded, continuous, and has a continuous first derivative. If the two end points are fixed as shown in Fig. 7.1, then $\eta(x)$ has to satisfy the boundary conditions $\eta(x_1) = \eta(x_2) = 0$. Replacing y with Y in (7.1), we have

$$I(\alpha) = \int_{x_1}^{x_2} F(Y, Y', x)\mathrm{d}x. \tag{7.3}$$

Now we have the values of the integral along a family of curves passing through the two end points, each of the curves is labeled by the variable α. The curve which makes I stationary has the label $\alpha = 0$.

A necessary, but not sufficient, condition for minimum is the vanishing of the first derivative. Thus we require

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0, \quad \text{for all } \eta(x). \tag{7.4}$$

Since α does not depend on x, differentiation can be carried out under the integral sign

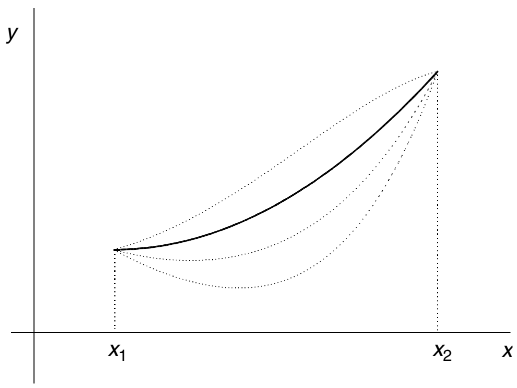


Fig. 7.1. The *solid line* is the curve $y(x)$ along which the integral is stationary. The *broken lines* are curves $y(x) + \alpha\eta(x)$ representing small variations from the solid path. They all pass the two end points

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\alpha} + \frac{\partial F}{\partial Y'}\frac{\mathrm{d}Y'}{\mathrm{d}\alpha} \right] \mathrm{d}x. \tag{7.5}$$

It is clear from (7.2) that

$$Y'(x) = y'(x) + \alpha\eta'(x),$$

$$\frac{\mathrm{d}Y}{\mathrm{d}\alpha} = \eta(x), \quad \frac{\mathrm{d}Y'}{\mathrm{d}\alpha} = \eta'(x).$$

Setting $\alpha = 0$ is equivalent to setting $Y(x) = y(x),\ Y'(x) = y'(x)$. Therefore after α is set to zero, (7.5) becomes

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x) \right] \mathrm{d}x = 0. \tag{7.6}$$

The second term can be integrated by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'(x)\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{\mathrm{d}\eta}{\mathrm{d}x}\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\mathrm{d}\eta$$

$$= \left.\frac{\partial F}{\partial y'}\eta(x)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\mathrm{d}x.$$

The integrated term is zero because $\eta(x_2) = \eta(x_1) = 0$. Therefore we have

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) \right] \eta(x)\mathrm{d}x = 0.$$

Since $\eta(x)$ is an arbitrary function, our intuition tells us that

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{7.7}$$

This equation was derived by Euler in 1744. It is known as Euler–Lagrange equation, because it is also the basis for Lagrange's formulation of classical mechanics.

If we expand the total derivative with respect to x,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\frac{\mathrm{d}y'}{\mathrm{d}x}$$

$$= \frac{\partial^2 F}{\partial x \partial y'} + \frac{\partial^2 F}{\partial y \partial y'}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial^2 F}{\partial y'^2}\frac{\mathrm{d}y'}{\mathrm{d}x},$$

the Euler–Lagrange equation becomes

$$\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - \frac{\partial^2 F}{\partial y \partial y'}y' - \frac{\partial^2 F}{\partial y'^2}y'' = 0$$

a second-order differential equation. Since the form of F is given, this equation can be solved to give the desired extremum function $y(x)$.

The condition of (7.4) is only a necessary condition for a minimum, the solution $y(x)$ could also produce a maximum or even a point of inflection of the function $I(\alpha)$ at $\alpha = 0$. To mathematically decide the nature of the extremum, one has to investigate the sign of higher derivatives of $I(\alpha)$. While this can be done, but it is rather complicated. Fortunately, for most applications, Euler–Lagrange equation by itself is enough to give a complete solution of the problem, because the existence and the nature of an extremum are often clear from the physical or geometrical meaning of the problem.

7.1.2 Fundamental Theorem of Variational Calculus

The Euler–Lagrange equation is the center piece of calculus of variation. One can establish it with mathematical rigor by the following theorem, known as the fundamental theorem of variational calculus.

Theorem 7.1.1. *If $f(x)$ is a continuous function on the interval (x_1, x_2), and if*

$$\int_{x_1}^{x_2} f(x)\eta(x)\mathrm{d}x = 0$$

for every continuously differentiable function $\eta(x)$ that satisfies the boundary conditions $\eta(x_1) = \eta(x_2) = 0$, then $f(x) = 0$ in the interval (x_1, x_2).

Proof. Let us assume that at some point ξ in the interval (x_1, x_2), $f(\xi) \neq 0$. Assume $f(\xi) > 0$. Since $f(x)$ is continuous, there must be a region around ξ, within which $f(x) > 0$. This region is a subinterval in (x_1, x_2). This means that we can find two numbers a and b in (x_1, x_2) such that inside $a < x < b$, $f(x) > 0$. Now it is readily verified that the function $\eta(x)$ represented by

$$\eta(x) = \begin{cases} 0 & x_1 \le x \le a \\ (x-a)^2(x-b)^2 & a \le x \le b \\ 0 & b \le x \le x_2 \end{cases}$$

is continuously differentiable on (x_1, x_2) and satisfies the boundary conditions $\eta(x_1) = \eta(x_2) = 0$. For this particular $\eta(x)$, we have

$$\int_{x_1}^{x_2} f(x)\eta(x)dx = \int_a^b f(x)(x-a)^2(x-b)^2\mathrm{d}x > 0$$

which contradicts the given condition. This eliminates the possibility that $f(\xi) > 0$ at some ξ inside (x_1, x_2). Similar argument shows that it is also not possible for $f(\xi) < 0$ at some ξ inside (x_1, x_2). Thus the theorem is established. □

This shows that for I to be stationary, F must satisfy the Euler–Lagrange equation. Since F is a given function, the Euler–Lagrange equation is a differential equation for $y(x)$.

Before we proceed further, let us illustrate how to use the Euler–Lagrange equation with a simple example.

Example 7.1.1. Shortest distance between two points in a plane.
Find the equation $y = y(x)$ of a curve joining two points (x_1, y_1) and (x_2, y_2) in the plane so that the distance between them measured along the curve is a minimum.

Solution 7.1.1. Let $\mathrm{d}s$ be the length of a small arc in a plane, then

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

or

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x.$$

The total length of any curve going between the two points is

$$I = \int_{x_1}^{x_2} \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\mathrm{d}x.$$

This equation is in the form of (7.1) with

$$F = \sqrt{1 + y'^2}.$$

The condition that the curve be the shortest path is that I be a minimum. Thus F must satisfy the Euler–Lagrange equation. Substituting it into (7.7) with

$$\frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{y'}{\sqrt{1 + y'^2}} = 0,$$

or

$$\frac{y'}{\sqrt{1 + y'^2}} = c,$$

where c is a integration constant. Squaring the equation and solving for y', we get

$$y' = \frac{c}{\sqrt{1 - c^2}}.$$

In fact, by inspection we can conclude directly that

$$y' = a,$$

where a is another constant. But this is clearly the equation of a straight line

$$y = ax + b,$$

where b is another constant of integration. The constants a and b are determined by the condition that the curve passes through the two end points, $(x_1, y_1),\ (x_2, y_2)$.

Strictly speaking, the straight line has only been proved to be an extremum path, but for this problem it is obviously also a minimum.

7.1.3 Variational Notation

In literature on calculus of variation, the symbol δ is often found as a differential operator. It is defined in the following way.

Expanding (7.3) as a Taylor series in α, we have

$$I(\alpha) = I(0) + \left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} \alpha + \mathrm{O}(\alpha^2).$$

The variation of I due to the variation of $y(x)$ expressed in (7.2) is simply

$$I(\alpha) - I(0) = \left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} \alpha + \mathrm{O}(\alpha^2).$$

The first-order variation of I is denoted as δI, which is just the first term on the right-hand side of this equation

$$\delta I = \left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} \alpha.$$

Using (7.6), we have

$$\delta I = \alpha \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x) \right] \mathrm{d}x. \tag{7.8}$$

With the family of curves defined in (7.2), the variations of $y(x)$ and $y'(x)$ are given by

$$\delta y(x) = Y(x) - y(x) = \alpha\eta(x),$$
$$\delta y'(x) = Y'(x) - y'(x) = \alpha\eta'(x).$$

Expanding F in a Taylor series

$$F(y+\delta y, y'+\delta y', x) = F(y, y', x) + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \cdots \cdot,$$

we see that the first-order variation of F is given by

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'.$$

These equations enable us to write δI as

$$\begin{aligned}
\delta I &= \delta \int_{x_1}^{x_2} F\,\mathrm{d}x = \int_{x_1}^{x_2} \delta F\,\mathrm{d}x \\
&= \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right]\mathrm{d}x \\
&= \alpha \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\right]\mathrm{d}x,
\end{aligned}$$

which is identical to (7.8).

Since α, as a parameter, cannot be identically equal to zero, the condition

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$$

can be replaced by the statement

$$\delta I = 0.$$

Although the δ symbol is not much used any more in mathematics, but it does enable us to operate on functionals in a formal way. Therefore it still appears frequently in applications.

7.1.4 Special Cases

In certain special cases, the Euler–Lagrange equation can be reduced to a first-order differential equation.

Integrand Does Not Depend on y Explicitly

In this case

$$\frac{\partial F}{\partial y} = 0.$$

The Euler–Lagrange equation is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

Therefore

$$\frac{\partial F}{\partial y'} = c,$$

where c is a constant. This is a first-order differential equation which does not depend on y. Solving for y', we obtain an equation of the form

$$y' = f(x, c)$$

from which $y(x)$ can be found.

Example 7.1.2. Find the curve $y = y(x)$ passing through $(1, 0)$ and $(2, 1)$ that renders the following functional stationary:

$$I = \int_1^2 \frac{1}{x}\sqrt{1 + y'^2}\mathrm{d}x.$$

Solution 7.1.2. The integrand does not contain y, therefore

$$\frac{\partial F}{\partial y'} = c,$$

or

$$\frac{\partial}{\partial y'}\left(\frac{1}{x}\sqrt{1 + y'^2}\right) = \frac{y'}{x\sqrt{1 + y'^2}} = c,$$

so that

$$y' = cx\sqrt{1 + y'^2}.$$

Squaring and solving for y', we get

$$y'^2 = c^2x^2(1 + y'^2),$$

or

$$y' = \frac{cx}{\sqrt{1 - c^2x^2}}$$

from which it follows that:

$$y = \int \frac{cx\,\mathrm{d}x}{\sqrt{1 - c^2x^2}} = -\frac{1}{c}\sqrt{1 - c^2x^2} + c',$$

where c' is another constant of integration. Hence

$$(y - c')^2 = \frac{1}{c^2} - x^2.$$

Since the curve passes through $(1, 0)$ and $(2, 1)$, so we have

$$c'^2 = \frac{1}{c^2} - 1, \quad (1 - c')^2 = \frac{1}{c^2} - 4.$$

It follows that:

$$c' = 2, \quad c = \frac{1}{\sqrt{5}}.$$

Thus the curve is given by

$$x^2 + (y-2)^2 = 5.$$

Integrand Does Not Depend on x Explicitly

In this case

$$\frac{\partial F}{\partial x} = 0$$

and F is a function of y and y'. Since $y = y(x)$, so F must still implicitly depend on x through y and y', i.e.,

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\ &= \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''.\end{aligned}$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right).$$

Subtracting one from the other, we have

$$\frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}y' - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right),$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]y'.$$

The quantity in the bracket of the right-hand side is equal to zero because of the Euler–Lagrange equation, therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0.$$

Thus

$$F - y'\frac{\partial F}{\partial y'} = c, \tag{7.9}$$

where c is constant. This is a first-order differential equation which can be solved for $y(x)$.

Example 7.1.3. Find $y(x)$ so that the following integral is stationary

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{1+y} \mathrm{d}x.$$

Solution 7.1.3. The integrand does not depend on x, hence

$$\frac{\sqrt{1+y'^2}}{1+y} - y' \frac{\partial}{\partial y'} \frac{\sqrt{1+y'^2}}{1+y} = c$$

or

$$\frac{\sqrt{1+y'^2}}{1+y} - \frac{y'^2}{(1+y)\sqrt{1+y'^2}} = \frac{1}{(1+y)\sqrt{1+y'^2}} = c.$$

Thus

$$(1+y)^2(1+y'^2) = \frac{1}{c^2}.$$

It follows that:

$$y'^2 = \frac{1}{c^2(1+y)^2} - 1 = \frac{1-c^2(1+y)^2}{c^2(1+y)^2}$$

and

$$y' = \frac{\sqrt{1-c^2(1+y)^2}}{c(1+y)}.$$

Thus

$$\frac{c(1+y)}{\sqrt{1-c^2(1+y)^2}} \mathrm{d}y = \mathrm{d}x.$$

Integrating both sides, we get

$$-\frac{1}{c}\sqrt{1-c^2(1+y)^2} = x + c'$$

or

$$1 - c^2(1+y)^2 = c^2(x+c')^2.$$

So the solution is given by the circle

$$(x+c')^2 + (1+y)^2 = \frac{1}{c^2},$$

where c and c' are two constants.

7.2 Constrained Variation

Very often we want to find the curve $y = y(x)$ that not only renders the integral

$$I = \int_{x_1}^{x_2} F(y, y', x)\mathrm{d}x \tag{7.10}$$

an extremum, but also makes the second integral

$$J = \int_{x_1}^{x_2} G(y, y', x)\mathrm{d}x \tag{7.11}$$

equal to a prescribed value. The curve is required to pass through the two end points (x_1, y_1) and (x_2, y_2), and the given functions F and G are supposed to be twice differentiable.

In essence we follow the same procedure as before by letting $y(x)$ denote the actual extremizing function and introducing a family of "neighboring" functions $Y(x)$ with respect to which we carry out the extremization. We cannot, however, express $Y(x)$ as functions depending on a single parameter as shown in (7.2), because the constant value of J would determine that parameter, which would, in turn, determine I. That would make it impossible to extremize I. For this reason we introduce a family of two parameters

$$Y(x) = y(x) + \alpha_1\eta_1(x) + \alpha_2\eta_2(x), \tag{7.12}$$

where $\eta_1(x)$ and $\eta_2(x)$ are arbitrary differentiable functions for which

$$\eta_1(x_1) = \eta_1(x_2) = 0, \tag{7.13}$$

$$\eta_2(x_1) = \eta_2(x_2) = 0. \tag{7.14}$$

These conditions ensure that every curve in the family passes through (x_1, y_1) and (x_2, y_2) for all values of α_1 and α_2.

We replace $y(x)$ by $Y(x)$ in (7.10) and (7.11) so as to form

$$I(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} F(Y, Y', x)\mathrm{d}x,$$

$$J(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} G(Y, Y', x)\mathrm{d}x.$$

Clearly the parameters α_1 and α_2 are not independent, because J is to be maintained at a constant value.

We proceed by forming a combination

$$K(\alpha_1, \alpha_2) = I(\alpha_1, \alpha_2) + \lambda J(\alpha_1, \alpha_2) = \int_{x_1}^{x_2} H(Y, Y', x)\mathrm{d}x,$$

where

$$H = F + \lambda G.$$

The yet undetermined constant λ is known as the Lagrange multiplier. Now if I is stationary, and J is constant, then K must also be stationary. The conditions for K to be stationary are

$$\frac{\partial K}{\partial \alpha_1} = 0, \quad \frac{\partial K}{\partial \alpha_2} = 0,$$

where the partial derivatives are to be evaluated at $\alpha_1 = 0$ and $\alpha_2 = 0$.

Calculating the two partial derivatives, we have

$$\frac{\partial K}{\partial \alpha_i} = \int_{x_1}^{x_2} \left[\frac{\partial H}{\partial Y} \frac{\partial Y}{\partial \alpha_i} + \frac{\partial H}{\partial Y'} \frac{\partial Y'}{\partial \alpha_i} \right] \mathrm{d}x, \quad i = 1, 2.$$

It is clear from (7.12) that

$$\frac{\partial Y}{\partial \alpha_i} = \eta_i(x), \quad \frac{\partial Y'}{\partial \alpha_i} = \eta_i'(x).$$

Thus

$$\frac{\partial K}{\partial \alpha_i} = \int_{x_1}^{x_2} \left[\frac{\partial H}{\partial Y} \eta_i + \frac{\partial H}{\partial Y'} \eta_i' \right] \mathrm{d}x, \quad i = 1, 2.$$

Integrating by parts the second term of the integrand, we get

$$\frac{\partial K}{\partial \alpha_i} = \eta_i(x) \frac{\partial H}{\partial Y'} \bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial H}{\partial Y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial H}{\partial Y'} \right) \right] \eta_i \mathrm{d}x, \quad i = 1, 2.$$

The integrated term can be dropped because of the boundary conditions (7.13) and (7.14). Now we set $\alpha_1 = 0$ and $\alpha_2 = 0$, so Y and Y' are replaced by y and y'. For the two partial derivatives of K to vanish, we must have

$$\int_{x_1}^{x_2} \left[\frac{\partial H}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial H}{\partial y'} \right) \right] \eta_i(x) \mathrm{d}x = 0, \quad i = 1, 2.$$

Because $\eta_1(x)$ and $\eta_2(x)$ are both arbitrary functions, the two relations embodied in this equation are essentially one. By the fundamental theorem of calculus of variation, we conclude that

$$\frac{\partial H}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial H}{\partial y'} \right) = 0.$$

This is the same Euler–Lagrange equation except with F replaced by H which is equal to $F + \lambda G$. It is a second-order differential equation that $y(x)$ must satisfy in order to keep J at a constant value and render I stationary.

Solution of this equation yields $y(x)$ that has three undetermined quantities: two constants of integration and the Lagrange multiplier λ. These quantities can be fixed by the boundary conditions at the two end points and by the requirement that J must be kept at its prescribed value.

Example 7.2.1. A curve of length L passes through x_1 and x_2 on the x-axis. Find the shape of the curve so that the area enclosed by the curve and the x-axis is the largest possible.

Solution 7.2.1. The area is given by

$$I = \int_{x_1}^{x_2} y \, \mathrm{d}x$$

and the length of the curve is

$$J = \int_{x_1}^{x_2} \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \mathrm{d}x.$$

We want to maximize I subject to the condition that J is equal to the constant L. So we solve the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial H}{\partial y'} - \frac{\partial H}{\partial y} = 0,$$

where

$$H = y + \lambda \sqrt{1 + y'^2}.$$

Since H does not explicitly depend on x, so

$$y' \frac{\partial H}{\partial y'} - H = c_1$$

or

$$\frac{\lambda y'^2}{\sqrt{1 + y'^2}} - y - \lambda \sqrt{1 + y'^2} = c_1.$$

This can be simplified to

$$(c_1 + y) \sqrt{1 + y'^2} = -\lambda,$$

or

$$1 + y'^2 = \frac{\lambda^2}{(c_1 + y)^2}$$

from which we have

$$y' = \frac{\sqrt{\lambda^2 - (c_1 + y)^2}}{c_1 + y}.$$

Thus

$$\frac{(c_1 + y) \mathrm{d}y}{\sqrt{\lambda^2 - (c_1 + y)^2}} = \mathrm{d}x.$$

Let $z = \lambda^2 - (c_1 + y)^2, \quad \mathrm{d}z = -2(c_1 + y)\mathrm{d}y$, so

$$\int \frac{(c_1 + y)\mathrm{d}y}{\sqrt{\lambda^2 - (c_1 + y)^2}} = -\int \frac{1}{2} z^{-1/2}\mathrm{d}z = -\int \mathrm{d}z^{1/2} = -\int \mathrm{d}\sqrt{\lambda^2 - (c_1 + y)^2}.$$

Hence

$$-\sqrt{\lambda^2 - (c_1 + y)^2} = x + c_2,$$

where c_1 and c_2 are two constants of integration. Squaring both sides, we get

$$(x + c_2)^2 + (c_1 + y)^2 = \lambda^2.$$

So the curve should be an arc of a circle passing through the two given points. The constants $c_1,\ c_2$, and λ may be fixed by requiring that the curve passes through the appropriate end points, and has the required length between these points.

7.3 Solutions to Some Famous Problems

7.3.1 The Brachistochrone Problem

Suppose that a bead is sliding down a wire without friction as shown in Fig. 7.2. We learnt in physics class that at the point (x, y), the kinetic energy of the bead is $\frac{1}{2}mv^2$ and the potential energy is $-mgy$, if we take $y = 0$ as our reference level. Because of the conservation of energy, the sum of these two energies must be equal to zero

$$\frac{1}{2}mv^2 - mhy = 0,$$

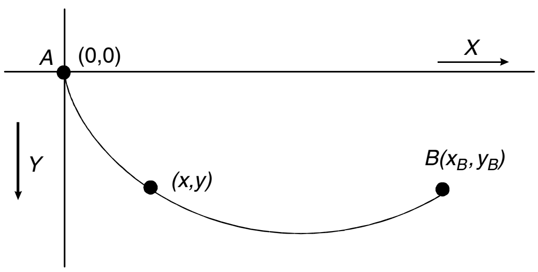


Fig. 7.2. The Brachistochrone problem. A bead sliding down on a wire from A to B without friction, the problem is to find the shape of the wire so that it takes the least amount of time

since initially at $(0,0)$, both kinetic and potential energies are equal to zero. Therefore the velocity at that point is

$$v = \sqrt{2gy}.$$

Thus the time it takes from A to B is

$$\begin{aligned} T &= \int \mathrm{d}t = \int \frac{\mathrm{d}s}{v} = \int \frac{\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}}{v} \\ &= \frac{1}{\sqrt{2g}} \int_0^{x_B} \frac{\sqrt{1+y'^2}}{\sqrt{y}} \mathrm{d}x. \end{aligned} \tag{7.15}$$

Now the question is: what should be the shape of the wire so that the time it takes from A to B is the shortest? This famous problem is known as the Brachistochrone problem (from the Greek words meaning shortest time).

In 1696, Johann Bernoulli proposed this problem and he addressed it "to the shrewdest mathematicians in all the world" and allowed 6 months for anyone to come up with a solution. This marks the beginning of general interest in the calculus of variation. Five correct solutions were submitted – by Newton, Leibniz, L'Hospital, himself, and his brother Jakob Bernoulli. They independently with different methods arrived at the same answer. The required shape is a cycloid, the curve traced by a point on the rim of a wheel as it rolls on a horizontal surface.

We can answer the question by minimizing the integral in (7.15). Since the integrand does not explicitly depend on x, so the Euler–Lagrange equation can be written in the form of (7.9)

$$F - y' \frac{\partial F}{\partial y'} = c$$

with

$$F = \frac{\sqrt{1+y'^2}}{\sqrt{y}}.$$

Thus

$$\frac{\sqrt{1+y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1+y'^2}} = c$$

or

$$\frac{1}{\sqrt{y}\sqrt{1+y'^2}} = c.$$

Squaring both sides, we have

$$y(1+y'^2) = \frac{1}{c^2}.$$

It follows that:

$$y' = \sqrt{\frac{1-c^2y}{c^2y}}.$$

With $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, this equation can be written as

$$\sqrt{\frac{c^2y}{1-c^2y}}\mathrm{d}y = \mathrm{d}x.$$

To find $y(x)$, we have to integrate both sides of this equation. We can carry out the integration by a change of variable. Let

$$c^2y = \frac{1}{2}(1-\cos\theta) \tag{7.16}$$

so

$$1 - c^2y = \frac{1}{2}(1+\cos\theta)$$

and

$$\mathrm{d}y = \frac{1}{2c^2}\sin\theta\,\mathrm{d}\theta.$$

Since

$$\cos\theta = \cos 2\frac{\theta}{2} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2},$$
$$\sin\theta = \sin 2\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},$$

we can write

$$1-\cos\theta = 2\sin^2\frac{\theta}{2},$$

$$1+\cos\theta = 2\cos^2\frac{\theta}{2},$$

$$\mathrm{d}y = \frac{1}{c^2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathrm{d}\theta.$$

Therefore

$$\sqrt{\frac{c^2y}{1-c^2y}}\mathrm{d}y = \frac{1}{c^2}\frac{\sin(\theta/2)}{\cos(\theta/2)}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathrm{d}\theta = \frac{1}{c^2}\sin^2\frac{\theta}{2}\mathrm{d}\theta$$
$$= \frac{1}{2c^2}(1-\cos\theta)\mathrm{d}\theta.$$

Hence

$$\int\sqrt{\frac{c^2y}{1-c^2y}}\mathrm{d}y = \int\mathrm{d}x$$

becomes

$$\frac{1}{2c^2}\int(1-\cos\theta)\mathrm{d}\theta = \int \mathrm{d}x,$$

or

$$\frac{1}{2c^2}(\theta-\sin\theta) = x + c'.$$

Now the curve goes through $(0,0)$, i.e., when $x = 0$, y must also be zero. But, by (7.16), $y = (1-\cos\theta)/2c^2$, so when $y = 0$, θ must be equal to zero. Therefore $x = \theta = 0$ must satisfy the above equation. Thus $c' = 0$. The remaining constant is fixed by the condition that the curve should pass the other end point.

As a result, the required curve is given by the parametric equations

$$x = \frac{1}{2c^2}(\theta-\sin\theta),$$
$$y = \frac{1}{2c^2}(1-\cos\theta).$$

These are the equations of a cycloid that can be seen as follows. Let a circle of radius r rolls along the x-axis as shown in Fig. 7.3. The origin is so chosen that the point P makes contact with x-axis at the origin. When the circle has revolved through an angle of θ radians, it will have rolled a distance $OC = r\theta$ from the origin as shown in Fig. 7.3. Hence the center of the circle will be at the point $(r\theta,\ r)$. It is clear from the figure that the x coordinate of P is

$$\begin{aligned} x &= r\theta + r\cos\phi = r\theta + r\cos\left(\frac{3\pi}{2}-\theta\right) \\ &= r\theta - r\sin\theta = r(\theta-\sin\theta) \end{aligned}$$

and the y coordinate of P is

$$\begin{aligned} y &= r + r\sin\phi = r + r\sin\left(\frac{3\pi}{2}-\theta\right) \\ &= r - r\cos\theta = r(1-\cos\theta). \end{aligned}$$

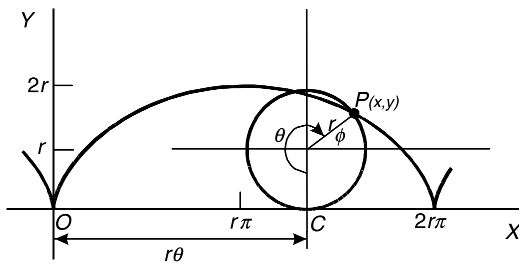


Fig. 7.3. Cycloid. The *curve* traced out by the point P on a *circle* while the *circle* is rolling on the x-axis

Therefore the solution of the Brachistochrone problem is a cycloid, since the parametric equations for the required curve are identical to the last two equations with $r = 1/2c^2$. Note that, since we have taken the downward direction of the y-axis as positive, the circle which generates the cycloid rolls along the under side of the x-axis.

The correct minimum path is shown in Fig. 7.2. It is somewhat surprising that going to the bottom of the curve and then back up to B actually takes less time than sliding on the straight line from A to B.

7.3.2 Isoperimetric Problems

The word isoperimetric means same perimeter. The most famous isoperimetric problem is to find the plane curve of given length which encloses the greatest possible area.

Let C be a closed plane curve as shown in Fig. 7.4. The area inside C can be found as follows. The shaded infinitesimal area is given by half of the product of its base and height

$$\mathrm{d}A = \frac{1}{2} r h = \frac{1}{2} r \, \mathrm{d}r \sin\theta = \frac{1}{2} \left| \mathbf{r} \times \mathrm{d}\mathbf{r} \right| .$$

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, so $\mathrm{d}\mathbf{r} = \mathrm{d}x\mathbf{i} + \mathrm{d}y\mathbf{j}$, and

$$\mathbf{r} \times \mathrm{d}\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ \mathrm{d}x & \mathrm{d}y & 0 \end{vmatrix} = (x \, \mathrm{d}y - y \, \mathrm{d}x)\mathbf{k}.$$

Thus

$$\mathrm{d}A = \frac{1}{2} \left| \mathbf{r} \times \mathrm{d}\mathbf{r} \right| = \frac{1}{2}(x \, \mathrm{d}y - y \, \mathrm{d}x).$$

It follows that the area enclosed by C is:

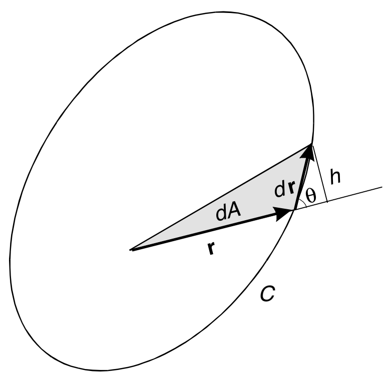


Fig. 7.4. Area inside a *closed curve* C. The *infinitesimal shaded area* is given by $\left| \mathbf{r} \times \mathrm{d}\mathbf{r} \right| / 2 = (x \, \mathrm{d}y - y \, \mathrm{d}x) / 2$

$$A = \oint_C \mathrm{d}A = \oint_C \frac{1}{2}(x\,\mathrm{d}y - y\,\mathrm{d}x) = \oint_C \frac{1}{2}(xy' - y)\mathrm{d}x.$$

The length of the curve is

$$L = \oint_C \mathrm{d}s = \oint_C (\mathrm{d}x^2 + \mathrm{d}y^2)^{1/2} = \oint_C (1 + y'^2)^{1/2}\mathrm{d}x.$$

Now we want to maximize A and keep L constant. According to the theory of constrained variation, we have to solve the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial H}{\partial y'} - \frac{\partial H}{\partial y} = 0$$

with

$$H = \frac{1}{2}(xy' - y) + \lambda(1 + y'^2)^{1/2}.$$

Since

$$\frac{\partial H}{\partial y'} = \frac{1}{2}x + \frac{\lambda y'}{(1 + y'^2)^{1/2}} \quad \text{and} \quad \frac{\partial H}{\partial y} = -\frac{1}{2},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}x + \frac{\lambda y'}{(1 + y'^2)^{1/2}}\right] + \frac{1}{2} = 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\lambda y'}{(1 + y'^2)^{1/2}}\right] = -1.$$

Integrating once

$$\frac{\lambda y'}{(1 + y'^2)^{1/2}} = -x + c_1.$$

Squaring

$$(\lambda y')^2 = (x - c_1)^2(1 + y'^2),$$

and solving for y',

$$y' = \frac{\pm(x - c_1)}{\left[\lambda^2 - (x - c_1)^2\right]^{1/2}}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\lambda^2 - (x - c_1)^2\right]^{1/2} = -\frac{(x - c_1)}{\left[\lambda^2 - (x - c_1)^2\right]^{1/2}},$$

so we have

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \mp\frac{\mathrm{d}}{\mathrm{d}x}\left[\lambda^2 - (x - c_1)^2\right]^{1/2}.$$

Integrating again

$$y = \mp\left[\lambda^2 - (x - c_1)^2\right]^{1/2} + c_2$$

or

$$(y - c_2)^2 = \lambda^2 - (x - c_1)^2.$$

Clearly this is a circle of radius λ centered at (c_1, c_2)

$$(x - c_1)^2 + (y - c_2)^2 = \lambda^2.$$

The area enclosed

$$A = \pi\lambda^2 = \pi\left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi}$$

must be the maximum, since the minimum area is obviously zero when the curve is squeezed into a line.

7.3.3 The Catenary

The Latin word Catena means a chain. The catenary is a problem of hanging chain. Its history is parallel to that of Brachistochrone. In 1690, Jakob Bernoulli proposed the problem: "to find the curve assumed by a loose rope hung freely from two fixed points." One year later, three correct solutions were submitted – by Christian Huygens, Leibnitz, and Johann Bernoulli.

The rope will assume a shape for which its potential energy is a minimum. If ϱ is the mass per unit length of the rope, then the potential energy of an infinitesimal section of length $\mathrm{d}s$ due to gravity is $\varrho\,\mathrm{d}sgy$, where g is the gravitational constant. Let the two fixed points be A and B, the total potential energy of the rope is given by the functional

$$I = \int_A^B \varrho g y \,\mathrm{d}s = \varrho g \int_A^B y \,\mathrm{d}s.$$

On using $\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + y'^2}\mathrm{d}x$, and ignoring the constant factor ϱg, we have to minimize the functional

$$I = \int_A^B y\sqrt{1 + y'^2}\mathrm{d}x = \int_A^B F\left(y, y'\right)\mathrm{d}x. \tag{7.17}$$

Since F is independent of x, the Euler–Lagrange equation is given by

$$F - y'\frac{\partial F}{\partial y'} = c \tag{7.18}$$

or

$$y\sqrt{1 + y'^2} - y'\frac{yy'}{\sqrt{1 + y'^2}} = c.$$

Thus

$$y\left(1 + y'^2\right) - yy'^2 = c\sqrt{1 + y'^2},$$

or

$$y = c\sqrt{1 + y'^2}.$$

It follows that:

$$y' = \sqrt{\frac{y^2}{c^2} - 1},$$

so

$$\frac{dy}{\sqrt{(y/c)^2 - 1}} = \mathrm{d}x. \tag{7.19}$$

Recall

$$\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}), \quad \sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}),$$

$$\cosh^2 x - \sinh^2 x = 1, \quad \frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x.$$

With the substitution

$$\frac{y}{c} = \cosh z$$

and

$$\mathrm{d}y = c \sinh z \, \mathrm{d}z, \quad \sqrt{(y/c)^2 - 1} = \sqrt{\cosh^2 z - 1} = \sinh z,$$

we can write (7.19) as

$$c \, \mathrm{d}z = \mathrm{d}x.$$

Integrating once again, we have

$$cz = x + b,$$

$$z = \frac{x + b}{c}.$$

Hence

$$y = c \cosh \frac{x + b}{c}. \tag{7.20}$$

The constants c and b can be determined if the coordinates of the fixed points are known. For example, if the coordinates of A and B are $(-x_0, y_0)$ and (x_0, y_0), respectively, then

$$y_0 = c \cosh \frac{-x_0 + b}{c} = c \cosh \frac{x_0 + b}{c}.$$

Since $\cosh(-x) = \cosh(x)$, it follows that $b = 0$. In this case,

$$y = c \cosh \frac{x}{c}. \tag{7.21}$$

This is the equation of the catenary. The shape of the catenary is shown in Fig. 7.5

It should be noted that if the length of the rope L is given

$$L = \int_A^B \mathrm{d}s = \int_A^B \sqrt{1 + y'^2}\mathrm{d}x,$$

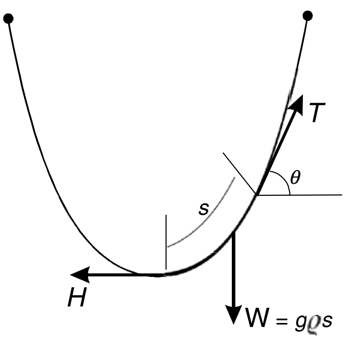


Fig. 7.5. The catenary. The shape of a hanging chain is given by $y = c\cosh(x/c)$, known as the catenary

then L must be kept constant. According the theory of constrained variation, we have to minimize the functional

$$K = \int_A^B \varrho g y\sqrt{1+y'^2}\mathrm{d}x + \lambda\int_A^B \sqrt{1+y'^2}\mathrm{d}x.$$

In this case, F in (7.18) is given by

$$F = \varrho g y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2}.$$

Following the same procedure as above, one can show that

$$y = \frac{c_1}{\varrho g}\cosh\frac{\varrho g(x+c_2)}{c_1} - \frac{\lambda}{\varrho g}.$$

The constants c_1, c_2, and λ are to be determined by the length of the rope L and the coordinates of A and B. As we see that the character of the catenary is not changed.

This problem can also be solved with the "ordinary calculus." Let $x = 0$ be the lowest point of the rope, and H be the tension at that point. Clearly, H acts horizontally. Let T be the tension at the other end of the section of length s, as shown in Fig. 7.5. The weight of rope of that section is $w = \varrho g s$. Since it is in equilibrium, the forces must be balanced in both x and y directions,

$$T\sin\theta = \varrho g s,$$
$$T\cos\theta = H.$$

The ratio of these two equations is

$$\tan\theta = \frac{\varrho g}{H}s,$$

which is the slope of the curve. That is

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = \frac{\varrho g}{H} s.$$

It follows:

$$\frac{\mathrm{d}}{\mathrm{d}x} y' = \frac{\varrho g}{H} \frac{\mathrm{d}s}{\mathrm{d}x}.$$

Since $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + y'^2}\mathrm{d}x$,

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + y'^2}.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}x} y' = \frac{\varrho g}{H} \sqrt{1 + y'^2}$$

or

$$\frac{\mathrm{d}y'}{\sqrt{1 + y'^2}} = \frac{\varrho g}{H} \mathrm{d}x.$$

Let

$$y' = \sinh z,$$

the last equation becomes

$$\frac{\mathrm{d}\sinh z}{\sqrt{1 + \sinh^2 z}} = \frac{\cosh z \, \mathrm{d}z}{\cosh z} = \mathrm{d}z = \frac{\varrho g}{H} \mathrm{d}x.$$

Integrating once

$$z = \frac{\varrho g}{H} x + c'$$

or

$$y' = \sinh\left(\frac{\varrho g}{H} x + c'\right).$$

At $x = 0$, $y' = 0$, therefore $c' = 0$, since $\sinh(0) = 0$. Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\left(\frac{\varrho g}{H} x\right)$$

or

$$\mathrm{d}y = \sinh\left(\frac{\varrho g}{H} x\right) \mathrm{d}x.$$

Integrating again, we get

$$y = \frac{H}{\varrho g} \cosh\left(\frac{\varrho g}{H} x\right) + b.$$

If we adjust the y-axis scale in such a way that $y(0) = H/\varrho g$, then $b = 0$ since $\cosh(0) = 1$. Thus

$$y(x) = \frac{H}{\varrho g} \cosh\left(\frac{\varrho g}{H} x\right),$$

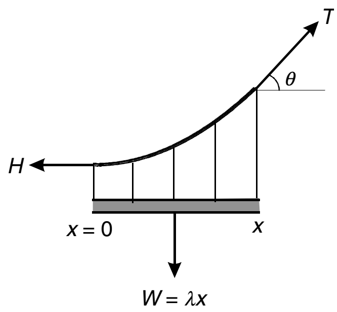


Fig. 7.6. The cable of a suspension bridge. The cable of a suspension bridge is in the form of parabola, which looks like a catenary but is not

which is in the same form as (7.21). In addition, we have given a physical meaning to the constant c of (7.21).

It is interesting to note that the shape of the cable of a suspension bridge is not a catenary, although it looks like one. Suppose that the cable is supporting a uniform roadway, and the weight of the cable is negligible compared to the weight of the roadway. A section of the cable is shown in Fig. 7.6. We take $x = 0$, $y = 0$ at the center of the span. This section supports a portion of the roadway whose weight W is proportional to the distance x, since the roadway is assumed uniform, i.e.,

$$W = \lambda x,$$

where λ is the weight per unit length. It is clear from Fig. 7.6 that

$$T \sin\theta = \lambda x,$$
$$T \cos\theta = H.$$

Thus

$$\tan\theta = \frac{\lambda}{H} x.$$

Since $\tan\theta$ is the slope of the cable

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = \frac{\lambda}{H} x$$

or

$$\mathrm{d}y = \frac{\lambda}{H} x \, \mathrm{d}x.$$

Integrating, we get

$$y = \frac{1}{2} \frac{\lambda}{H} x^2 + c.$$

This is a parabola. While the catenary certainly looks like a parabola, we see that they are fundamentally different. The parabola is algebraic, while the catenary is transcendental.

For a long time, the shape of a hanging chain was thought as a parabola. In 1646, Huygens (at age 17) proved that it could not possibly be a parabola. But it was not until 1691, after the invention of calculus, the catenary was correctly described as a hyperbolic cosine function.

7.3.4 Minimum Surface of Revolution

The catenary is also the solution of the problem of the minimum surface of revolution passing through two given points A and B.

The area of the surface of revolution generated by rotating the curve $y = y(x)$ about the x-axis is

$$I = 2\pi \int_A^B y \, \mathrm{d}s = 2\pi \int_A^B y\sqrt{1 + y'^2}\mathrm{d}x.$$

The integral we seek to minimize is the same as (7.17). Therefore the required curve is a catenary given by (7.20)

$$y(x) = c \cosh \frac{x + b}{c}.$$

The surface generated by the rotation of the catenary is called *catenoid.* The values of the arbitrary constants c and b are determined by the conditions

$$y(x_A) = y_A, \quad y(x_B) = y_B.$$

Unfortunately this is only an incomplete solution, because the two points A and B must satisfy certain condition in order to have a curve of the form (7.17) passing through them. In other words, if the condition is not satisfied, there is no surface in the class of smooth surfaces of revolution which achieves the minimum area.

We will find this condition for the following problem. Let the coordinates of A be $(-x_0, y_0)$ and of B be (x_0, y_0). As shown in (7.21), in this case the catenary can be written as

$$y = c \cosh \frac{x}{c}.$$

The constant c is determined by the ratio y_0/x_0. Now if we define

$$u = \frac{x}{c}, \quad v = \frac{y}{c}$$

and

$$u_0 = \frac{x_0}{c}, \quad v_0 = \frac{y_0}{c},$$

we can write (7.21) as

$$v = \cosh u. \tag{7.22}$$

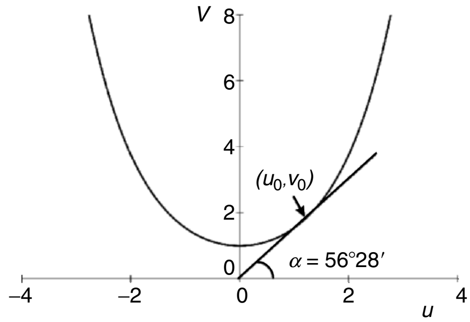


Fig. 7.7. To have a minimum surface of revolution the curve $v = \cosh u$ and $v = \frac{y_0}{x_0}u$ must intersect

On the other hand, we can take another curve

$$\frac{v}{u} = \frac{y}{x} = \frac{y_0}{x_0}$$

or

$$v = \frac{y_0}{x_0}u \tag{7.23}$$

which is a straight line. Since (u_0, v_0) satisfies both (7.22) and (7.23), it must be the point of intersection the two curves expressed by these equations. Figure 7.7 shows the curve $v = \cosh u$ and a straight line $v = (\tan\alpha)u$ that is tangent to this curve. It is clear that if $(y_0/x_0) < \tan\alpha$, then the two curves will not intersect, and no catenary can be drawn from A to B.

The angle α can be found by noting that at the point u_0 where the straight line is the tangent to the curve, we have the following relationship:

$$\frac{v}{u} = \frac{\mathrm{d}v}{\mathrm{d}u}.$$

Since $v = \cosh u$ and $\mathrm{d}v/\mathrm{d}u = \sinh u$, so we have

$$\frac{\cosh u}{u} - \sinh u = 0.$$

This equation can be solved numerically. For example, this book is written with the computer software "Scientific WorkPlace." It came with a computer algebra system "MuPAD." After this equation is typed in the math mode, click the "compute" and "solve" button, the program will return with an answer:

$$u = 1.1997.$$

This means that $(u_0, v_0) = (1.1997,\ \cosh 1.1997)$. It follows that:

$$\alpha = \tan^{-1}\left(\frac{\cosh 1.1997}{1.1997}\right) = 0.9885\ \text{radians}\ (56^o28').$$

Thus, if

$$\frac{y_0}{x_0} < \tan\alpha = 1.5089$$

the straight line $v = \dfrac{y_0}{x_0}u$ and the curve $v = \cosh u$ will not meet, and there is no twice-differentiable minimizing curve.

This limitation can be illustrated with a soap film experiment. Because of surface tension, a soap film will form a surface of minimum energy, which happens to be the surface of minimum area with the frame as the boundary.

A soap film will form a catenoid between two parallel rings of radius y_0 with their centers $2x_0$ apart on an axis perpendicular to the rings, as shown in Fig. 7.8, if y_0/x_0 is greater than 1.5089. We can increase x_0. When y_0/x_0 becomes less than 1.5089, the catenoid will no longer be formed, and the soap film will cover only the two rings to give a surface area of $2\pi y_0^2$. Clearly the solution is discontinuous and beyond the scope of the variational theory.

Example 7.3.1. Find the area of the minimum surface of revolution shown in Fig. 7.8 with $x_0 = 1$ and $y_0 = 2$.

Solution 7.3.1. The area of the catenoid is given by

$$A = 2\pi \int_{-1}^{1} y\sqrt{1+y'^2}\mathrm{d}x,$$

where

$$y = c\cosh\frac{x}{c}.$$

Since

$$y\sqrt{1+y'^2} = c\cosh\frac{x}{c}\sqrt{1+\sinh^2\frac{x}{c}} = c\cosh^2\frac{x}{c},$$

the area can be written as

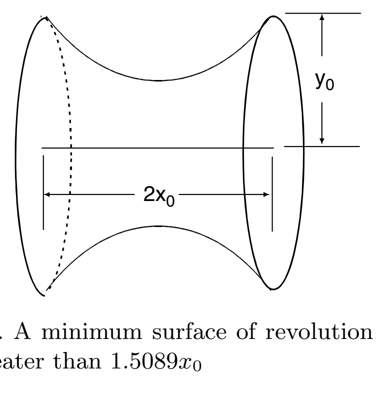


Fig. 7.8. The catenoid. A minimum surface of revolution can be formed between the two rings if y_0 is greater than $1.5089x_0$

$$A = 2\pi c \int_{-1}^{1} \cosh^2 \frac{x}{c} \mathrm{d}x = \pi c \int_{-1}^{1} \left(1 + \cosh \frac{2x}{c}\right) \mathrm{d}x$$
$$= \pi c^2 \left(\frac{2}{c} + \sinh \frac{2}{c}\right).$$

Since
$$\frac{y_0}{x_0} = 2 > 1.5089,$$
there are two intersection points of
$$v = \cosh u \quad \text{and} \quad v = 2u.$$
The equation
$$\cosh u - 2u = 0$$
can be solved numerically to give
$$u_0 = \begin{cases} 0.5894, \\ 2.1268. \end{cases}$$
Recall $u_0 = x_0/c$, so
$$c = \frac{x_0}{u_0} = \begin{cases} \frac{1}{0.5894} = 1.6967, \\ \frac{1}{2.1268} = 0.4702. \end{cases}$$
Therefore
$$A = \begin{cases} \pi\, (1.6967)^2 \left(\frac{2}{1.6967} + \sinh \frac{2}{1.6967}\right) = 23.968, \\ \pi\, (0.4702)^2 \left(\frac{2}{0.4702} + \sinh \frac{2}{0.4702}\right) = 27.382. \end{cases}$$
Thus, rotating
$$y = 1.6967 \cosh \frac{x}{1.6967},$$
around the x-axis will generate the minimum surface of revolution with an area of 23.968.

7.3.5 Fermat's Principle

In the 1650s Pierre de Fermat adopted the view espoused by Aristotelians that nature always chooses the shortest path, and formulated a "principle of least time" for geometrical optics. It says that a ray of light traveling from one point to another takes the path which requires the shortest time.

If the velocity of the ray of light in a medium is v, then the time it takes from A to B is

$$T=\int\frac{\mathrm{d}s}{v}=\int\frac{1}{v}\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}=\int_A^B\frac{1}{v}(1+y'^2)^{1/2}\mathrm{d}x,$$

where $y(x)$ is the path of the ray. In optics, a very useful quantity is the index of refraction n, defined as

$$n=\frac{c}{v},$$

where c is the speed of light in vacuum, which is a constant. To have the shortest time is to require the following integral to be stationary:

$$I=\int_A^B n(1+y'^2)^{1/2}\mathrm{d}x.$$

Let us assume that n is not a function of x. In that case, the integrand

$$F=n(1+y'^2)^{1/2}$$

does not explicitly contain the independent variable x. Therefore

$$F-y'\frac{\partial}{\partial y'}F=k,$$

where k is a constant. Carrying out the differentiation, we have

$$n(1+y'^2)^{1/2}-y'n\frac{y'}{(1+y'^2)^{1/2}}=k$$

or

$$n(1+y'^2)-ny'^2=k(1+y'^2)^{1/2}.$$

Thus

$$n=k(1+y'^2)^{1/2}.$$

Since y' is the slope of $y(x)$, so $y'=\tan\phi$ where ϕ is the angle between the instantaneous direction of the ray and the x-axis. Thus the last equation can be written as

$$n=k(1+\tan^2\phi)^{1/2}=k\left(1+\frac{\sin^2\phi}{\cos^2\phi}\right)^{1/2}=k\frac{1}{\cos\phi}.$$

Therefore the general result is in the form of

$$n\cos\phi=k.$$

If n is not changing in space, then ϕ must be a constant since k is a constant. This means that the ray of light is traveling in a straight line. This is certainly the case.

Suppose that the ray travels from one medium with index of refraction n_1 to another medium with index of refraction n_2, and the interface between

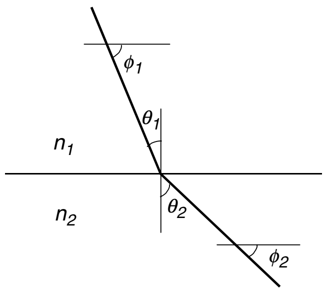


Fig. 7.9. Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$. If $n_1 > n_2$, then $\theta_1 < \theta_2$

them is a plane as shown in Fig. 7.9. Since k is constant along the whole path, we must have

$$n_1 \cos\phi_1 = n_2 \cos\phi_2.$$

This is the well-known Snell's law. Usually the Snell's law is written as

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where θ_1 and θ_2 are the angles between the ray and the normal of the interface as shown in Fig. 7.9. Since $\theta_1 + \phi_1 = \theta_2 + \phi_2 = \pi/2$, the last two equations are identical.

If the ray is going from medium 1 to medium 2, θ_1 is known as the angle of incidence $\theta_{\rm i}$ and θ_2, the angle of transmission (refraction) $\theta_{\rm t}$. If $\theta_{\rm t} = \pi/2$, the incident angle is known as the critical angle $\theta_{\rm c}$

$$\theta_{\rm c} = \sin^{-1}\frac{n_2}{n_1}.$$

If the angle of incidence is greater than the critical angle, the ray is reflected back. The angle between the reflected ray and the normal is known as the angle of reflection $\theta_{\rm r}$. In that case, $n_1 \sin\theta_{\rm i} = n_1 \sin\theta_{\rm r}$. Therefore

$$\theta_{\rm i} = \theta_{\rm r},$$

another well-known fact in geometrical optics.

In fact, the entire geometrical optics can be derived from the Fermat principle.

Suppose that the light is going through a series of mediums as shown in Fig. 7.10. If $n_1 > n_2 > n_3 > n_4$, then $\theta_1 < \theta_2 < \theta_3$. If θ_3 is greater than the critical angle, then the ray will be reflected back as shown in the figure. It is clear that if the index of refraction is decreasing continuously (this is equivalent to say that if the velocity of the light v is increasing continuously), the path of the light will become a continuous curve. If v is proportional to $\sqrt{y}$ (with the positive y-axis directed downward), the path of the light will become the Brachistochrone curve shown in Fig. 7.2. In fact, Johann Bernoulli first solved the Brachistochrone problem with this optical analogy.

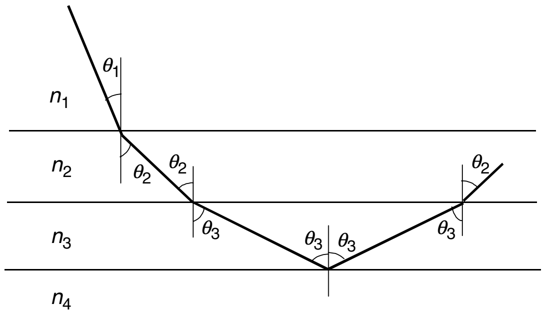


Fig. 7.10. Light going through a series of mediums with increasing index of refraction

This is also the reason for the mirage that one often sees while driving in hot roads. One sees "water" on the road, but when one gets there, it is dry. The explanation is this. The air is very hot just above the road and is cooler up higher. Light travels faster in the hot region because the air is more expanded and therefore thinner. So the light from the sky, heading for the road, is going faster and faster. As a consequence, it follows a curved path, like the Brachistochrone curve shown in Fig. 7.2. When it ended in our eyes, we thought it was reflected from the water on the road.

7.4 Some Extensions

Often the functionals contain higher derivatives, or several independent or dependent variables. The Euler–Lagrange equations for these problems can be derived in a similar way.

7.4.1 Functionals with Higher Derivatives

Consider the functional

$$I = \int_{x_1}^{x_2} F(y, y', y'', x)\mathrm{d}x, \tag{7.24}$$

where the values of y and y' are specified at the end points

$$\begin{aligned} y(x_1) &= A_0, \quad y'(x_1) = A_1, \\ y(x_2) &= B_0, \quad y'(x_2) = B_1. \end{aligned}$$

Among all possible functions that satisfy these boundary conditions, we want to find the function $y(x)$ for which the functional I has an extremum.

To solve this problem, we follow the previous procedure. We define a family of curves that satisfies these boundary conditions

$$Y(x) = y(x) + \alpha\eta(x), \quad Y'(x) = y' + \alpha\eta', \quad Y'' = y'' + \alpha\eta'',$$

where $\eta(x)$ is a twice-differentiable arbitrary function that satisfies the boundary conditions

$$\eta(x_1) = \eta(x_2) = 0, \quad \eta'(x_1) = \eta'(x_2) = 0.$$

Replacing y by Y in (7.24), we have

$$I(\alpha) = \int_{x_1}^{x_2} F(Y, Y', Y'', x)\mathrm{d}x.$$

A necessary condition for it to be an extremum is that

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0.$$

Carrying the differentiation inside the integral, we have

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' + \frac{\partial F}{\partial y''}\eta''\right] \mathrm{d}x.$$

We have already shown that

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\eta'\mathrm{d}x = -\int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial y'}\right] \eta\, \mathrm{d}x.$$

Similarly, with integration by parts

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\eta''\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\frac{\mathrm{d}\eta'}{\mathrm{d}x}\mathrm{d}x = \int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\mathrm{d}\eta'$$

$$= \left.\frac{\partial F}{\partial y''}\eta'\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y''}\right)\eta'\mathrm{d}x.$$

The integrated part is zero because of the boundary conditions of $\eta'(x)$. With integration by parts again, the last term becomes

$$-\int_{x_1}^{x_2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y''}\right)\eta'\mathrm{d}x = -\left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y''}\right)\eta\right|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right)\eta\,\mathrm{d}x.$$

Again the integrated part vanishes because of the boundary conditions of $\eta(x)$. Therefore

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right)\right]\eta\,\mathrm{d}x = 0.$$

It follows that the function $y(x)$, for which I is stationary, must satisfy the differential equation:

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right) = 0.$$

Notice the alternating sign in this equation. Clearly, the function $y(x)$ that minimizes the functional

$$I = \int_{x_1}^{x_2} F(y, y', y'', \ldots, y^n, x)\mathrm{d}x$$

is the solution of

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) + \cdots (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\partial F}{\partial y^n}\right) = 0.$$

7.4.2 Several Dependent Variables

Consider the integral

$$I = \int_{t_1}^{t_2} F(x, y, x', y', t)\mathrm{d}t,$$

where x and y are twice-differentiable functions of the independent variable t. Their derivatives with respect to t are, respectively, x' and y'. The values of $x(t_1)$, $y(t_1)$ and $x(t_2)$, $y(t_2)$ are specified. We want to find the differential equations that x and y must satisfy so that the value of I is stationary. We can solve this problem by the same procedure as in the case of one dependent variable.

Let $x(t)$ and $y(t)$ be the actual curve along which I is stationary. We denote the family of curves that go through the two fixed points at t_1 and t_2 as

$$X(t) = x(t) + \alpha\varepsilon(t), \qquad Y(t) = y(t) + \alpha\eta(t),$$

where $\varepsilon(t)$ and $\eta(t)$ are arbitrary differentiable functions for which

$$\varepsilon(t_1) = \varepsilon(t_2) = 0, \quad \eta(t_1) = \eta(t_2) = 0.$$

These boundary conditions ensure that every curve in the family goes through the two end points. The parameter α specifies each individual curve and the actual curve that minimizes I is labeled as $\alpha = 0$. As a consequence

$$X' = x' + \alpha\varepsilon', \quad Y' = y' + \alpha\eta'.$$

Replacing x and y by X and Y, respectively, in the integral I, it becomes a function of α

$$I(\alpha) = \int_{t_1}^{t_2} F(X, Y, X', Y', t)\mathrm{d}t.$$

A necessary condition for I to be stationary is

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0.$$

Since α does not depend on t, the differentiation can carry out inside the integral,

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}\alpha} &= \int_{t_1}^{t_2}\left[\frac{\partial F}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}\alpha} + \frac{\partial F}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\alpha} + \frac{\partial F}{\partial X'}\frac{\mathrm{d}X'}{\mathrm{d}\alpha} + \frac{\partial F}{\partial Y'}\frac{\mathrm{d}Y'}{\mathrm{d}\alpha}\right]\mathrm{d}t \\ &= \int_{t_1}^{t_2}\left[\frac{\partial F}{\partial X}\varepsilon + \frac{\partial F}{\partial Y}\eta + \frac{\partial F}{\partial X'}\varepsilon' + \frac{\partial F}{\partial Y'}\eta'\right]\mathrm{d}t.\end{aligned}$$

Setting $\alpha = 0$ is equivalent to replace X and Y by x and y. Thus

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{t_1}^{t_2}\left[\frac{\partial F}{\partial x}\varepsilon + \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial x'}\varepsilon' + \frac{\partial F}{\partial y'}\eta'\right]\mathrm{d}t.$$

This relation must hold for all possible choices of $\varepsilon(t)$ and $\eta(t)$, as long as they satisfy the boundary conditions. In particular, it holds for the special choice in which $\varepsilon(t)$ is identically equal to zero and $\eta(t)$ is still arbitrary. For this choice, the last equation becomes

$$\left.\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{t_1}^{t_2}\left[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right]\mathrm{d}t.$$

This equation is identical to (7.6) with the independent variable x replaced by t. Following the same procedure, we obtain

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial y'} = 0.$$

Similarly,

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial x'} = 0.$$

Therefore for this system, we have two separate but simultaneous Euler–Lagrange equations for $x(t)$ and $y(t)$. Clearly, if we have n dependent variables, the analysis will lead to n separate but simultaneous equations.

This method is easily generalized to cases with more than one constraint. If we wish to find the stationary value of the integral I, which has n dependent variables, subject to multiple constraints that the values of the integrals J_j be held constant for $i = 1, 2, \ldots m$, then we simply find the unconstrained stationary value of the new integral K,

$$K = I + \sum_{j=1}^{m}\lambda_j J_j.$$

With

$$I = \int_{t_1}^{t_2} F(x_1, \ldots, x_n, x_1', \ldots, x_n', t)\mathrm{d}t,$$

$$J_j = \int_{t_1}^{t_2} G_j(x_1, \ldots, x_n, x_1', \cdot\cdot, x_n', t)\mathrm{d}t, \quad i = 1, 2, \ldots, m.$$

Following the same procedure, one can obtain the set of Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial x_i'} - \frac{\partial H}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n,$$

where

$$H = F + \sum_{j=1}^{m} \lambda_i G_j.$$

This is a set of n coupled differential equations.

7.4.3 Several Independent Variables

For problems in more than one dimension, we need to consider functionals that depend on more than one independent variables. Let us consider the following double integral of x and y over some region R

$$I = \iint_R F(u, u_x', u_y', x, y)\mathrm{d}x\mathrm{d}y, \tag{7.25}$$

where u is a function of x and y, and

$$u_x' = \frac{\partial u(x,y)}{\partial x}, \quad u_y' = \frac{\partial u(x,y)}{\partial y}.$$

Let the region R be bounded by the curve C. The values of $u(x, y)$ are specified on C. We assume F is continuous and twice differentiable. We wish to determine the function $u(x, y)$ for which I is stationary with respect to small changes of u.

Procedures analogous to the one for one-dimensional problems can be used to solve this two-dimensional problem. Let $u(x, y)$ be the function for which the integral I is stationary, and the trial functions be of the form

$$U(x, y) = u(x, y) + \alpha\eta(x, y),$$

where $\eta(x, y) = 0$ on C. We will use the following notations to express partial derivatives

$$U_x' = \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} + \alpha\frac{\partial \eta}{\partial x} = u_x' + \alpha\eta_x',$$

$$U_y' = \frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} + \alpha\frac{\partial \eta}{\partial y} = u_y' + \alpha\eta_y'.$$

It follows that:

$$\frac{\mathrm{d}U}{\mathrm{d}\alpha} = \eta, \quad \frac{\mathrm{d}U_x'}{\mathrm{d}\alpha} = \eta_x', \quad \frac{\mathrm{d}U_y'}{\mathrm{d}\alpha} = \eta_y'.$$

Replacing $u(x,y)$ with $U(x,y)$ in (7.25), we have

$$I(\alpha) = I = \iint_R F(U, U_x', U_y', x, y)\mathrm{d}x\mathrm{d}y.$$

A necessary condition for $u(x,y)$ to be the function for which I is an extremum is that the derivative of I must vanish at $\alpha = 0$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} I(\alpha)\right|_{\alpha=0} = 0.$$

Since α does not depend on x or y, the differentiation can be carried out under the integral sign

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha} I(\alpha) &= \iint_R \left[\frac{\partial F}{\partial U}\frac{\mathrm{d}U}{\mathrm{d}\alpha} + \frac{\partial F}{\partial U_x'}\frac{\mathrm{d}U_x'}{\mathrm{d}\alpha} + \frac{\partial F}{\partial U_y'}\frac{\mathrm{d}U_y'}{\mathrm{d}\alpha}\right]\mathrm{d}x\mathrm{d}y \\ &= \iint_R \left[\frac{\partial F}{\partial U}\eta + \frac{\partial F}{\partial U_x'}\eta_x' + \frac{\partial F}{\partial U_y'}\eta_y'\right]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

In the limit of $\alpha \to 0$, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} I(\alpha)\right|_{\alpha=0} = \iint_R \left[\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial u_x'}\eta_x' + \frac{\partial F}{\partial u_y'}\eta_y'\right]\mathrm{d}x\mathrm{d}y.$$

The second term on the right-hand side can be written as

$$\iint_R \frac{\partial F}{\partial u_x'}\eta_x'\mathrm{d}x\mathrm{d}y = \int_{y_1}^{y_2}\left[\int_{x=C_1(y)}^{x=C_2(y)} \frac{\partial F}{\partial u_x'}\frac{\partial \eta}{\partial x}\mathrm{d}x\right]\mathrm{d}y,$$

where $C_1(y)$ and $C_2(y)$ are shown in Fig. 7.11.

With a given y, we can use integration by parts to write

$$\int_{x=C_1(y)}^{x=C_2(y)} \frac{\partial F}{\partial u_x'}\frac{\partial \eta}{\partial x}\mathrm{d}x = \left[\frac{\partial F}{\partial u_x'}\eta\right]_{x=C_1(y)}^{x=C_2(y)} - \int_{x=C_1(y)}^{x=C_2(y)} \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x'}\right)\mathrm{d}x.$$

The integrated part is equal to zero because on the boundary $\eta(x,y) = 0$. Therefore

$$\begin{aligned}\iint_R \frac{\partial F}{\partial u_x'}\eta_x'\mathrm{d}x\mathrm{d}y &= -\int_{y_1}^{y_2}\left[\int_{x=C_1(y)}^{x=C_2(y)} \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x'}\right)\mathrm{d}x\right]\mathrm{d}y \\ &= -\iint_R \eta\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x'}\right)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

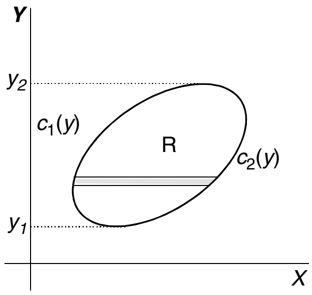


Fig. 7.11. Double integral. A double integral over the region R can be carried out by first integrating x from the left boundary $x_1 = C_1(y)$ to the right boundary $x_2 = C_2(y)$ with a fixed y, then integrating y from y_1 to y_2

Similarly,

$$\iint_R \frac{\partial F}{\partial u'_y}\eta'_y \mathrm{d}x\mathrm{d}y = -\iint_R \eta\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u'_y}\right)\mathrm{d}x\mathrm{d}y.$$

Thus,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}I\left(\alpha\right)\right|_{\alpha=0} = \iint_R \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u'_y}\right)\right]\eta\left(x,y\right)\mathrm{d}x\mathrm{d}y = 0.$$

Since $\eta\left(x,y\right)$ is arbitrary except at the boundary, we conclude that the term in the bracket must be equal to zero,

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u'_y}\right) = 0.$$

This is the Euler–Lagrange equation in two dimensions. Extension to three and higher dimensions is straightforward.

7.5 Sturm–Liouville Problems and Variational Principles

7.5.1 Variational Formulation of Sturm–Liouville Problems

Suppose we seek a function $y = y(x)$ in the range of $x_1 \leq x \leq x_2$ which satisfies the boundary condition

$$y(x_1) = 0, \quad y(x_2) = 0$$

and makes the value of the following integral stationary:

$$I = \int_{x_1}^{x_2} \left[p(x)y'^2 - q(x)y^2\right]\mathrm{d}x, \tag{7.26}$$

where $p(x)$ and $q(x)$ are continuous differential functions of x. In addition, we require the integral

$$J = \int_{x_1}^{x_2} w(x) y^2 \mathrm{d}x \tag{7.27}$$

to equal to a prescribed value with a given positive function $w(x)$.

According to the constrained variational theory, the answer is given by the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial H}{\partial y'} - \frac{\partial H}{\partial y} = 0$$

with

$$H = \left[p(x) y'^2 - q(x) y^2 \right] - \lambda w(x) y^2.$$

(The sign of λ is immaterial, since it is an undetermined multiplier. We use $-\lambda$ instead of positive to conform with the sign convention of the Sturm–Liouville problem, see below.) Since

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial H}{\partial y'} &= 2 \frac{\mathrm{d}}{\mathrm{d}x} (p(x) y'), \\ \frac{\partial H}{\partial y} &= -2q(x) y - 2\lambda w(x) y, \end{aligned}$$

the Euler–Lagrange equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x} (p(x) y') + q(x) y + \lambda w(x) y = 0. \tag{7.28}$$

This is a Sturm–Liouville equation with eigenvalue λ.

This opens up a relation between the calculus of variation and eigenvalue problems.

Note that J of (7.27) is just a normalization integral. If we want $y(x)$ to be normalized to one with respect to the weight function $w(x)$, then $y(x)$ should be replaced by $y(x) J^{-1/2}$. Replace $y(x)$ in (7.26) by $y(x) J^{-1/2}$, it becomes

$$K\left[y(x) \right] = \frac{\displaystyle\int_{x_1}^{x_2} \left[p(x) y'^2 - q(x) y^2 \right] \mathrm{d}x}{\displaystyle\int_{x_1}^{x_2} w(x) y^2 \mathrm{d}x} = \frac{I}{J}. \tag{7.29}$$

Since the denominator J is constant, the stationary value of I corresponds to the stationary value of K. That is, the solution of the Sturm–Liouville equation (7.28) is still the function that minimizes the functional $K[y(x)]$.

Integrating the first term in the numerator of $K[y(x)]$ by parts, we get

$$\begin{aligned} \int_{x_1}^{x_2} p(x) y'^2 \mathrm{d}x &= \int_{x_1}^{x_2} p(x) y' \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_1}^{x_2} p(x) y' \mathrm{d}y \\ &= p(x) y'(x) y(x) \big|_{x=x_1}^{x=x_2} - \int_{x_1}^{x_2} y \, \mathrm{d}\left[p(x) y' \right]. \end{aligned}$$

The integrated part is equal to zero because of the boundary conditions of $y(x)$. Thus

$$\int_{x_1}^{x_2} p(x)y'^2 \mathrm{d}x = -\int_{x_1}^{x_2} y \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]\mathrm{d}x.$$

It follows that:

$$K[y(x)] = \frac{-\displaystyle\int_{x_1}^{x_2} y\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right] + q(x)y\right\}\mathrm{d}x}{\displaystyle\int_{x_1}^{x_2} w(x)y^2 \mathrm{d}x}. \tag{7.30}$$

If $y(x)$ is the ith eigenfunction of (7.28), then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y_i'\right] + q(x)y_i = -\lambda w(x)y_i. \tag{7.31}$$

Substituting it into (7.30), we get

$$K\left[y_i(x)\right] = \frac{I}{J} = \frac{\lambda_i \int_{x_1}^{x_2} y_i w(x) y_i \mathrm{d}x}{\int_{x_1}^{x_2} w(x)y_i^2 \mathrm{d}x} = \lambda_i. \tag{7.32}$$

Thus, the eigenvalue λ, introduced originally as the undetermined multiplier, is the stationary value of the functional $K[y(x)]$. The function $y(x)$ that minimizes $K[y(x)]$ is the corresponding eigenfunction.

7.5.2 Variational Calculations of Eigenvalues and Eigenfunctions

The advantage of the variational formulation of the Sturm–Liouville equation is that one can use (7.29) or (7.30) to make systematic estimates of the eigenvalues and eigenfunctions of such equations.

The value of the functional $K\left[y(x)\right]$ can be calculated for any function of $y(x)$. There is a theorem which says that the functional $K\left[\phi(x)\right]$ of (7.29) evaluated with any function $\phi(x)$ that satisfies the same boundary conditions as given in the eigenvalue problem will be greater or equal to the smallest eigenvalue.

Let $\{y_i(x)\}$ be the set of eigenfunctions of the Sturm–Liouville problem. We may not know what they are, but we know that they are orthogonal and can be made orthonormal

$$\int_{x_1}^{x_2} y_i(x)y_j(x)w(x)\mathrm{d}x = \delta_{ij}, \tag{7.33}$$

and they form a complete set. Therefore $\phi(x)$ can be expressed as

$$\phi(x) = \sum_i c_i y_i(x).$$

Substituting this expression into the functional of (7.30)

$$K[\phi(x)] = \frac{-\displaystyle\int_{x_1}^{x_2} \phi \left\{ \frac{\mathrm{d}}{\mathrm{d}x} \left[p(x)\phi' \right] + q(x)\phi \right\} \mathrm{d}x}{\displaystyle\int_{x_1}^{x_2} w(x)\phi^2 \mathrm{d}x},$$

and using (7.31) and (7.33), we have

$$K[\phi(x)] = \frac{\sum_i c_i^2 \lambda_i}{\sum_i c_i^2}.$$

Let λ_1 be the smallest eigenvalue, then

$$K[\phi(x)] - \lambda_1 = \frac{\sum_i c_i^2 \lambda_i}{\sum_i c_i^2} - \lambda_1 = \frac{\sum_i c_i^2 (\lambda_i - \lambda_1)}{\sum_i c_i^2}.$$

Since every λ_i, $i \neq 1$, is greater than λ_1, therefore $K[\phi(x)] - \lambda_1 > 0$, and

$$K[\phi(x)] \geq \lambda_1.$$

The equal sign holds only if $\phi(x) = y_1(x)$, the ground state eigenfunction. (The ground state is the state with the smallest eigenvalue.) This is often called the Rayleigh–Ritz variational principle.

Now we can approximate $y_1(x)$ with any reasonable trial function $\phi(x)$ that satisfies the boundary conditions. The eigenvalue obtained from (7.29)

$$\lambda_u = K[\phi(x)] = \frac{\displaystyle\int_{x_1}^{x_2} \left[p(x)\phi'^2 - q(x)\phi^2 \right] \mathrm{d}x}{\displaystyle\int_{x_1}^{x_2} w(x)\phi^2 \mathrm{d}x} \tag{7.34}$$

is always greater than or equal to λ_1. We can include parameters in $\phi(x)$, these parameters can be varied to minimize $K[\phi(x)]$ and thereby improve the estimate of the ground state eigenvalue.

As an illustration, let us consider the equation

$$y'' + \lambda y = 0$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 0.$$

This is a Sturm–Livouville problem with $p(x) = 1$, $q(x) = 0$, and $w(x) = 1$. This problem is simple enough for all of us to know the exact solutions

$$y = \sin \sqrt{\lambda} x,$$
$$\lambda = n^2\pi^2, \quad n = 1, 2, \dots .$$

Therefore the lowest eigenvalue is

$$\lambda_1 = \pi^2 = 9.8696.$$

Now let us use the Rayleigh–Ritz method to approximate λ_1. We may use the simple function

$$\phi(x) = x(1-x)$$

as our trial function, since it satisfies the boundary conditions $\phi(0) = \phi(1) = 0$. Substituting this function into (7.34), with $\phi'(x) = 1-2x$, $p(x) = 1$, $q(x) = 0$, and $w(x) = 1$, we have

$$\lambda_u = K[\phi(x)] = \frac{\displaystyle\int_{x_1}^{x_2} (1-2x)^2 \mathrm{d}x}{\displaystyle\int_{x_1}^{x_2} x^2(1-x)^2 \mathrm{d}x} = \frac{1/3}{1/30} = 10,$$

which is only 1.3% in error compared to the exact value of π^2.

To calculate the eigenvalue, it is not necessary to use a normalized trial function, because of the denominator in $K[\phi(x)]$. However, it should be kept in mind that the trial function is an approximation of the eigenfunction only within a multiplicative constant. The comparison between the normalized trial function $\sqrt{30}x(1-x)$ and the normalized exact eigenfunction $\sqrt{2}\sin\pi x$ is shown in Fig. 7.12.

The result can be improved by introducing more terms with parameters. These parameters can be adjusted to minimize $K[\phi(x)]$, since no matter what these parameters are, the results are always upper bounds to λ_1. For example, we may use

$$\phi_1(x) = x(1-x) + cx^2(1-x)^2$$

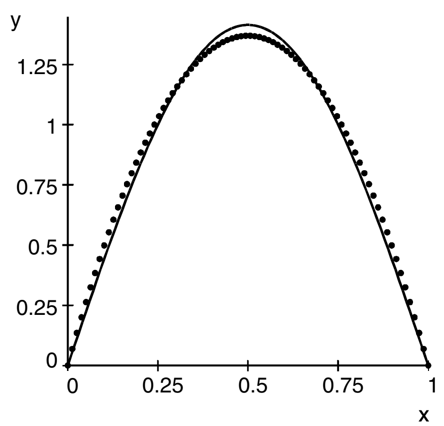


Fig. 7.12. Comparison between the normalized exact eigenfunction $\sqrt{2}\sin\pi x$ *(solid line)* and the normalized trial function $\sqrt{30}x(1-x)$ *(dotted line)*

as the trial function. As a consequence, $K[\phi_1(x)]$ becomes a function of c. Take the derivative of $K[\phi_1(x)]$ with respect to c and set it to zero, we find $c = 1.1353$. Use this value of c, we find

$$\lambda_u = K[\phi_1(x)] = 9.8697,$$

which is very close to the exact value of $\pi^2 = 9.8696$. When normalized, this trial function becomes $4.404x(1-x)+4.990x^2(1-x)^2$. Plotted against x, this function is indistinguishable from the exact eigenfunction in the scale shown in Fig. 7.12. This suggests that if the eigenvalue calculated from the trial function is very good, the trial function is probably also a good approximation of the eigenfunction.

This method can be extended to the second and higher eigenvalues by imposing additional restrictions of the trial functions to only those that are orthogonal to the eigenfunctions corresponding to the lower eigenvalues.

For example, we may use a trial function in the form of

$$\phi(x) = c_1 f_1(x) + c_2 f_2(x),$$

where $f_1(x)$ and $f_2(x)$ are known as basis. We will show that in minimizing the functional

$$K[\phi(x)] = \frac{I[\phi(x)]}{J[\phi(x)]},$$

we will obtain two "eigenvalues." If $f_2(x)$ is orthogonal to $f_1(x)$

$$\int_{x_1}^{x_2} f_1(x) f_2(x) w(x) \mathrm{d}x = 0,$$

then they may approximate the first two true eigenvalues of the Sturm–Liouville problem.

To minimize $K[\phi(x)]$, we need to set both $\partial K/\partial c_1$ and $\partial K/\partial c_2$ to zero. Since

$$\frac{\partial K}{\partial c_i} = \frac{\partial I/\partial c_i}{J} - I\frac{\partial J/\partial c_i}{J^2} = \frac{1}{J}\left[\frac{\partial I}{\partial c_i} - \frac{I}{J}\frac{\partial J}{\partial c_i}\right] = 0$$

and $J > 0$, this means

$$\frac{\partial I}{\partial c_i} - \frac{I}{J}\frac{\partial J}{\partial c_i} = 0, \quad i = 1, 2. \tag{7.35}$$

Carrying out the differentiation

$$\begin{aligned}
\frac{\partial I}{\partial c_1} &= \frac{\partial}{\partial c_1} I[c_1 f_1 + c_2 f_2] = \frac{\partial}{\partial c_1}\int_{x_1}^{x_2} [p(c_1 f_1' + c_2 f_2')^2 - q(c_1 f_1 + c_2 f_2)^2]\mathrm{d}x \\
&= 2\int_{x_1}^{x_2} [p(c_1 f_1' + c_2 f_2')f_1' - q(c_1 f_1 + c_2 f_2)f_1]\mathrm{d}x \\
&= 2c_1 \int_{x_1}^{x_2} [p f_1' f_1' - q f_1 f_1]\mathrm{d}x + 2c_2 \int_{x_1}^{x_2} [p f_2' f_1' - q f_2 f_1]\mathrm{d}x.
\end{aligned}$$

Clearly we can write both derivatives as

$$\frac{\partial I}{\partial c_i} = 2\sum_{j=1}^{2} c_j a_{ji}, \quad i = 1, 2,$$

$$a_{ji} = \int_{x_1}^{x_2} [p f_j' f_i' - q f_j f_i] \mathrm{d}x.$$

Similarly,

$$\frac{\partial J}{\partial c_i} = 2\sum_{j=1}^{2} c_j b_{ji}, \quad k = 1, 2,$$

$$b_{ji} = \int_{x_1}^{x_2} f_j f_i w \, \mathrm{d}x.$$

According to (7.32), our "approximate eigenvalue" is given by

$$K = \frac{I}{J} = \lambda.$$

Therefore (7.35) can be written as

$$\sum_{j=1}^{2} c_j (a_{ji} - \lambda b_{ji}) = 0, \quad i = 1, 2.$$

Since f_1 and f_2 are orthogonal, so $b_{ij} = 0$ for $i \neq j$. Thus we have

$$(a_{11} - \lambda b_{11}) c_1 + a_{21} c_2 = 0,$$
$$a_{12} c_1 + (a_{22} - \lambda b_{22}) c_2 = 0.$$

For a nonzero solution, λ must satisfy the secular equation

$$\begin{vmatrix} a_{11} - \lambda b_{11} & a_{21} \\ a_{12} & a_{22} - \lambda b_{22} \end{vmatrix} = 0.$$

This is a quadratic equation, λ will have two roots. They may approximate the first two eigenvalues of the problem.

To illustrate the procedure, let us try to approximate the first two eigenvalues of the previous problem

$$y'' + \lambda y = 0, \quad y(0) = y(1) = 0.$$

We choose the following trial function:

$$\phi(x) = c_1 f_1(x) + c_2 f_2(x)$$

with

$$f_1(x) = x(1-x),$$
$$f_2(x) = x(1-x)(1+ax).$$

Both $f_1(x)$ and $f_2(x)$ satisfy the boundary conditions. The constant a is determined from the orthogonal condition

$$\int_0^1 f_1(x) f_2(x) \mathrm{d}x = 0$$

to be -2. With $f_1' = 1 - 2x,\ f_2' = 1 - 6x + 6x^2$, one can readily find

$$a_{11} = \int_0^1 f_1'^2 \mathrm{d}x = \frac{1}{3}, \quad a_{22} = \int_0^1 f_2'^2 \mathrm{d}x = \frac{1}{5},$$
$$a_{12} = a_{21} = \int_0^1 f_1' f_2' \mathrm{d}x = 0,$$
$$b_{11} = \int_0^1 f_1^2 \mathrm{d}x = \frac{1}{30}, \quad b_{22} = \int_0^1 f_2^2 \mathrm{d}x = \frac{1}{210}.$$

Therefore the secular equation is

$$\begin{vmatrix} \frac{1}{3} - \frac{1}{30}\lambda & 0 \\ 0 & \frac{1}{5} - \frac{1}{210}\lambda \end{vmatrix} = 0$$

which has two roots

$$\lambda_1 = 10, \quad \lambda_2 = 42.$$

These two roots are to be compared with the first two exact eigenvalues

$$\lambda_1 = \pi^2 = 9.8696, \quad \lambda_2 = 4\pi^2 = 39.48.$$

7.6 Rayleigh–Ritz Methods for Partial Differential Equations

The Euler–Lagrange equations for functionals with more than one independent variables are partial differential equations. The minimizing function of the functional will be the solution of the corresponding partial differential equations.

Consider the functional

$$I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} F(u, u_x', u_x'', u_y', u_y'', x, y) \mathrm{d}x \mathrm{d}y,$$

where x and y are two independent variables and

$$u = u(x, y),$$
$$u'_x = \frac{\partial u}{\partial x}, \quad u''_x = \frac{\partial^2 u}{\partial x^2},$$
$$u'_y = \frac{\partial u}{\partial y}, \quad u''_y = \frac{\partial^2 u}{\partial y^2}.$$

With the methods developed for functionals with two independent variables and with higher derivatives, one can easily show that the Euler–Lagrange equation for the functional is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u''_x}\right)$$
$$-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u'_y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u''_y}\right) = 0.$$

Many important partial differential equations in mathematical physics can be put in this form. In what follows, we will interpret these partial differential equations as the Euler–Lagrange equations of some functionals, then use the Rayleigh–Ritz method to approximate the minimizing functions of these functionals. The minimizing functions will then be the solutions of these partial differential equations.

7.6.1 Laplace's Equation

To find the Euler–Lagrange equation for the following two-dimensional functional

$$I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] \mathrm{d}x\mathrm{d}y,$$

we can write the integrand as

$$F = u'^2_x + u'^2_y.$$

Thus

$$\frac{\partial F}{\partial u} = 0, \quad \frac{\partial F}{\partial u'_x} = 2u'_x = 2\frac{\partial u}{\partial x}, \quad \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'_x}\right) = 2\frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial F}{\partial u'_y} = 2u'_y = 2\frac{\partial u}{\partial y}, \quad \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u'_y}\right) = 2\frac{\partial^2 u}{\partial y^2}.$$

Therefore the Euler–Lagrange equation for this functional

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u'_y}\right) = 0$$

is the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We have developed this relation in two dimensions, the extension to three dimension is obvious. Now we will use the notation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

$$|\boldsymbol{\nabla} u|^2 = \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u = \left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y}\right) \cdot \left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2,$$

so that all the results will be automatically valid for three dimensions.

What we have shown is that the function u that minimizes the functional

$$I = \int\int |\boldsymbol{\nabla} u|^2 \, \mathrm{d}x\mathrm{d}y$$

will also be the solution of the Laplace equation

$$\nabla^2 u = 0.$$

Now we turn it around, saying that to solve the Laplace equation with some boundary conditions is to find the function satisfying the same boundary conditions that minimizes the functional.

One way of finding the minimizing function is first to approximate it with a trial function with many terms

$$u(x,y) = f_0(x,y) + c_1 f_1(x,y) + \ldots + c_n f_n(x,y). \tag{7.36}$$

Then adjust the coefficients $c_1, c_2, \ldots, c_n$ so that the functional is as small as possible. Note that, as long as the trial function satisfies the boundary conditions, one additional term will make it closer to the true minimizing function. This is because the trial function that has the term $c_{n+1} f_{n+1}(x,y)$ automatically includes all the previous terms. If the minimizing process cannot make the functional smaller than the previous minimum, it will make $c_{n+1} = 0$ and settle with the previous minimum. Therefore by including more and more nonzero terms, the trial function will get closer and closer to the true solution.

To illustrate this process, let us take three terms

$$u(x,y) = f_0(x,y) + c_1 f_1(x,y) + c_2 f_2(x,y).$$

Putting it in the functional, we have

$$I = \int\int F \, \mathrm{d}x\mathrm{d}y,$$

$$\begin{aligned} F &= \boldsymbol{\nabla}(f_0 + c_1 f_1 + c_2 f_2) \cdot \boldsymbol{\nabla}(f_0 + c_1 f_1 + c_2 f_2) \\ &= |\boldsymbol{\nabla} f_0|^2 + c_1^2 |\boldsymbol{\nabla} f_1|^2 + c_2^2 |\boldsymbol{\nabla} f_2|^2 + 2c_1 \boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla} f_1 \\ &\quad + 2c_2 \boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla} f_1 + 2c_1 c_2 \boldsymbol{\nabla} f_1 \cdot \boldsymbol{\nabla} f_2. \end{aligned}$$

To minimize it, we have to set the following derivatives to zero.

$$\frac{\partial I}{\partial c_1} = \int\int \left\{ \mathbf{2}c_1 \left|\boldsymbol{\nabla} f_1\right|^2 + 2\boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla} f_1 + 2c_2 \boldsymbol{\nabla} f_1 \cdot \boldsymbol{\nabla} f_2 \right\} \mathrm{d}x\mathrm{d}y = 0,$$

$$\frac{\partial I}{\partial c_2} = \int\int \left\{ \mathbf{2}c_2 \left|\boldsymbol{\nabla} f_2\right|^2 + 2\boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla} f_2 + 2c_1 \boldsymbol{\nabla} f_1 \cdot \boldsymbol{\nabla} f_2 \right\} \mathrm{d}x\mathrm{d}y = 0.$$

These two equations can be put in the form of

$$a_{11}c_1 + a_{12}c_2 = b_1,$$
$$a_{21}c_1 + a_{22}c_2 = b_2$$

with

$$a_{11} = \int\int \left|\boldsymbol{\nabla} f_1\right|^2 \mathrm{d}x\mathrm{d}y, \qquad a_{12} = \int\int \boldsymbol{\nabla} f_1 \cdot \boldsymbol{\nabla} f_2 \mathrm{d}x\mathrm{d}y,$$

$$a_{21} = \int\int \boldsymbol{\nabla} f_1 \cdot \boldsymbol{\nabla} f_2 \mathrm{d}x\mathrm{d}y, \qquad a_{22} = \int\int \left|\boldsymbol{\nabla} f_2\right|^2 \mathrm{d}x\mathrm{d}y,$$

$$b_1 = -\int\int \boldsymbol{\nabla} f_1 \cdot \boldsymbol{\nabla} f_0 \mathrm{d}x\mathrm{d}y, \qquad b_2 = -\int\int \boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla} f_2 \mathrm{d}x\mathrm{d}y.$$

Therefore

$$c_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

It is clear that with the trial function given by (7.36), the coefficients c_i are determined by the system of n linear equations

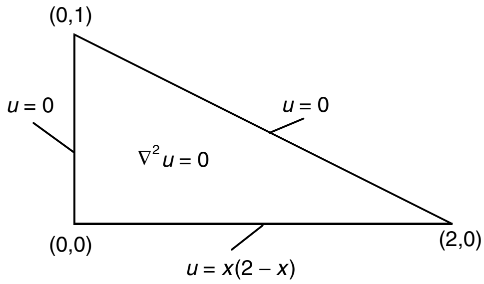
$$\sum_{j=1}^{n} a_{ij}c_j = b_i, \quad i = 1, 2, \ldots, n,$$

where

$$a_{ij} = \int\int \boldsymbol{\nabla} f_i \cdot \boldsymbol{\nabla} f_j \mathrm{d}x\mathrm{d}y, \quad b_i = -\int\int \boldsymbol{\nabla} f_0 \cdot \boldsymbol{\nabla} f_i \mathrm{d}x\mathrm{d}y.$$

The true solution can be approached by ever larger n. With a computer, such calculations are not difficult to carry out.

Example 7.6.1. Find a three term approximation to the solution of the Laplace's equation with the boundary conditions in the region shown in the following figure.



Solution 7.6.1. The equation for the straight line going through $(0,1)$ and $(2,0)$ is $x = 2-2y$. Therefore the region is bounded by the lines $x = 0,\ y = 0$ and $x = 2-2y$. the boundary conditions are

$$u(0,y) = 0, \quad u(x,0) = x(2-x), \quad u(2-2y,y) = 0.$$

It is readily seen that the following simple function satisfies these boundary conditions:

$$f_0(x,y) = x(2-x-2y).$$

It is clear that functions in the form of

$$u(x,y) = x(2-x-2y)(1+c_1y+c_2y^2)$$

will also satisfy the boundary conditions. Writing it in the form of

$$u(x,y) = f_0(x,y) + c_1f_1(x,y) + c_2f_2(x,y),$$

we have

$$f_1(x,y) = yx(2-x-2y), \quad f_2(x,y) = y^2x(2-x-2y).$$

Carrying out the integration, we find

$$a_{11} = \int_0^1 \int_0^{2-2y} |\nabla f_1|^2 \,\mathrm{d}x\mathrm{d}y = \frac{2}{9},$$

$$a_{12} = a_{21} = \int_0^1 \int_0^{2-2y} \nabla f_1 \cdot \nabla f_2 \mathrm{d}x\mathrm{d}y = \frac{22}{315},$$

$$a_{22} = \int_0^1 \int_0^{2-2y} |\nabla f_2|^2 \,\mathrm{d}x\mathrm{d}y = \frac{11}{315},$$

$$b_1 = -\int_0^1 \int_0^{2-2y} \nabla f_0 \cdot \nabla f_1 \mathrm{d}x\mathrm{d}y = -\frac{2}{15},$$

$$b_2 = -\int_0^1 \int_0^{2-2y} \nabla f_0 \cdot \nabla f_2 \mathrm{d}x\mathrm{d}y = -\frac{28}{143}.$$

Therefore

$$c_1 = \frac{\begin{vmatrix} -\frac{2}{15} & \frac{22}{315} \\ -\frac{28}{143} & \frac{11}{315} \end{vmatrix}}{\begin{vmatrix} \frac{2}{9} & \frac{22}{315} \\ \frac{22}{315} & \frac{11}{315} \end{vmatrix}} = -\frac{7}{13}, \quad c_2 = \frac{\begin{vmatrix} \frac{2}{9} & -\frac{2}{15} \\ \frac{22}{315} & -\frac{28}{143} \end{vmatrix}}{\begin{vmatrix} \frac{2}{9} & \frac{22}{315} \\ \frac{22}{315} & \frac{11}{315} \end{vmatrix}} = -\frac{28}{143}.$$

Thus, the three term approximation to the solution of the Laplace equation is

$$u(x, y) = x(2 - x - 2y)\left(1 - \frac{7}{13}y - \frac{28}{143}y^2\right).$$

7.6.2 Poisson's Equation

It is easy to show that the Poisson's equation

$$\nabla^2 u = \rho$$

is the Euler–Lagrange equation of the functional

$$I = \int\int \left[|\boldsymbol{\nabla} u|^2 + 2u\rho\right] \mathrm{d}x\mathrm{d}y. \tag{7.37}$$

Since the integrand of this functional is

$$F = u_x'^2 + u_y'^2 + 2u\rho,$$

the Euler–Lagrange equation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x'}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y'}\right) = 0$$

is clearly

$$2\rho - 2\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y^2} = 0$$

which is identical to the Poisson's equation $\nabla^2 u = \rho$.

Therefore to solve this Poisson's equation with some boundary conditions is equivalent to finding the function that satisfies the same boundary conditions and minimizes the corresponding functional (7.37).

Again we can approximate the solution with a trial function

$$u(x, y) = c_1 f_1(x, y) + c_2 f_2(x, y) + \cdots + c_n f_n(x, y). \tag{7.38}$$

The same method as we used in solving the Laplace's equation can be used to determine the coefficients $c_1, c_2, \ldots, c_n$. However, There is one difference.

Note that in (7.38) there is no $f_0(x,y)$ term with $c_0 = 1$. This is because any constant times the solution of a Laplace equation is still a solution. In other words, the coefficients in (7.36) are determined only up to a multiplicative constant. Therefore we can arbitrarily assign the coefficient of $f_0(x,y)$ to be one. We have no such freedom in solving the Poisson's equation because of the presence of the nonhomogeneous term $\rho(x,y)$. Therefore the trial function for the Poisson's equation has to start with the term $c_1 f_1(x,y)$.

For example, suppose we want to solve the following problem:

$$\begin{aligned} \nabla^2 u &= \rho(x,y), \quad 0 \le x \le 1,\ 0 \le y \le 1, \\ u &= 0, \quad \text{on the boundary of the square.} \end{aligned}$$

We may choose a trial function in the form of

$$u(x,y) = xy(1-x)(1-y)(c_1 + c_2 x + c_3 y + c_4 x^2 + \cdots),$$

which clearly satisfies the boundary conditions. This function can be written in the form (7.38) with

$$f_1 = xy(1-x)(1-y), \quad f_2 = x f_1, \quad f_3 = y f_1, \quad f_4 = x^2 f_1, \ldots .$$

Put it into the functional

$$I = \int_0^1 \int_0^1 \left[\left| \nabla \sum_{j=1}^{n} c_j f_j \right|^2 + 2 \left(\sum_{j=1}^{n} c_j f_j \right) \rho \right] \mathrm{d}x \mathrm{d}y.$$

To minimize it, we set the derivatives with respect to c_j to zero

$$\frac{\partial I}{\partial c_1} = 0, \frac{\partial I}{\partial c_2} = 0, \ldots, \quad \frac{\partial I}{\partial c_n} = 0.$$

The result is a system of n linear equations

$$\sum_{j=1}^{n} a_{ij} c_j = b_i, \quad i = 1, 2, \ldots, n,$$

where

$$a_{ij} = \int_0^1 \int_0^1 \boldsymbol{\nabla} f_i \cdot \boldsymbol{\nabla} f_j \mathrm{d}x \mathrm{d}y, \quad b_i = -\int_0^1 \int_0^1 f_i(x,y) \rho(x,y) \mathrm{d}x \mathrm{d}y.$$

This matrix equation can be solved for the set of coefficients $c_1, c_2, \ldots,\ c_n$. One strategy would be to calculate the functional with larger and larger n, until it is stabilized. This way we can get the approximation as close to the true solution as we want. Such calculations would have to be carried out on a computer.

7.6.3 Helmholtz's Equation

The Helmholtz's equation with boundary conditions is an eigenvalue problem. The Rayleigh–Ritz method developed for Sturm–Liouville problems can be used to obtain the solution. Consider the two-dimensional problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda u. \tag{7.39}$$

Multiplying both sides by u from the left and integrating, we have

$$\int\int u \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u \, \mathrm{d}x\mathrm{d}y = \int\int \lambda u^2 \mathrm{d}x\mathrm{d}y.$$

If u is not an eigenfunction, we can show that

$$\lambda = \frac{\displaystyle\int\int u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \mathrm{d}x\mathrm{d}y}{\int\int u^2 \mathrm{d}x\mathrm{d}y} = \frac{\int\int u \nabla^2 u \, \mathrm{d}x\mathrm{d}y}{\int\int u^2 \mathrm{d}x\mathrm{d}y} \tag{7.40}$$

is an upper bound of the lowest eigenvalue by expanding u in terms of the eigenfunctions, as we did in the one-dimensional case.

In the context of variational principle, $\lambda[u]$ is a functional. One can show that to minimize $\lambda[u]$ is equivalent to minimizing the following functional (see exercise 11):

$$K[u] = \int\int \left[u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \lambda u^2 \right] \mathrm{d}x\mathrm{d}y.$$

The integrand F of this functional is

$$F = u(u''_x + u''_y) - \lambda u^2,$$

so

$$\frac{\partial F}{\partial u} = \left(u''_x + u''_y \right) - 2\lambda u,$$

$$\frac{\partial F}{\partial u'_x} = \frac{\partial F}{\partial u'_y} = 0, \quad \frac{\partial F}{\partial u''_x} = \frac{\partial F}{\partial u''_y} = u.$$

Therefore the Euler–Lagrange equation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u'_x} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial u''_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u'_y} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial u''_y} \right) = 0$$

for this functional becomes

$$(u''_x + u''_y) - 2\lambda u + u''_x + u''_y = 0,$$

which is identical to the original equation

$$\nabla^2 u = \lambda u.$$

It is interesting to note that the Euler–Lagrange equation for another functional

$$K[u] = \int\int \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \lambda u^2 \right] \mathrm{d}x\mathrm{d}y \tag{7.41}$$

is also the Helmholtz's equation (7.39). The integrand F of this functional is

$$F = (u'_x)^2 + (u'_x)^2 + \lambda u^2.$$

Since

$$\frac{\partial F}{\partial u} = 2\lambda u, \quad \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'_x}\right) = 2u''_x, \quad \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u'_y}\right) = 2u''_y,$$

the Euler–Lagrange equation becomes

$$2\lambda u - 2u''_x - 2u''_y = 0,$$

which is identical to (7.39).

Since to minimize the functional of (7.41) is equivalent to minimizing

$$\lambda = \frac{-\int\int \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] \mathrm{d}x\mathrm{d}y}{\int\int u^2 \mathrm{d}x\mathrm{d}y} = \frac{-\int\int |\boldsymbol{\nabla} u|^2 \, \mathrm{d}x\mathrm{d}y}{\int\int u^2 \mathrm{d}x\mathrm{d}y}, \tag{7.42}$$

we can approximate the ground state energy by either (7.40) or (7.42).

This is not surprising. When the boundary values of u are specified, one can use divergence theorem to show that $\int\int u\nabla^2 u \, \mathrm{d}x\mathrm{d}y$ and $-\int\int |\boldsymbol{\nabla} u|^2 \, \mathrm{d}x\mathrm{d}y$ differ at most by a constant. Since

$$\int\int \boldsymbol{\nabla} \cdot (u\boldsymbol{\nabla} u)\mathrm{d}x\mathrm{d}y = \oint u\boldsymbol{\nabla} u \cdot \mathbf{n} \, \mathrm{d}l = \text{constant},$$

where the line integral is along the boundary of the area. But

$$\boldsymbol{\nabla} \cdot (u\boldsymbol{\nabla} u) = \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u + u\nabla^2 u.$$

Therefore

$$\int\int u\nabla^2 u \, \mathrm{d}x\mathrm{d}y = -\int\int |\boldsymbol{\nabla} u|^2 \, \mathrm{d}x\mathrm{d}y + \text{ constant}.$$

Thus if u minimizes (7.40), it must also minimize (7.42). Usually it is (7.42) that is more convenient.

Example 7.6.2. Use the variational method to estimate the lowest vibrational frequency of circular membrane of radius c.

Solution 7.6.2. As we have learned in Chap. 6 that the vibration is governed by the wave equation

$$\nabla^2\phi = \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\phi.$$

The time-dependent part can be separated out to give $T(t) = \cos\omega t$, where $\omega = 2\pi\nu$ and ν is the frequency. The space part is then governed by the Helmholtz's equation

$$\nabla^2 u(x,y) = -\frac{\omega^2}{a^2}u(x,y).$$

Therefore the frequency of the vibration of any normal mode is determined by the eigenvalue of this equation. The boundary condition of u is that it is zero on the rim of the circular membrane. According to (7.42),

$$\frac{\omega^2}{a^2} = \frac{\int\int |\boldsymbol{\nabla} u|^2 \,\mathrm{d}x\mathrm{d}y}{\int\int u^2\mathrm{d}x\mathrm{d}y}.$$

Any u, as long as it satisfies the boundary condition will give an upper limit to the lowest frequency. For a circular membrane, clearly it is more convenient to do the integration in the polar coordinates. Written in polar coordinates, the boundary condition of $u(r,\theta)$ is

$$u(c,\theta) = 0.$$

The simplest trial function satisfying this boundary condition is

$$u(r,\theta) = r - c.$$

In polar coordinates

$$|\boldsymbol{\nabla} u|^2 = \left(\frac{\partial}{\partial r}(r-c)\right)^2 = 1.$$

Thus

$$\int\int |\boldsymbol{\nabla} u|^2 \,\mathrm{d}x\mathrm{d}y = \int_0^c\int_0^{2\pi} 1\cdot r\,\mathrm{d}\theta\mathrm{d}r = \pi c^2,$$

$$\int\int u^2\mathrm{d}x\mathrm{d}y = \int_0^c\int_0^{2\pi} (r-c)^2 r\,\mathrm{d}\theta\mathrm{d}r = \frac{1}{6}\pi c^4.$$

Hence

$$\frac{\omega^2}{a^2} = \frac{6}{c^2}, \quad \omega = 2.449\frac{a}{c}.$$

We have shown in Chap. 6, the exact value of numerical factor is given by the first zero of $J_0(x)$ which is 2.405. It is seen that even with such a simple trial function, we still can get a reasonable estimate.

7.7 Hamilton's Principle

The Euler–Lagrange equation for the functional

$$I = \int_{t_1}^{t_2} F(y, y', t)\mathrm{d}t = \int_{t_1}^{t_2} (py'^2 - qy^2)\mathrm{d}t \tag{7.43}$$

is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 2py'' + 2qy = 0. \tag{7.44}$$

Let

$$p = \frac{1}{2}m, \quad q = \frac{1}{2}k,$$

then (7.44) becomes

$$my'' + ky = 0,$$

which we recognize as the equation of a harmonic oscillator with mass m and spring constant k. Put the same values of p and q into (7.43), we find

$$I = \int_{t_1}^{t_2} \left(\frac{1}{2}my'^2 - \frac{1}{2}ky^2\right) \mathrm{d}t.$$

It is readily seen the first term is the kinetic energy T and the second term is the potential energy V of the harmonic oscillator

$$T = \frac{1}{2}my'^2, \quad V = \frac{1}{2}ky^2.$$

Thus the functional can be written as

$$I = \int_{t_1}^{t_2} (T - V)\mathrm{d}t. \tag{7.45}$$

Therefore for an harmonic oscillator, we have found that the Newton's equation of motion is identical with the Euler–Lagrange equation for the functional of (7.45). This is just a special case of a general principle known as Hamilton's principle. It was first announced in 1834 by the brilliant Irish mathematician William Rowan Hamilton.

The difference between kinetic energy and potential energy $T - V$ is denoted by L and is called the Lagrangian. Hamilton's principle states that the motion of a system from t_1 to t_2 is such that the time integral of the Lagrangian L, (7.45), known as "action," has a stationary value. The Lagrangian is specified by a set of "generalized" coordinates $q_1, q_2, \ldots, q_n$ and their time derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$. From here on, we will follow the convention in mechanics, a dot on top means derivative with respect to time (Newton's notation). The Euler–Lagrange equations for the action functional (7.45) are usually called simply Lagrangian equations,

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, \ldots, n.$$

These are a set of n simultaneous second-order differential equations. These equations were independently developed by Joseph L. Lagrange (1736–1816). They are equivalent to Newton's equations of motion. The Lagrange equations deal only with scalar variables, whereas Newton's equations are intrinsically vector equations. In many situations it is much easier to generate the correct differential equations of motion from the Lagrangian equations than from Newton's equation.

For example, suppose that a particle of mass m is moving in an arbitrary potential field. Then

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2), \quad V = V(x_1, x_2, x_3)$$

and

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(x_1, x_2, x_3).$$

The Lagrangian equations give us

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = -\frac{\partial V}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}_i) = 0, \quad i = 1, 2, 3$$

or

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i}.$$

Since $-\partial V/\partial x_i$ is the force on the particle in the x_i direction, this is simply Newton's second law which may be written in vector form as

$$m\ddot{\mathbf{r}} = \mathbf{F}.$$

From the following examples, we will see that Hamilton's principle is equally valid for system of continuum, and the "generalized" q_is need not be any standard coordinate set. They can be selected to match the conditions of the physical problem.

Example 7.7.1. Use Hamilton's principle to derive the wave equation for small transverse oscillations of a taut string.

Solution 7.7.1. Let ρ and τ be the linear density and tension of the oscillating string shown in Fig. 7.13.

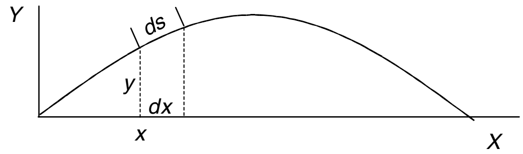


Fig. 7.13. Small oscillations of a taut string

Since the transverse speed of any part of the string is $\partial y/\partial t$, we can easily determine the kinetic energy of the vibration. It is

$$T = \int_0^L \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x.$$

The potential energy V is found by considering the increase of length of the element $\mathrm{d}x$. This element has increased its length from $\mathrm{d}x$ to $\mathrm{d}s$. We have therefore done an amount of work $\tau(\mathrm{d}s - \mathrm{d}x)$. Since the potential energy is equal to the work done, summing all the work done along the line, we have

$$V = \int_0^L \tau(\mathrm{d}s - \mathrm{d}x).$$

But

$$\mathrm{d}s - \mathrm{d}x = [(\mathrm{d}x)^2 + (\mathrm{d}y)^2]^{1/2} - \mathrm{d}x = \left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{1/2} \mathrm{d}x - \mathrm{d}x,$$

and

$$\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{1/2} = 1 + \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \cdots.$$

Since it is a small oscillation, we will take the first two terms. Thus

$$\mathrm{d}s - \mathrm{d}x = \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x$$

and

$$V = \int_0^L \frac{1}{2}\tau \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x.$$

Therefore the Lagrangian is given by

$$L = T - V = \int_0^L \left[\frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] \mathrm{d}x$$

and the action integral becomes

$$I = \int_{t_1}^{t_2} \int_0^L \left[\frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] \mathrm{d}x\mathrm{d}t.$$

This is a two-dimensional functional. The independent variables are x and t. The integrand L can be written in the form of

$$L = \frac{1}{2}\rho y_t'^2 - \frac{1}{2}\tau y_x'^2.$$

The Lagrangian equation is

$$\frac{\partial L}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_t'}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial y_x'}\right) = 0,$$

which becomes

$$-\rho\frac{\partial}{\partial t}y_t' + \tau\frac{\partial}{\partial x}y_x' = -\rho\frac{\partial}{\partial t}\frac{\partial y}{\partial t} + \tau\frac{\partial}{\partial x}\frac{\partial y}{\partial x} = 0.$$

This is exactly the same wave equation we derived before

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 y}{\partial t^2}.$$

Example 7.7.2. (a) Find the angular acceleration of a pendulum of length l. (b) A bead of mass m slides freely on a frictionless circular wire of radius r that rotates in a horizontal plane about a point on the circular wire with a constant angular velocity ω. Show that the bead oscillates as a pendulum of length $l = g/\omega^2$ about the line joining the center of rotation and the center of the circle.

Solution 7.7.2. (a) It is clear from Fig. 7.14a that the coordinates of m are given by

$$x = l\cos\theta, \quad y = l\sin\theta.$$

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2(\sin^2\theta + \cos^2\theta)\dot{\theta}^2 = \frac{1}{2}ml^2\dot{\theta}^2.$$

Choosing the reference level for potential energy at distance l below the point of suspension, we have

$$V = mgl(1 - \cos\theta).$$

Thus the Lagrangian L is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta).$$

So we have only one independent variable θ, which is our "generalized" coordinate. Hence we have the Lagrangian equation

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

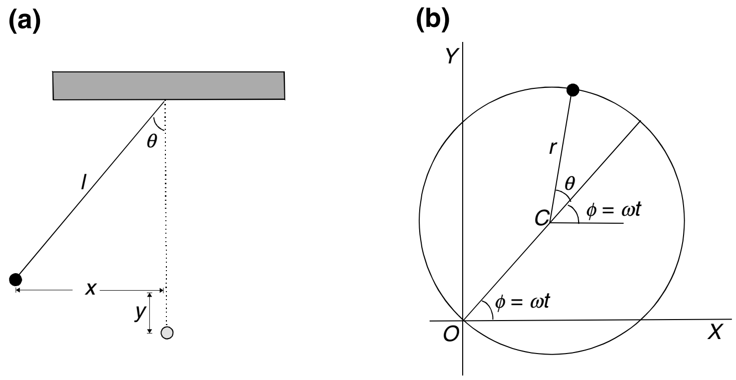


Fig. 7.14. Motions of pendulums

or

$$-mgl\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left(ml^2\dot{\theta}\right) = 0.$$

Therefore

$$\ddot{\theta} = -\frac{g}{l}\sin\theta.$$

(b) Let C be the center of the circular wire, and the angles θ and ϕ are indicated in the Fig. 7.14b. As the wire rotates counterclockwise with an angular velocity ω, so $\phi = \omega t$. The coordinates x and y of the bead are seen to be

$$x = r\cos\omega t + r\cos(\theta + \omega t),$$
$$y = r\sin\omega t + r\sin(\theta + \omega t).$$

Since the motion is taken place in a horizontal plan, the potential energy can be taken as zero. The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\ &= \frac{1}{2}mr^2\left[\omega^2 + \left(\dot{\theta}+\omega\right)^2 + 2\omega\left(\dot{\theta}+\omega\right)\cos\theta\right], \end{aligned}$$

which is also the Lagrangian L, since $V = 0$. Thus

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

becomes

$$mr^2\left[-\omega\left(\dot{\theta}+\omega\right)\sin\theta - \ddot{\theta} + \omega\sin\theta\dot{\theta}\right] = 0,$$

or

$$\ddot{\theta} = -\omega^2\sin\theta.$$

Thus we see that the bead oscillates about the line joining the center of rotation and the center of the circular wire like a pendulum of length $l = g/\omega^2$.

Hamilton's principle and Fermat's principle are only examples showing that the physical universe follows paths through space and time based on extrema principles. Almost in all branches of physics, one can find such a principle. Why the nature operates in accordance with this principle of economy is a question for philosophers and theologians. As scientists, we can just enjoy the elegance of the theory. However this is not to say that variation principle is merely a device to provide an alternative derivation of known results. In fact its impact on the development of science cannot be overemphasized. When the basic physics is not yet known, a postulated variational principle can be very useful. A shining example is Richard Feynman's formulation of quantum electrodynamics which is based on the principle of least action. For his achievements, he was awarded a 1965 Nobel prize in physics.

Variational principle as a computation tool is also very important. With variational methods, energy levels of all kinds of molecules can now be calculated to a high degree of accuracy. John Pople codified such calculations in a computer program known as GAUSSIAN. He was awarded a 1998 Nobel prize in chemistry.

Exercises

1. Find the Euler–Lagrange equation for

$$\text{(a) } F = x^2y^2 - y'^2, \quad \text{(b) } F = \sqrt{xy} + y'^2.$$

Ans. (a) $y'' + x^2y = 0$, (b) $\dfrac{1}{4}\sqrt{\dfrac{x}{y}} - y'' = 0.$

2. Find the curve $y(x)$ that will make the following functional stationary

$$\text{(a) } I = \int_a^b (y^2 + y'^2 + 2y\mathrm{e}^x)\mathrm{d}x,$$

$$\text{(b) } I = \int_a^b \frac{y'^2}{x^3}\mathrm{d}x.$$

Ans. (a) $y = \frac{1}{2}x\mathrm{e}^x + c_1\mathrm{e}^x + c_2\mathrm{e}^{-x}$, (b) $y = c_1x^4 + c_2$.

3. Find the function $y(x)$ that passes through the points $(0,0)$ and $(1,1)$ and minimizes

$$I(y) = \int_0^1 (y^2 + y'^2)\mathrm{d}x.$$

Ans. $y(x) = 0.42\mathrm{e}^x - 0.42\mathrm{e}^{-x}$.

4. Find the function $y(x)$ that passes through the points $(0,0)$ and $(\pi/2,1)$ and minimizes

$$I(y) = \int_0^1 (y'^2 - y^2)\mathrm{d}x.$$

Ans. $y(x) = \sin x$.

5. What would be the functional corresponding to the following problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1, \quad 0 < x < 1,\ 0 < y < 1,$$
$$u = 0, \quad \text{on the boundary}.$$

Ans. $I(u) = \int_0^1 \int_0^1 \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2u \right] \mathrm{d}x\mathrm{d}y.$

6. Show that if the integrand of the following integral:

$$I = \int_{t_1}^{t_2} F(x, y, x', y')\mathrm{d}t$$

does not explicitly contain the independent variable t, then the Euler–Lagrange equations lead to

$$F - x'\frac{\partial F}{\partial x'} - y'\frac{\partial F}{\partial y'} = C,$$

where C is a constant.

7. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 (yy'' + 4y)\mathrm{d}x.$$

Ans. $y'' + 2 = 0$.

8. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 (-y'^2 + 4y)\mathrm{d}x.$$

Ans. $y'' + 2 = 0$.

9. Show that the Euler–Lagrange equation for the three-dimensional functional

$$I = \iiint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

is given by the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

10. Estimate the lowest vibrational frequency of a circular drum-head with radius a, using the functional

$$\frac{\omega^2}{v^2} = \frac{-\int\int u\nabla^2 u \, \mathrm{d}x\mathrm{d}y}{\int\int u^2 \mathrm{d}x\mathrm{d}y}$$

and the trial function

$$u(r) = r - a.$$

Ans. $\omega = 2.449v/a$.

11. If $I[u]$ and $J[u]$ are both two-dimensional functionals and

$$\lambda[u] = \frac{I[u]}{J[u]},$$

show that to minimize $\lambda[u]$ is equivalent to minimizing the functional $K[u]$

$$K[u] = I[u] - \lambda J[u].$$

Hint: Replace $u(x, y)$ by $U(x, y) + \alpha\eta(x, y)$, and show that $\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$ leads to $\left[\frac{\mathrm{d}I}{\mathrm{d}\alpha} - \lambda\frac{\mathrm{d}J}{\mathrm{d}\alpha}\right]_{\alpha=0} = 0.$

12. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 xy'^2 \mathrm{d}x$$

subject to the constraint

$$\int_0^1 xy^2 \mathrm{d}x = 1.$$

Ans. $xy'' + y' - \lambda xy = 0$.

13. Find the Euler–Lagrange equation for the functional

$$I = \int_0^1 (py'^2 - qy^2)\mathrm{d}x$$

subject to the constraint

$$\int_0^1 ry^2 \mathrm{d}x = 1.$$

Ans. $\frac{\mathrm{d}}{\mathrm{d}x}(py') + (q - \lambda r)y = 0$.

14. Show the equivalence of the following two forms of Euler–Lagrange equations:

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0.$$

15. Approximate the solution of the problem

$$y'' + \left(\frac{\pi}{2}\right)^2 y = 0,$$
$$y(0) = 1, \quad y(1) = 0$$

with a trial function

$$y = 1 - x^2.$$

With this trial function, find the eigenvalue and compare it with the exact value.

Ans. $\lambda = 2.5, \quad \lambda/\lambda_{\mathrm{exat}} = 1.013.$

16. In the previous problem, use a trial function

$$y = 1 - x^n.$$

Find the optimum value of n. With that n, what is $\lambda/\lambda_{\mathrm{exat}}$?

Ans. $n = 1.7247, \quad \lambda/\lambda_{\mathrm{exat}} = 1.003.$

17. Find the function $y(x)$ that will extremize the integral

$$I = \int_0^a y'^2 \mathrm{d}x$$

subject to the constraint

$$\int_0^a y^2 \mathrm{d}x = 1, \quad y(0) = 0, \quad y(a) = 0.$$

Ans. $y(x) = \left(\frac{2}{a}\right)^{1/2} \sin \frac{n\pi}{a} x.$

18. Use the Fermat principle to find the path followed by a light ray if the index of refraction is proportional to

$$\text{(a)} \;\; y^{-1}, \quad \text{(b)} \;\; y.$$

Ans. (a) $(x - c_1)^2 + y^2 = c_2^2$, (b) $y = c_1 \cosh \frac{x - c_2}{c_1}$.

19. Use a trial function of the form

$$u = (r - c) + b(r - c)^2$$

to calculate the lowest frequency of the vibration of a circular membrane of radius c.

Ans. $\omega = 2.4203\ a/c$.

20. **Conservation of energy.** If

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{q}_i^2, \quad V = V(q_1, q_2, \dots, q_n)$$

use Hamilton's principle to show that

$$T + V = \text{constant}.$$

Hint: From the fact that the independent variable t does not appear explicitly in the integrand, show that

$$L - \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \text{constant}.$$

21. Derive Lagrangian equation of motion for a particle in a gravitation field constraint to be on a circle of radius c in a fixed vertical plane.

Ans. $\dfrac{\mathrm{d}}{\mathrm{d}t}(mc^2\dot{\theta}) + mgc\cos\theta = 0.$